Complex Analysis and Operator Theory (2024) 18:109
https://doi.org/10.1007/s11785-024-01545-x

Complex Analysis and Operator Theory

The Schur–Potapov Algorithm in the General Matrix Case and Its Application to the Matricial Schur Problem

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Received: 11 June 2023 / Accepted: 29 April 2024

Abstract

This paper is a generalization of the topic handled in Bogner et al. (Oper Theory 1(1):55–95, 2007a, Oper Theory 1(2):235–278, 2007b) where the Schur–Potapov algorithm (SP-algorithm) was handled in the context of non-degenerate $p \times q$ Schur sequences and non-degenerate $p \times q$ Schur functions. In particular, the interplay between both types of algorithms was intensively studied there. This was itself a generalization of the classical Schur algorithm (Schur in J Reine Angew Math 148:122–145, 1918) to the non-degenerate matrix case. In treating the matrix case a result due to Potapov (Potapov in Trudy Moskov Mat Obšč 4:125–236, 1955) concerning particular linear fractional transformations of contractive $p \times q$ matrices was used. For this reason, the notation SP-algorithm was already chosen in Dubovoj et al. (Matricial version of the classical Schur problem, volume 129 of Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992). We are going to introduce both types of SP-algorithms as well for arbitrary $p \times q$ Schur sequences as for arbitrary $p \times q$ Schur functions. Again we will intensively discuss the interplay between both types of algorithms. Applying the SP-algorithm, a complete treatment of the matricial Schur problem in the most general case is established. A one-step extension problem for finite $p \times q$ Schur sequences is considered. Central $p \times q$ Schur sequences are studied under the view of SP-parameters.

Keywords Schur–Potapov algorithm · Schur–Potapov transform · Schur parameters · Matricial Schur problem

Mathematics Subject Classification 30E05 · 47A57

Dedicated to the memory of V. E. Katsnelson.

Communicated by Daniel Alpay.

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1 Introduction

In this paper, a topic will be studied again, which was studied in the former work on Schur analysis methods by the first three authors (see [11, 12, 14]). In the background of these considerations was the discussion of a matricial version of the classical Schur problem. The most complete result could be achieved in that time for the so-called non-degenerate case. The main goal of this paper is a treatment of the general matrix case by an appropriate adaption of the classical algorithm due to I. Schur [28, 29] and its matricial generalization going back to ideas of V. P. Potapov [27]. We are guided by our former investigations on matricial versions of truncated power moment problems. The essential feature of this concept can be described as a detailed study of the structure of the sequence of moment matrices using Schur type algorithms on the one side combined with the construction of concordant Schur type algorithms for various classes of holomorphic matrix-valued functions in several domains which are determined by the choice of the moment problem under consideration. This method enabled a simultaneous treatment of both non-degenerate and degenerate cases of the moment or interpolation problem under consideration. By a careful analysis of the interplay between two versions of Schur algorithm a complete description of the solution set of the moment problem via Stieltjes transformation could be achieved. Roughly speaking, some features of this approach are already contained in the famous landmark papers [28, 29] by I. Schur who more concentrated on the function-theoretic version of the algorithm named after him, however also sketched some ideas on the algebraic version. In the non-degenerate case, a first systematic treatment of both types of Schur algorithms and their interplay was established in [6, 7]. It should be mentioned that the matricial version of the Schur algorithm for strict Schur functions was also considered in Cedzich [8, formulas (4.1), (4.2)] under the view of generalizing fundamental relations found in the scalar case by S. V. Khrushchev (see [24–26]) to the matrix case.

The main goal of this paper is to extend these methods for arbitrary matricial Schur functions defined on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. Roughly speaking, the content of this paper can be summarized as follows. In Sect. 2, we introduce some notation. In particular, we state some facts on matricial $p \times q$ Schur sequences and matricial Schur functions. In Sect. 3, we define a Schur–Potapov transform (shortly SP-transform) for arbitrary sequences of complex $p \times q$ matrices. As in [6, 7], we consider first as well a right as a left version of the SP-transform. Although we will prove later that both versions coincide (see Proposition 3.19), both representations prove to be useful. An essential aspect is that the SP-transform transforms $p \times q$ Schur sequences into $p \times q$ Schur sequences (see Proposition 3.24). This will be used in Sect. 4 in order to iterate the SP-transform of $p \times q$ Schur sequences. This leads us to a SP-algorithm for $p \times q$ Schur sequences. Intimately connected with this SP-algorithm is the explicitly constructed sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ of SP-parameters of a $p \times q$ Schur sequence $(A_j)_{j=0}^{\kappa}$ (see Definition 4.7). In Sect. 5, we discuss an inverse SP-transform for sequences of complex matrices. We consider again first a left version and right version of inverse SP-transforms before we see that both versions coincide (see Proposition 5.9). Observe that both representations prove to be useful for further considerations. The inverse SP-transform maps $p \times q$ Schur sequences into $p \times q$ Schur sequences (see Proposition 5.11). Section 6 is aimed to work out a convenient parametrization of finite matricial Schur sequences (see Theorem 6.20). In Sect. 7, we introduce the SP-transform for matricial Schur functions. Section 8 is aimed to recognize the concordance between SP-transforms of matricial Schur functions and SP-transforms of matricial Schur sequences (see Theorem 8.6). In Sect. 9, we introduce a SP-algorithm for $p \times q$ Schur functions. We show that the SP-parameter sequences of a $p \times q$ Schur functions and the SP-parameter sequences of its Taylor coefficient sequence coincide (see Proposition 9.7). In Sect. 10 we discuss the inverse SP-transform for Schur functions. In Sect. 11, we prove that there is a complete concordance of the inverse SP-transform of $p \times q$ Schur functions and of the inverse SP-transform of infinite $p \times q$ Schur sequences (see Propositions 11.2 and 11.4). In Sects. 12 and 13, we apply the preceding considerations on the SP-algorithm to the matricial Schur problem in order to parametrize the solution set of this interpolation problem (see Theorem 12.7). We rewrite the description of the solution set of the matricial Schur problem in terms of linear fractional transformations of matrices (see Theorems 13.3 and 13.5). In Sect. 14, we express the Taylor coefficients of a $p \times q$ Schur functions only in terms of its SP-parameters. In Sect. 15, we turn our attention to the extension problem for finite $p \times q$ Schur sequences. In [11, 14], we described the solution set of this problem as a closed matrix ball which is given in terms of Taylor coefficients. Now we obtain a description of the solution set as a closed matrix ball which is written with the aid of the SP-parameter sequences (see Theorem 15.23). In Theorem 16.3 we present explicit formulas between the SP-parameters and the choice sequence (see Definition 15.4) corresponding to a $p \times q$ Schur sequence. The final Sects. 17 and 18 are dedicated to the characterization of central and completely degenerate matricial Schur functions and sequences, respectively, in terms of their SP-parameters.

At the end of the paper some appendices on several results about matrices, linear subspaces, and linear fractional transformations of matrices are given.

2 Preliminaries

Throughout this paper, let p and q be positive integers. We will use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ to denote the set of all complex numbers, the set of all integers, the set of all non-negative integers, and the set of all positive integers, respectively. Further, let $\mathbb{D}$ be the open unit disk of the complex plane, i. e., $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. If $\upsilon, \omega \in \mathbb{Z} \cup \{-\infty, \infty\}$, then $\mathbb{Z}_{\upsilon,\omega}$ designates the set of all integers n which fulfill $\upsilon \le n \le \omega$. If $\mathfrak{X}$ is a non-empty set, then $\mathfrak{X}^{p\times q}$ denotes the set of all $p \times q$ matrices each entry of which belongs to $\mathfrak{X}$. The notation $O_{p\times q}$ stands for the null matrix which belongs to the set $\mathbb{C}^{p\times q}$ of all complex $p \times q$ matrices and the identity matrix which belongs to $\mathbb{C}^{q\times q}$ will be designated by I_q. If the size of an identity matrix or a null matrix is obvious, then we will omit the indices. Let $\mathbb{C}_{\mathrm{H}}^{q\times q}$ (resp., $\mathbb{C}_{\succcurlyeq}^{q\times q}$) be the set of all Hermitian (resp., non-negative Hermitian) complex $q \times q$ matrices. As usual, we write $A \succcurlyeq B$ or $B \preccurlyeq A$ if A and B are Hermitian complex $q \times q$ matrices fulfilling $A - B \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. For each $A \in \mathbb{C}^{p\times q}$, let $\mathcal{R}(A)$ be the range of A, let $\mathcal{N}(A)$ be the null space of A, let $\operatorname{rank}(A)$ be the rank of A, let $\|A\|$ be the operator norm of A, and let $\|A\|_{\mathrm{E}}$ be the Euclidean norm (or Frobenius norm) of A. A complex $p \times q$ matrix A is said to be contractive (resp., strictly contractive) if $\|A\| \leq 1$ (resp., $\|A\| < 1$) holds true. Observe that a complex $p \times q$ matrix A is contractive (resp., strictly contractive) if and only if $I - A^*A$ is non-negative Hermitian (resp., positive Hermitian). We use $\mathbb{K}_{p\times q}$ (resp., $\mathbb{D}_{p\times q}$) to denote the set of all contractive (resp., strictly contractive) complex $p \times q$ matrices. If $A \in \mathbb{C}^{q\times q}$, then $\det A$ stands for the determinant of A. For each matrix $A \in \mathbb{C}^{p\times q}$, let $A^\dagger$ be the Moore–Penrose inverse of A, i.e., the unique complex $q \times p$ matrix X, satisfying the four equations

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX, \qquad \text{and} \qquad (XA)^* = XA. \tag{2.1}$$

For all $x, y \in \mathbb{C}^q$, by $\langle x, y\rangle_{\mathrm{E}}$ we denote the (left-hand side) Euclidean inner product of x and y, i.e., we have $\langle x, y\rangle_{\mathrm{E}} := y^*x$. If $\mathcal{M}$ is a non-empty subset of $\mathbb{C}^q$, then let $\mathcal{M}^\perp$ be the set of all vectors in $\mathbb{C}^q$ which are orthogonal to $\mathcal{M}$ (with respect to $\langle ., .\rangle_{\mathrm{E}}$). If $\mathcal{U}$ is a linear subspace of $\mathbb{C}^q$, then let $\mathbb{P}_{\mathcal{U}}$ be the orthogonal projection matrix onto $\mathcal{U}$ (see also Remark A.3).

Throughout this paper, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Considering an arbitrary sequence $(A_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices, we use some further notation: We associate with $(A_j)_{j=0}^{\kappa}$ a collection of matrices. For each $n \in \mathbb{Z}_{0,\kappa}$, we define

$$\mathbf{S}_n := \begin{bmatrix} A_0 & O & \dots & O \\ A_1 & A_0 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ A_n & A_{n-1} & \dots & A_0 \end{bmatrix} \qquad \text{and} \qquad \mathring{\mathbf{S}}_n := \begin{bmatrix} O_{p\times(n+1)q} & O_{p\times q} \\ \mathbf{S}_n & O_{(n+1)p\times q} \end{bmatrix} \tag{2.2}$$

as well as the left and right defect matrices corresponding to $\mathbf{S}_n$, namely

$$L_n := I_{(n+1)p} - \mathbf{S}_n\mathbf{S}_n^* \qquad \text{and} \qquad R_n := I_{(n+1)q} - \mathbf{S}_n^*\mathbf{S}_n. \tag{2.3}$$

Further, let

$$m_{-1} := O_{p\times q}, \qquad m_0 := O_{p\times q}, \tag{2.4}$$

let

$$l_{-1} := I_p, \qquad l_0 := I_p - A_0A_0^*, \qquad r_{-1} := I_q, \qquad r_0 := I_q - A_0^*A_0, \tag{2.5}$$

and, if $\kappa \geq 1$, let

$$y_n := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}, \qquad z_n := [A_n, A_{n-1}, \dots, A_1], \tag{2.6}$$

let

$$m_n := -z_n \mathbf{S}_{n-1}^* L_{n-1}^\dagger y_n, \tag{2.7}$$

and let

$$l_n := I_p - A_0 A_0^* - z_n R_{n-1}^\dagger z_n^*, \qquad r_n := I_q - A_0^* A_0 - y_n^* L_{n-1}^\dagger y_n. \tag{2.8}$$

In view of (2.2), (2.3), and (2.5), we have $\mathbf{S}_0 = A_0$ as well as

$$L_0 = I_p - A_0 A_0^* = l_0 \qquad \text{and} \qquad R_0 = I_q - A_0^* A_0 = r_0. \tag{2.9}$$

Let

$$P_0 := I_p - l_0 l_0^\dagger \qquad \text{and} \qquad Q_0 := I_q - r_0^\dagger r_0. \tag{2.10}$$

The matrices P_0 and Q_0 are orthoprojections. Indeed, because of Remarks A.6, A.4 and A.2, we have

$$P_0 = \mathbb{P}_{\mathcal{R}(l_0)^\perp} \qquad \text{and} \qquad Q_0 = \mathbb{P}_{\mathcal{N}(r_0)}. \tag{2.11}$$

A finite sequence $(A_j)_{j=0}^n$ of complex $p \times q$ matrices with some $n \in \mathbb{N}_0$ is said to be a $p \times q$ *Schur sequence* (resp., *non-degenerate* $p \times q$ *Schur sequence*) if the block Toeplitz matrix $\mathbf{S}_n$ given by (2.2) is contractive (resp., strictly contractive). Obviously, if $n \in \mathbb{N}_0$ and if $(A_j)_{j=0}^n$ is a $p \times q$ Schur sequence (resp., non-degenerate $p \times q$ Schur sequence), then $(A_j)_{j=0}^k$ is a $p \times q$ Schur sequence (resp., non-degenerate $p \times q$ Schur sequence) for all $k \in \mathbb{Z}_{0,n}$ as well. A sequence $(A_j)_{j=0}^\infty$ of complex $p \times q$ matrices is said to be a $p \times q$ Schur sequence (resp., non-degenerate $p \times q$ Schur sequence) if for every non-negative integer n the sequence $(A_j)_{j=0}^n$ is a $p \times q$ Schur sequence (resp., non-degenerate $p \times q$ Schur sequence). We will use $\mathscr{S}_{p\times q;\kappa}$ to denote the set of all $p \times q$ Schur sequences $(A_j)_{j=0}^\kappa$. From Lemma A.15 one can see obviously that, if $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$, then $L_n \succcurlyeq O$ and $R_n \succcurlyeq O$ for all $n \in \mathbb{Z}_{0,\kappa}$. Conversely, Lemma A.15 also yields that if $m \in \mathbb{N}_0$ and if $(A_j)_{j=0}^m$ is such that $L_m \succcurlyeq O$ or $R_m \succcurlyeq O$, then $(A_j)_{j=0}^m$ belongs to $\mathscr{S}_{p\times q;m}$. If $(A_j)_{j=0}^\kappa$ is a sequence of complex $p \times q$ matrices then it is easily checked that $(A_j)_{j=0}^\kappa$ is a $p \times q$ Schur sequence (resp., non-degenerate $p \times q$ Schur sequence) if and only if $(A_j^*)_{j=0}^\kappa$ is a $q \times p$ Schur sequence (resp., non-degenerate $q \times p$ Schur sequence). A function F whose domain is a region $\mathcal{G}$ of $\mathbb{C}$ and whose values lie in $\mathbb{C}^{p\times q}$ is called $p \times q$ *Schur function (in* $\mathcal{G}$*)* if F is a holomorphic matrix-valued function the values of which are contractive $p \times q$ matrices. The class of all $p \times q$ Schur functions (in $\mathcal{G}$) is denoted by $\mathscr{S}_{p\times q}(\mathcal{G})$. We mainly consider the particular domain $\mathcal{G} = \mathbb{D}$, where $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ is the open unit disk of $\mathbb{C}$. In particular, we consider functions belonging to $[\mathcal{H}(\mathbb{D})]^{p\times q}$ where $\mathcal{H}(\mathbb{D})$ is the set of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$. If $F(w) = \sum_{j=0}^\infty w^j A_j$ for all $w \in \mathbb{D}$ is the Taylor series representation of a function $F \in [\mathcal{H}(\mathbb{D})]^{p\times q}$, then we call $(A_j)_{j=0}^\infty$ the Taylor coefficient sequence of F.

There is an intimate connection between $p \times q$ Schur sequences $(A_j)_{j=0}^{\infty}$ and the $p \times q$ Schur class $\mathscr{S}_{p\times q}(\mathbb{D})$. More precisely, note that a function $F\colon \mathbb{D} \to \mathbb{C}^{p\times q}$ which is holomorphic in $\mathbb{D}$ with Taylor series representation $F(w) = \sum_{j=0}^{\infty} w^j A_j$ for all $w \in \mathbb{D}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ if and only if $(A_j)_{j=0}^{\infty}$ is a $p \times q$ Schur sequence (see, e. g., [11, Thm. 3.1.1]). Let $f\colon \mathbb{D} \to \mathbb{C}^{p\times q}$. Then $f \in \mathscr{S}_{p\times q}(\mathbb{D})$ if and only if $f^{\vee} \in \mathscr{S}_{q\times p}(\mathbb{D})$, where $f^{\vee}\colon \mathbb{D} \to \mathbb{C}^{q\times p}$ is defined by $f^{\vee}(w) := [f(\overline{w})]^*$. The matricial version of the classical Schur problem can be formulated as follows:

> Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^{n}$ be a sequence of complex $p \times q$ matrices. Parametrize the set $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ of all $p \times q$ Schur functions F (in $\mathbb{D}$) such that $(j!)^{-1} F^{(j)}(0) = A_j$ is satisfied for all $j \in \mathbb{Z}_{0,n}$, where $F^{(j)}(0)$ is the j-th derivative of F at the point $w = 0$.

It is well known that if $n \in \mathbb{N}_0$ and if $(A_j)_{j=0}^{n}$ is a sequence of complex $p \times q$ matrices, then the set $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ is non-empty if and only if $(A_j)_{j=0}^{n}$ is a $p \times q$ Schur sequence (see, e. g., [11, Thm. 3.5.2]). In the case of a given non-degenerate $p \times q$ Schur sequence $(A_j)_{j=0}^{n}$, i. e., that the block Toeplitz matrix $\mathbf{S}_n$ given by (2.2) is even strictly contractive, there are various parametrizations of $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ via appropriately constructed linear fractional transformations (see, e. g., [2, 3, 13] or [11, Theorems 3.9.1 and 5.3.2]). The study of the degenerate case where the associated block Pick matrix is non-negative Hermitian and singular was started in [12]. The main goal of [18] was to present an approach to the matricial version of the classical Schur problem in both non-degenerate and degenerate cases where an explicit representation of the central matrix-valued Schur function associated with a finite $p \times q$ Schur sequence (see [15]) was used as reference function for a proof by mathematical induction. This strategy was already applied in the case of the matricial version of the classical Carathéodory problem (see [16, 17]). In [6, 7] a SP-algorithm for sequences of complex $p \times q$ matrices was constructed which is directed to later applications to non-degenerate $p \times q$ Schur sequences. In this paper, we are going to extend the construction of [6, 7] to broader classes of sequences of complex $p \times q$ matrices which include arbitrary $p \times q$ Schur sequences. The main results of this paper present a generalization of the classical Schur algorithm [28], which provides in particular parametrizations of the set $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ in the case of an arbitrarily given $p \times q$ Schur sequence $(A_j)_{j=0}^{n}$.

3 The SP-transform for Sequences of Complex $p \times q$ matrices

In order to generalize the SP-algorithm for non-degenerate $p \times q$ Schur sequences, which was constructed in [6, 7] to classes of sequences of complex $p \times q$ matrices including $p \times q$ Schur sequences, we first discuss which classes of sequences of complex $p \times q$ matrices we have in mind. Since we are going to treat simultaneously both the non-degenerate and the degenerate cases of the considered interpolation problem, a whole series of technical considerations arise. For this reason, it is convenient to work out results for special classes of matrix sequences.

Notation 3.1 Let $\mathscr{K}_{p\times q;\kappa}$ be the set of all sequences $(A_j)_{j=0}^{\kappa}$ with $A_0 \in \mathbb{K}_{p\times q}$, let $\mathscr{KR}_{p\times q;0} := \mathscr{K}_{p\times q;0}$, and let $\mathscr{KN}_{p\times q;0} := \mathscr{K}_{p\times q;0}$. If $\kappa \geq 1$, then let $\mathscr{KR}_{p\times q;\kappa}$ be the set of all sequences $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$ fulfilling $\sum_{j=1}^{\kappa} \mathcal{R}(A_j) \subseteq \mathcal{R}(l_0)$, whereas we use $\mathscr{KN}_{p\times q;\kappa}$ to denote the set of all sequences $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$ such that $\mathcal{N}(r_0) \subseteq \bigcap_{j=1}^{\kappa} \mathcal{N}(A_j)$ holds true. Furthermore, let $\mathscr{KRN}_{p\times q;\kappa} := \mathscr{KR}_{p\times q;\kappa} \cap \mathscr{KN}_{p\times q;\kappa}$ and let $\mathscr{D}_{p\times q;\kappa}$ be the set of all sequences $(A_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices such that $\sum_{j=0}^{\kappa} \mathcal{R}(A_j) \subseteq \mathcal{R}(A_0)$ and $\mathcal{N}(A_0) \subseteq \bigcap_{j=0}^{\kappa} \mathcal{N}(A_j)$ hold true.

Remark 3.2 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $j \in \mathbb{Z}_{1,\kappa}$. Then

$$l_0 - A_j A_j^* \succcurlyeq I_p - \sum_{\ell=0}^{j} A_\ell A_\ell^* = [O_{p\times np}, I_p] L_j [O_{p\times np}, I_p]^* \succcurlyeq O_{p\times p}$$

and, consequently, $O_{p\times p} \preccurlyeq A_j A_j^* \preccurlyeq l_0$, $\mathcal{R}(A_j) \subseteq \mathcal{R}(l_0)$, and $\mathcal{N}(l_0) \subseteq \mathcal{N}(A_j^*)$. Analogously, one gets $O_{q\times q} \preccurlyeq A_j^* A_j \preccurlyeq r_0$, $\mathcal{R}(A_j^*) \subseteq \mathcal{R}(r_0)$, and $\mathcal{N}(r_0) \subseteq \mathcal{N}(A_j)$.

The classes introduced in Notation 3.1 will play an important role in our further considerations. This is caused by the following simple observations.

Remark 3.3 Remark 3.2 and Notation 3.1 imply the inclusions $\mathscr{S}_{p\times q;\kappa} \subseteq \mathscr{KRN}_{p\times q;\kappa} \subseteq \mathscr{KR}_{p\times q;\kappa} \cup \mathscr{KN}_{p\times q;\kappa} \subseteq \mathscr{K}_{p\times q;\kappa}$.

We recall now the definition of the reciprocal sequence corresponding to a given sequence $(A_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices (see [22]). If $(A_j)_{j=0}^{\kappa}$ is a sequence of complex $p \times q$ matrices, then the sequence $(A_j^\sharp)_{j=0}^{\kappa}$ recursively defined by

$$A_0^\sharp := A_0^\dagger \qquad \text{and} \qquad A_j^\sharp := -A_0^\dagger \sum_{\ell=0}^{j-1} A_{j-\ell} A_\ell^\sharp \qquad \text{for all } j \in \mathbb{Z}_{1,\kappa} \tag{3.1}$$

is said to be the *reciprocal sequence corresponding to* $(A_j)_{j=0}^{\kappa}$. For each (block) matrix X built from the sequence $(A_j)_{j=0}^{\kappa}$, we denote by $X^\sharp$ the corresponding matrix built from the reciprocal sequence $(A_j^\sharp)_{j=0}^{\kappa}$ corresponding to $(A_j)_{j=0}^{\kappa}$ instead of the sequence $(A_j)_{j=0}^{\kappa}$. To emphasize that a certain (block) matrix X_n is built from a sequence $(A_j)_{j=0}^{\kappa}$, we sometimes write $X_{A;n}$ for X_n. If $n \in \mathbb{N}_0$ and if $(A_j)_{j=0}^{n}$ is a sequence of complex $p \times q$ matrices, then $(A_j)_{j=0}^{n}$ is called *invertible* if there is a sequence $(B_j)_{j=0}^{n}$ of complex $q \times p$ matrices such that $\mathbf{S}_{A;n}^\dagger = \mathbf{S}_{B;n}$. In this case, $\mathbf{S}_{A;m}^\dagger = \mathbf{S}_{B;m}$ for all $m \in \mathbb{Z}_{0,n}$. A sequence $(A_j)_{j=0}^{\infty}$ of complex $p \times q$ matrices is said to be *invertible* if there is a sequence $(B_j)_{j=0}^{\infty}$ of complex $q \times p$ matrices such that $\mathbf{S}_{A;m}^\dagger = \mathbf{S}_{B;m}$ for all $m \in \mathbb{N}_0$. We will use $\mathscr{I}_{p\times q;\kappa}$ to denote the set of all invertible sequences $(A_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices. One can easily see that if $(A_j)_{j=0}^{\kappa} \in \mathscr{I}_{p\times q;\kappa}$, then there is a unique sequence $(B_j)_{j=0}^{\kappa}$ of complex $q \times p$ matrices such that $\mathbf{S}_{A;m}^\dagger = \mathbf{S}_{B;m}$ for all $m \in \mathbb{Z}_{0,\kappa}$, the so-called inverse sequence corresponding to $(A_j)_{j=0}^{\kappa}$. In [22], one can find several results on invertible sequences of complex $p \times q$ matrices. In particular, $\mathscr{I}_{p\times q;\kappa} = \mathscr{D}_{p\times q;\kappa}$ is proved and, moreover, if $(A_j)_{j=0}^{\kappa}$ belongs to $\mathscr{I}_{p\times q;\kappa}$, then one obtains that $(A_j^{\sharp})_{j=0}^{\kappa}$ is the unique sequence $(B_j)_{j=0}^{\kappa}$ of complex $q \times p$ matrices which fulfills $\mathbf{S}_{B;m} = \mathbf{S}_{A;m}^{\dagger}$ for all $m \in \mathbb{Z}_{0,\kappa}$ (see [22, Thm. 4.21, Rem. 2.3]).

We introduce now one of the central objects of this paper. This object has two forms, namely a left one and a right one. At the end of this section (see Proposition 3.19), for sequences belonging to $\mathscr{KRN}_{p\times q;\kappa}$, we will see that both forms indeed coincide, a result which will be proved to be essential for our considerations.

Definition 3.4 Suppose $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. Let $W_{A;0} := \sqrt{l_0}$ and let $Y_{A;0} := \sqrt{r_0}$. If $\kappa \geq 1$, then:

(a) Let $W_{A;j} := -A_j A_0^* \sqrt{l_0}^{\dagger}$ for all $j \in \mathbb{Z}_{1,\kappa}$ and let $X_{A;j} := A_{j+1}\sqrt{r_0}^{\dagger}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Then the sequence $(A_j^{(1)})_{j=0}^{\kappa-1}$ defined by

$$A_j^{(1)} := \sum_{\ell=0}^{j} W_{A;j-\ell}^{\sharp} X_{A;\ell}$$

is called the *left SP-transform of* $(A_j)_{j=0}^{\kappa}$.

(b) Let $Y_{A;j} := -\sqrt{r_0}^{\dagger} A_0^* A_j$ for all $j \in \mathbb{Z}_{1,\kappa}$ and let $Z_{A;j} := \sqrt{l_0}^{\dagger} A_{j+1}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Then the sequence $(A_j^{[1]})_{j=0}^{\kappa-1}$ defined by

$$A_j^{[1]} := \sum_{\ell=0}^{j} Z_{A;\ell} Y_{A;j-\ell}^{\sharp}$$

is called the *right SP-transform of* $(A_j)_{j=0}^{\kappa}$.

Observe that Definition 3.4 is a natural generalization of [6, Def. 3.1] for sequences $(A_j)_{j=0}^{\kappa}$ that only satisfy $\|A_0\| \leq 1$ instead of $\|A_0\| < 1$, by replacing inverses with Moore–Penrose inverses.

For each matrix X built from the sequence $(A_j)_{j=0}^{\kappa}$, we denote (if possible) by $X^{(1)}$ (resp., $X^{[1]}$) the corresponding matrix built from the left (resp., right) SP-transform $(A_j^{(1)})_{j=0}^{\kappa-1}$ (resp., $(A_j^{[1]})_{j=0}^{\kappa-1}$) of $(A_j)_{j=0}^{\kappa}$ instead of $(A_j)_{j=0}^{\kappa}$.

Remark 3.5 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$, let $(A_j^{(1)})_{j=0}^{\kappa-1}$ (resp., $(A_j^{[1]})_{j=0}^{\kappa-1}$) be the left (resp., right) SP-transform of $(A_j)_{j=0}^{\kappa}$. For each $n \in \mathbb{Z}_{1,\kappa}$, then one can easily see that $(A_j)_{j=0}^{n}$ belongs to $\mathscr{K}_{p\times q;n}$ and that $(A_j^{(1)})_{j=0}^{n-1}$ (resp., $(A_j^{[1]})_{j=0}^{n-1}$) is the left (resp., right) SP-transform of $(A_j)_{j=0}^{n}$.

Example 3.6 Suppose $\kappa \geq 1$. Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be defined by $A_0 := E$ and, for all $j \in \mathbb{Z}_{1,\kappa}$ by $A_j := O_{p\times q}$. Then $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and $A_j^{[1]} = O_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.

Lemma 3.7 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$ with left SP-transform $(B_j)_{j=0}^{\kappa-1}$ and right SP-transform $(C_j)_{j=0}^{\kappa-1}$. Then $(A_j^*)_{j=0}^{\kappa}$ belongs to $\mathscr{K}_{q\times p;\kappa}$ and has left SP-transform $(C_j^*)_{j=0}^{\kappa-1}$ and right SP-transform $(B_j^*)_{j=0}^{\kappa-1}$.*

Proof Lemma A.15 shows $A_0^* \in \mathbb{K}_{q\times p}$, so that $(A_j^*)_{j=0}^{\kappa} \in \mathscr{K}_{q\times p;\kappa}$. Denote by $(\Lambda_j)_{j=0}^{\kappa}$ and $(\Upsilon_j)_{j=0}^{\kappa}$ the reciprocal sequence corresponding to $(W_{A;j})_{j=0}^{\kappa}$ and $(Y_{A;j})_{j=0}^{\kappa}$, respectively. According to Definition 3.4, we have then $B_j = \sum_{\ell=0}^{j} \Lambda_{j-\ell} X_{A;\ell}$ and $C_j = \sum_{\ell=0}^{j} Z_{A;\ell} \Upsilon_{j-\ell}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Let $(\Delta_j)_{j=0}^{\kappa}$ and $(\Theta_j)_{j=0}^{\kappa}$ be defined by $\Delta_j := W_{A;j}^*$ and $\Theta_j := Y_{A;j}^*$, respectively. From [20, Prop. 3.13] we can infer $(\Lambda_j^*)_{j=0}^{\kappa} = (\Delta_j^\sharp)_{j=0}^{\kappa}$ and $(\Upsilon_j^*)_{j=0}^{\kappa} = (\Theta_j^\sharp)_{j=0}^{\kappa}$, so that

$$B_j^* = \sum_{\ell=0}^{j} X_{A;\ell}^* \Delta_{j-\ell}^\sharp \qquad \text{and} \qquad C_j^* = \sum_{\ell=0}^{j} \Theta_{j-\ell}^\sharp Z_{A;\ell}^*$$

for all $j \in \mathbb{Z}_{0,\kappa-1}$ follow. Let $(T_j)_{j=0}^{\kappa}$ be defined by $T_j := A_j^*$. By virtue of Definition 3.4 and (2.5), we have $W_{A;0}^* = \sqrt{I_p - A_0 A_0^*} = \sqrt{I_p - T_0^* T_0} = Y_{T;0}$ and $Y_{A;0}^* = \sqrt{I_q - A_0^* A_0} = \sqrt{I_q - T_0 T_0^*} = W_{T;0}$. Using Remark A.8, in view of Definition 3.4, we obtain then $W_{A;j}^* = -(W_{A;0}^*)^\dagger A_0 A_j^* = -Y_{T;0}^\dagger T_0^* T_j = Y_{T;j}$ and $Y_{A;j}^* = -A_j^* A_0 (Y_{A;0}^*)^\dagger = -T_j T_0^* W_{T;0}^\dagger = W_{T;j}$ for all $j \in \mathbb{Z}_{1,\kappa}$ as well as $X_{A;j}^* = (Y_{A;0}^*)^\dagger A_{j+1}^* = W_{T;0}^\dagger T_{j+1} = Z_{T;j}$ and $Z_{A;j}^* = A_{j+1}^* (W_{A;0}^*)^\dagger = T_{j+1} Y_{T;0}^\dagger = X_{T;j}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. In particular, we have shown that $(Y_{T;j})_{j=0}^{\kappa} = (\Delta_j)_{j=0}^{\kappa}$ and $(W_{T;j})_{j=0}^{\kappa} = (\Theta_j)_{j=0}^{\kappa}$. Taking additionally into account Definition 3.4, we get then $T_j^{(1)} = \sum_{\ell=0}^{j} W_{T;j-\ell}^\sharp X_{T;\ell} = \sum_{\ell=0}^{j} \Theta_{j-\ell}^\sharp Z_{A;\ell}^* = C_j^*$ and $T_j^{[1]} = \sum_{\ell=0}^{j} Z_{T;\ell} Y_{T;j-\ell}^\sharp = \sum_{\ell=0}^{j} X_{A;\ell}^* \Delta_{j-\ell}^\sharp = B_j^*$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. □

Notation 3.8 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. Then, for all $n \in \mathbb{Z}_{0,\kappa}$, let $\mathbf{W}_n := \mathbf{S}_{W_A;n}$ and $\mathbf{Y}_n := \mathbf{S}_{Y_A;n}$ as well as $\mathbf{W}_n^\sharp := \mathbf{S}_{W_A^\sharp;n}$ and $\mathbf{Y}_n^\sharp := \mathbf{S}_{Y_A^\sharp;n}$. Furthermore, if $\kappa \geq 1$, then, for all $n \in \mathbb{Z}_{0,\kappa-1}$, let $\mathbf{X}_n := \mathbf{S}_{X_A;n}$ and $\mathbf{Z}_n := \mathbf{S}_{Z_A;n}$ as well as $\mathring{\mathbf{X}}_n := \mathring{\mathbf{S}}_{X_A;n}$ and $\mathring{\mathbf{Z}}_n := \mathring{\mathbf{S}}_{Z_A;n}$.

Given an arbitrary $n \in \mathbb{N}$ and arbitrary rectangular complex matrices $A_1, A_2, \ldots, A_n$, we use $\operatorname{diag}((A_j)_{j=1}^n)$ or $\operatorname{diag}(A_1, A_2, \ldots, A_n)$ to denote the block diagonal matrix with diagonal blocks $A_1, A_2, \ldots, A_n$. Furthermore, for arbitrarily given $A \in \mathbb{C}^{p\times q}$ and $m \in \mathbb{N}_0$, we write

$$\langle\!\langle A \rangle\!\rangle_m := \operatorname{diag}((A)_{j=0}^m). \tag{3.2}$$

Now we give some identities, which can be easily checked by virtue of Remark A.7 and Lemma A.16(e).

Remark 3.9 Let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices.

(a) Suppose $(A_j)_{j=0}^{\kappa} \in \mathscr{KR}_{p\times q;\kappa}$. For each $n \in \mathbb{Z}_{0,\kappa}$, then

$$\langle\langle l_0 l_0^\dagger \rangle\rangle_n \mathbf{S}_n = \mathbf{S}_n - \langle\langle P_0 A_0 \rangle\rangle_n, \qquad \mathbf{S}_n^* \langle\langle l_0 l_0^\dagger \rangle\rangle_n \mathbf{S}_n = \mathbf{S}_n^* \mathbf{S}_n - \langle\langle Q_0 \rangle\rangle_n,$$

and

$$\langle\langle r_0^\dagger r_0 \rangle\rangle_n - \mathbf{S}_n^* \langle\langle l_0 l_0^\dagger \rangle\rangle_n \mathbf{S}_n = R_n. \tag{3.3}$$

(b) Suppose $(A_j)_{j=0}^{\kappa} \in \mathscr{KN}_{p\times q;\kappa}$. For each $n \in \mathbb{Z}_{0,\kappa}$, then

$$\mathbf{S}_n \langle\langle r_0^\dagger r_0 \rangle\rangle_n = \mathbf{S}_n - \langle\langle A_0 Q_0 \rangle\rangle_n, \qquad \mathbf{S}_n \langle\langle r_0^\dagger r_0 \rangle\rangle_n \mathbf{S}_n^* = \mathbf{S}_n \mathbf{S}_n^* - \langle\langle P_0 \rangle\rangle_n,$$

and

$$\langle\langle l_0 l_0^\dagger \rangle\rangle_n - \mathbf{S}_n \langle\langle r_0^\dagger r_0 \rangle\rangle_n \mathbf{S}_n^* = L_n. \tag{3.4}$$

Remark 3.10 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. In view of (2.5) and Remark A.10(d), for all $n \in \mathbb{Z}_{0,\kappa}$, then $\mathbf{W}_n = [I_{(n+1)p} - \mathbf{S}_n \langle\langle A_0^* \rangle\rangle_n] \langle\langle \sqrt{l_0}^\dagger \rangle\rangle_n$ and $\mathbf{Y}_n = \langle\langle \sqrt{r_0}^\dagger \rangle\rangle_n [I_{(n+1)q} - \langle\langle A_0^* \rangle\rangle_n \mathbf{S}_n]$.

Remark 3.11 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. For each $n \in \mathbb{Z}_{1,\kappa}$, then $\mathring{\mathbf{X}}_{n-1} = [\mathbf{S}_n - \langle\langle A_0 \rangle\rangle_n] \langle\langle \sqrt{r_0}^\dagger \rangle\rangle_n$ and $\mathbf{S}_{n-1}^{(1)} = \mathbf{W}_{n-1}^\sharp \mathbf{X}_{n-1}$ as well as $\mathring{\mathbf{Z}}_{n-1} = \langle\langle \sqrt{l_0}^\dagger \rangle\rangle_n [\mathbf{S}_n - \langle\langle A_0 \rangle\rangle_n]$ and $\mathbf{S}_{n-1}^{[1]} = \mathbf{Z}_{n-1} \mathbf{Y}_{n-1}^\sharp$.

Using Remarks A.9 and A.10(a), we can obtain the following result:

Remark 3.12 If $(A_j)_{j=0}^{\kappa} \in \mathscr{KR}_{p\times q;\kappa}$, then $(W_{A;j})_{j=0}^{\kappa} \in \mathscr{D}_{p\times p;\kappa}$. Moreover, if $(A_j)_{j=0}^{\kappa} \in \mathscr{KN}_{p\times q;\kappa}$, then $(Y_{A;j})_{j=0}^{\kappa} \in \mathscr{D}_{q\times q;\kappa}$.

Remark 3.13 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{KR}_{p\times q;\kappa}$. In view of Remark 3.12 and [22, Prop. 4.20], for all $n \in \mathbb{Z}_{0,\kappa}$, then $\mathbf{W}_n^\sharp = \mathbf{W}_n^\dagger$. Moreover, if $\kappa \geq 1$, then Remark 3.12 and [22, Lem. 4.18] show that $\mathbf{W}_n^\dagger = \begin{bmatrix} * & O_{p\times np} \\ * & \mathbf{W}_{n-1}^\dagger \end{bmatrix}$ is valid for all $n \in \mathbb{Z}_{1,\kappa}$.

Remark 3.14 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{KN}_{p\times q;\kappa}$. In view of Remark 3.12 and [22, Prop. 4.20], for all $n \in \mathbb{Z}_{0,\kappa}$, then $\mathbf{Y}_n^\sharp = \mathbf{Y}_n^\dagger$. Moreover, if $\kappa \geq 1$, then Remark 3.12 and [22, Lem. 4.18] show that $\mathbf{Y}_n^\dagger = \begin{bmatrix} \mathbf{Y}_{n-1}^\dagger & O_{nq\times q} \\ * & * \end{bmatrix}$ is valid for all $n \in \mathbb{Z}_{1,\kappa}$.

Remark 3.15 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. In view of Remarks A.9 and A.10(a), then $\sum_{j=0}^{\kappa-1} \mathcal{R}(Z_{A;j}) \subseteq \mathcal{R}(l_0)$. Moreover, if $(A_j)_{j=0}^{\kappa} \in \mathscr{KR}_{p\times q;\kappa}$, then $\sum_{j=0}^{\kappa-1} \mathcal{R}(X_{A;j}) \subseteq \mathcal{R}(l_0)$.

Remark 3.16 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. In view of Remarks A.10(a) and A.9, then $\mathcal{N}(r_0) \subseteq \bigcap_{j=0}^{\kappa-1} \mathcal{N}(X_{A;j})$. Moreover, if $(A_j)_{j=0}^{\kappa} \in \mathscr{KN}_{p\times q;\kappa}$, then $\mathcal{N}(r_0) \subseteq \bigcap_{j=0}^{\kappa-1} \mathcal{N}(Z_{A;j})$.

Using Remark 3.12, [22, Thm. 4.21(a), Lem. 3.6], and Remark A.10(c), we can obtain the following result:

Remark 3.17 If $(A_j)_{j=0}^{\kappa} \in \mathscr{KR}_{p\times q;\kappa}$ then

$$\mathbf{W}_n\mathbf{W}_n^\dagger = \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n = \langle\!\langle \sqrt{l_0}\sqrt{l_0}^\dagger \rangle\!\rangle_n = \langle\!\langle \sqrt{l_0}^\dagger\sqrt{l_0} \rangle\!\rangle_n = \langle\!\langle l_0^\dagger l_0 \rangle\!\rangle_n = \mathbf{W}_n^\dagger\mathbf{W}_n \quad (3.5)$$

for all $n \in \mathbb{Z}_{0,\kappa}$. Moreover if $(A_j)_{j=0}^{\kappa} \in \mathscr{KN}_{p\times q;\kappa}$, then

$$\mathbf{Y}_n\mathbf{Y}_n^\dagger = \langle\!\langle r_0 r_0^\dagger \rangle\!\rangle_n = \langle\!\langle \sqrt{r_0}\sqrt{r_0}^\dagger \rangle\!\rangle_n = \langle\!\langle \sqrt{r_0}^\dagger\sqrt{r_0} \rangle\!\rangle_n = \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n = \mathbf{Y}_n^\dagger\mathbf{Y}_n \quad (3.6)$$

for all $n \in \mathbb{Z}_{0,\kappa}$.

The following result plays an important role in the proof of Proposition 3.19.

Lemma 3.18 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{KRN}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then*

$$\mathbf{X}_{n-1}\mathbf{Y}_{n-1} = \mathbf{W}_{n-1}\mathbf{Z}_{n-1}. \quad (3.7)$$

Proof Remarks 3.11 and 3.10 yield

$$\begin{aligned} \mathring{\mathbf{X}}_{n-1}\mathbf{Y}_n &= [\mathbf{S}_n - \langle\!\langle A_0 \rangle\!\rangle_n]\langle\!\langle r_0^\dagger \rangle\!\rangle_n[I_{(n+1)q} - \langle\!\langle A_0^* \rangle\!\rangle_n\mathbf{S}_n] \\ &= \mathbf{S}_n\langle\!\langle r_0^\dagger \rangle\!\rangle_n - \mathbf{S}_n\langle\!\langle r_0^\dagger A_0^* \rangle\!\rangle_n\mathbf{S}_n - \langle\!\langle A_0 r_0^\dagger \rangle\!\rangle_n + \langle\!\langle A_0 r_0^\dagger A_0^* \rangle\!\rangle_n\mathbf{S}_n \end{aligned} \quad (3.8)$$

and, analogously,

$$\mathbf{W}_n\mathring{\mathbf{Z}}_{n-1} = \langle\!\langle l_0^\dagger \rangle\!\rangle_n\mathbf{S}_n - \langle\!\langle l_0^\dagger A_0 \rangle\!\rangle_n - \mathbf{S}_n\langle\!\langle A_0^* l_0^\dagger \rangle\!\rangle_n\mathbf{S}_n + \mathbf{S}_n\langle\!\langle A_0^* l_0^\dagger A_0 \rangle\!\rangle_n. \quad (3.9)$$

According to parts , (c), (b), (a) and (d) of Lemma (A.16), we have $r_0^\dagger - A_0^* l_0^\dagger A_0 = r_0^\dagger r_0$ and $l_0^\dagger - A_0 r_0^\dagger A_0^* = l_0 l_0^\dagger$ as well as $A_0^* l_0^\dagger = r_0^\dagger A_0^*$, $l_0^\dagger A_0 = A_0 r_0^\dagger$ and $P_0 A_0 = A_0 Q_0$. Using (3.8), (3.9), Remark 3.9, and (2.10), we can conclude then

$$\begin{aligned} &\mathring{\mathbf{X}}_{n-1}\mathbf{Y}_n - \mathbf{W}_n\mathring{\mathbf{Z}}_{n-1} \\ &= \mathbf{S}_n\langle\!\langle r_0^\dagger - A_0^* l_0^\dagger A_0 \rangle\!\rangle_n - \mathbf{S}_n\langle\!\langle r_0^\dagger A_0^* - A_0^* l_0^\dagger \rangle\!\rangle_n\mathbf{S}_n \\ &\quad - \langle\!\langle A_0 r_0^\dagger - l_0^\dagger A_0 \rangle\!\rangle_n + \langle\!\langle A_0 r_0^\dagger A_0^* - l_0^\dagger \rangle\!\rangle_n\mathbf{S}_n \\ &= \mathbf{S}_n\langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n - \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n\mathbf{S}_n = \mathbf{S}_n - \langle\!\langle A_0 Q_0 \rangle\!\rangle_n - [\mathbf{S}_n - \langle\!\langle P_0 A_0 \rangle\!\rangle_n] \\ &= \langle\!\langle P_0 A_0 - A_0 Q_0 \rangle\!\rangle_n = O. \end{aligned}$$

Regarding Notation 3.8 and (2.2), this implies finally

$$\begin{aligned} O_{(n+1)p\times(n+1)q} &= \mathring{\mathbf{X}}_{n-1}\mathbf{Y}_n - \mathbf{W}_n\mathring{\mathbf{Z}}_{n-1} \\ &= \begin{bmatrix} O_{p\times nq} & O_{p\times q} \\ \mathbf{X}_{n-1} & O_{np\times q} \end{bmatrix}\begin{bmatrix} \mathbf{Y}_{n-1} & O_{nq\times q} \\ * & * \end{bmatrix} - \begin{bmatrix} * & O_{p\times np} \\ * & \mathbf{W}_{n-1} \end{bmatrix}\begin{bmatrix} O_{p\times nq} & O_{p\times q} \\ \mathbf{Z}_{n-1} & O_{np\times q} \end{bmatrix} \\ &= \left[\begin{array}{c:c} O_{p\times nq} & O_{p\times q} \\ \hdashline \mathbf{X}_{n-1}\mathbf{Y}_{n-1} - \mathbf{W}_{n-1}\mathbf{Z}_{n-1} & O_{np\times q} \end{array}\right]. \end{aligned}$$

□

Now we obtain that the left and the right SP-transforms coincide.

Proposition 3.19 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa}$. Then $(A_j^{(1)})_{j=0}^{\kappa-1} = (A_j^{[1]})_{j=0}^{\kappa-1}$.*

Proof We consider an arbitrary $n \in \mathbb{Z}_{1,\kappa}$. First we observe that Remarks 3.11 and 3.13 yield $\mathbf{S}_{n-1}^{(1)} = \mathbf{W}_{n-1}^{\sharp}\mathbf{X}_{n-1} = \mathbf{W}_{n-1}^{\dagger}\mathbf{X}_{n-1}$. Similarly, Remarks 3.11 and 3.14 yield $\mathbf{S}_{n-1}^{[1]} = \mathbf{Z}_{n-1}\mathbf{Y}_{n-1}^{\sharp} = \mathbf{Z}_{n-1}\mathbf{Y}_{n-1}^{\dagger}$. For each $j \in \mathbb{Z}_{0,\kappa-1}$, by virtue of Remarks 3.15 and 3.16, we have $\mathcal{R}(Z_{A;j}) \subseteq \mathcal{R}(l_0)$ and $\mathcal{N}(r_0) \subseteq \mathcal{N}(X_{A;j})$, which, because of Remark A.7, implies $l_0 l_0^{\dagger} Z_{A;j} = Z_{A;j}$ and $X_{A;j} r_0^{\dagger} r_0 = X_{A;j}$. Regarding Notation 3.8, (2.2), and (3.2), hence $\langle\!\langle l_0 l_0^{\dagger}\rangle\!\rangle_n \mathbf{Z}_{n-1} = \mathbf{Z}_{n-1}$ and $\mathbf{X}_{n-1}\langle\!\langle r_0^{\dagger} r_0\rangle\!\rangle_n = \mathbf{X}_{n-1}$ follow. Thus, combining the obtained equations, we get $\mathbf{S}_{n-1}^{(1)} = \mathbf{W}_{n-1}^{\dagger}\mathbf{X}_{n-1}\langle\!\langle r_0^{\dagger} r_0\rangle\!\rangle_n$ and $\mathbf{S}_{n-1}^{[1]} = \langle\!\langle l_0 l_0^{\dagger}\rangle\!\rangle_n \mathbf{Z}_{n-1}\mathbf{Y}_{n-1}^{\dagger}$. Because of Remark 3.17, then

$$\mathbf{S}_{n-1}^{(1)} = \mathbf{W}_{n-1}^{\dagger}\mathbf{X}_{n-1}\mathbf{Y}_{n-1}\mathbf{Y}_{n-1}^{\dagger} \quad \text{and} \quad \mathbf{S}_{n-1}^{[1]} = \mathbf{W}_{n-1}^{\dagger}\mathbf{W}_{n-1}\mathbf{Z}_{n-1}\mathbf{Y}_{n-1}^{\dagger} \tag{3.10}$$

follow. Lemma 3.18 gives (3.7). Thus, summarizing (3.10) and (3.7), we get finally $\mathbf{S}_{n-1}^{(1)} = \mathbf{S}_{n-1}^{[1]}$. □

Remark 3.20 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}_{p\times q;\kappa}$. Taking into account Remarks 3.13 and 3.11, for all $n \in \mathbb{Z}_{1,\kappa}$, then

$$\mathbf{W}_n^{\dagger}\mathring{\mathbf{X}}_{n-1} = \begin{bmatrix} * & O_{p\times np} \\ * & \mathbf{W}_{n-1}^{\dagger} \end{bmatrix}\begin{bmatrix} O_{p\times nq} & O_{p\times q} \\ \mathbf{X}_{n-1} & O_{np\times q} \end{bmatrix} = \mathring{\mathbf{S}}_{n-1}^{(1)}. \tag{3.11}$$

Remark 3.21 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}_{p\times q;\kappa}$. Because of Definition 3.4 and Remarks 3.15 and 3.16, then $\sum_{j=0}^{\kappa-1}\mathcal{R}(A_j^{[1]}) \subseteq \mathcal{R}(l_0)$ and $\mathcal{N}(r_0) \subseteq \bigcap_{j=0}^{\kappa-1}\mathcal{N}(A_j^{(1)})$.

Remark 3.22 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa}$. In view of Proposition 3.19 and Remark 3.21, then $\sum_{j=0}^{\kappa-1}\mathcal{R}(A_j^{(1)}) \subseteq \mathcal{R}(l_0)$ and $\mathcal{N}(r_0) \subseteq \bigcap_{j=0}^{\kappa-1}\mathcal{N}(A_j^{[1]})$.

