Unitary fermionic topological field theory

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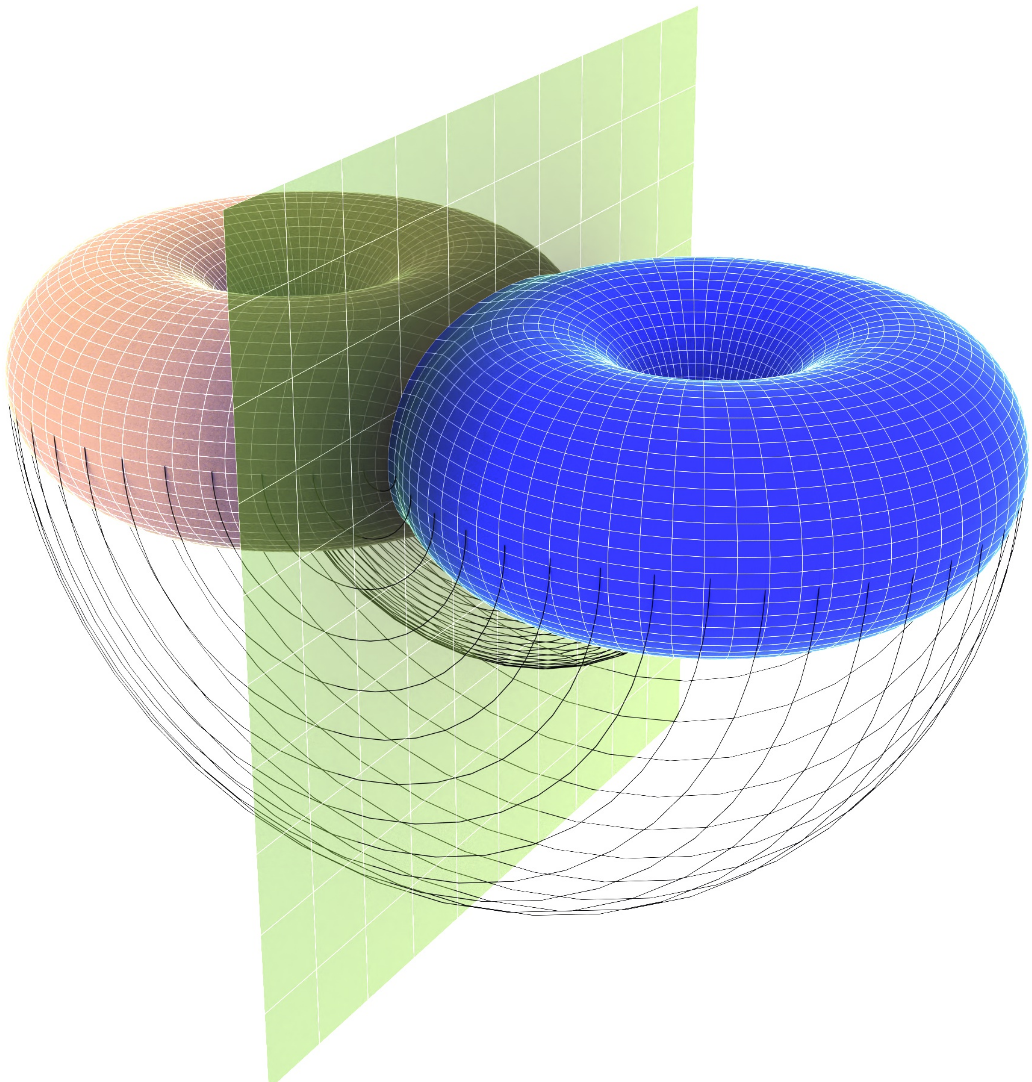

Figure 0.1: The double of the blue manifold with boundary is nullbordant. This can be seen by following the movement of a rotation by 180° to its reflection along the hyperplane as pictured. This traces out a manifold one dimension higher bounding the blue and pink manifold glued together.

Contents

"As we navigate the illuminated landscapes of these mathematical abstractions, the mathematical tapestry that unfurls is akin to a kaleidoscopic spectacle, inviting contemplation of a realm that defies the limits of conventional human cognition. It is within this transcendental transcendence that we encounter the apex of this research, a vertex that bristles with an audacious vivacity that both tantalizes and bewilders the reader, defying the very fabric of mathematical propriety and leaving us, the humble researchers, at the precipice of epistemic reckoning." - ChatGPT

Chapter 1

Introduction

Quantum field theory is a main pillar of theoretical physics; effectively all modern theories in physics that involve appropriate quantum phenomena are quantum field theories. Finding suitable mathematical formulations of quantum field theory has been one of the main motivations for mathematical physicists from the beginning of the 20th century until now. Even though most aspects of nonperturbative quantum field theory remain mathematically mysterious, a rigorous formulation of topological quantum field theory has been developed in the past few decades [2]. It started with the insight of Graeme Segal that a spacetime with two distinguished time slices called 'now' and 'soon' can be modeled by a bordism [62]. The assignment of the Hilbert space of states to a time slice and the time evolution operator to a spacetime ideally satisfy a couple of axioms providing compatibility between gluing spacetimes. From another perspective, this attempts to axiomatize desirable properties of the Feynman path integral without relying on measures on the space of fields [50]. These axioms can be neatly summarized using category theory by building a bordism category $\mathrm{Bord}_{n,n-1}$ in which objects are closed oriented $(n-1)$-dimensional manifolds and morphisms are oriented bordisms. Physically one can think of the morphisms as spacetimes and objects as time slices; closed codimension one submanifolds of n-dimensional spacetime. A topological field theory[1] can then be defined as a symmetric monoidal functor from $\mathrm{Bord}_{n,n-1}$ equipped with disjoint union as its monoidal product to complex vector spaces Vect with the tensor product [2]. The latter monoidal structure corresponds to the familiar source of entanglement in physics; the operation of putting two physical systems together without allowing them to interact — known as 'stacking' — requires one to take the tensor products of their state spaces. A crucial consequence is that there exists states in the composite system, which are not tensor products of states in the original two systems.

The approach to axiomatize quantum field theory through functors from bordism categories to algebraic categories over the complex numbers is called 'functorial field theory'. The program has been successful for understanding quantum field theories that are topological, which means the theory is insensitive to the Lorentzian metric on spacetime and only depends on its topology. Topological field theory has found applications to knot theory [75], the classification of anomalies [18, 52], topological phases of matter (such as invertible phases [20] and topological order). The functorial field theory approach has been applied to nontopological theories as well [70, 71, 43], but nontrivial and physically relevant examples are scarce. The main exceptions are free theories, one-dimensional theories (quantum mechanics), two-dimensional conformal field theories [48, 26, 33] and two-dimensional area-dependent theories [19, 55].

[1] We use the term 'field theory' independently of whether the theory is classical or quantized.

Topological theories lack any dynamics whatsoever and so are prime examples of mathematical toy models for more complicated quantum field theories. Toy models have the advantage of allowing some of the constructions familiar from the physics literature with less technical complications. For example, under certain finiteness assumptions it is possible to talk about gauging a symmetry, or more generally to quantize topological action functionals of fields with some fixed target space [14, 23, 59]. Except in a small number of cases, such as free theories, quantization procedures for nontopological quantum field theories require perturbative methods and renormalization, which lose a lot of information about the global structure. In topological field theory, it is similarly possible to talk about anomalies that obstruct the quantization topologically [18]. Even though functorial field theory is usually formulated in the state-focused Schrödinger picture, functorial field theory from the observable-focused Heisenberg picture has also been considered [28].

This thesis is concerned with the formulation of another important aspect of quantum field theory in the topological case: unitarity. In the usual formulation of topological field theory, state spaces are finite-dimensional vector spaces and time evolution is a linear map. For unitary theories, state spaces become Hilbert spaces and time evolution is unitary. There are several ways to implement this in the functorial formalism. We argue that the most natural starting point is the observation that the category of Hilbert spaces and continuous linear maps is a dagger category. This approach is well-studied in quantum mechanics, especially in quantum information theory [1, 64, 32]. Here, a dagger category is defined to be a category $\mathcal{D}$ equipped with a contravariant endofunctor $\dagger\colon \mathcal{D} \to \mathcal{D}^{\mathrm{op}}$ that is the identity on objects and squares to the identity functor:

$$x^\dagger = x \qquad f^{\dagger\dagger} = f$$

for all objects x and all morphisms f in $\mathcal{D}$. A dagger functor between dagger categories is a functor that satisfies $F(f^\dagger) = F(f)^\dagger$. In a dagger category, it is possible to talk about unitary morphisms, which are morphisms $f\colon x_1 \to x_2$ for which the equations

$$f^\dagger f = \mathrm{id}_{x_1} \qquad f f^\dagger = \mathrm{id}_{x_2}$$

hold. A symmetric monoidal dagger category has both a symmetric monoidal and a dagger structure, such that the associator and braiding are unitary and

$$\otimes\colon \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

is a dagger functor. The category of Hilbert spaces is a symmetric monoidal dagger category through the Hermitian adjoint operation. The oriented bordism category can also be made into a symmetric monoidal dagger category in which the adjoint is given by taking a bordism, reversing its direction and taking its orientation reversal. In this situation, we could ask a topological field theory to send the adjoint of a bordism to the adjoint of the corresponding linear map [3, 4] [73, Appendix G]. More precisely, a unitary topological field theory is a symmetric monoidal dagger functor from the bordism symmetric monoidal dagger category to the symmetric monoidal dagger category of Hilbert spaces in the sense of Definition 2.4.1. The goal of this thesis is to generalize this dagger categorical definition of unitarity to include fermions. With enough care, we show how this leads to a form of the spin-statistics theorem for topological field theory.

We briefly digress to compare with another way to make unitarity precise. This is the approach using covariant $\mathbb{Z}/2$-actions instead of dagger categories, as suggested in [20]. Namely, we can equip the bordism category $\mathrm{Bord}_{n,n-1}$ with the $\mathbb{Z}/2$-action given by orientation-reversal $Y \mapsto \overline{Y}$ and the category of vector spaces with the $\mathbb{Z}/2$-action given by complex conjugation $V \mapsto \overline{V}$.[2] We

[2]We discuss actions on categories and equivariant functors in Appendix A.3.

emphasize one major difference between $\mathbb{Z}/2$-actions and dagger categories; the $\mathbb{Z}/2$-action includes the extra datum of an isomorphism $Y \cong \overline{\overline{Y}}$ specifying exactly how the functor $Y \mapsto \overline{Y}$ squares to the identity functor. In this setting, it is possible to ask topological field theories to be $\mathbb{Z}/2$-equivariant functors, i.e. to come equipped with the data of an isomorphism $Z(\overline{Y}) \cong \overline{Z(Y)}$ for every time slice $Y \in \operatorname{ob} \mathrm{Bord}_{n,n-1}$ satisfying some conditions. Such $\mathbb{Z}/2$-equivariance data is called a *reflection structure* on a topological field theory, see Definition 6.2.1.

To connect with the approach using dagger categories, we first observe that both the bordism category and the category of finite-dimensional vector spaces admit dual objects, see Appendix A.1 for the basics of duality in monoidal categories. For a finite-dimensional vector space V this is the usual linear dual V^* and for a time slice Y we prefer to think of Y^* as the same time slice with its direction of time reversed, also see the discussion in Section 5.1. Now note that a Hermitian form (a not necessarily positive definite but still nondegenerate Hilbert space structure) on V is equivalent to an isomorphism $h\colon V \cong \overline{V}^*$ satisfying the usual skew-linear symmetry property. The latter property turns out to be equivalent to requiring that the diagram

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[rr, bend right, "h"'] & \overline{\overline{\overline{V}^*}}^* \arrow[r, "\overline{h}^*"] & \overline{V}^* \end{tikzcd} \tag{1.1}$$

commutes. Analogously, we define a Hermitian form on a time slice Y to be an isomorphism $h_Y : Y \cong \overline{Y}^*$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\cong"] \arrow[rr, bend right, "h_Y"'] & \overline{\overline{\overline{Y}^*}}^* \arrow[r, "\overline{h}_Y^*"] & \overline{Y}^* \end{tikzcd} \tag{1.2}$$

commutes. Here the isomorphism $\overline{\overline{\overline{Y}^*}}^* \cong Y$ is provided by the isomorphisms $\overline{\overline{Y}} \cong Y$ given by the $\mathbb{Z}/2$-action, $Y^{**} \cong Y$ given by the braiding and $\overline{Y}^* \cong \overline{Y^*}$ given by the fact that $\overline{(.)}$ is a monoidal functor, which preserves duals. Now, if Z is a $\mathbb{Z}/2$-equivariant functor and h_Y a Hermitian form on Y, then $Z(Y)$ has a canonical Hermitian form by the composition

$$Z(Y) \xrightarrow{Z(h_Y)} Z(\overline{Y}^*) \to Z(\overline{Y})^* \to \overline{Z(Y)}^*.$$

Here in the second arrow we used that the symmetric monoidal functor Z preserves duals. For unitarity, we can then ask whether this Hermitian form is positive definite. As we will soon see, this formulation is closely related to the formulation using dagger categories.

However, we first highlight an important subtlety in the last paragraph we have swept under the rug: the above definition is strongly dependent on the choice of h_Y. In the simple case of the oriented bordism categories or in specific concrete models of more complicated bordism categories, this problem is often overlooked, because typically there is a "canonical" Hermitian form on objects of the bordism category. However, for many quantum field theories, more complicated geometric structures on the bordisms are a necessity, such as principal bundles with connection for gauge theory or spin structures for spinors. It can then happen that time slices have many automorphisms $\phi\colon Y \to Y$, which potentially induce many Hermitian pairings by precomposing a Hermitian pairing with ϕ. One immediate reaction to this problem might be to ask that all Hermitian pairings should map to positive definite inner products. But this would require $Z(\phi)$ to be a positive operator for all ϕ. We will soon elaborate on why this is too strong of a requirement in general. Therefore, we will need to record the data of the Hermitian pairings h_Y we would like to allow. As we will soon see, it turns out that recording this data is exactly equivalent to making the bordism category into a dagger category. With this in mind, we would like to argue that expressing unitarity through dagger functors is categorically more convenient than expressing it through $\mathbb{Z}/2$-equivariance. One disadvantage of dagger categories, however, is that they are not well-behaved under equivalences of categories, but this disadvantage gets resolved precisely by presenting them by categories with Hermitian pairings, as shown in the joint work [68] of the author with Jan Steinebrunner.

To explain these subtleties further, we get to the second topic of study in this thesis: fermions, spinors, and their relationship. Implementing fermions in topological field theory is straightforward: in a fermionic theory, state spaces are $\mathbb{Z}/2$-graded by the fermion parity operator $(-1)^F$. This operator is always a symmetry of the physical system and so all other operations are required to commute with it (we will not discuss supersymmetry). The further crucial property that distinguishes odd from even states in this grading is their statistics; bosonic states have Bose statistics and fermionic states have Fermi statistics. This is implemented by requiring the exchange operation between two fermions to come with a minus sign. This is familiar to mathematicians; we have to equip the monoidal category of $\mathbb{Z}/2$-graded vector spaces with the braiding that satisfies the usual Koszul sign rule. We will, from now on, refer to this symmetric monoidal category as the category of complex super vector spaces sVect.

How do we implement spinors? To define relevant mathematical machinery, such as spinor bundles, Dirac operators et cetera, we need to equip our bordisms with a spin structure. Therefore, the barebone structure of a toplogical field theory in n spacetime dimensions with spinors and fermions is a symmetric monoidal functor

$$Z\colon \mathrm{Bord}_{n,n-1}^{\mathrm{Spin}} \to \mathrm{sVect}$$

from the symmetric monoidal category of spin bordisms. In this introduction, we will refer to such a functor as a fermionic topological field theory.[3] The special nontrivial central element $c \in \mathrm{Spin}_n$ in the kernel of the double cover to SO_n gives a distinguished automorphism Y_c of any time slice Y^{n-1} of the spin bordism category. This automorphism will be called the spin flip. It can be thought of as rotating particles by 360°. As such, the spin flip should map integer spin particles to themselves and map particles with half-integer spin to their negative.

This setting is convenient to formulate the connection between spin and statistics commonly assumed in quantum field theory. Namely, given a (not necessarily unitary) fermionic topological field theory Z and a time slice Y, the state space $Z(Y)$ comes equipped with two commuting $\mathbb{Z}/2$-gradings. One is the supergrading $(-1)^F_{Z(Y)}$, while the other one is the involution $Z(Y_c)$ induced by the spin flip on the spin manifold Y. For a general fermionic topological field theory, there is no reason for these to agree. We will argue in Section 6.3 that the connection between spin and statistics for fermionic topological field theory should say that these two gradings are equal. Hence, we say Z has a *spin-statistics connection* if $(-1)^F_{Z(Y)} = Z(Y_c)$ for all time slices, see Definition 6.3.1. The famous spin-statistics theorem for unitary quantum field theory says that every unitary quantum field theory should have a connection between spin and statistics. Therefore, a guiding principle to resolve the question of which Hermitian forms h_Y we allow in the definition of unitary fermionic topological field theory, is to ensure that every unitary fermionic topological field theory has a spin-statistics connection. A major insight of this thesis is that some elementary requirements on h_Y will give abstract category-theoretical properties to the spin bordism dagger category, which will ensure the spin-statistics theorem holds.

[3] We allow for many more types of geometric structures on bordisms (see Section 4.1). This results in Definition 6.1.1 of a fermionic topological field theory with arbitrary internal symmetry group. For simplicity, we will postpone further discussion until the end of the introduction.

The first step to translate this formulation of the spin-statistics theorem to a more categorical language, is to realize that the connection expresses equivariance with respect to the 2-group $B\mathbb{Z}/2$, as observed in [38]. We refer the reader to Appendix A.2 for an introduction to 2-groups. More precisely, both $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ and sVect come equipped with $B\mathbb{Z}/2$-actions, induced by the spin flip involution and the grading involution respectively. We prefer to think of these $B\mathbb{Z}/2$-actions physically as expressing the fermionic nature of these categories. A fermionic topological field theory has a spin-statistics connection if and only if it is $B\mathbb{Z}/2$-equivariant:

$$\begin{tikzcd} \mathrm{Bord}^{\mathrm{Spin}}_{n,n-1} \arrow[loop above, "\text{Spin } B\mathbb{Z}/2", "Y\mapsto Y_c"'] \arrow[r, "Z"] & \mathrm{sVect}\,. \arrow[loop above, "\text{Statistics } B\mathbb{Z}/2", "V\mapsto (-1)^F_V"'] \end{tikzcd}$$

This motivates the abstract study of $B\mathbb{Z}/2$-actions on dagger categories and $B\mathbb{Z}/2$-equivariant functors between them, see Sections 2.6 and 2.9.

We will now illustrate how the problem of choosing Hermitian pairings on objects of bordism categories we explained before gets even worse for fermionic topological field theory. Firstly, in the spin bordism category, it is much less straightforward to define an orientation-reversal $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$. For example, if $P \to X$ is a Spin_n-principal bundle, we could define its orientation-reversed Spin_n-principal bundle as follows. Consider the pin group $\mathrm{Pin}_n^+ \subseteq Cl_{+n}$, which is a double cover of the orthogonal group O_n, for which lifts of reflections have square equal to 1. It has the two connected components $\mathrm{Pin}_n^+ = \mathrm{Spin}_n \sqcup (\mathrm{Pin}_n^+)_{odd}$ induced by the supergrading of the Clifford algebra, or equivalently the determinant of the underlying element of O_n. Since the odd part of the pin group is a Spin_n-torsor, the associated Pin_n^+-bundle $P \times_{\mathrm{Spin}_n} \mathrm{Pin}_n^+$ decomposes into two Spin_n-bundles

$$P \cong P \times_{\mathrm{Spin}_n} \mathrm{Spin}_n \quad \text{ and } \quad \overline{P} := P \times_{\mathrm{Spin}_n} (\mathrm{Pin}_n^+)_{odd}.$$

In analogy with using SO_n and O_n instead of Spin_n and Pin_n^+, we obtain $\overline{P}$ as a candidate for the orientation-reversal of P. This construction not only works for principal Spin_n-bundles, but also gives a $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$ on the category of spin structures on a fixed manifold, see Section 4.5 for details. This in turn will induce a $\mathbb{Z}/2$-action on the bordism category, as we show in Corollary 5.2.7.