Proposition 3.23 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then $L_n = \mathbf{W}_n \cdot \operatorname{diag}(I_p, L_{n-1}^{[1]}) \cdot \mathbf{W}_n^*$ and*

$$\operatorname{diag}(I_p, L_{n-1}^{[1]}) = \langle\!\langle P_0\rangle\!\rangle_n + \mathbf{W}_n^{\dagger} L_n (\mathbf{W}_n^{\dagger})^*. \tag{3.12}$$

Proof One can easily check that

$$I_{(n+1)p} - \mathring{\mathbf{S}}_{n-1}^{(1)}(\mathring{\mathbf{S}}_{n-1}^{(1)})^* = \operatorname{diag}(I_p, L_{n-1}^{(1)}). \tag{3.13}$$

Remark 3.20 yields (3.11). From Remark 3.15 and Remark A.7(a) we can infer $l_0 l_0^\dagger X_{A;j} = X_{A;j}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Taking into account Remark 3.17, (3.2), Notation 3.8, and (2.2), then $\mathbf{W}_n \mathbf{W}_n^\dagger \mathring{\mathbf{X}}_{n-1} = \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n \mathring{\mathbf{X}}_{n-1} = \mathring{\mathbf{X}}_{n-1}$ follows. Using additionally (3.11), we obtain consequently

$$\mathbf{W}_n \mathring{\mathbf{S}}_{n-1}^{(1)}(\mathring{\mathbf{S}}_{n-1}^{(1)})^* \mathbf{W}_n^* = \mathring{\mathbf{X}}_{n-1}\mathring{\mathbf{X}}_{n-1}^*. \tag{3.14}$$

By virtue of Remark A.10(b), we have moreover $\langle\!\langle \sqrt{l_0}^\dagger \rangle\!\rangle_n \langle\!\langle \sqrt{l_0}^\dagger \rangle\!\rangle_n^* = \langle\!\langle l_0^\dagger \rangle\!\rangle_n$ and $\langle\!\langle \sqrt{r_0}^\dagger \rangle\!\rangle_n$
$\langle\!\langle \sqrt{r_0}^\dagger \rangle\!\rangle_n^* = \langle\!\langle r_0^\dagger \rangle\!\rangle_n$. Applying Remark 3.10, we get then

$$\begin{aligned}\mathbf{W}_n \mathbf{W}_n^* &= (I_{(n+1)p} - \mathbf{S}_n \langle\!\langle A_0^* \rangle\!\rangle_n)\langle\!\langle l_0^\dagger \rangle\!\rangle_n (I_{(n+1)p} - \langle\!\langle A_0 \rangle\!\rangle_n \mathbf{S}_n^*)\\ &= \langle\!\langle l_0^\dagger \rangle\!\rangle_n - \langle\!\langle l_0^\dagger A_0 \rangle\!\rangle_n \mathbf{S}_n^* - \mathbf{S}_n \langle\!\langle A_0^* l_0^\dagger \rangle\!\rangle_n + \mathbf{S}_n \langle\!\langle A_0^* l_0^\dagger A_0 \rangle\!\rangle_n \mathbf{S}_n^*. \end{aligned} \tag{3.15}$$

Similarly, from Remark 3.11 we conclude

$$\mathring{\mathbf{X}}_{n-1}\mathring{\mathbf{X}}_{n-1}^* = \mathbf{S}_n \langle\!\langle r_0^\dagger \rangle\!\rangle_n \mathbf{S}_n^* - \mathbf{S}_n \langle\!\langle r_0^\dagger A_0^* \rangle\!\rangle_n - \langle\!\langle A_0 r_0^\dagger \rangle\!\rangle_n \mathbf{S}_n^* + \langle\!\langle A_0 r_0^\dagger A_0^* \rangle\!\rangle_n. \tag{3.16}$$

Parts (a), (b), and (c) of Lemma A.16 yield $l_0^\dagger A_0 = A_0 r_0^\dagger$ and $A_0^* l_0^\dagger = r_0^\dagger A_0^*$ as well as $l_0^\dagger - A_0 r_0^\dagger A_0^* = l_0 l_0^\dagger$ and $r_0^\dagger - A_0^* l_0^\dagger A_0 = r_0^\dagger r_0$. Remark 3.9(b) provides (3.4). Using (3.13), (3.14), (3.15), (3.16), and (3.4), we get then

$$\begin{aligned}\mathbf{W}_n \cdot \operatorname{diag}(I_p, L_{n-1}^{(1)}) \cdot \mathbf{W}_n^* &= \mathbf{W}_n \mathbf{W}_n^* - \mathbf{W}_n \mathring{\mathbf{S}}_{n-1}^{(1)}(\mathring{\mathbf{S}}_{n-1}^{(1)})^* \mathbf{W}_n^*\\ &= \mathbf{W}_n \mathbf{W}_n^* - \mathring{\mathbf{X}}_{n-1}\mathring{\mathbf{X}}_{n-1}^*\\ &= \langle\!\langle l_0^\dagger \rangle\!\rangle_n - \langle\!\langle l_0^\dagger A_0 \rangle\!\rangle_n \mathbf{S}_n^* - \mathbf{S}_n \langle\!\langle A_0^* l_0^\dagger \rangle\!\rangle_n + \mathbf{S}_n \langle\!\langle A_0^* l_0^\dagger A_0 \rangle\!\rangle_n \mathbf{S}_n^*\\ &\quad - \mathbf{S}_n \langle\!\langle r_0^\dagger \rangle\!\rangle_n \mathbf{S}_n^* + \mathbf{S}_n \langle\!\langle r_0^\dagger A_0^* \rangle\!\rangle_n + \langle\!\langle A_0 r_0^\dagger \rangle\!\rangle_n \mathbf{S}_n^* - \langle\!\langle A_0 r_0^\dagger A_0^* \rangle\!\rangle_n\\ &= \langle\!\langle l_0^\dagger - A_0 r_0^\dagger A_0^* \rangle\!\rangle_n + \mathbf{S}_n \langle\!\langle A_0^* l_0^\dagger A_0 - r_0^\dagger \rangle\!\rangle_n \mathbf{S}_n^*\\ &= \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n - \mathbf{S}_n \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n \mathbf{S}_n^* = L_n. \end{aligned} \tag{3.17}$$

By virtue of Proposition 3.19, thus $L_n = \mathbf{W}_n \cdot \operatorname{diag}(I_p, L_{n-1}^{[1]}) \cdot \mathbf{W}_n^*$ follows. Remarks 3.22 and A.7(a) yield $l_0 l_0^\dagger A_j^{(1)} = A_j^{(1)}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Regarding (3.2) and (2.2), hence $\langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^{(1)} = \mathring{\mathbf{S}}_{n-1}^{(1)}$. Using (3.17), (3.5), (3.13), Remark A.24(b), and (2.1), we get then

$$\begin{aligned}\mathbf{W}_n^\dagger L_n(\mathbf{W}_n^\dagger)^* &= \mathbf{W}_n^\dagger \mathbf{W}_n \cdot \operatorname{diag}(I_p, L_{n-1}^{(1)}) \cdot (\mathbf{W}_n^\dagger \mathbf{W}_n)^* \\ &= \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n [I_{(n+1)p} - \mathring{\mathbf{S}}_{n-1}^{(1)} (\mathring{\mathbf{S}}_{n-1}^{(1)})^*] \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n^* \\ &= \langle\!\langle l_0 l_0^\dagger (l_0 l_0^\dagger)^* \rangle\!\rangle_n - \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^{(1)} (\mathring{\mathbf{S}}_{n-1}^{(1)})^* \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n^* = \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n - \mathring{\mathbf{S}}_{n-1}^{(1)} (\mathring{\mathbf{S}}_{n-1}^{(1)})^*\end{aligned}$$

and, in view of (3.2), (2.10) and (3.13), consequently

$$\begin{aligned}\langle\!\langle P_0 \rangle\!\rangle_n + \mathbf{W}_n^\dagger L_n(\mathbf{W}_n^\dagger)^* &= I_{(n+1)p} - \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n + \mathbf{W}_n^\dagger L_n(\mathbf{W}_n^\dagger)^* \\ &= I_{(n+1)p} - \mathring{\mathbf{S}}_{n-1}^{(1)} (\mathring{\mathbf{S}}_{n-1}^{(1)})^* = \operatorname{diag}(I_p, L_{n-1}^{(1)}).\end{aligned}$$

By virtue of Proposition 3.19, thus (3.12) follows. □

The next result contains the essential observation that the SP-transform maps the class $\mathscr{S}_{p\times q;\kappa}$ into the class $\mathscr{S}_{p\times q;\kappa-1}$.

Proposition 3.24 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. Then $(A_j^{[1]})_{j=0}^{\kappa-1} \in \mathscr{S}_{p\times q;\kappa-1}$.*

Proof We consider an arbitrary $n \in \mathbb{Z}_{1,\kappa}$. Remark 3.3 provides $(A_j)_{j=0}^{\kappa} \in \mathscr{KRN}_{p\times q;\kappa}$. Thus, Proposition 3.23 yields (3.12). Regarding (2.11), from Remark A.4 we can infer $P_0 \in \mathbb{C}_{\succcurlyeq}^{p\times p}$. In view of (3.2), then $\langle\!\langle P_0 \rangle\!\rangle_n \succcurlyeq O$ follows. Since $(A_j)_{j=0}^{\kappa}$ belongs to $\mathscr{S}_{p\times q;\kappa}$, we also have $L_n \succcurlyeq O$. Thus, from (3.12) we see that $\operatorname{diag}(I_p, L_{n-1}^{[1]}) \succcurlyeq O$ and, consequently, that $L_{n-1}^{[1]} \succcurlyeq O$. Hence, $(A_j^{[1]})_{j=0}^{\kappa-1} \in \mathscr{S}_{p\times q;\kappa-1}$. □

Now we are going to derive a right version of Proposition 3.23. For this we need a little preparation.

Remark 3.25 Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{KN}_{p\times q;\kappa}$. Taking into account Remarks 3.14 and 3.11, for all $n \in \mathbb{Z}_{1,\kappa}$, then

$$\mathring{\mathbf{Z}}_{n-1}\mathbf{Y}_n^\dagger = \begin{bmatrix} O_{p\times nq} & O_{p\times q} \\ \mathbf{Z}_{n-1} & O_{np\times q}\end{bmatrix}\begin{bmatrix}\mathbf{Y}_{n-1}^\dagger & O_{nq\times q} \\ * & *\end{bmatrix} = \mathring{\mathbf{S}}_{n-1}^{[1]}. \tag{3.18}$$

Proposition 3.26 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{KRN}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then*

$$R_n = \mathbf{Y}_n^* \cdot \operatorname{diag}(R_{n-1}^{[1]}, I_q) \cdot \mathbf{Y}_n \tag{3.19}$$

and

$$\operatorname{diag}(R_{n-1}^{[1]}, I_q) = \langle\!\langle Q_0 \rangle\!\rangle_n + (\mathbf{Y}_n^\dagger)^* R_n \mathbf{Y}_n^\dagger.$$

Proof. One can easily check that

$$I_{(n+1)q} - (\mathring{\mathbf{S}}^{[1]}_{n-1})^* \mathring{\mathbf{S}}^{[1]}_{n-1} = \operatorname{diag}(R^{[1]}_{n-1}, I_q). \tag{3.20}$$

Remark 3.25 yields (3.18). From Remark 3.16 and Remark A.7(b) we can infer $Z_{A;j} r_0^\dagger r_0 = Z_{A;j}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Taking into account Remark 3.17, Notation 3.8, (2.2), and (3.2), then $\mathring{\mathbf{Z}}_{n-1} \mathbf{Y}_n^\dagger \mathbf{Y}_n = \mathring{\mathbf{Z}}_{n-1} \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n = \mathring{\mathbf{Z}}_{n-1}$ follows. Using additionally (3.18), we obtain consequently

$$\mathbf{Y}_n^* (\mathring{\mathbf{S}}^{[1]}_{n-1})^* \mathring{\mathbf{S}}^{[1]}_{n-1} \mathbf{Y}_n = \mathring{\mathbf{Z}}^*_{n-1} \mathring{\mathbf{Z}}_{n-1}. \tag{3.21}$$

By virtue of Remark A.10(b), we have moreover $\langle\!\langle \sqrt{r_0}^\dagger \rangle\!\rangle_n^* \langle\!\langle \sqrt{r_0}^\dagger \rangle\!\rangle_n = \langle\!\langle r_0^\dagger \rangle\!\rangle_n$ and $\langle\!\langle \sqrt{l_0}^\dagger \rangle\!\rangle_n^*$
$\langle\!\langle \sqrt{l_0}^\dagger \rangle\!\rangle_n = \langle\!\langle l_0^\dagger \rangle\!\rangle_n$. Applying Remark 3.10, we get then

$$\begin{aligned} \mathbf{Y}_n^* \mathbf{Y}_n &= (I_{(n+1)q} - \mathbf{S}_n^* \langle\!\langle A_0 \rangle\!\rangle_n) \langle\!\langle r_0^\dagger \rangle\!\rangle_n [I_{(n+1)q} - \langle\!\langle A_0^* \rangle\!\rangle_n \mathbf{S}_n] \\ &= \langle\!\langle r_0^\dagger \rangle\!\rangle_n - \langle\!\langle r_0^\dagger A_0^* \rangle\!\rangle_n \mathbf{S}_n - \mathbf{S}_n^* \langle\!\langle A_0 r_0^\dagger \rangle\!\rangle_n + \mathbf{S}_n^* \langle\!\langle A_0 r_0^\dagger A_0^* \rangle\!\rangle_n \mathbf{S}_n. \end{aligned} \tag{3.22}$$

Similarly, from Remark 3.11 we conclude

$$\mathring{\mathbf{Z}}^*_{n-1} \mathring{\mathbf{Z}}_{n-1} = \mathbf{S}_n^* \langle\!\langle l_0^\dagger \rangle\!\rangle_n \mathbf{S}_n - \mathbf{S}_n^* \langle\!\langle l_0^\dagger A_0 \rangle\!\rangle_n - \langle\!\langle A_0^* l_0^\dagger \rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle A_0^* l_0^\dagger A_0 \rangle\!\rangle_n. \tag{3.23}$$

Parts (b), (a), and (c) of Lemma A.16 yield $A_0^* l_0^\dagger = r_0^\dagger A_0^*$ and $l_0^\dagger A_0 = A_0 r_0^\dagger$ as well as $r_0^\dagger - A_0^* l_0^\dagger A_0 = r_0^\dagger r_0$ and $l_0^\dagger - A_0 r_0^\dagger A_0^* = l_0 l_0^\dagger$. Remark 3.9(a) provides (3.3). Applying (3.20), (3.21), (3.22), (3.23), and (3.3), we get then

$$\begin{aligned} \mathbf{Y}_n^* \cdot \operatorname{diag}(R^{[1]}_{n-1}, I_q) \cdot \mathbf{Y}_n &= \mathbf{Y}_n^* [I_{n+1} - (\mathring{\mathbf{S}}^{[1]}_{n-1})^* \mathring{\mathbf{S}}^{[1]}_{n-1}] \mathbf{Y}_n \\ &= \mathbf{Y}_n^* \mathbf{Y}_n - \mathring{\mathbf{Z}}^*_{n-1} \mathring{\mathbf{Z}}_{n-1} \\ &= \langle\!\langle r_0^\dagger \rangle\!\rangle_n - \langle\!\langle r_0^\dagger A_0^* \rangle\!\rangle_n \mathbf{S}_n - \mathbf{S}_n^* \langle\!\langle A_0 r_0^\dagger \rangle\!\rangle_n + \mathbf{S}_n^* \langle\!\langle A_0 r_0^\dagger A_0^* \rangle\!\rangle_n \mathbf{S}_n \\ &\quad - \mathbf{S}_n^* \langle\!\langle l_0^\dagger \rangle\!\rangle_n \mathbf{S}_n + \mathbf{S}_n^* \langle\!\langle l_0^\dagger A_0 \rangle\!\rangle_n + \langle\!\langle A_0^* l_0^\dagger \rangle\!\rangle_n \mathbf{S}_n - \langle\!\langle A_0^* l_0^\dagger A_0 \rangle\!\rangle_n \\ &= \langle\!\langle r_0^\dagger - A_0^* l_0^\dagger A_0 \rangle\!\rangle_n + \mathbf{S}_n^* \langle\!\langle A_0 r_0^\dagger A_0^* - l_0^\dagger \rangle\!\rangle_n \mathbf{S}_n \\ &= \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n - \mathbf{S}_n^* \langle\!\langle l_0 l_0^\dagger \rangle\!\rangle_n \mathbf{S}_n = R_n, \end{aligned}$$

i.e., (3.19). Remarks 3.22 and A.7(b) yield $A_j^{[1]} r_0^\dagger r_0 = A_j^{[1]}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Regarding (2.2) and (3.2), hence $\mathring{\mathbf{S}}^{[1]}_{n-1} \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n = \mathring{\mathbf{S}}^{[1]}_{n-1}$. Using (3.19), (3.6), (3.20), Remark A.24(b), and (2.1), we get then

$$\begin{aligned}(\mathbf{Y}_n^\dagger)^* R_n \mathbf{Y}_n^\dagger &= (\mathbf{Y}_n \mathbf{Y}_n^\dagger)^* \operatorname{diag}(R_{n-1}^{[1]}, I_q) \mathbf{Y}_n \mathbf{Y}_n^\dagger \\ &= \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n^* [I_{(n+1)q} - (\mathring{\mathbf{S}}_{n-1}^{[1]})^* \mathring{\mathbf{S}}_{n-1}^{[1]}] \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n \\ &= \langle\!\langle (r_0^\dagger r_0)^* r_0^\dagger r_0 \rangle\!\rangle_n - \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n^* (\mathring{\mathbf{S}}_{n-1}^{[1]})^* \mathring{\mathbf{S}}_{n-1}^{[1]} \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n \\ &= \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n - (\mathring{\mathbf{S}}_{n-1}^{[1]})^* \mathring{\mathbf{S}}_{n-1}^{[1]}\end{aligned}$$

and, in view of (3.2), (2.10) and (3.20), consequently

$$\begin{aligned}\langle\!\langle Q_0 \rangle\!\rangle_n + (\mathbf{Y}_n^\dagger)^* R_n \mathbf{Y}_n^\dagger &= I_{(n+1)p} - \langle\!\langle r_0^\dagger r_0 \rangle\!\rangle_n + (\mathbf{Y}_n^\dagger)^* R_n \mathbf{Y}_n^\dagger \\ &= I_{(n+1)p} - (\mathring{\mathbf{S}}_{n-1}^{[1]})^* \mathring{\mathbf{S}}_{n-1}^{[1]} = \operatorname{diag}(R_{n-1}^{[1]}, I_q). \quad \square\end{aligned}$$

4 The SP-Algorithm for $p \times q$ Schur sequences

Regarding Propositions 3.24 and 3.19, we are able to generalize the notions of the left and the right SP-transforms of a sequence of complex $p \times q$ matrices, introduced in Definition 3.4 (see also Remark 4.2 below).

Definition 4.1 Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$. Then let the sequence $(A_j^{(0)})_{j=0}^\kappa$ (resp., $(A_j^{[0]})_{j=0}^\kappa$) be defined by $A_j^{(0)} := A_j$ (resp., $A_j^{[0]} := A_j$) for all $j \in \mathbb{Z}_{0,\kappa}$. Furthermore, if $\kappa \geq 1$, for all $k \in \mathbb{Z}_{1,\kappa}$, let the sequence $(A_j^{(k)})_{j=0}^{\kappa-k}$ (resp., $(A_j^{[k]})_{j=0}^{\kappa-k}$) be recursively defined to be the left SP-transform of $(A_j^{(k-1)})_{j=0}^{\kappa-(k-1)}$ (resp., right SP-transform of $(A_j^{[k-1]})_{j=0}^{\kappa-(k-1)}$). For all $k \in \mathbb{Z}_{0,\kappa}$, then the sequence $(A_j^{(k)})_{j=0}^{\kappa-k}$ (resp., $(A_j^{[k]})_{j=0}^{\kappa-k}$) is called the *k-th left SP-transform of* $(A_j)_{j=0}^\kappa$. (resp., *k-th right SP-transform of* $(A_j)_{j=0}^\kappa$).

Remark 4.2 Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$. We emphasize explicitly that, in Definition 4.1, we used $\mathscr{S}_{p\times q;\kappa} \subseteq \mathscr{K}_{p\times q;\kappa}$ and the following: By virtue of Propositions 3.24 and 3.19, one can easily verify by induction that $(A_j^{(k)})_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$ and $(A_j^{[k]})_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

Now we obtain that the left and the right SP-transforms coincide.

Proposition 4.3 *Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$. Then $A_j^{(k)} = A_j^{[k]}$ for every choice of $k \in \mathbb{Z}_{0,\kappa}$ and $j \in \mathbb{Z}_{0,\kappa-k}$.*

Proof In view of Definition 4.1, there is an $m \in \mathbb{Z}_{0,\kappa}$ such that $(A_j^{(k)})_{j=0}^{\kappa-k} = (A_j^{[k]})_{j=0}^{\kappa-k}$ for all $k \in \mathbb{Z}_{0,m}$. Consequently, Remark 3.3 provides $(A_j^{[m]})_{j=0}^{\kappa-m} \in \mathscr{KRN}_{p\times q;\kappa-m}$. If $m < \kappa$, then, in view of Definition 4.1, the application of Proposition 3.19 yields $(A_j^{(m+1)})_{j=0}^{\kappa-(m+1)} = (A_j^{[m+1]})_{j=0}^{\kappa-(m+1)}$. $\square$

Remark 4.4 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and, for each $k \in \mathbb{Z}_{0,\kappa}$, let $(A_j^{(k)})_{j=0}^{\kappa-k}$ (resp., $(A_j^{[k]})_{j=0}^{\kappa-k}$) be the k-th left (resp., right) SP-transform of $(A_j)_{j=0}^{\kappa}$. For every choice of $n \in \mathbb{Z}_{0,\kappa}$ and $k \in \mathbb{Z}_{0,n}$, one can see then from Definition 4.1, Remarks 4.2 and 3.5, and Proposition 4.3 that $(A_j)_{j=0}^{n}$ belongs to $\mathscr{S}_{p\times q;n}$ and that $(A_j^{(k)})_{j=0}^{n-k}$ (resp., $(A_j^{[k]})_{j=0}^{n-k}$) is the k-th left (resp., right) SP-transform of $(A_j)_{j=0}^{n}$.

Example 4.5 Let $(A_j)_{j=0}^{\kappa}$ be given by $A_j := O_{p\times q}$. From Example 3.6 and Definition 4.1 one can easily see then that $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and $A_j^{[k]} = O_{p\times q}$ for every choice of $k \in \mathbb{Z}_{0,\kappa}$ and $j \in \mathbb{Z}_{0,\kappa-k}$.

Lemma 4.6 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. Then $(T_j)_{j=0}^{\kappa}$ defined by $T_j := A_j^*$ belongs to $\mathscr{S}_{q\times p;\kappa}$ and, for all $k \in \mathbb{Z}_{0,\kappa}$, the k-th right SP-transform of $(T_j)_{j=0}^{\kappa}$ coincides with $(B_j^*)_{j=0}^{\kappa-k}$, where $(B_j)_{j=0}^{\kappa-k}$ denotes the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$.*

Proof Clearly, $(T_j)_{j=0}^{\kappa} \in \mathscr{S}_{q\times p;\kappa}$. Denote by $(C_j)_{j=0}^{\kappa}$ the 0-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. In view of Definition 4.1, then $(C_j)_{j=0}^{\kappa} = (A_j)_{j=0}^{\kappa}$ and hence $(T_j^{[0]})_{j=0}^{\kappa} = (T_j)_{j=0}^{\kappa} = (A_j^*)_{j=0}^{\kappa} = (C_j^*)_{j=0}^{\kappa}$. In the case $\kappa = 0$, the proof is complete. Now suppose $\kappa \geq 1$. Now denote by $(C_j)_{j=0}^{\kappa-1}$ the first right SP-transform of $(A_j)_{j=0}^{\kappa}$. In view of Remark 3.3, we can apply Proposition 3.19 to get $(T_j^{(1)})_{j=0}^{\kappa-1} = (T_j^{[1]})_{j=0}^{\kappa-1}$. Regarding Remark 3.3 again, we can apply Lemma 3.7 to obtain $(T_j^{(1)})_{j=0}^{\kappa-1} = (C_j^*)_{j=0}^{\kappa-1}$. Summarizing, we have $(T_j^{[1]})_{j=0}^{\kappa-1} = (C_j^*)_{j=0}^{\kappa-1}$. In the case $\kappa = 1$, the proof is complete. Now suppose $\kappa \geq 2$. Then there exists an $n \in \mathbb{Z}_{1,\kappa-1}$ such that for all $k \in \mathbb{Z}_{1,n}$ the following statement holds true:

(I)$_k$ $(T_j^{[k]})_{j=0}^{\kappa-k} = (B_j^*)_{j=0}^{\kappa-k}$, where $(B_j)_{j=0}^{\kappa-k}$ denotes the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$.

Let $(S_j)_{j=0}^{\kappa-n}$ be defined by $S_j := T_j^{[n]}$. According to Remark 4.2, then $(S_j)_{j=0}^{\kappa-n} \in \mathscr{S}_{q\times p;\kappa-n}$. In view of Remark 3.3, we can thus apply Proposition 3.19 to get $(S_j^{(1)})_{j=0}^{(\kappa-n)-1} = (S_j^{[1]})_{j=0}^{(\kappa-n)-1}$. According to Remark 4.2, the n-th right SP-transform $(D_j)_{j=0}^{\kappa-n}$ of $(A_j)_{j=0}^{\kappa}$ belongs to $\mathscr{S}_{p\times q;\kappa-n}$. Now denote by $(C_j)_{j=0}^{(\kappa-n)-1}$ the first right SP-transform of $(D_j)_{j=0}^{\kappa-n}$. Regarding Remark 3.3 and that (I)$_k$ for $k = n$ shows $(S_j)_{j=0}^{\kappa-n} = (D_j^*)_{j=0}^{\kappa-n}$, we can apply Lemma 3.7 to the sequence $(D_j)_{j=0}^{\kappa-n}$ to obtain $(S_j^{(1)})_{j=0}^{(\kappa-n)-1} = (C_j^*)_{j=0}^{(\kappa-n)-1}$. Taking additionally into account Definition 4.1, we obtain $(T_j^{[n+1]})_{j=0}^{\kappa-(n+1)} = (S_j^{[1]})_{j=0}^{(\kappa-n)-1} = (S_j^{(1)})_{j=0}^{(\kappa-n)-1} = (C_j^*)_{j=0}^{(\kappa-n)-1}$. Since Definition 4.1 implies that $(C_j)_{j=0}^{\kappa-(n+1)}$ is the $(n+1)$-th right SP-transform of $(A_j)_{j=0}^{\kappa}$, thus (I)$_k$ holds true for $k = n+1$. Therefore, the assertion is inductively proved. □

Definition 4.7 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. Then the sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ given by $\mathfrak{e}_j := A_0^{[j]}$ for all $j \in \mathbb{Z}_{0,\kappa}$ is called the *sequence of Schur–Potapov parameters* (short *SP-parameter sequence*) *of* $(A_j)_{j=0}^{\kappa}$.

One can easily convince oneself that in the scalar case $p = q = 1$ (see [28]) the parameters given in Definition 4.7 are exactly the classical Schur parameters .

Remark 4.8 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For all $k \in \mathbb{Z}_{0,\kappa}$, according to Remark 4.2 and Definitions 4.1 and 4.7, then $(A_j^{[k]})_{j=0}^{\kappa-k}$ belongs to $\mathscr{S}_{p\times q;\kappa-k}$ and has SP-parameter sequence $(\mathfrak{e}_{j+k})_{j=0}^{\kappa-k}$.

Remark 4.9 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. In view of Definition 4.7 and Remark 4.4, then $(A_j)_{j=0}^{n}$ belongs to $\mathscr{S}_{p\times q;n}$ and has SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n}$.

Lemma 4.10 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. Then $(A_j^*)_{j=0}^{\kappa}$ belongs to $\mathscr{S}_{q\times p;\kappa}$ and has SP-parameter sequence $(\mathfrak{e}_j^*)_{j=0}^{\kappa}$.*

Proof Regarding Definition 4.7, this follows from Lemma 4.6. □

Notation 4.11 Let $(\mathfrak{e}_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $j \in \mathbb{Z}_{0,\kappa}$, then let $\mathfrak{l}_j := I_p - \mathfrak{e}_j\mathfrak{e}_j^*$ and $\mathfrak{r}_j := I_q - \mathfrak{e}_j^*\mathfrak{e}_j$.

Remark 4.12 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For each $j \in \mathbb{Z}_{0,\kappa}$, in view of Remark 4.2, then $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ and hence $\mathfrak{l}_j \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $\mathfrak{r}_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.

Notation 4.13 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $k \in \mathbb{Z}_{0,\kappa}$. For each matrix X built from the sequence $(A_j)_{j=0}^{\kappa}$, we denote (if possible) by $X^{[k]}$ the corresponding matrix built from the k-th right SP-transform $(A_j^{[k]})_{j=0}^{\kappa-k}$ of $(A_j)_{j=0}^{\kappa}$ instead of $(A_j)_{j=0}^{\kappa}$.

Remark 4.14 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. In view of Notation 4.13, (2.5), Definition 4.7, and Notation 4.11, then $l_0^{[j]} = \mathfrak{l}_j$ and $r_0^{[j]} = \mathfrak{r}_j$ for all $j \in \mathbb{Z}_{0,\kappa}$. In particular, $\mathfrak{l}_0 = l_0$ and $\mathfrak{r}_0 = r_0$.

5 The Inverse SP-Transformation for Sequences of Complex Matrices

The main goal of this section is to generalize the notion of the inverse SP-transform of a sequence of complex $p \times q$ matrices with respect to a given contractive complex $p \times q$ matrix E. In [6, Def. 3.4], such considerations are carried out for the special case that the matrix E is strictly contractive. Taking into account the nature of the objects under consideration, we start again by considering a left and a right version of the inverse SP-transform of a sequence $(A_j)_{j=0}^{\kappa}$ with respect to a given $E \in \mathbb{K}_{p\times q}$. For each $E \in \mathbb{C}^{p\times q}$, let

$$l := I_p - EE^* \qquad \text{and} \qquad r := I_q - E^*E \tag{5.1}$$

as well as

$$P := I_p - ll^\dagger \qquad \text{and} \qquad Q := I_q - r^\dagger r. \tag{5.2}$$

Because of Remarks A.6, A.4 and A.2, we have then

$$P = \mathbb{P}_{\mathcal{R}(l)^\perp} \qquad \text{and} \qquad Q = \mathbb{P}_{\mathcal{N}(r)}. \tag{5.3}$$

If $E \in \mathbb{K}_{p\times q}$, in view of Remark A.10(c), furthermore

$$P = I_p - \sqrt{l}\sqrt{l}^{\dagger} \qquad \text{and} \qquad Q = I_q - \sqrt{r}^{\dagger}\sqrt{r}. \tag{5.4}$$

Definition 5.1 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then:

(a) Let $R_{E,A;0} := E$ and $T_{E,A;0} := I_p$, and, for all $j \in \mathbb{Z}_{1,\kappa+1}$, let $R_{E,A;j} := \sqrt{l}A_{j-1}\sqrt{r}^{\dagger}$ and $T_{E,A;j} := R_{E,A;j}E^*$. Then the sequence $(A_j^{(-1;E)})_{j=0}^{\kappa+1}$ defined by

$$A_j^{(-1;E)} := \sum_{\ell=0}^{j} T_{E,A;j-\ell}^{\sharp} R_{E,A;\ell}$$

is called the *left E-inverse SP-transform of* $(A_j)_{j=0}^{\kappa}$.

(b) Let $U_{E,A;0} := E$ and $V_{E,A;0} := I_q$, and, for all $j \in \mathbb{Z}_{1,\kappa+1}$, moreover let $U_{E,A;j} := \sqrt{l}^{\dagger} A_{j-1}\sqrt{r}$ and $V_{E,A;j} := E^* U_{E,A;j}$. Then the sequence $(A_j^{[-1;E]})_{j=0}^{\kappa+1}$ defined by

$$A_j^{[-1;E]} := \sum_{\ell=0}^{j} U_{E,A;\ell} V_{E,A;j-\ell}^{\sharp}$$

is called the *right E-inverse SP-transform of* $(A_j)_{j=0}^{\kappa}$.

Definition 5.1 is a generalization of [6, Definitions 3.4 and 3.10]. We will establish (see Proposition 5.9) that the left and right inverse SP-transform indeed coincide.

For each matrix X built from the sequence $(A_j)_{j=0}^{\kappa}$, we denote (if possible) by $X^{(-1;E)}$ (resp., $X^{[-1;E]}$) the corresponding matrix built from the left (resp., right) E-inverse SP-transform $(A_j^{(-1;E)})_{j=0}^{\kappa+1}$ (resp., $(A_j^{[-1;E]})_{j=0}^{\kappa+1}$) of $(A_j)_{j=0}^{\kappa}$ instead of $(A_j)_{j=0}^{\kappa}$.

Remark 5.2 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of Definition 5.1 and (3.1), we have $A_0^{(-1;E)} = E$ and $A_0^{[-1;E]} = E$.

Remark 5.3 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $k \in \mathbb{Z}_{0,\kappa}$, then the sequence $(A_j^{(-1;E)})_{j=0}^{k+1}$ (resp., $(A_j^{[-1;E]})_{j=0}^{k+1}$) is the left (resp., right) E-inverse SP-transform of $(A_j)_{j=0}^{k}$.

Lemma 5.4 *Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with left E-inverse SP-transform $(B_j)_{j=0}^{\kappa+1}$ and right E-inverse SP-transform $(C_j)_{j=0}^{\kappa+1}$. Then $E^* \in \mathbb{K}_{q\times p}$ and $(A_j^*)_{j=0}^{\kappa}$ has left E^*-inverse SP-transform $(C_j^*)_{j=0}^{\kappa+1}$ and right E^*-inverse SP-transform $(B_j^*)_{j=0}^{\kappa+1}$.*

Proof Lemma A.15 shows $E^* \in \mathbb{K}_{q\times p}$. Denote by $(\Lambda_j)_{j=0}^{\kappa+1}$ and $(\Upsilon_j)_{j=0}^{\kappa+1}$ the reciprocal sequence corresponding to $(T_{E,A;j})_{j=0}^{\kappa+1}$ and $(V_{E,A;j})_{j=0}^{\kappa+1}$, respectively. According to Definition 5.1, we have then $B_j = \sum_{\ell=0}^{j} \Lambda_{j-\ell} R_{E,A;\ell}$ and $C_j = \sum_{\ell=0}^{j} U_{E,A;\ell} \Upsilon_{j-\ell}$ for all $j \in \mathbb{Z}_{0,\kappa+1}$. Let $(\Delta_j)_{j=0}^{\kappa+1}$ and $(\Theta_j)_{j=0}^{\kappa+1}$ be defined by $\Delta_j := T_{E,A;j}^*$ and $\Theta_j := V_{E,A;j}^*$, respectively. From [20, Prop. 3.13] we can infer $(\Lambda_j^*)_{j=0}^{\kappa+1} = (\Delta_j^\sharp)_{j=0}^{\kappa+1}$ and $(\Upsilon_j^*)_{j=0}^{\kappa+1} = (\Theta_j^\sharp)_{j=0}^{\kappa+1}$, so that

$$B_j^* = \sum_{\ell=0}^{j} R_{E,A;\ell}^* \Delta_{j-\ell}^\sharp \qquad \text{and} \qquad C_j^* = \sum_{\ell=0}^{j} \Theta_{j-\ell}^\sharp U_{E,A;\ell}^*$$

for all $j \in \mathbb{Z}_{0,\kappa+1}$ follow. Let $F := E^*$ and let $(S_j)_{j=0}^{\kappa}$ be defined by $S_j := A_j^*$. By virtue of Definition 5.1, we have $R_{E,A;0}^* = E^* = F = U_{F,S;0}$ and $U_{E,A;0}^* = E^* = F = R_{F,S;0}$ as well as $T_{E,A;0}^* = I_p = V_{F,S;0}$ and $V_{E,A;0}^* = I_q = T_{F,S;0}$. Using Remark A.8, we obtain, for all $j \in \mathbb{Z}_{1,\kappa+1}$, in view of (5.1) and Definition 5.1, furthermore $R_{E,A;j}^* = \sqrt{r}^\dagger A_{j-1}^* \sqrt{l} = \sqrt{I_q - FF^*}^\dagger S_{j-1} \sqrt{I_p - F^*F} = U_{F,S;j}$ and $U_{E,A;j}^* = \sqrt{r} A_{j-1}^* \sqrt{l}^\dagger = \sqrt{I_q - FF^*} S_{j-1} \sqrt{I_p - F^*F}^\dagger = R_{F,S;j}$ as well as $T_{E,A;j}^* = E R_{E,A;j}^* = F^* U_{F,S;j} = V_{F,S;j}$ and $V_{E,A;j}^* = U_{E,A;j}^* E = R_{F,S;j} F^* = T_{F,S;j}$. In particular, we have shown that $(V_{F,S;j})_{j=0}^{\kappa+1} = (\Delta_j)_{j=0}^{\kappa+1}$ and $(T_{F,S;j})_{j=0}^{\kappa+1} = (\Theta_j)_{j=0}^{\kappa+1}$. Taking additionally into account Definition 5.1, we get then $S_j^{(-1;F)} = \sum_{\ell=0}^{j} T_{F,S;j-\ell}^\sharp R_{F,S;\ell} = \sum_{\ell=0}^{j} \Theta_{j-\ell}^\sharp U_{E,A;\ell}^* = C_j^*$ and $S_j^{[-1;F]} = \sum_{\ell=0}^{j} U_{F,S;\ell} V_{F,S;j-\ell}^\sharp = \sum_{\ell=0}^{j} R_{E,A;\ell}^* \Delta_{j-\ell}^\sharp = B_j^*$ for all $j \in \mathbb{Z}_{0,\kappa+1}$. □

Notation 5.5 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then, for all $n \in \mathbb{Z}_{0,\kappa+1}$, let $\mathbf{R}_n := \mathbf{S}_{R_{E,A};n}$ and $\mathbf{U}_n := \mathbf{S}_{U_{E,A};n}$ as well as $\mathbf{T}_n := \mathbf{S}_{T_{E,A};n}$ and $\mathbf{V}_n := \mathbf{S}_{V_{E,A};n}$ and furthermore $\mathbf{T}_n^\sharp := \mathbf{S}_{T_{E,A}^\sharp;n}$ and $\mathbf{V}_n^\sharp := \mathbf{S}_{V_{E,A}^\sharp;n}$.

Remark 5.6 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then

$$\mathbf{R}_n = \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n, \qquad \mathbf{T}_n = \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n + I_{(n+1)p},$$
$$\mathbf{U}_n = \langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n, \qquad \mathbf{V}_n = \langle\!\langle E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + I_{(n+1)q}$$

for all $n \in \mathbb{Z}_{1,\kappa+1}$ as well as $\mathbf{S}_n^{(-1;E)} = \mathbf{T}_n^\sharp \mathbf{R}_n$ and $\mathbf{S}_n^{[-1;E]} = \mathbf{U}_n \mathbf{V}_n^\sharp$ for all $n \in \mathbb{Z}_{0,\kappa+1}$ can be checked by straightforward calculation.

Now we use the notation introduced in Notation A.18.

Lemma 5.7 *Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $(T_{E,A;j})_{j=0}^{\kappa+1} \in \mathscr{D}_{p\times p;\kappa+1}$ and $(V_{E,A;j})_{j=0}^{\kappa+1} \in \mathscr{D}_{q\times q;\kappa+1}$. For each $n \in \mathbb{Z}_{0,\kappa+1}$, moreover, $\mathbf{T}_n \in \mathscr{L}_{p,n}$ and $\mathbf{V}_n \in \mathscr{L}_{q,n}$. In particular, $\det \mathbf{T}_n = 1$ and $\mathbf{T}_n^\sharp = \mathbf{T}_n^{-1}$ as well as $\det \mathbf{V}_n = 1$ and $\mathbf{V}_n^\sharp = \mathbf{V}_n^{-1}$ for all $n \in \mathbb{Z}_{0,\kappa+1}$.*

Proof First observe that $T_{E,A;0} = I_p$ and $V_{E,A;0} = I_q$. Consequently, $(T_{E,A;j})_{j=0}^{\kappa+1} \in \mathscr{D}_{p\times p;\kappa+1}$ and $(V_{E,A;j})_{j=0}^{\kappa+1} \in \mathscr{D}_{q\times q;\kappa+1}$ follow. Now we consider an arbitrary $n \in \mathbb{Z}_{0,\kappa+1}$. Regarding Notations A.18 and 5.5 and (2.2), then $\mathbf{T}_n \in \mathscr{L}_{p,n}$ and $\mathbf{V}_n \in \mathscr{L}_{q,n}$. In particular, $\det \mathbf{T}_n = 1$ and $\det \mathbf{V}_n = 1$. According to [22, Prop. 4.20], furthermore $\mathbf{T}_n^\sharp = \mathbf{T}_n^{-1}$ and $\mathbf{V}_n^\sharp = \mathbf{V}_n^{-1}$. □

Lemma 5.8 *Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{Z}_{0,\kappa+1}$, then $\mathbf{R}_n\mathbf{V}_n = \mathbf{T}_n\mathbf{U}_n$.*

Proof We have $\mathbf{R}_0\mathbf{V}_0 = E \cdot I_q = I_p \cdot E = \mathbf{T}_0\mathbf{U}_0$. Now suppose $\kappa \geq 1$ and consider an arbitrary $n \in \mathbb{Z}_{1,\kappa+1}$. Remarks 5.6 and A.24(b) yield

$$\begin{aligned}\mathbf{R}_n\mathbf{V}_n &= \left[\langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n\right]\left[\langle\!\langle E^*\sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + I_{(n+1)q}\right]\\ &= \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger \rangle\!\rangle_n\\ &\quad + \langle\!\langle EE^*\sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n\end{aligned}$$

and

$$\begin{aligned}\mathbf{T}_n\mathbf{U}_n &= \left[\langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n + I_{(n+1)p}\right]\left[\langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n\right]\\ &= \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger E^* E \rangle\!\rangle_n\\ &\quad + \langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n.\end{aligned}$$

Remark A.17(a) shows $l \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. We can thus apply Remark A.10(d) to obtain with (5.1) then

$$\sqrt{r}^\dagger - \sqrt{r}^\dagger E^*E = \sqrt{r}^\dagger (I_q - E^*E) = \sqrt{r}^\dagger r = \sqrt{r} \tag{5.5}$$

and

$$\sqrt{l}^\dagger - EE^*\sqrt{l}^\dagger = (I_p - EE^*)\sqrt{l}^\dagger = l\sqrt{l}^\dagger = \sqrt{l}. \tag{5.6}$$

Using additionally Remark A.24(b), we can conclude then $\mathbf{R}_n\mathbf{V}_n - \mathbf{T}_n\mathbf{U}_n = \langle\!\langle \sqrt{l} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}^\dagger - \sqrt{r}^\dagger E^*E \rangle\!\rangle_n + \langle\!\langle EE^*\sqrt{l}^\dagger - \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n = O$. □

Now we are able to verify that, for each matrix $E \in \mathbb{K}_{p\times q}$, the left and right E-inverse SP-transforms of a sequence $(A_j)_{j=0}^{\kappa}$ from $\mathbb{C}^{p\times q}$ coincide. This is a generalization of [6, Prop. 3.11].

Proposition 5.9 *Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $(A_j^{(-1;E)})_{j=0}^{\kappa+1} = (A_j^{[-1;E]})_{j=0}^{\kappa+1}$.*

Proof. We consider an arbitrary $n \in \mathbb{Z}_{0,\kappa+1}$. Remark 5.6 shows $\mathbf{S}_n^{(-1;E)} = \mathbf{T}_n^\sharp \mathbf{R}_n$ and $\mathbf{S}_n^{[-1;E]} = \mathbf{U}_n \mathbf{V}_n^\sharp$. Lemma 5.7 yields $\det \mathbf{T}_n \neq 0$ and $\mathbf{T}_n^\sharp = \mathbf{T}_n^{-1}$ as well as $\det \mathbf{V}_n \neq 0$ and $\mathbf{V}_n^\sharp = \mathbf{V}_n^{-1}$. Using additionally Lemma 5.8, we obtain

$$\begin{aligned}\mathbf{S}_n^{(-1;E)} - \mathbf{S}_n^{[-1;E]} &= \mathbf{T}_n^\sharp \mathbf{R}_n - \mathbf{U}_n \mathbf{V}_n^\sharp = \mathbf{T}_n^{-1}\mathbf{R}_n - \mathbf{U}_n \mathbf{V}_n^{-1}\\ &= \mathbf{T}_n^{-1}(\mathbf{R}_n\mathbf{V}_n - \mathbf{T}_n\mathbf{U}_n)\mathbf{V}_n^{-1} = O.\end{aligned}$$

□

In order to show that the inverse SP-transform with respect to given $E \in \mathbb{K}_{p\times q}$ maps the class $\mathscr{S}_{p\times q;\kappa}$ into the class $\mathscr{S}_{p\times q;\kappa+1}$, we prove the following result.

Lemma 5.10 *Let $E \in \mathbb{K}_{p\times q}$, let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $n \in \mathbb{Z}_{1,\kappa+1}$. Then* $\det \mathbf{V}_n \neq 0$ *and*

$$R_n^{[-1;E]} = \mathbf{V}_n^{-*} \operatorname{diag}(\langle\!\langle \sqrt{r} \rangle\!\rangle_{n-1}(I_{nq} - \mathbf{S}_{n-1}^* \langle\!\langle l l^\dagger \rangle\!\rangle_{n-1} \mathbf{S}_{n-1}) \langle\!\langle \sqrt{r} \rangle\!\rangle_{n-1}, r)\mathbf{V}_n^{-1}.$$

Proof Remark 5.6 shows $\mathbf{S}_n^{[-1;E]} = \mathbf{U}_n \mathbf{V}_n^\sharp$. Lemma 5.7 provides $\det \mathbf{V}_n \neq 0$ and $\mathbf{V}_n^\sharp = \mathbf{V}_n^{-1}$. Regarding (2.3), we can infer then

$$R_n^{[-1;E]} = I_{(n+1)q} - (\mathbf{S}_n^{[-1;E]})^* \mathbf{S}_n^{[-1;E]} = \mathbf{V}_n^{-*}(\mathbf{V}_n^*\mathbf{V}_n - \mathbf{U}_n^*\mathbf{U}_n)\mathbf{V}_n^{-1}.$$

Remark A.17(a) shows $l \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Remarks 5.6 and A.24 yield

$$\begin{aligned}\mathbf{V}_n^*\mathbf{V}_n &= \Big[\langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n + I_{(n+1)q}\Big]\Big[\langle\!\langle E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + I_{(n+1)q}\Big]\\ &= \langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger E E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n\\ &\quad + \langle\!\langle E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + I_{(n+1)q}\end{aligned}$$

and

$$\begin{aligned}\mathbf{U}_n^*\mathbf{U}_n &= \Big[\langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n + \langle\!\langle E^* \rangle\!\rangle_n\Big]\Big[\langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n\Big]\\ &= \langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n\\ &\quad + \langle\!\langle E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E^* E \rangle\!\rangle_n.\end{aligned}$$

Using (5.1) and Remark A.10(e), we get

$$\sqrt{l}^\dagger \sqrt{l}^\dagger - \sqrt{l}^\dagger E E^* \sqrt{l}^\dagger = \sqrt{l}^\dagger (I_p - E E^*) \sqrt{l}^\dagger = \sqrt{l}^\dagger l \sqrt{l}^\dagger = l l^\dagger. \tag{5.7}$$

Taking additionally into account Remark A.24(b), (3.2), (5.1), and (2.2), we can conclude then

$$\begin{aligned}
&\mathbf{V}_n^*\mathbf{V}_n - \mathbf{U}_n^*\mathbf{U}_n \\
&\quad = \langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle \sqrt{l}^\dagger E E^* \sqrt{l}^\dagger - \sqrt{l}^\dagger \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle I_q - E^* E \rangle\!\rangle_n \\
&\quad = \langle\!\langle r \rangle\!\rangle_n - \langle\!\langle \sqrt{r} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}^* \langle\!\langle l l^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n \\
&\quad = \operatorname{diag}(\langle\!\langle r \rangle\!\rangle_{n-1} - \langle\!\langle \sqrt{r} \rangle\!\rangle_{n-1} \mathbf{S}_{n-1}^* \langle\!\langle l l^\dagger \rangle\!\rangle_{n-1} \mathbf{S}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_{n-1}, r) \\
&\quad = \operatorname{diag}(\langle\!\langle \sqrt{r} \rangle\!\rangle_{n-1} (I_{nq} - \mathbf{S}_{n-1}^* \langle\!\langle l l^\dagger \rangle\!\rangle_{n-1} \mathbf{S}_{n-1}) \langle\!\langle \sqrt{r} \rangle\!\rangle_{n-1}, r),
\end{aligned}$$

which completes the proof. □

Now we are able to verify the result announced above, which is a generalization of [6, Prop. 3.6(d)].

Proposition 5.11 *Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. Then $(A_j^{[-1;E]})_{j=0}^{\kappa+1} \in \mathscr{S}_{p\times q;\kappa+1}$.*

Proof We consider an arbitrary $n \in \mathbb{Z}_{1,\kappa+1}$. Then $R_{n-1} \succcurlyeq O$. From Remarks A.6 and A.4 we can infer $ll^\dagger \preccurlyeq I_p$. In view of (3.2), then $\langle\!\langle ll^\dagger \rangle\!\rangle_{n-1} \preccurlyeq I_{np}$ follows, implying $\mathbf{S}_{n-1}^* \langle\!\langle ll^\dagger \rangle\!\rangle_{n-1} \mathbf{S}_{n-1} \preccurlyeq \mathbf{S}_{n-1}^* \mathbf{S}_{n-1}$. Taking additionally into account (2.3), we thus obtain $I_{nq} - \mathbf{S}_{n-1}^* \langle\!\langle ll^\dagger \rangle\!\rangle_{n-1} \mathbf{S}_{n-1} \succcurlyeq I_{nq} - \mathbf{S}_{n-1}^* \mathbf{S}_{n-1} = R_{n-1} \succcurlyeq O$. Since Remark A.17(a) shows $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, we can conclude from Lemma 5.10 then $R_n^{[-1;E]} \succcurlyeq O$. Hence, $(A_j^{[-1;E]})_{j=0}^{\kappa+1} \in \mathscr{S}_{p\times q;\kappa+1}$. □

The goal of our next considerations is to explain why we have chosen the terminology "inverse SP-transform". For this we still need some preparation.

Remark 5.12 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of Definition 5.1 and Remarks A.10(a) and A.9, then $\sum_{j=1}^{\kappa+1} \mathcal{R}(R_{E,A;j}) \subseteq \mathcal{R}(l)$ and $\mathcal{N}(r) \subseteq \bigcap_{j=1}^{\kappa+1} \mathcal{N}(R_{E,A;j})$ as well as $\sum_{j=1}^{\kappa+1} \mathcal{R}(U_{E,A;j}) \subseteq \mathcal{R}(l)$ and $\mathcal{N}(r) \subseteq \bigcap_{j=1}^{\kappa+1} \mathcal{N}(U_{E,A;j})$.

Lemma 5.13 *Let $L \in \mathbb{C}^{p\times p}$, let $R \in \mathbb{C}^{q\times p}$, and let $(M_j)_{j=1}^{\kappa+1}$ be a sequence of complex $p \times q$ matrices. Let the sequence $(C_j)_{j=0}^{\kappa+1}$ be defined by $C_0 := I_p$ and $C_j := LM_jR$ for all $j \in \mathbb{Z}_{1,\kappa+1}$. Then $C_0^\sharp = I_p$ and, for all $j \in \mathbb{Z}_{1,\kappa+1}$, there exists a matrix $N_j \in \mathbb{C}^{p\times q}$ such that $C_j^\sharp = LN_jR$.*

Proof Obviously, $C_0^\dagger = I_p$. Using [20, Thm. 3.9], we obtain then $C_0^\sharp = I_p$ and, for all $j \in \mathbb{Z}_{1,\kappa+1}$, furthermore $C_j^\sharp = \sum_{\ell=1}^{j} (-1)^\ell \sum_{(k_1,k_2,\dots,k_\ell)\in\mathcal{G}_{\ell,j}} C_{k_1} C_{k_2} \cdots C_{k_\ell}$, where $\mathcal{G}_{\ell,j} := \{[k_1, k_2, \dots, k_\ell] \in \mathbb{N}^{1\times\ell} : k_1 + k_2 + \cdots + k_\ell = j\}$. The assertion now follows from this representation, since $C_j = LM_jR$ for all $j \in \mathbb{Z}_{1,\kappa+1}$. □

Regarding Definition 5.1, from Lemma 5.13 we can infer the following:

Remark 5.14 Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $T_{E,A;0}^{\sharp} = I_p$ and $V_{E,A;0}^{\sharp} = I_q$. Furthermore, for all $j \in \mathbb{Z}_{1,\kappa+1}$, there exist matrices $M_j, N_j \in \mathbb{C}^{p\times q}$ such that $T_{E,A;j}^{\sharp} = \sqrt{l} M_j \sqrt{r}^{\dagger} E^*$ and $V_{E,A;j}^{\sharp} = E^* \sqrt{l}^{\dagger} N_j \sqrt{r}$.

Lemma 5.15 *Let $E \in \mathbb{K}_{p\times q}$ and let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $\sum_{j=1}^{\kappa+1} \mathcal{R}(A_j^{[-1;E]}) \subseteq \mathcal{R}(l)$ and $\mathcal{N}(r) \subseteq \bigcap_{j=1}^{\kappa+1} \mathcal{N}(A_j^{[-1;E]})$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{1,\kappa+1}$. According to Remark 5.14, there exist matrices $M_j, N_j \in \mathbb{C}^{p\times q}$ such that $T_{E,A;j}^{\sharp} = \sqrt{l} M_j \sqrt{r}^{\dagger} E^*$ and $V_{E,A;j}^{\sharp} = E^* \sqrt{l}^{\dagger} N_j \sqrt{r}$. By virtue of Definition 5.1, then $U_{E,A;0} V_{E,A;j}^{\sharp} = E E^* \sqrt{l}^{\dagger} N_j \sqrt{r}$ and $T_{E,A;j}^{\sharp} R_{E,A;0} = \sqrt{l} M_j \sqrt{r}^{\dagger} E^* E$, so that

$$A_j^{[-1;E]} = E E^* \sqrt{l}^{\dagger} N_j \sqrt{r} + \sum_{\ell=1}^{j} U_{E,A;\ell} V_{E,A;j-\ell}^{\sharp}$$

and

$$A_j^{(-1;E)} = \sqrt{l} M_j \sqrt{r}^{\dagger} E^* E + \sum_{\ell=1}^{j} T_{E,A;j-\ell}^{\sharp} R_{E,A;\ell}.$$

Using parts (a) and (b) of Lemma A.16, we can infer $l l^{\dagger} E E^* = E E^* l l^{\dagger}$ and $E^* E r^{\dagger} r = r^{\dagger} r E^* E$. Regarding Remark A.17(a), we can apply Remark A.10(c) to obtain $l l^{\dagger} = \sqrt{l}^{\dagger} \sqrt{l}$ and $r^{\dagger} r = \sqrt{r} \sqrt{r}^{\dagger}$. Taking additionally into account (2.1), then $l l^{\dagger} E E^* \sqrt{l}^{\dagger} = E E^* \sqrt{l}^{\dagger}$ and $\sqrt{r}^{\dagger} E^* E r^{\dagger} r = \sqrt{r}^{\dagger} E^* E$ follow. For each $\ell \in \mathbb{Z}_{1,j}$, by virtue of Remark 5.12, furthermore $\mathcal{R}(U_{E,A;\ell}) \subseteq \mathcal{R}(l)$ and $\mathcal{N}(r) \subseteq \mathcal{N}(R_{E,A;\ell})$, which, because of Remark A.7, implies $l l^{\dagger} U_{E,A;\ell} = U_{E,A;\ell}$ and $R_{E,A;\ell} r^{\dagger} r = R_{E,A;\ell}$. Summarizing, we can infer that $l l^{\dagger} A_j^{[-1;E]} = A_j^{[-1;E]}$ and $A_j^{(-1;E)} r^{\dagger} r = A_j^{(-1;E)}$. Consequently, $\mathcal{R}(A_j^{[-1;E]}) \subseteq \mathcal{R}(l)$ and $\mathcal{N}(r) \subseteq \mathcal{N}(A_j^{(-1;E)})$ follow. By virtue of Proposition 5.9, the proof is complete. □

Lemma 5.16 *Let $E \in \mathbb{K}_{p\times q}$, let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ such that $\mathcal{R}(E) \subseteq \mathcal{M}$ and $\sum_{j=0}^{\kappa} \mathcal{R}(A_j) \subseteq \mathcal{M}$. Then $\sum_{j=0}^{\kappa+1} \mathcal{R}(A_j^{[-1;E]}) \subseteq \mathcal{M}$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{0,\kappa+1}$. The assumption $\sum_{j=0}^{\kappa} \mathcal{R}(A_j) \subseteq \mathcal{M}$ implies $\mathcal{R}(A_{\ell-1} \sqrt{r}) \subseteq \mathcal{M}$ for all $\ell \in \mathbb{Z}_{1,\kappa+1}$. Taking additionally into account the assumption $\mathcal{R}(E) \subseteq \mathcal{M}$, we can thus apply Lemma B.8 to get $\mathcal{R}(\sqrt{l}^{\dagger} A_{\ell-1} \sqrt{r}) \subseteq \mathcal{M}$ for all $\ell \in \mathbb{Z}_{1,\kappa+1}$. Regarding Definition 5.1(b) and again $\mathcal{R}(E) \subseteq \mathcal{M}$, we hence get $\mathcal{R}(U_{E,A;\ell}) \subseteq \mathcal{M}$ for all $\ell \in \mathbb{Z}_{0,\kappa+1}$, so that $\mathcal{R}(A_j^{[-1;E]}) \subseteq \mathcal{M}$ follows. □

Lemma 5.17 *Let $E \in \mathbb{K}_{p\times q}$, let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ such that $\mathcal{Q} \subseteq \mathcal{N}(E)$ and $\mathcal{Q} \subseteq \bigcap_{j=0}^{\kappa} \mathcal{N}(A_j)$. Then $\mathcal{Q} \subseteq \bigcap_{j=0}^{\kappa+1} \mathcal{N}(A_j^{[-1;E]})$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{0,\kappa+1}$. The assumption $\mathcal{Q} \subseteq \bigcap_{j=0}^{\kappa} \mathcal{N}(A_j)$ implies $\mathcal{Q} \subseteq \mathcal{N}(\sqrt{l} A_{\ell-1})$ for all $\ell \in \mathbb{Z}_{1,\kappa+1}$. Taking additionally into account the assumption $\mathcal{Q} \subseteq \mathcal{N}(E)$, we can thus apply Lemma B.13 to get $\mathcal{Q} \subseteq \mathcal{N}(\sqrt{l} A_{\ell-1} \sqrt{r}^{\dagger})$ for all $\ell \in \mathbb{Z}_{1,\kappa+1}$. Regarding Definition 5.1(a) and again $\mathcal{Q} \subseteq \mathcal{N}(E)$, we hence get $\mathcal{Q} \subseteq \mathcal{N}(R_{E,A;\ell})$ for all $\ell \in \mathbb{Z}_{0,\kappa+1}$, so that $\mathcal{Q} \subseteq \mathcal{N}(A_j^{(-1;E)})$ follows. By virtue of Proposition 5.9, the proof is complete. □

Lemma 5.18 *Let* $E \in \mathbb{K}_{p\times q}$, *let* $(A_j)_{j=0}^{\kappa}$ *be a sequence of complex* $p \times q$ *matrices, and let* $n \in \mathbb{Z}_{1,\kappa+1}$. *Then*

$$\mathbf{U}_n \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n = \langle\!\langle \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle r^{\dagger} r \rangle\!\rangle_n + \langle\!\langle E(\sqrt{r}^{\dagger} + Q) \rangle\!\rangle_n$$

and

$$\mathbf{V}_n \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n = \langle\!\langle E^* \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle r^{\dagger} r \rangle\!\rangle_n + \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n. \tag{5.8}$$

Proof Regarding Remark A.17(a), we can apply Remark A.10(c) to obtain $\sqrt{r}\sqrt{r}^{\dagger} = r^{\dagger} r$. Taking into account (5.4) and (2.1), we get then

$$\sqrt{r}(\sqrt{r}^{\dagger} + Q) = \sqrt{r}\sqrt{r}^{\dagger} + \sqrt{r} Q = r^{\dagger} r + \sqrt{r}(I_q - \sqrt{r}^{\dagger}\sqrt{r}) = r^{\dagger} r. \tag{5.9}$$

By virtue of Remark A.24(b), we can thus conclude

$$\mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n = \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r}(\sqrt{r}^{\dagger} + Q) \rangle\!\rangle_n = \mathring{\mathbf{S}}_{n-1} \langle\!\langle r^{\dagger} r \rangle\!\rangle_n.$$

Consequently, using Remark 5.6 and again Remark A.24(b), we get finally

$$\begin{aligned} \mathbf{U}_n \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n &= \Big[\langle\!\langle \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n \Big] \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n \\ &= \langle\!\langle \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1} \langle\!\langle r^{\dagger} r \rangle\!\rangle_n + \langle\!\langle E(\sqrt{r}^{\dagger} + Q) \rangle\!\rangle_n \end{aligned}$$

and, analogously, (5.8). □

Lemma 5.19 *Let* $E \in \mathbb{K}_{p\times q}$. *Then* $(\sqrt{r} + Q)(\sqrt{r}^{\dagger} + Q) = I_q$, *where* Q *is given in* (5.2).

Proof First observe that Remark A.17(a) shows $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Thus, we can apply Remark A.10(c) to obtain $\sqrt{r}\sqrt{r}^{\dagger} = \sqrt{r}^{\dagger}\sqrt{r}$. In view of (5.4) and (2.1), we have $\sqrt{r} Q = \sqrt{r} - \sqrt{r}\sqrt{r}^{\dagger}\sqrt{r} = O$ and $Q\sqrt{r}^{\dagger} = \sqrt{r}^{\dagger} - \sqrt{r}^{\dagger}\sqrt{r}\sqrt{r}^{\dagger} = O$. Regarding (5.3) and Remark A.3, we see $Q^2 = Q$. Taking again into account (5.4), we finally conclude $(\sqrt{r} + Q)(\sqrt{r}^{\dagger} + Q) = \sqrt{r}^{\dagger}\sqrt{r} + Q = I_q$. □

Lemma 5.20 *Let $E \in \mathbb{K}_{p\times q}$, let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $n \in \mathbb{Z}_{1,\kappa+1}$. Then* $\det(\langle\!\langle E^*\sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n) \neq 0$ *and*

$$\mathbf{S}_n^{[-1;E]} = \left[\langle\!\langle \sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle E(\sqrt{r}^{\dagger} + Q)\rangle\!\rangle_n\right] \times\left[\langle\!\langle E^*\sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n\right]^{-1}.$$

***Proof*.** Remark 5.6 shows $\mathbf{S}_n^{[-1;E]} = \mathbf{U}_n\mathbf{V}_n^{\sharp}$. Lemma 5.7 yields $\det \mathbf{V}_n \neq 0$ and $\mathbf{V}_n^{\sharp} = \mathbf{V}_n^{-1}$. From Lemma 5.19 we infer $\det(\sqrt{r}^{\dagger} + Q) \neq 0$. Regarding (3.2), then $\det\langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n \neq 0$ follows. Taking additionally into account Lemma 5.18, we can conclude $\det(\langle\!\langle E^*\sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n) \neq 0$ and $[\langle\!\langle E^*\sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n]^{-1} = \langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n^{-1}\mathbf{V}_n^{-1}$. Using again Lemma 5.18, we finally get

$$\begin{aligned}&\left[\langle\!\langle \sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle E(\sqrt{r}^{\dagger} + Q)\rangle\!\rangle_n\right]\\ &\quad\times\left[\langle\!\langle E^*\sqrt{l}^{\dagger}\rangle\!\rangle_n \mathring{\mathbf{S}}_{n-1}\langle\!\langle r^{\dagger}r\rangle\!\rangle_n + \langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n\right]^{-1}\\ &= \mathbf{U}_n\langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n\langle\!\langle \sqrt{r}^{\dagger} + Q\rangle\!\rangle_n^{-1}\mathbf{V}_n^{-1} = \mathbf{U}_n\mathbf{V}_n^{-1} = \mathbf{U}_n\mathbf{V}_n^{\sharp} = \mathbf{S}_n^{[-1;E]}.\end{aligned}$$ □

Lemma 5.21 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{N}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. Then $\mathcal{R}(\mathbf{Y}_n) = \mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n)$ and $\mathcal{N}(\mathbf{Y}_n) = \mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)$ as well as* $\det(-\langle\!\langle \sqrt{r_0}^{\dagger}A_0^*\rangle\!\rangle_n\mathbf{S}_n + \langle\!\langle \sqrt{r_0}^{\dagger} + Q_0\rangle\!\rangle_n) \neq 0$ *and* $\mathbf{Y}_n^{\dagger} + \langle\!\langle Q_0\rangle\!\rangle_n = [-\langle\!\langle \sqrt{r_0}^{\dagger}A_0^*\rangle\!\rangle_n\mathbf{S}_n + \langle\!\langle \sqrt{r_0}^{\dagger} + Q_0\rangle\!\rangle_n]^{-1}$.

Proof Using Remarks 3.17 and A.24, we get $\mathbf{Y}_n\mathbf{Y}_n^{\dagger} = \langle\!\langle r_0r_0^{\dagger}\rangle\!\rangle_n = \langle\!\langle r_0\rangle\!\rangle_n\langle\!\langle r_0\rangle\!\rangle_n^{\dagger}$ and $\mathbf{Y}_n^{\dagger}\mathbf{Y}_n = \langle\!\langle r_0^{\dagger}r_0\rangle\!\rangle_n = \langle\!\langle r_0\rangle\!\rangle_n^{\dagger}\langle\!\langle r_0\rangle\!\rangle_n$ as well as $\mathbf{Y}_n\mathbf{Y}_n^{\dagger} = \mathbf{Y}_n^{\dagger}\mathbf{Y}_n$. From Remark A.6 we can infer then $\mathbb{P}_{\mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n)} = \mathbb{P}_{\mathcal{R}(\mathbf{Y}_n)} = \mathbb{P}_{\mathcal{R}(\mathbf{Y}_n^*)} = \mathbb{P}_{\mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n^*)}$, implying $\mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n) = \mathcal{R}(\mathbf{Y}_n)$ and $\mathcal{R}(\mathbf{Y}_n) = \mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n^*) = \mathcal{R}(\mathbf{Y}_n^*)$. By virtue of Remark A.2, then also $\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n) = \mathcal{N}(\mathbf{Y}_n)$ follows. Furthermore, we can apply Lemma A.11 to obtain $\det(\mathbf{Y}_n + \mathbb{P}_{\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)}) \neq 0$ and

$$\mathbf{Y}_n^{\dagger} = (\mathbf{Y}_n + \mathbb{P}_{\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)})^{-1} - \mathbb{P}_{\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)}.$$

According to Remark A.2, we have $\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)^{\perp} = \mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n^*)$. Using Remarks A.4, A.6 and A.24, (3.2), and (2.10), we obtain then

$$\mathbb{P}_{\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)} = I_{(n+1)q} - \mathbb{P}_{\mathcal{R}(\langle\!\langle r_0\rangle\!\rangle_n^*)} = I_{(n+1)q} - \langle\!\langle r_0\rangle\!\rangle_n^{\dagger}\langle\!\langle r_0\rangle\!\rangle_n = \langle\!\langle Q_0\rangle\!\rangle_n.$$

Remark 3.10 shows $\mathbf{Y}_n = \langle\!\langle \sqrt{r_0}^{\dagger}\rangle\!\rangle_n[I_{(n+1)q} - \langle\!\langle A_0^*\rangle\!\rangle_n\mathbf{S}_n]$. Taking into account Remark A.24(b), we can conclude then

$$\begin{aligned}\mathbf{Y}_n + \mathbb{P}_{\mathcal{N}(\langle\!\langle r_0\rangle\!\rangle_n)} &= \langle\!\langle \sqrt{r_0}^{\dagger}\rangle\!\rangle_n - \langle\!\langle \sqrt{r_0}^{\dagger}\rangle\!\rangle_n\langle\!\langle A_0^*\rangle\!\rangle_n\mathbf{S}_n + \langle\!\langle Q_0\rangle\!\rangle_n\\ &= -\langle\!\langle \sqrt{r_0}^{\dagger}A_0^*\rangle\!\rangle_n\mathbf{S}_n + \langle\!\langle \sqrt{r_0}^{\dagger} + Q_0\rangle\!\rangle_n.\end{aligned}$$

Thus, the remaining assertions follow. □

Lemma 5.22 *Suppose* $\kappa \geq 1$. *Let* $(A_j)_{j=0}^{\kappa} \in \mathcal{KN}_{p\times q;\kappa}$ *and let* $n \in \mathbb{Z}_{1,\kappa}$. *Then* $\det(-\langle\!\langle \sqrt{r_0}^{\dagger} A_0^* \rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle \sqrt{r_0}^{\dagger} + Q_0 \rangle\!\rangle_n) \neq 0$ *and*

$$\mathring{\mathbf{S}}_{n-1}^{[1]} = \left[\langle\!\langle \sqrt{l_0}^{\dagger} \rangle\!\rangle_n \mathbf{S}_n - \langle\!\langle \sqrt{l_0}^{\dagger} A_0 \rangle\!\rangle_n\right]\left[-\langle\!\langle \sqrt{r_0}^{\dagger} A_0^* \rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle \sqrt{r_0}^{\dagger} + Q_0 \rangle\!\rangle_n\right]^{-1}.$$

Proof From Remark 3.16 and Remark A.7(b) we can infer $Z_{A;j} r_0^{\dagger} r_0 = Z_{A;j}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Regarding Notation 3.8, (2.2), (2.10), and (3.2), then $\mathbf{Z}_{n-1}\langle\!\langle Q_0 \rangle\!\rangle_{n-1} = \mathbf{Z}_{n-1}\langle\!\langle I_q - r_0^{\dagger} r_0 \rangle\!\rangle_n = O_{np\times nq}$ follows. Taking additionally into account Remark 3.25, Notation 3.8, (2.2), and (3.2), we thus get

$$\begin{aligned} \mathring{\mathbf{Z}}_{n-1}[\mathbf{Y}_n^{\dagger} + \langle\!\langle Q_0 \rangle\!\rangle_n] &= \mathring{\mathbf{Z}}_{n-1}\mathbf{Y}_n^{\dagger} + \mathring{\mathbf{Z}}_{n-1}\langle\!\langle Q_0 \rangle\!\rangle_n \\ &= \mathring{\mathbf{S}}_{n-1}^{[1]} + \begin{bmatrix} O_{p\times nq} & O_{p\times q} \\ \mathbf{Z}_{n-1}\langle\!\langle Q_0 \rangle\!\rangle_{n-1} & O_{np\times q} \end{bmatrix} = \mathring{\mathbf{S}}_{n-1}^{[1]}. \end{aligned}$$

Using Remarks A.24(b) and 3.11 then $\mathring{\mathbf{Z}}_{n-1} = \langle\!\langle \sqrt{l_0}^{\dagger} \rangle\!\rangle_n \mathbf{S}_n - \langle\!\langle \sqrt{l_0}^{\dagger} A_0 \rangle\!\rangle_n$ follows. Consequently, by virtue of Lemma 5.21, the proof is complete. □

The next result provides a key observation for the realization of our aim formulated before Remark 5.12.

Lemma 5.23 *Let* $E \in \mathbb{K}_{p\times q}$ *and let* $(B_j)_{j=0}^{\kappa}$ *be a sequence of complex* $p \times q$ *matrices. Denote by* $(A_j)_{j=0}^{\kappa+1}$ *the right* E*-inverse SP-transform of* $(B_j)_{j=0}^{\kappa}$. *Then* $A_0 = E$ *and* $A_j^{[1]} = ll^{\dagger} B_j r^{\dagger} r$ *for all* $j \in \mathbb{Z}_{0,\kappa}$.

Proof First observe that Remark A.17(a) shows $l \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. According to Remark 5.2, we have $A_0 = E$. In particular, $A_0 \in \mathbb{K}_{p\times q}$. Regarding (2.5) and (5.1), furthermore $l_0 = l$ and $r_0 = r$. By virtue of (2.10) and (5.2), hence $P_0 = P$ and $Q_0 = Q$ follow. We now consider an arbitrary $n \in \mathbb{Z}_{1,\kappa+1}$. Lemma 5.15 provides then $\mathcal{N}(r_0) \subseteq \bigcap_{j=1}^{\kappa+1} \mathcal{N}(A_j)$. Consequently, $(A_j)_{j=0}^{\kappa+1} \in \mathcal{KN}_{p\times q;\kappa+1}$. Thus, we can apply Lemma 5.22 to obtain $\det(-\langle\!\langle \sqrt{r}^{\dagger} E^* \rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n) \neq 0$ and

$$\mathring{\mathbf{S}}_{n-1}^{[1]} = \left[\langle\!\langle \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathbf{S}_n - \langle\!\langle \sqrt{l}^{\dagger} E \rangle\!\rangle_n\right]\left[-\langle\!\langle \sqrt{r}^{\dagger} E^* \rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle \sqrt{r}^{\dagger} + Q \rangle\!\rangle_n\right]^{-1}.$$

Let $\mathbf{C}_n := \mathbf{S}_{U_{E,B};n}$ and $\mathbf{D}_n := \mathbf{S}_{V_{E,B};n}$ as well as $\mathbf{D}_n^{\sharp} := \mathbf{S}_{V_{E,B}^{\sharp};n}$. Regarding Notation 5.5, then Remark 5.6 shows

$$\mathbf{C}_n = \langle\!\langle \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle E \rangle\!\rangle_n, \quad \mathbf{D}_n = \langle\!\langle E^* \sqrt{l}^{\dagger} \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + I_{(n+1)q},$$

and $\mathbf{S}_n = \mathbf{C}_n \mathbf{D}_n^\sharp$. Lemma 5.7 yields $\det \mathbf{D}_n \neq 0$ and $\mathbf{D}_n^\sharp = \mathbf{D}_n^{-1}$. Summarizing, we can infer $\det(-\langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n \mathbf{C}_n + \langle\!\langle \sqrt{r}^\dagger + Q \rangle\!\rangle_n \mathbf{D}_n) \neq 0$ and

$$\begin{aligned}\mathring{\mathbf{S}}_{n-1}^{[1]} &= \Big[\langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathbf{C}_n \mathbf{D}_n^\sharp - \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n\Big]\Big[-\langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n \mathbf{C}_n \mathbf{D}_n^\sharp + \langle\!\langle \sqrt{r}^\dagger + Q \rangle\!\rangle_n\Big]^{-1} \\ &= \Big[\langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathbf{C}_n - \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n \mathbf{D}_n\Big]\Big[-\langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n \mathbf{C}_n + \langle\!\langle \sqrt{r}^\dagger + Q \rangle\!\rangle_n \mathbf{D}_n\Big]^{-1}.\end{aligned}$$

Using Remark A.24(b), we get

$$\begin{aligned}\langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathbf{C}_n &= \langle\!\langle \sqrt{l}^\dagger \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n, \\ \langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n \mathbf{C}_n &= \langle\!\langle \sqrt{r}^\dagger E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{r}^\dagger E^* E \rangle\!\rangle_n\end{aligned}$$

and

$$\begin{aligned}\langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n \mathbf{D}_n &= \langle\!\langle \sqrt{l}^\dagger E E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n, \\ \langle\!\langle \sqrt{r}^\dagger + Q \rangle\!\rangle_n \mathbf{D}_n &= \langle\!\langle (\sqrt{r}^\dagger + Q) E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{r}^\dagger + Q \rangle\!\rangle_n.\end{aligned}$$

From (5.1) and parts (e) and (d) of Remark A.10 we obtain

$$\sqrt{l}^\dagger \sqrt{l}^\dagger - \sqrt{l}^\dagger E E^* \sqrt{l}^\dagger = \sqrt{l}^\dagger (I_p - E E^*) \sqrt{l}^\dagger = \sqrt{l}^\dagger l \sqrt{l}^\dagger = l l^\dagger \tag{5.10}$$

and

$$\sqrt{r}^\dagger (I_q - E^* E) = \sqrt{r}^\dagger r = \sqrt{r}. \tag{5.11}$$

Using Remark A.17(c), (5.4), and (2.1), we get furthermore

$$Q E^* \sqrt{l}^\dagger = Q \sqrt{r}^\dagger E^* = (I_q - \sqrt{r}^\dagger \sqrt{r}) \sqrt{r}^\dagger E^* = O. \tag{5.12}$$

Taking additionally into account Remark A.24(b), then

$$\begin{aligned}\langle\!\langle \sqrt{l}^\dagger \rangle\!\rangle_n \mathbf{C}_n - \langle\!\langle \sqrt{l}^\dagger E \rangle\!\rangle_n \mathbf{D}_n &= \langle\!\langle \sqrt{l}^\dagger \sqrt{l}^\dagger - \sqrt{l}^\dagger E E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n \\ &= \langle\!\langle l l^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n\end{aligned}$$

and

$$\begin{aligned}&- \langle\!\langle \sqrt{r}^\dagger E^* \rangle\!\rangle_n \mathbf{C}_n + \langle\!\langle \sqrt{r}^\dagger + Q \rangle\!\rangle_n \mathbf{D}_n \\ &= \langle\!\langle -\sqrt{r}^\dagger E^* \sqrt{l}^\dagger + (\sqrt{r}^\dagger + Q) E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n \\ &\quad + \langle\!\langle -\sqrt{r}^\dagger E^* E + \sqrt{r}^\dagger + Q \rangle\!\rangle_n \\ &= \langle\!\langle Q E^* \sqrt{l}^\dagger \rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1} \langle\!\langle \sqrt{r} \rangle\!\rangle_n + \langle\!\langle \sqrt{r}^\dagger (I_q - E^* E) + Q \rangle\!\rangle_n = \langle\!\langle \sqrt{r} + Q \rangle\!\rangle_n\end{aligned}$$

follow. Consequently, we have $\det\langle\!\langle\sqrt{r}+Q\rangle\!\rangle_n \neq 0$ and $\mathring{\mathbf{S}}^{[1]}_{n-1} = \langle\!\langle ll^\dagger\rangle\!\rangle_n \mathring{\mathbf{S}}_{B;n-1}\langle\!\langle\sqrt{r}\rangle\!\rangle_n \langle\!\langle\sqrt{r}+Q\rangle\!\rangle_n^{-1}$. Regarding (3.2), then in particular, $\det(\sqrt{r}+Q)\neq 0$. Using Remark A.10(c), (5.4), and (2.1), we conclude

$$r^\dagger r(\sqrt{r}+Q) = r^\dagger r\sqrt{r} + r^\dagger r Q = \sqrt{r}\sqrt{r}^\dagger\sqrt{r} + \sqrt{r}^\dagger\sqrt{r}(I_q - \sqrt{r}^\dagger\sqrt{r}) = \sqrt{r},$$

so that $\sqrt{r}(\sqrt{r}+Q)^{-1} = r^\dagger r$. Regarding Remark A.24, hence $\langle\!\langle\sqrt{r}\rangle\!\rangle_n\langle\!\langle\sqrt{r}+Q\rangle\!\rangle_n^{-1} = \langle\!\langle r^\dagger r\rangle\!\rangle_n$ follows. Thus, we obtain $\mathring{\mathbf{S}}^{[1]}_{n-1} = \langle\!\langle ll^\dagger\rangle\!\rangle_n\mathring{\mathbf{S}}_{B;n-1}\langle\!\langle r^\dagger r\rangle\!\rangle_n$. Taking into account (2.2) and (3.2), therefore $\mathbf{S}^{[1]}_{n-1} = \langle\!\langle ll^\dagger\rangle\!\rangle_{n-1}\mathbf{S}_{B;n-1}\langle\!\langle r^\dagger r\rangle\!\rangle_{n-1}$ and, in particular, $A^{[1]}_{n-1} = ll^\dagger B_{n-1}r^\dagger r$. Since $n\in\mathbb{Z}_{1,\kappa+1}$ was arbitrarily chosen, the proof is complete. □

Proposition 5.24 *Suppose $\kappa\geq 1$. Let $(A_j)_{j=0}^{\kappa}\in\mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa}$ and let $E := A_0$. Denote by $(B_j)_{j=0}^{\kappa-1}$ the right SP-transform of $(A_j)_{j=0}^{\kappa}$. Then $B_j^{[-1;E]} = A_j$ for all $j\in\mathbb{Z}_{0,\kappa}$.*

Proof First observe that $E\in\mathbb{K}_{p\times q}$, so that Remark A.17(a) shows $l\in\mathbb{C}^{p\times p}_{\succcurlyeq}$ and $r\in\mathbb{C}^{q\times q}_{\succcurlyeq}$. Regarding (2.5) and (5.1), we see that $l_0 = l$ and $r_0 = r$. By virtue of (2.10) and (5.2), hence $P_0 = P$ and $Q_0 = Q$ follow. Denote by $(C_j)_{j=0}^{\kappa}$ the right E-inverse SP-transform of $(B_j)_{j=0}^{\kappa-1}$. According to Remark 5.2, we have then $C_0 = E = A_0$. We now consider an arbitrary $n\in\mathbb{Z}_{1,\kappa}$. Then $\mathring{\mathbf{S}}_{B;n-1} = \mathring{\mathbf{S}}^{[1]}_{n-1}$, so that Lemma 5.20 shows the inequality $\det(\langle\!\langle E^*\sqrt{l}^\dagger\rangle\!\rangle_n\mathring{\mathbf{S}}^{[1]}_{n-1}\langle\!\langle r^\dagger r\rangle\!\rangle_n + \langle\!\langle\sqrt{r}^\dagger+Q\rangle\!\rangle_n)\neq 0$ and

$$\mathbf{S}_{C;n} = \left[\langle\!\langle\sqrt{l}^\dagger\rangle\!\rangle_n\mathring{\mathbf{S}}^{[1]}_{n-1}\langle\!\langle r^\dagger r\rangle\!\rangle_n + \langle\!\langle E(\sqrt{r}^\dagger+Q)\rangle\!\rangle_n\right] \times\left[\langle\!\langle E^*\sqrt{l}^\dagger\rangle\!\rangle_n\mathring{\mathbf{S}}^{[1]}_{n-1}\langle\!\langle r^\dagger r\rangle\!\rangle_n + \langle\!\langle\sqrt{r}^\dagger+Q\rangle\!\rangle_n\right]^{-1}.$$

Remarks 3.22 and A.7(b) yield $A_j^{[1]}r^\dagger r = A_j^{[1]}$ for all $j\in\mathbb{Z}_{0,\kappa-1}$. Regarding (2.2) and (3.2), hence $\mathring{\mathbf{S}}^{[1]}_{n-1}\langle\!\langle r^\dagger r\rangle\!\rangle_n = \mathring{\mathbf{S}}^{[1]}_{n-1}$. Setting

$$\mathbf{F}_n := \langle\!\langle\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{S}_n - \langle\!\langle\sqrt{l}^\dagger E\rangle\!\rangle_n, \qquad \mathbf{G}_n := -\langle\!\langle\sqrt{r}^\dagger E^*\rangle\!\rangle_n\mathbf{S}_n + \langle\!\langle\sqrt{r}^\dagger+Q\rangle\!\rangle_n,$$

we can furthermore apply Lemma 5.22 to obtain $\det\mathbf{G}_n\neq 0$ and $\mathring{\mathbf{S}}^{[1]}_{n-1} = \mathbf{F}_n\mathbf{G}_n^{-1}$. Summarizing, we infer

$$\begin{aligned}\mathbf{S}_{C;n} &= \left[\langle\!\langle\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{F}_n\mathbf{G}_n^{-1} + \langle\!\langle E(\sqrt{r}^\dagger+Q)\rangle\!\rangle_n\right]\left[\langle\!\langle E^*\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{F}_n\mathbf{G}_n^{-1} + \langle\!\langle\sqrt{r}^\dagger+Q\rangle\!\rangle_n\right]^{-1}\\ &= \left[\langle\!\langle\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{F}_n + \langle\!\langle E(\sqrt{r}^\dagger+Q)\rangle\!\rangle_n\mathbf{G}_n\right]\left[\langle\!\langle E^*\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{F}_n + \langle\!\langle\sqrt{r}^\dagger+Q\rangle\!\rangle_n\mathbf{G}_n\right]^{-1}.\end{aligned}$$

Using Remark A.24(b), we get

$$\langle\!\langle\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{F}_n = \langle\!\langle\sqrt{l}^\dagger\sqrt{l}^\dagger\rangle\!\rangle_n\mathbf{S}_n - \langle\!\langle\sqrt{l}^\dagger\sqrt{l}^\dagger E\rangle\!\rangle_n,$$

$$\langle\!\langle E^*\sqrt{l}^\dagger\rangle\!\rangle_n \mathbf{F}_n = \langle\!\langle E^*\sqrt{l}^\dagger\sqrt{l}^\dagger\rangle\!\rangle_n \mathbf{S}_n - \langle\!\langle E^*\sqrt{l}^\dagger\sqrt{l}^\dagger E\rangle\!\rangle_n$$

and

$$\langle\!\langle E(\sqrt{r}^\dagger + Q)\rangle\!\rangle_n \mathbf{G}_n = -\langle\!\langle E(\sqrt{r}^\dagger + Q)\sqrt{r}^\dagger E^*\rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle E(\sqrt{r}^\dagger + Q)^2\rangle\!\rangle_n,$$
$$\langle\!\langle \sqrt{r}^\dagger + Q\rangle\!\rangle_n \mathbf{G}_n = -\langle\!\langle (\sqrt{r}^\dagger + Q)\sqrt{r}^\dagger E^*\rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle (\sqrt{r}^\dagger + Q)^2\rangle\!\rangle_n.$$

Furthermore, we can use parts (b) and (c) of Remark A.10 to obtain

$$\sqrt{l}^\dagger\sqrt{l}^\dagger = l^\dagger, \qquad \sqrt{r}^\dagger\sqrt{r}^\dagger = r^\dagger, \qquad \text{and} \qquad \sqrt{r}^\dagger\sqrt{r} = \sqrt{r}\sqrt{r}^\dagger.$$

In view of (5.4) and (2.1), we thus obtain

$$\sqrt{r}^\dagger Q = \sqrt{r}^\dagger(I_q - \sqrt{r}\sqrt{r}^\dagger) = O, \qquad Q\sqrt{r}^\dagger = (I_q - \sqrt{r}^\dagger\sqrt{r})\sqrt{r}^\dagger = O.$$

Regarding (5.3) and Remark A.3, we see $Q^2 = Q$. Hence, we can conclude

$$(\sqrt{r}^\dagger + Q)^2 = \sqrt{r}^\dagger\sqrt{r}^\dagger + \sqrt{r}^\dagger Q + Q\sqrt{r}^\dagger + Q^2 = r^\dagger + Q.$$

Using parts (c), (a), and (b) of Lemma A.16 as well as (5.2), we get then

$$\sqrt{l}^\dagger\sqrt{l}^\dagger - E(\sqrt{r}^\dagger + Q)\sqrt{r}^\dagger E^* = l^\dagger - E\sqrt{r}^\dagger\sqrt{r}^\dagger E^* = l^\dagger - Er^\dagger E^* = ll^\dagger, \tag{5.13}$$

$$E(\sqrt{r}^\dagger + Q)^2 - \sqrt{l}^\dagger\sqrt{l}^\dagger E = E(r^\dagger + Q) - l^\dagger E = EQ, \tag{5.14}$$

$$E^*\sqrt{l}^\dagger\sqrt{l}^\dagger - (\sqrt{r}^\dagger + Q)\sqrt{r}^\dagger E^* = E^* l^\dagger - \sqrt{r}^\dagger\sqrt{r}^\dagger E^* = E^* l^\dagger - r^\dagger E^* = O, \tag{5.15}$$

and

$$(\sqrt{r}^\dagger + Q)^2 - E^*\sqrt{l}^\dagger\sqrt{l}^\dagger E = r^\dagger + Q - E^* l^\dagger E = r^\dagger r + Q = I_q. \tag{5.16}$$

Taking additionally into account Remark A.24(b) and (3.2), then

$$\begin{aligned}
&\langle\!\langle \sqrt{l}^\dagger\rangle\!\rangle_n \mathbf{F}_n + \langle\!\langle E(\sqrt{r}^\dagger + Q)\rangle\!\rangle_n \mathbf{G}_n \\
&\quad = \langle\!\langle \sqrt{l}^\dagger\sqrt{l}^\dagger - E(\sqrt{r}^\dagger + Q)\sqrt{r}^\dagger E^*\rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle E(\sqrt{r}^\dagger + Q)^2 - \sqrt{l}^\dagger\sqrt{l}^\dagger E\rangle\!\rangle_n \\
&\quad = \langle\!\langle ll^\dagger\rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle EQ\rangle\!\rangle_n
\end{aligned}$$

and

$$\begin{aligned}
&\langle\!\langle E^*\sqrt{l}^\dagger\rangle\!\rangle_n \mathbf{F}_n + \langle\!\langle \sqrt{r}^\dagger + Q\rangle\!\rangle_n \mathbf{G}_n \\
&\quad = \langle\!\langle E^*\sqrt{l}^\dagger\sqrt{l}^\dagger - (\sqrt{r}^\dagger + Q)\sqrt{r}^\dagger E^*\rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle (\sqrt{r}^\dagger + Q)^2 - E^*\sqrt{l}^\dagger\sqrt{l}^\dagger E\rangle\!\rangle_n \\
&\quad = \langle\!\langle O_{q\times p}\rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle I_q\rangle\!\rangle_n = I_{(n+1)q}
\end{aligned}$$

follow. Consequently, we have $\mathbf{S}_{C;n} = \langle\!\langle ll^\dagger \rangle\!\rangle_n \mathbf{S}_n + \langle\!\langle EQ \rangle\!\rangle_n$. The assumption $(A_j)_{j=0}^{\kappa} \in \mathscr{KRN}_{p\times q;\kappa}$ and Remark A.7(a) yield $ll^\dagger A_j = A_j$ for all $j \in \mathbb{Z}_{1,\kappa}$. Regarding (2.2), (3.2) and $n \geq 1$, therefore $C_n = ll^\dagger A_n = A_n$. Since $n \in \mathbb{Z}_{1,\kappa}$ was arbitrarily chosen, the proof is complete. □

Proposition 5.24 yields immediately a generalization of [6, Prop. 3.7].

Corollary 5.25 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. Denote by $(B_j)_{j=0}^{\kappa-1}$ the right SP-transform of $(A_j)_{j=0}^{\kappa}$ and let $E := A_0$. Then $B_j^{[-1;E]} = A_j$ for all $j \in \mathbb{Z}_{0,\kappa}$.*

Proof Remark 3.3 yields $\mathscr{S}_{p\times q;\kappa} \subseteq \mathscr{KRN}_{p\times q;\kappa}$. Consequently, applying Proposition 5.24 completes the proof. □

Lemma 5.26 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times p;\kappa}$ with right SP-transform $(B_j)_{j=0}^{\kappa-1}$. Then the following statements are equivalent:*

(i) $A_0^* = A_0$ *and* $(B_j^*)_{j=0}^{\kappa-1} = (B_j)_{j=0}^{\kappa-1}$
(ii) $(A_j^*)_{j=0}^{\kappa} = (A_j)_{j=0}^{\kappa}$.

Proof "(i) ⇒ (ii)": The assumption $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ implies that $E := A_0$ belongs to $\mathbb{K}_{p\times q}$. Corollary 5.25 shows $(B_j^{[-1;E]})_{j=0}^{\kappa} = (A_j)_{j=0}^{\kappa}$. Applying Lemma 5.4 to the sequence $(B_j)_{j=0}^{\kappa-1}$ yields that $F := E^*$ belongs to $\mathbb{K}_{q\times p}$ and that $(S_j)_{j=0}^{\kappa-1}$ defined by $S_j := B_j^*$ has left F-inverse SP-transform $(A_j^*)_{j=0}^{\kappa}$, i.e. $(A_j^*)_{j=0}^{\kappa} = (S_j^{(-1;F)})_{j=0}^{\kappa}$. Because of (i), we have $(S_j^{(-1;F)})_{j=0}^{\kappa} = (B_j^{(-1;E)})_{j=0}^{\kappa}$. Proposition 5.9 yields $(B_j^{(-1;E)})_{j=0}^{\kappa} = (B_j^{[-1;E]})_{j=0}^{\kappa}$. Consequently, (ii) holds true.

"(ii) ⇒ (i)": Regarding Remark 3.3, we can apply Lemma 3.7 to see that $(T_j)_{j=0}^{\kappa}$ defined by $T_j := A_j^*$ belongs to $\mathscr{K}_{q\times p;\kappa}$ and has left SP-transform $(B_j^*)_{j=0}^{\kappa-1}$, i.e. $(B_j^*)_{j=0}^{\kappa-1} = (T_j^{(1)})_{j=0}^{\kappa-1}$. Because of (ii), we have $A_0^* = A_0$ and $(T_j)_{j=0}^{\kappa} = (A_j)_{j=0}^{\kappa}$, implying $(T_j^{(1)})_{j=0}^{\kappa-1} = (A_j^{(1)})_{j=0}^{\kappa-1}$. In view of Remark 3.3, we can apply Proposition 3.19 to get $(A_j^{(1)})_{j=0}^{\kappa-1} = (B_j)_{j=0}^{\kappa-1}$. Consequently, (i) holds true. □

6 Parametrization of the Class $\mathscr{S}_{p\times q;\kappa}$

In this section, we are going to determine which sequences $(\mathfrak{e}_j)_{j=0}^{\kappa}$ occur really as SP-parameter sequence of a sequence $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. First we introduce two sequences of linear subspaces which will turn out to be essential for our further considerations.

Notation 6.1 Let $(\mathfrak{e}_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then let $\mathcal{M}_{-1} := \mathbb{C}^p$ and $\mathcal{Q}_{-1} := \{O_{q\times 1}\}$. Furthermore, in view of Notation 4.11, for all $j \in \mathbb{Z}_{0,\kappa}$, let $\mathcal{M}_j := \bigcap_{\ell=0}^{j} \mathcal{R}(\mathfrak{l}_\ell)$ and $\mathcal{Q}_j := \sum_{\ell=0}^{j} \mathcal{N}(\mathfrak{r}_\ell)$.

The set introduced in the following notation will turn out as one of the most important objects occurring in this paper.

Notation 6.2 Let $\mathscr{E}_{p\times q;\kappa}$ be the set of all sequences $(\mathfrak{e}_j)_{j=0}^{\kappa}$ of complex $p\times q$ matrices which, for all $j\in\mathbb{Z}_{0,\kappa}$, fulfill $\mathfrak{e}_j\in\mathbb{K}_{p\times q}$ as well as $\mathcal{R}(\mathfrak{e}_j)\subseteq\mathcal{M}_{j-1}$ and $\mathcal{Q}_{j-1}\subseteq\mathcal{N}(\mathfrak{e}_j)$.

The following observation corresponds to the description of all SP-parameter sequences of non-degenerate $p\times q$ Schur sequences.

Remark 6.3 Let $(\mathfrak{e}_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{D}_{p\times q}$. In view of Notations 6.1 and 4.11, then $\mathcal{M}_j=\mathbb{C}^p$ and $\mathcal{Q}_j=\{O_{q\times 1}\}$ for all $j\in\mathbb{Z}_{-1,\kappa}$, so that $(\mathfrak{e}_j)_{j=0}^{\kappa}\in\mathscr{E}_{p\times q;\kappa}$.

Notation 6.4 Let $(\mathfrak{e}_j)_{j=0}^{\kappa}$ be a sequence of contractive complex $p\times q$ matrices. Then let $\mathfrak{M}_{-1}:=I_p$ and $\mathfrak{Q}_{-1}:=I_q$. Furthermore, using Notation 4.11, for all $j\in\mathbb{Z}_{0,\kappa}$, let

$$\mathfrak{M}_j:=\sqrt{\mathfrak{l}_j}^{\dagger}\sqrt{\mathfrak{l}_{j-1}}^{\dagger}\cdots\sqrt{\mathfrak{l}_0}^{\dagger}\qquad\text{and}\qquad\mathfrak{Q}_j:=\sqrt{\mathfrak{r}_0}^{\dagger}\sqrt{\mathfrak{r}_1}^{\dagger}\cdots\sqrt{\mathfrak{r}_j}^{\dagger}.$$

Remark 6.5 Let $(A_j)_{j=0}^{\kappa}\in\mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For each $j\in\mathbb{Z}_{0,\kappa}$, then $\mathcal{R}(\mathfrak{M}_j)=\sqrt{\mathfrak{l}_j}^{\dagger}\mathcal{R}(\mathfrak{M}_{j-1})$ and $\mathcal{N}(\mathfrak{Q}_j)=\{v\in\mathbb{C}^q:\sqrt{\mathfrak{r}_j}^{\dagger}v\in\mathcal{N}(\mathfrak{Q}_{j-1})\}$.

Now we will see that the matrices introduced in Notation 6.4 are closely related to the SP-algorithm for a $p\times q$ Schur sequence $(A_j)_{j=0}^{\kappa}$.

Proposition 6.6 *Let $(A_j)_{j=0}^{\kappa}\in\mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For every choice of $k\in\mathbb{Z}_{0,\kappa}$ and $j\in\mathbb{Z}_{0,\kappa-k}$, then*

$$\mathcal{R}(A_j^{[k]})\subseteq\mathcal{R}(\mathfrak{M}_{k-1})\qquad\textit{and}\qquad\mathcal{N}(\mathfrak{Q}_{k-1})\subseteq\mathcal{N}(A_j^{[k]}).\tag{6.1}$$

Proof Regarding Notation 6.4, we see that the assertion holds true obviously in the case $k=0$. Now we work inductively and assume that $\kappa\geq 1$, that $m\in\mathbb{Z}_{0,\kappa-1}$, and that (6.1) is valid for every choice of $k\in\mathbb{Z}_{0,m}$ and $j\in\mathbb{Z}_{0,\kappa-k}$. Denote by $(C_j)_{j=0}^{\kappa-m}$ the m-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. Remark 4.2 yields then $(C_j)_{j=0}^{\kappa-m}\in\mathscr{S}_{p\times q;\kappa-m}$, which, by virtue of Remark 3.3, implies $(C_j)_{j=0}^{\kappa-m}\in\mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa-m}$. In view of Definition 3.4 and Remark 4.14, we have then

$$X_{C;\ell}=C_{\ell+1}\sqrt{\mathfrak{r}_m}^{\dagger}\qquad\text{and}\qquad Z_{C;\ell}=\sqrt{\mathfrak{l}_m}^{\dagger}C_{\ell+1}\qquad\text{for all }\ell\in\mathbb{Z}_{0,\kappa-m-1}.\tag{6.2}$$

Consider now an arbitrary $v\in\mathcal{N}(\mathfrak{Q}_m)$. Remark 6.5 yields then $\sqrt{\mathfrak{r}_m}^{\dagger}v\in\mathcal{N}(\mathfrak{Q}_{m-1})$. Since we assume that (6.1) is valid for $k=m$ and all $j\in\mathbb{Z}_{0,\kappa-m}$, we get then $\sqrt{\mathfrak{r}_m}^{\dagger}v\in\mathcal{N}(C_j)$ for all $j\in\mathbb{Z}_{0,\kappa-m}$ and, by virtue of (6.2), consequently, $X_{C;\ell}v=O$ for all $\ell\in\mathbb{Z}_{0,\kappa-m-1}$. Hence,

$$\mathcal{N}(\mathfrak{Q}_m)\subseteq\mathcal{N}(X_{C;\ell})\qquad\text{for all }\ell\in\mathbb{Z}_{0,\kappa-m-1}\tag{6.3}$$

is proved. Analogously, using (6.2), the assumption that (6.1) holds true for $k=m$ and all $j\in\mathbb{Z}_{0,\kappa-m}$, and Remark 6.5, we can infer

$$\mathcal{R}(Z_{C;\ell})\subseteq\mathcal{R}(\mathfrak{M}_m)\qquad\text{for all }\ell\in\mathbb{Z}_{0,\kappa-m-1}.\tag{6.4}$$

In view of Proposition 3.19 and Definition 4.1, we have $C_j^{(1)} = C_j^{[1]} = A_j^{[m+1]}$ for all $j \in \mathbb{Z}_{0,\kappa-m-1}$. Taking additionally into account Definition 3.4, for all $j \in \mathbb{Z}_{0,\kappa-m-1}$, from (6.3) we can conclude

$$\mathcal{N}(\mathfrak{Q}_m) \subseteq \mathcal{N}\left(\sum_{\ell=0}^{j} W_{C;j-\ell}^{\sharp} X_{C;\ell}\right) = \mathcal{N}(C_j^{(1)}) = \mathcal{N}(A_j^{[m+1]})$$

and from (6.4) moreover

$$\mathcal{R}(A_j^{[m+1]}) = \mathcal{R}(C_j^{[1]}) = \mathcal{R}\left(\sum_{\ell=0}^{j} Z_{C;\ell} Y_{C;j-\ell}^{\sharp}\right) \subseteq \mathcal{R}(\mathfrak{M}_m).$$

Thus, (6.1) is valid for $k = m+1$ and all $j \in \mathbb{Z}_{0,\kappa-(m+1)}$. Consequently, the assertion is proved inductively. □

Corollary 6.7 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For each $j \in \mathbb{Z}_{0,\kappa}$, then there exists a matrix $M_j \in \mathbb{C}^{p\times q}$ such that $\mathfrak{e}_j = \mathfrak{M}_{j-1} M_j \mathfrak{Q}_{j-1}$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{0,\kappa}$. According to Definition 4.7, we have $\mathfrak{e}_j = A_0^{[j]}$. Proposition 6.6 yields $\mathcal{R}(A_0^{[j]}) \subseteq \mathcal{R}(\mathfrak{M}_{j-1})$ and $\mathcal{N}(\mathfrak{Q}_{j-1}) \subseteq \mathcal{N}(A_0^{[j]})$. Consequently, $\mathcal{R}(\mathfrak{e}_j) \subseteq \mathcal{R}(\mathfrak{M}_{j-1})$ and $\mathcal{N}(\mathfrak{Q}_{j-1}) \subseteq \mathcal{N}(\mathfrak{e}_j)$. The application of Remark A.7 completes the proof. □

Now we are going to show that the SP-parameter sequence of a sequence $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ belongs to $\mathscr{E}_{p\times q;\kappa}$. This requires some preparations.

Lemma 6.8 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For each $j \in \mathbb{Z}_{0,\kappa}$, then $\mathcal{R}(I_p - \sqrt{\mathfrak{l}_j}^{\dagger}) \subseteq \mathcal{R}(\mathfrak{M}_{j-1})$ and $\mathcal{N}(\mathfrak{Q}_{j-1}) \subseteq \mathcal{N}(I_q - \sqrt{\mathfrak{r}_j}^{\dagger})$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{0,\kappa}$. Remark 4.12 shows $\mathfrak{l}_j \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $\mathfrak{r}_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Because of Corollary 6.7, there exists a matrix $M_j \in \mathbb{C}^{p\times q}$ such that $\mathfrak{e}_j = \mathfrak{M}_{j-1} M_j \mathfrak{Q}_{j-1}$. Consequently, $\mathfrak{e}_j^* = \mathfrak{Q}_{j-1}^* M_j^* \mathfrak{M}_{j-1}^*$. We consider an arbitrary $x \in \mathcal{N}(\mathfrak{M}_{j-1}^*)$. Then $\mathfrak{e}_j^* x = O$. Thus, in view of Notation 4.11, we obtain $\mathfrak{l}_j x = x$. Using Remark A.13, we conclude $\sqrt{\mathfrak{l}_j} x = x$. Remark A.12 provides then $\sqrt{\mathfrak{l}_j}^{\dagger} x = x$. Consequently, $x \in \mathcal{N}(I_p - \sqrt{\mathfrak{l}_j}^{\dagger})$. Thus, $\mathcal{N}(\mathfrak{M}_{j-1}^*) \subseteq \mathcal{N}(I_p - \sqrt{\mathfrak{l}_j}^{\dagger})$ is proved. Applying Remarks A.8 and A.2, we get then

$$\mathcal{R}(I_p - \sqrt{\mathfrak{l}_j}^{\dagger}) = \mathcal{R}((I_p - \sqrt{\mathfrak{l}_j}^{\dagger})^*) = \mathcal{N}(I_p - \sqrt{\mathfrak{l}_j}^{\dagger})^{\perp} \subseteq \mathcal{N}(\mathfrak{M}_{j-1}^*)^{\perp} = \mathcal{R}(\mathfrak{M}_{j-1}).$$

Now we consider an arbitrary $y \in \mathcal{N}(\mathfrak{Q}_{j-1})$. From $\mathfrak{e}_j = \mathfrak{M}_{j-1} M_j \mathfrak{Q}_{j-1}$ we see then that $\mathfrak{e}_j y = O$. Thus, in view of Notation 4.11, we obtain $\mathfrak{r}_j y = y$. Using Remark A.13, we conclude $\sqrt{\mathfrak{r}_j} y = y$. Remark A.12 provides then $\sqrt{\mathfrak{r}_j}^{\dagger} y = y$. Consequently, $y \in \mathcal{N}(I_q - \sqrt{\mathfrak{r}_j}^{\dagger})$. Thus, $\mathcal{N}(\mathfrak{Q}_{j-1}) \subseteq \mathcal{N}(I_q - \sqrt{\mathfrak{r}_j}^{\dagger})$ is checked as well. □

The following observation plays a key role in proving that the SP-parameter sequence of an arbitrary sequence $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ belongs to $\mathscr{E}_{p\times q;\kappa}$. For our considerations, it is essential that the spaces on the left sides of the equations in (6.5) below can be represented via the spaces on the right sides.

Lemma 6.9 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For each $j \in \mathbb{Z}_{-1,\kappa}$, then*

$$\mathcal{R}(\mathfrak{M}_j) = \mathcal{M}_j \qquad \text{and} \qquad \mathcal{N}(\mathfrak{Q}_j) = \mathcal{Q}_j. \tag{6.5}$$

Proof Our proof works inductively. According to Notations 6.4 and 6.1, we have $\mathcal{R}(\mathfrak{M}_{-1}) = \mathbb{C}^p = \mathcal{M}_{-1}$ and $\mathcal{N}(\mathfrak{Q}_{-1}) = \{O_{q\times 1}\} = \mathcal{Q}_{-1}$. Now assume that $m \in \mathbb{Z}_{-1,\kappa-1}$ and that (6.5) is valid for all $j \in \mathbb{Z}_{-1,m}$. From Lemma 6.8 we know that $\mathcal{R}(I_p - \sqrt{\mathfrak{l}_{m+1}}^{\dagger}) \subseteq \mathcal{R}(\mathfrak{M}_m)$ and $\mathcal{N}(\mathfrak{Q}_m) \subseteq \mathcal{N}(I_q - \sqrt{\mathfrak{r}_{m+1}}^{\dagger})$. Applying Lemma B.3, we get then $\mathcal{R}(\sqrt{\mathfrak{l}_{m+1}}^{\dagger}) \cap \mathcal{R}(\mathfrak{M}_m) = \mathcal{R}(\sqrt{\mathfrak{l}_{m+1}}^{\dagger}\mathfrak{M}_m)$, whereas Lemma B.2 yields $\mathcal{N}(\mathfrak{Q}_m) + \mathcal{N}(\sqrt{\mathfrak{r}_{m+1}}^{\dagger}) = \mathcal{N}(\mathfrak{Q}_m\sqrt{\mathfrak{r}_{m+1}}^{\dagger})$. Using Remarks A.9 and A.10(a), we can infer $\mathcal{R}(\sqrt{\mathfrak{l}_{m+1}}^{\dagger}) = \mathcal{R}(\mathfrak{l}_{m+1})$ and $\mathcal{N}(\sqrt{\mathfrak{r}_{m+1}}^{\dagger}) = \mathcal{N}(\mathfrak{r}_{m+1})$. Thus, since (6.5) holds true for $j = m$, from Notations 6.4 and 6.1 we can conclude $\mathcal{R}(\mathfrak{M}_{m+1}) = \mathcal{R}(\sqrt{\mathfrak{l}_{m+1}}^{\dagger}\mathfrak{M}_m) = \mathcal{R}(\sqrt{\mathfrak{l}_{m+1}}^{\dagger}) \cap \mathcal{R}(\mathfrak{M}_m) = \mathcal{R}(\mathfrak{l}_{m+1}) \cap \mathcal{M}_m = \mathcal{M}_{m+1}$ and $\mathcal{N}(\mathfrak{Q}_{m+1}) = \mathcal{N}(\mathfrak{Q}_m\sqrt{\mathfrak{r}_{m+1}}^{\dagger}) = \mathcal{N}(\mathfrak{Q}_m) + \mathcal{N}(\sqrt{\mathfrak{r}_{m+1}}^{\dagger}) = \mathcal{Q}_m + \mathcal{N}(\mathfrak{r}_{m+1}) = \mathcal{Q}_{m+1}$. Thus, the assertion is inductively proved. □

Proposition 6.10 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. Then $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{0,\kappa}$. Remark 4.12 shows $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$. Proposition 6.6 provides $\mathcal{R}(A_0^{[j]}) \subseteq \mathcal{R}(\mathfrak{M}_{j-1})$ and $\mathcal{N}(\mathfrak{Q}_{j-1}) \subseteq \mathcal{N}(A_0^{[j]})$, whereas Lemma 6.9 yields $\mathcal{R}(\mathfrak{M}_{j-1}) = \mathcal{M}_{j-1}$ and $\mathcal{N}(\mathfrak{Q}_{j-1}) = \mathcal{Q}_{j-1}$. Taking additionally into account Definition 4.7, we can infer then $\mathcal{R}(\mathfrak{e}_j) \subseteq \mathcal{M}_{j-1}$ and $\mathcal{Q}_{j-1} \subseteq \mathcal{N}(\mathfrak{e}_j)$. Thus, by virtue of Notation 6.2, we get $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$. □

Remark 6.11 In view of Proposition 6.10, the mapping $\phi_{p\times q;\kappa}\colon \mathscr{S}_{p\times q;\kappa} \to \mathscr{E}_{p\times q;\kappa}$ defined by $\phi_{p\times q;\kappa}((A_j)_{j=0}^{\kappa}) := (\mathfrak{e}_j)_{j=0}^{\kappa}$, where $(\mathfrak{e}_j)_{j=0}^{\kappa}$ is the SP-parameter sequence of $(A_j)_{j=0}^{\kappa}$, is well defined.

Now we are going to prove that the mapping $\phi_{p\times q;n}$ defined in Remark 6.11 is even a bijection between $\mathscr{S}_{p\times q;n}$ and $\mathscr{E}_{p\times q;n}$. In particular, we have to show that each sequence $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$ is indeed the SP-parameter sequence of some $p \times q$ Schur sequence $(A_j)_{j=0}^{\kappa}$.

Notation 6.12 Let $(\mathfrak{e}_j)_{j=0}^{\kappa}$ be a sequence of contractive complex $p \times q$ matrices and let $n \in \mathbb{Z}_{0,\kappa}$. For each $k \in \mathbb{Z}_{0,n}$, then let $(D_{n,k;j})_{j=0}^{k}$ be defined recursively by $D_{n,0;0} := \mathfrak{e}_n$ and, for all $k \in \mathbb{Z}_{1,n}$, by

$$(D_{n,k;j})_{j=0}^{k} := (D_{n,k-1;j}^{[-1;\mathfrak{e}_{n-k}]})_{j=0}^{k}.$$

Remark 6.13 Let $(\mathfrak{e}_j)_{j=0}^{\kappa}$ be a sequence of contractive complex $p \times q$ matrices and let $n \in \mathbb{Z}_{0,\kappa}$. Regarding Notation 6.12, from Remark 5.2 we get immediately $D_{n,k;0} = \mathfrak{e}_{n-k}$ for all $k \in \mathbb{Z}_{0,n}$.