There was a subtle choice, however, in the definition of this orientation-reversal; we could have used the other pin group Pin_n^- in the definition, compare Remark 3.2.2. This would have resulted in a possibly different notion of orientation reversal $Y \mapsto Y'$. It turns out that there is a canonical spin diffeomorphism $\overline{Y} \cong Y'$. Unfortunately, this diffeomorphism does not induce an equivalence of $\mathbb{Z}/2$-actions, because it does not commute with the identifications $Y \cong \overline{\overline{Y}}$ and $Y \cong Y''$. In fact these two identifications differ by the spin flip $B\mathbb{Z}/2$-action, see Corollary 5.2.10. As a consequence, Hermitian pairings for the $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$ will not correspond to Hermitian pairings for the $\mathbb{Z}/2$-action $Y \mapsto Y'$. In fact, we will see in Section 4.6 that while $Y \mapsto \overline{Y}$ admits Hermitian pairings on all objects, Hermitian pairings typically do not exist at all for $Y \mapsto Y'$, also see Example 5.2.19.

The $\mathbb{Z}/2$-action $Y \mapsto Y'$ does enjoy some favourable properties that make it useful for other considerations. For example, the an action $Y \mapsto Y'$ can be defined in very high generality for vector bundles with G_n-structures, as long as there exists what we will call a strict geometric stabilization

G_{n+1} of G_n, see Definition 4.3.10. Under some extra assumptions, we provide an intuitive construction of an isomorphism $Y \cong (Y')^*$ in Corollary 4.6.7. Geometrically this isomorphism is given by a rotation by π in the plane spanned by the time direction e_n and the extra orthogonal direction e_{n+1}. However, as already was clear in the spin case, these isomorphisms are not Hermitian pairings in general. We mention in passing that this setting is convenient to show that the double of an n-dimensional manifold with G-structure with boundary bounds an $(n+1)$-dimensional manifold with a G_{n+1}-structure, see Figure 0.1. Here, we conveniently define the double of a manifold X with boundary Y as $X^\dagger X$, where we considered X as a morphism $\emptyset \to Y$ in a bordism dagger category, see Definition 5.2.20. Note that this definition depends on the dagger-categorical structure.

We highlight more subtleties by considering the concrete case of fermionic topological field theory in one spacetime dimension. Note that Spin_1 is a group of order two and so a spin structure on a one-dimensional manifold simply consists of an orientation and a double cover. The spin flip operation is given by exchanging the two sheets of the double cover. There are two isomorphism classes of zero-dimensional connected spin manifolds in the spin bordism category: the positively oriented $+$ and the negatively oriented point $-$.[4] Let us assume we have defined some notion of orientation reversal $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$ on the spin bordism category, so that we can talk about Hermitian pairings on zero-dimensional spin manifolds. We can then discuss unitarity using the definition with equivariant functors. So suppose h is a Hermitian pairing on the positively oriented point $+$ and Z a $\mathbb{Z}/2$-equivariant fermionic topological field theory. We can then ask whether Z maps h to a positive definite Hermitian pairing on $Z(+)$ and we would like to call Z unitary in that case. However, we can also compose h with the spin flip operation to get another Hermitian pairing. In a theory in which the spin flip acts nontrivially, this other Hermitian pairing will never have the same signature as h. Note that any theory that does not factor through the oriented bordism category – i.e. one that is not essentially without spinors – satisfies this requirement. Many such theories exist. In fact, up to isomorphism, the data of Z is exactly given by the super vector space $Z(+)$ together with a further even involution given by the spin flip. Moreover, any finite-dimensional super vector space with involution like this gives a one-dimensional fermionic topological field theory. Therefore, we do not want to ask both h and $h \circ +_c$ to be mapped to Hilbert space structures on $Z(+)$ in general.

The conclusion of the discussion above is that we have to be really careful about which Hermitian pairings on the bordism category we allow. We will call this preferred subset of allowed Hermitian pairings the "positive" ones, similarly to how in the dagger category of Hilbert spaces we only allow positive definite pairings. Then unitary topological field theories are those that send positive Hermitian pairings in the bordism category to positive Hermitian pairings in the category of vector spaces. We will now further abstract the situation and discuss how to reformulate recording "positive Hermitian pairings" into the language of category theory. After the setup is clear, we will discuss the desiderata the spin-statistics theorem imposes.

Let $\mathcal{C}$ be a symmetric monoidal category admitting duals. Assume it comes equipped with a symmetric monoidal involution $x \mapsto \overline{x}$, i.e. a $\mathbb{Z}/2$-action. We pick a symmetric monoidal dual functor $x \mapsto x^*$, which turns out to be unique up to canonical monoidal natural isomorphism. We show in Section 2.2 that this equips $\mathcal{C}$ with a canonical symmetric monoidal anti-involution $dx := \overline{x}^*$. Here, by anti-involution we mean the weak version of a dagger category: a contravariant functor $d\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$, equipped with data telling us how it squares to the identity functor, see Definition 2.2.4. The theory of categories equipped with anti-involutions is not the same as dagger structures, but in every dagger category the $\dagger$ is also an anti-involution. This defines a canonical functor from the 2-category of dagger categories to the 2-category of categories with anti-involution. We

[4] We discuss subtleties of orientations on zero-dimensional manifolds in Section 4.2.

refer to [41] for an introduction to 2-categories. Confusingly, the underlying anti-involution of a dagger category does not retain all desired information of the dagger category if it is only taken up to anti-involutive equivalence in the sense of Definition 2.2.5. For example, it turns out that the inclusion $\mathrm{Hilb} \hookrightarrow \mathrm{Herm}_{\mathbb{C}}$ of finite-dimensional Hilbert spaces into finite-dimensional Hermitian vector spaces of arbitrary signature is an anti-involutive equivalence. This is certainly undesirable from the perspective of quantum physics, because it forgets all the information of positivity of transition probabilities. Luckily, the inclusion is not an equivalence of dagger categories. So our goal is to remember the extra information the dagger category Hilb contains and see it as an category equipped with an anti-involution and extra data.

The data we will want to remember are analogous to the Hermitian pairings in the sense discussed above: isomorphisms $h\colon x \to dx$ satisfying a condition, see Definition 2.3.4. Namely, given a morphism between objects of $\mathcal{C}$ equipped with Hermitian pairings, one can talk about the Hermitian adjoint, see the formula (2.4). The category in which objects are pairs (x, h), consisting of an object of $\mathcal{C}$ and a Hermitian pairing on it, is a symmetric monoidal dagger category that we call the Hermitian completion $\mathrm{Herm}\,\mathcal{C}$, see Definition 2.3.8. For example, when applying the construction to $\mathcal{C} = \mathrm{Vect}$, we obtain the dagger category of Hermitian vector spaces $\mathrm{Herm}_{\mathbb{C}}$, and for $\mathcal{C} = \mathrm{sVect}$, we obtain super Hermitian vector spaces $\mathrm{sHerm}_{\mathbb{C}}$. Here, by super Hermitian vector space we mean a finite-dimensional super vector space V, together with a nondegenerate sesquilinear pairing $\langle ., . \rangle\colon V \times V \to \mathbb{C}$ such that

$$\langle v, w \rangle = (-1)^{|v||w|}\overline{\langle w, v \rangle}$$

for all homogeneous $v, w \in V$. The Koszul sign in the above definition unfortunately does not agree with what is usually considered a $\mathbb{Z}/2$-graded Hilbert space in the literature, but it is forced on us by the categorical setup. We set up our non-standard conventions for super Hilbert spaces in Section 2.1. We will also argue that with enough effort it is possible to be persistent in assuming $\mathbb{Z}/2$-graded Hilbert spaces have their inner product violating the Koszul sign rule

$$\langle v, w \rangle = \overline{\langle w, v \rangle}.$$

However, this is not our preferred convention, as it will introduce subtle signs at other places. One such subtle sign is in the construction of dual functors compatible with the † structure, see Example 2.7.15.

We move back to the general setting of symmetric monoidal anti-involutive categories, i.e. symmetric monoidal categories with symmetric monoidal anti-involution. To obtain smaller dagger categories, we can take the full subcategory $\mathcal{C}_P$ of the Hermitian completion $\mathrm{Herm}\,\mathcal{C}$ on an arbitrary subset P of Hermitian pairings $h\colon x \to dx$. The resulting category will be a symmetric monoidal dagger category again, if we require the subset to be closed under the tensor product. For example, we might restrict the Hermitian pairings on sVect to those that are positive definite in the sense of Definition 2.1.4, so that we recover the dagger category sHilb of finite-dimensional super Hilbert spaces. It can happen that different subsets P of Hermitian pairings give equivalent symmetric monoidal dagger categories. We provide the concrete operation of "closing under transfers" on a collection of Hermitian pairings P, see Definition 2.3.7. Two subsets P_1, P_2 of Hermitian pairings give equivalent symmetric monoidal dagger categories $\mathcal{C}_{P_1} \cong \mathcal{C}_{P_2}$ covering the identity on $\mathcal{C}$ if and only if they have the same closure under transfers, compare Lemma 2.3.25. We call an equivalence class $[P]$ of such Hermitian pairings a positivity structure on $(\mathcal{C}, d)$, see Definition 2.3.20. It turns out that symmetric monoidal anti-involutive categories with positivity structures contain exactly the same information as symmetric monoidal dagger categories.

Theorem 1.0.1 (Theorem (2.4.14))**.** *The Hermitian completion construction is a right adjoint of the canonical functor*

$$\begin{tikzcd} \mathrm{Cat}^{\dagger}_{\mathbb{E}_\infty} \arrow[r, bend left] & \mathrm{aICat}_{\mathbb{E}_\infty} \arrow[l, bend left, dashed, "\mathrm{Herm}"] \end{tikzcd}$$

and induces an equivalence of 2-categories

$$\mathrm{Cat}^{\dagger}_{\mathbb{E}_\infty} \cong (\mathrm{aICat}_{\mathbb{E}_\infty})_P.$$

The above theorem is the symmetric monoidal analogue of the main theorem of [68], which was joint work of the author with Jan Steinebrunner. In the above theorem, $\mathrm{Cat}^{\dagger}_{\mathbb{E}_\infty}$ denotes the 2-category of symmetric monoidal dagger categories, in which 1-morphisms are symmetric monoidal dagger functors and 2-morphisms are symmetric monoidal unitary natural transformations. Instead, $\mathrm{aICat}_{\mathbb{E}_\infty}$ denotes the 2-category of symmetric monoidal categories with anti-involution, in which 1-morphisms are symmetric monoidal anti-involutive functors and 2-morphisms are symmetric monoidal anti-involutive natural transformations. $(\mathrm{aICat}_{\mathbb{E}_\infty})_P$ denotes the 2-category of symmetric monoidal categories with anti-involution and positivity structure, in which 1-morphisms are additionally required to preserve the positivity structures.

This theorem has many corollaries that are useful for comparing different formulations of unitarity of topological field theories. For example, as the Hermitian completion is a right adjoint, we obtain an equivalence of categories between symmetric monoidal anti-involutive functors

$$\mathcal{D} \to \mathrm{sVect}$$

and symmetric monoidal dagger functors

$$\mathcal{D} \to \mathrm{sHerm}_{\mathbb{C}},$$

if $\mathcal{D}$ is a symmetric monoidal dagger category. Also, it turns out that Herm is almost a one-sided inverse. Namely, if $\mathcal{C}$ is a symmetric monoidal anti-involutive category, then there is an equivalence of anti-involutive categories $\mathrm{Herm}\,\mathcal{C} \cong \mathcal{C}^{\exists \mathrm{herm}}$, where $\mathcal{C}^{\exists \mathrm{herm}} \subseteq \mathcal{C}$ is the full subcategory on objects that admit a Hermitian pairing, see Lemma 2.3.18. We therefore obtain:

Corollary 1.0.2. *Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category for which every object admits a Hermitian pairing. There is an equivalence between the category of symmetric monoidal anti-involutive functors*

$$\mathcal{C} \to \mathrm{sVect}$$

and the category of symmetric monoidal dagger functors

$$\mathrm{Herm}\,\mathcal{C} \to \mathrm{sHerm}_{\mathbb{C}}\,.$$

If the category has enough duals, we expect $\mathbb{Z}/2$-actions and anti-involutions to be equivalent concepts through composition with a dual functor. In Section 2.2, we show the following result:

Theorem 1.0.3 (Theorem 2.2.24)**.** *There is a 2-functor from the 2-category of symmetric monoidal rigid anti-involutive categories to the 2-category symmetric monoidal rigid involutive categories, which is an equivalence on hom-categories.*

We summarize the comparison between different notions of involutions on categories we have now established in the following diagram:

$$\begin{tikzcd}
\overline{(.)} \arrow[r, mapsto] & d = \overline{(.)}^* \arrow[r, rightsquigarrow, "\text{choices of } h\colon x\to dx"] & \dagger \\
\text{involutions} \arrow[r, "\sim"] & \text{anti-involutions} \arrow[r, dashed, bend right, "\text{Herm}"'] & \text{daggers} \arrow[l]
\end{tikzcd} . \tag{1.3}$$

Combining the above results we obtain:

Corollary 1.0.4 (Theorem 6.2.5, Theorem 6.2.7)**.** *Assume the bordism category* $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ *comes equipped with a symmetric monoidal involution* $\overline{(.)}$*. If every time slice admits a Hermitian pairing, there is an equivalence between the category of* $\mathbb{Z}/2$*-equivariant functors*

$$\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1} \to \mathrm{sVect}$$

and the category of dagger functors

$$\mathrm{Herm}(\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}) \to \mathrm{sHerm}_{\mathbb{C}} .$$

We will call the latter Hermitian *fermionic topological field theories (Definition 6.2.4) and so we will prove that fermionic topological field theories with reflection structure are equivalent to Hermitian fermionic topological field theories Moreover, if* P *is a positivity structure on* $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ *given by a collection of Hermitian pairings* $h_Y\colon Y \cong \overline{Y}^*$ *on objects of the bordism category, then there is an equivalence between the category of dagger functors*

$$(\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1})_P \to \mathrm{sHilb}$$

and the category of $\mathbb{Z}/2$*-equivariant functors*

$$Z\colon\ \mathrm{Bord}^{\mathrm{Spin}}_{n,n-1} \to \mathrm{sVect}$$

such that the induced inner products $Z(h_Y)\colon Z(Y) \to \overline{Z(Y)}^*$ *on state spaces are positive definite if* $h_Y \in P$*.*

We would now like to highlight the core properties of symmetric monoidal bordism †-category that make the spin-statistics theorem hold. Namely, let $\mathcal{C}$ be a symmetric monoidal category admitting duals equipped with a $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action. The most important examples to keep in mind are

1. $\mathcal{C} = \mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ with the orientation reversal $\mathbb{Z}/2$-action and the spin flip $B\mathbb{Z}/2$-action;
2. $\mathcal{C} = \mathrm{sVect}$ with the complex conjugation $\mathbb{Z}/2$-action and the $B\mathbb{Z}/2$-action given by the grading.

We consider the induced anti-involution $dx := \overline{x}^*$. For the spin-statistics theorem, it turns out to be relevant to know which positivity structures on this anti-involutive category are suitable to make 'doubles of the macaroni' nullbordant. Here, by the macaroni on an object Y^{n-1} of the bordism category, we mean the bordism ev_Y, which looks like $Y \times [0,1]$, but both ends are considered as incoming, see Figure 1.1. The relation between this requirement and the spin-statistics theorem is explained in Remark 6.3.8. Suppose $x \in \mathcal{C}$ is an object equipped with duality data $\mathrm{ev}_x\colon x^* \otimes x \to 1$

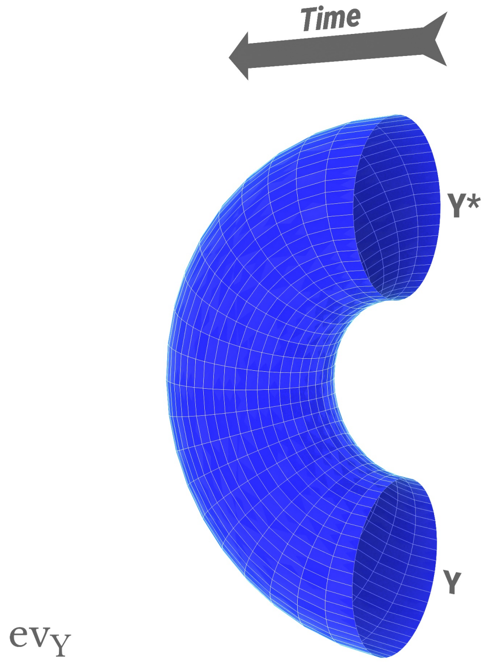

Figure 1.1: The macaroni bordism from the disjoint union of the space Y and its dual Y^*. It is the evaluation map ev_Y realizing Y^* as the dual of Y in the bordism category.

and $\mathrm{coev}_x : 1 \to x \otimes x^*$. If we are given Hermitian pairings $h\colon x \to dx$ and $h^\vee\colon x^* \to d(x^*)$ on both x and x^*, then we can talk about this 'double of the macaroni' $\mathrm{ev}_x \circ \mathrm{ev}_x^\dagger$. We would like this 'double' to behave like the 'bounding torus' $Y \times S^1_{ap}$ in the bordism category, where S^1_{ap} is the circle with the antiperiodic/Neveu-Schwarz spin structure.[5] For this, we need to compare $\mathrm{ev}_x \circ \mathrm{ev}_x^\dagger$ to the closed manifold $Y \times S^1_{per}$, where S^1_{per} is the periodic/Ramond circle S^1_{per}. This manifold can be written as a composition of bordisms $\mathrm{ev}_Y \circ \sigma_{Y,Y^*} \circ \mathrm{coev}_Y$, where σ denotes the symmetric braiding. This composition makes sense in a general symmetric monoidal category and so we want to compare

$$\mathrm{ev}_x \circ \mathrm{ev}_x^\dagger \text{ with } \mathrm{ev}_x \circ \sigma_{x,x^*} \circ \mathrm{coev}_x,$$

and hope they differ by the $B\mathbb{Z}/2$-action. The former is the dimension of x in the pivotal structure

$$(\mathrm{coev}_x^\dagger \otimes \mathrm{id}_{x^{\vee\vee}}) \circ (\mathrm{id}_x \otimes \mathrm{coev}_{x^\vee}),$$

i.e. the trace of the identity morphism. This pivotal structure arises on any monoidal dagger category equipped with a monoidal dual functor $(.)^\vee$ by Theorem 2.8.1 of Dave Penneys [58]. The other notion of dimension uses the canonical pivotal structure induced by the symmetric braiding given by the

[5]In the main text we consider more general geometric structures, such as Pin^+, for which both the periodic and the antiperiodic circle are bounding. Therefore, from the perspective of making doubles bounding, it does not matter whether $\mathrm{ev}_Y \circ \mathrm{ev}_Y^\dagger$ for Y a single Pin^+-point is the periodic or the antiperiodic circle for Pin^+. It is however still important to make the spin-statistics theorem hold.

composition given by (A.2). We refer the reader to Appendix A.1 for a review on duality and pivotal structures in monoidal categories.

One subtlety with the above is that the formula of the pivotal structure through $\mathrm{coev}_x^\dagger$ depends on the choice of dagger dual functor. In this thesis where all relevant dagger categories are constructed as full subcategories of a Hermitian completion through a choice of positivity structure, this manifests itself by the choices of Hermitian pairing h and $h^\vee$ on x and x^* in the codomain of coev_x. Changing these Hermitian pairings can change what the double of a macaroni will be. However, if two Hermitian pairings are related by a transfer, then the result will only change by a positive automorphism of the monoida unit, which suffices for our purposes.

Note that given h, there is a canonical choice h' for $h^\vee$ given by the composition

$$x^* \xrightarrow{h^{*-1}} (dx)^* \cong d(x^*),$$

where for the second isomorphism we used that the symmetric monoidal functor d preserves duals. We will call this Hermitian pairing the *dual of* h, see Definition 2.7.12. We show how to deal with the subtle interaction between dagger and dual functors in Section 2.7. The essential point is that for the dagger categories of fermionic nature in this thesis, we do *not* want to take the canonical choice $h' = h^\vee$, but instead take its composition with $(-1)^F$. This will exactly ensure that the double of the macaroni ev_Y is the 'antiperiodic mapping torus' $Y \times S^1_{ap}$. In the following definition we require the dual of a positive Hermitian pairing composed with $(-1)^F$ to be positive again:

Definition 1.0.5 (Definition 2.9.2)**.** Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category with duals equipped with a monoidal positivity structure P and an anti-involutive $B\mathbb{Z}/2$-action $x \mapsto (-1)^F_x \in \operatorname{Aut} x$ in the sense of Definition 2.6.1. Then $\mathcal{C}_P$ is called *weak fermionically dagger compact* if for all $(h\colon x \to dx) \in P$ we have $h' \circ (-1)^F_x \in P$.

Often we want to require something stronger, namely that the double of *every* possible macaroni is a periodic circle:

Definition 1.0.6. $\mathcal{C}_P$ is called *strong fermionically dagger compact* if for all $(h\colon x \to dx) \in P$ and $(h^\vee\colon x^* \to d(x^*)) \in P$ we have $h' \circ (-1)^F_x = h^\vee$.