Proposition 6.14 *Let $n \in \mathbb{N}_0$ and let $(\mathfrak{e}_j)_{j=0}^{n}$ be a sequence of contractive complex $p \times q$ matrices. For each $k \in \mathbb{Z}_{0,n}$, then $(D_{n,k;j})_{j=0}^{k}$ belongs to $\mathscr{S}_{p\times q;k}$.*

Proof Regarding Notation 6.12 and $\mathfrak{e}_n \in \mathbb{K}_{p\times q}$, we have $(D_{n,0;j})_{j=0}^{0} \in \mathscr{S}_{p\times q;0}$. Now we work inductively and assume that $n \geq 1$, that $m \in \mathbb{Z}_{1,n}$, and that $(D_{n,k;j})_{j=0}^{k} \in \mathscr{S}_{p\times q;k}$ is valid for all $k \in \mathbb{Z}_{0,m-1}$. Taking into account $\mathfrak{e}_{n-m} \in \mathbb{K}_{p\times q}$ and Notation 6.12, then Proposition 5.11 yields $(D_{n,m;j})_{j=0}^{m} \in \mathscr{S}_{p\times q;m}$. Thus, the assertion is proved inductively. □

Corollary 6.15 *Let $n \in \mathbb{N}_0$. Then $\chi_{p\times q;n} : \mathscr{E}_{p\times q;n} \to \mathscr{S}_{p\times q;n}$ defined by $\chi_{p\times q;n}((\mathfrak{e}_j)_{j=0}^{n}) := (D_{n,n;j})_{j=0}^{n}$, where $(D_{n,n;j})_{j=0}^{n}$ is given via Notation 6.12, is well defined.*

Proof Use Notation 6.2 and apply Proposition 6.14. □

Lemma 6.16 *Let $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. For each $k \in \mathbb{Z}_{0,n}$, then*

$$\mathcal{R}(D_{n,k;\ell}) \subseteq \mathcal{M}_{n-k-1} \quad \textit{and} \quad \mathcal{Q}_{n-k-1} \subseteq \mathcal{N}(D_{n,k;\ell}) \quad \textit{for all } \ell \in \mathbb{Z}_{0,k}. \tag{6.6}$$

Proof First observe that Notation 6.2 implies $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. Our proof works inductively. In view of Notation 6.1, the case $n = 0$ is trivial. Suppose now $\kappa \geq 1$ and $n \geq 1$ and assume that $m \in \mathbb{Z}_{0,n-1}$ is such that (6.6) is fulfilled for all $k \in \mathbb{Z}_{0,m}$. We consider an arbitrary $\ell \in \mathbb{Z}_{0,m+1}$. According to Notation 6.2, we have $\mathcal{R}(\mathfrak{e}_{n-m-1}) \subseteq \mathcal{M}_{n-m-2}$ and $\mathcal{Q}_{n-m-2} \subseteq \mathcal{N}(\mathfrak{e}_{n-m-1})$. From Notation 6.1 we can infer $\mathcal{M}_{n-m-1} \subseteq \mathcal{M}_{n-m-2}$ and $\mathcal{Q}_{n-m-2} \subseteq \mathcal{Q}_{n-m-1}$. Taking additionally into account that (6.6) holds true for $k = m$, then $\mathcal{R}(D_{n,m;j}) \subseteq \mathcal{M}_{n-m-2}$ and $\mathcal{Q}_{n-m-2} \subseteq \mathcal{N}(D_{n,m;j})$ for all $j \in \mathbb{Z}_{0,m}$ follow. Thus, we can apply Lemmas 5.16 and 5.17 to obtain $\mathcal{R}(D_{n,m;\ell}^{[-1;\mathfrak{e}_{n-m-1}]}) \subseteq \mathcal{M}_{n-m-2}$ and $\mathcal{Q}_{n-m-2} \subseteq \mathcal{N}(D_{n,m;\ell}^{[-1;\mathfrak{e}_{n-m-1}]})$. Since Notation 6.12 shows $D_{n,m;\ell}^{[-1;\mathfrak{e}_{n-m-1}]} = D_{n,m+1;\ell}$ and $\ell \in \mathbb{Z}_{0,m+1}$ was arbitrarily chosen, hence (6.6) is valid for $k = m + 1$. Consequently, the assertion is proved inductively. □

Lemma 6.17 *Suppose $\kappa \geq 1$. Let $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then*

$$(D_{n,k;j}^{[1]})_{j=0}^{k-1} = (D_{n,k-1;j})_{j=0}^{k-1} \qquad \textit{for all } k \in \mathbb{Z}_{1,n}. \tag{6.7}$$

Proof First observe that Notation 6.2 implies $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. We consider an arbitrary $k \in \mathbb{Z}_{1,n}$. From Notation 6.12 we see, that $(D_{n,k;j})_{j=0}^{k}$ is the right $\mathfrak{e}_{n-k}$-inverse SP-transform of $(D_{n,k-1;j})_{j=0}^{k-1}$. Regarding (5.1) and Notation 4.11, the application of Lemma 5.23 yields then

$$D_{n,k;j}^{[1]} = \mathfrak{l}_{n-k}\mathfrak{l}_{n-k}^{\dagger} D_{n,k-1;j}\mathfrak{r}_{n-k}^{\dagger}\mathfrak{r}_{n-k} \qquad \text{for all } j \in \mathbb{Z}_{0,k-1}. \tag{6.8}$$

Because of Lemma 6.16, we have $\mathcal{R}(D_{n,k-1;j}) \subseteq \mathcal{M}_{n-k}$ and $\mathcal{Q}_{n-k} \subseteq \mathcal{N}(D_{n,k-1;j})$ for all $j \in \mathbb{Z}_{0,k-1}$. By virtue of Notation 6.1, we see that $\mathcal{M}_{n-k} \subseteq \mathcal{R}(\mathfrak{l}_{n-k})$ and $\mathcal{N}(\mathfrak{r}_{n-k}) \subseteq \mathcal{Q}_{n-k}$. For each $j \in \mathbb{Z}_{0,k-1}$, thus $\mathcal{R}(D_{n,k-1;j}) \subseteq \mathcal{R}(\mathfrak{l}_{n-k})$ and $\mathcal{N}(\mathfrak{r}_{n-k}) \subseteq \mathcal{N}(D_{n,k-1;j})$ follow. Consequently, the application of Remark A.7 to (6.8) completes the proof. □

Lemma 6.18 *Let $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. For each $m \in \mathbb{Z}_{0,n}$, then $(D_{n,n;j}^{[m]})_{j=0}^{n-m} = (D_{n,n-m;j})_{j=0}^{n-m}$ and, in particular, $D_{n,n;0}^{[m]} = \mathfrak{e}_m$.*

Proof First observe that Notation 6.2 implies $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. Since Lemma 6.17 yields (6.7) provided that $\kappa \geq 1$ and $n \geq 1$, we can, in view of Definition 4.1, infer inductively $(D_{n,n;j}^{[m]})_{j=0}^{n-m} = (D_{n,n-m;j})_{j=0}^{n-m}$ for all $m \in \mathbb{Z}_{0,n}$. In particular, $D_{n,n;0}^{[m]} = D_{n,n-m;0}$ for all $m \in \mathbb{Z}_{0,n}$. Furthermore, from Remark 6.13 we can infer finally $D_{n,n-m;0} = \mathfrak{e}_m$ for all $m \in \mathbb{Z}_{0,n}$. □

Proposition 6.19 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. For each $k \in \mathbb{Z}_{0,n}$, then*

$$(D_{n,k;j})_{j=0}^{k} = (A_j^{[n-k]})_{j=0}^{k}. \tag{6.9}$$

Proof First observe that Proposition 6.10 and Notation 6.2 imply $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. Taking into account Notation 6.12 and Definition 4.7, we have $D_{n,0;0} = \mathfrak{e}_n = A_0^{[n]}$, i.e., (6.9) holds true for $k = 0$. Now suppose $\kappa \geq 1$ and $n \geq 1$ and assume that $m \in \mathbb{Z}_{1,n}$ is such that (6.9) is fulfilled for all $k \in \mathbb{Z}_{0,m-1}$. Denote by $(C_j)_{j=0}^{m}$ the $(n-m)$-th right SP-transform of $(A_j)_{j=0}^{n}$ and by $(B_j)_{j=0}^{m-1}$ the right SP-transform of $(C_j)_{j=0}^{m}$. According to Definition 4.1, we have then $(A_j^{[(n-m)+1]})_{j=0}^{m-1} = (B_j)_{j=0}^{m-1}$. Since (6.9) is assumed to be valid for $k = m-1$, thus $(D_{n,m-1;j})_{j=0}^{m-1} = (B_j)_{j=0}^{m-1}$ follows. In view of Definition 4.7, we have $\mathfrak{e}_{n-m} = A_0^{[n-m]} = C_0$. Since Remark 4.2 shows $(C_j)_{j=0}^{m} \in \mathscr{S}_{p\times q;m}$, we can apply Corollary 5.25 to get $(B_j^{[-1;C_0]})_{j=0}^{m} = (C_j)_{j=0}^{m}$. Thus, Notation 6.12 and our foregoing consideration provide

$$(D_{n,m;j})_{j=0}^{m} = (D_{n,m-1;j}^{[-1;\mathfrak{e}_{n-m}]})_{j=0}^{m} = (B_j^{[-1;C_0]})_{j=0}^{m} = (C_j)_{j=0}^{m} = (A_j^{[n-m]})_{j=0}^{m},$$

i.e., equation (6.9) is fulfilled for $k = m$ as well. Consequently, (6.9) is inductively proved all $k \in \mathbb{Z}_{0,n}$. □

In particular, the next theorem contains an explicit description of the set of all possible sequences of Schur parameters.

Theorem 6.20 *Let $n \in \mathbb{N}_0$, let $\phi_{p\times q;n} \colon \mathscr{S}_{p\times q;n} \to \mathscr{E}_{p\times q;n}$ be defined by $\phi_{p\times q;n}((A_j)_{j=0}^{n}) := (\mathfrak{e}_j)_{j=0}^{n}$, where $(\mathfrak{e}_j)_{j=0}^{n}$ is the SP-parameter sequence of $(A_j)_{j=0}^{n}$, and let $\chi_{p\times q;n} \colon \mathscr{E}_{p\times q;n} \to \mathscr{S}_{p\times q;n}$ be defined by $\chi_{p\times q;n}((\mathfrak{e}_j)_{j=0}^{n}) := (D_{n,n;j})_{j=0}^{n}$, where $(D_{n,n;j})_{j=0}^{n}$ is given via Notation 6.12. Then $\phi_{p\times q;n}$ and $\chi_{p\times q;n}$ are well-defined, bijective, and mutual inverses.*

Proof According to Remark 6.11 and Corollary 6.15, the mappings $\phi_{p\times q;n}$ and $\chi_{p\times q;n}$ are well defined. In the following, our proof is divided into two parts.
Part 1: In order to check that $\chi_{p\times q;n} \circ \phi_{p\times q;n} = \mathrm{id}_{\mathscr{S}_{p\times q;n}}$, we consider an arbitrary sequence $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$. Denote by $(\mathfrak{e}_j)_{j=0}^n$ the SP-parameter sequence of $(A_j)_{j=0}^n$. Observe that Proposition 6.10 yields $(\mathfrak{e}_j)_{j=0}^n \in \mathscr{E}_{p\times q;n}$, so that Notation 6.2 implies $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,n}$. Proposition 6.19 yields (6.9) for all $k \in \mathbb{Z}_{0,n}$. Regarding Definition 4.1, we have in particular $(D_{n,n;j})_{j=0}^n = (A_j^{[0]})_{j=0}^n = (A_j)_{j=0}^n$. Therefore, we conclude

$$\chi_{p\times q;n}\big(\phi_{p\times q;n}\big((A_j)_{j=0}^n\big)\big) = \chi_{p\times q;n}\big((\mathfrak{e}_j)_{j=0}^n\big) = (D_{n,n;j})_{j=0}^n = (A_j)_{j=0}^n$$

and, consequently, $\chi_{p\times q;n} \circ \phi_{p\times q;n} = \mathrm{id}_{\mathscr{S}_{p\times q;n}}$.
Part 2: In order to check that $\phi_{p\times q;n} \circ \chi_{p\times q;n} = \mathrm{id}_{\mathscr{E}_{p\times q;n}}$, we consider an arbitrary sequence $(\mathfrak{e}_j)_{j=0}^n \in \mathscr{E}_{p\times q;n}$. Observe that Notation 6.2 implies $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,n}$, so that Proposition 6.14 yields $(D_{n,n;j})_{j=0}^n \in \mathscr{S}_{p\times q;n}$. Because of Remark 6.13, we get $D_{n,j;0} = \mathfrak{e}_{n-j}$ for all $j \in \mathbb{Z}_{0,n}$. Regarding Definition 4.7, we have then $\phi_{p\times q;n}((D_{n,n;j})_{j=0}^n) = (D_{n,n;0}^{[j]})_{j=0}^n$. From Lemma 6.18, we get $D_{n,n;0}^{[m]} = \mathfrak{e}_m$ for all $m \in \mathbb{Z}_{0,n}$. Consequently, we obtain

$$\phi_{p\times q;n}\big(\chi_{p\times q;n}\big((\mathfrak{e}_j)_{j=0}^n\big)\big) = \phi_{p\times q;n}\big((D_{n,n;j})_{j=0}^n\big) = (D_{n,n;0}^{[j]})_{j=0}^n = (\mathfrak{e}_j)_{j=0}^n.$$

Thus, $\phi_{p\times q;n} \circ \chi_{p\times q;n} = \mathrm{id}_{\mathscr{E}_{p\times q;n}}$ is proved as well. □

7 The SP-transform for Matricial Schur Functions

In [6, Sec. 7], we discussed the SP-transformation for functions F belonging to $\mathscr{S}_{p\times q,0}(\mathbb{D}) := \{F \in \mathscr{S}_{p\times q}(\mathbb{D}) : \|F(0)\| < 1\}$. In particular, right and left versions of the SP-transform for functions from $\mathscr{S}_{p\times q,0}(\mathbb{D})$ were introduced. There is verified that the right and left versions of SP-transforms for functions from $\mathscr{S}_{p\times q,0}(\mathbb{D})$ coincide (see [6, Prop. 7.6]). In this section, we want to extend the notion of SP-transform to arbitrary functions belonging to $\mathscr{S}_{p\times q}(\mathbb{D})$. Similar as in [6], we consider first as well a right version as a left version. In Proposition 7.11 below, we show then that both versions coincide. Let us turn our attention to the right SP-transform for matricial Schur functions. We later will generalize the classical Schur algorithm (see [28]) for contractive complex-valued functions holomorphic in the open unit disk $\mathbb{D}$ to the case of contractive matrix-valued functions holomorphic in $\mathbb{D}$. We first consider the first step.

Let $\varepsilon : \mathbb{D} \to \mathbb{C}$ be defined by $\varepsilon(z) := z$.

Definition 7.1 Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and let

$$\Phi := \sqrt{l}^{\dagger}(F - E) \qquad \text{and} \qquad \Psi := \sqrt{r}^{\dagger}(I_q - E^*F), \tag{7.1}$$

where $E := F(0)$. Then

$$F^{[\![1]\!]} := \frac{1}{\varepsilon}\Phi\Psi^{\dagger}$$

is called the *right SP-transform of F*.

In the following, we continue to use the notations introduced in Definition 7.1. Observe that $E \in \mathbb{K}_{p\times q}$ and that, because of $\Phi(0) = \sqrt{l}^{\dagger}[F(0) - E] = O_{p\times q}$, the matrix-valued function $\frac{1}{\varepsilon}\Phi$ belongs to $[\mathcal{H}(\mathbb{D})]^{p\times q}$.

Lemma 7.2 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and let $S := \Phi\Psi^{\dagger}$. For all $z \in \mathbb{D}$, then*

$$[\Psi(z)]^*\Psi(z) - [\Phi(z)]^*\Phi(z) = I_q - [F(z)]^*F(z) \tag{7.2}$$

as well as

$$I_q - [S(z)]^*S(z) = \left(I_q - \Psi(z)[\Psi(z)]^{\dagger}\right) + \left([\Psi(z)]^{\dagger}\right)^*(I_q - [F(z)]^*F(z))[\Psi(z)]^{\dagger}$$

and, in particular, $I_q - [S(z)]^*S(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.

Proof We consider an arbitrary $z \in \mathbb{D}$. First observe that $E := F(0)$ belongs to $\mathbb{K}_{p\times q}$. Regarding Remark A.17(a), we can thus apply Remark A.10(b) to obtain $(\sqrt{l}^{\dagger})^*\sqrt{l}^{\dagger} = l^{\dagger}$ and $(\sqrt{r}^{\dagger})^*\sqrt{r}^{\dagger} = r^{\dagger}$. In view of (7.1), we get then

$$\begin{aligned}
&[\Psi(z)]^*\Psi(z) - [\Phi(z)]^*\Phi(z)\\
&\quad = r^{\dagger} - r^{\dagger}E^*F(z) - [F(z)]^*Er^{\dagger} + [F(z)]^*Er^{\dagger}E^*F(z)\\
&\qquad - ([F(z)]^*l^{\dagger}F(z) - [F(z)]^*l^{\dagger}E - E^*l^{\dagger}F(z) + E^*l^{\dagger}E)\\
&\quad = (r^{\dagger} - E^*l^{\dagger}E) - (r^{\dagger}E^* - E^*l^{\dagger})F(z)\\
&\qquad - [F(z)]^*(Er^{\dagger} - l^{\dagger}E) + [F(z)]^*(Er^{\dagger}E^* - l^{\dagger})F(z).
\end{aligned}$$

Parts (b) and (a) of Lemma A.16 show $E^*l^{\dagger} = r^{\dagger}E^*$ and $l^{\dagger}E = Er^{\dagger}$. Using additionally Lemmas A.16(c) and D.3(b), we conclude

$$\begin{aligned}
[\Psi(z)]^*\Psi(z) - [\Phi(z)]^*\Phi(z) &= (r^{\dagger} - E^*l^{\dagger}E) + [F(z)]^*(Er^{\dagger}E^* - l^{\dagger})F(z)\\
&= r^{\dagger}r - [F(z)]^*ll^{\dagger}F(z) = I_q - [F(z)]^*F(z),
\end{aligned}$$

i.e., (7.2). By virtue of (2.1), we see

$$\Psi(z)[\Psi(z)]^{\dagger} = \left(\Psi(z)[\Psi(z)]^{\dagger}\right)^*\Psi(z)[\Psi(z)]^{\dagger} = \left([\Psi(z)]^{\dagger}\right)^*[\Psi(z)]^*\Psi(z)[\Psi(z)]^{\dagger}.$$

Thus, taking additionally into account (7.2), we get

$$
\begin{aligned}
I_q - [S(z)]^* S(z) &= I_q - \left([\Psi(z)]^\dagger\right)^* [\Phi(z)]^* \Phi(z) [\Psi(z)]^\dagger \\
&= I_q - \Psi(z)[\Psi(z)]^\dagger + \left([\Psi(z)]^\dagger\right)^* ([\Psi(z)]^* \Psi(z) - [\Phi(z)]^* \Phi(z)) [\Psi(z)]^\dagger \\
&= \left(I_q - \Psi(z)[\Psi(z)]^\dagger\right) + \left([\Psi(z)]^\dagger\right)^* (I_q - [F(z)]^* F(z)) [\Psi(z)]^\dagger .
\end{aligned}
$$

From Remarks A.6 and A.4, we can infer $I_q - \Psi(z)[\Psi(z)]^\dagger \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Regarding $F \in \mathscr{S}_{p\times q}(\mathbb{D})$, Lemma A.15 implies $I_q - [F(z)]^* F(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Consequently, $I_q - [S(z)]^* S(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ follows. □

Lemma 7.3 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and let $E := F(0)$. For all $z \in \mathbb{D}$, then $\mathcal{R}(\Phi(z)) \subseteq \mathcal{R}(l)$ and $\mathcal{N}(r) \subseteq \mathcal{N}(\Phi(z))$ as well as $\mathcal{R}(\Psi(z)) = \mathcal{R}(r)$ and $\mathcal{N}(\Psi(z)) = \mathcal{N}(r)$.*

Proof We consider an arbitrary $z \in \mathbb{D}$. First observe that $E \in \mathbb{K}_{p\times q}$. Regarding Remark A.17(a), we can thus apply Remark A.10(a) to obtain $\mathcal{R}(\sqrt{l}) = \mathcal{R}(l)$ and $\mathcal{R}(\sqrt{r}) = \mathcal{R}(r)$. Taking additionally into account (7.1) and Remark A.9, we then conclude $\mathcal{R}(\Phi(z)) \subseteq \mathcal{R}(\sqrt{l}^\dagger) \subseteq \mathcal{R}(l)$ and $\mathcal{R}(\Psi(z)) \subseteq \mathcal{R}(\sqrt{r}^\dagger) \subseteq \mathcal{R}(r)$. From Lemma D.3(a) and (7.1) we see that $\mathcal{N}(r) \subseteq \mathcal{N}(F(z) - E) \subseteq \mathcal{N}(\Phi(z))$. For each $w \in \mathbb{D}$, let $F(w) = \sum_{j=0}^\infty w^j A_j$ be the Taylor series representation of F. Then $A_0 = E$, so that $r_0 = r$ by (2.5) and (5.1). Theorem D.2 yields $(A_j)_{j=0}^\infty \in \mathscr{S}_{p\times q;\infty}$. Lemma D.1 provides $I_q - E^* F(z) = r_0 - \sum_{j=1}^\infty z^j A_0^* A_j$. For all $j \in \mathbb{N}$, Remark 3.2 shows $\mathcal{N}(r_0) \subseteq \mathcal{N}(A_j)$, so that $\mathcal{N}(r_0) \subseteq \mathcal{N}(I_q - E^* F(z))$ follows. Consequently, in view of $r_0 = r$ and (7.1), we get $\mathcal{N}(r) \subseteq \mathcal{N}(I_q - E^* F(z)) \subseteq \mathcal{N}(\Psi(z))$. Lemma 7.2 yields (7.2), which implies

$$
[\Psi(z)]^* \Psi(z) - (I_q - [F(z)]^* F(z)) = [\Phi(z)]^* \Phi(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}.
$$

Taking additionally into account $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and Lemma A.15, we can conclude then $[\Psi(z)]^* \Psi(z) \succcurlyeq I_q - [F(z)]^* F(z) \succcurlyeq O_{q\times q}$. Remark A.14 then provides $\mathcal{N}([\Psi(z)]^* \Psi(z)) \subseteq \mathcal{N}(I_q - [F(z)]^* F(z))$. Since $\mathcal{N}([\Psi(z)]^* \Psi(z)) = \mathcal{N}(\Psi(z))$ and Lemma D.4 shows $\mathcal{N}(I_q - [F(z)]^* F(z)) = \mathcal{N}(r)$, we thus get $\mathcal{N}(\Psi(z)) \subseteq \mathcal{N}(r)$. Therefore, $\mathcal{N}(\Psi(z)) = \mathcal{N}(r)$ is proved. In particular, we see $\dim \mathcal{R}(r) = q - \dim \mathcal{N}(r) = q - \dim \mathcal{N}(\Psi(z)) = \dim \mathcal{R}(\Psi(z)) < \infty$. Using additionally $\mathcal{R}(\Psi(z)) \subseteq \mathcal{R}(r)$, we finally get $\mathcal{R}(\Psi(z)) = \mathcal{R}(r)$. □

Now we want to rewrite the function S introduced in Lemma 7.2 in form of a linear fractional transformation of matrices.

Proposition 7.4 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and let $S := \Phi\Psi^\dagger$. If Q is given in (5.2), then*

$$
\Psi_\bullet := \Psi + Q \tag{7.3}
$$

fulfills $\det \Psi_\bullet(z) \neq 0$ *and* $S(z) = \Phi(z)[\Psi_\bullet(z)]^{-1}$ *for all* $z \in \mathbb{D}$.

Proof Consider an arbitrary $z \in \mathbb{D}$. First observe that $E := F(0)$ belongs to $\mathbb{K}_{p\times q}$, so that Remark A.17(a) yields $r^* = r$. Lemma 7.3 provides $\mathcal{R}(\Psi(z)) = \mathcal{R}(r)$ and $\mathcal{N}(\Psi(z)) = \mathcal{N}(r)$. Using Remark A.2, then $\mathcal{R}([\Psi(z)]^*) = \mathcal{R}(r^*)$ follows. Summarizing, we get $\mathcal{R}(\Psi(z)) = \mathcal{R}(r^*) = \mathcal{R}([\Psi(z)]^*)$. Regarding (7.3) and (5.3), we can thus apply Lemma A.11 to obtain $\det \Psi_\bullet(z) \neq 0$ and $[\Psi(z)]^\dagger = [\Psi_\bullet(z)]^{-1} - \mathbb{P}_{\mathcal{N}(r)}$. Since Lemma 7.3 yields $\mathcal{N}(r) \subseteq \mathcal{N}(\Phi(z))$, we have $\Phi(z)\mathbb{P}_{\mathcal{N}(r)} = O_{p\times q}$. Consequently, $S(z) = \Phi(z)[\Psi(z)]^\dagger = \Phi(z)([\Psi_\bullet(z)]^{-1} - \mathbb{P}_{\mathcal{N}(r)}) = \Phi(z)[\Psi_\bullet(z)]^{-1}$ follows. □

Remark 7.5 Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$. In view of Definition 7.1 and Proposition 7.4, then $\det \Psi_\bullet(z) \neq 0$ for all $z \in \mathbb{D}$ and $F^{[\![1]\!]} = \frac{1}{\varepsilon}\Phi\Psi_\bullet^{-1}$.

Notation 7.6 If $E \in \mathbb{K}_{p\times q}$, then let $\mathcal{W}_E : \mathbb{D} \to \mathbb{C}^{(p+q)\times(p+q)}$ be defined by

$$\mathcal{W}_E(z) := \left[\begin{array}{c:c} \sqrt{l}^{\dagger} & -\sqrt{l}^{\dagger} E \\ \hdashline -z\sqrt{r}^{\dagger} E^* & z(\sqrt{r}^{\dagger} + Q) \end{array}\right].$$

The preceding considerations provide us the following representation of $F^{[\![1]\!]}$ in form of a usual linear fractional transformation of matrices.

Proposition 7.7 *Let* $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ *and let* $E := F(0)$. *Denote by* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ *the block representation of* $\mathcal{W}_E$ *with* $p \times p$ *block* a. *For all* $z \in \mathbb{D} \setminus \{0\}$, *then* $\det(c(z)F(z) + d(z)) \neq 0$ *and* $F^{[\![1]\!]}(z) = [a(z)F(z) + b(z)][c(z)F(z) + d(z)]^{-1}$.

Proof We consider an arbitrary $z \in \mathbb{D} \setminus \{0\}$. In view of (7.1), (7.3), and Notation 7.6, we have $\Phi(z) = a(z)F(z) + b(z)$ and $z\Psi_\bullet(z) = c(z)F(z) + d(z)$. Regarding $z \neq 0$, from Remark 7.5 we can conclude then $\det(c(z)F(z) + d(z)) = z^q \det \Psi_\bullet(z) \neq 0$ and $[a(z)F(z)+b(z)][c(z)F(z)+d(z)]^{-1} = \Phi(z)[z\Psi_\bullet(z)]^{-1} = \frac{1}{z}\Phi(z)[\Psi_\bullet(z)]^{-1} = F^{[\![1]\!]}(z)$. □

Now we carry out analogous considerations for the left SP-transform for functions from $\mathscr{S}_{p\times q}(\mathbb{D})$.

Definition 7.8 Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and let

$$\Theta := (F - E)\sqrt{r}^{\dagger} \qquad \text{and} \qquad \Xi := (I_p - FE^*)\sqrt{l}^{\dagger}, \tag{7.4}$$

where $E := F(0)$. Then

$$F^{(\!(1)\!)} := \frac{1}{\varepsilon}\Xi^{\dagger}\Theta$$

is called the *left SP-transform of* F.

Observe that $E \in \mathbb{K}_{p\times q}$ and that, because of $\Theta(0) = [F(0) - E]\sqrt{r}^{\dagger} = O_{p\times q}$, the matrix-valued function $\frac{1}{\varepsilon}\Theta$ belongs to $[\mathcal{H}(\mathbb{D})]^{p\times q}$.

Proposition 7.9 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and let $S := \Xi^\dagger\Theta$. If P is given in (5.2), then*

$$\Xi_\bullet := \Xi + P \tag{7.5}$$

fulfills $\det \Xi_\bullet(z) \neq 0$ *and* $S(z) = [\Xi_\bullet(z)]^{-1}\Theta(z)$ *for all* $z \in \mathbb{D}$.

Proof This can be proved analogous to Proposition 7.4. We omit the details. □

Remark 7.10 Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$. In view of Definition 7.8 and Proposition 7.9, then $\det \Xi_\bullet(z) \neq 0$ for all $z \in \mathbb{D}$ and $F^{⦅1⦆} = \frac{1}{\varepsilon}\Xi_\bullet^{-1}\Theta$.

Now we are able to verify that, for each function $F \in \mathscr{S}_{p\times q}(\mathbb{D})$, the left and right SP-transforms coincide.

Proposition 7.11 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then $F^{⦅1⦆} = F^{[\![1]\!]}$.*

Proof First observe that $E := F(0)$ belongs to $\mathbb{K}_{p\times q}$. Regarding Remark A.17(a), we can thus use parts (b) and (c) of Remark A.10 to obtain $\sqrt{l}^\dagger\sqrt{l}^\dagger = l^\dagger$ and $\sqrt{r}^\dagger\sqrt{r}^\dagger = r^\dagger$ as well as $\sqrt{l}\sqrt{l}^\dagger = \sqrt{l}^\dagger\sqrt{l}$ and $\sqrt{r}^\dagger\sqrt{r} = \sqrt{r}\sqrt{r}^\dagger$. By virtue of (5.4) and (2.1), we can conclude $P\sqrt{l}^\dagger = (I_p - \sqrt{l}^\dagger\sqrt{l})\sqrt{l}^\dagger = O_{p\times p}$ and $\sqrt{r}^\dagger Q = \sqrt{r}^\dagger(I_q - \sqrt{r}\sqrt{r}^\dagger) = O_{q\times q}$. According to parts (a), (b), and (c) of Lemma A.16, we have $l^\dagger E = Er^\dagger$ and $E^*l^\dagger = r^\dagger E^*$ as well as $l^\dagger - Er^\dagger E^* = ll^\dagger$ and $r^\dagger - E^*l^\dagger E = r^\dagger r$. Regarding (5.2), from Lemma D.3(a), we can infer $ll^\dagger F(z) = F(z) - PE$ and $F(z)r^\dagger r = F(z) - EQ$ for all $z \in \mathbb{D}$. Lemma A.16(d) yields $PE = EQ$. In view of (7.5), (7.1), (7.4), and (7.3), we consequently obtain

$$\begin{aligned}
&\Xi_\bullet(z)\Phi(z) - \Theta(z)\Psi_\bullet(z)\\
&\quad= [\Xi(z) + P]\sqrt{l}^\dagger[F(z) - E] - [F(z) - E]\sqrt{r}^\dagger[\Psi(z) + Q]\\
&\quad= [I_p - F(z)E^*]l^\dagger[F(z) - E] - [F(z) - E]r^\dagger[I_q - E^*F(z)]\\
&\quad= l^\dagger F(z) - l^\dagger E - F(z)E^*l^\dagger F(z) + F(z)E^*l^\dagger E\\
&\qquad - [F(z)r^\dagger - F(z)r^\dagger E^*F(z) - Er^\dagger + Er^\dagger E^*F(z)]\\
&\quad= l^\dagger F(z) + F(z)E^*l^\dagger E - F(z)r^\dagger - Er^\dagger E^*F(z)\\
&\quad= (l^\dagger - Er^\dagger E^*)F(z) - F(z)(r^\dagger - E^*l^\dagger E)\\
&\quad= ll^\dagger F(z) - F(z)r^\dagger r = [F(z) - PE] - [F(z) - EQ] = EQ - PE = O_{p\times q}
\end{aligned}$$

for all $z \in \mathbb{D}$. Taking additionally into account Remarks 7.10 and 7.5, then $F^{⦅1⦆} = F^{[\![1]\!]}$ follows. □

8 On the Concordance Between SP-transforms of $\mathscr{S}_{p\times q}(\mathbb{D})$ and $\mathscr{S}_{p\times q;\infty}$

In this section, we verify that there is a complete concordance between SP-transforms of $p \times q$ Schur functions and infinite $p \times q$ Schur sequences. This correspondence will be established by inspection of Taylor coefficient sequences.

Notation 8.1 Let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$. Then let $\mathscr{S}_{p\times q}\langle\mathbb{D};\mathcal{M},\mathcal{Q}\rangle$ be the set of all $G\in\mathscr{S}_{p\times q}(\mathbb{D})$ such that $\mathcal{R}(G(z))\subseteq\mathcal{M}$ and $\mathcal{Q}\subseteq\mathcal{N}(G(z))$ are valid for all $z\in\mathbb{D}$.

Remark 8.2 $\mathscr{S}_{p\times q}\langle\mathbb{D};\mathbb{C}^p,\{O_{q\times 1}\}\rangle=\mathscr{S}_{p\times q}(\mathbb{D})$.

Remark 8.3 Let $\theta_{p\times q}:\mathbb{D}\to\mathbb{C}^{p\times q}$ be given by $\theta_{p\times q}(z):=O_{p\times q}$. Then:

(a) $\mathscr{S}_{p\times q}\langle\mathbb{D};\mathcal{M},\mathbb{C}^q\rangle=\{\theta_{p\times q}\}$ for each linear subspace $\mathcal{M}$ of $\mathbb{C}^p$.
(b) $\mathscr{S}_{p\times q}\langle\mathbb{D};\{O_{p\times 1}\},\mathcal{Q}\rangle=\{\theta_{p\times q}\}$ for each linear subspace $\mathcal{Q}$ of $\mathbb{C}^q$.

Lemma 8.4 *Let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ with $\mathcal{M}\neq\{O_{p\times 1}\}$, let $m:=\dim\mathcal{M}$, let $u_1,u_2,\dots,u_m$ be an orthonormal basis of $\mathcal{M}$, and let $U:=[u_1,u_2,\dots,u_m]$. Furthermore, let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ with $\mathcal{Q}\neq\mathbb{C}^q$, let $t:=q-\dim\mathcal{Q}$, let $v_1,v_2,\dots,v_t$ be an orthonormal basis of $\mathcal{Q}^\perp$, and let $V:=[v_1,v_2,\dots,v_t]$. Then:*

(a) *Let $S\in\mathscr{S}_{m\times t}(\mathbb{D})$. Then $G:=USV^*$ belongs to $\mathscr{S}_{p\times q}\langle\mathbb{D};\mathcal{M},\mathcal{Q}\rangle$.*
(b) *For all $G\in\mathscr{S}_{p\times q}\langle\mathbb{D};\mathcal{M},\mathcal{Q}\rangle$, there exists a unique $S\in\mathscr{S}_{m\times t}(\mathbb{D})$ such that $G=USV^*$, namely $S=U^*GV$.*

Proof First observe that $U^*U=I_m$ and $V^*V=I_t$. By virtue of Remark A.5, furthermore $UU^*=\mathbb{P}_{\mathcal{M}}$ and $VV^*=\mathbb{P}_{\mathcal{Q}^\perp}$.

(a) Clearly, G is holomorphic in $\mathbb{D}$. For all $z\in\mathbb{D}$, because of Lemma A.15, we have $I_m-S(z)[S(z)]^*\in\mathbb{C}^{m\times m}_{\succcurlyeq}$, so that Remark A.4 yields

$$\begin{aligned}I_p-G(z)[G(z)]^*&=I_p-US(z)V^*V[S(z)]^*U^*=I_p-US(z)[S(z)]^*U^*\\&=I_p-UU^*+U(I_m-S(z)[S(z)]^*)U^*\succcurlyeq I_p-UU^*\\&=I_p-\mathbb{P}_{\mathcal{M}}\succcurlyeq O_{p\times p}.\end{aligned}$$

In view of Lemma A.15, then $G\in\mathscr{S}_{p\times q}(\mathbb{D})$ follows. For all $z\in\mathbb{D}$, we see that $\mathcal{R}(G(z))\subseteq\mathcal{R}(U)=\mathcal{M}$ holds true. From $\mathcal{Q}^\perp=\mathcal{R}(V)$ and Remark A.2 we obtain $\mathcal{Q}=\mathcal{R}(V)^\perp=\mathcal{N}(V^*)\subseteq\mathcal{N}(USV^*)\subseteq\mathcal{N}(G(z))$ for all $z\in\mathbb{D}$. According to Notation 8.1, consequently, $G\in\mathscr{S}_{p\times q}\langle\mathbb{D};\mathcal{M},\mathcal{Q}\rangle$.

(b) Let $G\in\mathscr{S}_{p\times q}\langle\mathbb{D};\mathcal{M},\mathcal{Q}\rangle$. We consider an arbitrary $z\in\mathbb{D}$. According to Notation 8.1, we have then $G\in\mathscr{S}_{p\times q}(\mathbb{D})$ as well as $\mathcal{R}(G(z))\subseteq\mathcal{M}$ and $\mathcal{Q}\subseteq\mathcal{N}(G(z))$. Thus, we get $UU^*G(z)=\mathbb{P}_{\mathcal{M}}G(z)=G(z)$ and $G(z)VV^*=G(z)\mathbb{P}_{\mathcal{Q}^\perp}=G(z)$. Clearly, $S:=U^*GV$ is holomorphic in $\mathbb{D}$ and fulfills then $USV^*=UU^*GVV^*=G$. From $G\in\mathscr{S}_{p\times q}(\mathbb{D})$ as well as $I_m-S(z)[S(z)]^*=U^*(I_p-G(z)VV^*[G(z)]^*)U=U^*(I_p-G(z)[G(z)]^*)U$ and Lemma A.15 we conclude $S(z)\in\mathbb{K}_{m\times t}$. Since $z\in\mathbb{D}$ was arbitrarily chosen, hence $S\in\mathscr{S}_{m\times t}(\mathbb{D})$. If $\tilde{S}$ is an arbitrary function belonging to $\mathscr{S}_{m\times t}(\mathbb{D})$ and fulfilling $G=U\tilde{S}V^*$, then $S=U^*GV=U^*U\tilde{S}V^*V=\tilde{S}$ follows. □

In the following, for each $F\in[\mathcal{H}(\mathbb{D})]^{p\times q}$, we denote by $(C_{F;j})_{j=0}^{\infty}$ the Taylor coefficient sequence of F, given by $C_{F;j}:=(j!)^{-1}F^{(j)}(0)$. In the sequel, we continue to use the notation given in Definition 3.4 and (2.5).

Lemma 8.5 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor coefficient sequence $(A_j)_{j=0}^{\infty}$. Then $\Psi \in [\mathcal{H}(\mathbb{D})]^{q\times q}$ and $(Y_{A;j})_{j=0}^{\infty}$ is the Taylor coefficient sequence of Ψ. Moreover, $\Psi^{\dagger} \in [\mathcal{H}(\mathbb{D})]^{q\times q}$ and $(Y^{\sharp}_{A;j})_{j=0}^{\infty}$ is the Taylor coefficient sequence of $\Psi^{\dagger}$.*

Proof First observe that $E := F(0)$ belongs to $\mathbb{K}_{p\times q}$ and fulfills $E = A_0$. In view of (5.1) and (2.5), then $r = r_0$. Regarding Remark A.17(a), we apply Remark A.10(d) to obtain $\sqrt{r_0}^{\dagger} r_0 = \sqrt{r_0}$. From (7.1) we get $\Psi \in [\mathcal{H}(\mathbb{D})]^{q\times q}$ and, using additionally Lemma D.1 and Definition 3.4, furthermore

$$\Psi(z) = \sqrt{r_0}^{\dagger}\left(r_0 - \sum_{j=1}^{\infty} z^j A_0^* A_j\right) = \sqrt{r_0} - \sum_{j=1}^{\infty} z^j \sqrt{r_0}^{\dagger} A_0^* A_j = \sum_{j=0}^{\infty} z^j Y_{A;j}$$

for all $z \in \mathbb{D}$. Consequently, $(C_{\Psi;j})_{j=0}^{\infty} = (Y_{A;j})_{j=0}^{\infty}$. Lemma 7.3 implies $\mathcal{R}(\Psi(z)) = \mathcal{R}(\Psi(0))$ and $\mathcal{N}(\Psi(z)) = \mathcal{N}(\Psi(0))$ for all $z \in \mathbb{D}$. Thus, we can apply Lemma D.5 to see that $\Pi := \Psi^{\dagger}$ belongs to $[\mathcal{H}(\mathbb{D})]^{q\times q}$ and that $(C_{\Pi;j})_{j=0}^{\infty} = (C^{\sharp}_{\Psi;j})_{j=0}^{\infty}$. Consequently, $(C_{\Pi;j})_{j=0}^{\infty} = (Y^{\sharp}_{A;j})_{j=0}^{\infty}$ follows. □

Theorem 8.6 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor coefficient sequence $(A_j)_{j=0}^{\infty}$. Then $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $(A_j^{[1]})_{j=0}^{\infty}$ is the Taylor coefficient sequence of $F^{[\![1]\!]}$.*

Proof First observe that $E := F(0)$ belongs to $\mathbb{K}_{p\times q}$ and fulfills $E = A_0$. In view of (5.1) and (2.5), then $l = l_0$. From (7.1) we see $\Phi \in [\mathcal{H}(\mathbb{D})]^{p\times q}$. Lemma 8.5 shows that $\Pi := \Psi^{\dagger}$ belongs to $[\mathcal{H}(\mathbb{D})]^{q\times q}$. Consequently, $S := \Phi\Pi$ belongs to $[\mathcal{H}(\mathbb{D})]^{p\times q}$. Lemma 7.2 yields $I_q - [S(z)]^* S(z) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ for all $z \in \mathbb{D}$. By virtue of Lemma A.15, then $S \in \mathscr{S}_{p\times q}(\mathbb{D})$ follows. Regarding (7.1) and $E = F(0)$, moreover $\Phi(0) = O_{p\times q}$, implying $S(0) = O_{p\times q}$. Thus, we can conclude $\frac{1}{\varepsilon} S \in \mathscr{S}_{p\times q}(\mathbb{D})$, where $\varepsilon \colon \mathbb{D} \to \mathbb{C}$ is defined by $\varepsilon(z) := z$ (see, e. g., [11, Lem. 2.3.1]). Because of (7.1) and $E = F(0)$ as well as Definition 3.4(b), we also get that $\Delta := \frac{1}{\varepsilon}\Phi$ belongs to $[\mathcal{H}(\mathbb{D})]^{p\times q}$ and that $C_{\Delta;k} = \sqrt{l}^{\dagger} C_{F;k+1} = \sqrt{l_0}^{\dagger} A_{k+1} = Z_{A;k}$ for all $k \in \mathbb{N}_0$. Lemma 8.5 yields $C_{\Pi;k} = Y^{\sharp}_{A;k}$ for all $k \in \mathbb{N}_0$. Taking additionally into account Definition 3.4(b), we conclude $\Delta\Pi \in [\mathcal{H}(\mathbb{D})]^{p\times q}$ and $C_{\Delta\Pi;j} = \sum_{\ell=0}^{j} C_{\Delta;\ell} C_{\Pi;j-\ell} = \sum_{\ell=0}^{j} Z_{A;\ell} Y^{\sharp}_{A;j-\ell} = A_j^{[1]}$ for all $j \in \mathbb{N}_0$. Since $\frac{1}{\varepsilon} S = \frac{1}{\varepsilon}\Phi\Pi = \Delta\Pi$ and Remark 7.1 show $\Delta\Pi = \frac{1}{\varepsilon}\Phi\Psi^{\dagger} = F^{[\![1]\!]}$, the proof is complete. □

Corollary 8.7 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{\kappa}]$. Then $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[1]})_{j=0}^{\kappa-1}]$.*

Proof If $\kappa < \infty$, then let $A_j := C_{F;j}$ for all $j \in \mathbb{Z}_{\kappa+1,\infty}$. Consequently, $(A_j)_{j=0}^{\infty}$ is the Taylor coefficient sequence of F. Taking additionally into account $F \in \mathscr{S}_{p\times q}(\mathbb{D})$, we can thus apply Theorem 8.6 to get that $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}(\mathbb{D})$ and that $(A_j^{[1]})_{j=0}^{\infty}$ is the Taylor coefficient sequence of $F^{[\![1]\!]}$. Regarding Remark 3.5, in particular $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[1]})_{j=0}^{\kappa-1}]$ follows. □

Our next considerations are aimed at examining the interplay between both types of SP-algorithms and the objects introduced in Notation 8.1. We again use the notations introduced in Definition 4.7 and Notations 4.11, 6.1 and 6.4.

Proposition 8.8 *Let* $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ *with SP-parameter sequence* $(\mathfrak{e}_j)_{j=0}^{\kappa}$, *let* $k \in \mathbb{Z}_{0,\kappa}$, *and let* $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{\kappa-k}] \cap \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$. *Then* $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k\rangle$.

Proof Theorem 8.6 provides $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}(\mathbb{D})$. According to Definition 4.7, we have

$$\mathfrak{e}_k = A_0^{[k]} = C_{F;0} = F(0). \tag{8.1}$$

By virtue of Notation 4.11 and Remark 4.12, moreover

$$\mathfrak{l}_k = I_p - \mathfrak{e}_k\mathfrak{e}_k^* \in \mathbb{C}_{\succcurlyeq}^{p\times p} \qquad \text{and} \qquad \mathfrak{r}_k = I_q - \mathfrak{e}_k^*\mathfrak{e}_k \in \mathbb{C}_{\succcurlyeq}^{q\times q}. \tag{8.2}$$

We are now going to show

$$\mathcal{R}(F^{[\![1]\!]}(z)) \subseteq \mathcal{M}_k \qquad \text{and} \qquad \mathcal{Q}_k \subseteq \mathcal{N}(F^{[\![1]\!]}(z)) \qquad \text{for all } z \in \mathbb{D}\setminus\{0\}. \tag{8.3}$$

To this end, we consider an arbitrary $z \in \mathbb{D}\setminus\{0\}$. From Definitions 7.1 and 7.8 we conclude $\mathcal{R}(F^{[\![1]\!]}(z)) \subseteq \mathcal{R}(\Phi(z))$ and $\mathcal{N}(\Theta(z)) \subseteq \mathcal{N}(F^{(\!(1)\!)}(z))$. Lemma 6.9 yields (6.5) for all $j \in \mathbb{Z}_{-1,\kappa}$. Taking into account $F \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$, Notation 8.1, and (6.5) for $j = k-1$, the application of Remark A.7 provides $\mathfrak{M}_{k-1}\mathfrak{M}_{k-1}^{\dagger}F(w) = F(w)$ and $F(w)\mathfrak{Q}_{k-1}^{\dagger}\mathfrak{Q}_{k-1} = F(w)$ for all $w \in \mathbb{D}$. Taking into account (8.1), (8.2), (5.1), (7.1), and (7.4), we infer then $\sqrt{\mathfrak{l}_k}^{\dagger}\mathfrak{M}_{k-1}\mathfrak{M}_{k-1}^{\dagger}[F(z) - \mathfrak{e}_k] = \Phi(z)$ and $[F(z) - \mathfrak{e}_k]\mathfrak{Q}_{k-1}^{\dagger}\mathfrak{Q}_{k-1}\sqrt{\mathfrak{r}_k}^{\dagger} = \Theta(z)$. In particular, $\mathcal{R}(\Phi(z)) \subseteq \mathcal{R}(\sqrt{\mathfrak{l}_k}^{\dagger}\mathfrak{M}_{k-1})$ and $\mathcal{N}(\mathfrak{Q}_{k-1}\sqrt{\mathfrak{r}_k}^{\dagger}) \subseteq \mathcal{N}(\Theta(z))$. From Notation 6.4 we see that $\sqrt{\mathfrak{l}_k}^{\dagger}\mathfrak{M}_{k-1} = \mathfrak{M}_k$ and $\mathfrak{Q}_{k-1}\sqrt{\mathfrak{r}_k}^{\dagger} = \mathfrak{Q}_k$. Using (6.5) for $j = k$, we get

$$\mathcal{R}(F^{[\![1]\!]}(z)) \subseteq \mathcal{R}(\Phi(z)) \subseteq \mathcal{R}(\sqrt{\mathfrak{l}_k}^{\dagger}\mathfrak{M}_{k-1}) = \mathcal{R}(\mathfrak{M}_k) = \mathcal{M}_k$$

and

$$\mathcal{Q}_k = \mathcal{N}(\mathfrak{Q}_k) = \mathcal{N}(\mathfrak{Q}_{k-1}\sqrt{\mathfrak{r}_k}^{\dagger}) \subseteq \mathcal{N}(\Theta(z)) \subseteq \mathcal{N}(F^{(\!(1)\!)}(z)).$$

Regarding Proposition 7.11, hence (8.3) is proved. Since $F^{[\![1]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$, from (8.3) we conclude that

$$\mathbb{P}_{\mathcal{M}_k}F^{[\![1]\!]}(0) = \lim_{z\to 0}\mathbb{P}_{\mathcal{M}_k}F^{[\![1]\!]}(z) = \lim_{z\to 0}F^{[\![1]\!]}(z) = F^{[\![1]\!]}(0)$$

and

$$F^{[\![1]\!]}(0)\mathbb{P}_{\mathcal{Q}_k} = \lim_{z\to 0}F^{[\![1]\!]}(z)\mathbb{P}_{\mathcal{Q}_k} = O_{p\times q},$$

implying $\mathcal{R}(F^{[\![1]\!]}(0)) \subseteq \mathcal{M}_k$ and $\mathcal{Q}_k \subseteq \mathcal{N}(F^{[\![1]\!]}(0))$. Taking additionally into account $F^{[\![1]\!]} \in \mathcal{S}_{p\times q}(\mathbb{D})$ and (8.3), according to Notation 8.1, then $F^{[\![1]\!]} \in \mathcal{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k\rangle$ follows. □

Proposition 8.9 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathcal{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$, let $k \in \mathbb{Z}_{0,\kappa-1}$, and let $F \in \mathcal{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{\kappa-k}] \cap \mathcal{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$. Then $F^{[\![1]\!]} \in \mathcal{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{\kappa-(k+1)}] \cap \mathcal{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k\rangle$.*

Proof Denote by $(B_j)_{j=0}^{\kappa-k}$ the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. Remark 4.2 yields then $(B_j)_{j=0}^{\kappa-k} \in \mathcal{S}_{p\times q;\kappa-k}$. By assumption, furthermore $F \in \mathcal{S}_{p\times q}[\mathbb{D}; (B_j)_{j=0}^{\kappa-k}]$. Thus, we can apply Corollary 8.7 to obtain $F^{[\![1]\!]} \in \mathcal{S}_{p\times q}[\mathbb{D}; (B_j^{[1]})_{j=0}^{(\kappa-k)-1}]$. According to Definition 4.1, we have $(B_j^{[1]})_{j=0}^{(\kappa-k)-1} = (A_j^{[k+1]})_{j=0}^{\kappa-(k+1)}$, so that $F^{[\![1]\!]} \in \mathcal{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{\kappa-(k+1)}]$. The application of Proposition 8.8 completes the proof. □

9 The SP-Algorithm for $p \times q$ Schur Functions

In view of Theorem 8.6, now we are going to generalize Definitions 7.1 and 7.8. One can easily convince oneself that it is a direct generalization of the classical algorithm developed by I. Schur in [28] for complex-valued contractive functions holomorphic in $\mathbb{D}$. In view of Remark 9.2 below, first we introduce the following notion.

Definition 9.1 Let $F \in \mathcal{S}_{p\times q}(\mathbb{D})$. Then let $F^{(\!(0)\!)} := F$ (resp., $F^{[\![0]\!]} := F$). Furthermore, for all $k \in \mathbb{N}$, let $F^{(\!(k)\!)}$ (resp., $F^{[\![k]\!]}$) be recursively defined to be the left SP-transform of $F^{(\!(k-1)\!)}$ (resp., right SP-transform of $F^{[\![k-1]\!]}$). For all $k \in \mathbb{N}_0$, then $F^{(\!(k)\!)}$ (resp., $F^{(\!(k)\!)}$) is called the *k-th left SP-transform of F* (resp., *k-th right SP-transform of F*).

Remark 9.2 Let $F \in \mathcal{S}_{p\times q}(\mathbb{D})$. We emphasize that, in Definition 9.1, we used the following: By virtue of Theorem 8.6 and Proposition 7.11, one can easily verify by mathematical induction that $F^{[\![k]\!]} \in \mathcal{S}_{p\times q}(\mathbb{D})$ and $F^{(\!(k)\!)} \in \mathcal{S}_{p\times q}(\mathbb{D})$ for all $k \in \mathbb{N}_0$.

Proposition 9.3 *Let $F \in \mathcal{S}_{p\times q}(\mathbb{D})$. For all $k \in \mathbb{N}_0$, then $F^{(\!(k)\!)} = F^{[\![k]\!]}$.*

Proof In view of Definition 9.1, there is an $m \in \mathbb{N}_0$ such that $F^{(\!(k)\!)} = F^{[\![k]\!]}$ for all $k \in \mathbb{Z}_{0,m}$. According to Remark 9.2, we have $F^{[\![m]\!]} \in \mathcal{S}_{p\times q}(\mathbb{D})$. In view of Definition 9.1, the application of Proposition 7.11 yields $F^{(\!(m+1)\!)} = F^{[\![m+1]\!]}$. □

Lemma 9.4 *Let $F \in \mathcal{S}_{p\times q}(\mathbb{D})$ with Taylor coefficient sequence $(A_j)_{j=0}^{\infty}$. For all $k \in \mathbb{N}_0$, then $F^{[\![k]\!]} \in \mathcal{S}_{p\times q}(\mathbb{D})$ and $(A_j^{[k]})_{j=0}^{\infty}$ is the Taylor coefficient sequence of $F^{[\![k]\!]}$.*

Proof Regarding Definitions 9.1 and 4.1, this can be proved inductively, using Theorem 8.6. □

Definition 9.5 Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then the sequence $(\gamma_j)_{j=0}^{\infty}$ given by $\gamma_j := F^{[\![j]\!]}(0)$ for all $j \in \mathbb{N}_0$ is called the *sequence of Schur–Potapov parameters* (short *SP-parameter sequence*) *of* F.

Remark 9.6 Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with SP-parameter sequence $(\gamma_j)_{j=0}^{\infty}$. For all $k \in \mathbb{N}_0$, according to Remark 9.2 and Definitions 9.1 and 9.5, then $F^{[\![k]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and has SP-parameter sequence $(\gamma_{j+k})_{j=0}^{\infty}$.

Proposition 9.7 *Let* $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ *with Taylor coefficient sequence* $(A_j)_{j=0}^{\infty}$ *and SP-parameter sequence* $(\gamma_j)_{j=0}^{\infty}$. *Then* $(A_j)_{j=0}^{\infty} \in \mathscr{S}_{p\times q;\infty}$ *and the SP-parameter sequence* $(\mathfrak{e}_j)_{j=0}^{\infty}$ *of* $(A_j)_{j=0}^{\infty}$ *coincides with* $(\gamma_j)_{j=0}^{\infty}$.

Proof From Theorem D.2 we can infer $(A_j)_{j=0}^{\infty} \in \mathscr{S}_{p\times q;\infty}$. We consider an arbitrary $k \in \mathbb{N}_0$. According to Definition 4.7, we have $\mathfrak{e}_k = A_0^{[k]}$. By virtue of Definitions 9.1 and 4.1 and Theorem 8.6, we can use mathematical induction to see that $F^{[\![k]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and has Taylor coefficient sequence $(A_j^{[k]})_{j=0}^{\infty}$. In particular, $F^{[\![k]\!]}(0) = A_0^{[k]}$. Taking additionally into account Definition 9.5, we obtain summarizing $\mathfrak{e}_k = A_0^{[k]} = F^{[\![k]\!]}(0) = \gamma_k$. □

10 The *E*-inverse SP-transform for Matricial Schur Functions

This section can be considered as analogue of Sect. 5 for matricial Schur functions. In this section, we want to extend the notions of E-inverse SP-transform to arbitrary functions belonging to $\mathscr{S}_{p\times q}(\mathbb{D})$. Similar as in Sect. 7 we consider as well a right version as a left version. In Proposition 10.10, we show that both versions coincide. Recall that $\varepsilon\colon \mathbb{D} \to \mathbb{C}$ is defined by $\varepsilon(z) := z$.