The terminology is motivated by the case where the $B\mathbb{Z}/2$-action is trivial, in which case the above definition is closely related to what is usually called a dagger compact category, see Definitions 2.7.23 and 2.7.29. In Section 2.8, we will see that dual functor on a monoidal dagger category which is additionally a monoidal dagger functor, corresponds uniquely to a pivotal structure. The induced notion of dimension is the double of its macaroni

$$\dim_\dagger x = \mathrm{ev}_x \circ \mathrm{ev}_x^\dagger .$$

For a strong fermionically dagger compact category, $\dim_\dagger x$ will differ from $\dim x$ by the automorphism $(-1)^F_x$ of x corresponding to the $B\mathbb{Z}/2$-action up to an *inv-positive morphism* in the sense of Definition 2.3.30, see Section 2.9. For example, in the symmetric monoidal dagger category of super Hilbert spaces $\dim \mathcal{H} = \dim_s \mathcal{H}$ is the superdimension. On the other hand, it has a unique dagger dual functor so that $\dim_\dagger \mathcal{H} = \dim_{ungr} \mathcal{H}$ is the ungraded dimension. In the dagger category $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ it turns out that $\dim Y = Y \times S^1_{per}$, while $\dim_\dagger Y = Y \times S^1_{ap}$. These abstract observations are a convenient way to conceptualize the spin-statistics theorem: suppose $Z\colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal dagger functor between weak fermionically dagger compact categories. Given an object $x \in \mathcal{C}$, we can try and compare $(-1)^F_x$ with $(-1)^F_{Z(x)}$ by comparing $\dim x, \dim_\dagger x, \dim Z(x)$

and $\dim_\dagger Z(x)$. This gives some information about the extent to which Z is $B\mathbb{Z}/2$-equivariant. In fact, when the target $\mathcal{D}$ is sHilb, we show that Z is always $B\mathbb{Z}/2$-equivariant in Corollary 2.9.12. A crucial ingredient to make this work is that sHilb is strong fermionically dagger compact, see Corollary 2.9.14. In particular, when $\mathcal{C} = (\mathrm{Bord}_{n,n-1})_P$ is a weak fermionically dagger compact bordism category, it implies the spin-statistics theorem holds for symmetric monoidal dagger functors from $\mathcal{C}$ to sHilb. With these abstract categorical considerations, it becomes a primary goal to construct orientation-reversal $\mathbb{Z}/2$-actions and Hermitian pairings on fermionic bordism categories that make them weak fermionically †-compact.

We will establish such properties not just for spin bordism categories, but for bordisms equipped with more general G-structures. Motivated by physics, our main focus will be on structure groups $G = G_n(K)$ that are constructed from certain kinds of internal symmetry groups K by a specific construction given in Definition 3.2.1. One can think of $G_n(K)$ roughly as a product

$$G_n(K) \sim K \times SO_n$$

of K with the Euclidean signature Lorentz group SO_n. In general both time-reversal and fermions will be allowed in the symmetry group, which makes the construction of $G_n(K)$ more like a graded tensor product. We formalize the situation by asking K to be a fermionic group in the sense of Definition 3.1.1: it comes equipped with a group homomorphism $\theta\colon K \to \mathbb{Z}/2$ and a distinguished central element $c \in K$ which squares to one, such that $\theta(c) = 0$. The element c should be thought of as an abstraction of the spin flip $c \in \mathrm{Spin}_n$ and so the spin-statistics connection for a topological field theory with $G_n(K)$-structure requires c to act by $(-1)^F$ on state spaces. Our hope is that the theory of fermionic groups we develop in Chapter 3.1 will be useful more generally in quantum physics, for example for time-reversal symmetries of symmetry-protected topological phases in condensed matter, or in the study of ‘t Hooft anomalies. All results above generalize to bordism categories in which all manifolds $\mathrm{Bord}_{n,n-1}^{G(K)}$ are equipped with $G_n(K)$-structures defined in Section 5.1, in particular the relationship between dagger functors and $\mathbb{Z}/2$-equivariant functors of Corollary 1.0.4. For any fermionic internal symmetry group K we identify a specific positivity structure P on the bordism category $\mathrm{Bord}_{n,n-1}^{G_n(K)}$, so that we obtain a construction of a symmetric monoidal dagger structure on it in Definition 5.2.18. This positivity structure makes this bordism category weak fermionically dagger compact, as we show in Theorem 5.2.25.

Definition 1.0.7 (Definition 6.2.4)**.** A *unitary (fermionic) topological field theory with internal fermionic symmetry group* K is a dagger functor $(\mathrm{Bord}_{n,n-1}^{G_n(K)})_P \to \mathrm{sHilb}_{\mathbb{C}}$.

The main theorem of this thesis is the spin-statistics theorem for fermionic topological field theories:

Theorem 1.0.8 (Theorem 6.3.5)**.** *Every unitary fermionic topological field theory with internal fermioinic symmetry group K satisfies*

$$Z(Y_c) = (-1)^F_{Z(Y)}.$$

Our proof in Section 6.3 is a direct consequence of the fact that $(\mathrm{Bord}_{n,n-1}^{G_n(K)})_P$ is weak fermionically dagger-compact, see Corollary 2.9.12 and Theorem 5.2.25. We are unsure whether $(\mathrm{Bord}_{n,n-1}^{G_n(K)})_P$ is strong fermionically dagger-compact.

We also prove the following result, which should be familiar to physicists.

Proposition 1.0.9 (Proposition 6.3.13)**.** *Let $Z\colon \mathrm{Bord}_{n,n-1}^{G_n(K)} \to \mathrm{sHilb}_{\mathbb{C}}$ be a unitary topological field theory with symmetry K and let Y^{n-1} be a time slice with trivial principal $G_n(K)$-bundle. Then the super Hilbert space $Z(Y)$ comes equipped with a representation of K for which $(-1)^F$ acts by the grading and $k \in K$ acts unitarily when $\theta(k) = 0$ and anti-unitarily when $\theta(k) = 1$.*

We emphasize that mathematically the above result is somewhat surprising, as it strongly relies on unitarity through the spin-statistics theorem. Namely, for a Hermitian topological field theory with internal fermionic symmetry group K, there is still a collection of (anti-)unitary operators $\rho(k)\colon Z(Y) \to Z(Y)$ on state spaces for internal symmetries $k \in K$, but if $k_1, k_2 \in K$ are time-reversing then

$$\rho(k_1)\rho(k_2) = (-1)^F_{Z(Y)}\rho(c)\rho(k_1k_2).$$

1.1 Outlook

In this thesis, we have argued that unitarity of topological field theories is most naturally formulated by enhancing symmetric monoidal functors to symmetric monoidal dagger functors. On the other hand, it has been increasingly appreciated in recent years that fully extended topological field theories are mathematically the best behaved, because of the cobordism hypothesis, and physically the most relevant, because of locality in quantum field theory. Since fully extended topological field theories are symmetric monoidal functors between symmetric monoidal (∞, n)-categories, this motivates the quest to understand dagger (∞, n)-categories. For this purpose, a workshop was organized in June 2023, in which several definitions were proposed in a similar spirit to the approach using Hermitian pairings, introduced in [68] and applied in this thesis. A short report on our findings is expected to appear in the near future [6].

One insight of the workshop was that so-called pivotal structures on 2-categories are special cases of higher dagger structures, while the corresponding higher anti-involutive structure only gives a so-called weak pivotal structure. For example, if the 2-category is a monoidal category $\mathcal{C}$ considered as a 2-category $B\mathcal{C}$ with one object, then a pivotal structure $B\mathcal{C}$ is a pivotal structure on $\mathcal{C}$ in the usual sense. A weak pivotal structure on the other hand, is an isomorphism $c \cong \theta_c \otimes c^{**} \otimes \theta_c^{-1}$ satisfying some conditions, where $\theta_c \in \mathcal{C}$ is an invertible object. We expect that using the relationship between dagger dual functors and unitary pivotal structures on dagger categories, these observations will give a higher dagger-categorical interpretation for the condition of being strong fermionically dagger-compact.

The report is expected to include a generalization of the construction of the bordism dagger category $\mathrm{Bord}_{n,n-1}$ of Definition 5.2.18 to the $(\infty, 1)$-setting. In other words, the anti-involution is defined using the involution $Y \mapsto Y'$ through a single stabilization of the structure group and the construction of Hermitian pairings involves a rotation by π in the plane spanned by the nth coordinate (time) and the $(n+1)$th coordinate (the extra stabilization). We expect such constructions to generalize to a higher dagger categorical structure on the (∞, n)-category Bord_n, when the structure group is fully stable. This will solve the open question of defining unitary extended topological field theories.

However, in general we do not know the answer to the question what the right target higher dagger category is, a question that is already subtle in the non-unitary setting. There is a reasonable amount of evidence that the suitable target for once extended topological field theories is the bicategory $\mathrm{sAlg}_{\mathbb{C}}$ of superalgebras. Suggestions for constructions of universal targets for extended topological field theories generalizing sVect in category level one and $\mathrm{sAlg}_{\mathbb{C}}$ in category level two, are work in progress independently by Freed–Scheimbauer–Teleman and Johnson-Freyd–Reutter. For once extended unitary topological field theory, we therefore have to look for a suitable dagger bicategory of superalgebras. We have been exploring this proposal for two-dimensional topological field theories with reflection structure (which should be equivalent to anti-involutive categories without positivity structure) in [53], together with Lukas Müller. In joint work, we aim to include Hermitian pairings in this discussion, in order to study once extended unitary topological field theory. For some indication of how this would work, note that the bicategory of superalgebras has an anti-involution $A \mapsto \overline{A}^{\mathrm{op}}$ analogous to the anti-involution $V \mapsto \overline{V}^*$ on finite-dimensional super vector spaces. Hermitian pairings $A \cong \overline{A}^{\mathrm{op}}$ for this anti-involution are stellar algebras, the Morita-invariant analogue of $*$-algebras, see [53, section 5.2]. From the perspective of algebraic quantum mechanics, it is unsurprising that $*$-algebras come up in defining unitarity for once extended topological field theories. Given two stellar algebras A, B, there is an induced anti-involution on the hom-categories[6] for which the Hermitian pairings are stellar bimodules. These are a purely algebraic, not necessarily positive definite analogue of Hilbert bimodules between C^*-algebras. To make this anti-involutive bicategory into a dagger bicategory, we have to choose a positivity structure, which now amounts to Hermitian pairings both on objects and on 1-morphisms. For example, we could take those stellar algebras that are induced by C^*-algebras and those stellar bimodules that are induced by Hilbert bimodules. Further research is required to discover which of these positivity structures on this anti-involutive bicategory recover expected results, such as the spin-statistics theorem for once extended unitary topological field theories.

Another result that definitions of extended unitary topological field theories can be compared to, is the classification of unitary invertible field theories by bordism groups [20, Theorem 1.1]. We will now compare our analysis of the 1-categorical properties required for the spin-statistics theorem with what is required for this classification, namely that doubles are nullbordant. First, we emphasize that even though folklore in physics tells us that invertible topological field theories are classified by bordism groups, this is not quite the case for non-unitary theories, see [45]:

Theorem 1.1.1. *Taking the partition function of a functor*

$$\mathrm{Bord}_{n,n-1}^{G_n(K)} \to \mathrm{sLine}$$

to the category of super lines (one-dimensional super vector spaces) induces an isomorphism between equivalence classes of such functors and homomorphisms

$$SKK_n^{G_n(K)} \to \mathbb{C}^\times,$$

from the controlled cut and paste group of manifolds with $G_n(K)$-structure to $\mathbb{C}^\times$.

Under minor assumptions that are satisfied for the structure groups $G_n(K)$, the controlled cut and paste groups are closely related to bordism groups, but they are not isomorphic. A typical example of an invertible topological field theory that is not a bordism invariant is an Euler theory corresponding to a fixed nonzero complex number λ, which has partition function

$$Z(X) = \lambda^{\chi(X)}.$$

However, under the assumption of unitarity these unstable invariants disappear.

[6] Actually this is only a lax involution in the sense of Remark 2.3.13 unless we are willing to restrict to invertible bimodules.

Theorem 1.1.2 ([77, 44]). *Isomorphism classes of nonextended unitary invertible topological field theories with target superlines are given by homomorphisms*

$$\mathrm{Hom}(\Omega_n^{G_n(K)}, U(1)),$$

if n is odd and

$$\mathrm{Hom}(\Omega_n^{G_n(K)}, U(1)) \times \mathbb{R}_{>0}$$

otherwise.

The above is a nonextended analogue of the main result of Freed and Hopkins [20].[7] In future work, we aim to study Hermitian and unitary invertible topological field theories by applying the theory of Hermitian pairings to the case where the symmetric monoidal category is a Picard groupoid. Our analysis is preceded by a completely algebraic characterization of dagger Picard groupoids (and † 2-groups) and symmetric monoidal dagger functors between them, in the spirit of Sính [67]. This, in particular, recovers the work in progress [44].

The crucial input to theorems stating that symmetric monoidal dagger functors from a bordism category to the dagger category of one-dimensional super Hilbert spaces are given by bordism invariants, is that the double of a manifold with boundary is nullbordant. Note that this is physically very surprising: to know when two closed n-dimensional spacetimes are bordant you need to know something about $(n+1)$-dimensional manifolds, even though your quantum field theory is of dimension n. In particular, when manifolds come equipped with extra geometry, there might be a choice how to extend your notion of geometry to $(n+1)$-manifolds. For example, the structure of an orientation and a double cover is the same as a spin structure in spacetime dimension one, but not in spacetime dimension two. Even though in both scenarios the one-dimensional bordism group is $\mathbb{Z}/2$ generated by a circle, which circle is nullbordant is different. This is yet another way to argue that the Hermitian pairings we have to pick for the one-dimensional structure groups Spin_1 and $SO_1 \times \mathbb{Z}/2$ must be different, even though their bordism categories are identical, see Example 5.2.27.

We prefer to think of doubles being nullbordant as a remnant of higher categorical structure as follows. Let $X^n \colon \emptyset \to Y^{n-1}$ be a morphism in a bordism category $\mathrm{Bord}_{n,n-1}$ on which we have constructed a dagger, for example by requiring a preferred Hermitian pairing on Y. Recall that the double of X is defined as the endomorphism of $\emptyset \in \mathrm{Bord}_{n,n-1}$ given by $X^\dagger \circ X$. A nullbordism of the double is a 2-morphism $W \colon X^\dagger \circ X \to \emptyset$ in the bordism bicategory $\mathrm{Bord}_{n+1,n,n-1}$. We can think of W in two ways. On the one hand W provides the 'unitarity data', specifying that X is 'weakly' unitary, or rather an isometry. On the other hand, we can note that † specifies an involution on the bordism bicategory $\mathrm{Bord}_{n+1,n,n-1}$, which is the identity on objects, but not on 1- and 2-morphisms. In that sense, † behaves more like a weak involution between hom-categories. The 2-morphism W gives us a Hermitian pairing on the 1-morphism X, which allows us to define a † 2-functor, which is additionally the identity on 1-morphisms. For this, we will need W to be a nondegenerate pairing in a suitable sense. We hope that this description of the nullbordism of the double is amenable to higher-categorical generalizations in the juvenile, but exciting field of higher dagger categories. Unlike for the spin-statistics theorem, we do not claim to have a more conceptual description of the 1-categorical remnants that the above structure gives us on the dagger category $\mathrm{Bord}_{n,n-1}$.

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[7] Freed-Hopkins moreover mainly work with 'topological*' theories (target spectrum the Anderson dual of the sphere), while we work with 'discrete theories' (target the Brown-Commenetz dual of the sphere).

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Chapter 2

Involutions on categories

In the introduction we have seen that in order to understand Hermitian and unitary topological field theories, we have to understand $\mathbb{Z}/2$-equivariant functors between certain categories with involution or dagger functors between dagger categories. With this goal in mind we first study the abstract theory of symmetric monoidal categories with different notions of involution and their relationship with symmetric monoidal dagger categories. We start with the easiest example of the type of category we will be interested in, in which most of the main features are already present: the category of super vector spaces.

2.1 Hermitian forms on super vector spaces

In this section, we will study in detail the notions of duality, complex conjugation and Hermitian pairings in sVect, the symmetric monoidal category of finite-dimensional super vector spaces together with grading-preserving (even) linear maps. We will use the convention that "super vector spaces" refers to vector spaces with $\mathbb{Z}/2$-grading $V = V_0 \oplus V_1$, in which the braiding is given by the Koszul sign rule, while "$\mathbb{Z}/2$-graded vector spaces" refers to the same monoidal category with trivial braiding $v \otimes w \mapsto w \otimes v$. If W is a vector space, we denote the associated purely even super vector space again by W and the purely odd super vector space by ΠW.

The dual V^* of a super vector space V is given by all linear functionals $V \to \mathbb{C}$ decomposed into even maps $V_0 \to \mathbb{C}$ and odd maps $V_1 \to \mathbb{C}$. If V is finite-dimensional, this is a dual in the categorical sense as we now describe, see Appendix A.1 for a brief review of duality in monoidal categories. The evaluation map $V^* \otimes V \to \mathbb{C}$ is given by evaluating functionals, which is an even map. Since V is finite-dimensional, the triangle identities hold for the coevaluation map $\mathbb{C} \to V^* \otimes V$ which sends 1 to the finite sum

$$\sum_i \epsilon^i \otimes e_i$$

where $\{e_i\}$ is a basis of V with dual basis $\{\epsilon^i\}$. Using that sVect is symmetric, we get a canonical spherical pivotal structure $\Phi_V : V \to V^{**}$, which expresses that both V and V^{**} are duals of V^*. One can check that it is given by $\Phi_V(v)(f) = (-1)^{|v||f|} f(v)$, for example using formula (A.2). In particular, the trace associated to the pivotal structure is the supertrace $f \mapsto \mathrm{tr}_{V_0} f - \mathrm{tr}_{V_1} f$. The reader may refer to Appendix A.1 for a review on pivotal structures, sphericality and traces in them.

If V is a complex super vector space, we denote the *complex conjugate super vector space* by $\overline{V} = \{\bar{v} : v \in V\}$. This vector space is equal to $(V, +)$ as an abelian group, but with scalar multiplication $z\bar{v} := \overline{\bar{z}v}$. The operation $V \mapsto \overline{V}$ defines a functor if we set $\overline{T}(\bar{v}) := \overline{T(v)}$ for a linear map $T : V \to W$. Given our definitions, we have that $\overline{\overline{V}} = V$ and $\overline{V \otimes W} = \overline{V} \otimes \overline{W}$ on the nose. It is straightforward to verify that this is a symmetric monoidal $\mathbb{Z}/2$-action on sVect in the sense of Appendix A.3. Analogous statements are true for Vect. There is room to change these $\mathbb{Z}/2$-actions by introducing several signs and we will comment on these near the end of this section. This is not merely an act of self-flagellation; the sign subtleties in fermionic topological field theory are unforgiving.

Since $V \mapsto \overline{V}$ is monoidal, it preserves duals. Using the formula for uniqueness of duals (A.1), one can compute that the induced isomorphism $\phi : \overline{V^*} \cong \overline{V}^*$ is the canonical one:

$$\phi(\bar{f})(\bar{v}) = \overline{f(v)}.$$

From now on we typically identify $\overline{V^*}$ and $\overline{V}^*$ directly.

There are two perspectives on sesquilinear forms on a complex vector space V; a sesquilinear map $V \times V \to \mathbb{C}$ or a complex-linear map $\overline{V} \to V^*$ (which is equivalent to a complex anti-linear map $V \to V^*$ and a complex-linear map $V \to \overline{V}^*$). We develop both perspectives for super vector spaces here.

Definition 2.1.1. Let V be a super vector space. A *sesquilinear* form on V is an even linear map $h : V \to \overline{V}^*$. It is called *nondegenerate* if h is an isomorphism. A *(super) Hermitian form* is a nondegenerate sesquilinear form that is symmetric in the sense that

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d] & \overline{V}^* \\ \overline{\overline{V}^*}^* \arrow[ur, "\overline{h}^*"'] & \end{tikzcd}$$

commutes. Here the vertical arrow uses the canonical isomorphisms $V \to V^{**}$, $\overline{\overline{V}} \cong V$ and $\overline{V^*} \cong \overline{V}^*$.

Remark 2.1.2. In the above convention, the induced pairing

$$\langle v, w\rangle = h(v)(\bar{w})$$

will be conjugate linear in the right variable. Since we prefer to use inner products that are conjugate linear in the left variable we often implicitly apply the functor $V \mapsto \overline{V}$ to h in the rest of this section. However, the convention in the above definition will be more convenient for the abstract framework of Hermitian forms that will be developed in the coming sections.

Note that our definition of a nondegenerate pairing $V \times W \to \mathbb{C}$ does not extend well to infinite-dimensional vector spaces. The reason is that we also require for all $f \in W^*$ the existence of an element $v \in V$ such that pairing with v gives f. This is not much of a loss because we will only study topological field theories in this thesis. The reader is referred to Remarks 2.3.13 and 2.3.14 for a brief discussion on how to deal with the dagger category of infinite-dimensional Hilbert spaces in the context of this thesis.

Proposition 2.1.3. *A sesquilinear form h on V is equivalent to a sesquilinear pairing $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ such that*

$$\langle v, w\rangle = 0$$

if v, w are homogeneous of different degree. Moreover, h is nondegenerate if and only if the pairing is nondegenerate. It is Hermitian if and only if

$$\langle v, w\rangle = (-1)^{|v||w|}\overline{\langle w, v\rangle}$$

for all $v, w \in V$ homogeneous.

Proof. Given $h : \overline{V} \to V^*$, define the pairing

$$\langle v, w\rangle = h(\bar{v})(w).$$

Note that h is an even map if and only if $V_0 \perp V_1$. Clearly h is bijective if and only if $\langle \cdot, \cdot\rangle$ is nondegenerate. For the complex-conjugate symmetry, we compute for $v, w \in V$ homogeneous that

$$h^*(\Phi_V(v))(\bar{w}) = \Phi_V(v)(h(\bar{w})) = (-1)^{|v||w|}h(\bar{w})(v) = (-1)^{|v||w|}\langle w, v\rangle,$$

while under the isomorphism $\overline{V^*} \cong \overline{V}^*$, the element $\overline{h(\bar{v})} \in \overline{V^*}$ is mapped to the functional $\overline{V} \to \mathbb{C}$ given as

$$\overline{w} \mapsto \overline{h(\bar{v})(w)} = \overline{\langle v, w\rangle}.$$

So $\bar{h}$ is equal to the required composition if and only if the pairing satisfies

$$\langle v, w\rangle = (-1)^{|v||w|}\overline{\langle w, v\rangle}$$

for all $v, w \in V$ homogeneous. □

Note that if h is a Hermitian form and $v \in V$ is homogeneous, then

$$\langle v, v\rangle = (-1)^{|v|}\overline{\langle v, v\rangle}$$

is real when v is even and imaginary when v is odd. In defining positive definiteness, we decide to call imaginary numbers of the form ai where $a \in \mathbb{R}_{>0}$ positive:

Definition 2.1.4. A Hermitian form is *positive (definite)* if for all even $v \in V$

$$\langle v, v\rangle \geq 0$$

and for all odd $v \in V$

$$\frac{\langle v, v\rangle}{i} \geq 0.$$

We also say that (V, h) is a (finite-dimensional) super Hilbert space. More generally, the sign of $\langle v, v\rangle$ either as a real or imaginary number will be refered to as the *inner product sign.*

Let $\{e_1, \dots, e_n\} \subset V$ be a graded basis with dual basis $\{\epsilon^1, \dots, \epsilon^n\}$. We can decompose h in components as $h(\bar{e}_i)(e_j) = h_{ij}$ so that $h_{ji} = (-1)^{|e_i||e_j|}\overline{h_{ij}}$. Note that $h_{ij} = 0$ if e_i and e_j are of different degree. In matrix notation

$$h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A is self-adjoint and B is skew-adjoint. The form h is positive if and only if A and $\frac{B}{i}$ are positive matrices in the ordinary sense.

Definition 2.1.5. An *orthonormal basis* with respect to a super Hermitian form on V is a homogeneous basis $(e_1, \ldots, e_n) \subseteq V$ such that

$$\langle e_i, e_j \rangle = \delta_{ij} \langle e_j, e_j \rangle,$$

where $\langle e_j, e_j \rangle \in \{\pm 1, \pm i\}$ will be called the *norm* of the orthonormal vector. The quadruple (p_1, p_2, p_3, p_4) containing the number of js for which the norm $\langle e_j, e_j \rangle$ are respectively $+1, -1, +i$ and $-i$ is called the *signature* of the form.

Note that the signature is well-defined. Unless stated otherwise, we will assume the basis has been ordered so that the first p_1 elements have norm $+1$, the p_2 after have norm -1, the next p_3 have norm i and the final p_4 vectors have norm $-i$. In particular $e_1, \ldots, e_{p_1+p_2}$ are even and $e_{p_1+p_2+1}, \ldots, e_n$ are odd. It is easy to show by a Gram-Schmidt process that orthonormal bases exist and that the signature is well-defined.

Proposition 2.1.6. *Let (V, h) be a super Hermitian vector space with signature (p_1, p_2, p_3, p_4). Then V^* is a super Hermitian vector space by $(h^*)^{-1} : \overline{V^*} \to V^{**}$. It has signature (p_1, p_2, p_4, p_3).*

Proof. This follows by a computation in coordinates. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis and denote the dual basis by $\{\epsilon^1, \ldots, \epsilon^n\}$, so that $\epsilon^i(e_j) = \delta^i_j$. The claim is that $\{\epsilon^1, \ldots, \epsilon^n\}$ is again an orthonormal basis. We have to compute

$$\langle \epsilon^i, \epsilon^j \rangle_{V^*} = (h^*)^{-1}(\overline{\epsilon^i})(\epsilon^j) = \overline{\epsilon^i}(h^{-1}(\epsilon^j)).$$

Now note that $h^{-1}(\epsilon^j) = \langle e_j, e_j \rangle^{-1} \overline{e_j}$ because

$$h(\langle e_j, e_j \rangle^{-1} \overline{e_j})(e_i) = \langle \overline{\langle e_j, e_j \rangle^{-1}} e_j, e_i \rangle = \langle e_j, e_j \rangle^{-1} \langle e_j, e_i \rangle = \delta_{ij}.$$

Therefore we can now compute

$$\langle \epsilon^i, \epsilon^j \rangle_{V^*} = \overline{\epsilon^i}(\langle e_j, e_j \rangle^{-1} \overline{e_j}) = \langle e_j, e_j \rangle^{-1} \delta^i_j.$$

Since $(\pm i)^{-1} = \mp i$ and $(\pm 1)^{-1} = \pm 1$ this ends the proof. □

Remark 2.1.7. Given a super Hermitian vector space $(V, \langle \cdot, \cdot \rangle)$, one can define an ordinary Hermitian pairing on V by

$$(v, w) := \begin{cases} \langle v, w \rangle & v \text{ even}, \\ \frac{\langle v, w \rangle}{i} & v \text{ odd}. \end{cases}$$

This will give a bijection between super Hermitian forms on V and ordinary Hermitian forms on V such that $V_0 \perp V_1$. Moreover, this bijection preserves positivity and so super Hilbert spaces as in this document are in bijection with $\mathbb{Z}/2$-graded Hilbert spaces in the usual sense:

$$\langle v, w \rangle = \overline{\langle w, v \rangle}.$$

In practice, it is often conceptually easier to work with $\mathbb{Z}/2$-graded Hilbert spaces, but when working in super vector spaces it is less natural from a categorical perspective. Namely, their symmetry property violates the Koszul sign rule and so is difficult to unify with the usual nontrivial braiding on the category of super vector spaces at first glance. However, there are ways to fix this as we will discuss at the end of the section.

Definition 2.1.8. let $T : (V, \langle ., . \rangle_V) \to (W, \langle ., . \rangle_W)$ be a complex-linear map of homogeneous degree between super Hermitian vector spaces. Then the *Hermitian adjoint* $T^\dagger : (W, \langle ., . \rangle_W) \to (V, \langle ., . \rangle_V)$ is defined to be the unique complex-linear map such that

$$\langle Tv, w \rangle_W = (-1)^{|T||v|} \langle v, T^\dagger w \rangle_V$$

for all homogeneous $v \in V$ and $w \in W$.

The proof that Hermitian adjoints exist and are unique is the same as in the ungraded case. In fact, if $T^\ddagger : (W, (., .)_W) \to (V, (., .)_V)$ denotes the ungraded Hermitian adjoint of the corresponding ungraded Hermitian forms on V and W under the remark above, then by a straightforward computation it follows that

$$T^\dagger = \begin{cases} T^\ddagger & T \text{ even,} \\ iT^\ddagger & T \text{ odd.} \end{cases}$$

It is easy to show that $T^{\dagger\dagger} = T$ and $(T_1 T_2)^\dagger = (-1)^{|T_1||T_2|} T_2^\dagger T_1^\dagger$, i.e. the superalgebra $\mathrm{End}(V)$ becomes a super $*$-algebra. Note that for the usual ungraded definition of Hermitian adjoints, we would have $(T_1 T_2)^\ddagger = T_2^\ddagger T_1^\ddagger$ without the sign. In other words, this would define what will be called a $\mathbb{Z}/2$-graded $*$-algebra instead, in which the law $(ab)^* = b^* a^*$ violates the Koszul sign rule. To see what $\dagger$ looks like concretely in coordinates, we consider the following proposition.

Proposition 2.1.9. *Let $(V, \langle ., . \rangle_V)$ and $(W, \langle ., . \rangle_W)$ be super Hermitian vector spaces and let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ be orthonormal bases. Denote elementary matrices by E^i_α for $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$, so $E^i_\alpha e_j = \delta^i_j f_\alpha$. Then $(E^i_\alpha)^\dagger = p_{i,\alpha} E^\alpha_i$ where $p_{i,\alpha} \in \{\pm 1, \pm i\}$. Moreover, $p_{i,\alpha}$ is real if and only if e_i and f_α have the same degree and $p_{i,\alpha}$ has a $+$-sign if and only if e_i and f_α have the same inner product sign.*

Proof. Consider the computations

$$\langle E^j_\beta e_i, f_\alpha \rangle_W = \delta^j_i \langle f_\beta, f_\alpha \rangle_W = \delta^j_i \delta_{\beta\alpha} \langle f_\beta, f_\beta \rangle_W$$

and

$$(-1)^{|e_i||E^\beta_j|} \langle e_i, p_{j,\beta} E^\beta_j f_\alpha \rangle_V = (-1)^{|e_i||E^\beta_j|} \delta^\beta_\alpha \delta_{ij} p_{j,\beta} \langle e_i, e_j \rangle_V = (-1)^{|e_j||E^\beta_j|} \delta^\beta_\alpha \delta_{ij} p_{j,\beta} \langle e_j, e_j \rangle_V.$$

If these are to be equal, then

$$p_{j,\beta} = \frac{\langle f_\beta, f_\beta \rangle_W}{\langle e_j, e_j \rangle_V} (-1)^{|e_j||E^\beta_j|}.$$

We now check all cases. If f_β and e_j have the same degree, E^β_j is even and $p_{j,\beta}$ is indeed the difference in inner product sign between f_β and e_j. If f_β and e_j are of different degree, E^β_j is odd. Suppose e_j is even. Then the sign is $+1$ and hence

$$p_{j,\beta} = \frac{\langle f_\beta, f_\beta \rangle_W}{\langle e_j, e_j \rangle_V}$$

is i times the difference in inner product sign as desired. If instead e_m is odd, then the sign is -1, which cancels the sign coming from the fact that the imaginary unit is now in the denominator. $\square$

From the proposition it follows that for any Hermitian form $\langle\cdot,\cdot\rangle_V$ on V, the induced super $*$-algebra on $\operatorname{End} V$ becomes the following $*$-structure on $M_n(\mathbb{C})$ after a choice of orthonormal basis $\{e_1,\dots,e_{p+q}\}$:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}^{\dagger} = \begin{pmatrix} A_{11}^{\dagger} & -A_{21}^{\dagger} & iA_{31}^{\dagger} & -iA_{41}^{\dagger} \\ -A_{12}^{\dagger} & A_{22}^{\dagger} & -iA_{32}^{\dagger} & iA_{42}^{\dagger} \\ iA_{13}^{\dagger} & -iA_{23}^{\dagger} & A_{33}^{\dagger} & -A_{43}^{\dagger} \\ -iA_{14}^{\dagger} & iA_{24}^{\dagger} & -A_{34}^{\dagger} & A_{44}^{\dagger} \end{pmatrix}$$

where on the right hand side $\dagger$ denotes the usual Hermitian adjoint of (possibly nonsquare) matrices.

We briefly put some of the above analysis in a more abstract language that will be used in the coming sections, also to facilitate the discussion of sign conventions. For this, note that the contavariant functor $dV := \overline{V}^*$ squares to the identity in the following sense. There is a natural map $\eta_V : V \to d^2V$ induced by ϕ and Φ and it is straightforward to check that η_{dV} and $d\eta_V$ are inverses if V is finite-dimensional. One can then rephrase the notion of a Hermitian form as an isomorphism $h : V \to dV$ such that $dh \circ \eta_V$ is the inverse of h. This motivates the abstract notion of a Hermitian pairing for what we will call a category with anti-involution, as will be discussed in Section 2.3, see Definition 2.3.4. Note that d naturally reverses the direction of the tensor product, because $V \mapsto \overline{V}$ is monoidal and $V \mapsto V^*$ reverses the direction. Here we used the fact uniqueness of duals to get the isomorphism $V^* \otimes W^* \cong (W \otimes V)^*$. It is explicitly given by the composition

$$W^* \otimes V^* \xrightarrow{\mathrm{coev}_{V\otimes W}} W^* \otimes V^* \otimes V \otimes W \otimes (V \otimes W)^* \xrightarrow{\mathrm{ev}_V} W^* \otimes W \otimes (V \otimes W)^* \xrightarrow{\mathrm{coev_W}} (V \otimes W)^*$$

which in this case yields the isomorphism

$$V^* \otimes W^* \cong (W \otimes V)^* \quad f \otimes g \mapsto (v \otimes w \mapsto f(v)g(w)).$$

We warn the reader that depending on context, this isomorphism can violate the Koszul sign rule. We will use the braiding to make $d : \mathrm{sVect} \to \mathrm{sVect}^{\mathrm{op}}$ monoidal, which will introduce signs in some formulas:

Definition 2.1.10. The tensor product of super Hermitian vector spaces $(V_1, h_1), (V_2, h_2)$ has Hermitian pairing

$$\langle v_1 \otimes v_2, w_1 \otimes w_2\rangle := (-1)^{|v_2||w_1|}\langle v_1, v_2\rangle\langle w_1, w_2\rangle.$$

In terms of maps $V \to dV$, this tensor product corresponds to the composition

$$V_1 \otimes V_2 \xrightarrow{h_1 \otimes h_2} dV_1 \otimes dV_2 \cong d(V_2 \otimes V_1) \cong d(V_1 \otimes V_2).$$

This formula motivates the more general definitions of tensor products of Hermitian pairings in monoidal categories with anti-involution, which we will consider in Section 2.4.

Example 2.1.11. Somewhat surprisingly, the tensor product of an odd Hermitian superline with itself is the positive definite even line. More generally, the tensor product of two super Hilbert spaces is again a super Hilbert space.

We now discuss ways to modify the $\mathbb{Z}/2$-action $V \mapsto \overline{V}$ and anti-involution $V \mapsto dV$ by natural automorphisms of sVect. The following lemma is easy to prove.

Lemma 2.1.12. *Natural automorphisms of the identity in* Vect *are classified by $\mathbb{C}^\times$, where the transformation corresponding to $\lambda \in \mathbb{C}^\times$ is given by mapping V to $\lambda\, \mathrm{id}_V$. Only the identity natural automorphism is monoidal. Natural automorphisms of the identity in* sVect *are classified by a pair $(\lambda_0, \lambda_1) \in \mathbb{C}^\times \times \mathbb{C}^\times$ given by multiplying with λ_0 on the even part and λ_1 on the odd part. Let $\lambda_V^F : V \to V$ for $\lambda \in \mathbb{C}^\times$ denote the natural automorphism of* $\mathrm{id}_{\mathrm{sVect}}$ *given by multiplying by λ on the odd part. Only the identity natural automorphism and $(-1)^F$ are monoidal.*

One way to change our notions of Hermitian form on a super vector space is to change the anti-involution (d, η) on sVect. Using the above lemma we will now contemplate our options.

Remark 2.1.13. Let us try and modify the natural isomorphism $V \cong d^2V$ with a natural automorphism of the identity. It is easy to verify that changing this isomorphism with (λ_0, λ_1) as in Lemma 2.1.12 will result in nondegenerate sesquilinear pairings $V \times V \to \mathbb{C}$ satisfying

$$\langle v_1, v_2 \rangle = \lambda_0 \overline{\langle v_2, v_1 \rangle}$$

if v_1 and v_2 are even and

$$\langle v_1, v_2 \rangle = -\lambda_1 \overline{\langle v_2, v_1 \rangle}$$

if v_1 and v_2 are odd. In particular, changing $d^2V \cong V$ by $(-1)^F$ results exactly in the usual notion of $\mathbb{Z}/2$-graded Hermitian forms. The bijection explained in Remark 2.1.7 identifies these two conventions. For a more precise statement and generalizations see Section 2.6. If we change $V \cong d^2V$ by any other natural automorphism, it will no longer be monoidal by Lemma 2.1.12. Therefore the tensor product of two such 'twisted' notions of Hermitian pairing will not be a 'twisted' Hermitian pairing of the same type. This is undesirable if we want Hermitian pairings of a fixed twist to assemble into a monoidal category.

Remark 2.1.14. If we change the natural isomorphism $d(V \otimes W) \cong dW \otimes dV$ specifying how d is monoidal, we do not change the notion of Hermitian pairing, but we do change the notion of tensor product. This is at first sight a very bad idea. For example, changing the isomorphism by $v \otimes w \mapsto (-1)^{|v||w|} v \otimes w$, we get that the tensor square of the odd line $\Pi\mathbb{C}$ with any of its two Hermitian pairings is the even line with a negative definite Hermitian inner product. Positive definite not being closed under tensor product is undesirable for the definition of unitary topological field theory.

However, suppose we have already changed $d^2V \cong V$ by a $(-1)^F$ so that Hermitian pairings will correspond to $\mathbb{Z}/2$-graded Hermitian vector spaces as discussed in the previous remark. In order to make sure that the tensor product of $\mathbb{Z}/2$-graded Hilbert spaces is again a Hilbert space, we then want to define the tensor product of Hermitian forms as

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle.$$

unlike in Definition 2.1.10. To achieve this we deduce that we also need to change the isomorphism $d(V \otimes W) \cong dW \otimes dV$. We conclude that to obtain the monoidal category of finite-dimensional $\mathbb{Z}/2$-graded Hilbert spaces in the usual sense, we need to change $d^2V \cong V$ by a sign to get the ungraded symmetric notion of a pairing and then additionally change $d(V \otimes W) \cong dW \otimes dV$ by $v \otimes w \mapsto (-1)^{|v||w|} v \otimes w$ to get the correct monoidal structure.

Remark 2.1.15. We translate the considerations above into modifying the natural isomorphisms associated to the covariant symmetric monoidal $\mathbb{Z}/2$-action $V \mapsto \overline{V}$. Similarly to the contravariant case, we can change the natural isomorphisms $V \cong \overline{\overline{V}}$ and $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$. There is however one subtlety in comparing these modifications with modifying the anti-involution d: if we change the isomorphism $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$, we also modify the data $\overline{V}^* \cong \overline{V^*}$ saying that $V \mapsto \overline{V}$ preserves duals. It is straightforward to verify that changing the former isomorphism by $v \otimes w \mapsto (-1)^{|v||w|} v \otimes w$ changes the latter by $(-1)^F$. Consequently, just changing the monoidal data of the complex conjugation functor and not the squaring-to-one isomorphism will result in changing both the monoidal data of d and the isomorphism $d^2 V \cong V$. In particular, if we want to arrive at the convention in which Hermitian pairings will be ungraded symmetric, we can either change $V \cong \overline{\overline{V}}$ or $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$ but not both. If we then additionally want the category of Hilbert spaces to be closed under tensor product, we will need to change only $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$ and not $V \cong \overline{\overline{V}}$.[1] Suitable adaptations to the theory have to be made in this convention. For example, in Proposition 2.1.6 we implicitly used the isomorphism $\overline{V}^* \cong \overline{V^*}$. Because of this, when we take the category-theoretic considerations seriously, we again get that the signature of the Hermitian pairing on the dual vector space has its signature negated on the odd part in this convention.

We will study the sign choices coming from the above remarks in more detail and higher generality in Section 2.6. We will find that there are only ad hoc ways to get rid of the sign on the dual Hermitian vector space in a precise sense. This is because the †-category sHilb of finite-dimensional super Hilbert spaces is not quite a †-compact category but only up to the grading automorphism $(-1)^F$ as will be discussed in Section 2.9.