Definition 10.1 Let $E \in \mathbb{K}_{p\times q}$, let $G\colon \mathbb{D} \to \mathbb{C}^{p\times q}$ be a matrix-valued function, and let

$$\Gamma := E + \varepsilon\sqrt{l}^{\dagger} G\sqrt{r} \qquad \text{and} \qquad \Lambda := I_q + \varepsilon E^{*}\sqrt{l}^{\dagger} G\sqrt{r}. \tag{10.1}$$

Then

$$G^{[\![-1;E]\!]} := \Gamma\Lambda^{\dagger}$$

is called the *right E-inverse SP-transform of* G.

Now we are going to rewrite, for arbitrarily given $G \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $E \in \mathbb{K}_{p\times q}$, the function $G^{[\![-1;E]\!]}$ as linear fractional transformation of matrices. This requires some preparations.

Lemma 10.2 *Let* $E \in \mathbb{K}_{p\times q}$ *and let* $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. *For all* $z \in \mathbb{D}$, *then* $\det \Lambda(z) \neq 0$.

Proof We consider an arbitrary $z \in \mathbb{D}$. Let $v \in \mathcal{N}(\Lambda(z))$. Then (10.1) implies

$$v = -zE^*\sqrt{l}^\dagger G(z)\sqrt{r}v. \tag{10.2}$$

Since Remark A.17(c) provides $\sqrt{r}E^* = E^*\sqrt{l}$, consequently $\sqrt{r}v = -zE^*\sqrt{l}\sqrt{l}^\dagger G(z)\sqrt{r}v$ follows. Hence,

$$\|\sqrt{r}v\|_\mathrm{E} \le \rho(z)\|\sqrt{r}v\|_\mathrm{E}, \tag{10.3}$$

where $\rho(z) := \|-zE^*\sqrt{l}\sqrt{l}^\dagger G(z)\|$. Lemma A.15 shows $E^* \in \mathbb{K}_{q\times p}$. From Remark A.6 we can infer $\sqrt{l}\sqrt{l}^\dagger \in \mathbb{K}_{p\times p}$. Taking additionally into account $G \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $z \in \mathbb{D}$, we get then

$$\rho(z) \le |-z| \cdot \|E^*\| \cdot \|\sqrt{l}\sqrt{l}^\dagger\| \cdot \|G(z)\| \le |z| < 1. \tag{10.4}$$

If $\|\sqrt{r}v\|_\mathrm{E} \neq 0$, then (10.3) provides $\rho(z) \ge 1$, contradicting (10.4). Thus, $\|\sqrt{r}v\|_\mathrm{E} = 0$, i.e., $\sqrt{r}v = O_{q\times 1}$. Hence, from (10.2) we obtain $v = O_{q\times 1}$. Summarizing, we have proved $\mathcal{N}(\Lambda(z)) \subseteq \{O_{q\times 1}\}$, implying $\det \Lambda(z) \neq 0$. □

Lemma 10.3 *Let $E \in \mathbb{K}_{p\times q}$, let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$, let $F := G^{[\![-1;E]\!]}$, and let $S := \varepsilon G$. For all $z \in \mathbb{D}$, then* $\det \Lambda(z) \neq 0$ *and*

$$\begin{aligned} &I_q - [F(z)]^*F(z) \\ &\quad = (\sqrt{r}[\Lambda(z)]^{-1})^*(I_q - [S(z)]^*[S(z)] + [S(z)]^*P[S(z)])\sqrt{r}[\Lambda(z)]^{-1} \end{aligned} \tag{10.5}$$

as well as, in particular, $I_q - [F(z)]^*F(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.

Proof We consider an arbitrary $z \in \mathbb{D}$. Definition 10.1 shows $F(z) = \Gamma(z)[\Lambda(z)]^\dagger$. Lemma 10.2 yields $\det \Lambda(z) \neq 0$. Consequently, we infer

$$I_q - [F(z)]^*F(z) = [\Lambda(z)]^{-*}([\Lambda(z)]^*[\Lambda(z)] - [\Gamma(z)]^*[\Gamma(z)])[\Lambda(z)]^{-1}. \tag{10.6}$$

We can apply Remark A.8 to obtain $(\sqrt{l}^\dagger)^* = \sqrt{l}^\dagger$ and $(\sqrt{r}^\dagger)^* = \sqrt{r}^\dagger$. Since (10.1) yields $\Gamma = E + \sqrt{l}^\dagger S\sqrt{r}$ and $\Lambda = I_q + E^*\sqrt{l}^\dagger S\sqrt{r}$, we get then

$$\begin{aligned} [\Lambda(z)]^*\Lambda(z) &= (I_q + \sqrt{r}[S(z)]^*\sqrt{l}^\dagger E)[I_q + E^*\sqrt{l}^\dagger S(z)\sqrt{r}] = I_q \\ &\quad + E^*\sqrt{l}^\dagger S(z)\sqrt{r} + \sqrt{r}[S(z)]^*\sqrt{l}^\dagger E + \sqrt{r}[S(z)]^*\sqrt{l}^\dagger EE^*\sqrt{l}^\dagger S(z)\sqrt{r} \end{aligned}$$

and

$$\begin{aligned} [\Gamma(z)]^*\Gamma(z) &= (E^* + \sqrt{r}[S(z)]^*\sqrt{l}^\dagger)[E + \sqrt{l}^\dagger S(z)\sqrt{r}] = E^*E \\ &\quad + E^*\sqrt{l}^\dagger S(z)\sqrt{r} + \sqrt{r}[S(z)]^*\sqrt{l}^\dagger E + \sqrt{r}[S(z)]^*\sqrt{l}^\dagger\sqrt{l}^\dagger S(z)\sqrt{r}. \end{aligned}$$

As in the proof of Lemma 5.10 we can obtain (5.7). Using (5.1), (5.7), and (5.2), we conclude then

$$\begin{aligned}
&[\Lambda(z)]^*\Lambda(z) - [\Gamma(z)]^*\Gamma(z)\\
&\quad = (I_q - E^*E) - \sqrt{r}[S(z)]^*(\sqrt{l}^\dagger\sqrt{l}^\dagger - \sqrt{l}^\dagger EE^*\sqrt{l}^\dagger)S(z)\sqrt{r}\\
&\quad = r - \sqrt{r}[S(z)]^* ll^\dagger S(z)\sqrt{r} = \sqrt{r}(I_q - [S(z)]^* ll^\dagger S(z))\sqrt{r}\\
&\quad = \sqrt{r}(I_q - [S(z)]^*[S(z)] + [S(z)]^* P[S(z)])\sqrt{r},
\end{aligned}$$

which, inserted in (10.6), gives (10.5). Since G belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$, we have $\|zG(z)\| \leq 1$, i.e., $S(z)$ belongs to $\mathbb{K}_{p\times q}$. In view of Lemma A.15, then $I_q - [S(z)]^*[S(z)] \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ follows. Regarding (5.3), Remark A.4 yields $P \in \mathbb{C}_{\succcurlyeq}^{p\times p}$, so that $[S(z)]^* P[S(z)] \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Hence, we infer $[\Lambda(z)]^*\Lambda(z) - [\Gamma(z)]^*\Gamma(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Taking additionally into account (10.5), then $I_q - [F(z)]^*F(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. □

Lemma 10.4 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then $\Gamma(\sqrt{r}^\dagger + Q) = \varepsilon\sqrt{l}^\dagger Gr^\dagger r + E(\sqrt{r}^\dagger + Q)$ and $\Lambda(\sqrt{r}^\dagger + Q) = \varepsilon E^*\sqrt{l}^\dagger Gr^\dagger r + (\sqrt{r}^\dagger + Q)$.*

***Proof*.** As in the proof of Lemma 5.18 we can obtain (5.9). Taking additionally into account (10.1), we get then

$$\Gamma(\sqrt{r}^\dagger + Q) = (E + \varepsilon\sqrt{l}^\dagger G\sqrt{r})(\sqrt{r}^\dagger + Q) = \varepsilon\sqrt{l}^\dagger Gr^\dagger r + E(\sqrt{r}^\dagger + Q)$$

and

$$\Lambda(\sqrt{r}^\dagger + Q) = (I_q + \varepsilon E^*\sqrt{l}^\dagger G\sqrt{r})(\sqrt{r}^\dagger + Q) = \varepsilon E^*\sqrt{l}^\dagger Gr^\dagger r + (\sqrt{r}^\dagger + Q). \quad \square$$

Notation 10.5 If $E \in \mathbb{K}_{p\times q}$, then let $\mathcal{V}_E\colon \mathbb{D} \to \mathbb{C}^{(p+q)\times(p+q)}$ be defined by

$$\mathcal{V}_E(z) := \left[\begin{array}{c:c} z\sqrt{l}^\dagger & E(\sqrt{r}^\dagger + Q) \\ \hdashline zE^*\sqrt{l}^\dagger & \sqrt{r}^\dagger + Q \end{array}\right].$$

Proposition 10.6 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Denote by $\left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right]$ the block representation of $\mathcal{V}_E$ with $p\times p$ block α. For all $z \in \mathbb{D}$, then $\det(\gamma(z)G(z)r^\dagger r + \delta(z)) \neq 0$ and $G^{[\![-1;E]\!]}(z) = [\alpha(z)G(z)r^\dagger r + \beta(z)][\gamma(z)G(z)r^\dagger r + \delta(z)]^{-1}$.*

***Proof*.** We consider an arbitrary $z \in \mathbb{D}$. In view of Lemma 10.4 and Notation 10.5, we have $\Gamma(z)(\sqrt{r}^\dagger + Q) = \alpha(z)G(z)r^\dagger r + \beta(z)$ and $\Lambda(z)(\sqrt{r}^\dagger + Q) = \gamma(z)G(z)r^\dagger r + \delta(z)$. Lemma 10.2 yields $\det\Lambda(z) \neq 0$. From Lemma 5.19 we infer $\det(\sqrt{r}^\dagger + Q) \neq 0$. Thus, we can conclude $\det(\gamma(z)G(z)r^\dagger r + \delta(z)) \neq 0$ and $[\gamma(z)G(z)r^\dagger r + \delta(z)]^{-1} = (\sqrt{r}^\dagger + Q)^{-1}[\Lambda(z)]^{-1}$. Definition 10.1 shows $G^{[\![-1;E]\!]}(z) = \Gamma(z)[\Lambda(z)]^\dagger$. Hence, we finally get

$$[\alpha(z)G(z)r^\dagger r + \beta(z)][\gamma(z)G(z)r^\dagger r + \delta(z)]^{-1}$$

$$= \Gamma(z)(\sqrt{r}^{\dagger} + Q)(\sqrt{r}^{\dagger} + Q)^{-1}[\Lambda(z)]^{-1}$$
$$= \Gamma(z)[\Lambda(z)]^{-1} = \Gamma(z)[\Lambda(z)]^{\dagger} = G^{[\![-1;E]\!]}(z). \qquad \square$$

We now carry out analogous considerations for left E-inverse SP-transforms of matrix-valued Schur functions.

Definition 10.7 Let $E \in \mathbb{K}_{p\times q}$, let $G\colon \mathbb{D} \to \mathbb{C}^{p\times q}$ be a matrix-valued function, and let

$$\Upsilon := E + \varepsilon\sqrt{l}G\sqrt{r}^{\dagger} \qquad \text{and} \qquad \Omega := I_p + \varepsilon\sqrt{l}G\sqrt{r}^{\dagger}E^*. \tag{10.7}$$

Then

$$G^{(\!(-1;E)\!)} := \Omega^{\dagger}\Upsilon$$

is called the *left E-inverse SP-transform of* G.

Lemma 10.8 *Let* $E \in \mathbb{K}_{p\times q}$ *and let* $G \in \mathscr{S}_{p\times q}(\mathbb{D})$*. For all* $z \in \mathbb{D}$*, then* $\det \Omega(z) \neq 0$.

Proof We consider an arbitrary $z \in \mathbb{D}$. Using Remark A.8, from (10.7) we infer $[\Omega(z)]^* = I_p + \bar{z}E\sqrt{r}^{\dagger}[G(z)]^*\sqrt{l}$. Let $v \in \mathcal{N}([\Omega(z)]^*)$. Then we obtain

$$v = -\bar{z}E\sqrt{r}^{\dagger}[G(z)]^*\sqrt{l}v. \tag{10.8}$$

Since Remark A.17(b) provides $\sqrt{l}E = E\sqrt{r}$, consequently $\sqrt{l}v = -\bar{z}E\sqrt{r}\sqrt{r}^{\dagger}[G(z)]^*\sqrt{l}v$ follows. Hence,

$$\|\sqrt{l}v\|_{\mathrm{E}} \leq \sigma(z)\|\sqrt{l}v\|_{\mathrm{E}}, \tag{10.9}$$

where $\sigma(z) := \|-\bar{z}E\sqrt{r}\sqrt{r}^{\dagger}[G(z)]^*\|$. From Remark A.6 we conclude $\sqrt{r}\sqrt{r}^{\dagger} \in \mathbb{K}_{q\times q}$. Lemma A.15 shows $[G(z)]^* \in \mathbb{K}_{q\times p}$. Taking additionally into account $E \in \mathbb{K}_{p\times q}$ and $z \in \mathbb{D}$, we get then

$$\sigma(z) \leq |-\bar{z}| \cdot \|E\| \cdot \|\sqrt{r}\sqrt{r}^{\dagger}\| \cdot \|[G(z)]^*\| \leq |z| < 1. \tag{10.10}$$

If $\|\sqrt{l}v\|_{\mathrm{E}} \neq 0$, then (10.9) provides $\sigma(z) \geq 1$, contradicting (10.10). Thus, $\|\sqrt{l}v\|_{\mathrm{E}} = 0$, i. e., $\sqrt{l}v = O_{p\times 1}$. Hence, from (10.8) we obtain $v = O_{p\times 1}$. Summarizing, we have proved $\mathcal{N}([\Omega(z)]^*) \subseteq \{O_{p\times 1}\}$. Therefore, $\det([\Omega(z)]^*) \neq 0$, implying $\det \Omega(z) \neq 0$. $\square$

Lemma 10.9 *Let* $E \in \mathbb{K}_{p\times q}$ *and let* $G\colon \mathbb{D} \to \mathbb{C}^{p\times q}$ *be a matrix-valued function. For all* $z \in \mathbb{D}$*, then* $\Upsilon(z)\Lambda(z) = \Omega(z)\Gamma(z)$.

Proof. We consider an arbitrary $z \in \mathbb{D}$. In view of (10.7) and (10.1), we have

$$\Upsilon(z)\Lambda(z) = [E + z\sqrt{l}G(z)\sqrt{r}^{\dagger}][I_q + zE^*\sqrt{l}^{\dagger}G(z)\sqrt{r}]$$

$$= E + zEE^*\sqrt{l}^\dagger G(z)\sqrt{r} + z\sqrt{l}G(z)\sqrt{r}^\dagger + z^2\sqrt{l}G(z)\sqrt{r}^\dagger E^*\sqrt{l}^\dagger G(z)\sqrt{r}$$

and

$$\begin{aligned}\Omega(z)\Gamma(z) &= [I_p + z\sqrt{l}G(z)\sqrt{r}^\dagger E^*][E + z\sqrt{l}^\dagger G(z)\sqrt{r}]\\ &= E + z\sqrt{l}^\dagger G(z)\sqrt{r} + z\sqrt{l}G(z)\sqrt{r}^\dagger E^*E + z^2\sqrt{l}G(z)\sqrt{r}^\dagger E^*\sqrt{l}^\dagger G(z)\sqrt{r}.\end{aligned}$$

As in the proof of Lemma 5.8, we can obtain (5.5) and (5.6). Using this, we conclude then

$$\begin{aligned}&\Upsilon(z)\Lambda(z) - \Omega(z)\Gamma(z)\\ &\quad= zEE^*\sqrt{l}^\dagger G(z)\sqrt{r} + z\sqrt{l}G(z)\sqrt{r}^\dagger - z\sqrt{l}^\dagger G(z)\sqrt{r} - z\sqrt{l}G(z)\sqrt{r}^\dagger E^*E\\ &\quad= z\sqrt{l}G(z)(\sqrt{r}^\dagger - \sqrt{r}^\dagger E^*E) - z(\sqrt{l}^\dagger - EE^*\sqrt{l}^\dagger)G(z)\sqrt{r}\\ &\quad= z\sqrt{l}G(z)\sqrt{r} - z\sqrt{l}G(z)\sqrt{r} = O. \quad \square\end{aligned}$$

Now we are able to verify that, for arbitrarily given $G \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $E \in \mathbb{K}_{p\times q}$, the right and left E-inverse SP-transforms coincide.

Proposition 10.10 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then $G^{(\!(-1;E)\!)} = G^{[\![-1;E]\!]}$.*

Proof. We consider an arbitrary $z \in \mathbb{D}$. Definitions 10.7 and 10.1 show $G^{(\!(-1;E)\!)}(z) = [\Omega(z)]^\dagger\Upsilon(z)$ and $G^{[\![-1;E]\!]}(z) = \Gamma(z)[\Lambda(z)]^\dagger$. Lemmas 10.8 and 10.2 yield $\det\Omega(z) \neq 0$ and $\det\Lambda(z) \neq 0$. Using additionally Lemma 10.9, we obtain

$$\begin{aligned}G^{(\!(-1;E)\!)}(z) - G^{[\![-1;E]\!]}(z) &= [\Omega(z)]^\dagger\Upsilon(z) - \Gamma(z)[\Lambda(z)]^\dagger\\ &= [\Omega(z)]^{-1}\Upsilon(z) - \Gamma(z)[\Lambda(z)]^{-1}\\ &= [\Omega(z)]^{-1}[\Upsilon(z)\Lambda(z) - \Omega(z)\Gamma(z)][\Lambda(z)]^{-1} = O. \quad \square\end{aligned}$$

11 On the Concordance Between *E*-inverse SP-transforms for $\mathscr{S}_{p\times q}(\mathbb{D})$ and $\mathscr{S}_{p\times q;\infty}$

In this section, we verify that there is a complete concordance between E-inverse SP-transforms for $p \times q$ Schur functions and infinite $p \times q$ Schur sequences. This correspondence will be established by inspection of Taylor coefficient sequences. In view of Definition 5.1(b), first we get the following:

Lemma 11.1 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor coefficient sequence $(A_j)_{j=0}^\infty$. If Λ is given by (10.1), then $\Lambda \in [\mathcal{H}(\mathbb{D})]^{q\times q}$ and $(V_{E,A;j})_{j=0}^\infty$ is the Taylor coefficient sequence of Λ. Moreover, $\Lambda^\dagger \in [\mathcal{H}(\mathbb{D})]^{q\times q}$ and $(V^\sharp_{E,A;j})_{j=0}^\infty$ is the Taylor coefficient sequence of $\Lambda^\dagger$.*

Proof From (10.1) we see $\Lambda \in [\mathcal{H}(\mathbb{D})]^{q\times q}$ and, in view of Definition 5.1(b), furthermore

$$\Lambda(z) = I_q + \sum_{k=0}^{\infty} z^{k+1} E^* \sqrt{l}^{\dagger} A_k \sqrt{r} = I_q + \sum_{j=1}^{\infty} z^j E^* \sqrt{l}^{\dagger} A_{j-1} \sqrt{r} = \sum_{j=0}^{\infty} z^j V_{E,A;j}$$

for all $z \in \mathbb{D}$. Consequently, $(C_{\Lambda;j})_{j=0}^{\infty} = (V_{E,A;j})_{j=0}^{\infty}$. Lemma 10.2 provides $\det \Lambda(z) \neq 0$ for all $z \in \mathbb{D}$. In particular, $\mathcal{R}(\Lambda(z)) = \mathbb{C}^q$ and $\mathcal{N}(\Lambda(z)) = \{O_{q\times 1}\}$ for all $z \in \mathbb{D}$. Thus, we can apply Lemma D.5 to see that $\Delta := \Lambda^{-1}$ belongs to $[\mathcal{H}(\mathbb{D})]^{q\times q}$ and that $(C_{\Delta;j})_{j=0}^{\infty} = (C^{\sharp}_{\Lambda;j})_{j=0}^{\infty}$. Consequently, $(C_{\Delta;j})_{j=0}^{\infty} = (V^{\sharp}_{E,A;j})_{j=0}^{\infty}$ follows. □

Proposition 11.2 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor coefficient sequence $(A_j)_{j=0}^{\infty}$. Then $G^{[\![-1;E]\!]} \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $(A_j^{[-1;E]})_{j=0}^{\infty}$ is the Taylor coefficient sequence of $G^{[\![-1;E]\!]}$.*

Proof From (10.1) we see $\Gamma \in [\mathcal{H}(\mathbb{D})]^{p\times q}$ and, in view of Definition 5.1(b), furthermore

$$\Gamma(z) = E + \sum_{k=0}^{\infty} z^{k+1} \sqrt{l}^{\dagger} A_k \sqrt{r} = E + \sum_{j=1}^{\infty} z^j \sqrt{l}^{\dagger} A_{j-1} \sqrt{r} = \sum_{j=0}^{\infty} z^j U_{E,A;j}$$

for all $z \in \mathbb{D}$. Consequently, $(C_{\Gamma;j})_{j=0}^{\infty} = (U_{E,A;j})_{j=0}^{\infty}$. Lemma 11.1 shows that $\Delta := \Lambda^{\dagger}$ belongs to $[\mathcal{H}(\mathbb{D})]^{q\times q}$ and that $(C_{\Delta;j})_{j=0}^{\infty} = (V^{\sharp}_{E,A;j})_{j=0}^{\infty}$. According to Definition 10.1, we have $G^{[\![-1;E]\!]} = \Gamma\Delta$. In particular, $F := G^{[\![-1;E]\!]}$ belongs to $[\mathcal{H}(\mathbb{D})]^{p\times q}$ with $C_{F;j} = \sum_{\ell=0}^{j} U_{E,A;\ell} V^{\sharp}_{E,A;j-\ell}$ for all $j \in \mathbb{N}_0$. Taking into account Definition 5.1(b), we get then $C_{F;j} = A_j^{[-1;E]}$ for all $j \in \mathbb{N}_0$. Hence, $(A_j^{[-1;E]})_{j=0}^{\infty}$ is the Taylor coefficient sequence of $G^{[\![-1;E]\!]}$. Lemma 10.3 yields $I_q - [F(z)]^* F(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $z \in \mathbb{D}$. By virtue of Lemma A.15, then $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ follows. □

Lemma 11.3 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then $F := G^{[\![-1;E]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and fulfills $F(0) = E$.*

Proof Denote by $(A_j)_{j=0}^{\infty}$ the Taylor coefficient sequence of G. Using Proposition 11.2 and Remark 5.2, we can infer then $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $F(0) = C_{F;0} = A_0^{[-1;E]} = E$. □

Proposition 11.4 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $G \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[1]})_{j=0}^{\kappa-1}]$. Then $A_0 \in \mathbb{K}_{p\times q}$ and $G^{[\![-1;A_0]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{\kappa}]$.*

Proof The assumption $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ implies that $E := A_0$ belongs to $\mathbb{K}_{p\times q}$. Denote by $(B_j)_{j=0}^{\infty}$ the Taylor coefficient sequence of G. From Proposition 11.2 we can infer then $G^{[\![-1;E]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (B_j^{[-1;E]})_{j=0}^{\infty}]$. Because of $G \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[1]})_{j=0}^{\kappa-1}]$, we have $B_j = A_j^{[1]}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Regarding Remark 5.3, then the application of Corollary 5.25 yields $B_j^{[-1;E]} = A_j$ for all $j \in \mathbb{Z}_{0,\kappa}$. This shows that $G^{[\![-1;A_0]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{\kappa}]$. □

In the sequel, we use again the linear subspaces introduced in Notation 6.1.

Lemma 11.5 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$, let $k \in \mathbb{Z}_{0,\kappa}$, and let $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k\rangle$. Then $A_0^{[k]} \in \mathbb{K}_{p\times q}$ and $G^{[\![-1;A_0^{[k]}]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{0}] \cap \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$.*

Proof By virtue of Definition 4.7 and Remark 4.12, we see that $E := A_0^{[k]}$ fulfills $E = \mathfrak{e}_k \in \mathbb{K}_{p\times q}$. In view of (5.1) and Notation 4.11, in particular we get $l = \mathfrak{l}_k$ and $r = \mathfrak{r}_k$. According to Notation 8.1, we have $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Thus, we can apply Lemma 11.3 to obtain that $F := G^{[\![-1;E]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and fulfills $F(0) = E$. This shows $G^{[\![-1;A_0^{[k]}]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{0}]$. Now we consider an arbitrary $z \in \mathbb{D}$. Regarding that Proposition 10.10 shows $F(z) = G^{(\!(-1;E)\!)}(z)$, from Definitions 10.1 and 10.7 we can conclude $\mathcal{R}(F(z)) \subseteq \mathcal{R}(\Gamma(z))$ and $\mathcal{N}(\Upsilon(z)) \subseteq \mathcal{N}(F(z))$. According to (10.1) and (10.7), we have $\Gamma(z) = \mathfrak{e}_k + z\sqrt{\mathfrak{l}_k}^{\dagger}G(z)\sqrt{\mathfrak{r}_k}$ and $\Upsilon(z) = \mathfrak{e}_k + z\sqrt{\mathfrak{l}_k}G(z)\sqrt{\mathfrak{r}_k}^{\dagger}$. Proposition 6.10 yields $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$. By virtue of Notation 6.2, we then see that $\mathcal{R}(\mathfrak{e}_k) \subseteq \mathcal{M}_{k-1}$ and $\mathcal{Q}_{k-1} \subseteq \mathcal{N}(\mathfrak{e}_k)$. According to Notation 8.1, we have $\mathcal{R}(G(z)) \subseteq \mathcal{M}_k$ and $\mathcal{Q}_k \subseteq \mathcal{N}(G(z))$. From Notation 6.1 we can infer $\mathcal{M}_k \subseteq \mathcal{M}_{k-1}$ and $\mathcal{Q}_{k-1} \subseteq \mathcal{Q}_k$. Lemma 6.9 yields (6.5) for all $j \in \mathbb{Z}_{-1,\kappa}$. Taking into account (6.5) for $j = k-1$, we obtain then $\mathcal{R}(G(z)) \subseteq \mathcal{M}_k \subseteq \mathcal{M}_{k-1} = \mathcal{R}(\mathfrak{M}_{k-1})$ and $\mathcal{N}(\mathfrak{Q}_{k-1}) = \mathcal{Q}_{k-1} \subseteq \mathcal{Q}_k \subseteq \mathcal{N}(G(z))$. Consequently, Remark A.7 provides $\mathfrak{M}_{k-1}\mathfrak{M}_{k-1}^{\dagger}G(z) = G(z)$ and $G(z)\mathfrak{Q}_{k-1}^{\dagger}\mathfrak{Q}_{k-1} = G(z)$. From Notation 6.4 we conclude $\sqrt{\mathfrak{l}_k}^{\dagger}\mathfrak{M}_{k-1} = \mathfrak{M}_k$ and $\mathfrak{Q}_{k-1}\sqrt{\mathfrak{r}_k}^{\dagger} = \mathfrak{Q}_k$. Hence, $\sqrt{\mathfrak{l}_k}^{\dagger}G(z) = \mathfrak{M}_k\mathfrak{M}_{k-1}^{\dagger}G(z)$ and $G(z)\sqrt{\mathfrak{r}_k}^{\dagger} = G(z)\mathfrak{Q}_{k-1}^{\dagger}\mathfrak{Q}_k$ follow, implying $\mathcal{R}(\sqrt{\mathfrak{l}_k}^{\dagger}G(z)) \subseteq \mathcal{R}(\mathfrak{M}_k)$ and $\mathcal{N}(\mathfrak{Q}_k) \subseteq \mathcal{N}(G(z)\sqrt{\mathfrak{r}_k}^{\dagger})$. Since (6.5) is valid for $j = k$, thus $\mathcal{R}(\sqrt{\mathfrak{l}_k}^{\dagger}G(z)) \subseteq \mathcal{M}_k$ and $\mathcal{Q}_k \subseteq \mathcal{N}(G(z)\sqrt{\mathfrak{r}_k}^{\dagger})$ hold true. Summarizing, we obtain

$$\mathcal{R}(F(z)) \subseteq \mathcal{R}(\Gamma(z)) \subseteq \mathcal{R}(\mathfrak{e}_k) + \mathcal{R}(\sqrt{\mathfrak{l}_k}^{\dagger}G(z)) \subseteq \mathcal{M}_{k-1} + \mathcal{M}_k = \mathcal{M}_{k-1}$$

and

$$\mathcal{Q}_{k-1} = \mathcal{Q}_{k-1} \cap \mathcal{Q}_k \subseteq \mathcal{N}(\mathfrak{e}_k) \cap \mathcal{N}(G(z)\sqrt{\mathfrak{r}_k}^{\dagger}) \subseteq \mathcal{N}(\Upsilon(z)) \subseteq \mathcal{N}(F(z)).$$

Taking additionally into account $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and that $z \in \mathbb{D}$ was arbitrarily chosen, according to Notation 8.1, then $F \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$ follows. Thus, $G^{[\![-1;A_0^{[k]}]\!]}$ belongs to $\mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$. □

Proposition 11.6 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$, let $k \in \mathbb{Z}_{0,\kappa-1}$, and let $G \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{\kappa-(k+1)}] \cap \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k\rangle$. Then $A_0^{[k]} \in \mathbb{K}_{p\times q}$ and $G^{[\![-1;A_0^{[k]}]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{\kappa-k}] \cap \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$.*

Proof Denote by $(B_j)_{j=0}^{\kappa-k}$ the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. From Lemma 11.5 we can infer then $B_0 = A_0^{[k]} \in \mathbb{K}_{p\times q}$ and $G^{[\![-1;A_0^{[k]}]\!]} \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$. Remark 4.2 yields $(B_j)_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$. According to Definition 4.1, we have $B_j^{[1]} = A_j^{[k+1]}$ for all $j \in \mathbb{Z}_{0,\kappa-k-1}$, so that $G \in \mathscr{S}_{p\times q}[\mathbb{D}; (B_j^{[1]})_{j=0}^{(\kappa-k)-1}]$. Thus, we can apply Proposition 11.4 to get $G^{[\![-1;B_0]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (B_j)_{j=0}^{\kappa-k}]$. Therefore, $G^{[\![-1;A_0^{[k]}]\!]} \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{\kappa-k}]$. □

12 Parametrization of the Set of All Solutions of the Matricial Schur Problem

In this section, we use the preceding results on the SP-transform to treat the matricial Schur problem connected with an arbitrarily given finite $p \times q$ Schur sequence $(A_j)_{j=0}^{n}$. We again use the function $\varepsilon\colon \mathbb{D} \to \mathbb{C}$ defined by $\varepsilon(z) := z$.

Lemma 12.1 *Let $E \in \mathbb{K}_{p\times q}$. Using Notations 7.6 and 10.5, then*

$$\mathcal{W}_E\mathcal{V}_E = \varepsilon\,\mathrm{diag}(ll^\dagger, I_q) \qquad \textit{and} \qquad \mathcal{V}_E\mathcal{W}_E = \varepsilon\begin{bmatrix} ll^\dagger & EQ \\ O_{q\times p} & I_q \end{bmatrix}. \tag{12.1}$$

Proof Denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the block representation of $\mathcal{W}_E$ with $p \times p$ block a and by $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ the block representation of $\mathcal{V}_E$ with $p \times p$ block α. Obviously,

$$\mathcal{W}_E\mathcal{V}_E = \left[\begin{array}{c:c} a\alpha + b\gamma & a\beta + b\delta \\ \hdashline c\alpha + d\gamma & c\beta + d\delta \end{array}\right] \quad \text{and} \quad \mathcal{V}_E\mathcal{W}_E = \left[\begin{array}{c:c} \alpha a + \beta c & \alpha b + \beta d \\ \hdashline \gamma a + \delta c & \gamma b + \delta d \end{array}\right].$$

By virtue of Notations 7.6 and 10.5, we see

$$a\beta + b\delta = \sqrt{l}^\dagger E(\sqrt{r}^\dagger + Q) + (-\sqrt{l}^\dagger E)(\sqrt{r}^\dagger + Q) = O_{p\times q}.$$

As in the proof of Lemma 5.23, we can obtain (5.10), (5.12), and (5.11). Regarding Notations 7.6 and 10.5, from these identities and Lemma 5.19 we can infer then

$$\begin{aligned} a\alpha + b\gamma &= \sqrt{l}^\dagger(\varepsilon\sqrt{l}^\dagger) + (-\sqrt{l}^\dagger E)(\varepsilon E^*\sqrt{l}^\dagger) \\ &= \varepsilon(\sqrt{l}^\dagger\sqrt{l}^\dagger - \sqrt{l}^\dagger EE^*\sqrt{l}^\dagger) = \varepsilon ll^\dagger, \\ c\alpha + d\gamma &= (-\varepsilon\sqrt{r}^\dagger E^*)(\varepsilon\sqrt{l}^\dagger) + \varepsilon(\sqrt{r}^\dagger + Q)(\varepsilon E^*\sqrt{l}^\dagger) = \varepsilon^2 QE^*\sqrt{l}^\dagger = O_{q\times p}, \end{aligned}$$

and

$$\begin{aligned} c\beta + d\delta &= (-\varepsilon\sqrt{r}^\dagger E^*)E(\sqrt{r}^\dagger + Q) + \varepsilon(\sqrt{r}^\dagger + Q)(\sqrt{r}^\dagger + Q) \\ &= \varepsilon\Big[\sqrt{r}^\dagger(I_q - E^*E) + Q\Big](\sqrt{r}^\dagger + Q) = \varepsilon(\sqrt{r} + Q)(\sqrt{r}^\dagger + Q) = \varepsilon I_q. \end{aligned}$$

Thus, the first identity in (12.1) is verified. As in the proof of Proposition 5.24, we can obtain (5.13)–(5.16). Regarding Notations 10.5 and 7.6, from these identities we conclude

$$\begin{aligned}
\alpha a + \beta c &= \varepsilon \sqrt{l}^{\dagger} \sqrt{l}^{\dagger} + E(\sqrt{r}^{\dagger} + Q)(-\varepsilon \sqrt{r}^{\dagger} E^{*}) \\
&= \varepsilon \Big[\sqrt{l}^{\dagger} \sqrt{l}^{\dagger} - E(\sqrt{r}^{\dagger} + Q)\sqrt{r}^{\dagger} E^{*} \Big] = \varepsilon l l^{\dagger}, \\
\alpha b + \beta d &= \varepsilon \sqrt{l}^{\dagger} (-\sqrt{l}^{\dagger} E) + E(\sqrt{r}^{\dagger} + Q)[\varepsilon(\sqrt{r}^{\dagger} + Q)] \\
&= \varepsilon \Big[E(\sqrt{r}^{\dagger} + Q)^2 - \sqrt{l}^{\dagger} \sqrt{l}^{\dagger} E \Big] = \varepsilon E Q, \\
\gamma a + \delta c &= \varepsilon E^{*} \sqrt{l}^{\dagger} \sqrt{l}^{\dagger} + (\sqrt{r}^{\dagger} + Q)(-\varepsilon \sqrt{r}^{\dagger} E^{*}) \\
&= \varepsilon \Big[E^{*} \sqrt{l}^{\dagger} \sqrt{l}^{\dagger} - (\sqrt{r}^{\dagger} + Q)\sqrt{r}^{\dagger} E^{*} \Big] = O_{q \times p},
\end{aligned}$$

and

$$\begin{aligned}
\gamma b + \delta d &= \varepsilon E^{*} \sqrt{l}^{\dagger} (-\sqrt{l}^{\dagger} E) + (\sqrt{r}^{\dagger} + Q)\Big[\varepsilon(\sqrt{r}^{\dagger} + Q)\Big] \\
&= \varepsilon \Big[(\sqrt{r}^{\dagger} + Q)^2 - E^{*} \sqrt{l}^{\dagger} \sqrt{l}^{\dagger} E \Big] = \varepsilon I_q.
\end{aligned}$$

Consequently, the second identity in (12.1) is verified as well. □

Lemma 12.2 *Let $E \in \mathbb{K}_{p\times q}$ and let $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then $F := G^{[\![-1;E]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and fulfills $F^{[\![1]\!]} = ll^{\dagger} G r^{\dagger} r$.*

Proof We consider an arbitrary $z \in \mathbb{D} \setminus \{0\}$. Denote by $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ the block representation of $\mathcal{V}_E$ with $p \times p$ block α. Proposition 10.6 then yields $\det(\gamma(z)G(z)r^{\dagger}r + \delta(z)) \neq 0$ and $F(z) = [\alpha(z)G(z)r^{\dagger}r + \beta(z)][\gamma(z)G(z)r^{\dagger}r + \delta(z)]^{-1}$. In particular, Remark C.1 shows $\operatorname{rank}([\gamma(z), \delta(z)]) = q$. Proposition 11.2 and Remark 5.2 imply $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $F(0) = C_{F;0} = A_0^{[-1;E]} = E$. Denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the block representation of $\mathcal{W}_E$ with $p \times p$ block a. Proposition 7.7 then yields $\det(c(z)F(z) + d(z)) \neq 0$ and $F^{[\![1]\!]}(z) = [a(z)F(z) + b(z)][c(z)F(z) + d(z)]^{-1}$. In particular, Remark C.1 provides $\operatorname{rank}([c(z), d(z)]) = q$. Regarding that Lemma 12.1 shows $\mathcal{W}_E(z)\mathcal{V}_E(z) = \operatorname{diag}(zll^{\dagger}, zI_q)$, we can thus apply Proposition C.2 to obtain $F^{[\![1]\!]}(z) = [zll^{\dagger}G(z)r^{\dagger}r](zI_q)^{-1} = ll^{\dagger}G(z)r^{\dagger}r$. In view of $F \in \mathscr{S}_{p\times q}(\mathbb{D})$, from Theorem 8.6 we see that $F^{[\![1]\!]} \in \mathscr{S}_{p\times q}(\mathbb{D})$. Consequently, $F^{[\![1]\!]}, G \in [\mathcal{H}(\mathbb{D})]^{p\times q}$, so that the Identity Theorem for holomorphic functions yields $F^{[\![1]\!]} = ll^{\dagger} G r^{\dagger} r$. □

Lemma 12.3 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$, let $k \in \mathbb{Z}_{0,\kappa}$, and let $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k \rangle$. Then $E := A_0^{[k]}$ belongs to $\mathbb{K}_{p\times q}$ and $F := G^{[\![-1;E]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and fulfills $F^{[\![1]\!]} = G$.*

Proof By virtue of Definition 4.7 and Remark 4.12, we get $E = \mathfrak{e}_k \in \mathbb{K}_{p\times q}$. In view of (5.1) and Notation 4.11, in particular $l = \mathfrak{l}_k$ and $r = \mathfrak{r}_k$. Taking additionally into account, that Notation 8.1 shows $G \in \mathscr{S}_{p\times q}(\mathbb{D})$, we can thus apply Lemma 12.2 to obtain $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ and $F^{[\![1]\!]} = \mathfrak{l}_k \mathfrak{l}_k^{\dagger} G \mathfrak{r}_k^{\dagger} \mathfrak{r}_k$. According to Notations 8.1 and 6.1, we have $\mathcal{R}(G(z)) \subseteq \mathcal{M}_k \subseteq \mathcal{R}(\mathfrak{l}_k)$ and $\mathcal{N}(\mathfrak{r}_k) \subseteq \mathcal{Q}_k \subseteq \mathcal{N}(G(z))$ for all $z \in \mathbb{D}$. From Remark A.7, for all $z \in \mathbb{D}$, then $\mathfrak{l}_k \mathfrak{l}_k^{\dagger} G(z) = G(z)$ and $G(z)\mathfrak{r}_k^{\dagger}\mathfrak{r}_k = G(z)$ follow. Consequently, we get $F^{[\![1]\!]} = \mathfrak{l}_k \mathfrak{l}_k^{\dagger} G \mathfrak{r}_k^{\dagger} \mathfrak{r}_k = G$. □

Lemma 12.4 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$. Then $E := F(0)$ belongs to $\mathbb{K}_{p\times q}$ and $G := F^{[\![1]\!]}$ fulfills $G^{[\![-1;E]\!]} = F$.*

Proof Because of $F \in \mathscr{S}_{p\times q}(\mathbb{D})$, we have $E \in \mathbb{K}_{p\times q}$. We consider an arbitrary $z \in \mathbb{D} \setminus \{0\}$. Denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the block representation of $\mathcal{W}_E$ with $p \times p$ block a. Proposition 7.7 then yields $\det(c(z)F(z) + d(z)) \neq 0$ and $G(z) = [a(z)F(z) + b(z)][c(z)F(z) + d(z)]^{-1}$. In particular, Remark C.1 shows that $\operatorname{rank}([c(z), d(z)]) = q$. Theorem 8.6 provides $G \in \mathscr{S}_{p\times q}(\mathbb{D})$. Denote by $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ the block representation of $\mathcal{V}_E$ with $p \times p$ block α. Proposition 10.6 then yields $\det(\gamma(z)G(z)r^{\dagger}r + \delta(z)) \neq 0$ and $G^{[\![-1;E]\!]}(z) = [\alpha(z)G(z)r^{\dagger}r + \beta(z)][\gamma(z)G(z)r^{\dagger}r + \delta(z)]^{-1}$. In particular, Remark C.1 provides $\operatorname{rank}([\gamma(z), \delta(z)]) = q$. Regarding that Lemma 12.1 implies $\mathcal{V}_E(z)\mathcal{W}_E(z) = \begin{bmatrix} zll^{\dagger} & zEQ \\ O_{q\times p} & zI_q \end{bmatrix}$, we can thus apply Proposition C.2 to obtain $G^{[\![-1;E]\!]}(z) = [zll^{\dagger}F(z) + zEQ](zI_q)^{-1} = ll^{\dagger}F(z) + EQ$. Regarding (5.2), from Lemma D.3(a), we can infer $ll^{\dagger}F(z) = F(z) - PE$. Lemma A.16(d) yields $PE = EQ$. Summarizing, we get $G^{[\![-1;E]\!]}(z) = [F(z) - PE] + EQ = F(z)$. In view of $E \in \mathbb{K}_{p\times q}$ and $G \in \mathscr{S}_{p\times q}(\mathbb{D})$, from Proposition 11.2 we see that $G^{[\![-1;E]\!]} \in \mathscr{S}_{p\times q}(\mathbb{D})$. Consequently, $G^{[\![-1;E]\!]}, F \in [\mathcal{H}(\mathbb{D})]^{p\times q}$, so that the Identity Theorem for holomorphic functions yields $G^{[\![-1;E]\!]} = F$. □

Proposition 12.5 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^{n} \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n}$. Then $\psi : \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle \to \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[n]})_{j=0}^{0}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{n-1}, \mathcal{Q}_{n-1}\rangle$ given by $\psi(G) := G^{[\![-1;A_0^{[n]}]\!]}$ is a well-defined bijection with inverse ψ^{-1} given by $\psi^{-1}(F) = F^{[\![1]\!]}$ for all $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[n]})_{j=0}^{0}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{n-1}, \mathcal{Q}_{n-1}\rangle$.*

Proof According to Lemma 11.5, the mapping ψ is well defined. Using Proposition 8.8 for $\kappa = n$ and $k = n$, we see that $\chi : \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[n]})_{j=0}^{0}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{n-1}, \mathcal{Q}_{n-1}\rangle \to \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ given by $\chi(F) := F^{[\![1]\!]}$ is also well defined. Applying Lemma 12.3, we get

$$(\chi \circ \psi)(G) = \chi(\psi(G)) = \chi(G^{[\![-1;A_0^{[n]}]\!]}) = G \tag{12.2}$$

for all $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$. Consequently, the mapping ψ is injective with $\psi^{-1} = \chi$. We now consider an arbitrary $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[n]})_{j=0}^{0}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{n-1}, \mathcal{Q}_{n-1}\rangle$. Using Proposition 8.8 with $\kappa = n$ and $k = n$, we see then that $G := \chi(F)$ belongs to $\mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$. Regarding $F(0) = A_0^{[n]}$, Lemma 12.4 yields $\psi(G) = F$. Thus, ψ is also surjective. □

Proposition 12.6 *Let $n \in \mathbb{N}$, let $(A_j)_{j=0}^{n} \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n}$, and let $k \in \mathbb{Z}_{0,n-1}$. Then $\psi : \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{n-(k+1)}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_k, \mathcal{Q}_k\rangle \to$*

$\mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{n-k}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$ given by $\psi(G) := G^{[\![-1;A_0^{[k]}]\!]}$ is a well-defined bijection with inverse ψ^{-1} given by $\psi^{-1}(F) = F^{[\![1]\!]}$ for all $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{n-k}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$.

Proof According to Proposition 11.6, the mapping ψ is well defined. Using Proposition 8.9, we see that $\chi\colon \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{n-k}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle \to \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{n-(k+1)}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k}, \mathcal{Q}_{k}\rangle$ given by $\chi(F) := F^{[\![1]\!]}$ is also well defined. From Lemma 12.3, we get (12.2) for all $G \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{n-(k+1)}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k}, \mathcal{Q}_{k}\rangle$. Therefore, ψ is injective with $\psi^{-1} = \chi$. We now consider an arbitrary $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{n-k}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$. Using Proposition 8.9, we see then that $G := \chi(F)$ belongs to $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k+1]})_{j=0}^{n-(k+1)}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k}, \mathcal{Q}_{k}\rangle$. Regarding $F(0) = A_0^{[k]}$, Lemma 12.4 yields $\psi(G) = F$. Thus, ψ is surjective as well. □

Now we are able to prove a first parametrization of the solution set of the matricial Schur problem, where we in particular use the notations introduced in Notations 4.11 and 6.1. Observe that the parameters still depend on the given data.

Theorem 12.7 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^n$. For all $k \in \mathbb{Z}_{0,n}$, let $\mathscr{U}_k := \mathscr{S}_{p\times q}[\mathbb{D}; (A_j^{[k]})_{j=0}^{n-k}] \cap \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{k-1}, \mathcal{Q}_{k-1}\rangle$. Let $\psi_n\colon \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle \to \mathscr{U}_n$ be defined by $\psi_n(G) := G^{[\![-1;A_0^{[n]}]\!]}$. In the case $n \geq 1$, for all $k \in \mathbb{Z}_{0,n-1}$, let furthermore $\psi_k\colon \mathscr{U}_{k+1} \to \mathscr{U}_k$ be given by $\psi_k(G) := G^{[\![-1;A_0^{[k]}]\!]}$. Then $\Psi_n\colon \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle \to \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$ defined by $\Psi_n(G) := (\psi_0 \circ \psi_1 \circ \cdots \circ \psi_n)(G)$ is a well-defined bijection with inverse Ψ_n^{-1} given by $\Psi_n^{-1}(F) = F^{[\![n+1]\!]}$ for all $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$.*

Proof Since Notation 6.1 and Remark 8.2 show that $\mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_{-1}, \mathcal{Q}_{-1}\rangle = \mathscr{S}_{p\times q}(\mathbb{D})$ is valid, from Definition 4.1 we see $\mathscr{U}_0 = \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$. According to Proposition 12.5, the mapping ψ_n is well defined and bijective with inverse ψ_n^{-1} given by $\psi_n^{-1}(F) = F^{[\![1]\!]}$ for all $F \in \mathscr{U}_n$. If $n = 0$, then we have $\Psi_n = \psi_n$ and $\mathscr{U}_n = \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$, so that the proof is complete.

Now suppose $n \geq 1$. We already know that there is an $m \in \mathbb{Z}_{1,n}$ such that, for all $k \in \mathbb{Z}_{m,n}$, the following statement holds true:

(I_k) The mapping $\rho_k := \psi_k \circ \psi_{k+1} \circ \cdots \circ \psi_n$ is a bijective mapping from $\mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ onto $\mathscr{U}_k$ with inverse ρ_k^{-1} fulfilling $\rho_k^{-1}(F) = F^{[\![n-k+1]\!]}$ for all $F \in \mathscr{U}_k$.

Taking into account Proposition 12.6, we see then that $\rho_{m-1} := \psi_{m-1} \circ \rho_m$ is a bijective mapping from $\mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ onto $\mathscr{U}_{m-1}$, where Definition 9.1 provides $\rho_{m-1}^{-1}(F) = (\rho_m^{-1} \circ \psi_{m-1}^{-1})(F) = \rho_m^{-1}(\psi_{m-1}^{-1}(F)) = \rho_m^{-1}(F^{[\![1]\!]}) = (F^{[\![1]\!]})^{[\![n-m+1]\!]} = F^{[\![n-m+2]\!]}$ for all $F \in \mathscr{U}_{m-1}$. Thus, we proved inductively that statement (I_k) holds true for all $k \in \mathbb{Z}_{0,n}$. Consequently, because of $\Psi_n = \rho_0$, we checked that Ψ_n is a bijective mapping from $\mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ onto $\mathscr{U}_0$ with inverse mapping Ψ_n^{-1} fulfilling $\Psi_n^{-1}(F) = F^{[\![n+1]\!]}$ for all $F \in \mathscr{U}_0$. In view of $\mathscr{U}_0 = \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$, the proof is complete. □

13 Description via Linear Fractional Transformation

In this section, we rewrite the result of Theorem 12.7 in form of a linear fractional transformation of matrices. This enables us to construct a parametrization of the solution set of an arbitrary matricial Schur problem by parameters which are independent of the given data.

Notation 13.1 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. Regarding Remark 4.12 and Notation 10.5, then, for all $n \in \mathbb{Z}_{0,\kappa}$, let $\mathfrak{V}_n := \mathcal{V}_{\mathfrak{e}_0}\mathcal{V}_{\mathfrak{e}_1}\cdots\mathcal{V}_{\mathfrak{e}_n}$.

Lemma 13.2 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^{n} \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n}$. Denote by $\begin{bmatrix} \mathfrak{w}_n & \mathfrak{x}_n \\ \mathfrak{y}_n & \mathfrak{z}_n \end{bmatrix}$ the block representation of $\mathfrak{V}_n$ with $p\times p$ block $\mathfrak{w}_n$. Let $\Psi_n : \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle \to \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ be given as in Theorem 12.7. For every choice of $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ and $z \in \mathbb{D}$, then $\det(\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)) \neq 0$ and*

$$[\Psi_n(G)](z) = [\mathfrak{w}_n(z)G(z) + \mathfrak{x}_n(z)][\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)]^{-1}.$$

Proof For all $k \in \mathbb{Z}_{0,n}$, let $\mathscr{U}_k$ and ψ_k be given as in Theorem 12.7. According to Remark 4.12, we have $\mathfrak{e}_0, \ldots, \mathfrak{e}_n \in \mathbb{K}_{p\times q}$. In view of Notation 10.5, for all $k \in \mathbb{Z}_{0,n}$, we can thus define $\mathfrak{U}_k := \mathcal{V}_{\mathfrak{e}_k}\mathcal{V}_{\mathfrak{e}_{k+1}}\cdots\mathcal{V}_{\mathfrak{e}_n}$. For all $k \in \mathbb{Z}_{0,n}$, let $\left[\begin{smallmatrix} \mathfrak{s}_k & \mathfrak{t}_k \\ \mathfrak{u}_k & \mathfrak{v}_k \end{smallmatrix}\right]$ be the block representation of $\mathfrak{U}_k$ with $p\times p$ block $\mathfrak{s}_k$.
Part 1: In the proof of Theorem 12.7, we verified $\mathscr{U}_0 = \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ and that, for all $k \in \mathbb{Z}_{0,n}$, the mapping $\rho_k := \psi_k \circ \psi_{k+1} \circ \cdots \circ \psi_n$ is a bijective mapping from $\mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ onto $\mathscr{U}_k$ with inverse ρ_k^{-1} fulfilling $\rho_k^{-1}(F) = F^{[\![n-k+1]\!]}$ for all $F \in \mathscr{U}_k$. Now we will work inductively.
Part 2: First we consider the function ρ_n. We set $E := \mathfrak{e}_n$. Then $E \in \mathbb{K}_{p\times q}$. By virtue of Notation 4.11 and (5.1), we get moreover $\mathfrak{r}_n = r$. Let $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ and $z \in \mathbb{D}$ be arbitrarily chosen. In particular, then $G \in \mathscr{S}_{p\times q}(\mathbb{D})$, according to Notation 8.1. Regarding $\mathfrak{U}_n = \mathcal{V}_E$, we can thus apply Proposition 10.6 to obtain $\det(\mathfrak{u}_n(z)G(z)\mathfrak{r}_n^{\dagger}\mathfrak{r}_n + \mathfrak{v}_n(z)) \neq 0$ and $G^{[\![-1;E]\!]}(z) = [\mathfrak{s}_n(z)G(z)\mathfrak{r}_n^{\dagger}\mathfrak{r}_n + \mathfrak{t}_n(z)][\mathfrak{u}_n(z)G(z)\mathfrak{r}_n^{\dagger}\mathfrak{r}_n + \mathfrak{v}_n(z)]^{-1}$. According to Notation 8.1, we have $\mathcal{Q}_n \subseteq \mathcal{N}(G(z))$. By virtue of Notation 6.1, we see that $\mathcal{N}(\mathfrak{r}_n) \subseteq \mathcal{Q}_n$. Hence, $\mathcal{N}(\mathfrak{r}_n) \subseteq \mathcal{N}(G(z))$, so that Remark A.7(b) yields $G(z)\mathfrak{r}_n^{\dagger}\mathfrak{r}_n = G(z)$. Consequently, $\det(\mathfrak{u}_n(z)G(z) + \mathfrak{v}_n(z)) \neq 0$ and $G^{[\![-1;E]\!]}(z) = [\mathfrak{s}_n(z)G(z) + \mathfrak{t}_n(z)][\mathfrak{u}_n(z)G(z) + \mathfrak{v}_n(z)]^{-1}$ follow. Regarding that Definition 4.7 yields $E = A_0^{[n]}$, summarizing we get

$$\begin{aligned}[\rho_n(G)](z) = [\psi_n(G)](z) &= G^{[\![-1;A_0^{[n]}]\!]}(z) = G^{[\![-1;E]\!]}(z) \\ &= [\mathfrak{s}_n(z)G(z) + \mathfrak{t}_n(z)][\mathfrak{u}_n(z)G(z) + \mathfrak{v}_n(z)]^{-1}.\end{aligned}$$

If $n = 0$, then $\mathscr{U}_n = \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^{n}]$ and $\Psi_n = \rho_n$ as well as $\mathfrak{V}_n = \mathfrak{U}_n$, so that $\mathfrak{w}_n = \mathfrak{s}_n$, $\mathfrak{x}_n = \mathfrak{t}_n$, $\mathfrak{y}_n = \mathfrak{u}_n$, $\mathfrak{z}_n = \mathfrak{v}_n$, which completes the proof in this case.
Part 3: Now suppose $n \geq 1$. According to Part 2 of the proof, there exists an $m \in \mathbb{Z}_{0,n-1}$ such that, for all $k \in \mathbb{Z}_{m+1,n}$ the following statement holds true:

(I)$_k$ If $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$, then

$$\det(\mathfrak{u}_k(z)G(z) + \mathfrak{v}_k(z)) \neq 0 \tag{13.1}$$

and

$$[\rho_k(G)](z) = [\mathfrak{s}_k(z)G(z) + \mathfrak{t}_k(z)][\mathfrak{u}_k(z)G(z) + \mathfrak{v}_k(z)]^{-1} \tag{13.2}$$

for all $z \in \mathbb{D}$.

We set $E := \mathfrak{e}_m$. Then $E \in \mathbb{K}_{p\times q}$. By virtue of Notation 4.11 and (5.1), we get moreover $\mathfrak{r}_m = r$. Let $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ and $z \in \mathbb{D}$ be arbitrarily chosen. In view of Part 1 of the proof, then $H := \rho_{m+1}(G)$ belongs to $\mathscr{U}_{m+1}$. In particular, $H \in \mathscr{S}_{p\times q}(\mathbb{D})$, according to Notation 8.1. Denoting by $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ the block representation of $\mathcal{V}_E$ with $p \times p$ block α, we can thus apply Proposition 10.6 to obtain $\det(\gamma(z)H(z)\mathfrak{r}_m^\dagger\mathfrak{r}_m + \delta(z)) \neq 0$ and $H^{[\![-1;E]\!]}(z) = [\alpha(z)H(z)\mathfrak{r}_m^\dagger\mathfrak{r}_m + \beta(z)][\gamma(z)H(z)\mathfrak{r}_m^\dagger\mathfrak{r}_m + \delta(z)]^{-1}$. According to Notation 8.1, we have $\mathcal{Q}_m \subseteq \mathcal{N}(H(z))$. By virtue of Notation 6.1, we see that $\mathcal{N}(\mathfrak{r}_m) \subseteq \mathcal{Q}_m$. Hence, $\mathcal{N}(\mathfrak{r}_m) \subseteq \mathcal{N}(H(z))$, so that Remark A.7(b) yields $H(z)\mathfrak{r}_m^\dagger\mathfrak{r}_m = H(z)$. Consequently, $\det(\gamma(z)H(z)+\delta(z)) \neq 0$ and $H^{[\![-1;E]\!]}(z) = [\alpha(z)H(z)+\beta(z)][\gamma(z)H(z)+\delta(z)]^{-1}$. In particular, Remark C.1 provides $\operatorname{rank}([\gamma(z), \delta(z)]) = q$. In view of (I)$_{m+1}$, from Remark C.1 we can infer $\operatorname{rank}([\mathfrak{u}_{m+1}(z), \mathfrak{v}_{m+1}(z)]) = q$. Taking additionally into account $\mathfrak{U}_m = \mathcal{V}_E\mathfrak{U}_{m+1}$, the application of Proposition C.2 yields that (13.1) and (13.2) hold true for $k = m$ as well. Thus, we proved inductively that (I)$_k$ is fulfilled for all $k \in \mathbb{Z}_{0,n}$. Since $\Psi_n = \rho_0$ and $\mathfrak{V}_n = \mathfrak{U}_0$ are valid, now the assertions in the considered case $n \geq 1$ follow from (I)$_0$. □

Now we are able to prove a first variant of a reformulation of Theorem 12.7 in form of a linear fractional transformation of matrices. We note that the parameters still depend on the given data.

Theorem 13.3 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^n$. Denote by $\begin{bmatrix} \mathfrak{w}_n & \mathfrak{x}_n \\ \mathfrak{y}_n & \mathfrak{z}_n \end{bmatrix}$ the block representation of the matrix-valued function $\mathfrak{V}_n$ given by Notation 13.1 with $p \times p$ block $\mathfrak{w}_n$. Then:*

(a) *If $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$, then $\det(\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)) \neq 0$ for all $z \in \mathbb{D}$ and the function $F : \mathbb{D} \to \mathbb{C}^{p\times q}$ defined by*

$$F(z) := [\mathfrak{w}_n(z)G(z) + \mathfrak{x}_n(z)][\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)]^{-1}$$

belongs to $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$.

(b) *For all $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$, there exists a unique $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ such that the function $\det(\mathfrak{y}_n G + \mathfrak{z}_n)$ does not vanish identically and that F admits the representation $F = (\mathfrak{w}_n G + \mathfrak{x}_n)(\mathfrak{y}_n G + \mathfrak{z}_n)^{-1}$, namely $G = F^{[\![n+1]\!]}$.*

Proof (a) Let $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$. Using Theorem 12.7 and the notations therein, we see that $\Psi_n(G) \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$. On the other hand, we know from

Lemma 13.2 that $\det(\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)) \neq 0$ and $[\Psi_n(G)](z) = F(z)$ hold true for all $z \in \mathbb{D}$.

(b) We consider an arbitrary $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$. Because of Theorem 12.7, there exists a unique $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ such that $\Psi_n(G) = F$, namely $G = F^{[\![n+1]\!]}$. From Lemma 13.2 we can infer then that $\det(\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)) \neq 0$ and $F(z) = [\mathfrak{w}_n(z)G(z)+\mathfrak{x}_n(z)][\mathfrak{y}_n(z)G(z)+\mathfrak{z}_n(z)]^{-1}$ hold true for all $z \in \mathbb{D}$. It remains to check that there is only one function $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ such that $\det(\mathfrak{y}_n G+\mathfrak{z}_n)$ does not vanish identically and $F = (\mathfrak{w}_n G+\mathfrak{x}_n)(\mathfrak{y}_n G+\mathfrak{z}_n)^{-1}$ is fulfilled. To this end, we assume that $\tilde{G}$ is an arbitrary function belonging to $\mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ such that $\det(\mathfrak{y}_n\tilde{G} + \mathfrak{z}_n)$ does not vanish identically and that $F = (\mathfrak{w}_n\tilde{G} + \mathfrak{x}_n)(\mathfrak{y}_n\tilde{G} + \mathfrak{z}_n)^{-1}$ holds true. By virtue of Lemma 13.2, then $\Psi_n(\tilde{G}) = F = \Psi_n(G)$ follows. Since Ψ_n is bijective, according to Theorem 12.7, we get finally $\tilde{G} = G = F^{[\![n+1]\!]}$. □

Now we are able to parametrize the set $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$ by parameters which are independent of the given data. We distinguish the following two cases:

(I) $1 \leq \dim \mathcal{M}_n$ and $\dim \mathcal{Q}_n \leq q - 1$.
(II) $\dim \mathcal{M}_n = 0$ or $\dim \mathcal{Q}_n = q$.

In the so-called non-degenerate case, which is a special case of case (I), we get immediately a corresponding result:

Theorem 13.4 *Let the assumptions of Theorem 13.3 be fulfilled where $\mathcal{M}_n = \mathbb{C}^p$ and $\mathcal{Q}_n = \{O_{q\times 1}\}$ are supposed. Then both statements (a) and (b) in Theorem 13.3 hold true with replacing the set $\mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ by the set $\mathscr{S}_{p\times q}(\mathbb{D})$.*

Proof Use Theorem 13.3 and Remark 8.2. □

Now we turn our attention to case (I) in general:

Theorem 13.5 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^n$. Denote by $\begin{bmatrix} \mathfrak{w}_n & \mathfrak{x}_n \\ \mathfrak{y}_n & \mathfrak{z}_n \end{bmatrix}$ the block representation of the matrix-valued function $\mathfrak{V}_n$ given by Notation 13.1 with $p\times p$ block $\mathfrak{w}_n$. Suppose that the linear subspaces $\mathcal{M}_n$ of $\mathbb{C}^p$ and $\mathcal{Q}_n$ of $\mathbb{C}^q$ given by Notation 6.1 are such that $\mathcal{M}_n \neq \{O_{p\times 1}\}$ and $\mathcal{Q}_n \neq \mathbb{C}^q$. Let $m := \dim \mathcal{M}_n$, let $u_1, u_2, \ldots, u_p$ be an orthonormal basis of $\mathbb{C}^p$ such that $u_1, u_2, \ldots, u_m$ is a basis of $\mathcal{M}_n$, let $U_\bullet := [u_1, u_2, \ldots, u_p]$, and let $U := [u_1, u_2, \ldots, u_m]$. Furthermore, let $t := q - \dim \mathcal{Q}_n$, let $v_1, v_2, \ldots, v_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $v_1, v_2, \ldots, v_t$ is a basis of $\mathcal{Q}_n^\perp$, let $V_\bullet := [v_1, v_2, \ldots, v_q]$, and let $V := [v_1, v_2, \ldots, v_t]$. For all $S \in \mathscr{S}_{m\times t}(\mathbb{D})$, let $S_\diamond\colon \mathbb{D} \to \mathbb{C}^{p\times q}$ be defined by*

$$S_\diamond(z) := \begin{cases} S(z), & \text{if } m = p \text{ and } t = q \\ [S(z), O_{p\times(q-t)}], & \text{if } m = p \text{ and } t < q \\ \begin{bmatrix} S(z) \\ O_{(p-m)\times q} \end{bmatrix}, & \text{if } m < p \text{ and } t = q \\ \begin{bmatrix} S(z) & O_{m\times(q-t)} \\ O_{(p-m)\times t} & O_{(p-m)\times(q-t)} \end{bmatrix}, & \text{if } m < p \text{ and } t < q \end{cases}. \tag{13.3}$$

Then:

(a) *Let* $S \in \mathscr{S}_{m\times t}(\mathbb{D})$. *Then* $\det(\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet) \neq 0$ *for all* $z \in \mathbb{D}$ *and* $F : \mathbb{D} \to \mathbb{C}^{p\times q}$ *defined by*

$$F(z) := [\mathfrak{w}_n(z)U_\bullet S_\diamond(z) + \mathfrak{x}_n(z)V_\bullet][\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet]^{-1}$$

belongs to $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$.

(b) *For all* $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$, *there exists a unique* $S \in \mathscr{S}_{m\times t}(\mathbb{D})$ *such that the function* $\det(\mathfrak{y}_n U_\bullet S_\diamond + \mathfrak{z}_n V_\bullet)$ *does not vanish identically and* $F = (\mathfrak{w}_n U_\bullet S_\diamond + \mathfrak{x}_n V_\bullet)(\mathfrak{y}_n U_\bullet S_\diamond + \mathfrak{z}_n V_\bullet)^{-1}$ *holds true, namely* $S = U^* F^{[\![n+1]\!]} V$.

Proof First observe that U is the left $p \times m$ block of $U_\bullet$, that V is the left $q \times t$ block of $V_\bullet$, that $U^*U = I_m$ and $V^*V = I_t$, and that the matrices $U_\bullet$ and $V_\bullet$ are unitary. According to our assumptions, we can apply Lemma 8.4 with $\mathcal{M} = \mathcal{M}_n$ and $\mathcal{Q} = \mathcal{Q}_n$.

(a) Let $G := U_\bullet S_\diamond V_\bullet^*$. Regarding that $V_\bullet$ is unitary, we have then

$$\mathfrak{w}_n(z)G(z) + \mathfrak{x}_n(z) = [\mathfrak{w}_n(z)U_\bullet S_\diamond(z) + \mathfrak{x}_n(z)V_\bullet]V_\bullet^* \tag{13.4}$$

and

$$\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z) = [\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet]V_\bullet^* \tag{13.5}$$

for all $z \in \mathbb{D}$. By virtue of (13.3), we see that $G = USV^*$, so that Lemma 8.4(a) yields $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$. Thus, we can apply Theorem 13.3(a) to get $\det(\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)) \neq 0$ for all $z \in \mathbb{D}$ and that $H : \mathbb{D} \to \mathbb{C}^{p\times q}$ defined by $H(z) := [\mathfrak{w}_n(z)G(z) + \mathfrak{x}_n(z)][\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)]^{-1}$ belongs to $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$. Taking additionally into account (13.5) and (13.4), for all $z \in \mathbb{D}$, then $\det(\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet) \neq 0$ and $H(z) = F(z)$ follow. In particular, $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$.

(b) Let $F \in \mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$. According to Theorem 13.3(b), then there exists a $G \in \mathscr{S}_{p\times q}\langle\mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$ such that the function $\det(\mathfrak{y}_n G + \mathfrak{z}_n)$ does not vanish identically and that $F = (\mathfrak{w}_n G + \mathfrak{x}_n)(\mathfrak{y}_n G + \mathfrak{z}_n)^{-1}$ holds true. From Theorem 13.3(a), thus $\det(\mathfrak{y}_n(z)G(z)+\mathfrak{z}_n(z)) \neq 0$ for all $z \in \mathbb{D}$ follows. Consequently, for all $z \in \mathbb{D}$, we have $F(z) = [\mathfrak{w}_n(z)G(z)+\mathfrak{x}_n(z)][\mathfrak{y}_n(z)G(z)+\mathfrak{z}_n(z)]^{-1}$. According to Lemma 8.4(b), there exists an $S \in \mathscr{S}_{m\times t}(\mathbb{D})$ such that $G = USV^*$. Taking additionally into account (13.3), we can conclude $U_\bullet S_\diamond V_\bullet^* = G$. Regarding that $V_\bullet$ is unitary, for all $z \in \mathbb{D}$, we have then (13.4) and (13.5). For all $z \in \mathbb{D}$, consequently, $\det(\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet) \neq 0$ and $F(z) = [\mathfrak{w}_n(z)U_\bullet S_\diamond(z) + \mathfrak{x}_n(z)V_\bullet][\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet]^{-1}$ follow. In particular, $\det(\mathfrak{y}_n U_\bullet S_\diamond + \mathfrak{z}_n V_\bullet)$ does not vanish identically and $F = (\mathfrak{w}_n U_\bullet S_\diamond + \mathfrak{x}_n V_\bullet)(\mathfrak{y}_n U_\bullet S_\diamond + \mathfrak{z}_n V_\bullet)^{-1}$.

Now we consider an arbitrary $S \in \mathscr{S}_{m\times t}(\mathbb{D})$ such that the function $\det(\mathfrak{y}_n U_\bullet S_\diamond + \mathfrak{z}_n V_\bullet)$ does not vanish identically and that $F = (\mathfrak{w}_n U_\bullet S_\diamond + \mathfrak{x}_n V_\bullet)(\mathfrak{y}_n U_\bullet S_\diamond + \mathfrak{z}_n V_\bullet)^{-1}$ holds true. Using part (a), we can infer then $\det(\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet) \neq 0$ for all $z \in \mathbb{D}$. Hence, for all $z \in \mathbb{D}$, we have $F(z) = [\mathfrak{w}_n(z)U_\bullet S_\diamond(z) + \mathfrak{x}_n(z)V_\bullet][\mathfrak{y}_n(z)U_\bullet S_\diamond(z) + \mathfrak{z}_n(z)V_\bullet]^{-1}$. Let $G := U_\bullet S_\diamond V_\bullet^*$. Regarding that $V_\bullet$ is unitary, for all $z \in \mathbb{D}$, we have then (13.4), (13.5), and, in particular, $\det(\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)) \neq$

0, so that $[\mathfrak{w}_n(z)G(z) + \mathfrak{x}_n(z)][\mathfrak{y}_n(z)G(z) + \mathfrak{z}_n(z)]^{-1} = F(z)$ follows. In particular, $\det(\mathfrak{y}_n G + \mathfrak{z}_n)$ does not vanish identically and $F = (\mathfrak{w}_n G + \mathfrak{x}_n)(\mathfrak{y}_n G + \mathfrak{z}_n)^{-1}$. By virtue of (13.3), we see $G = USV^*$, so that Lemma 8.4(a) yields $G \in \mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle$. Consequently, Theorem 13.3(b) provides $G = F^{[\![n+1]\!]}$, whereas Lemma 8.4(b) shows that $S = U^* G V$. Thus, we obtain $S = U^* F^{[\![n+1]\!]} V$. □

Now we turn our attention to case (II):

Theorem 13.6 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^n$ be such that $\mathcal{M}_n = \{O_{p\times 1}\}$ or $\mathcal{Q}_n = \mathbb{C}^q$. Denote by $\begin{bmatrix} \mathfrak{w}_n & \mathfrak{x}_n \\ \mathfrak{y}_n & \mathfrak{z}_n \end{bmatrix}$ the block representation of $\mathfrak{V}_n$ with $p \times p$ block $\mathfrak{w}_n$. Then $\det \mathfrak{z}_n(z) \neq 0$ for all $z \in \mathbb{D}$ and $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n] = \{\mathfrak{x}_n \mathfrak{z}_n^{-1}\}$.*

Proof Using Remark 8.3 and the notations therein, we can infer $\mathscr{S}_{p\times q}\langle \mathbb{D}; \mathcal{M}_n, \mathcal{Q}_n\rangle = \{\theta_{p\times q}\}$. Thus, the application of Theorem 13.3 completes the proof. □

14 Recovering the Taylor Coefficients from the Schur–Potapov Parameters

We reconsider in this section a topic which was already a central theme of Issai Schur in [28, §2] when he studied complex-valued holomorphic functions bounded by 1. Our main goal is to prove a parametrization of an arbitrarily given matricial Schur function by its SP-parameter sequence. In the context of the special case of non-degenerate $p \times q$ Schur sequences, the topic of this section was also handled in [11, Sec. 3.8]. In particular, [11, Prop. 3.8.1, Thm. 3.8.1, and Prop. 3.8.5] should be considered. We note that several results of [11, Sec. 3.8] could be obtained by applying relations between $p \times q$ Schur functions and non-negative Hermitian $(p+q) \times (p+q)$ Borel measures on the unit circle. Especially, the SP-algorithm is closely related to the Szegő recursion formulas for these non-negative Hermitian $(p+q) \times (p+q)$ measures. It should be mentioned that even in the context of complex Hilbert spaces, Constantinescu [9] also constructed a Schur-type algorithm in order to parametrize contractive operator matrices of the type $\mathbf{S}_n$ given by (2.2).

First we want to give a parametrization of an arbitrary given $p \times q$ Schur sequence by its SP-parameter sequence. With this in mind, we introduce the following notation.

Notation 14.1 Let $(\mathfrak{e}_j)_{j=0}^\kappa$ be a sequence of contractive complex $p \times q$ matrices. Then let $\mathfrak{L}_{-1} := I_p$ and $\mathfrak{R}_{-1} := I_q$. Furthermore, for all $j \in \mathbb{Z}_{0,\kappa}$, regarding Remark 4.12, let

$$\mathfrak{L}_j := \sqrt{\mathfrak{l}_0}\sqrt{\mathfrak{l}_1}\cdots\sqrt{\mathfrak{l}_j} \qquad \text{and} \qquad \mathfrak{R}_j := \sqrt{\mathfrak{r}_j}\cdots\sqrt{\mathfrak{r}_1}\sqrt{\mathfrak{r}_0}.$$

In the sequel, the sequence $(\mathfrak{e}_j)_{j=0}^\kappa$ of contractive complex $p \times q$ matrices mainly arises as the SP-parameter sequence of a $p \times q$ Schur sequence.

Notation 14.2 (cf. [11, p. 181]) Let $\Psi_0, \mu_0 \colon \mathbb{K}_{p\times q} \to \mathbb{C}^{p\times q}$ be defined by $\Psi_0(\mathfrak{e}_0) := \mathfrak{e}_0$ and $\mu_0(\mathfrak{e}_0) := O_{p\times q}$, respectively. For all $m \in \mathbb{N}$, let $\Psi_m, \mu_m \colon \mathbb{K}_{p\times q}^{m+1} \to \mathbb{C}^{p\times q}$ be recursively defined by

$$\Psi_m(\mathfrak{e}_0, \mathfrak{e}_1, \ldots, \mathfrak{e}_m) := \mu_{m-1}(\mathfrak{e}_0, \ldots, \mathfrak{e}_{m-1}) + \mathfrak{L}_{m-1}\mathfrak{e}_m\mathfrak{R}_{m-1} \tag{14.1}$$

and

$$\begin{aligned}\mu_m(\mathfrak{e}_0, \mathfrak{e}_1, \ldots, \mathfrak{e}_m) &:= \sqrt{\mathfrak{l}_0}\mu_{m-1}(\mathfrak{e}_1 \ldots, \mathfrak{e}_m)\sqrt{\mathfrak{r}_0} \\ &- \sum_{\ell=1}^{m} \sqrt{\mathfrak{l}_0}\Psi_{m-\ell}(\mathfrak{e}_1, \ldots, \mathfrak{e}_{m-\ell+1})\sqrt{\mathfrak{r}_0}^{\dagger}\mathfrak{e}_0^*\Psi_\ell(\mathfrak{e}_0, \ldots, \mathfrak{e}_\ell).\end{aligned} \tag{14.2}$$

Now we are able to describe how an arbitrary $p \times q$ Schur sequence can be recovered from its SP-parameters.

Theorem 14.3 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For all $k \in \mathbb{Z}_{0,\kappa}$, then $A_k = \Psi_k(\mathfrak{e}_0, \ldots, \mathfrak{e}_k)$.*

Proof First observe that Remark 4.12 shows $\mathfrak{e}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. According to Definitions 4.7 and 4.1, we have $\mathfrak{e}_0 = A_0^{[0]} = A_0$. In particular, $A_0 \in \mathbb{K}_{p\times q}$. Regarding (2.5), hence Remark A.17(a) shows $l_0 \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $r_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Thus, we can apply Remark A.10(d) to obtain with (2.5) then

$$-\sqrt{r_0}^{\dagger}A_0^*A_0 + \sqrt{r_0}^{\dagger} = \sqrt{r_0}^{\dagger}(I_q - A_0^*A_0) = \sqrt{r_0}^{\dagger}r_0 = \sqrt{r_0}. \tag{14.3}$$

According to Notation 14.2, we have $\Psi_0(\mathfrak{e}_0) = \mathfrak{e}_0 = A_0$.

Now assume $\kappa \geq 1$. According to Notations 14.2 and 14.1, and Definition 4.7, we have

$$\Psi_1(\mathfrak{e}_0, \mathfrak{e}_1) = \mathfrak{L}_0\mathfrak{e}_1\mathfrak{R}_0 + \mu_0(\mathfrak{e}_0) = \sqrt{\mathfrak{l}_0}A_0^{[1]}\sqrt{\mathfrak{r}_0}. \tag{14.4}$$

Remark 3.3 yields $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{N}_{p\times q;\kappa}$. Thus, we can apply Lemma 5.22 to obtain $\mathring{\mathbf{S}}_0^{[1]}[-\langle\!\langle\sqrt{r_0}^{\dagger}A_0^*\rangle\!\rangle_1\mathbf{S}_1 + \langle\!\langle\sqrt{r_0}^{\dagger} + Q_0\rangle\!\rangle_1] = \langle\!\langle\sqrt{l_0}^{\dagger}\rangle\!\rangle_1\mathbf{S}_1 - \langle\!\langle\sqrt{l_0}^{\dagger}A_0\rangle\!\rangle_1$. Comparing the lower left $p \times q$ block on both sides, in view of (2.2), and (3.2), then

$$[A_0^{[1]}, O_{p\times q}]\begin{bmatrix}-\sqrt{r_0}^{\dagger}A_0^*A_0 + (\sqrt{r_0}^{\dagger} + Q_0) \\ -\sqrt{r_0}^{\dagger}A_0^*A_1\end{bmatrix} = \sqrt{l_0}^{\dagger}A_1 \tag{14.5}$$

follows. Remark 3.3 yields $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa}$. Thus, from Remark 3.22 we can infer $\mathcal{N}(r_0) \subseteq \mathcal{N}(A_0^{[1]})$. In view of (2.11), hence $A_0^{[1]}Q_0 = O$. Taking additionally into account (14.3), from (14.5) we get then $A_0^{[1]}\sqrt{r_0} = \sqrt{l_0}^{\dagger}A_1$. Remark 3.3 yields $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}_{p\times q;\kappa}$. From Notation 3.1 we then see $\mathcal{R}(A_1) \subseteq \mathcal{R}(l_0)$. Hence, Remark A.7(a) provides $l_0l_0^{\dagger}A_1 = A_1$. Since Remark A.10(c) shows $l_0l_0^{\dagger} = \sqrt{l_0}\sqrt{l_0}^{\dagger}$, we obtain $\sqrt{l_0}A_0^{[1]}\sqrt{r_0} = \sqrt{l_0}\sqrt{l_0}^\dagger A_1 = l_0 l_0^\dagger A_1 = A_1$. Comparing this with (14.4) and regarding Remark 4.14, then $\Psi_1(\mathfrak{e}_0, \mathfrak{e}_1) = A_1$ follows.

Now assume $\kappa \geq 2$ and that there exists $n \in \mathbb{Z}_{1,\kappa-1}$ such that for all $m \in \mathbb{Z}_{0,n}$ the following statement holds true:

(I_m) For each $(B_j)_{j=0}^m \in \mathscr{S}_{p\times q;m}$ with SP-parameter sequence $(\mathfrak{p}_j)_{j=0}^m$, the identity $B_k = \Psi_k(\mathfrak{p}_0, \dots, \mathfrak{p}_k)$ is valid for all $k \in \mathbb{Z}_{0,m}$.

From Remark 4.9 we know that $(A_j)_{j=0}^n$ belongs to $\mathscr{S}_{p\times q;n}$ and has SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^n$, so that (I_n) yields

$$A_k = \Psi_k(\mathfrak{e}_0, \dots, \mathfrak{e}_k) \qquad \text{for all } k \in \mathbb{Z}_{0,n}. \tag{14.6}$$

Remark 4.8 shows that $(A_j^{[1]})_{j=0}^{\kappa-1}$ belongs to $\mathscr{S}_{p\times q;\kappa-1}$ and has SP-parameter sequence $(\mathfrak{e}_{j+1})_{j=0}^{\kappa-1}$. According to Remark 4.9, then $(A_j^{[1]})_{j=0}^n$ belongs to $\mathscr{S}_{p\times q;n}$ and has SP-parameter sequence $(\mathfrak{e}_{j+1})_{j=0}^n$, so that ($\mathrm{I}_n$) yields

$$A_k^{[1]} = \Psi_k(\mathfrak{e}_1, \dots, \mathfrak{e}_{k+1}) \qquad \text{for all } k \in \mathbb{Z}_{0,n}. \tag{14.7}$$

By virtue of (14.1) and Notations 14.1 and 4.11, we see $\Psi_n(\mathfrak{e}_1, \dots, \mathfrak{e}_{n+1}) = \sqrt{\mathfrak{l}_1}\cdots\sqrt{\mathfrak{l}_n}\mathfrak{e}_{n+1}\sqrt{\mathfrak{r}_n}\cdots\sqrt{\mathfrak{r}_1} + \mu_{n-1}(\mathfrak{e}_1, \dots, \mathfrak{e}_n)$. In view of (14.7) and Notation 14.1, then $\sqrt{\mathfrak{l}_0}A_n^{[1]}\sqrt{\mathfrak{r}_0} = \mathfrak{L}_n\mathfrak{e}_{n+1}\mathfrak{R}_n + \sqrt{\mathfrak{l}_0}\mu_{n-1}(\mathfrak{e}_1, \dots, \mathfrak{e}_n)\sqrt{\mathfrak{r}_0}$ follows. Since (14.1) shows $\Psi_{n+1}(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_{n+1}) = \mathfrak{L}_n\mathfrak{e}_{n+1}\mathfrak{R}_n + \mu_n(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_n)$, we can thus conclude $\Psi_{n+1}(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_{n+1}) - \sqrt{\mathfrak{l}_0}A_n^{[1]}\sqrt{\mathfrak{r}_0} = \mu_n(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_n) - \sqrt{\mathfrak{l}_0}\mu_{n-1}(\mathfrak{e}_1, \dots, \mathfrak{e}_n)\sqrt{\mathfrak{r}_0}$. By virtue of (14.2), then $\Psi_{n+1}(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_{n+1}) - \sqrt{\mathfrak{l}_0}A_n^{[1]}\sqrt{\mathfrak{r}_0} = -\sum_{\ell=1}^n \sqrt{\mathfrak{l}_0}\Psi_{n-\ell}(\mathfrak{e}_1, \dots, \mathfrak{e}_{n-\ell+1})\sqrt{\mathfrak{r}_0}^\dagger\mathfrak{e}_0^*\Psi_\ell(\mathfrak{e}_0, \dots, \mathfrak{e}_\ell)$ follows. Using (14.7) and (14.6), we thus get $\Psi_{n+1}(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_{n+1}) - \sqrt{\mathfrak{l}_0}A_n^{[1]}\sqrt{\mathfrak{r}_0} = -\sum_{\ell=1}^n \sqrt{\mathfrak{l}_0}A_{n-\ell}^{[1]}\sqrt{\mathfrak{r}_0}^\dagger\mathfrak{e}_0^* A_\ell$. Taking additionally into account $\mathfrak{e}_0 = A_0$, we can conclude

$$\Psi_{n+1}(\mathfrak{e}_0, \mathfrak{e}_1, \dots, \mathfrak{e}_{n+1}) = \sqrt{\mathfrak{l}_0}A_n^{[1]}\sqrt{\mathfrak{r}_0} - \sum_{\ell=1}^n \sqrt{\mathfrak{l}_0}A_{n-\ell}^{[1]}\sqrt{\mathfrak{r}_0}^\dagger A_0^* A_\ell. \tag{14.8}$$

Regarding $(A_j)_{j=0}^\kappa \in \mathscr{K}\mathscr{N}_{p\times q;\kappa}$, we can apply Lemma 5.22 to obtain $\mathring{\mathbf{S}}_n^{[1]}[-\langle\!\langle\sqrt{r_0}^\dagger A_0^*\rangle\!\rangle_{n+1}\mathbf{S}_{n+1} + \langle\!\langle\sqrt{r_0}^\dagger + Q_0\rangle\!\rangle_{n+1}] = \langle\!\langle\sqrt{l_0}^\dagger\rangle\!\rangle_{n+1}\mathbf{S}_{n+1} - \langle\!\langle\sqrt{l_0}^\dagger A_0\rangle\!\rangle_{n+1}$. Comparing the lower left $p \times q$ block on both sides, in view of (2.2), and (3.2), then

$$[A_n^{[1]}, A_{n-1}^{[1]}, \dots, A_0^{[1]}, O_{p\times q}]\begin{bmatrix} -\sqrt{r_0}^\dagger A_0^* A_0 + (\sqrt{r_0}^\dagger + Q_0) \\ -\sqrt{r_0}^\dagger A_0^* A_1 \\ \vdots \\ -\sqrt{r_0}^\dagger A_0^* A_n \\ -\sqrt{r_0}^\dagger A_0^* A_{n+1} \end{bmatrix} = \sqrt{l_0}^\dagger A_{n+1} \tag{14.9}$$

follows. Regarding $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa}$, Remark 3.22 yields $\mathcal{N}(r_0) \subseteq \mathcal{N}(A_n^{[1]})$. In view of (2.11), hence $A_n^{[1]} Q_0 = O$. Taking additionally into account (14.3), from (14.9) we get then $A_n^{[1]}\sqrt{r_0} - \sum_{\ell=1}^{n} A_{n-\ell}^{[1]}\sqrt{r_0}^{\dagger} A_0^* A_\ell = \sqrt{l_0}^{\dagger} A_{n+1}$. Regarding $(A_j)_{j=0}^{\kappa} \in \mathscr{K}\mathscr{R}_{p\times q;\kappa}$, from Notation 3.1 we see $\mathcal{R}(A_{n+1}) \subseteq \mathcal{R}(l_0)$. Hence, Remark A.7(a) provides $l_0 l_0^\dagger A_{n+1} = A_{n+1}$. Since $l_0 l_0^\dagger = \sqrt{l_0}\sqrt{l_0}^\dagger$, we obtain

$$\sqrt{l_0}\left(A_n^{[1]}\sqrt{r_0} - \sum_{\ell=1}^{n} A_{n-\ell}^{[1]}\sqrt{r_0}^{\dagger} A_0^* A_\ell\right) = \sqrt{l_0}\sqrt{l_0}^{\dagger} A_{n+1} = l_0 l_0^\dagger A_{n+1} = A_{n+1}.$$

Comparing this with (14.8) and regarding Remark 4.14, then $\Psi_{n+1}(\mathfrak{e}_0, \mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1}) = A_{n+1}$ follows. □

In view of Theorem 14.3 and the following corollary, it should be mentioned that an operator version for parametrizing lower triangular block Toeplitz contractions was worked out by Constantinescu. This was a far-reaching generalization of an idea of Schur [28, §2]. In particular, he obtained an operator version (see [9, Theorems 2.1 and 2.3]) of the following result:

Corollary 14.4 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For all $j \in \mathbb{Z}_{1,\kappa}$, then $A_j = \mu_{j-1}(\mathfrak{e}_0, \ldots, \mathfrak{e}_{j-1}) + \mathfrak{L}_{j-1}\mathfrak{e}_j\mathfrak{R}_{j-1}$.*

Proof Regarding (14.1), this is an immediate consequence of Theorem 14.3. □

Corollary 14.5 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $k \in \mathbb{Z}_{0,\kappa}$. For all $\ell \in \mathbb{Z}_{0,\kappa-k}$, then $A_\ell^{[k]} = \Psi_\ell(\mathfrak{e}_k, \ldots, \mathfrak{e}_{k+\ell})$.*

Proof Consider an arbitrary $\ell \in \mathbb{Z}_{0,\kappa-k}$. Remark 4.8 shows that $(A_j^{[k]})_{j=0}^{\kappa-k}$ belongs to $\mathscr{S}_{p\times q;\kappa-k}$ and has SP-parameter sequence $(\mathfrak{e}_{j+k})_{j=0}^{\kappa-k}$. Thus, we can apply Theorem 14.3 to obtain $A_\ell^{[k]} = \Psi_\ell(\mathfrak{e}_{0+k}, \ldots, \mathfrak{e}_{\ell+k})$. □

Corollary 14.6 *Let $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$. For every choice of $n \in \mathbb{Z}_{0,\kappa}$, $k \in \mathbb{Z}_{0,n}$, and $\ell \in \mathbb{Z}_{0,k}$, then $D_{n,k;\ell} = \Psi_\ell(\mathfrak{e}_{n-k}, \ldots, \mathfrak{e}_{n-k+\ell})$, where $(D_{n,k;j})_{j=0}^{k}$ is given via Notation 6.12.*

Proof Consider an arbitrary $n \in \mathbb{Z}_{0,\kappa}$. According to Notation 6.2, then $(\mathfrak{e}_j)_{j=0}^{n}$ is a sequence of contractive complex $p \times q$ matrices. Thus, we can apply Proposition 6.14, to see that $(A_j)_{j=0}^{n} := (D_{n,n;j})_{j=0}^{n}$ belongs to $\mathscr{S}_{p\times q;n}$. Theorem 6.20 shows that $(\mathfrak{e}_j)_{j=0}^{n}$ is the SP-parameter sequence of $(A_j)_{j=0}^{n}$. Consider an arbitrary $k \in \mathbb{Z}_{0,n}$. The application of Proposition 6.19 to the sequence $(A_j)_{j=0}^{n}$ then yields $(D_{n,k;j})_{j=0}^{k} = (A_j^{[n-k]})_{j=0}^{k}$. The application of Corollary 14.5 to the sequence $(A_j)_{j=0}^{n}$ provides $A_\ell^{[m]} = \Psi_\ell(\mathfrak{e}_m, \ldots, \mathfrak{e}_{m+\ell})$ for every choice of $m \in \mathbb{Z}_{0,n}$ and $\ell \in \mathbb{Z}_{0,n-m}$. Choosing $m = n-k$, for all $\ell \in \mathbb{Z}_{0,k}$, we thus obtain $D_{n,k;\ell} = A_\ell^{[n-k]} = \Psi_\ell(\mathfrak{e}_{n-k}, \ldots, \mathfrak{e}_{n-k+\ell})$. □

Proposition 14.7 *Let* $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$ *and let* $k \in \mathbb{Z}_{0,\kappa}$. *Then* $(\Psi_j(\mathfrak{e}_k, \ldots, \mathfrak{e}_{k+j}))_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$.

Proof Consider an arbitrary $n \in \mathbb{Z}_{0,\kappa-k}$. The application of Corollary 14.6 yields $D_{n+k,n;\ell} = \Psi_\ell(\mathfrak{e}_k, \ldots, \mathfrak{e}_{k+\ell})$ for all $\ell \in \mathbb{Z}_{0,n}$. From Notation 6.2 we see that $(\mathfrak{e}_j)_{j=0}^{n+k}$ is a sequence of contractive complex $p \times q$ matrices. Thus, we can apply Proposition 6.14, to see that $(D_{n+k,n;j})_{j=0}^{n}$ belongs to $\mathscr{S}_{p\times q;n}$. Summarizing, we obtain $(\Psi_j(\mathfrak{e}_k, \ldots, \mathfrak{e}_{k+j}))_{j=0}^{n} = (D_{n+k,n;j})_{j=0}^{n} \in \mathscr{S}_{p\times q;n}$ for all $n \in \mathbb{Z}_{0,\kappa-k}$, implying $(\Psi_j(\mathfrak{e}_k, \ldots, \mathfrak{e}_{k+j}))_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$. □

Remark 14.8 In view of Proposition 14.7, the mapping $\psi_{p\times q;\kappa} : \mathscr{E}_{p\times q;\kappa} \to \mathscr{S}_{p\times q;\kappa}$ defined by $\psi_{p\times q;\kappa}((\mathfrak{e}_j)_{j=0}^{\kappa}) := (\Psi_j(\mathfrak{e}_0, \ldots, \mathfrak{e}_j))_{j=0}^{\kappa}$ is well defined.

Now we obtain a useful parametrization of the set $\mathscr{S}_{p\times q;\kappa}$.

Theorem 14.9 *Let* $\phi_{p\times q;\kappa} : \mathscr{S}_{p\times q;\kappa} \to \mathscr{E}_{p\times q;\kappa}$ *be defined by* $\phi_{p\times q;\kappa}((A_j)_{j=0}^{\kappa}) := (\mathfrak{e}_j)_{j=0}^{\kappa}$, *where* $(\mathfrak{e}_j)_{j=0}^{\kappa}$ *is the SP-parameter sequence of* $(A_j)_{j=0}^{\kappa}$, *and let* $\psi_{p\times q;\kappa} : \mathscr{E}_{p\times q;\kappa} \to \mathscr{S}_{p\times q;\kappa}$ *be defined by* $\psi_{p\times q;\kappa}((\mathfrak{e}_j)_{j=0}^{\kappa}) := (\Psi_j(\mathfrak{e}_0, \ldots, \mathfrak{e}_j))_{j=0}^{\kappa}$, *where* Ψ_j *is given via Notation 14.2. Then* $\phi_{p\times q;\kappa}$ *and* $\psi_{p\times q;\kappa}$ *are well defined, bijective, and mutual inverses.*

Proof According to Remarks 6.11 and 14.8, the mappings $\phi_{p\times q;\kappa}$ and $\psi_{p\times q;\kappa}$ are well defined.

In order to check that $\psi_{p\times q;\kappa} \circ \phi_{p\times q;\kappa} = \mathrm{id}_{\mathscr{S}_{p\times q;\kappa}}$, we consider an arbitrary sequence $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$. Then $\phi_{p\times q;\kappa}((A_j)_{j=0}^{\kappa})$ is the SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ of $(A_j)_{j=0}^{\kappa}$ and belongs to $\mathscr{E}_{p\times q;\kappa}$ Theorem 14.3 yields $A_k = \Psi_k(\mathfrak{e}_0, \ldots, \mathfrak{e}_k)$ for all $k \in \mathbb{Z}_{0,\kappa}$. Therefore, we conclude

$$\psi_{p\times q;\kappa}(\phi_{p\times q;\kappa}((A_j)_{j=0}^{\kappa})) = \psi_{p\times q;\kappa}((\mathfrak{e}_j)_{j=0}^{\kappa}) = (\Psi_j(\mathfrak{e}_0, \ldots, \mathfrak{e}_j))_{j=0}^{\kappa} = (A_j)_{j=0}^{\kappa}.$$

Consequently, $\psi_{p\times q;\kappa} \circ \phi_{p\times q;\kappa} = \mathrm{id}_{\mathscr{S}_{p\times q;\kappa}}$.

In order to check that $\phi_{p\times q;\kappa} \circ \psi_{p\times q;\kappa} = \mathrm{id}_{\mathscr{E}_{p\times q;\kappa}}$, we consider an arbitrary sequence $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$. Then $(A_j)_{j=0}^{\kappa} := \psi_{p\times q;\kappa}((\mathfrak{e}_j)_{j=0}^{\kappa})$ belongs to $\mathscr{S}_{p\times q;\kappa}$. Denote by $(\mathfrak{p}_j)_{j=0}^{\kappa}$ the SP-parameter sequence of $(A_j)_{j=0}^{\kappa}$. Consider an arbitrary $n \in \mathbb{Z}_{0,\kappa}$. Remark 4.9 then shows that $(A_j)_{j=0}^{n}$ belongs to $\mathscr{S}_{p\times q;n}$ and has SP-parameter sequence $(\mathfrak{p}_j)_{j=0}^{n}$. Using the given notation for $\kappa = n$, we have then $(\mathfrak{p}_j)_{j=0}^{n} = \phi_{p\times q;n}((A_j)_{j=0}^{n})$ by definition. According to the definition of $(A_j)_{j=0}^{\kappa}$ and $\psi_{p\times q;\kappa}$, we have $A_k = \Psi_k(\mathfrak{e}_0, \ldots, \mathfrak{e}_k)$ for all $k \in \mathbb{Z}_{0,\kappa}$. Regarding $(\mathfrak{e}_j)_{j=0}^{\kappa} \in \mathscr{E}_{p\times q;\kappa}$, form Notation 6.2 we infer $(\mathfrak{e}_j)_{j=0}^{n} \in \mathscr{E}_{p\times q;n}$. Thus, we can use Corollary 6.15 and the notation therein as well as Corollary 14.6 to see that $(B_j)_{j=0}^{n} := \chi_{p\times q;n}((\mathfrak{e}_j)_{j=0}^{n})$ belongs to $\mathscr{S}_{p\times q;n}$ and fulfills $B_\ell = D_{n,n;\ell} = \Psi_\ell(\mathfrak{e}_0, \ldots, \mathfrak{e}_\ell)$ for all $\ell \in \mathbb{Z}_{0,n}$. Consequently, we conclude $A_j = B_j$ for all $j \in \mathbb{Z}_{0,n}$. Because of Theorem 6.20, we have $\phi_{p\times q;n}((B_j)_{j=0}^{n}) = \phi_{p\times q;n}(\chi_{p\times q;n}((\mathfrak{e}_j)_{j=0}^{n})) = (\mathfrak{e}_j)_{j=0}^{n}$. Summarizing, we obtain $(\mathfrak{p}_j)_{j=0}^{n} = \phi_{p\times q;n}((A_j)_{j=0}^{n}) = \phi_{p\times q;n}((B_j)_{j=0}^{n}) = (\mathfrak{e}_j)_{j=0}^{n}$. Since $n \in \mathbb{Z}_{0,\kappa}$ was arbitrarily chosen, then $(\mathfrak{p}_j)_{j=0}^{\kappa} = (\mathfrak{e}_j)_{j=0}^{\kappa}$, i.e., $(\mathfrak{e}_j)_{j=0}^{\kappa}$ is the SP-parameter sequence of $(A_j)_{j=0}^{\kappa}$. Taking additionally into account the definition of $(A_j)_{j=0}^{\kappa}$ and $\phi_{p\times q;\kappa}$, we get $\phi_{p\times q;\kappa}(\psi_{p\times q;\kappa}((\mathfrak{e}_j)_{j=0}^{\kappa})) = \phi_{p\times q;\kappa}((A_j)_{j=0}^{\kappa}) = (\mathfrak{e}_j)_{j=0}^{\kappa}$. Thus, $\phi_{p\times q;\kappa} \circ \psi_{p\times q;\kappa} = \mathrm{id}_{\mathscr{E}_{p\times q;\kappa}}$ is proved as well. □

Corollary 14.10 *Let* $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ *with SP-parameter sequence* $(\mathfrak{e}_j)_{j=0}^{\kappa}$. *Then* $A_j = O_{p\times q}$ *for all* $j \in \mathbb{Z}_{0,\kappa}$ *if and only if* $\mathfrak{e}_j = O_{p\times q}$ *for all* $j \in \mathbb{Z}_{0,\kappa}$.

Proof If $A_j = O_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$, then, from Definition 4.7 and Example 4.5, we can infer $\mathfrak{e}_j = A_0^{[j]} = O_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. Taking additionally into account Theorem 14.9, thus the asserted equivalence follows. □

Corollary 14.11 *Let* $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times p;\kappa}$ *with SP-parameter sequence* $(\mathfrak{e}_j)_{j=0}^{\kappa}$. *Then* $(A_j^*)_{j=0}^{\kappa} = (A_j)_{j=0}^{\kappa}$ *if and only if* $(\mathfrak{e}_j^*)_{j=0}^{\kappa} = (\mathfrak{e}_j)_{j=0}^{\kappa}$.

Proof Using the notation given in Theorem 14.9, we have $\phi_{p\times q;\kappa}((A_j)_{j=0}^{\kappa}) = (\mathfrak{e}_j)_{j=0}^{\kappa}$. Lemma 4.10 shows that $(A_j^*)_{j=0}^{\kappa}$ belongs to $\mathscr{S}_{q\times p;\kappa}$ and has SP-parameter sequence $(\mathfrak{e}_j^*)_{j=0}^{\kappa}$. Hence, $\phi_{p\times q;\kappa}((A_j^*)_{j=0}^{\kappa}) = (\mathfrak{e}_j^*)_{j=0}^{\kappa}$. Taking additionally into account that Theorem 14.9, in particular, implies that $\phi_{p\times q;\kappa}$ is injective, the asserted equivalence follows. □

Now we obtain a main result of this paper. We draw the reader's attention to the particular result $\sigma_{p\times q}(\mathscr{S}_{p\times q}(\mathbb{D})) = \mathscr{E}_{p\times q;\infty}$.

Theorem 14.12 *Let* $\sigma_{p\times q} : \mathscr{S}_{p\times q}(\mathbb{D}) \to \mathscr{E}_{p\times q;\infty}$ *be defined by* $\sigma_{p\times q}(F) := (\gamma_j)_{j=0}^{\infty}$, *where* $(\gamma_j)_{j=0}^{\infty}$ *is the SP-parameter sequence of* F. *Then* $\sigma_{p\times q}$ *is well defined and bijective.*

Proof Using Theorems D.2 and 14.9, and the notations given there, we see that $\tau_{p\times q} : \mathscr{S}_{p\times q}(\mathbb{D}) \to \mathscr{S}_{p\times q;\infty}$ and $\phi_{p\times q;\infty} : \mathscr{S}_{p\times q;\infty} \to \mathscr{E}_{p\times q;\infty}$ are well defined bijections. Furthermore, Proposition 9.7 provides $\sigma_{p\times q} = \phi_{p\times q;\infty} \circ \tau_{p\times q}$. □

15 An Extension Problem in $\mathscr{S}_{p\times q;\kappa}$

In this section, we are going to show how the preceding considerations can be used to get a description of the set

$$\mathcal{A}_{n+1} := \{A_{n+1} \in \mathbb{C}^{p\times q} : (A_j)_{j=0}^{n+1} \in \mathscr{S}_{p\times q;n+1}\},$$

where $n \in \mathbb{N}_0$ and $(A_j)_{j=0}^{n} \in \mathscr{S}_{p\times q;n}$ are arbitrarily given. Parametrizations of $\mathcal{A}_{n+1}$ are already given in [9], [14, Part I, Thm. 1], [10, Thm. 8], and [11, Thm. 3.5.1]. We will develop an explicit connection between the parameters used in [11, Thm. 3.5.1] and the Schur–Potapov parameters introduced in Definition 4.7. Recall that $\mathbb{K}_{p\times q}$ stands for the set of all contractive complex $p \times q$ matrices. In [30], Yu. L. Shmul'yan worked out the theory of operator balls. In the following, we use some of that results in the special case of complex matrices.

Notation 15.1 The set $\mathfrak{K}(M; A, B) := \{M + AKB : K \in \mathbb{K}_{p\times q}\}$ signifies the (closed) *matrix ball* with center M, left semi-radius A, and right semi-radius B with respect to arbitrarily given matrices $M \in \mathbb{C}^{p\times q}$, $A \in \mathbb{C}^{p\times p}$, and $B \in \mathbb{C}^{q\times q}$.

Note that Corollary 14.4 can be interpreted in the sense that A_j belongs to the matrix ball $\mathfrak{K}(\mu_{j-1}(\mathfrak{e}_0, \ldots, \mathfrak{e}_{j-1}); \mathfrak{L}_{j-1}, \mathfrak{R}_{j-1})$.

Theorem 15.2 (cf. [28], [11, Lem. 3.3.1, Thm. 3.5.1], Lemma A.15) *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$. In view of (2.4), (2.5), (2.7), and (2.8), then l_n and r_n are non-negative Hermitian and $\mathcal{A}_{n+1} = \mathfrak{K}(m_n; \sqrt{l_n}, \sqrt{r_n})$.*

Corollary 15.3 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$. Then $m_n \in \mathcal{A}_{n+1}$. In particular, there exists a sequence $(A_k)_{k=n+1}^\infty$ of complex $p \times q$ matrices such that $(A_j)_{j=0}^\infty \in \mathscr{S}_{p\times q;\infty}$.*

Proof This is a consequence of Theorem 15.2 and Notation 15.1. □

Definition 15.4 (*see also* [11, Def. 3.5.1]) If $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$, then the sequence $(\mathfrak{k}_j)_{j=0}^\kappa$ given by $\mathfrak{k}_0 := A_0$ and by $\mathfrak{k}_j := \sqrt{l_{j-1}}^\dagger (A_j - m_{j-1}) \sqrt{r_{j-1}}^\dagger$ for all $j \in \mathbb{Z}_{1,\kappa}$ is called the *choice sequence corresponding to* $(A_j)_{j=0}^\kappa$.

Proposition 15.5 (cf. [11, Thm. 3.5.1], Lemma A.15) *Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^\kappa$. For all $j \in \mathbb{Z}_{0,\kappa}$, then $\mathfrak{k}_j \in \mathbb{K}_{p\times q}$. Furthermore, $A_0 = \mathfrak{k}_0$ and $A_j = m_{j-1} + \sqrt{l_{j-1}}\mathfrak{k}_j\sqrt{r_{j-1}}$ for all $j \in \mathbb{Z}_{1,\kappa}$.*

Notation 15.6 Let $\mathscr{C}_{p\times q;\kappa}$ be the set of all sequences $(\mathfrak{k}_j)_{j=0}^\kappa$ of complex $p \times q$ matrices which fulfill $\mathfrak{k}_j \in \mathbb{K}_{p\times q}$ as well as $\mathcal{R}(\mathfrak{k}_j) \subseteq \mathcal{R}(l_{j-1})$ and $\mathcal{N}(r_{j-1}) \subseteq \mathcal{N}(\mathfrak{k}_j)$ for all $j \in \mathbb{Z}_{0,\kappa}$.

Proposition 15.7 *Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^\kappa$. Then $(\mathfrak{k}_j)_{j=0}^\kappa \in \mathscr{C}_{p\times q;\kappa}$.*

Proof Proposition 15.5 yields $\mathfrak{k}_j \in \mathbb{K}_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. In view of (2.5), clearly $\mathcal{R}(\mathfrak{k}_0) \subseteq \mathcal{R}(l_{-1})$ and $\mathcal{N}(r_{-1}) \subseteq \mathcal{N}(\mathfrak{k}_0)$ hold true. Now assume that $\kappa \geq 1$ and let $j \in \mathbb{Z}_{1,\kappa}$. Then, by virtue of Definition 15.4 and Remarks A.9 and A.10(a), we can conclude $\mathcal{R}(\mathfrak{k}_j) \subseteq \mathcal{R}(\sqrt{l_{j-1}}^\dagger) \subseteq \mathcal{R}(\sqrt{l_{j-1}}) = \mathcal{R}(l_{j-1})$ and $\mathcal{N}(r_{j-1}) = \mathcal{N}(\sqrt{r_{j-1}}) \subseteq \mathcal{N}(\sqrt{r_{j-1}}^\dagger) \subseteq \mathcal{N}(\mathfrak{k}_j)$. Thus, by virtue of Notation 15.6, we get $(\mathfrak{k}_j)_{j=0}^\kappa \in \mathscr{C}_{p\times q;\kappa}$. □

Remark 15.8 Let $(\mathfrak{e}_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of Notations 6.1 and 4.11 and Remarks A.1 and A.2, for all $j \in \mathbb{Z}_{-1,\kappa}$, then $\mathcal{M}_j^\perp = \sum_{\ell=0}^j \mathcal{N}(\mathfrak{l}_\ell)$ and $\mathcal{Q}_j^\perp = \bigcap_{\ell=0}^j \mathcal{R}(\mathfrak{r}_\ell)$.

Now we turn our attention to interesting relations between the matrices introduced in Notation 14.1 and the linear subspaces introduced in Notation 6.1.