Proposition 2.1.16. *Let (V, h) be a super Hermitian vector space. Equip V^* with the Hermitian vector space structure given by composing $(h^*)^{-1}$ and the grading automorphism $(-1)^F$ so that V^* has the same signature as V by Proposition 2.1.6. Then the following diagram commutes*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{ev}_V^\dagger"] \arrow[d, "\mathrm{coev}_V"] & V^* \otimes V \arrow[d, "\sigma_{V^*,V}"] \\
V \otimes V^* \arrow[r, "(-1)^F_V \otimes 1"] & V \otimes V^*
\end{tikzcd} .$$

Proof. We look in coordinates and apply Proposition 2.1.9. Let $\{e_i\}$ be an orthonormal basis of V and $\{e^i\}$ its dual basis. Then by the choice of Hermitian pairing on the dual, the vectors e_i and e^i have the same sign for the inner product. Since signs multiply under the tensor product of Hermitian vector spaces, we see that $e^i \otimes e_i \in V^* \otimes V$ is both even and positive definite in the inner product. Therefore,

$$\mathrm{ev}^\dagger(1) = \sum_i e^i \otimes e_i,$$

which gets mapped by $\sigma_{V^*,V}$ to

$$\sum_i (-1)^{|e^i||e_i|} e^i \otimes e_i = \sum_i e^i \otimes (-1)^F_V(e_i)$$

as desired. □

2.2 Involutive and anti-involutive categories

In Section 2.1, we abstracted the notion of a Hermitian pairing on a vector space V to an isomorphism $\overline{V}^* \cong V$ satisfying a symmetry property. The main idea to define unitary topological field theories, is

[1] This sign convention is analogous to the sign convention $\overline{\psi_1\psi_2} = -\overline{\psi_1}\,\overline{\psi_2}$ for Grassmann variables ψ_1, ψ_2, which is sometimes used in the physics literature.

to do this same construction for the bordism category and require topological field theories to preserve the appropriate structure of involutions and Hermitian pairings. In other words, we would like to study $\mathbb{Z}/2$-actions $Y \mapsto \overline{Y}$ on the bordism category defined by some orientation-reversal procedure and define Hermitian pairings $Y \cong \overline{Y}^*$ on the bordism category satisfying certain properties.

Before diving into the geometry in Chapter 4, we will first abstract the situation and work in general symmetric monoidal categories with duals and a symmetric monoidal $\mathbb{Z}/2$-action. In this section, we will introduce the basic definitions regarding involutions and anti-involutions. We expect these two concepts to be equivalent under composing with a chosen dual functor, see the left part of the diagram (1.3) in the introduction. Since we will not need this equivalence in general, the main goal of this section will be to give a technical proof of a slightly weaker statement in Theorem 2.2.24. We will use the following terminology:

Definition 2.2.1. An *involutive category* or a *category with involution* is a category $\mathcal{C}$ with $\mathbb{Z}/2$-action.

See Appendix A.3 for the definition of an action of a group on a category. Explicitly, the structure of an involution on a category consists of a functor $\overline{(.)} : \mathcal{C} \to \mathcal{C}$ and a natural isomorphism $\lambda_x : x \to \overline{\overline{x}}$ such that the two induced morphisms $\lambda_{\overline{x}}$ and $\overline{\lambda_x} : \overline{\overline{\overline{x}}} \cong \overline{x}$ agree. When $\mathcal{C}$ is symmetric monoidal, we require the action to be symmetric monoidal. This means that $\overline{(.)} : \mathcal{C} \to \mathcal{C}$ is symmetric monoidal and $\lambda : x \to \overline{\overline{x}}$ is monoidal. Involutive categories form a 2-category ICat using $\mathbb{Z}/2$-equivariant functors and $\mathbb{Z}/2$-equivariant natural transformations, see Definition A.3.6. Similarly, we sometimes refer to $\mathbb{Z}/2$-equivariant functors (natural transformations) as *involutive functors (natural transformations).*

Remark 2.2.2. The above terminology seems to be somewhat standard in the literature [8, 76, 37]. However, in the monoidal setting, involutions are sometimes required to be op-monoidal [15], i.e. reverse the direction of the tensor product.

Now suppose we are given a rigid involutive symmetric monoidal category. Recall here that a monoidal category is called *rigid* if every object has a dual. In this section, we combine the $\mathbb{Z}/2$-action $x \mapsto \overline{x}$ with the dual into a single operation $d(x) := \overline{x}^*$. This will be more convenient as it reduces the amount of data we have to carry around and as we saw in Section 2.1, abstract Hermitian pairings can be defined by only referring to d. This results in what we will call a category with anti-involution, which is closely related but not equivalent to the notion of a dagger category. The exact relationship between dagger categories and anti-involutions is provided in joint work with Jan Steinebrunner [68], which will be reviewed in Section 2.3. It turns out that the anti-involution contains exactly the same information as the involution, i.e. we will prove the following theorem:

Theorem 2.2.3 (Theorem 2.2.24)**.** *There is a* 2*-functor from the* 2*-category of rigid symmetric monoidal categories with involution* $\mathrm{ICat}^{fd}_{\mathbb{E}_\infty}$ *to the* 2*-category of rigid symmetric monoidal categories with anti-involution* $\mathrm{aICat}^{fd}_{\mathbb{E}_\infty}$ *given by mapping the involution* $\overline{(.)}$ *to the anti-involution* $d = \overline{(.)}^*$. *It is an equivalence on hom-categories.*

To make the theorem precise, we first work through all relevant definitions. An anti-involution on a category is a fixed point for the $\mathbb{Z}/2$-action $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ on the 2-category of categories. Explicitly writing out this definition using the notion of fixed points in bicategories [53, Appendix A] [31, Definition 3.9 and subsequent remarks], results in the following 2-category aICat of anti-involutive categories.[2]

[2]We refer to [47] for an introduction to bicategories.

Definition 2.2.4. An *anti-involution* on a category $\mathcal{C}$ consists of a functor $d : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ together with a natural isomorphism $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow d^2$ such that $d\eta_c$ is the inverse of η_{dc}. We call a category with anti-involution an *anti-involutive category.*

Definition 2.2.5. An *anti-involutive functor* between anti-involutive categories $(\mathcal{C}_1, d_1, \eta_1), (\mathcal{C}_2, d_2, \eta_2)$ consists of a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ and a natural isomorphism $\phi : F \circ d_1 \Rightarrow d_2 \circ F$ such that the following diagram commutes:

$$\begin{tikzcd} F(x) \arrow[r, "F((\eta_1)_x)"] \arrow[d, "(\eta_2)_{F(x)}"] & (F \circ d_1 \circ d_1)(x) \arrow[d, "\phi_{d_1(x)}"] \\ (d_2 \circ d_2 \circ F)(x) \arrow[r, "d_2(\phi_x)"] & (d_2 \circ F \circ d_1)(x) \end{tikzcd} . \tag{2.1}$$

An *anti-involutive equivalence* is an anti-involutive functor which is also an equivalence of categories.

Definition 2.2.6. An *anti-involutive natural transformation* u between anti-involutive functors (F, ϕ) and (G, ψ) is a natural transformation such that the diagram

$$\begin{tikzcd} (F \circ d_1)(x) \arrow[r, "u_{d_1 x}"] \arrow[d, "\phi"] & (G \circ d_1)(x) \arrow[d, "\psi"] \\ (d_2 \circ F)(x) & (d_2 \circ G)(x) \arrow[l, "d_2 u_x"] \end{tikzcd}$$

commutes for all objects x.

By [68, Lemma 2.5.], an anti-involutive functor is an an anti-involutive equivalence if and only if it admits an anti-involutive inverse up to anti-involutive natural transformation.

Remark 2.2.7. Calling $(\mathcal{C}, d, \eta)$ an anti-involutive category is not standard terminology in the literature. In the surgery theory literature, this structure is sometimes called a category with duality. However, we find this terminology confusing with the fact that in many examples relevant in this thesis, dc will not be the dual of c in the sense of monoidal categories. Our terminology is motivated by what is sometimes called an anti-involution on an algebra A; a linear map $d : A \to A$ such that $d^2 = \mathrm{id}_A$ and $d(ab) = db \cdot da$.

Example 2.2.8. Let $(\mathcal{C}, d, \eta)$ be an anti-involutive category. Then $\mathcal{C}^{\mathrm{op}}$ is an anti-involutive category with inverted η. This makes d into an anti-involutive functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.

Example 2.2.9. Let $\mathcal{C}$ be a symmetric monoidal category with duals. Then a choice of dual functor $x \mapsto x^*$ can be made into an anti-involution, see Lemma 2.2.16.

Example 2.2.10. In $\mathcal{C} = \mathrm{sVect}$ the functor $V \mapsto \overline{V}^*$ can be made into an anti-involution, as explained in Section 2.1.

We extend these notions to monoidal categories, for which we will introduce some notation. If $(\mathcal{C}, \otimes, 1)$ is a monoidal category, we will write the same category with opposite tensor product as $\mathcal{C}^{\otimes\mathrm{op}}$. If we want to emphasize that in the opposite category $\mathcal{C}^{\mathrm{op}}$ we only reverse composition of morphisms and not the tensor product, we will write $\mathcal{C}^{\circ\mathrm{op}}$ instead. In particular, $\mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ is the monoidal category in which both the order of composition and tensor product is reversed. For example, if we can choose duality data in a monoidal category, then we get an induced dual functor which will be a monoidal functor $(.)^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$, also see Appendix A.1. For the purpose of defining tensor products of Hermitian pairings it will be more convenient to define a monoidal anti-involution as a monoidal functor $d : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op}}$:

Definition 2.2.11. An anti-involution (d, η) on a monoidal category is called *monoidal* if d comes equipped with the data

$$\chi_{c_1,c_2} : dc_1 \otimes dc_2 \to d(c_1 \otimes c_2) \quad u : 1 \to d(1)$$

of being a monoidal functor $\mathcal{C} \to \mathcal{C}^{\circ\mathrm{op}}$ and η is a monoidal natural transformation.

Definition 2.2.12. A *monoidal anti-involutive functor* is an anti-involutive functor $(F, \phi) : (\mathcal{C}_1, d_1, \eta_1) \to (\mathcal{C}_2, d_2, \eta_2)$ which is also a monoidal functor such that the natural isomorphism $F \circ d \cong d \circ F$ of functors $\mathcal{C}_1 \to \mathcal{C}_2^{\mathrm{op}}$ is monoidal. Explicitly this means that the diagrams

$$\begin{tikzcd}
F(1_{\mathcal{C}_1}) \arrow[r, "F(u_{\mathcal{C}_1})"] & F(d1_{\mathcal{C}_1}) \arrow[r, "\phi_1"] & dF(1_{\mathcal{C}_1}) \arrow[d, "d\epsilon"] \\
1_{\mathcal{C}_2} \arrow[u, "\epsilon"] \arrow[rr, "u_{\mathcal{C}_2}"] & & d1_{\mathcal{C}_2}
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(dx) \otimes F(dx') \arrow[r, "\phi_x \otimes \phi_{x'}"] \arrow[d, "\mu_{dx,dx'}"] & dF(x) \otimes dF(x') \arrow[r, "\chi_{F(x)\otimes F(x')}"] & d(F(x) \otimes F(x')) \\
F(dx \otimes dx') \arrow[r, "F(\chi_{x,x'})"] & F(d(x \otimes x')) \arrow[r, "\phi_{x\otimes x'}"] & dF(x \otimes x') \arrow[u, "d\mu_{x,x'}"]
\end{tikzcd}$$

commute for all objects x, x' of $\mathcal{C}_1$. Here $\epsilon : 1_{\mathcal{C}_1} \to F(1_{\mathcal{C}_2})$ is data of F preserving the monoidal unit and $u_{\mathcal{C}_i} : 1_{\mathcal{C}_i} \to d1_{\mathcal{C}_i}$ the data of d preserving the monoidal unit for $i = 1, 2$.[3] A monoidal anti-involutive natural transformation is a natural transformation that are both anti-involutive and monoidal. We denote the 2-category of monoidal anti-involutive categories by $\mathrm{aICat}_{\mathbb{E}_1}$.

The second diagram is a consequence of the fact that if we want the two monoidal functors F and d commute, there is still an extra condition requiring that their monoidal data commute. Alternatively, we can interpret it as requiring the monoidal data of F to be an anti-involutive natural isomorphism.

We will also discuss anti-involutions for braided and symmetric monoidal categories. So let $(\mathcal{C}, \beta)$ be a braided monoidal category. Then $\mathcal{C}^{\circ\mathrm{op}}, \mathcal{C}^{\otimes\mathrm{op}}$ and $\mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ are braided, the first by $\beta^{-1}_{c_1,c_2}$, the second by β_{c_2,c_1} and the last by $\beta^{-1}_{c_2,c_1}$. Note that there are other obvious options (see [73, example 3.1.8]), for example the braiding on $\mathcal{C}^{\circ\mathrm{op}}$ would be its reverse β_{c_2,c_1}, but this turns out not to give the right type of braided anti-involutions for applications to braided dagger categories. With these definitions, the identity functors $\mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op}}$ and $\mathcal{C}^{\otimes\mathrm{op}\circ\mathrm{op}} \to \mathcal{C}^{\circ\mathrm{op}}$ equipped with the monoidal data β become braided monoidal equivalences. Indeed, the associativity condition on the monoidal data of these functors is equivalent to the diagram

$$\begin{tikzcd}
z \otimes (y \otimes x) \arrow[d, "\beta_{y,x}"] & (z \otimes y) \otimes x \arrow[l, "\alpha_{z,y,x}"'] \arrow[d, "\beta_{z\otimes y,x}"] \\
z \otimes (x \otimes y) \arrow[d, "\beta_{z,x\otimes y}"] & (y \otimes z) \otimes x \arrow[d, "\beta_{y\otimes z,x}"] \\
(x \otimes y) \otimes z \arrow[r, "\alpha_{x,y,z}"] & x \otimes (y \otimes z)
\end{tikzcd}$$

[3] We will from now on often assume without loss of generality that monoidal functors are strictly unital so that we can omit unitality data such as u and ϵ.

commuting, which follows by [39, Proposition 1 (B6)].[4] The equivalence is braided with this definition, since the diagram

$$\begin{tikzcd} c_1^{\otimes\mathrm{op}} \otimes c_2^{\otimes\mathrm{op}} \arrow[r, "(\beta_{\mathcal{C}^{\otimes\mathrm{op}}})_{c_1,c_2}"] \arrow[d, equal] & c_2^{\otimes\mathrm{op}} \otimes c_1^{\otimes\mathrm{op}} \arrow[d, equal] \\ (c_2 \otimes c_1)^{\otimes\mathrm{op}} \arrow[r, "\beta_{c_2,c_1}"] & (c_1 \otimes c_2)^{\otimes\mathrm{op}} \end{tikzcd}$$

commutes. If β is symmetric, then so are $\mathcal{C}^{\circ\mathrm{op}}, \mathcal{C}^{\otimes\mathrm{op}}$ and $\mathcal{C}^{\otimes\mathrm{op}\circ\mathrm{op}}$. In that case we will sometimes implicitly identify $\mathcal{C} \cong \mathcal{C}^{\otimes\mathrm{op}}$ by the monoidal functor $\mathrm{id}_{\mathcal{C}}$ equipped the monoidal data given by the braiding.

We now list the straightforward generalizations of anti-involutions to the braided and symmetric setting for reference.

Definition 2.2.13. A monoidal anti-involution (d, η, χ) on the braided category $\mathcal{C}$ is *braided* if $d : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op}}$ is a braided functor. A braided anti-involutive category is *symmetric* if its underlying braided category is symmetric. A *braided anti-involutive functor* is a monoidal anti-involutive functor for which the underlying functor is braided. A *braided anti-involutive natural transformation* between braided anti-involutive categories is simply a monoidal anti-involutive natural transformation. A *symmetric anti-involutive functor* between symmetric anti-involutive categories is simply a braided anti-involutive functor and a *symmetric anti-involutive natural tranformation* is again a monoidal anti-involutive natural transformation. We denote the 2-category of braided monoidal anti-involutive categories by $\mathrm{aICat}_{\mathbb{E}_2}$ and the 2-category of symmetric monoidal anti-involutive categories by $\mathrm{aICat}_{\mathbb{E}_\infty}$.

We will now get into the proof of Theorem 2.2.24. The main idea of the proof is to show that

1. every symmetric monoidal category with duals has a symmetric monoidal anti-involution $x \mapsto x^*$, which is canonical up to monoidal natural isomorphism;

2. symmetric monoidal anti-involutions on a monoidal category with duals $\mathcal{C}$ form a torsor over compatible symmetric monoidal involutions. In other words, given an anti-involution d' and an involution $\overline{(.)}$ that are compatible[5], we can make $d = d' \circ \overline{(.)}$ into an anti-involution and every anti-involution is obtained this way.[6]

3. for every symmetric monoidal involution $\overline{(.)}$ on a symmetric monoidal category with duals, $\overline{(.)}$ is compatible with $(.)^*$. For example, $\overline{(.)}$ is anti-involutive with respect to $(.)^*$ and $(.)^*$ is involutive with respect to $\overline{(.)}$.

We then apply the second point to the case where $d' = (.)^*$.

Remark 2.2.14. For the purposes of defining monoidal anti-involutions from involutions in a non-symmetric rigid monoidal category, it makes more sense to require the involution $\overline{(.)}$ to be an op-monoidal functor $\mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op}}$. In other words, $\mathcal{C}$ is a $\mathbb{Z}/2$-fixed point under the action $\mathcal{C} \mapsto \mathcal{C}^{\otimes\mathrm{op}}$ on the 2-category of monoidal categories. The reason for this strange op-monoidality is that a dual functor will both reverse the direction of tensor products and morphisms. If we eventually want to

[4] We will from now on typically apply Mac Lane's coherence theorem and assume without loss of generality that all monoidal categories are strictly unital and associative.

[5] We will need that d' is involutive with respect to $\overline{(.)}$ and $\overline{(.)}$ is anti-involutive with respect to d' using the same isomorphism $d\overline{x} \cong \overline{dx}$.

[6] We learned this perspective from David Reutter.

get a monoidal dagger category, we need the anti-involution d to reverse the direction of morphisms, but *not* of tensor products. Therefore, if $\mathcal{C}$ has an involution $\overline{(.)} : \mathcal{C} \to \mathcal{C}$ and we want $d = \overline{(.)}^*$ to be monoidal, we need $\overline{(.)}$ to be op-monoidal. The importance of op-monoidal involutions in the context of dagger categories has been noted before [7, 30, 15]. However, we will work only with symmetric monoidal categories for which this distinction is irrelevant. We expect the analogue of Theorem 2.2.24 to hold without braidings when taking the involution to be op-monoidal, but leave this question to future work.

Abstractly we think of this situation as follows. There is a $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on the 2-category of (symmetric) monoidal categories given by $\mathcal{C} \mapsto \mathcal{C}^{\circ\mathrm{op}}$ and $\mathcal{C} \mapsto \mathcal{C}^{\otimes\mathrm{op}}$. The 2-category of fixed points for the former $\mathbb{Z}/2$-action is exactly the 2-category of categories equipped with a (symmetric) monoidal anti-involution d. If a symmetric monoidal category has duals, a dual functor (which is unique up to monoidal equivalence) makes it canonically into a fixed point for the diagonal $\mathbb{Z}/2$-action. Moreover, 1- and 2-morphisms of symmetric monoidal categories preserve this fixed point structure. One can then ask whether the fixed points for the two $\mathbb{Z}/2$-actions assemble into a $\mathbb{Z}/2 \times \mathbb{Z}/2$-fixed point. In that case it would be reasonable to expect that we can exchange fixed points for $\mathcal{C} \mapsto \mathcal{C}^{\circ\mathrm{op}}$ and $\mathcal{C} \mapsto \mathcal{C}^{\otimes\mathrm{op}}$ by composing with the fixed dual functor.

Remark 2.2.15. For the convenience of the reader, we spell out what it means for monoidal categories with an op-monoidal involution $(\overline{(.)}, \lambda)$ and a monoidal anti-involution (d, η) to assemble into a $\mathbb{Z}/2 \times \mathbb{Z}/2$-fixed point using [53, Appendix A]. The result is that we need a monoidal natural isomorphism

$$\mu : \overline{dx} \to d\overline{x}$$

satisfying a compatibility between η and λ given by the diagram

$$\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[d, "\lambda_x"] & d^2x \arrow[r, "\lambda_{d^2x}"] & \overline{\overline{d^2x}} \arrow[d, "\overline{\mu_{dx}}"] \\
\overline{\overline{x}} \arrow[d, "\eta_{\overline{\overline{x}}}"] & & \overline{d\overline{dx}} \\
d^2\overline{\overline{x}} \arrow[r, "d\mu_{\overline{x}}"] & d\overline{d\overline{x}} & \overline{d^2\overline{x}} \arrow[l, "\mu_{d\overline{x}}"] \arrow[u, "\overline{d\mu_x}"']
\end{tikzcd}$$

Note that there are several equivalent ways to formulate this diagram depending on the preferred result by using the following diagrams that use naturality of μ, λ and η:

$$\begin{tikzcd}
\overline{d\overline{dx}} \arrow[d, "\mu_{\overline{dx}}"] & \overline{d^2\overline{x}} \arrow[l, "\overline{d\mu_x}"] \arrow[d, "\mu_{d\overline{x}}"] \\
d\overline{\overline{dx}} & d\overline{d\overline{x}} \arrow[l, "d\overline{\mu}_x"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[d, "\lambda_x"] & d^2x \arrow[d, "d^2\lambda_x"] \\
\overline{\overline{x}} \arrow[r, "\eta_{\overline{\overline{x}}}"] & d^2\overline{\overline{x}}
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[d, "\lambda_x"] & d^2x \arrow[d, "\xi_{d^2x}"] \\
\overline{\overline{x}} \arrow[r, "\overline{\overline{\eta_x}}"] & \overline{\overline{d^2x}}
\end{tikzcd}.$$

Unfortunately, it is not clear to us how this compatibility between μ and η above is used in our proof of Theorem 2.2.24.

We now prove that the dual gives an anti-involution.

Lemma 2.2.16. *Fix a dual functor $(.)^* : \mathcal{C}_1 \to \mathcal{C}_1^{\circ\mathrm{op}}$ on a symmetric monoidal category $\mathcal{C}_1$ and denote the isomorphism $x \to x^{**}$ induced by the symmetry by Φ_x. Then the pair $((.)^*, \Phi)$ defines a* *symmetric monoidal anti-involution on $\mathcal{C}_1$. If $\mathcal{C}_2$ is another symmetric monoidal category with duals and $F : \mathcal{C}_1 \to \mathcal{C}_2$ a symmetric monoidal functor, then F is anti-involutive for the anti-involution $x \mapsto x^*$ using the isomorphism $F(x)^* \cong F(x^*)$ expressing uniqueness of duals.*

Proof. Using the symmetry, we can assure that $(.)^*$ becomes a monoidal functor $(.)^* : \mathcal{C}_1 \to \mathcal{C}_1^{\text{oop}}$ instead of being op-monoidal. We have to show that Φ_{x^*} and Φ_x^* are inverses, which is proven in Lemma A.1.15. To show that the anti-involution is symmetric monoidal, the only thing left to show is that the natural isomorphism $\alpha_x : F(x^*) \cong F(x)^*$ is monoidal between monoidal functors $\mathcal{C}_1 \to \mathcal{C}_2^{\text{oop},\otimes\text{op}}$, but this follows from the fact that isomorphisms expressing uniqueness of duals are monoidal. Explicitly, this means that the diagram

$$\begin{tikzcd}
F((x_1 \otimes x_2)^*) \arrow[r, "\alpha_{x_1 \otimes x_2}"] & F(x_1 \otimes x_2)^* \arrow[d] \\
F(x_2^* \otimes x_1^*) \arrow[u] & (F(x_1) \otimes F(x_2))^* \\
F(x_2^*) \otimes F(x_1^*) \arrow[u] \arrow[r, "\alpha_{x_1} \otimes \alpha_{x_2}"] & F(x_2)^* \otimes F(x_1)^* \arrow[u]
\end{tikzcd}$$

commutes.

To show that a symmetric monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is anti-involutive for $x \mapsto x^*$, we have to show that the diagram

$$\begin{tikzcd}
F(x) \arrow[r, "F(\Phi_x)"] \arrow[d, "\Phi_{F(x)}"] & F(x^{**}) \\
F(x)^{**} & F(x^*)^* \arrow[l] \arrow[u]
\end{tikzcd}$$

commutes. This diagram is exactly saying that F is a pivotal functor for the pivotal structure Φ. Since any symmetric monoidal functor is pivotal for this pivotal structure the result follows, see Lemma A.1.14. $\square$

It follows in particular that if $\overline{(.)}$ is a symmetric monoidal $\mathbb{Z}/2$-action, then $\overline{(.)}$ is an anti-involutive functor with respect to the anti-involution $x \mapsto x^*$ using the isomorphism $\mu_x : \overline{x^*} \cong \overline{x}^*$. This can be seen by applying Lemma 2.2.16 to $F = \overline{(.)}$.

Remark 2.2.17. If $\mathcal{C}$ is a monoidal category with duals, the double dual need not be isomorphic to the original, see [66] for a discussion. Therefore $x \mapsto x^*$ will not define an anti-involution in general.

Lemma 2.2.18. *Let $\mathcal{C}_1, \mathcal{C}_2$ be symmetric monoidal categories with dual functors, let $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ be symmetric monoidal functors and $\gamma : F \Rightarrow G$ a monoidal natural isomorphism. Then γ is anti-involutive for the anti-involution $x \mapsto x^*$.*

Proof. This is exactly the content of Lemma A.1.9. $\square$

Let $\mathcal{C}$ be a symmetric monoidal category with involution consisting of $x \mapsto \overline{x}$ and the monoidal natural isomorphism $\lambda_x : x \to \overline{\overline{x}}$ such that $\lambda_{\overline{x}} = \overline{\lambda_x}$. Our goal is to make the functor $d := \overline{(.)}^* : \mathcal{C} \to \mathcal{C}^{\text{oop},\otimes\text{op}} \cong \mathcal{C}^{\text{oop}}$ into an anti-involution. We first study the compatibility of the isomorphism $\overline{x}^* \cong \overline{x^*}$ with the isomorphisms $\lambda_x : x \to \overline{\overline{x}}$ and $\Phi_x : x \mapsto x^{**}$.

Lemma 2.2.19. *The dual functor equipped with the isomorphism $\overline{x}^* \cong \overline{x^*}$ becomes $\mathbb{Z}/2$-equivariant as a non-monoidal functor.*

Proof. We have to show that the diagram

$$\begin{tikzcd}
x^* \arrow[d, "\lambda_{x^*}"] & \overline{\overline{x}}^* \arrow[l, "\lambda_x^*"'] \arrow[d] \\
\overline{\overline{x^*}} & \overline{\overline{x}^*} \arrow[l]
\end{tikzcd}$$

commutes, where the right arrow follows from the fact that both $\overline{\overline{x}}^*$ and $\overline{\overline{x}^*}$ are duals of $\overline{\overline{x}^*}^*$. The bottom arrow is the induced by the canonical isomorphism expressing that both $\overline{x}^*$ and $\overline{x^*}$ are duals of $\overline{x^*}^*$. Note that the composition of these arrows is the unique dual isomorphism saying that $\overline{\overline{x}}^*$ and $\overline{\overline{x^*}}$ are both dual to $\overline{\overline{x^*}}^*$. It therefore suffices to show that the composition of λ_{x^*} and the inverse of λ_x^* intertwines the appropriate evaluation maps. To do this, it sufficies to show the diagram

$$\begin{tikzcd}
\overline{\overline{x}}^* \otimes \overline{\overline{x}} \arrow[rr, "\lambda_x^*"] \arrow[dr, "\lambda_x^* \otimes \lambda_x^{-1}"'] \arrow[dd, "\mathrm{ev}_{\overline{\overline{x}}}"'] & & x^* \otimes \overline{\overline{x}} \arrow[rr, "\lambda_{x^*}"] & & \overline{\overline{x^*}} \otimes \overline{\overline{x}} \arrow[dd] \\
& & x^* \otimes x \arrow[u, "\lambda_x"] \arrow[urr, "\lambda_{x^*} \otimes \lambda_x"'] \arrow[dll, "\mathrm{ev}_x"] \arrow[drr, "\lambda_{x^* \otimes x}"] & & \\
1 & & & & \overline{\overline{x^* \otimes x}} \arrow[llll, "\overline{\overline{\mathrm{ev}_x}}"]
\end{tikzcd}$$

commutes. The lower triangle commutes because λ is natural, the right triangle commutes because it is monoidal and the left triangle commutes by definition of the dual morphism. $\square$

Next we show that if $\mathcal{C}$ is a symmetric monoidal category with duals and involution $\overline{(.)}$, then $\overline{(.)}^*$ is an anti-involution. The proof only uses that $x \mapsto x^*$ is an anti-involution, $x \mapsto \overline{x}$ is anti-involutive with respect to it and $x \mapsto x^*$ is involutive and so generalizes to that setting.

Proposition 2.2.20. $d := \overline{(.)}^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op}}$ *is a symmetric monoidal anti-involution.*

Proof. For $x \in \mathcal{C}$ we define η_x as the composition

$$x \xrightarrow{\lambda_x} \overline{\overline{x}} \xrightarrow{\Phi_{\overline{\overline{x}}}} \overline{\overline{x}}^{**} \xrightarrow{\mu_{\overline{x}}^*} \overline{\overline{x}^*}^*$$

where the last isomorphism is given by the fact that $\overline{\overline{x}}^*$ and $\overline{\overline{x}^*}$ both dual to $\overline{\overline{x}^*}^*$. We have to show that $\overline{\eta_x}^*$ is the inverse of $\eta_{\overline{x}^*}$. Consider the diagram

$$\begin{tikzcd}
& \overline{\overline{\overline{x}^*}} \arrow[rr, "\Phi_{\overline{\overline{x}^*}}"] \arrow[ddr] & & \overline{\overline{\overline{x}^*}}^{**} \arrow[d] \arrow[ddr, bend left] & \\
& & & \overline{\overline{\overline{x}}^{**}}^* \arrow[dd] \arrow[ddd, bend left] & \\
\overline{x}^* \arrow[uur, "\lambda_{\overline{x}^*}"] & & \overline{\overline{\overline{x}}^*} \arrow[ur, "\Phi_{\overline{\overline{\overline{x}}^*}}"] & & \overline{\overline{\overline{x}^*}^*}^* \arrow[ddl] \\
& & & \overline{\overline{\overline{x}}}^{***} \arrow[dll, bend left, "\Phi^*_{\overline{\overline{\overline{x}}}}"] & \\
& \overline{\overline{\overline{x}}}^* \arrow[uul, bend left, "\overline{\lambda_x}^*"] \arrow[uul, bend right, "\lambda^*_{\overline{x}}"'] \arrow[uur, bend left] \arrow[urr, bend left, "\Phi_{\overline{\overline{x}}^*}"] & & \overline{\overline{\overline{x}}^{**}}^* \arrow[ll, bend left, "\overline{\Phi_{\overline{\overline{x}}}}^*"] &
\end{tikzcd}$$

Going from $\overline{x}^*$ in the west to $\overline{\overline{\overline{x}^*}^*}^*$ in the east through the northern path means applying $\eta_{\overline{x}^*}$, while the southern path from east to west is $\overline{\eta_x}^*$ and so it suffices to show this diagram commutes. To show the commutativity of all individual parts of this diagram, we need that $\overline{\lambda_x} = \lambda_{\overline{x}}$, the fact that $\Phi^*_x = \Phi^{-1}_{x^*}$, the fact that $x \mapsto x^*$ is a $\mathbb{Z}/2$-equivariant functor, the fact that the functor $x \mapsto \overline{x}$ is anti-involutive for $x \mapsto x^*$ by Lemma 2.2.16 and we use naturality of Φ twice. Finally, for the rightmost part we need to show that

$$\begin{tikzcd}
\overline{\overline{y^*}}^* \arrow[dr] & \\
\overline{\overline{y}^*}^* \arrow[u] \arrow[d] & \overline{\overline{y^*}^*} \\
\overline{\overline{y}^{**}} \arrow[ur] &
\end{tikzcd}$$

commutes, which follows by Lemma A.1.10. We have thus shown that (d, η) is an anti-involution.

Since $\overline{(.)}$ and $(.)^*$ are symmetric monoidal functors, so is d. Explicitly, the monoidal data χ is given by

$$\overline{x_1 \otimes x_2}^* \cong (\overline{x_1} \otimes \overline{x_2})^* \cong \overline{x_2}^* \otimes \overline{x_1}^* \cong \overline{x_1}^* \otimes \overline{x_2}^*.$$

Here in the last line we used the braiding to convert the functor $\overline{(.)}^* : \mathcal{C} \to \mathcal{C}^{\mathrm{oop}, \otimes\mathrm{op}}$ to a functor $d : \mathcal{C} \to \mathcal{C}^{\mathrm{oop}}$. To show that (d, η) is symmetric monoidal we are therefore only left with showing η is a monoidal natural isomorphism. Since the $\mathbb{Z}/2$-action is monoidal, λ is monoidal. Because Φ_x is an isomorphism specifying uniqueness of duals between two monoidal functors, it is also monoidal, see Lemma A.1.5. This finishes the proof. $\square$

Theorem 2.2.21. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be symmetric monoidal categories with duals and symmetric monoidal $\mathbb{Z}/2$-actions $x \mapsto \overline{x}$ and denote the corresponding anti-involutions by $d(x) = \overline{x}^*$. Then a symmetric monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a monoidal involutive functor with equivariance* *data $F(\overline{x}) \cong \overline{F(x)}$ if and only if it is a monoidal anti-involutive functor with equivariance data $F(\overline{x}^*) \cong F(\overline{x})^* \cong \overline{F(x)}^*$.*

Proof. Let α denote the natural isomorphism $F(x)^* \cong F(x^*)$. By Lemma 2.2.16, F is anti-involutive for the anti-involution $(x \mapsto x^*, \Phi)$ with anti-involutive data α. A natural isomorphism $\xi : \overline{F(x)} \cong F(\overline{x})$ uniquely corresponds to a natural isomorphism $\overline{F(x)}^* \cong F(\overline{x}^*)$. Consider the diagram

$$
\begin{tikzcd}
F(x) \arrow[r, "F(\lambda_x)"] \arrow[d, "\lambda_{F(x)}"] \arrow[rrrd, bend left=40, "F(\eta_x)"] \arrow[rddd, bend right=70, "\eta_{F(x)}"'] & F(\overline{\overline{x}}) \arrow[r, "F(\Phi_{\overline{\overline{x}}})"] \arrow[d, "\xi_{\overline{x}}"] \arrow[rdd, "\Phi_{F(\overline{\overline{x}})}"] & F(\overline{\overline{x}}^{**}) \arrow[rd, "F(\mu^*_{\overline{x}})"] & \\
\overline{\overline{F(x)}} \arrow[r, "\overline{\xi_x}"] \arrow[dd, "\Phi_{\overline{\overline{F(x)}}}"'] & \overline{F(\overline{x})} \arrow[d, "\Phi_{\overline{F(x)}}"] & F(\overline{\overline{x}}^*)^* \arrow[u, "\alpha_{\overline{\overline{x}}^*}"'] \arrow[d, "\alpha^*_{\overline{\overline{x}}}"] \arrow[rd, "F(\mu_{\overline{x}})^*"'] & F(\overline{\overline{x}^*}^*) \\
 & \overline{F(\overline{x})}^{**} \arrow[rd, "\mu^*_{F(\overline{x})}"] & F(\overline{\overline{x}})^{**} \arrow[l, "\xi^{**}_{\overline{x}}"] & F(\overline{\overline{x}^*})^* \arrow[u, "\alpha_{\overline{\overline{x}^*}}"'] \arrow[d, "\xi^*_{\overline{x}^*}"] \\
\overline{\overline{F(x)}}^{**} \arrow[r, "\mu^*_{\overline{F(x)}}"] \arrow[ru, "\overline{\xi_x}^{**}"] & \overline{\overline{F(x)}^*}^* \arrow[r, "\overline{\xi_x^*}^*"'] & \overline{F(\overline{x})^*}^* & \overline{F(\overline{x}^*)}^* \arrow[l, "\overline{\alpha_{\overline{x}}}^*"]
\end{tikzcd}
,
$$

of which the outside two paths commute if and only if F is an anti-involutive functor. Note that the northwest square commutes if and only if F is $\mathbb{Z}/2$-equivariant. The southeast corner follows from the fact that ξ is a monoidal natural isomorphism between the functors $x \mapsto F(\overline{x})$ and $x \mapsto \overline{F(x)}$ by applying Lemma A.1.9. The northeast parallelogram commutes by Lemma A.1.10. For showing that the other parts of the diagram commute we need naturality of Φ, the fact that F is anti-involutive for $x \mapsto x^*$ and that μ is natural. We conclude that F is an anti-involutive functor if and only if it is $\mathbb{Z}/2$-equivariant.

To show that F is a monoidal anti-involutive functor if F is monoidally $\mathbb{Z}/2$-equivariant, we still have to show the natural isomorphism $F(\bar{c}^*) \cong \overline{F(c)}^*$ is monoidal, in other words F satisfies

$$\begin{tikzcd}
& F(\overline{c_2}^*) \otimes F(\overline{c_1}^*) \arrow[r] \arrow[dl] & F(\overline{c_2}^* \otimes \overline{c_1}^*) \arrow[dr] & \\
\overline{F(c_2)}^* \otimes \overline{F(c_1)}^* \arrow[dr] & & & F(\overline{c_1 \otimes c_2}^*) \arrow[dl] \\
& \overline{F(c_1) \otimes F(c_2)}^* \arrow[r] & \overline{F(c_1 \otimes c_2)}^* &
\end{tikzcd} .$$

Consider the following diagram of which the outside is exactly the compositions that we have to show are equal.

$$\begin{tikzcd}
\overline{F(c_2)}^* \otimes \overline{F(c_1)}^* \arrow[r] \arrow[d] & F(\overline{c_2})^* \otimes F(\overline{c_1})^* \arrow[r] \arrow[d] & F(\overline{c_2}^*) \otimes F(\overline{c_1}^*) \arrow[d] \\
(\overline{F(c_1)} \otimes \overline{F(c_2)})^* \arrow[r] \arrow[d] & (F(\overline{c_1}) \otimes F(\overline{c_2}))^* \arrow[d] & F(\overline{c_2}^* \otimes \overline{c_1}^*) \arrow[d] \\
\overline{F(c_1) \otimes F(c_2)}^* \arrow[d] & F(\overline{c_1} \otimes \overline{c_2})^* \arrow[r] \arrow[d] & F((\overline{c_1} \otimes \overline{c_2})^*) \arrow[d] \\
\overline{F(c_1 \otimes c_2)}^* \arrow[r] & F(\overline{c_1 \otimes c_2})^* \arrow[r] & F(\overline{c_1 \otimes c_2}^*)
\end{tikzcd}$$

The northwest square commutes because the dual functor is a monoidal functor so that the diagram

$$\begin{tikzcd}
x_1^* \otimes x_2^* \arrow[r] & (x_2 \otimes x_1)^* \\
y_1^* \otimes y_2^* \arrow[r] \arrow[u, "f_1^* \otimes f_2^*"] & (y_2 \otimes y_1)^* \arrow[u, "(f_1 \otimes f_2)^*"']
\end{tikzcd}$$

commutes for all morphisms $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$. The southeast square commutes by naturality of the isomorphism $F(c^*) \cong F(c)^*$ applied to the morphism $\overline{c_1 \otimes c_2} \cong \overline{c_1} \otimes \overline{c_2}$. The northeast rectangle commute because $F(c^*) \cong F(c)^*$ is a monoidal natural isomorphism. The southwest rectangle commutes if and only if $F(\bar{c}) \cong \overline{F(c)}$ is a monoidal natural isomorphism. We see that $F(\bar{c}) \cong \overline{F(c)}$ is monoidal if and only if $F(\bar{c}^*) \cong \overline{F(c)}^*$ is monoidal. □

Lemma 2.2.22. *Let $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ be symmetric monoidal $\mathbb{Z}/2$-equivariant functors between symmetric monoidal categories with $\mathbb{Z}/2$-action equipped with a dual functor. Then a monoidal natural transformation $\zeta : F \Rightarrow G$ is $\mathbb{Z}/2$-equivariant if and only if it is anti-involutive for $x \mapsto \overline{x}^*$.*

Proof. Denote the equivariance data of F and G by $\psi : \overline{F(x)} \cong F(\overline{x})$ and $\gamma : \overline{G(x)} \cong G(\overline{x})$ respectively. Consider the diagram

$$\begin{tikzcd}
F(\overline{x}^*) \arrow[r, "\zeta_{\overline{x}^*}"] \arrow[d] & G(\overline{x}^*) \arrow[d] \\
F(\overline{x})^* \arrow[d, "\psi_x^*"] & G(\overline{x})^* \arrow[l, "\zeta_{\overline{x}}^*"'] \arrow[d, "\gamma_x^*"] \\
\overline{F(x)}^* & \overline{G(x)}^* \arrow[l, "\overline{\zeta_x}^*"']
\end{tikzcd} .$$

The upper diagram commutes since ζ is anti-involutive for $x \mapsto x^{**}$ by Lemma 2.2.18. The lower diagram commutes if and only if ζ is $\mathbb{Z}/2$-equivariant. The combined diagram commutes if and only if ζ is anti-involutive. $\square$

Lemma 2.2.23. *Let $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$ be symmetric monoidal involutive functors between symmetric monoidal categories with symmetric monoidal $\mathbb{Z}/2$-actions. Then the anti-involutive equivariance data $FG(dx) \cong dFG(x)$ induced by the anti-involutive equivariance data $FG(\overline{x}) \cong \overline{FG(x)}$ agrees with the composition of the anti-involutive equivariance data $F(dx) \cong dF(x)$ and $G(dx) \cong dG(x)$.*

Proof. We have to show that

$$\begin{tikzcd}
\overline{F(G(x))}^* \arrow[r] \arrow[d, equal] & F(\overline{G(x)})^* \arrow[r] \arrow[d, equal] & F(G(\overline{x}))^* \arrow[r] & F(G(\overline{x})^*) \arrow[r] \arrow[d, equal] & F(G(\overline{x}^*)) \arrow[d, equal] \\
\overline{F(G(x))}^* \arrow[r] & F(\overline{G(x)})^* \arrow[r] & F(\overline{G(x)}^*) \arrow[r] & F(G(\overline{x})^*) \arrow[r] & F(G(\overline{x}^*))
\end{tikzcd}$$

commutes. The only interesting part is the middle, which commutes by the naturality of $F(x^*) \cong F(x)^*$ applied to $G(\overline{x}) \cong \overline{G(x)}$. $\square$

Theorem 2.2.24. *Let $\mathrm{ICat}^{fd}_{\mathbb{E}_\infty}$ be the 2-category of symmetric monoidal involutive categories with duals and $\mathrm{aICat}^{fd}_{\mathbb{E}_\infty}$ the 2-category of symmetric monoidal anti-involutive categories with duals. Then the operation of sending $x \mapsto \overline{x}$ to $x \mapsto \overline{x}^*$ defines a 2-functor*

$$\mathrm{ICat}^{fd}_{\mathbb{E}_\infty} \to \mathrm{aICat}^{fd}_{\mathbb{E}_\infty},$$

which is an equivalence on hom-categories.