Proposition 15.9 *Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^\kappa$. For each $j \in \mathbb{Z}_{-1,\kappa}$, then*

$$\mathcal{N}(\mathfrak{L}_j) = \mathcal{M}_j^\perp \qquad \textit{and} \qquad \mathcal{R}(\mathfrak{R}_j) = \mathcal{Q}_j^\perp. \tag{15.1}$$

Proof Our proof works inductively. According to Notations 14.1 and 6.1, we have $\mathcal{N}(\mathfrak{L}_{-1}) = \{O_{p\times 1}\} = \mathcal{M}_{-1}^\perp$ and $\mathcal{R}(\mathfrak{R}_{-1}) = \mathbb{C}^q = \mathcal{Q}_{-1}^\perp$. Now assume that $m \in$

$\mathbb{Z}_{-1,\kappa-1}$ and that (15.1) is valid for all $j \in \mathbb{Z}_{-1,m}$. Remark 4.12 shows $\mathfrak{l}_{m+1} \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $\mathfrak{r}_{m+1} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. We first prove that

$$\mathcal{N}(\mathfrak{L}_m) \subseteq \mathcal{N}(I_p - \sqrt{\mathfrak{l}_{m+1}}) \quad \text{and} \quad \mathcal{R}(I_q - \sqrt{\mathfrak{r}_{m+1}}) \subseteq \mathcal{R}(\mathfrak{R}_m). \tag{15.2}$$

We consider an arbitrary $x \in \mathcal{N}(\mathfrak{L}_m)$. According to (15.1) for $j = m$, then $x \in \mathcal{M}_m^{\perp}$. Proposition 6.10 and Notation 6.2 provide $\mathcal{R}(\mathfrak{e}_{m+1}) \subseteq \mathcal{M}_m$. Because of Remark A.2, then $\mathcal{M}_m^{\perp} \subseteq \mathcal{N}(\mathfrak{e}_{m+1}^*)$, so that $\mathfrak{e}_{m+1}^* x = O$ follows. In view of Notation 4.11, hence $\mathfrak{l}_{m+1} x = x$. Using Remark A.13, we conclude $\sqrt{\mathfrak{l}_{m+1}} x = x$. Consequently, $x \in \mathcal{N}(I_p - \sqrt{\mathfrak{l}_{m+1}})$. Thus, $\mathcal{N}(\mathfrak{L}_m) \subseteq \mathcal{N}(I_p - \sqrt{\mathfrak{l}_{m+1}})$ is proved. We now consider an arbitrary $y \in \mathcal{R}(\mathfrak{R}_m)^{\perp}$. According to (15.1) for $j = m$, then $y \in \mathcal{Q}_m$. Proposition 6.10 and Notation 6.2 provide $\mathcal{Q}_m \subseteq \mathcal{N}(\mathfrak{e}_{m+1})$, so that $\mathfrak{e}_{m+1} y = O$ follows. In view of Notation 4.11, hence $\mathfrak{r}_{m+1} y = y$. Using Remark A.13, we conclude $\sqrt{\mathfrak{r}_{m+1}} y = y$. Consequently, $y \in \mathcal{N}(I_q - \sqrt{\mathfrak{r}_{m+1}})$. Thus, $\mathcal{R}(\mathfrak{R}_m)^{\perp} \subseteq \mathcal{N}(I_q - \sqrt{\mathfrak{r}_{m+1}})$ is checked. Applying Remark A.2, we get then

$$\mathcal{R}(I_q - \sqrt{\mathfrak{r}_{m+1}}) = \mathcal{R}((I_q - \sqrt{\mathfrak{r}_{m+1}})^*) = \mathcal{N}(I_q - \sqrt{\mathfrak{r}_{m+1}})^{\perp} \subseteq \mathcal{R}(\mathfrak{R}_m).$$

Hence, (15.2) is proved. Thus, we can apply Lemmas B.2 and B.3 to obtain $\mathcal{N}(\mathfrak{L}_m) + \mathcal{N}(\sqrt{\mathfrak{l}_{m+1}}) = \mathcal{N}(\mathfrak{L}_m\sqrt{\mathfrak{l}_{m+1}})$ and $\mathcal{R}(\sqrt{\mathfrak{r}_{m+1}}) \cap \mathcal{R}(\mathfrak{R}_m) = \mathcal{R}(\sqrt{\mathfrak{r}_{m+1}}\mathfrak{R}_m)$. Using Remark A.10(a), we can infer $\mathcal{N}(\sqrt{\mathfrak{r}_{m+1}}) = \mathcal{N}(\mathfrak{r}_{m+1})$ and $\mathcal{R}(\sqrt{\mathfrak{l}_{m+1}}) = \mathcal{R}(\mathfrak{l}_{m+1})$. Thus, since (15.1) holds true for $j = m$, from Notation 14.1 and Remark 15.8 we can conclude $\mathcal{N}(\mathfrak{L}_{m+1}) = \mathcal{N}(\mathfrak{L}_m\sqrt{\mathfrak{l}_{m+1}}) = \mathcal{N}(\mathfrak{L}_m) + \mathcal{N}(\sqrt{\mathfrak{l}_{m+1}}) = \mathcal{M}_m^{\perp} + \mathcal{N}(\mathfrak{l}_{m+1}) = \mathcal{M}_{m+1}^{\perp}$ and $\mathcal{R}(\mathfrak{R}_{m+1}) = \mathcal{R}(\sqrt{\mathfrak{r}_{m+1}}\mathfrak{R}_m) = \mathcal{R}(\sqrt{\mathfrak{r}_{m+1}}) \cap \mathcal{R}(\mathfrak{R}_m) = \mathcal{R}(\mathfrak{r}_{m+1}) \cap \mathcal{Q}_m^{\perp} = \mathcal{Q}_{m+1}^{\perp}$. Thus, the assertion is inductively proved. □

Corollary 15.10 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. In view of Notations 14.1 and 6.4, for each $j \in \mathbb{Z}_{-1,\kappa}$, then $\mathfrak{L}_j^{\dagger}\mathfrak{L}_j = \mathfrak{M}_j\mathfrak{M}_j^{\dagger}$ and $\mathfrak{R}_j\mathfrak{R}_j^{\dagger} = \mathfrak{Q}_j^{\dagger}\mathfrak{Q}_j$.*

Proof We consider an arbitrary $j \in \mathbb{Z}_{-1,\kappa}$. Using Remarks A.6 and A.2 as well as Proposition 15.9 and Lemma 6.9, we have then $\mathfrak{L}_j^{\dagger}\mathfrak{L}_j = \mathbb{P}_{\mathcal{R}(\mathfrak{L}_j^*)} = \mathbb{P}_{\mathcal{N}(\mathfrak{L}_j)^{\perp}} = \mathbb{P}_{\mathcal{M}_j} = \mathbb{P}_{\mathcal{R}(\mathfrak{M}_j)} = \mathfrak{M}_j\mathfrak{M}_j^{\dagger}$ and $\mathfrak{R}_j\mathfrak{R}_j^{\dagger} = \mathbb{P}_{\mathcal{R}(\mathfrak{R}_j)} = \mathbb{P}_{\mathcal{Q}_j^{\perp}} = \mathbb{P}_{\mathcal{N}(\mathfrak{Q}_j)^{\perp}} = \mathbb{P}_{\mathcal{R}(\mathfrak{Q}_j^*)} = \mathfrak{Q}_j^{\dagger}\mathfrak{Q}_j$. □

Notation 15.11 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. Then, in view of Notations 14.1, 3.8 and 4.13, for every choice of $n \in \mathbb{Z}_{0,\kappa}$ and $k \in \mathbb{Z}_{0,n}$, let

$$\mathbf{W}_{\bullet,n;k} := \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k} \mathbf{W}_{n-k}^{[k]} \langle\!\langle \mathfrak{L}_k^{\dagger} \rangle\!\rangle_{n-k} + \langle\!\langle I_p - \mathfrak{L}_k\mathfrak{L}_k^{\dagger} \rangle\!\rangle_{n-k}$$

and

$$\mathbf{Y}_{\bullet,n;k} := \langle\!\langle \mathfrak{R}_k^{\dagger} \rangle\!\rangle_{n-k} \mathbf{Y}_{n-k}^{[k]} \langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k} + \langle\!\langle I_q - \mathfrak{R}_k^{\dagger}\mathfrak{R}_k \rangle\!\rangle_{n-k}.$$

Lemma 15.12 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$, let $n \in \mathbb{Z}_{0,\kappa}$, and let $k \in \mathbb{Z}_{0,n}$. Then $\mathbf{W}_{\bullet,n;k}$ is a block Toeplitz matrix belonging to $\mathscr{L}_{p,n-k}$ and $\mathbf{Y}_{\bullet,n;k}$ is a block Toeplitz matrix belonging to $\mathscr{L}_{q,n-k}$. In particular,* $\det \mathbf{W}_{\bullet,n;k} = 1$ *and* $\det \mathbf{Y}_{\bullet,n;k} = 1$.

Proof Denote by $(B_j)_{j=0}^{\kappa-k}$ the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. According to Remark 4.2, we have $(B_j)_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$. Hence, Remark 3.3 yields $(B_j)_{j=0}^{\kappa-k} \in \mathscr{K}_{p\times q;\kappa-k}$. In view of Definition 3.4, Remark 4.14, and Notation 14.1, we have

$$\mathfrak{L}_{k-1} W_{B;0} = \mathfrak{L}_{k-1}\sqrt{l_0^{[k]}} = \mathfrak{L}_{k-1}\sqrt{\mathfrak{l}_k} = \mathfrak{L}_k \tag{15.3}$$

and

$$Y_{B;0}\mathfrak{R}_{k-1} = \sqrt{r_0^{[k]}}\mathfrak{R}_{k-1} = \sqrt{\mathfrak{r}_k}\mathfrak{R}_{k-1} = \mathfrak{R}_k. \tag{15.4}$$

Consequently, $\mathfrak{L}_{k-1} W_{B;0}\mathfrak{L}_k^{\dagger} + (I_p - \mathfrak{L}_k\mathfrak{L}_k^{\dagger}) = I_p$ and $\mathfrak{R}_k^{\dagger} Y_{B;0}\mathfrak{R}_{k-1} + (I_q - \mathfrak{R}_k^{\dagger}\mathfrak{R}_k) = I_q$. Regarding Notations 15.11, 4.13, 3.8 and A.18, (3.2), and (2.2), thus the assertions follow. □

The next result indicates a connection between the matrices introduced in Notation 14.1 and the k-th right SP-transform of a $p \times q$ Schur sequence.

Lemma 15.13 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $k \in \mathbb{Z}_{0,\kappa}$. Denote by $(B_j)_{j=0}^{\kappa-k}$ the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. For all $j \in \mathbb{Z}_{0,\kappa-k}$, then $\mathcal{N}(\mathfrak{L}_k) \subseteq \mathcal{N}(\mathfrak{L}_{k-1} W_{B;j})$ and $\mathcal{R}(Y_{B;j}\mathfrak{R}_{k-1}) \subseteq \mathcal{R}(\mathfrak{R}_k)$.*

***Proof*.** As in the proof of Lemma 15.12, we can obtain $(B_j)_{j=0}^{\kappa-k} \in \mathscr{K}_{p\times q;\kappa-k}$ as well as (15.3) and (15.4), implying trivially $\mathcal{N}(\mathfrak{L}_k) \subseteq \mathcal{N}(\mathfrak{L}_{k-1} W_{B;0})$ and $\mathcal{R}(Y_{B;0}\mathfrak{R}_{k-1}) \subseteq \mathcal{R}(\mathfrak{R}_k)$. Now suppose $\kappa - k \geq 1$ and consider an arbitrary $j \in \mathbb{Z}_{1,\kappa-k}$. In view of Definition 4.7, we have $B_0 = \mathfrak{e}_k$. Therefore, Corollary 6.7 shows that there exists $M_k \in \mathbb{C}^{p\times q}$ such that $B_0 = \mathfrak{M}_{k-1} M_k \mathfrak{Q}_{k-1}$. Using Remark A.8, from Notation 6.4 we can infer $\mathfrak{M}_{k-1}^{*}\sqrt{\mathfrak{l}_k}^{\dagger} = \mathfrak{M}_k^{*}$ and $\sqrt{\mathfrak{r}_k}^{\dagger}\mathfrak{Q}_{k-1}^{*} = \mathfrak{Q}_k^{*}$. Regarding Definition 3.4 and Remark 4.14, we can conclude then

$$W_{B;j} = -B_j B_0^{*}\sqrt{l_0^{[k]}}^{\dagger} = -B_j\mathfrak{Q}_{k-1}^{*} M_k^{*}\mathfrak{M}_{k-1}^{*}\sqrt{\mathfrak{l}_k}^{\dagger} = -B_j\mathfrak{Q}_{k-1}^{*} M_k^{*}\mathfrak{M}_k^{*}$$

and

$$Y_{B;j} = -\sqrt{r_0^{[k]}}^{\dagger} B_0^{*} B_j = -\sqrt{\mathfrak{r}_k}^{\dagger}\mathfrak{Q}_{k-1}^{*} M_k^{*}\mathfrak{M}_{k-1}^{*} B_j = -\mathfrak{Q}_k^{*} M_k^{*}\mathfrak{M}_{k-1}^{*} B_j.$$

In particular, $\mathcal{N}(\mathfrak{M}_k^{*}) \subseteq \mathcal{N}(W_{B;j})$ and $\mathcal{R}(Y_{B;j}) \subseteq \mathcal{R}(\mathfrak{Q}_k^{*})$ follow. Proposition 15.9 shows $\mathcal{N}(\mathfrak{L}_k) = \mathcal{M}_k^{\perp}$ and $\mathcal{R}(\mathfrak{R}_k) = \mathcal{Q}_k^{\perp}$, whereas Lemma 6.9 provides $\mathcal{R}(\mathfrak{M}_k) = \mathcal{M}_k$ and $\mathcal{N}(\mathfrak{Q}_k) = \mathcal{Q}_k$. Using additionally Remark A.2, we get then

$$\mathcal{N}(\mathfrak{L}_k) = \mathcal{M}_k^{\perp} = \mathcal{R}(\mathfrak{M}_k)^{\perp} = \mathcal{N}(\mathfrak{M}_k^{*}) \subseteq \mathcal{N}(W_{B;j}) \subseteq \mathcal{N}(\mathfrak{L}_{k-1} W_{B;j})$$

and

$$\mathcal{R}(Y_{B;j}\mathfrak{R}_{k-1}) \subseteq \mathcal{R}(Y_{B;j}) \subseteq \mathcal{R}(\mathfrak{Q}_k^*) = \mathcal{N}(\mathfrak{Q}_k)^\perp = \mathcal{Q}_k^\perp = \mathcal{R}(\mathfrak{R}_k). \quad \square$$

Lemma 15.14 *Let* $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ *with SP-parameter sequence* $(\mathfrak{e}_j)_{j=0}^{\kappa}$, *let* $n \in \mathbb{Z}_{0,\kappa}$, *and let* $k \in \mathbb{Z}_{0,n}$. *Then* $\mathbf{W}_{\bullet,n;k}\langle\!\langle \mathfrak{L}_k \rangle\!\rangle_{n-k} = \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}\mathbf{W}_{n-k}^{[k]}$ *and* $\langle\!\langle \mathfrak{R}_k \rangle\!\rangle_{n-k}\mathbf{Y}_{\bullet,n;k} = \mathbf{Y}_{n-k}^{[k]}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k}$.

Proof. Denote by $(B_j)_{j=0}^{\kappa-k}$ the k-th right SP-transform of $(A_j)_{j=0}^{\kappa}$. For all $j \in \mathbb{Z}_{0,\kappa-k}$, then Lemma 15.13 shows $\mathcal{N}(\mathfrak{L}_k) \subseteq \mathcal{N}(\mathfrak{L}_{k-1}W_{B;j})$ and $\mathcal{R}(Y_{B;j}\mathfrak{R}_{k-1}) \subseteq \mathcal{R}(\mathfrak{R}_k)$, so that Remark A.7 yields $\mathfrak{L}_{k-1}W_{B;j}\mathfrak{L}_k^\dagger\mathfrak{L}_k = \mathfrak{L}_{k-1}W_{B;j}$ and $\mathfrak{R}_k\mathfrak{R}_k^\dagger Y_{B;j}\mathfrak{R}_{k-1} = Y_{B;j}\mathfrak{R}_{k-1}$. Regarding, (3.2) and (2.2), hence $\langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}\mathbf{S}_{W_B;n-k}\langle\!\langle \mathfrak{L}_k^\dagger\mathfrak{L}_k \rangle\!\rangle_{n-k} = \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}\mathbf{S}_{W_B;n-k}$ and $\langle\!\langle \mathfrak{R}_k\mathfrak{R}_k^\dagger \rangle\!\rangle_{n-k}\mathbf{S}_{Y_B;n-k}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k} = \mathbf{S}_{Y_B;n-k}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k}$ follow. According to Notations 3.8 and 4.13, we have $\mathbf{S}_{W_B;n-k} = \mathbf{W}_{n-k}^{[k]}$ and $\mathbf{S}_{Y_B;n-k} = \mathbf{Y}_{n-k}^{[k]}$. Using additionally Notation 15.11, Remark A.24(b), and (2.1), we obtain

$$\begin{aligned}\mathbf{W}_{\bullet,n;k}\langle\!\langle \mathfrak{L}_k \rangle\!\rangle_{n-k} &= \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}\mathbf{W}_{n-k}^{[k]}\langle\!\langle \mathfrak{L}_k^\dagger\mathfrak{L}_k \rangle\!\rangle_{n-k} + \langle\!\langle (I_p - \mathfrak{L}_k\mathfrak{L}_k^\dagger)\mathfrak{L}_k \rangle\!\rangle_{n-k} \\ &= \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}\mathbf{S}_{W_B;n-k}\langle\!\langle \mathfrak{L}_k^\dagger\mathfrak{L}_k \rangle\!\rangle_{n-k} = \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}\mathbf{W}_{n-k}^{[k]}\end{aligned}$$

and

$$\begin{aligned}\langle\!\langle \mathfrak{R}_k \rangle\!\rangle_{n-k}\mathbf{Y}_{\bullet,n;k} &= \langle\!\langle \mathfrak{R}_k\mathfrak{R}_k^\dagger \rangle\!\rangle_{n-k}\mathbf{Y}_{n-k}^{[k]}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k} + \langle\!\langle \mathfrak{R}_k(I_q - \mathfrak{R}_k^\dagger\mathfrak{R}_k) \rangle\!\rangle_{n-k} \\ &= \langle\!\langle \mathfrak{R}_k\mathfrak{R}_k^\dagger \rangle\!\rangle_{n-k}\mathbf{S}_{Y_B;n-k}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k} = \mathbf{Y}_{n-k}^{[k]}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k}. \quad \square\end{aligned}$$

Lemma 15.15 *Suppose* $\kappa \geq 1$. *Let* $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ *with SP-parameter sequence* $(\mathfrak{e}_j)_{j=0}^{\kappa}$, *let* $n \in \mathbb{Z}_{1,\kappa}$, *and let* $k \in \mathbb{Z}_{0,n-1}$. *Then*

$$\begin{aligned}&\langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}L_{n-k}^{[k]}\langle\!\langle \mathfrak{L}_{k-1}^* \rangle\!\rangle_{n-k} \\ &\quad = \mathbf{W}_{\bullet,n;k}\operatorname{diag}\left(\mathfrak{L}_k\mathfrak{L}_k^*, \langle\!\langle \mathfrak{L}_k \rangle\!\rangle_{n-k-1}L_{n-k-1}^{[k+1]}\langle\!\langle \mathfrak{L}_k^* \rangle\!\rangle_{n-k-1}\right)\mathbf{W}_{\bullet,n;k}^* \end{aligned} \tag{15.5}$$

and

$$\begin{aligned}&\langle\!\langle \mathfrak{R}_{k-1}^* \rangle\!\rangle_{n-k}R_{n-k}^{[k]}\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k} \\ &\quad = \mathbf{Y}_{\bullet,n;k}^*\operatorname{diag}\left(\langle\!\langle \mathfrak{R}_k^* \rangle\!\rangle_{n-k-1}R_{n-k-1}^{[k+1]}\langle\!\langle \mathfrak{R}_k \rangle\!\rangle_{n-k-1}, \mathfrak{R}_k^*\mathfrak{R}_k\right)\mathbf{Y}_{\bullet,n;k}. \end{aligned} \tag{15.6}$$

Proof According to Remark 4.2, we have $(A_j^{[k]})_{j=0}^{\kappa-k} \in \mathscr{S}_{p\times q;\kappa-k}$. Hence, Remark 3.3 yields $(A_j^{[k]})_{j=0}^{\kappa-k} \in \mathscr{K}\mathscr{R}\mathscr{N}_{p\times q;\kappa-k}$. Regarding Notation 4.13 and Definition 4.1, we can thus apply Propositions 3.23 and 3.26 to the sequence $(A_j^{[k]})_{j=0}^{\kappa-k}$ to obtain $L_{n-k}^{[k]} =$

$\mathbf{W}_{n-k}^{[k]} \operatorname{diag}(I_p, L_{n-k-1}^{[k+1]})(\mathbf{W}_{n-k}^{[k]})^*$ and $R_{n-k}^{[k]} = (\mathbf{Y}_{n-k}^{[k]})^* \operatorname{diag}(R_{n-k-1}^{[k+1]}, I_q)\mathbf{Y}_{n-k}^{[k]}$. Using Lemma 15.14, we can consequently conclude

$$\begin{aligned}&\langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k} L_{n-k}^{[k]} \langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k}^* \\ &\quad = \mathbf{W}_{\bullet,n;k} \langle\!\langle \mathfrak{L}_k \rangle\!\rangle_{n-k} \operatorname{diag}(I_p, L_{n-k-1}^{[k+1]}) \langle\!\langle \mathfrak{L}_k \rangle\!\rangle_{n-k}^* \mathbf{W}_{\bullet,n;k}^*\end{aligned}$$

and

$$\begin{aligned}&\langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k}^* R_{n-k}^{[k]} \langle\!\langle \mathfrak{R}_{k-1} \rangle\!\rangle_{n-k} \\ &\quad = \mathbf{Y}_{\bullet,n;k}^* \langle\!\langle \mathfrak{R}_k \rangle\!\rangle_{n-k}^* \operatorname{diag}(R_{n-k-1}^{[k+1]}, I_q) \langle\!\langle \mathfrak{R}_k \rangle\!\rangle_{n-k} \mathbf{Y}_{\bullet,n;k}.\end{aligned}$$

Regarding Remark A.24(a) and (3.2), then (15.5) and (15.6) follow. □

In the following results, we will use the equivalence relations "$\sim$" and "$\backsim$" introduced in Notation A.20 (see also Remark A.21). The next observation contains a relation between the matrices L_n and R_n introduced in (2.3) and the matrices introduced in Notation 14.1.

Lemma 15.16 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. If L_n and R_n are defined by (2.3), then*

$$L_n \sim \operatorname{diag}(\mathfrak{L}_0\mathfrak{L}_0^*, \mathfrak{L}_1\mathfrak{L}_1^*, \ldots, \mathfrak{L}_n\mathfrak{L}_n^*) \tag{15.7}$$

and

$$R_n \backsim \operatorname{diag}(\mathfrak{R}_n^*\mathfrak{R}_n, \mathfrak{R}_{n-1}^*\mathfrak{R}_{n-1}, \ldots, \mathfrak{R}_0^*\mathfrak{R}_0). \tag{15.8}$$

Proof First observe that Remark 4.12 shows $\mathfrak{l}_j \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ for all $j \in \mathbb{Z}_{0,\kappa}$. Using (2.9), Remark 4.14, and Notation 14.1, we get $L_0 = l_0 = \mathfrak{l}_0 = \mathfrak{L}_0\mathfrak{L}_0^*$. Regarding Notations A.18 and A.20(a), in particular, (15.7) holds true for $n = 0$. Now assume that $\kappa \geq 1$ and $n \in \mathbb{Z}_{1,\kappa}$. Lemma 15.15 then provides (15.5) for all $k \in \mathbb{Z}_{0,n-1}$. Remark 15.12 shows $\mathbf{W}_{\bullet,n;k} \in \mathscr{L}_{p,n-k}$ for all $k \in \mathbb{Z}_{0,n-1}$. By virtue of Notation A.18, we can thus infer $\mathbf{W}_{\bullet,n;k}^* \in \mathscr{U}_{p,n-k}$ for all $k \in \mathbb{Z}_{0,n-1}$. According to Notation A.20(a), for all $k \in \mathbb{Z}_{0,n-1}$, consequently (15.5) implies

$$\begin{aligned}&\langle\!\langle \mathfrak{L}_{k-1} \rangle\!\rangle_{n-k} L_{n-k}^{[k]} \langle\!\langle \mathfrak{L}_{k-1}^* \rangle\!\rangle_{n-k} \\ &\quad \sim \operatorname{diag}\left(\mathfrak{L}_k\mathfrak{L}_k^*, \langle\!\langle \mathfrak{L}_k \rangle\!\rangle_{n-k-1} L_{n-k-1}^{[k+1]} \langle\!\langle \mathfrak{L}_k^* \rangle\!\rangle_{n-k-1}\right).\end{aligned} \tag{15.9}$$

We now show, for all $\ell \in \mathbb{Z}_{1,n}$, inductively

$$L_n \sim \operatorname{diag}\left(\mathfrak{L}_0\mathfrak{L}_0^*, \mathfrak{L}_1\mathfrak{L}_1^*, \ldots, \mathfrak{L}_{\ell-1}\mathfrak{L}_{\ell-1}^*, \langle\!\langle \mathfrak{L}_{\ell-1} \rangle\!\rangle_{n-\ell} L_{n-\ell}^{[\ell]} \langle\!\langle \mathfrak{L}_{\ell-1}^* \rangle\!\rangle_{n-\ell}\right). \tag{15.10}$$

Using Definition 4.1, Notations 4.13 and 14.1, and (15.9) for $k = 0$, we can infer

$$L_n = L_n^{[0]} = \langle\!\langle \mathfrak{L}_{-1} \rangle\!\rangle_n L_n^{[0]} \langle\!\langle \mathfrak{L}_{-1}^* \rangle\!\rangle_n$$

$$\sim \operatorname{diag}\left(\mathfrak{L}_0\mathfrak{L}_0^*, \langle\!\langle \mathfrak{L}_0 \rangle\!\rangle_{n-1} L_{n-1}^{[1]} \langle\!\langle \mathfrak{L}_0^* \rangle\!\rangle_{n-1}\right).$$

Hence, (15.10) holds true for $\ell = 1$. Now assume $\kappa \geq 2$ and $n \geq 2$ and that $m \in \mathbb{Z}_{1,n-1}$ is such that (15.10) is valid for all $\ell \in \mathbb{Z}_{1,m}$. In view of Remark A.22(a), the combination of (15.10) for $\ell = m$ and (15.9) for $k = m$ yields that (15.10) is valid for $\ell = m + 1$. Consequently, we get inductively that (15.10) is fulfilled for all $\ell \in \mathbb{Z}_{1,n}$. Using (2.9), Notation 4.13, and Remark 4.14, we get furthermore $L_0^{[n]} = l_0^{[n]} = \mathfrak{l}_n$. Regarding (3.2) and Notation 14.1, we can thus conclude $\langle\!\langle \mathfrak{L}_{n-1} \rangle\!\rangle_0 L_0^{[n]} \langle\!\langle \mathfrak{L}_{n-1}^* \rangle\!\rangle_0 = \mathfrak{L}_{n-1}\mathfrak{l}_n\mathfrak{L}_{n-1}^* = \mathfrak{L}_n\mathfrak{L}_n^*$. Combining this with (15.10) for $\ell = n$, we get (15.7). Analogously, (15.8) can be proved. We omit the details. □

The following result can be embedded in a more general context (compare [14, Sec. 3], [11, Sec. 3.5]).

Lemma 15.17 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then l_n and r_n given by (2.8) admit the representations*

$$l_n = I_p - A_0 A_0^* - z_n (I + \mathbf{S}_{n-1}^* L_{n-1}^{\dagger} \mathbf{S}_{n-1}) z_n^* \tag{15.11}$$

and

$$r_n = I_q - A_0^* A_0 - y_n^* (I + \mathbf{S}_{n-1} R_{n-1}^{\dagger} \mathbf{S}_{n-1}^*) y_n, \tag{15.12}$$

respectively.

Proof Since $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$, we see from Lemma A.15 that the matrix $T_n := \left[\begin{smallmatrix} I & \mathbf{S}_n^* \\ \mathbf{S}_n & I \end{smallmatrix}\right]$ is non-negative Hermitian. Taking into account the block representation $\mathbf{S}_n = \left[\begin{smallmatrix} \mathbf{S}_{n-1} & O \\ z_n & A_0 \end{smallmatrix}\right]$ of $\mathbf{S}_n$, we see that the principal submatrix $\left[\begin{smallmatrix} I_{nq} & \mathbf{S}_{n-1}^* & z_n^* \\ \mathbf{S}_{n-1} & I_{np} & O \\ z_n & O & I_p \end{smallmatrix}\right]$ of T_n is non-negative Hermitian as well. Thus, we have $\mathcal{N}(T_{n-1}) \subseteq \mathcal{N}([z_n, O])$, i. e., there are matrices $X \in \mathbb{C}^{p\times nq}$ and $Y \in \mathbb{C}^{p\times np}$ such that

$$[X, Y] \begin{bmatrix} I_{nq} & \mathbf{S}_{n-1}^* \\ \mathbf{S}_{n-1} & I_{np} \end{bmatrix} = [z_n, O]. \tag{15.13}$$

Multiplying (15.13) from the right by $\left[\begin{smallmatrix} I_{nq} \\ -\mathbf{S}_{n-1} \end{smallmatrix}\right]$ and using (2.3), we get $[X, Y]\left[\begin{smallmatrix} R_{n-1} \\ O \end{smallmatrix}\right] = z_n$, i. e., $XR_{n-1} = z_n$. Regarding (2.1), thus $z_n R_{n-1}^{\dagger} R_{n-1} = z_n$. Using additionally Lemma A.16(c), then

$$z_n R_{n-1}^{\dagger} z_n^* = z_n (R_{n-1}^{\dagger} R_{n-1} + \mathbf{S}_{n-1}^* L_{n-1}^{\dagger} \mathbf{S}_{n-1}) z_n^* = z_n (I + \mathbf{S}_{n-1}^* L_{n-1}^{\dagger} \mathbf{S}_{n-1}) z_n^*$$

and, consequently, (15.11) follow. Analogously, (15.12) can be proved. □

Lemma 15.18 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. In view of (2.3), (2.5), and (2.8), then*

$$L_n \sim \operatorname{diag}(l_0, l_1, \ldots, l_n) \qquad \textit{and} \qquad R_n \backsim \operatorname{diag}(r_n, r_{n-1}, \ldots, r_0). \tag{15.14}$$

Proof According to (2.9), we have $L_0 = l_0$ and, by Notation A.20(a), especially $L_0 \sim l_0$. In particular, the first relation in (15.14) holds true for $n = 0$. Now assume $\kappa \geq 1$ and $n \in \mathbb{Z}_{1,\kappa}$. We consider an arbitrary $k \in \mathbb{Z}_{1,n}$. Regarding (2.3), (2.2), (2.6), and (2.8), we can see the block representation

$$L_k = \left[\begin{array}{c:c} L_{k-1} & -\mathbf{S}_{k-1} z_k^* \\ \hdashline -z_k \mathbf{S}_{k-1}^* & I_p - A_0 A_0^* - z_k z_k^* \end{array}\right].$$

Since $(A_j)_{j=0}^{\kappa} \in \mathcal{S}_{p\times q;\kappa}$ implies that the matrix L_k is non-negative Hermitian, then $\mathcal{R}(-\mathbf{S}_{k-1} z_k^*) \subseteq \mathcal{R}(L_{k-1})$ follows (see, e. g., [11, Lem. 1.1.9(a)]). Consequently, we can conclude

$$L_k = \begin{bmatrix} I_{pk} & O \\ -z_k \mathbf{S}_{k-1}^* L_{k-1}^\dagger & I_p \end{bmatrix} \operatorname{diag}(L_{k-1}, Z) \begin{bmatrix} I_{pk} & -L_{k-1}^\dagger \mathbf{S}_{k-1} z_k^* \\ O & I_p \end{bmatrix},$$

where $Z := I_p - A_0 A_0^* - z_k z_k^* - z_k \mathbf{S}_{k-1}^* L_{k-1}^\dagger \mathbf{S}_{k-1} z_k^*$ (see, e. g., [11, Lem. 1.1.7(a)]). According to Notations A.18 and A.20(a), therefore $L_k \sim \operatorname{diag}(L_{k-1}, Z)$. From Lemma 15.17 we know $Z = l_k$. Consequently, for all $k \in \mathbb{Z}_{1,n}$, we have

$$L_k \sim \operatorname{diag}(L_{k-1}, l_k). \tag{15.15}$$

We now show for all $\ell \in \mathbb{Z}_{1,n}$ inductively

$$L_n \sim \operatorname{diag}(L_{\ell-1}, l_\ell, \dots, l_n). \tag{15.16}$$

Using (15.15) for $k = n$, we can infer that (15.16) holds true for $\ell = n$. Now assume $\kappa \geq 2$ and $n \geq 2$ and that $m \in \mathbb{Z}_{2,n}$ is such that (15.16) is valid for all $\ell \in \mathbb{Z}_{m,n}$. In view of Remark A.22(a), the combination of (15.16) for $\ell = m$ and (15.15) for $k = m - 1$ yields that (15.16) is valid for $\ell = m - 1$. Consequently, we get inductively that (15.16) is fulfilled for all $\ell \in \mathbb{Z}_{1,n}$. Combining $L_0 = l_0$ with (15.16) for $\ell = 1$, we get the first relation in (15.14). Analogously, the second relation in (15.14) can be proved. □

Remark 15.19 (cf. [11, Lem. 1.1.7]) Let $(A_j)_{j=0}^{\kappa} \in \mathcal{S}_{p\times q;\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. Regarding Notations A.20 and A.18, from Lemma 15.18, one can easily see then that $\operatorname{rank} L_n = \sum_{\ell=0}^{n} \operatorname{rank} l_\ell$ and $\det L_n = \prod_{\ell=0}^{n} \det l_\ell$ as well as $\operatorname{rank} R_n = \sum_{\ell=0}^{n} \operatorname{rank} r_\ell$ and $\det R_n = \prod_{\ell=0}^{n} \det r_\ell$.

We derive now a useful relation between the sequences of left and right Schur complements of a $p \times q$ Schur sequence (see (2.8)) and the sequences of matrices introduced in Notation 14.1.

Lemma 15.20 *Let $(A_j)_{j=0}^{\kappa} \in \mathcal{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. In view of (2.5), (2.8), and Notation 14.1, for all $j \in \mathbb{Z}_{0,\kappa}$, then $l_j = \mathfrak{L}_j \mathfrak{L}_j^*$ and $r_j = \mathfrak{R}_j^* \mathfrak{R}_j$.*

Proof Taking into account Lemmas 15.18 and 15.16, the assertion can be obtained easily using Remark A.23. □

Remark 15.21 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. In view of Lemma 15.20 and Notation 14.1, for all $j \in \mathbb{Z}_{0,\kappa}$, then $l_j = \mathfrak{L}_{j-1}\mathfrak{l}_j\mathfrak{L}_{j-1}^*$ and $r_j = \mathfrak{R}_{j-1}^*\mathfrak{r}_j\mathfrak{R}_{j-1}$.

Remark 15.22 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. In view of Lemma 15.20, for all $j \in \mathbb{Z}_{0,\kappa}$, then $\mathcal{R}(\mathfrak{L}_j) = \mathcal{R}(l_j)$ and $\mathcal{N}(\mathfrak{R}_j) = \mathcal{N}(r_j)$.

The following result contains an answer to the extension problem for finite $p \times q$ Schur sequences in terms of SP-parameters. The solution set is again written as a closed matrix ball. However, the corresponding center and semi-radii are expressed in terms of SP-parameters. In the particular case of a non-degenerate $p \times q$ Schur sequence, this result appears already in [11, Thm. 3.8.1].

Theorem 15.23 *Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^{n} \in \mathscr{S}_{p\times q;n}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n}$. Taking into account Notations 14.2 and 14.1, then $\mathcal{A}_{n+1} = \mathfrak{K}(\mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n); \mathfrak{L}_n, \mathfrak{R}_n)$.*

Proof First we consider an arbitrary $A_{n+1} \in \mathbb{C}^{p\times q}$ such that $(A_j)_{j=0}^{n+1} \in \mathscr{S}_{p\times q;n+1}$. Denote by $(\mathfrak{p}_j)_{j=0}^{n+1}$ the SP-parameter sequence of $(A_j)_{j=0}^{n+1}$. According to Remark 4.9, then $\mathfrak{p}_j = \mathfrak{e}_j$ for all $j \in \mathbb{Z}_{0,n}$. Taking additionally into account Notations 14.1 and 4.11, we can infer from Corollary 14.4 then $A_{n+1} = \mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n) + \mathfrak{L}_n\mathfrak{p}_{n+1}\mathfrak{R}_n$. Since Remark 4.12 shows $\mathfrak{p}_{n+1} \in \mathbb{K}_{p\times q}$, in view of Notation 15.1, consequently $A_{n+1} \in \mathfrak{K}(\mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n); \mathfrak{L}_n, \mathfrak{R}_n)$ follows. Conversely, now consider an arbitrary $A_{n+1} \in \mathfrak{K}(\mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n); \mathfrak{L}_n, \mathfrak{R}_n)$. According to Notation 15.1, then there exists $K \in \mathbb{K}_{p\times q}$ such that $A_{n+1} = \mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n) + \mathfrak{L}_n K\mathfrak{R}_n$. Clearly, then $\mathfrak{e}_{n+1} := \mathbb{P}_{\mathcal{M}_n} K\mathbb{P}_{\mathcal{Q}_n^\perp}$ belongs to $\mathbb{K}_{p\times q}$ and fulfills $\mathcal{R}(\mathfrak{e}_{n+1}) \subseteq \mathcal{M}_n$ and $\mathcal{Q}_n \subseteq \mathcal{N}(\mathfrak{e}_{n+1})$. Since Proposition 6.10 shows $(\mathfrak{e}_j)_{j=0}^{n} \in \mathscr{E}_{p\times q;n}$, we can, by virtue of Notation 6.2, infer then $(\mathfrak{e}_j)_{j=0}^{n+1} \in \mathscr{E}_{p\times q;n+1}$. Thus, we can apply Theorem 6.20 to see that there exists a unique sequence $(B_j)_{j=0}^{n+1} \in \mathscr{S}_{p\times q;n+1}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n+1}$. Using Theorem 14.3, we can, for all $j \in \mathbb{Z}_{0,n}$, conclude $B_j = \Psi_j(\mathfrak{e}_0, \ldots, \mathfrak{e}_j) = A_j$. The application of Corollary 14.4 yields furthermore $B_{n+1} = \mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n) + \mathfrak{L}_n\mathfrak{e}_{n+1}\mathfrak{R}_n$. By virtue of Lemma 6.9, Proposition 15.9, and Remark A.6, we get $\mathbb{P}_{\mathcal{M}_n} = \mathfrak{M}_n\mathfrak{M}_n^\dagger$ and $\mathbb{P}_{\mathcal{Q}_n^\perp} = \mathfrak{R}_n\mathfrak{R}_n^\dagger$. Since Corollary 15.10 shows $\mathfrak{M}_n\mathfrak{M}_n^\dagger = \mathfrak{L}_n^\dagger\mathfrak{L}_n$, consequently $\mathfrak{e}_{n+1} = \mathfrak{L}_n^\dagger\mathfrak{L}_n K\mathfrak{R}_n\mathfrak{R}_n^\dagger$. Taking additionally into account (2.1), then $\mathfrak{L}_n\mathfrak{e}_{n+1}\mathfrak{R}_n = \mathfrak{L}_n K\mathfrak{R}_n$ follows. Therefore, $B_{n+1} = A_{n+1}$. Summarizing, we have $(A_j)_{j=0}^{n+1} = (B_j)_{j=0}^{n+1}$, implying $(A_j)_{j=0}^{n+1} \in \mathscr{S}_{p\times q;n+1}$. □

The considerations of Theorems 15.2 and 15.23 under the view of theory of matrix balls lead us to the following identity.

Corollary 15.24 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For all $n \in \mathbb{Z}_{0,\kappa}$, then $m_n = \mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n)$.*

Proof We consider an arbitrary $n \in \mathbb{Z}_{0,\kappa}$. From Remark 4.9 we know that $(A_j)_{j=0}^{n}$ belongs to $\mathscr{S}_{p\times q;n}$ and has SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{n}$. Thus, we can use Theorems 15.2 and 15.23 to see $\mathfrak{K}(m_n; \sqrt{l_n}, \sqrt{r_n}) = \mathfrak{K}(\mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n); \mathfrak{L}_n, \mathfrak{R}_n)$, which implies $m_n = \mu_n(\mathfrak{e}_0, \ldots, \mathfrak{e}_n)$ (see, e. g., [11, Cor. 1.5.1]). □

16 On an Explicit Connection Between Choice Sequences and SP-parameter Sequences

In this section, we consider a finite or infinite $p \times q$ Schur sequence. We are interested in obtaining explicit formulas describing the connections between choice sequence (see Definition 15.4) and SP-parameter sequences. Taking into account Theorem 15.2, as a first step in this direction, we introduce the following notation:

Notation 16.1 Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. For all $j \in \mathbb{Z}_{0,\kappa}$, let $U_j := \sqrt{l_j}^{\dagger}\mathfrak{L}_j$ and $V_j := \mathfrak{R}_j\sqrt{r_j}^{\dagger}$.

Recall that $W \in \mathbb{C}^{p\times q}$ is a partial isometry if and only if W^*W is idempotent or equivalently WW^* is idempotent. In this case, $\mathcal{R}(W^*W)$ and $\mathcal{R}(WW^*)$ are called initial and final subspace of W, respectively (see, e. g., [23]). We see now that two sequences of partial isometries are associated with a $p \times q$ Schur sequence.

Lemma 16.2 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $j \in \mathbb{Z}_{0,\kappa}$. Then $U_jU_j^* = \mathbb{P}_{\mathcal{R}(l_j)}$ and $U_j^*U_j = \mathbb{P}_{\mathcal{M}_j}$ as well as $V_j^*V_j = \mathbb{P}_{\mathcal{R}(r_j)}$ and $V_jV_j^* = \mathbb{P}_{\mathcal{Q}_j^{\perp}}$. In particular, U_j (resp., V_j) is a partial isometry with initial subspace $\mathcal{M}_j$ (resp., $\mathcal{R}(r_j)$) and final subspace $\mathcal{R}(l_j)$ (resp., $\mathcal{Q}_j^{\perp}$).*

Proof Using Notation 16.1, Remark A.8, Lemma 15.20, and Remarks A.10(e) and A.6, we get

$$U_jU_j^* = \sqrt{l_j}^{\dagger}\mathfrak{L}_j\mathfrak{L}_j^*\sqrt{l_j}^{\dagger} = \sqrt{l_j}^{\dagger}l_j\sqrt{l_j}^{\dagger} = l_jl_j^{\dagger} = \mathbb{P}_{\mathcal{R}(l_j)}$$

and, analogously, $V_j^*V_j = \mathbb{P}_{\mathcal{R}(r_j)}$. From Remark A.10(b) we can infer $(\sqrt{l_j}^{\dagger})^*\sqrt{l_j}^{\dagger} = l_j^{\dagger}$ and $\sqrt{r_j}^{\dagger}(\sqrt{r_j}^{\dagger})^* = r_j^{\dagger}$. Furthermore, we have $\mathfrak{L}_j^{\dagger} = \mathfrak{L}_j^*(\mathfrak{L}_j\mathfrak{L}_j^*)^{\dagger}$ and $\mathfrak{R}_j^{\dagger} = (\mathfrak{R}_j^*\mathfrak{R}_j)^{\dagger}\mathfrak{R}_j^*$ (see, e. g., [11, Prop. 1.1.2]). Taking additionally into account Notation 16.1, Lemma 15.20, Remarks A.6 and A.2, and Proposition 15.9, we get

$$U_j^*U_j = \mathfrak{L}_j^*l_j^{\dagger}\mathfrak{L}_j = \mathfrak{L}_j^*(\mathfrak{L}_j\mathfrak{L}_j^*)^{\dagger}\mathfrak{L}_j = \mathfrak{L}_j^{\dagger}\mathfrak{L}_j = \mathbb{P}_{\mathcal{R}(\mathfrak{L}_j^*)} = \mathbb{P}_{\mathcal{N}(\mathfrak{L}_j)^{\perp}} = \mathbb{P}_{\mathcal{M}_j}$$

and, analogously, $V_jV_j^* = \mathbb{P}_{\mathcal{Q}_j^{\perp}}$. □

Now we are able to present an explicit connection between the choice sequence and the SP-parameter sequence of an arbitrarily given $p \times q$ Schur sequence.

Theorem 16.3 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^{\kappa}$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$. Then $\mathfrak{k}_0 = \mathfrak{e}_0$. Moreover, if $\kappa \geq 1$, then $\mathfrak{k}_n = U_{n-1}\mathfrak{e}_nV_{n-1}$ and $\mathfrak{e}_n = U_{n-1}^*\mathfrak{k}_nV_{n-1}^*$ for all $n \in \mathbb{Z}_{1,\kappa}$.*

Proof. According to Definitions 15.4, 4.1 and 4.7, we have $\mathfrak{k}_0 = A_0 = A_0^{[0]} = \mathfrak{e}_0$. Now assume $\kappa \geq 1$ and consider an arbitrary $n \in \mathbb{Z}_{1,\kappa}$. Corollary 14.4 yields $A_n = \mu_{n-1}(\mathfrak{e}_0,\ldots,\mathfrak{e}_{n-1}) + \mathfrak{L}_{n-1}\mathfrak{e}_n\mathfrak{R}_{n-1}$. Corollary 15.24 provides $m_{n-1} =$

$\mu_{n-1}(\mathfrak{e}_0, \ldots, \mathfrak{e}_{n-1})$. Taking additionally into account Notation 16.1 and Definition 15.4, we get

$$\begin{aligned} U_{n-1}\mathfrak{e}_n V_{n-1} &= \sqrt{l_{n-1}}^\dagger \mathfrak{L}_{n-1}\mathfrak{e}_n \mathfrak{R}_{n-1}\sqrt{r_{n-1}}^\dagger \\ &= \sqrt{l_{n-1}}^\dagger [A_n - \mu_{n-1}(\mathfrak{e}_0, \ldots, \mathfrak{e}_{n-1})]\sqrt{r_{n-1}}^\dagger \\ &= \sqrt{l_{n-1}}^\dagger (A_n - m_{n-1})\sqrt{r_{n-1}}^\dagger = \mathfrak{k}_n. \end{aligned}$$

Proposition 6.10 shows $(\mathfrak{e}_j)_{j=0}^\kappa \in \mathscr{E}_{p\times q;\kappa}$. According to Notation 6.2, we have then $\mathcal{R}(\mathfrak{e}_n) \subseteq \mathcal{M}_{n-1}$ and $\mathcal{Q}_{n-1} \subseteq \mathcal{N}(\mathfrak{e}_n)$, so that $\mathbb{P}_{\mathcal{M}_{n-1}}\mathfrak{e}_n\mathbb{P}_{\mathcal{Q}_{n-1}^\perp} = \mathfrak{e}_n$. Lemma 16.2 provides $U_{n-1}^*U_{n-1} = \mathbb{P}_{\mathcal{M}_{n-1}}$ and $V_{n-1}V_{n-1}^* = \mathbb{P}_{\mathcal{Q}_{n-1}^\perp}$. Summarizing, we get

$$U_{n-1}^*\mathfrak{k}_n V_{n-1}^* = U_{n-1}^*U_{n-1}\mathfrak{e}_n V_{n-1}V_{n-1}^* = \mathbb{P}_{\mathcal{M}_{n-1}}\mathfrak{e}_n\mathbb{P}_{\mathcal{Q}_{n-1}^\perp} = \mathfrak{e}_n. \quad \square$$

Corollary 16.4 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^\kappa$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^\kappa$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then $\mathfrak{k}_n\mathfrak{k}_n^* = U_{n-1}\mathfrak{e}_n\mathfrak{e}_n^*U_{n-1}^*$ and $\mathfrak{k}_n^*\mathfrak{k}_n = V_{n-1}^*\mathfrak{e}_n^*\mathfrak{e}_n V_{n-1}$ as well as $\mathfrak{e}_n\mathfrak{e}_n^* = U_{n-1}^*\mathfrak{k}_n\mathfrak{k}_n^*U_{n-1}$ and $\mathfrak{e}_n^*\mathfrak{e}_n = V_{n-1}\mathfrak{k}_n^*\mathfrak{k}_n V_{n-1}^*$.*

Proof Lemma 16.2 provides

$$U_{n-1}U_{n-1}^* = \mathbb{P}_{\mathcal{R}(l_{n-1})}, \qquad U_{n-1}^*U_{n-1} = \mathbb{P}_{\mathcal{M}_{n-1}}, \tag{16.1}$$
$$V_{n-1}^*V_{n-1} = \mathbb{P}_{\mathcal{R}(r_{n-1})}, \quad \text{and} \quad V_{n-1}V_{n-1}^* = \mathbb{P}_{\mathcal{Q}_{n-1}^\perp}. \tag{16.2}$$

Proposition 6.10 shows $(\mathfrak{e}_j)_{j=0}^\kappa \in \mathscr{E}_{p\times q;\kappa}$. According to Notation 6.2, we have then $\mathcal{R}(\mathfrak{e}_n) \subseteq \mathcal{M}_{n-1}$ and $\mathcal{Q}_{n-1} \subseteq \mathcal{N}(\mathfrak{e}_n)$, so that $\mathbb{P}_{\mathcal{M}_{n-1}}\mathfrak{e}_n = \mathfrak{e}_n$ and $\mathfrak{e}_n\mathbb{P}_{\mathcal{Q}_{n-1}^\perp} = \mathfrak{e}_n$. Using additionally Theorem 16.3 and the second identity in (16.2) and (16.1), resp., we get $\mathfrak{k}_n\mathfrak{k}_n^* = U_{n-1}\mathfrak{e}_n V_{n-1}V_{n-1}^*\mathfrak{e}_n^*U_{n-1}^* = U_{n-1}\mathfrak{e}_n\mathfrak{e}_n^*U_{n-1}^*$ and $\mathfrak{k}_n^*\mathfrak{k}_n = V_{n-1}^*\mathfrak{e}_n^*U_{n-1}^*U_{n-1}\mathfrak{e}_n V_{n-1} = V_{n-1}^*\mathfrak{e}_n^*\mathfrak{e}_n V_{n-1}$. By virtue of Definition 15.4 we see $\mathcal{R}(\mathfrak{k}_n) \subseteq \mathcal{R}(\sqrt{l_{n-1}}^\dagger)$ and $\mathcal{R}(\mathfrak{k}_n^*) \subseteq \mathcal{R}((\sqrt{r_{n-1}}^\dagger)^*)$. Applying Remarks A.9, A.10(a), and A.8, we can conclude then $\mathcal{R}(\mathfrak{k}_n) \subseteq \mathcal{R}(\sqrt{l_{n-1}}) = \mathcal{R}(l_{n-1})$ and $\mathcal{R}(\mathfrak{k}_n^*) \subseteq \mathcal{R}(\sqrt{r_{n-1}}^\dagger) = \mathcal{R}(\sqrt{r_{n-1}}) = \mathcal{R}(r_{n-1})$, so that $\mathbb{P}_{\mathcal{R}(l_{n-1})}\mathfrak{k}_n = \mathfrak{k}_n$ and $\mathbb{P}_{\mathcal{R}(r_{n-1})}\mathfrak{k}_n^* = \mathfrak{k}_n^*$. Using additionally Theorem 16.3 and the first identity in (16.2) and (16.1), resp., we get $\mathfrak{e}_n\mathfrak{e}_n^* = U_{n-1}^*\mathfrak{k}_n V_{n-1}^*V_{n-1}\mathfrak{k}_n^*U_{n-1} = U_{n-1}^*\mathfrak{k}_n\mathfrak{k}_n^*U_{n-1}$ and $\mathfrak{e}_n^*\mathfrak{e}_n = V_{n-1}\mathfrak{k}_n^*U_{n-1}U_{n-1}^*\mathfrak{k}_n V_{n-1}^* = V_{n-1}\mathfrak{k}_n^*\mathfrak{k}_n V_{n-1}^*$. □

Corollary 16.5 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^\kappa$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^\kappa$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then $I_p - \mathfrak{k}_n\mathfrak{k}_n^* = \mathbb{P}_{\mathcal{N}(l_{n-1})} + U_{n-1}\mathfrak{l}_n U_{n-1}^*$ and $I_q - \mathfrak{k}_n^*\mathfrak{k}_n = \mathbb{P}_{\mathcal{N}(r_{n-1})} + V_{n-1}^*\mathfrak{r}_n V_{n-1}$ as well as $\mathfrak{l}_n = \mathbb{P}_{\mathcal{M}_{n-1}^\perp} + U_{n-1}^*(I_p - \mathfrak{k}_n\mathfrak{k}_n^*)U_{n-1}$ and $\mathfrak{r}_n = \mathbb{P}_{\mathcal{Q}_{n-1}} + V_{n-1}(I_q - \mathfrak{k}_n^*\mathfrak{k}_n)V_{n-1}^*$.*

***Proof*.** Lemma 16.2 provides (16.1) and (16.2). Regarding that the matrices l_{n-1} and r_{n-1} are Hermitian, using the first identity in (16.1) and (16.2), resp., and Remark A.2, we can infer $U_{n-1}U_{n-1}^* = \mathbb{P}_{\mathcal{R}(l_{n-1}^*)} = \mathbb{P}_{\mathcal{N}(l_{n-1})^\perp}$ and $V_{n-1}^*V_{n-1} =$

$\mathbb{P}_{\mathcal{R}(r_{n-1}^*)} = \mathbb{P}_{\mathcal{N}(r_{n-1})^\perp}$. Consequently, Remark A.4 yields $\mathbb{P}_{\mathcal{N}(l_{n-1})} + U_{n-1}U_{n-1}^* = I_p$ and $\mathbb{P}_{\mathcal{N}(r_{n-1})} + V_{n-1}^* V_{n-1} = I_q$. Taking additionally into account Corollary 16.4 and Notation 4.11, we obtain

$$I_p - \mathfrak{k}_n\mathfrak{k}_n^* = \mathbb{P}_{\mathcal{N}(l_{n-1})} + U_{n-1}U_{n-1}^* - U_{n-1}\mathfrak{e}_n\mathfrak{e}_n^*U_{n-1}^* = \mathbb{P}_{\mathcal{N}(l_{n-1})} + U_{n-1}\mathfrak{l}_n U_{n-1}^*$$

and

$$I_q - \mathfrak{k}_n^*\mathfrak{k}_n = \mathbb{P}_{\mathcal{N}(r_{n-1})} + V_{n-1}^* V_{n-1} - V_{n-1}^*\mathfrak{e}_n^*\mathfrak{e}_n V_{n-1} = \mathbb{P}_{\mathcal{N}(r_{n-1})} + V_{n-1}^*\mathfrak{r}_n V_{n-1}.$$

In view of the second identities in (16.1) and (16.2), resp., Remark A.4 yields $\mathbb{P}_{\mathcal{M}_{n-1}^\perp} + U_{n-1}^* U_{n-1} = I_p$ and $\mathbb{P}_{\mathcal{Q}_{n-1}} + V_{n-1}V_{n-1}^* = I_q$. Taking additionally into account Corollary 16.4 and Notation 4.11, we obtain

$$\mathbb{P}_{\mathcal{M}_{n-1}^\perp} + U_{n-1}^*(I_p - \mathfrak{k}_n\mathfrak{k}_n^*)U_{n-1} = I_p - U_{n-1}^*\mathfrak{k}_n\mathfrak{k}_n^*U_{n-1} = I_p - \mathfrak{e}_n\mathfrak{e}_n^* = \mathfrak{l}_n$$

and

$$\mathbb{P}_{\mathcal{Q}_{n-1}} + V_{n-1}(I_q - \mathfrak{k}_n^*\mathfrak{k}_n)V_{n-1}^* = I_q - V_{n-1}\mathfrak{k}_n^*\mathfrak{k}_n V_{n-1}^* = I_q - \mathfrak{e}_n^*\mathfrak{e}_n = \mathfrak{r}_n. \quad \square$$

17 Central Matricial Schur Functions

As already mentioned above, in [18] a reference function is used to obtain a parametrization of the solution set $\mathscr{S}_{p\times q}[\mathbb{D}; (A_j)_{j=0}^n]$ of a matricial Schur problem, namely the so-called central $p \times q$ Schur function corresponding to a given $p \times q$ Schur sequence $(A_j)_{j=0}^n$. We recall this notion which was introduced in [14, Part II, Def. 5]: If $n \in \mathbb{N}_0$ and if $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$, then the sequence $(A_j)_{j=0}^\infty$ given by $A_k := m_{k-1}$ for all $k \in \mathbb{Z}_{n+1,\infty}$ is called the *central $p \times q$ Schur sequence corresponding* $(A_j)_{j=0}^n$. A $p \times q$ Schur sequence $(A_j)_{j=0}^\kappa$ is said to be *$\mathscr{S}$-central* if there is an $n \in \mathbb{Z}_{1,\kappa}$ such that $A_j = m_{j-1}$ for all $j \in \mathbb{Z}_{n,\kappa}$. In this case, the smallest n with this property is called the corresponding *order* and $(A_j)_{j=0}^\kappa$ is called *$\mathscr{S}$-central of order n*. If $n \in \mathbb{N}_0$ and if $(A_j)_{j=0}^n \in \mathscr{S}_{p\times q;n}$, then $F\colon \mathbb{D} \to \mathbb{C}^{p\times q}$ given by $F(w) := \sum_{j=0}^\infty w^j A_j$ for all $w \in \mathbb{D}$, where $(A_j)_{j=0}^\infty$ is the central $p \times q$ Schur sequence corresponding to $(A_j)_{j=0}^n$, is said to be the *central $p \times q$ Schur function corresponding to* $(A_j)_{j=0}^n$. A function $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor series expansion

$$F(w) = \sum_{j=0}^\infty w^j A_j \qquad \text{for all } w \in \mathbb{D}, \tag{17.1}$$

is called a *central $p \times q$ Schur function* (resp., a *central $p \times q$ Schur function of order n*) if $(A_j)_{j=0}^\infty$ is a central $p \times q$ Schur sequence (resp., a central $p \times q$ Schur sequence of order n). In [15], explicit representations of central $p \times q$ Schur functions as rational matrix-valued functions constructed by the given $p \times q$ Schur sequence

$(A_j)_{j=0}^{n}$ are proved. Central $p \times q$ Schur functions are distinguished rational matrix-valued functions which have certain extremal properties (see, e. g., [1] or [14, Part II]). Moreover, recurrence formulas for the Taylor coefficients of central $p \times q$ Schur functions can be found in [11, Thm. 3.5.4].