Proof. We have already defined the 2-functor on objects, 1-morphisms and 2-morphisms. If $\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{ICat}^{fd}_{\mathbb{E}_\infty}$, then the map

$$\mathrm{Hom}_{\mathrm{ICat}^{fd}_{\mathbb{E}_\infty}}(\mathcal{C}_1, \mathcal{C}_2) \to \mathrm{Hom}_{\mathrm{aICat}^{fd}_{\mathbb{E}_\infty}}(\mathcal{C}_1, \mathcal{C}_2)$$

is clearly a functor. It is an equivalence of categories by Theorem 2.2.21 and Lemma 2.2.22. In Lemma 2.2.23 we have shown that the 2-functor strictly preserves composition of 1-morphisms. $\square$

Remark 2.2.25. We expect the above functor to be an equivalence of 2-categories. Namely, there should be an inverse 2-functor given by sending a symmetric monoidal anti-involution d to the symmetric monoidal involution $x \mapsto dx^*$.

2.3 Dagger categories versus anti-involutive categories

The goal of this section is to review joint work with Jan Steinebrunner on the relationship between dagger categories and anti-involutive categories using Hermitian pairings [68], i.e. we we will work on the right part of the diagram (1.3) we saw in the introduction. For simplicity we will not require a monoidal structure our categories in this section. In Section 2.4 we will discuss adaptations needed to handle the symmetric monoidal case which will be relevant for unitary topological field theories.

We start by reviewing the basic definitions of the theory of †-categories:

Definition 2.3.1. A *dagger category* or *†-category* is a category $\mathcal{D}$ equipped with a contravariant functor $\dagger : \mathcal{D} \to \mathcal{D}^{\mathrm{op}}$ which squares to one and is the identity on objects. An endomorphism $f : x \to x$ in a dagger category is called *self-adjoint* if $f^\dagger = f$. It is called *positive* if it is of the form $g^\dagger g$ for $g : x \to x$ an endomorphism. A morphism $f : x_1 \to x_2$ is *unitary* if $f^\dagger f$ and $f f^\dagger$ are the identity.

Note that every positive morphism is self-adjoint. We denote by $\pi_0(\mathcal{D})$ the set of isomorphism classes of objects of $\mathcal{D}$. We take $\pi_0^U(\mathcal{D})$ to be the unitary isomorphism classes, i.e. objects of $\mathcal{D}$ modulo the equivalence relation given by the existence of unitary isomorphism.

Example 2.3.2. The category $\mathrm{Hilb}_{\mathbb{C}}$ of finite-dimensional complex Hilbert spaces is a dagger category in which the functor † is given by the adjoint of a linear map. The notions of unitary and self-adjoint morphisms agree with the familiar notions. Since every finite-dimensional vector space admits a Hilbert space structure and any two such are isometrically isomorphic, we have that $\pi_0^U \mathrm{Hilb}_{\mathbb{C}} \cong \pi_0 \mathrm{Hilb}_{\mathbb{C}} \cong \mathbb{N}$ given by the dimension.

Remark 2.3.3. Motivated by the last example, it is often assumed in the literature that †-categories are $\mathbb{C}$-linear and the functor † is $\mathbb{C}$-antilinear. This excludes some interesting examples of †-categories, such as the category of complex vector spaces equipped with a nondegenerate symmetric bilinear form. Namely, the usual notion of adjoint/transpose makes this into a †-category in which † is a $\mathbb{C}$-linear functor. We will not require this restriction because we will be interested in making bordism categories into dagger categories, which are certainly not $\mathbb{C}$-linear.

A †-functor between †-categories is a functor F which strictly commutes with the functor †, so $F(f^\dagger) = F(f)^\dagger$ for all morphisms. A natural transformation is called *unitary* if it evaluates to a unitary morphism on every object. This makes †-categories into a 2-category. There is a canonical 2-functor

$$T : \dagger\,\mathrm{Cat} \to \mathrm{aICat}$$

from the 2-category of dagger categories to the 2-category of anti-involutive categories which assigns the trivial coherence data $\eta_x = \mathrm{id}_x$. This functor is not an equivalence, but it turns out it does admit a 2-right adjoint

$$\mathrm{Herm} : \mathrm{aICat} \to \dagger\,\mathrm{Cat}\,.$$

This 2-right adjoint is strict in the sense that the triangle identities hold on the nose and not up to a natural isomorphism. The right adjoint has a concrete description using what we call Hermitian pairings, the generalization of Hermitian pairings we saw in Section 2.1 to arbitrary anti-involutive categories. By suitably restricting the collection of ‘allowed’ Hermitian pairings in a similar way to how we can restrict the pairings on Hermitian vector spaces to the positive definite ones, we can construct all possible dagger categories with a fixed underlying anti-involution.

We move to the definition of a Hermitian pairing, which we learned from [20, Definition B.14.].

Definition 2.3.4. Let $(\mathcal{C}, d, \eta)$ be an anti-involutive category. A *Hermitian pairing* in $\mathcal{C}$ is defined to be an isomorphism $h : c \to dc$ such that

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[rr, bend right, "h"'] & d^2 c \arrow[r, "dh"] & dc \end{tikzcd} \tag{2.2}$$

commutes.

The most relevant case for us will be when the anti-involutive category comes from a symmetric monoidal anti-involutive category with chosen dual functor as explained in Section 2.2. Translating along the equivalence, we see that a Hermitian pairing is an isomorphism $h : c \to \overline{c}^*$ such that

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[rr, bend right, "h"'] & \overline{\overline{c}^*}^* \arrow[r, "\overline{h}^*"] & \overline{c}^* \end{tikzcd} \tag{2.3}$$

commutes. Recall that η_c is given by the composition

$$c \cong \overline{\overline{c}} \cong \overline{\overline{c}}^{**} \cong \overline{\overline{c}^*}^*,$$

where the first isomorphism is given by the data of the covariant involution, the second isomorphism is the double dual isomorphism coming from the braiding and the final isomorphism is given by the fact that $\overline{(.)}$ is monoidal and so preserves duals. We review the examples from Section 2.1:

Example 2.3.5. Let $\mathcal{C} = \text{Vect}$ with $\overline{(.)}$ the $\mathbb{Z}/2$-action given by the complex conjugate vector space and $\overline{\overline{V}} = V$. Then an isomorphism $h : V \to \overline{V}^*$ is equivalent to a nondegenerate sesquilinear pairing $\langle \cdot, \cdot \rangle$ on V. Condition (2.3) is equivalent to

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

The same relation holds true in super vector spaces, where we require the pairing to preserve the grading and put in the appropriate Koszul sign

$$\langle v, w \rangle = (-1)^{|v||w|} \overline{\langle w, v \rangle}.$$

Example 2.3.6. Recall from Remarks 2.1.14 and 2.1.15 that we can change the canonical sign choices for the symmetric monoidal involution $\overline{(.)}$ in two ways, giving a total of four nonequivalent symmetric monoidal involutions on sVect. By either changing the isomorphism $\overline{\overline{V}} \cong V$ or $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$, we change η by a $(-1)^F$. Changing both isomorphisms leaves the anti-involution untouched, at least when ignoring the monoidal structure. Therefore changing one of these two isomorphisms will result in Hermitian pairings satisfying the more common convention

$$\langle v, w \rangle = \overline{\langle w, v \rangle}.$$

Changing both isomorphisms instead, would recover the formula of Hermitian pairings on sVect from the last example.

We introduce the following operation of tranferring Hermitian pairings along isomorphisms.

Definition 2.3.7. Let $h : c \to dc$ be a Hermitian pairing and $g : c' \to c$ any isomorphism. Then $dg \circ h \circ g$ is a Hermitian pairing on c', which we will call the *transfer of h along g*, see [68, Definition 5.1].

Transferring a Hilbert space pairing on a vector space along a linear map $f : V_1 \to V_2$, amounts to modifying the Hermitian pairing with the positive operator $f^\dagger f$.

Definition 2.3.8. The *Hermitian completion* $\mathrm{Herm}\,\mathcal{C}$ of an anti-involutive category $\mathcal{C}$ is the category in which objects consists of Hermitian pairings $h : c \to dc$ and morphisms $(c_1, h_1) \to (c_2, h_2)$ are simply morphisms $c_1 \to c_2$ in $\mathcal{C}$.

The Hermitian completion becomes a dagger category if we define the dagger of a morphism $f : (c_1, h_1) \to (c_2, h_2)$ as the composition

$$f^\dagger : c_2 \xrightarrow{h_2} dc_2 \xrightarrow{df} dc_1 \xrightarrow{h_1^{-1}} c_1, \tag{2.4}$$

see [68, Lemma 3.4].

Remark 2.3.9. The fact that we required a morphism $(c_1, h_1) \to (c_2, h_2)$ in $\mathrm{Herm}\,\mathcal{C}$ to simply be a morphism $c_1 \to c_2$ in $\mathcal{C}$ can be counter-intuitive. This trivially implies that the canonical functor $\mathrm{Herm}\,\mathcal{C} \to \mathcal{C}$ is fully faithful. So when every object in $\mathcal{C}$ admits a Hermitian pairing, it is a confusing fact that it is an equivalence of categories (it is then even an equivalence of anti-involutive categories, see Lemma 2.3.18). The reader might have expected us to require the obvious compatibility condition with the h_i given by the diagram

$$\begin{tikzcd} c_1 \arrow[r, "f"] \arrow[d, "h_1"] & c_2 \arrow[d, "h_2"] \\ dc_1 & dc_2 \arrow[l, "df"] \end{tikzcd} .$$

This diagram is saying that $f^\dagger$ is a left inverse of f, in which case f is called *an isometry* in the dagger category $\mathrm{Herm}\,\mathcal{C}$. For example, here f is invertible if and only if f is unitary. In particular, the data of this 'fixed point category' can be recovered from the data of the dagger category $\mathrm{Herm}\,\mathcal{C}$. One can think of $\mathrm{Herm}\,\mathcal{C}$ as the universal dagger-enlargement of the anti-involutive category $\mathcal{C}$ that is still equivalent to $\mathcal{C}$, as long as it admits Hermitian pairings on all objects.

From now on we will often implicitly assume that every object of $\mathcal{C}$ admits a Hermitian pairing.

Example 2.3.10. If $\mathcal{C} = \mathrm{Vect}$, then the Hermitian completion is the category of finite-dimensional Hermitian vector spaces $\mathrm{Herm}_{\mathbb{C}}$. It is straightforward to verify that also the dagger structure corresponds to the usual adjoint of linear maps. In particular, the notions of unitary and self-adjoint morphisms in this dagger category give the usual notions of unitary and self-adjoint linear maps. The previous remark recovers the fact that $\mathrm{Herm}_{\mathbb{C}}$ is equivalent to Vect as a category.

Example 2.3.11. In the super setting $\mathcal{C} = \mathrm{sVect}$ we obtain the category of finite-dimensional super Hermitian vector spaces $\mathrm{sHerm}_{\mathbb{C}}$ in which super Hermitian pairings satisfy

$$\langle v, w \rangle = (-1)^{|v||w|} \overline{\langle w, v \rangle}.$$

The dagger of an even linear map $f : V \to W$ satisfies

$$\langle fv, w \rangle = \langle v, f^\dagger w \rangle.$$

Note that for an odd map there would be a Koszul sign, but we will only consider sVect to have even morphisms in this thesis. Taking the other anti-involution on sVect changes the symmetry equation the Hermitian pairings have to satisfy, but not the formula for the adjoint of even morphisms.

Example 2.3.12. Consider a dagger category $\mathcal{C}$ as an anti-involutive category with $d = \dagger$ and $\eta_c = \mathrm{id}_c$. The Hermitian completion of $\mathcal{C}$ has objects consisting of pairs of objects c of $\mathcal{C}$ together with a self-adjoint automorphism $h : c \to dc = c^\dagger = c$. The new dagger $\ddagger$ on $\mathrm{Herm}\,\mathcal{C}$ on a morphisms $f : (c_1, h_1) \to (c_2, h_2)$ is defined as $f^\ddagger = \tau_2 \circ f^\dagger \circ \tau_1^{-1}$. For example, starting with the dagger category of finite-dimensional Hilbert spaces, new objects are triples $(V, (\cdot,\cdot), A)$ consisting of a Hilbert space and a self-adjoint invertible linear operator on V. We can identify such triples with the not-necessarily-positive Hermitian pairing $(\cdot, A\cdot)$ to realize the equivalence of dagger categories between Herm(Hilb) and $\mathrm{Herm}_{\mathbb{C}}$.

Remark 2.3.13. We provide a generalization of the notion of anti-involutive category, which we expect could be useful for generalizations to infinite dimensions, in particular for understanding non-topological unitary quantum field theory. First we ponder what happens when we weaken the condition that η is an isomorphism in the definition, resulting in something one might call a *lax anti-involution*. For example, we might want to take the category of not necessarily finite-dimensional vector spaces with $dV = \overline{V}^*$. Then, there is still a natural map $\eta_V : V \to d^2V$ such that $d\eta_V \circ \eta_{dV} = \mathrm{id}_{dV}$, but it is only an isomorphism if V is finite-dimensional. In particular, $\eta_{dV} \circ d\eta_V \neq \mathrm{id}_{d^3V}$ in that case. In the setting of lax anti-involutions, it is natural to define Hermitian pairings as maps $h : c \to dc$ for which $dh \circ h = \eta_c$. Note that it no longer makes sense to ask h to be an isomorphism. Also note that we could have independently asked h not to be an isomorphism, even when η_c is. Given a morphism $f : (c_1, h_1) \to (c_2, h_2)$ between such Hermitian pairings and looking at the definition given by the composition (2.4), we call f adjointable if there exists a morphism $f^\dagger : c_2 \to c_1$ such that the following diagram commutes

$$\begin{tikzcd} c_1 \arrow[d, "h_1"] & c_2 \arrow[l, "f^\dagger"'] \arrow[d, "h_2"] \\ dc_1 & dc_2 \arrow[l, "df"] \end{tikzcd} .$$

To obtain uniqueness of $f^\dagger$, it is reasonable to assume h_1 is a monomorphism. This is automatically the case when h_1 has a left inverse. This in turn is always the case when η_{c_1} has a left inverse, for example when c_1 is of the form dc' for some object c'. For example, in the category of infinite-dimensional vector spaces h will be a (not necessarily positive definite) inner product. If h_1, h_2 are Hilbert space structures on infinite-dimensional V_1 and V_2 respectively, it is a well-known fact in functional analysis that a linear map $f : V_1 \to V_2$ is adjointable if and only if it is continuous. In that case $f^\dagger$ is given by the usual Hilbert space adjoint.

Remark 2.3.14. Another approach to defining infinite-dimensional Hilbert spaces in this setting that still allows η to be an isomorphism is to work in the category of reflexive topological vector spaces. If we use the continuous dual with the strong topology to define d, then the reflexivity implies that η is still an isomorphism, and so we obtain an anti-involution. Of course, we then have to restrict both the allowed objects and Hermitian pairings further if we want to obtain the dagger category of Hilbert spaces. Here, we do not just require the pairing to be positive definite and complete, but also need to make sure that the topology induced by the pairing is the same as the original one on the topological vector spaces.

It turns out that the Hermitian completion construction has very nice categorical properties.

Theorem 2.3.15. *[68, Lemma 3.13] The Hermitian completion extends to a 2-functor*

$$\mathrm{Herm} : \mathrm{aICat} \to \dagger\,\mathrm{Cat},$$

which is right adjoint to the canonical functor $\dagger\,\mathrm{Cat} \to \mathrm{aICat}$.

The above theorem can be reformulated as a universal property of the Hermitian completion, saying that it is the 'cofree dagger category generated by a category with anti-involution':

Corollary 2.3.16. *Let $\mathcal{C}$ be an anti-involutive category and $\mathcal{D}$ a dagger category. Then there is an equivalence of categories*

$$\mathrm{Fun}_{\mathrm{aICat}}(\mathcal{D},\mathcal{C}) \cong \mathrm{Fun}_{\dagger}(\mathcal{D}, \mathrm{Herm}\,\mathcal{C}).$$

Remark 2.3.17. For later use we record the unit and counit of the adjunction in Theorem 2.3.15. The unit is given by sending the dagger category $\mathcal{D}$ to the dagger functor

$$U_{\mathcal{D}} : \mathcal{D} \to \mathrm{Herm}\,\mathcal{D}, \tag{2.5}$$

which includes $\mathcal{D}$ into its Hermitian completion by taking the Hermitian pairing on $x \in \mathcal{D}$ to be $h = \mathrm{id}_x : x \to x^\dagger = x$. The counit is given by sending the anti-involutive category $\mathcal{C}$ to the anti-involutive functor

$$K_{\mathcal{C}} : \mathrm{Herm}\,\mathcal{C} \to \mathcal{C}, \tag{2.6}$$

which forgets the Hermitian pairing and has anti-involutive data $h_\bullet : K_{\mathcal{C}} \circ \dagger \cong d \circ K_{\mathcal{C}}$ given by $(x, h) \mapsto h : x \to dx$. The unit and counit satisfy the triangle identities strictly, see [68, Theorem 4.9].

The following is [68, Lemma 4.6.].

Lemma 2.3.18. *Let $\mathcal{C}$ be an anti-involutive category and let $\mathcal{C}^{\exists\mathrm{Herm}} \subseteq \mathcal{C}$ be the full subcategory on objects that admit a Hermitian pairing. Then $K_{\mathcal{C}}$ is an equivalence of anti-involutive categories onto $\mathcal{C}^{\exists\mathrm{Herm}}$.*

Hermitian completions are in some sense 'maximal' dagger categories, but to understand unitarity we are more interested in 'smaller' dagger categories which contain very few Hermitian pairings. Especially small are 'minimal' dagger categories like Hilb, in which unitary isomorphism classes agree with usual isomorphism classes.

Definition 2.3.19. A dagger category $\mathcal{D}$ is called *minimal* if the map $\pi_0^U(\mathcal{D}) \to \pi_0(\mathcal{D})$ is bijective. A dagger category is called *maximal* if is unitarily equivalent to a Hermitian completion.

We will now restrict the collection of Hermitian pairings on $\mathrm{Herm}\,\mathcal{C}$ by calling a specific subset of objects positive:

Definition 2.3.20. A *positivity structure* on a category with involution is a collection P of objects of $\mathrm{Herm}\,\mathcal{C}$ that surjects onto the objects of $\mathcal{C}$ under the forgetful map. We will call elements $h \in P$ *positive pairings*. Given a positivity structure P we denote by $\mathcal{C}_P \subseteq \mathrm{Herm}\,\mathcal{C}$ the full dagger subcategory on those objects $(c, h) \in \mathrm{Herm}\,\mathcal{C}$ such that $h \in P$. A positivity structure is *closed* if it is closed under transfer in the sense of Example 2.3.7: for every $(h : c \to dc) \in P$ and every isomorphism $g : c' \to c$ also $dg \circ h \circ g \in P$. Two positivity structures are *equivalent* if they have the same closure.