In this section, we study $\mathscr{S}$-central $p \times q$ Schur sequences under the view of the SP-algorithm. For this reason, we recall the following:

Remark 17.1 (cf. [11, Rem. 3.5.3]) Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^{\kappa}$. For all $j \in \mathbb{Z}_{-1,\kappa-1}$, then

$$O \preccurlyeq l_{j+1} = \sqrt{l_j}(I_p - \mathfrak{k}_{j+1}\mathfrak{k}_{j+1}^*)\sqrt{l_j} = l_j - (A_{j+1} - m_j)r_j^\dagger(A_{j+1} - m_j)^* \preccurlyeq l_j$$

and

$$O \preccurlyeq r_{j+1} = \sqrt{r_j}(I_q - \mathfrak{k}_{j+1}^*\mathfrak{k}_{j+1})\sqrt{r_j} = r_j - (A_{j+1} - m_j)^* l_j^\dagger(A_{j+1} - m_j) \preccurlyeq r_j.$$

We give now several characterizations of the fact that a particular element of a given $p \times q$ Schur sequence coincides with the center of the corresponding matrix ball. Some of them are formulated in terms of the choice sequence and thus already known. The other ones formulated in terms of SP-parameters seem to be new.

Lemma 17.2 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^{\kappa}$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $j \in \mathbb{Z}_{0,\kappa-1}$. Then the following statements are equivalent:*

(i) $A_{j+1} = m_j$.
(ii) $l_{j+1} = l_j$.
(iii) $r_{j+1} = r_j$.
(iv) $\mathfrak{k}_{j+1} = O_{p\times q}$.
(v) $\mathfrak{e}_{j+1} = O_{p\times q}$.
(vi) $\mathfrak{l}_{j+1} = I_p$.
(vii) $\mathfrak{r}_{j+1} = I_q$.
(viii) $\mathfrak{L}_{j+1} = \mathfrak{L}_j$.
(ix) $\mathfrak{R}_{j+1} = \mathfrak{R}_j$.

Proof According to Definition 15.4, statement (i) implies (iv). Remark 17.1 shows that (iv) is sufficient for (ii) and (iii). If (ii) is fulfilled, then Remark 17.1 yields $\sqrt{l_j}\mathfrak{k}_{j+1} = O$ and, in view of Definition 15.4, consequently $\mathfrak{k}_{j+1} = \sqrt{l_j}^\dagger\sqrt{l_j}\mathfrak{k}_{j+1} = O$, i. e., (iv). Analogously, if (iii) holds true, then Remark 17.1 and Definition 15.4 provide $\mathfrak{k}_{j+1}\sqrt{r_j} = O$ and, thus, $\mathfrak{k}_{j+1} = \mathfrak{k}_{j+1}\sqrt{r_j}\sqrt{r_j}^\dagger = O$, i. e., (iv). Obviously, applying Proposition 15.5, we get that (iv) implies (i). Because of Theorem 16.3, the statements (iv) and (v) are equivalent. In view of Notation 4.11, the statements (v) and (vi) as well as the statements (v) and (vii) are equivalent. From Notation 14.1 we see that (viii) is necessary for (vi) as well as that (ix) is necessary for (vii). Finally, by virtue of Lemma 15.20, we see that (viii) implies (ii) and that (ix) is sufficient for (iii). □

Proposition 17.3 *Suppose $\kappa \geq 1$. Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^{\kappa}$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $n \in \mathbb{Z}_{1,\kappa}$. Then the following statements are equivalent:*

(i) *$(A_j)_{j=0}^{\kappa}$ is $\mathscr{S}$-central of order n.*
(ii) *$\mathfrak{k}_k = O_{p\times q}$ for all $k \in \mathbb{Z}_{n,\kappa}$.*
(iii) *$\mathfrak{e}_k = O_{p\times q}$ for all $k \in \mathbb{Z}_{n,\kappa}$.*
(iv) *$A_j^{[n]} = O_{p\times q}$ for all $j \in \mathbb{Z}_{0,\kappa-n}$.*
(v) *$A_j^{[k]} = O_{p\times q}$ for every choice of $k \in \mathbb{Z}_{n,\kappa}$ and $j \in \mathbb{Z}_{0,\kappa-k}$.*

Proof "(i) ⇔ (ii) ⇔ (iii)": Apply Lemma 17.2.
"(iii) ⇔ (iv)": Regarding Remark 4.8, this follows from Corollary 14.10.
"(iv) ⇒ (v)": Regarding Definition 4.1, use Example 4.5.
"(v) ⇒ (iv)": This implication holds true obviously. □

Proposition 17.4 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with SP-parameter sequence $(\gamma_j)_{j=0}^{\infty}$ and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) F is $\mathscr{S}$-central of order n.
(ii) $\gamma_k = O_{p\times q}$ for all $k \in \mathbb{Z}_{n,\infty}$.
(iii) $F^{[\![n]\!]}(z) = O_{p\times q}$ for all $z \in \mathbb{D}$.
(iv) $F^{[\![k]\!]}(z) = O_{p\times q}$ for every choice of $k \in \mathbb{Z}_{n,\infty}$ and $z \in \mathbb{D}$.

Proof Denote by $(A_j)_{j=0}^{\infty}$ the Taylor coefficient sequence of F. Proposition 9.7 shows then $(A_j)_{j=0}^{\infty} \in \mathscr{S}_{p\times q;\infty}$ and that the SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\infty}$ of $(A_j)_{j=0}^{\infty}$ coincides with $(\gamma_j)_{j=0}^{\infty}$. By virtue of Lemma 9.4, we see that, for all $k \in \mathbb{N}_0$, furthermore $F^{[\![k]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$ and has Taylor coefficient sequence $(A_j^{[k]})_{j=0}^{\infty}$. Now the asserted equivalences follow from Proposition 17.3. □

18 Completely Degenerate Matricial Schur Functions

In view of Lemma 17.2 and Remark 17.1, we discussed the case that the semi-radii l_n and r_n are maximal if $n \in \mathbb{N}$ and $(A_j)_{j=0}^{n-1} \in \mathscr{S}_{p\times q;n-1}$ are given. In this section, we study the other extremal situation, namely that $l_n = O_{p\times p}$ or $r_n = O_{q\times q}$ holds true.

A sequence $(A_j)_{j=0}^{\kappa}$ belonging to $\mathscr{S}_{p\times q;\kappa}$ is said to be *completely left $\mathscr{S}$-degenerate* (resp., *completely right $\mathscr{S}$-degenerate*) if there exists an $n \in \mathbb{Z}_{0,\kappa}$ such that $l_n = O_{p\times p}$ (resp., $r_n = O_{q\times q}$) holds true. In this case, the smallest n with this property is called the corresponding *order* and $(A_j)_{j=0}^{\kappa}$ is said to be *completely left $\mathscr{S}$-degenerate of order n* (resp., *completely right $\mathscr{S}$-degenerate of order n*). A function $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor series expansion (17.1) is called *completely left $\mathscr{S}$-degenerate* (resp., *completely left $\mathscr{S}$-degenerate*) if $(A_j)_{j=0}^{\infty}$ is completely left $\mathscr{S}$-degenerate (resp., completely right $\mathscr{S}$-degenerate). A function $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with Taylor series expansion (17.1) is said to be *completely left $\mathscr{S}$-degenerate of order n* (resp., *completely right $\mathscr{S}$-degenerate of order n*) if $(A_j)_{j=0}^{\infty}$ is completely left $\mathscr{S}$-degenerate of order n (resp., completely right $\mathscr{S}$-degenerate of order n).

Remark 18.1 Let $n \in \mathbb{Z}_{0,\kappa}$ and let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ be completely left $\mathscr{S}$-degenerate of order n or completely right $\mathscr{S}$-degenerate of order n. From Remark 17.1 and Proposition 15.5 one can easily see then that there exists an integer $k \in \mathbb{Z}_{0,n}$ such that $(A_j)_{j=0}^{\kappa}$ is $\mathscr{S}$-central of order $k+1$.

Proposition 18.2 *Let $(A_j)_{j=0}^{\kappa} \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^{\kappa}$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^{\kappa}$ and let $n \in \mathbb{Z}_{0,\kappa}$. Then the following statements are equivalent:*

(i) $(A_j)_{j=0}^{\kappa}$ *is completely left $\mathscr{S}$-degenerate of order n.*
(ii) $\mathfrak{L}_n = O_{p\times p}$.
(iii) $\mathfrak{M}_n = O_{p\times p}$.
(iv) $\mathcal{M}_n = \{O_{p\times 1}\}$.
(v) $\mathcal{M}_{n-1} \cap \mathcal{R}(\mathfrak{l}_n) = \{O_{p\times 1}\}$.
(vi) $\mathfrak{k}_n$ *is a partial isometry with final subspace $\mathcal{R}(l_{n-1})$.*
(vii) $\mathfrak{e}_n$ *is a partial isometry with final subspace $\mathcal{M}_{n-1}$.*

Proof "(i) $\Leftrightarrow$ (ii)": This is an immediate consequence of Remark 15.22.

"(ii) $\Leftrightarrow$ (iii)": Using (2.1), this can be seen from Corollary 15.10.

"(iii) $\Leftrightarrow$ (iv)": This is an immediate consequence of Lemma 6.9.

"(iv) $\Leftrightarrow$ (v)": This can be seen from Notation 6.1.

"(vi) $\Leftrightarrow$ (vii)": Since Theorem 16.3 shows $\mathfrak{k}_0 = \mathfrak{e}_0$ and Notation 6.1 and (2.5) yield $\mathcal{M}_{-1} = \mathcal{R}(I_p) = \mathcal{R}(l_{-1})$, the case $n = 0$ is trivial. Now suppose $\kappa \geq 1$ and let $n \in \mathbb{Z}_{1,\kappa}$. Lemma 16.2 provides (16.1). Corollary 16.4 yields

$$\mathfrak{k}_n\mathfrak{k}_n^* = U_{n-1}\mathfrak{e}_n\mathfrak{e}_n^*U_{n-1}^* \qquad \text{and} \qquad \mathfrak{e}_n\mathfrak{e}_n^* = U_{n-1}^*\mathfrak{k}_n\mathfrak{k}_n^*U_{n-1}. \tag{18.1}$$

First assume (vi). In view of Remark A.3, then $\mathfrak{k}_n\mathfrak{k}_n^* = \mathbb{P}_{\mathcal{R}(l_{n-1})}$. Using additionally (18.1) and (16.1), we consequently get

$$\mathfrak{e}_n\mathfrak{e}_n^* = U_{n-1}^*\mathbb{P}_{\mathcal{R}(l_{n-1})}U_{n-1} = U_{n-1}^*U_{n-1}U_{n-1}^*U_{n-1} = \mathbb{P}_{\mathcal{M}_{n-1}}^2 = \mathbb{P}_{\mathcal{M}_{n-1}},$$

which, because of Remark A.3, implies (vii).

Now assume (vii). Then Remark A.3 yields again $\mathfrak{e}_n\mathfrak{e}_n^* = \mathbb{P}_{\mathcal{M}_{n-1}}$. Using additionally (18.1) and (16.1), we consequently get

$$\mathfrak{k}_n\mathfrak{k}_n^* = U_{n-1}\mathbb{P}_{\mathcal{M}_{n-1}}U_{n-1}^* = U_{n-1}U_{n-1}^*U_{n-1}U_{n-1}^* = \mathbb{P}_{\mathcal{R}(l_{n-1})}^2 = \mathbb{P}_{\mathcal{R}(l_{n-1})},$$

which implies (vi).

"(vi) $\Leftrightarrow$ (i)": We first consider the case $n = 0$. By virtue of (2.5) and Definition 15.4, we see $l_0 = I_p - \mathfrak{k}_0\mathfrak{k}_0^*$. Consequently, $(A_j)_{j=0}^{\kappa}$ is completely left $\mathscr{S}$-degenerate of order 0 if and only if $\mathfrak{k}_0\mathfrak{k}_0^* = I_p$, which is equivalent to $\mathfrak{k}_0$ being a partial isometry with final subspace $\mathcal{R}(l_{-1})$. Now suppose $\kappa \geq 1$ and let $n \in \mathbb{Z}_{1,\kappa}$. From Remark 17.1 we can infer

$$l_n = l_{n-1} - \sqrt{l_{n-1}}\mathfrak{k}_n\mathfrak{k}_n^*\sqrt{l_{n-1}}. \tag{18.2}$$

First assume (vi). Then Remark A.3 yields $\mathfrak{k}_n\mathfrak{k}_n^* = \mathbb{P}_{\mathcal{R}(l_{n-1})}$. Because of Remark A.10(a), thus $\mathfrak{k}_n\mathfrak{k}_n^* = \mathbb{P}_{\mathcal{R}(\sqrt{l_{n-1}})}$, so that $\mathfrak{k}_n\mathfrak{k}_n^*\sqrt{l_{n-1}} = \sqrt{l_{n-1}}$. From (18.2), consequently $l_n = O_{p\times p}$ follows. Thus, (i) holds true. Conversely, now assume (i), i. e., $l_n = O_{p\times p}$. Regarding Definition 15.4 and (2.1), we have $\sqrt{l_{n-1}}^\dagger\sqrt{l_{n-1}}\mathfrak{k}_n = \mathfrak{k}_n$. Using Remark A.8, then $\mathfrak{k}_n^*\sqrt{l_{n-1}}\sqrt{l_{n-1}}^\dagger = \mathfrak{k}_n^*$ follows. Because of Remarks A.10(e) and A.6, we furthermore get $\sqrt{l_{n-1}}^\dagger l_{n-1}\sqrt{l_{n-1}}^\dagger = l_{n-1}l_{n-1}^\dagger = \mathbb{P}_{\mathcal{R}(l_{n-1})}$. From (18.2), we hence obtain $\sqrt{l_{n-1}}^\dagger l_n\sqrt{l_{n-1}}^\dagger = \mathbb{P}_{\mathcal{R}(l_{n-1})} - \mathfrak{k}_n\mathfrak{k}_n^*$. In view of $l_n = O_{p\times p}$, then $\mathfrak{k}_n\mathfrak{k}_n^* = \mathbb{P}_{\mathcal{R}(l_{n-1})}$ follows, which, by virtue of Remark A.3, implies (vi). □

Proposition 18.3 *Let $(A_j)_{j=0}^\kappa \in \mathscr{S}_{p\times q;\kappa}$ with choice sequence $(\mathfrak{k}_j)_{j=0}^\kappa$ and SP-parameter sequence $(\mathfrak{e}_j)_{j=0}^\kappa$ and let $n \in \mathbb{Z}_{0,\kappa}$. Then the following statements are equivalent:*

(i) $(A_j)_{j=0}^\kappa$ *is completely right $\mathscr{S}$-degenerate of order n.*
(ii) $\mathfrak{R}_n = O_{q\times q}$.
(iii) $\mathfrak{Q}_n = O_{q\times q}$.
(iv) $\mathcal{Q}_n^\perp = \{O_{q\times 1}\}$.
(v) $\mathcal{Q}_{n-1}^\perp \cap \mathcal{R}(\mathfrak{r}_n) = \{O_{q\times 1}\}$.
(vi) $\mathfrak{k}_n$ *is a partial isometry with initial subspace $\mathcal{R}(r_{n-1})$.*
(vii) $\mathfrak{e}_n$ *is a partial isometry with initial subspace $\mathcal{Q}_{n-1}^\perp$.*

Proof This can be proved analogous to Proposition 18.2. □

Let us observe that, using Remarks 17.1 and 15.22, Corollary 15.10, Lemma 6.9 as well as Propositions 6.10, 17.2 and 6.6, one can easily obtain further conditions for the complete left and right $\mathscr{S}$-degeneracy of a Schur sequence, respectively, which are implied by the statements formulated in Propositions 18.2 and 18.3. We omit the details.

Proposition 18.4 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with SP-parameter sequence $(\gamma_j)_{j=0}^\infty$ and let $n \in \mathbb{N}_0$. Let $\mathcal{M}_{n-1}$ be given by Notation 6.1, where $\mathfrak{e}_j := \gamma_j$ for all $j \in \mathbb{N}_0$. Then the following statements are equivalent:*

(i) *F is completely left $\mathscr{S}$-degenerate of order n.*
(ii) *γ_n is a partial isometry with final subspace $\mathcal{M}_{n-1}$.*
(iii) *There exists a partial isometry W with final subspace $\mathcal{M}_{n-1}$ such that $F^{[\![n]\!]}(z) = W$ for all $z \in \mathbb{D}$.*

Proof Denote by $(A_j)_{j=0}^\infty$ the Taylor coefficient sequence of F. Proposition 9.7 then shows that $(A_j)_{j=0}^\infty$ belongs to $\mathscr{S}_{p\times q;\infty}$ and has SP-parameter sequence $(\gamma_j)_{j=0}^\infty$, i. e., $(\mathfrak{e}_j)_{j=0}^\infty$ is the SP-parameter sequence of $(A_j)_{j=0}^\infty$. Now the equivalence (i) ⇔ (ii) follows from the equivalence (i) ⇔ (vii) in Proposition 18.2. Furthermore, according to Definition 9.5, we have $\gamma_n = F^{[\![n]\!]}(0)$, so that (iii) implies (ii).

Now suppose (i). Lemma 9.4 shows that $H := F^{[\![n]\!]}$ belongs to $\mathscr{S}_{p\times q}(\mathbb{D})$. Thus, we can apply Lemma 12.4 to see that $E := H(0)$ belongs to $\mathbb{K}_{p\times q}$ and that $G := H^{[\![1]\!]}$ fulfills $G^{[\![-1;E]\!]} = H$. According to Definition 9.1, we have $G = F^{[\![n+1]\!]}$. In view of (i), the sequence $(A_j)_{j=0}^\infty$ is completely left $\mathscr{S}$-degenerate of order n. Taking additionally into account $(A_j)_{j=0}^{\infty} \in \mathscr{S}_{p\times q;\infty}$, then Remark 18.1 shows that there exists an integer $k \in \mathbb{Z}_{0,n}$ such that $(A_j)_{j=0}^{\infty}$ is $\mathscr{S}$-central of order $k+1$. Thus, F is $\mathscr{S}$-central of order $k+1$. Consequently, Proposition 17.4 provides $F^{[\![n+1]\!]}(z) = O_{p\times q}$ for all $z \in \mathbb{D}$. Hence, $G(z) = O_{p\times q}$ for all $z \in \mathbb{D}$. By virtue of Definition 10.1, then $G^{[\![-1;E]\!]}(z) = E$ for all $z \in \mathbb{D}$ follows. Summarizing, for all $z \in \mathbb{D}$, we get

$$F^{[\![n]\!]}(z) = H(z) = G^{[\![-1;E]\!]}(z) = E = H(0) = F^{[\![n]\!]}(0) = \gamma_n.$$

Using additionally that (i) also implies (ii) , then (iii) follows. □

Proposition 18.5 *Let $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ with SP-parameter sequence $(\gamma_j)_{j=0}^{\infty}$ and let $n \in \mathbb{N}_0$. Let $\mathcal{Q}_{n-1}$ be given by Notation 6.1, where $\mathfrak{e}_j := \gamma_j$ for all $j \in \mathbb{N}_0$. Then the following statements are equivalent:*

(i) F is completely right $\mathscr{S}$-degenerate of order n.
(ii) γ_n is a partial isometry with initial subspace $\mathcal{Q}_{n-1}^{\perp}$.
(iii) There exists a partial isometry W with initial subspace $\mathcal{Q}_{n-1}^{\perp}$ such that $F^{[\![n]\!]}(z) = W$ for all $z \in \mathbb{D}$.

Proof This can be proved analogous to Proposition 18.4. □

Appendix A. Some Facts from Matrix Theory

Remark A.1 Let $\mathcal{U}$ and $\mathcal{V}$ be linear subspaces of the unitary space $\mathbb{C}^p$. Then $(\mathcal{U}+\mathcal{V})^{\perp} = \mathcal{U}^{\perp} \cap \mathcal{V}^{\perp}$ and $(\mathcal{U} \cap \mathcal{V})^{\perp} = \mathcal{U}^{\perp} + \mathcal{V}^{\perp}$.

Remark A.2 If $A \in \mathbb{C}^{p\times q}$, then $\mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$ and $\mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$.

Remark A.3 Let $\mathcal{U}$ be a linear subspace of the unitary space $\mathbb{C}^p$. Then $\mathbb{P}_{\mathcal{U}}$ is the unique complex $p \times p$ matrix satisfying the three equations $\mathbb{P}_{\mathcal{U}}^2 = \mathbb{P}_{\mathcal{U}}$, $\mathbb{P}_{\mathcal{U}}^* = \mathbb{P}_{\mathcal{U}}$, and $\mathcal{R}(\mathbb{P}_{\mathcal{U}}) = \mathcal{U}$.

Remark A.4 Let $\mathcal{U}$ be a linear subspace of the unitary space $\mathbb{C}^p$. Then $O_{p\times p} \preccurlyeq \mathbb{P}_{\mathcal{U}} \preccurlyeq I_p$ and $\mathbb{P}_{\mathcal{U}} + \mathbb{P}_{\mathcal{U}^{\perp}} = I_p$.

Remark A.5 If $\mathcal{U}$ is a linear subspace of the unitary space $\mathbb{C}^p$ with dimension $d := \dim \mathcal{U} \geq 1$ and some orthonormal basis $u_1, u_2, \ldots, u_d$, then $\mathbb{P}_{\mathcal{U}} = UU^*$, where $U := [u_1, u_2, \ldots, u_d]$.

Remark A.6 If $A \in \mathbb{C}^{p\times q}$, then $AA^{\dagger} = \mathbb{P}_{\mathcal{R}(A)}$ and $A^{\dagger}A = \mathbb{P}_{\mathcal{R}(A^*)}$.

Remark A.7 Let $A \in \mathbb{C}^{p\times q}$. In view of $AA^{\dagger}A = A$, we have:

(a) Let $B \in \mathbb{C}^{p\times m}$. Then $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if $AA^{\dagger}B = B$.
(b) Let $C \in \mathbb{C}^{n\times q}$. Then $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ if and only if $CA^{\dagger}A = C$.

Remark A.8 If $A \in \mathbb{C}^{p\times q}$, then $(A^{\dagger})^* = (A^*)^{\dagger}$.

Remark A.9 If $A \in \mathbb{C}^{p\times q}$, then $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$ and $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$.

Remark A.10 Let $A \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ and let $Q := \sqrt{A}$. Then:

(a) $\mathcal{R}(Q) = \mathcal{R}(A)$ and $\mathcal{N}(Q) = \mathcal{N}(A)$.
(b) $A^\dagger \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ and $\sqrt{A^\dagger} = Q^\dagger$.
(c) $QQ^\dagger = AA^\dagger = A^\dagger A = Q^\dagger Q$.
(d) $Q^\dagger A = Q$ and $AQ^\dagger = Q$.
(e) $Q^\dagger A Q^\dagger = AA^\dagger$.

Lemma A.11 (cf. [21, Lem. A.19]) *Let $A \in \mathbb{C}^{p\times q}$ and let $B \in \mathbb{C}^{q\times q}$ be such that $\mathcal{R}(B) \subseteq \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$. Then the matrix $B + \mathbb{P}_{\mathcal{N}(A)}$ is invertible and $B^\dagger = (B + \mathbb{P}_{\mathcal{N}(A)})^{-1} - \mathbb{P}_{\mathcal{N}(A)}$.*

Remark A.12 (e. g., combine [4, Theorems 4.4 and 4.6]) Let $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$, let $\lambda \in \mathbb{R}$, and let $x \in \mathbb{C}^q$ be such that $Ax = \lambda x$. Then $A^\dagger x = \lambda^\dagger x$.

Remark A.13 Let $A \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, let $\lambda \in [0, \infty)$, and let $x \in \mathbb{C}^q$ be such that $Ax = \lambda x$. Then $\sqrt{A}x = \sqrt{\lambda}x$.

Remark A.14 If $A, B \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ fulfill $O_{q\times q} \preccurlyeq A \preccurlyeq B$, then $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.

Lemma A.15 (cf. [11, Lem. 1.1.12]) *If $K \in \mathbb{C}^{p\times q}$, then the following statements are equivalent:*

(i) K is contractive.
(ii) $I_q - K^*K$ is non-negative Hermitian.
(iii) $\begin{bmatrix} I_p & K \\ K^* & I_q \end{bmatrix}$ is non-negative Hermitian.
(iv) $I_p - KK^*$ is non-negative Hermitian.
(v) K^* is contractive.

Lemma A.16 *Let $E \in \mathbb{C}^{p\times q}$ and let the matrices l, r and P, Q be given by (5.1) and (5.2), respectively. Then:*

(a) *$lE = Er$ and $l^\dagger E = Er^\dagger$.*
(b) *$E^*l = rE^*$ and $E^*l^\dagger = r^\dagger E^*$.*
(c) *$l^\dagger - Er^\dagger E^* = ll^\dagger$ and $r^\dagger - E^*l^\dagger E = r^\dagger r$.*
(d) *$PE = EQ$ and $E^*P = QE^*$.*
(e) *$E^*PE = Q$ and $EQE^* = P$.*

Proof (a) Regarding (5.1), we obtain $lE = (I_p - EE^*)E = E - EE^*E = E(I_q - E^*E) = Er$. Using (5.1) and the singular value decomposition, one can prove $l^\dagger E = Er^\dagger$ as well.

(b) Regarding (5.1), we obtain $E^*l = E^*(I_p - EE^*) = E^* - E^*EE^* = (I_q - E^*E)E^* = rE^*$. Using (5.1) and the singular value decomposition, one can prove $E^*l^\dagger = r^\dagger E^*$ as well.

(c) Using (b) and (5.1), we get $l^\dagger - Er^\dagger E^* = l^\dagger - EE^*l^\dagger = (I_p - EE^*)l^\dagger = ll^\dagger$ and $r^\dagger - E^*l^\dagger E = r^\dagger - r^\dagger E^*E = r^\dagger(I_q - E^*E) = r^\dagger r$.

(d) Using (a), we can infer $ll^\dagger E = lEr^\dagger = Err^\dagger$, whereas the application of (b) yields $E^*ll^\dagger = rE^*l^\dagger = rr^\dagger E^*$. By virtue of (5.1), we see $r^* = r$, so that Remark A.6 implies $r^\dagger r = rr^\dagger$. Taking additionally into account (5.2), we get then $PE = (I_p - ll^\dagger)E = E(I_q - rr^\dagger) = E(I_q - r^\dagger r) = EQ$ and $E^*P = E^*(I_p - ll^\dagger) = (I_q - rr^\dagger)E^* = (I_q - r^\dagger r)E^* = QE^*$.

(e) From (5.2) and (2.1) we can infer $rQ = O_{q\times q}$ and $Pl = O_{p\times p}$. Using additionally (d) and (5.1), we get then $E^*PE = E^*EQ = (I_q - r)Q = Q$ and $EQE^* = PEE^* = P(I_p - l) = P$. □

Remark A.17 (see, e. g., Lemma A.15 and [11, Lem. 1.1.12(c)]) Let $E \in \mathbb{K}_{p\times q}$ and let l and r be given by (5.1). Then:

(a) $l \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ and $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.
(b) $\sqrt{l}E = E\sqrt{r}$ and $\sqrt{l}^\dagger E = E\sqrt{r}^\dagger$.
(c) $E^*\sqrt{l} = \sqrt{r}E^*$ and $E^*\sqrt{l}^\dagger = \sqrt{r}^\dagger E^*$.

Notation A.18 For all $n \in \mathbb{N}_0$ denote by $\mathscr{L}_{p,n}$ (resp., $\mathscr{U}_{p,n}$) the set of all lower (resp., upper) $p \times p$ block triangular matrices belonging to $\mathbb{C}^{(n+1)p\times(n+1)p}$ with matrices I_p on its block main diagonal.

Remark A.19 ([20, Rem. A.20]) For all $n \in \mathbb{N}_0$ the sets $\mathscr{L}_{p,n}$ and $\mathscr{U}_{p,n}$ are both subgroups of the general linear group of invertible complex $(n+1)p \times (n+1)p$ matrices.

Notation A.20 Let $n \in \mathbb{N}_0$ and let $\mathbf{A}, \mathbf{B}$ be two complex $(n+1)p \times (n+1)q$ matrices.

(a) We write $\mathbf{A} \sim_{n,p\times q} \mathbf{B}$ if there exist matrices $\mathbf{L} \in \mathscr{L}_{p,n}$ and $\mathbf{U} \in \mathscr{U}_{q,n}$ such that $\mathbf{B} = \mathbf{LAU}$. If the corresponding (block) sizes are clear from the context, we will omit the indices and write $\mathbf{A} \sim \mathbf{B}$.
(b) We write $\mathbf{A} \backsim_{n,p\times q} \mathbf{B}$ if there exist matrices $\mathbf{V} \in \mathscr{U}_{p,n}$ and $\mathbf{M} \in \mathscr{L}_{q,n}$ such that $\mathbf{B} = \mathbf{VAM}$. If the corresponding (block) sizes are clear from the context, we will omit the indices and write $\mathbf{A} \backsim \mathbf{B}$.

Remark A.21 (cf. [20, Rem. A.25]) Let $n \in \mathbb{N}_0$. Then the relations $\sim_{n,p\times q}$ and $\backsim_{n,p\times q}$ are both equivalence relations on the set of complex $(n+1)p \times (n+1)q$ matrices.

Remark A.22 (cf. [20, Rem. A.26]) Let $\ell, m \in \mathbb{N}_0$, let $\mathbf{A}$ and $\mathbf{B}$ be complex $(\ell+1)p \times (\ell+1)q$ matrices, let $\mathbf{X}$ and $\mathbf{Y}$ be complex $(m+1)p \times (m+1)q$ matrices, and let $n := \ell + m + 1$. Then

(a) If $\mathbf{A} \sim_{\ell,p\times q} \mathbf{B}$ and $\mathbf{X} \sim_{m,p\times q} \mathbf{Y}$, then $\operatorname{diag}(\mathbf{A}, \mathbf{X}) \sim_{n,p\times q} \operatorname{diag}(\mathbf{B}, \mathbf{Y})$.
(b) If $\mathbf{A} \backsim_{\ell,p\times q} \mathbf{B}$ and $\mathbf{X} \backsim_{m,p\times q} \mathbf{Y}$, then $\operatorname{diag}(\mathbf{A}, \mathbf{X}) \backsim_{n,p\times q} \operatorname{diag}(\mathbf{B}, \mathbf{Y})$.

Remark A.23 (cf. [19, Lem. A.3]) Let $n \in \mathbb{N}_0$ and let $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_n$ be complex $p \times q$ matrices such that $\operatorname{diag}(A_j)_{j=0}^n \sim \operatorname{diag}(B_j)_{j=0}^n$ or $\operatorname{diag}(A_j)_{j=0}^n \backsim \operatorname{diag}(B_j)_{j=0}^n$. Then $A_j = B_j$ for all $j \in \mathbb{Z}_{0,n}$.

In view of (3.2), we state the following:

Remark A.24 Let $A, B \in \mathbb{C}^{p\times q}$, let $C \in \mathbb{C}^{q\times m}$, and let $n \in \mathbb{N}_0$. Then:

(a) $\langle\!\langle A\rangle\!\rangle_n^* = \langle\!\langle A^*\rangle\!\rangle_n$ and $\langle\!\langle A\rangle\!\rangle_n^\dagger = \langle\!\langle A^\dagger\rangle\!\rangle_n$.
(b) $\langle\!\langle A+B\rangle\!\rangle_n = \langle\!\langle A\rangle\!\rangle_n + \langle\!\langle B\rangle\!\rangle_n$ and $\langle\!\langle AC\rangle\!\rangle_n = \langle\!\langle A\rangle\!\rangle_n \langle\!\langle C\rangle\!\rangle_n$.

Appendix B. Some Technical Results on Linear Subspaces

Lemma B.1 *Let $L \in \mathbb{C}^{p\times p}$ and let $M \in \mathbb{C}^{p\times q}$ be such that $\mathcal{R}(I_p - L) \subseteq \mathcal{R}(M)$. Then $\mathcal{R}(L) \cap \mathcal{R}(M) \subseteq \mathcal{R}(LM)$.*

Proof We consider an arbitrary $y \in \mathcal{R}(L) \cap \mathcal{R}(M)$. Using Remark A.6, we can infer then $LL^\dagger y = y$ and $MM^\dagger y = y$. Remark A.7(b) yields $MM^\dagger(I_p - L) = I_p - L$, implying $I_p - MM^\dagger = (I_p - MM^\dagger)L$. Summarizing, we get $(I_p - MM^\dagger)L^\dagger y = (I_p - MM^\dagger)LL^\dagger y = (I_p - MM^\dagger)y = O$ and hence $L^\dagger y = MM^\dagger L^\dagger y$. Thus, $y = LL^\dagger y = LMM^\dagger L^\dagger y$. In particular, $y \in \mathcal{R}(LM)$. □

Lemma B.2 *Let $Q \in \mathbb{C}^{p\times q}$ and let $R \in \mathbb{C}^{q\times q}$ be such that $\mathcal{N}(Q) \subseteq \mathcal{N}(I_q - R)$. Then $\mathcal{N}(Q) + \mathcal{N}(R) = \mathcal{N}(QR)$.*

Proof From our assumption we get $\mathcal{N}(I_q - R)^\perp \subseteq \mathcal{N}(Q)^\perp$. Using Remark A.2, we can infer then that the matrices $L := R^*$ and $M := Q^*$ fulfill $\mathcal{R}(I_q - L) \subseteq \mathcal{R}(M)$. Thus, we can apply Lemma B.1 to obtain $\mathcal{R}(L) \cap \mathcal{R}(M) \subseteq \mathcal{R}(LM)$, implying $\mathcal{R}(LM)^\perp \subseteq [\mathcal{R}(L) \cap \mathcal{R}(M)]^\perp = \mathcal{R}(L)^\perp + \mathcal{R}(M)^\perp$. Using Remark A.2 again, then $\mathcal{N}(QR) = \mathcal{N}((LM)^*) \subseteq \mathcal{N}(L^*) + \mathcal{N}(M^*) = \mathcal{N}(R) + \mathcal{N}(Q)$ follows. It remains to prove $\mathcal{N}(Q) + \mathcal{N}(R) \subseteq \mathcal{N}(QR)$. To this end, let $v \in \mathcal{N}(Q) + \mathcal{N}(R)$, i. e., $v = x + y$ with certain $x \in \mathcal{N}(Q)$ and $y \in \mathcal{N}(R)$. Then, $x \in \mathcal{N}(I_q - R)$ by our assumption, implying $Rx = x$. Consequently, $QRv = QRx + QRy = Qx = O$. □

Lemma B.3 *Let $L \in \mathbb{C}^{p\times p}$ and let $M \in \mathbb{C}^{p\times q}$ be such that $\mathcal{R}(I_p - L) \subseteq \mathcal{R}(M)$. Then $\mathcal{R}(L) \cap \mathcal{R}(M) = \mathcal{R}(LM)$.*

Proof In view of Lemma B.1, it remains to prove $\mathcal{R}(LM) \subseteq \mathcal{R}(L) \cap \mathcal{R}(M)$. From our assumption we get $\mathcal{R}(M)^\perp \subseteq \mathcal{R}(I_p - L)^\perp$. Using Remark A.2, we can infer then that the matrices $Q := M^*$ and $R := L^*$ fulfill $\mathcal{N}(Q) \subseteq \mathcal{N}(I_p - R)$. Thus, we can apply Lemma B.2 to obtain $\mathcal{N}(Q) + \mathcal{N}(R) = \mathcal{N}(QR)$, implying $\mathcal{N}(QR)^\perp = [\mathcal{N}(Q) + \mathcal{N}(R)]^\perp = \mathcal{N}(Q)^\perp \cap \mathcal{N}(R)^\perp$. Using Remark A.2 again, then $\mathcal{R}(LM) = \mathcal{R}((QR)^*) \subseteq \mathcal{R}(Q^*) \cap \mathcal{R}(R^*) = \mathcal{R}(M) \cap \mathcal{R}(L)$ follows. □

In the sequel, we continue to use the notations given in (5.1).

Remark B.4 Let $E \in \mathbb{C}^{p\times q}$, let $B \in \mathbb{C}^{p\times m}$, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ such that $\mathcal{R}(E) + \mathcal{R}(B) \subseteq \mathcal{M}$. In view of (5.1), then $\mathcal{R}(lB) \subseteq \mathcal{R}(B) + \mathcal{R}(EE^*B) \subseteq \mathcal{M}$.

Remark B.5 Let $E \in \mathbb{C}^{p\times q}$, let $B \in \mathbb{C}^{p\times m}$, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ such that $\mathcal{R}(E) + \mathcal{R}(B) \subseteq \mathcal{M}$. Using Remark B.4, one can easily prove $\mathcal{R}(l^k B) \subseteq \mathcal{M}$ for all $k \in \mathbb{N}_0$ by mathematical induction.

Lemma B.6 *Let $E \in \mathbb{K}_{p\times q}$, let $B \in \mathbb{C}^{p\times m}$, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ such that $\mathcal{R}(E) + \mathcal{R}(B) \subseteq \mathcal{M}$. Then $\mathcal{R}(\sqrt{l}B) \subseteq \mathcal{M}$.*

Proof Remark A.17(a) shows $l \in \mathbb{C}^{p\times p}_{\succcurlyeq}$. Thus, we can choose a sequence $(\rho_n)_{n=1}^\infty$ of polynomials fulfilling $\sqrt{l} = \lim_{n\to\infty} \rho_n(l)$. From Remark B.5 we can infer $(I_p - \mathbb{P}_{\mathcal{M}})l^k B = O$ for all $k \in \mathbb{N}_0$. Consequently, $(I_p - \mathbb{P}_{\mathcal{M}})\rho_n(l)B = O$ for all $n \in \mathbb{N}$ follows. Passing to the limit $n \to \infty$, we obtain $(I_p - \mathbb{P}_{\mathcal{M}})\sqrt{l}B = O$. In particular, $\mathcal{R}(\sqrt{l}B) \subseteq \mathcal{M}$. □

Remark B.7 Let $E \in \mathbb{K}_{p\times q}$, let $B \in \mathbb{C}^{p\times m}$, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ such that $\mathcal{R}(E) + \mathcal{R}(B) \subseteq \mathcal{M}$. Using Lemma B.6, one can easily prove $\mathcal{R}(\sqrt{l}^k B) \subseteq \mathcal{M}$ for all $k \in \mathbb{N}_0$ by mathematical induction.

Lemma B.8 *Let $E \in \mathbb{K}_{p\times q}$, let $B \in \mathbb{C}^{p\times m}$, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^p$ such that $\mathcal{R}(E) + \mathcal{R}(B) \subseteq \mathcal{M}$. Then $\mathcal{R}(\sqrt{l}^\dagger B) \subseteq \mathcal{M}$.*

Proof First observe that $l \in \mathbb{C}_{\succcurlyeq}^{p\times p}$ by Remark A.17(a). Since the matrix $\sqrt{l}$ is Hermitian, there exists (see, e. g., [4, Cor. 4.3]) a polynomial π fulfilling $\sqrt{l}^\dagger = \pi(\sqrt{l})$. From Remark B.7 we can infer $(I_p - \mathbb{P}_{\mathcal{M}})\sqrt{l}^k B = O$ for all $k \in \mathbb{N}_0$. Consequently, $(I_p - \mathbb{P}_{\mathcal{M}})\pi(\sqrt{l})B = O$ follows, i. e., $(I_p - \mathbb{P}_{\mathcal{M}})\sqrt{l}^\dagger B = O$. In particular, $\mathcal{R}(\sqrt{l}^\dagger B) \subseteq \mathcal{M}$. □

Remark B.9 Let $E \in \mathbb{C}^{q\times q}$, let $B \in \mathbb{C}^{m\times q}$, and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ such that $\mathcal{Q} \subseteq \mathcal{N}(E) \cap \mathcal{N}(B)$. In view of (5.1), then $\mathcal{Q} \subseteq \mathcal{N}(B) \cap \mathcal{N}(BE^*E) \subseteq \mathcal{N}(Br)$.

Remark B.10 Let $E \in \mathbb{C}^{q\times q}$, let $B \in \mathbb{C}^{m\times q}$, and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ such that $\mathcal{Q} \subseteq \mathcal{N}(E) \cap \mathcal{N}(B)$. Using Remark B.9, one can easily prove $\mathcal{Q} \subseteq \mathcal{N}(Br^k)$ for all $k \in \mathbb{N}_0$ by mathematical induction.

Lemma B.11 *Let $E \in \mathbb{K}_{p\times q}$, let $B \in \mathbb{C}^{m\times q}$, and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ such that $\mathcal{Q} \subseteq \mathcal{N}(E) \cap \mathcal{N}(B)$. Then $\mathcal{Q} \subseteq \mathcal{N}(B\sqrt{r})$.*

Proof Remark A.17(a) shows $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Thus, we can choose a sequence $(\rho_n)_{n=1}^\infty$ of polynomials fulfilling $\sqrt{r} = \lim_{n\to\infty} \rho_n(r)$. From Remark B.10 we can infer $Br^k\mathbb{P}_{\mathcal{Q}} = O$ for all $k \in \mathbb{N}_0$. Consequently, $B\rho_n(r)\mathbb{P}_{\mathcal{Q}} = O$ for all $n \in \mathbb{N}$ follows. Passing to the limit $n \to \infty$, we obtain $B\sqrt{r}\mathbb{P}_{\mathcal{Q}} = O$. In particular, $\mathcal{Q} \subseteq \mathcal{N}(B\sqrt{r})$. □

Remark B.12 Let $E \in \mathbb{K}_{p\times q}$, let $B \in \mathbb{C}^{m\times q}$, and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ such that $\mathcal{Q} \subseteq \mathcal{N}(E) \cap \mathcal{N}(B)$. Using Lemma B.11, one can easily prove $\mathcal{Q} \subseteq \mathcal{N}(B\sqrt{r}^k)$ for all $k \in \mathbb{N}_0$ by mathematical induction.

Lemma B.13 *Let $E \in \mathbb{K}_{p\times q}$, let $B \in \mathbb{C}^{m\times q}$, and let $\mathcal{Q}$ be a linear subspace of $\mathbb{C}^q$ such that $\mathcal{Q} \subseteq \mathcal{N}(E) \cap \mathcal{N}(B)$. Then $\mathcal{Q} \subseteq \mathcal{N}(B\sqrt{r}^\dagger)$.*

Proof First observe that $r \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ by Remark A.17(a). Since the matrix $\sqrt{r}$ is Hermitian, there exists (see, e. g., [4, Cor. 4.3]) a polynomial π fulfilling $\sqrt{r}^\dagger = \pi(\sqrt{r})$. From Remark B.12 we can infer $B\sqrt{r}^k\mathbb{P}_{\mathcal{Q}} = O$ for all $k \in \mathbb{N}_0$. Consequently, $B\pi(\sqrt{r})\mathbb{P}_{\mathcal{Q}} = O$ follows, i. e., $B\sqrt{r}^\dagger\mathbb{P}_{\mathcal{Q}} = O$. In particular, $\mathcal{Q} \subseteq \mathcal{N}(B\sqrt{r}^\dagger)$. □

Appendix C. Linear Fractional Transformations of Matrices

Remark C.1 (see, e. g. [11, Lem. 1.6.1]) Let $c \in \mathbb{C}^{q\times p}$ and $d \in \mathbb{C}^{q\times q}$. Then the set $\mathcal{Q}(c, d) := \{x \in \mathbb{C}^{p\times q} : \det(cx+d) \neq 0\}$ is non-empty if and only if $\operatorname{rank}([c, d]) = q$.

Let M be a complex $(p+q)\times(p+q)$ matrix and let $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the block representation of M with $p\times p$ block a. Suppose that $\mathcal{Q}(c,d)\neq\emptyset$. Then let the linear fractional transformation $\mathcal{T}_M^{(p,q)}\colon \mathcal{Q}(c,d)\to\mathbb{C}^{p\times q}$ be defined by $\mathcal{T}_M^{(p,q)}(x) := (ax+b)(cx+d)^{-1}$.

Proposition C.2 ([11, Prop. 1.6.1]) *Let M_1 and M_2 be complex $(p+q)\times(p+q)$ matrices, let $M := M_2M_1$, and let*

$$M_1=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \qquad M_2=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \qquad \text{and} \qquad M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the block representations with $p\times p$ blocks a_1, a_2, and a of M_1, M_2, and M, respectively. Suppose that $\operatorname{rank}([c_1,d_1])=q$ and $\operatorname{rank}([c_2,d_2])=q$ hold true. Then $\mathcal{Q}(c,d)\cap\mathcal{Q}(c_1,d_1)=\{x\in\mathcal{Q}(c_1,d_1)\colon \mathcal{T}_{M_1}^{(p,q)}(x)\in\mathcal{Q}(c_2,d_2)\}$. Furthermore, if $\mathcal{Q}(c,d)\cap\mathcal{Q}(c_1,d_1)\neq\emptyset$, then $\mathcal{T}_{M_2}^{(p,q)}(\mathcal{T}_{M_1}^{(p,q)}(x))=\mathcal{T}_M^{(p,q)}(x)$ for all $x\in\mathcal{Q}(c,d)\cap\mathcal{Q}(c_1,d_1)$.

Example C.3 ([5, Beispiel B.11]) Let $M := M_1M_2$ where $M_1 = \left[\begin{smallmatrix} O_{q\times q} & O_{q\times q} \\ O_{q\times q} & I_q \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} O_{q\times q} & O_{q\times q} \\ I_q & O_{q\times q} \end{smallmatrix}\right]$. Let the block representations of M_1, M_2, and M be given as in Proposition C.2. Then $\mathcal{Q}(c_1,d_1)=\mathbb{C}^{q\times q}$, $\mathcal{Q}(c_2,d_2)=\{x\in\mathbb{C}^{q\times q}\colon \det x\neq 0\}$, and $\mathcal{Q}(c,d)=\emptyset$.

Appendix D. Some Facts on the Class $\mathscr{S}_{p\times q}(\mathbb{D})$

Lemma D.1 *Let $F\in[\mathcal{H}(\mathbb{D})]^{p\times q}$ with Taylor coefficient sequence $(A_j)_{j=0}^{\infty}$ and let $E := F(0)$. For all $z\in\mathbb{D}$, then $F(z)-E=\sum_{j=1}^{\infty} z^j A_j$ and $I_q - E^*F(z) = r_0 - \sum_{j=1}^{\infty} z^j A_0^* A_j$, where r_0 is given by (2.5).*

Proof. We consider an arbitrary $z\in\mathbb{D}$. Clearly, $F(z)=\sum_{j=0}^{\infty} z^j A_j$ and $A_0=E$. Therefore, $F(z)-E=\sum_{j=1}^{\infty} z^j A_j$ and, using (2.5), moreover

$$I_q - E^*F(z) = I_q - A_0^*\left(A_0 + \sum_{j=1}^{\infty} z^j A_j\right) = r_0 - \sum_{j=1}^{\infty} z^j A_0^* A_j. \quad \square$$

Theorem D.2 *Let $\tau_{p\times q}\colon \mathscr{S}_{p\times q}(\mathbb{D})\to\mathscr{S}_{p\times q;\infty}$ be defined by $\tau_{p\times q}(F) := (A_j)_{j=0}^{\infty}$, where $(A_j)_{j=0}^{\infty}$ is the Taylor coefficient sequence of F. Then $\tau_{p\times q}$ is well defined and bijective.*

Proof Well-definedness and surjectivity follow from [11, Thm. 3.1.1], whereas injectivity is clear. $\square$

Lemma D.3 *Let $F\in\mathscr{S}_{p\times q}(\mathbb{D})$, let $E := F(0)$, and let l and r be given by (5.1). For all $z\in\mathbb{D}$, then:*

(a) $ll^\dagger[F(z) - E] = F(z) - E$ *and* $[F(z) - E]r^\dagger r = F(z) - E$.
(b) $r^\dagger r - [F(z)]^* ll^\dagger F(z) = I_q - [F(z)]^* F(z)$ *and* $ll^\dagger - F(z)r^\dagger r[F(z)]^* = I_p - F(z)[F(z)]^*$.

Proof We consider an arbitrary $z \in \mathbb{D}$.

(a) Denote by $(A_j)_{j=0}^\infty$ the Taylor coefficient sequence of F. Lemma D.1 provides $F(z)-E = \sum_{j=1}^\infty z^j A_j$. Theorem D.2 yields $(A_j)_{j=0}^\infty \in \mathscr{S}_{p\times q;\infty}$. Hence, Remark 3.2 shows $\mathcal{R}(A_j) \subseteq \mathcal{R}(l_0)$ and $\mathcal{N}(r_0) \subseteq \mathcal{N}(A_j)$ for all $j \in \mathbb{N}$. Applying Remark A.7, we thus infer $l_0 l_0^\dagger A_j r_0^\dagger r_0 = A_j$ for all $j \in \mathbb{N}$. Clearly, $A_0 = E$, so that $r_0 = r$ and $l_0 = l$ by (2.5) and (5.1). Summarizing, we get $F(z) - E = \sum_{j=1}^\infty z^j l_0 l_0^\dagger A_j r_0^\dagger r_0 = l(\sum_{j=1}^\infty z^j l_0^\dagger A_j r_0^\dagger)r$. Using additionally (2.1), then $ll^\dagger[F(z) - E] = F(z) - E$ and $[F(z) - E]r^\dagger r = F(z) - E$ follow.

(b) Regarding (5.2), from (a) we can infer $PF(z) = PE$ and $F(z)Q = EQ$. Regarding (5.3) and Remark A.3, we see $P^* = P$ and $Q^* = Q$. Using additionally Lemma A.16(e), we conclude

$$[F(z)]^* PF(z) = [F(z)]^* PE = [PF(z)]^* E = (PE)^* E = E^* PE = Q$$

and, analogously, $F(z)Q[F(z)]^* = P$. Regarding again (5.2), consequently

$$\begin{aligned} r^\dagger r - [F(z)]^* ll^\dagger F(z) &= r^\dagger r + [F(z)]^* PF(z) - [F(z)]^* F(z) \\ &= r^\dagger r + Q - [F(z)]^* F(z) = I_q - [F(z)]^* F(z) \end{aligned}$$

and, analogously, $ll^\dagger - F(z)r^\dagger r[F(z)]^* = I_p - F(z)[F(z)]^*$. □

Lemma D.4 (see, e. g., [11, Lem. 2.1.5]) *Let* $F \in \mathscr{S}_{p\times q}(\mathbb{D})$ *and let* $E := F(0)$. *For all* $z \in \mathbb{D}$, *then* $\mathcal{N}(I_q - [F(z)]^* F(z)) = \mathcal{N}(r)$, *where* r *is given by (5.1).*

Lemma D.5 *Let* $F \in [\mathcal{H}(\mathbb{D})]^{p\times q}$ *be such that* $\mathcal{R}(F(z)) = \mathcal{R}(F(0))$ *and* $\mathcal{N}(F(z)) = \mathcal{N}(F(0))$ *for all* $z \in \mathbb{D}$. *Then* $G := F^\dagger$ *belongs to* $[\mathcal{H}(\mathbb{D})]^{q\times p}$ *and* $(C_{G;j})_{j=0}^\infty$ *is exactly the reciprocal sequence corresponding to* $(C_{F;j})_{j=0}^\infty$.

Proof From [22, Prop. 8.4] we know that G belongs to $[\mathcal{H}(\mathbb{D})]^{q\times p}$, whereas [22, Thm. 8.9] yields $(C_{F;j})_{j=0}^\infty \in \mathscr{I}_{p\times q;\infty}$ and that $(C_{G;j})_{j=0}^\infty$ is exactly the inverse sequence corresponding to $(C_{F;j})_{j=0}^\infty$. Using [22, Thm. 4.21], we see then that $(C_{F;j})_{j=0}^\infty$ belongs to $\mathscr{D}_{p\times q;\infty}$ and that $(C_{G;j})_{j=0}^\infty$ coincides with the reciprocal sequence corresponding to $(C_{F;j})_{j=0}^\infty$. □

Acknowledgements The first author would like to express special thanks to Professor Tatjana Eisner for her generous support. He also thanks the Max Planck Institute for Human Cognitive and Brain Sciences, in particular Professor Christian Doeller, Dr Christina Schroeder, and Dr Sebastian Ziegaus.

Author Contributions Conceptualization: Fritzsche, Kirstein, Mädler; Funding acquisition: Müller; Investigation: Dubovoy, Fritzsche, Kirstein, Mädler, Müller; Project administration: Fritzsche, Kirstein, Mädler; Validation: Fritzsche, Kirstein; Writing-original draft: Fritzsche, Mädler; Writing-review & editing: Fritzsche, Kirstein, Mädler.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors declare no competing interests.

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