Example 2.3.21. Consider the Hermitian completion of the category (equivalent to Vect) in which objects are $\mathbb{C}^n$ and morphisms are given by matrices. Consider the positivity structure given by only allowing the standard inner product $\langle\cdot,\cdot\rangle_{st}$, giving us the dagger category $\mathcal{D} \subseteq \mathrm{Hilb}$ of Hilbert spaces of the form $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_{st})$. The inclusion is an equivalence of dagger categories. However, this positivity structure is not closed. Its closure adds all inner products of the form $\langle ., A^\dagger A. \rangle_{st}$ for A an invertible matrix, which are in fact all positive definite inner products on $\mathbb{C}^n$.

Remark 2.3.22. In Definition 2.3.20 of a positivity structure, we could have required the collection P to only essentially surject onto the objects of $\mathcal{C}$. However, it is convenient in practice when every object of $\mathcal{C}$ admits at least one positive pairing. Moreover, this distinction is irrelevant for equivalence classes of positivity structures.

Let $\mathcal{C}$ be a symmetric monoidal category with duals and let $d = (.)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}} \cong \mathcal{C}^{\otimes\mathrm{op}}$ be the induced anti-involution we considered in Lemma 2.2.16. Then a Hermitian pairing on an object c is the same as a self-duality $\mathrm{ev}_c : c \otimes c \to 1$, which is symmetric in the sense that it stays invariant under the braiding $\sigma_{c,c}$. We provide two subexamples of this.

Example 2.3.23. Let Vect_k be the category of finite-dimensional vector spaces over a field. A self-duality of a vector space V is a nondegenerate bilinear form. We obtain that $\mathrm{Herm}\,\mathrm{Vect}_k$ is the dagger category of symmetric nondegenerate bilinear forms in which the dagger is given by the adjoint. Similarly $\mathrm{Herm}\,\mathrm{sVect}_k$ is given by graded-symmetric even nondegenerate bilinear forms.

Example 2.3.24. Let $\mathcal{C}$ be a category with pullbacks and let $\mathbf{Span}\mathcal{C}$ be the category of spans. This category has objects $\mathrm{obj}\,\mathcal{C}$ and morphisms from x to y are spans which are pairs of morphisms (f, g) of the following shape

$$x \xleftarrow{f} z \xrightarrow{g} y.$$

Composition of spans is given by pullback. This category is symmetric monoidal under the cartestian product and every object x is canonically symmetrically self-dual. Indeed for the evaluation we can take $z = x$ with f the diagonal map and g the identity, and similar for the coevaluation. The dagger category obtained by taking these self-dualities as the positivity structure, is the dagger category of spans with

$$(x \xleftarrow{f} z \xrightarrow{g} y)^\dagger = y \xleftarrow{g} z \xrightarrow{f} x.$$

Taking the opposite category of a category with pushouts, we obtain the dagger category of cospans.

Define the Hermitian isomorphism classes of an anti-involutive category $\mathcal{C}$ as $\pi_0^h(\mathcal{C}) := \pi_0^U(\mathrm{Herm}\,\mathcal{C})$.

Lemma 2.3.25. *[68, Corollary 5.7, Theorem 5.14] Let $\mathcal{C}$ be an anti-involutive category. There is a bijection between*

1. *equivalence classes $[P]$ of positivity structures;*
2. *dagger categories $\mathcal{C}_P$ such that $\mathcal{C}_P \cong \mathcal{C}$ as categories with anti-involution modulo the following equivalence relation. We say $\mathcal{C}_P$ and $\mathcal{C}_{P'}$ are equivalent when there exists an equivalence $\mathcal{C}_P \cong \mathcal{C}_{P'}$ of dagger categories such that the triangle of anti-involutive functors*

$$\begin{tikzcd} \mathcal{C}_P \arrow[rr] \arrow[dr] & & \mathcal{C}_{P'} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

can be filled by an anti-involutive natural isomorphism;

3. *subsets $\pi_0^U(\mathcal{C}_P) \subseteq \pi_0^h(\mathcal{C})$ such that the composition*

$$\pi_0^U(\mathcal{C}_P) \subseteq \pi_0^h(\mathcal{C}) \to \pi_0(\mathcal{C})$$

is surjective.

With the above lemma in mind, we will from now on assume all positivity structures are closed. Because every surjective map of sets admits a noncanonical splitting, we also immediately obtain the following.

Corollary 2.3.26. *In an anti-involutive category $\mathcal{C}$ every object admits a Hermitian pairing if and only if it admits a (noncanonical) positivity structure P such that $\mathcal{C}_P$ is minimal.*

Example 2.3.27. If $\mathcal{C} = \text{sVect}$ we have $\pi_0(\mathcal{C}) \cong \mathbb{N} \times \mathbb{N}$ given by the superdimension. Super Hermitian vector spaces will be unitarily equivalent if and only if they have the same signature in the sense of Definition 2.1.5 and so we obtain $\pi_0^h \mathcal{C} \cong \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The map $\pi_0^h \mathcal{C} \to \pi_0 \mathcal{C}$ is given by

$$\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \quad (p_1, p_2, p_3, p_4) \mapsto (p_1 + p_2, p_3 + p_4).$$

Similarly to Example 2.3.29, there are many positivity structures on sVect and many of them are minimal. However, if we restrict to those notions that are preserved under direct sum, everything is fixed once we decide on which Hermitian forms we allow on the even and the odd line. In particular, there are 4 minimal dagger categories with this property, depending on whether we allow positive definite or negative definite forms on the even part and odd part separately.

This example is unchanged for the ungraded symmetric convention of Hermitian forms on super vector spaces, in which we have changed η by $(-1)^F$. There are two equivalences of dagger categories between these two notions of Hermitian pairings given by mapping $h: V \to \overline{V}^*$ to $h \circ (\pm i)_V^F : V \to \overline{V}^*$. These two equivalences are not unitarily naturally isomorphic. This follows because they map the odd super Hermitian line $\Pi\mathbb{C}$ such that $\langle e, e \rangle = +i$ to either the positive definite odd graded Hermitian line $\Pi\mathbb{C}$ with $\langle e, e \rangle = 1$ or the negative definite line. These two objects are not unitarily isomorphic. Recall our convention that odd vectors are positive in a super Hermitian vector space when their norm is a positive multiple of i. This picks out one of the two equivalences if we want to relate $\mathbb{Z}/2$-graded Hilbert spaces and super Hilbert spaces. We will explore this equivalence between the two conventions for graded Hilbert spaces in a more general setting in Section 2.6.

Definition 2.3.28. Let $F : (\mathcal{C}_1, d, P_1) \to (\mathcal{C}_2, d, P_2)$ be an anti-involutive functor between anti-involutive categories equipped with positivity structures. Then F is said to *preserve the positivity structures* if for all $(h : c \to dc) \in P_1$ the composition

$$F(c) \xrightarrow{F(h)} F(dc) \xrightarrow{\phi_c} dF(c)$$

is in P_2.

In general, anti-involutive functors always send a Hermitian pairing to some Hermitian pairing and so $F : (\mathcal{C}_1, d) \to (\mathcal{C}_2, d)$ will map a subset P of $\pi_0^h(\mathcal{C}_1)$ to some subset of $\pi_0^h(\mathcal{C}_2)$. In particular, F preserves positivity structures if P_1 will be mapped to a subset of P_2 under this procedure.

Next we provide some explicit equivalent conditions for the dagger category $\mathcal{C}_P$ to be minimal or maximal. First note that there are typically many choices for P that make $\mathcal{C}_P$ minimal:

Example 2.3.29. Let $\mathcal{C} = \text{Vect}$ come equipped with the anti-involution $V \mapsto \overline{V}^*$. For every $d \in \mathbb{Z}_{\geq 0}$, pick a pair of integers (p, q) such that $p + q = d$. We could then call the d-dimensional Hermitian vector space $\mathbb{C}^d$ with signature (p, q) positive. This will result in a minimal dagger category. Note that at this stage, there is no condition forcing signatures of vector spaces of different dimensions to be compatible. Also note that some of these dagger categories are equivalent while some are not. For example, the dagger category of finite-dimensional Hilbert spaces is equivalent to the dagger category of finite-dimensional negative definite Hermitian vector spaces. Note that this does not contradict Lemma 2.3.25, because the equivalence of dagger categories between negative definite and positive definite Hermitian vector spaces does not cover the identity anti-involutive functor on $(\mathrm{Vect}, d = \overline{(.)}^*)$. Namely, this equivalence is realized by the identity functor equipped with the nontrivial anti-involutive structure, given by the natural automorphism $-\mathrm{id}_V : V \to V$.

To study minimality, we will now introduce certain kinds of automorphisms of dagger categories that behave similar to positive matrices. The term 'positive' in a dagger category is already taken to mean an *endomorphism* $f : c \to c$ for which there exists an *endomorphism* $g : c \to c$ such that $f = g^\dagger g$. This is rather unfortunate, because such endomorphisms are more closely related to nonnegative numbers or more generally, positive semidefinite matrices. We introduce analogous terminology for automorphisms:

Definition 2.3.30. An automorphism $f : c \to c$ is called *inv-positive* if there exists an automorphism $g : c \to c$ such that $f = g^\dagger g$.

There is also an analogous weaker notion of positivity, which allows the domain and range of g to be different. We introduce the following terminology.

Definition 2.3.31. An endomorphism $f : c \to c$ in a dagger category is called *weakly positive* if there exists a morphism $g : c \to c'$ such that $f = g^\dagger g$. An automorphism $f : c \to c$ in a dagger category is called *weakly inv-positive* if there exists an isomorphism $g : c \to c'$ such that $f = g^\dagger g$.

Example 2.3.32. Let $\mathcal{D} = \mathrm{Hilb}$ be the dagger category of finite-dimensional Hilbert spaces. Then an automorphism of a Hilbert space $\mathcal{H}$ is weakly inv-positive if and only if it is inv-positive if and only if it is a positive definite operator. A positive endomorphism corresponds instead to a positive semidefinite (i.e. nonnegative) operator. In the dagger category $\mathcal{D} = \mathrm{Herm}$ of finite-dimensional Hermitian vector spaces, not all inv-positives are positive definite operators and all self-adjoint automorphisms are weakly inv-positive.

Example 2.3.32 generalizes to the following statement.

Lemma 2.3.33.

- *A dagger category is maximal if and only if every self-adjoint automorphism is weakly inv-positive;*
- *A dagger category is minimal if and only if every weakly inv-positive is inv-positive.*

In other words, the notion of weakly inv-positive in a dagger category can vary all the way between self-adjoint automorphism and inv-positive, depending on how large its positivity structure is. Clearly we have the following implications

$$\begin{tikzcd}
\text{inv-positive} \arrow[r] \arrow[d] & \text{weakly inv-positive} \arrow[d] & \\
\text{positive} \arrow[r] & \text{weakly positive} \arrow[r] & \text{self-adjoint}
\end{tikzcd}$$

Confusingly, it is not true in general that every positive isomorphism is inv-positive. In a $\mathbb{C}$-linear setting this issue usually does not occur because of spectral theory:

Example 2.3.34. Let $a \in A$ be an element of a C^*-algebra such that a^*a is invertible. Then a^*a has spectrum inside the positive reals. By functional calculus, there exists an element $b = \sqrt{a^*a}$, which again has spectrum inside the positive reals. In particular, it is invertible and self-adjoint, so that

$$b^*b = b^2 = a^*a.$$

In the special case where $A = B(\mathcal{H})$ is the C^*-algebra of bounded operators on a Hilbert space, we can conclude that every positive isomorphism in the †-category of (potentially infinite-dimensional) Hilbert spaces is inv-positive.

Example 2.3.35. Even though the difference between inv-positive and positive automorphisms does not occur in reasonable $\mathbb{C}$-linear settings, infinite-dimensional settings still allow for the existence of non-invertible morphisms $g : c \to c$ such that $g^\dagger g$ is invertible. To show this, consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ and consider $A = B(\mathcal{H})$ as a ($\mathbb{C}$-linear) dagger category with one object. Let $a \in A$ be the right shift operator. Then a^* is the left shift operator and so $a^*a = 1$ is invertible. However, a is not surjective and so not invertible. This makes clear that there can exist isomorphisms $g : c \to c$ such that $g^\dagger g = g'^\dagger g'$, where $g' : c \to c$ not an isomorphism. Note that this does not contradict Example 2.3.34.

The situation of Example 2.3.35 can never arise in a dagger category enriched over finite-dimensional vector spaces:

Proposition 2.3.36. *Let $\mathcal{D}$ be a $\mathbb{C}$-linear dagger category such that hom-spaces are finite-dimensional. Suppose $g : c \to c$ is an endomorphism such that $g^\dagger g$ is an isomorphism. Then g is an isomorphism.*

Proof. If g would not be an isomorphism, then $g^\dagger g$ is not an isomorphism because

$$\det g^\dagger g = \det g^\dagger \det g = 0.$$

□

Proposition 2.3.36 tells us that the noninvertible analogue of being a minimal dagger category implies the invertible version.

Corollary 2.3.37. *Let $\mathcal{D}$ be a $\mathbb{C}$-linear dagger category such that hom-spaces are finite-dimensional. Suppose that for every weakly positive endomorphism $f : c \to c$ is positive. Then $\mathcal{D}$ is minimal.*

Proof. By Lemma 2.3.33, it suffices to show that every weakly inv-positive automorphism $f : c \to c$ is inv-positive. If $f : c \to c$ is a weakly inv-positive automorphism, is is also weakly positive. By assumption, there exists an endomorphism $g : c \to c$ such that $g^\dagger g = f$. By Proposition 2.3.36, g is invertible and so f is inv-positive. □

Remark 2.3.38. The converse of the above corollary is false: as explained in Example 2.3.29, there are many positivity structures on Vect resulting in a minimal dagger category. However, most of these do not satisfy that every weakly positive endomorphism $f : V \to V$ is positive. Indeed, suppose P is a positivity structure, for which there exist V_1 and V_2 which are not completely of the same signature (i.e. both positive definite or both negative definite). Let $v_1 \in V_1$ and $v_2 \in V_2$ be vectors in some orthonormal basis of opposite norm. Let $T : V_1 \to V_2$ be the linear map sending v_1 to v_2 and all vectors orthogonal to v_1 to zero. Then $T^\dagger T$ is the negative of the orthogonal projection onto the line spanned by v_1, which is not a positive operator. Therefore there are only two positivity structures on Vect so that every weakly inv-positive automorphism is inv-positive; the positive definite inner products and the negative definite inner products.

Remark 2.3.39. The minimality condition that every weakly positive morphism is positive, is one axiom required for C^*-categories.

The following example explains further why we call P a positivity structure:

Example 2.3.40. If $\mathcal{D}$ is a †-category considered as an anti-involutive category, recall that a Hermitian pairing is simply given by self adjoint automorphisms $h : c \to c^\dagger = c$. This anti-involutive category has a canonical positivity structure given by

$$P = \{h : c \xrightarrow{\sim} c : h \text{ is inv-positive}\}.$$

This positivity structure reproduces $\mathcal{D}$ inside its Hermitian completion. More precisely, the †-functor $U_{\mathcal{D}}$ of Remark 2.3.17 induces an equivalence of †-categories $\mathcal{D} \cong \mathcal{D}_P$. In particular, any †-category can up to †-equivalence be presented in the form $\mathcal{C}_P$, where P is a positivity structure on an anti-involutive category $\mathcal{C}$.

Theorem 2.3.41. *[68, Theorem 5.14] The* 2*-category of* †*-categories is equivalent to the* 2*-category of anti-involutive categories equipped with a closed positivity structure. The inverse functors are given by sending a dagger category to its canonical corresponding anti-involutive category with the positivity structure discussed in Example 2.3.40 and sending an anti-involutive category $\mathcal{C}$ with positivity structure P to $\mathcal{C}_P$. Here anti-involutive functors F between anti-involutive categories with positivity structures are required to preserve positivity structures.*

Example 2.3.42. Consider the case where the category $\mathcal{C}$ has one object $*$ and let $M = \mathrm{Hom}(*, *)$ be the corresponding monoid. Then an anti-involution on $\mathcal{C}$ is equivalent to a monoid map $\dagger : M \to M^{\mathrm{op}}$ squaring to one. In particular, $\mathcal{C}$ is automatically a dagger category. Let $\mathrm{sa} = \mathrm{sa}(M) \subseteq M$ denote the set of self-adjoint elements. It is a submonoid if M is abelian, but in general it is not (for example, when $M = GL_n(\mathbb{C})$ equipped with the usual Hermitian adjoint for $n > 1$). Let $\mathrm{Pos} \subseteq \mathrm{sa}$ denote the subset of positive endomorphisms.

A Hermitian pairing on the unique object of $\mathcal{C}$ is equivalent to an invertible self-adjoint $h \in M$. The new dagger of a morphism $m : (*, h_1) \to (*, h_2)$ in the Hermitian completion is $m^* = h_2^{-1} m^\dagger h_1$. Two Hermitian pairings are related by a transfer if and only if there is an invertible $m \in M$ such that $h_2 = m^\dagger h_1 m$. In particular, $h \sim 1$ if and only if $h = m^\dagger \cdot m$ is inv-positive. Note that $\pi_0^h(\mathcal{C})$ is in general not a monoid.

2.4 Symmetric monoidal generalizations

In this section we prove analogous results in the symmetric monoidal setting to what we discussed in Section 2.3. In the first part of this section we work monoidally. At the end we will include braidings.

Since the cartesian product of dagger categories is canonically a dagger category, we can define monoidal dagger categories as monoid objects in the 2-category of dagger categories:

Definition 2.4.1. A *monoidal dagger category* is a dagger category that is also a monoidal dagger category such that $\otimes$ is is a †-functor and the unitor and the associator are unitary. A (strong) monoidal functor between monoidal dagger categories is called a *(strong) monoidal dagger functor* if it is a dagger functor and $F(c_1) \otimes F(c_2) \to F(c_1 \otimes c_2)$ and $1_{\mathcal{C}_2} \to F(1_{\mathcal{C}_1})$ are unitary. A *monoidal unitary transformation* between monoidal dagger functors is a natural transformation that is both unitary and monoidal. Let $\mathrm{Cat}^\dagger_{\mathbb{E}_1}$ denote the 2-category of monoidal dagger categories.

Let $(\mathcal{C}, d, \eta, \chi)$ be a monoidal anti-involutive category. The main idea to make $\mathrm{Herm}\,\mathcal{C}$ into a monoidal dagger category is to take the tensor product of Hermitian pairings $h_1 : c_1 \to dc_1$ and $h_2 : c_2 \to dc_2$ by

$$c_1 \otimes c_2 \xrightarrow{h_1 \otimes h_2} dc_1 \otimes dc_2 \xrightarrow{\chi_{c_1, c_2}} d(c_1 \otimes c_2). \tag{2.7}$$

In this way the Hermitian completion will become a monoidal dagger category.

Example 2.4.2. Recall that sVect has two possible anti-involutions; the canonical one and the one in which η is modified by the grading operator. They each have two ways to make them into a monoidal anti-involution, depending on whether we want to give the isomorphism $\overline{V \otimes W}^* \cong \overline{V}^* \otimes \overline{W}^*$ a sign. This isomorphism will influence the notion of tensor product of super Hermitian vector spaces. For example, if we change the sign, the tensor product of two positive definite odd Hermitian vector spaces will become a negative definite even Hermitian vector space.

Example 2.4.3. We consider Example 2.3.42 for the case that $\mathcal{C}$ is monoidal. This is equivalent to the monoid M being commutative. Then $M = M^{\text{op}}$ and involutions on $\mathcal{C}$ are the same as anti-involutions. Note that Pos $\subseteq$ sa is a submonoid and the quotient monoid sa / Pos is an abelian group in which every element has order at most two, because if f is self-adjoint, then $f^2 = f \cdot f^\dagger$ is positive. Similarly, the quotient of invertible self-adjoints by inv-positives is an abelian group, in which every element has order at most two. By definition, the latter abelian group is equal to $\pi_0^h(\mathcal{C})$ as a set.

As a subexample, let $M = C(X, \mathbb{C})$ be the $*$-algebra of continuous functions on a topological space X. Then $\pi_0^h(\mathcal{C})$ is the group of real-valued functions on X modulo those that are positive, so there is a group isomorphism

$$\pi_0^h(\mathcal{C}) \cong C(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^{\pi_0(X)}.$$

All statements of the last section will now generalize in a straightforward way, with the main subtlety being that the analogue of Corollary 2.3.26 does not hold. Our first goal is to show

Theorem 2.4.4. Herm *extends to a* 2*-functor*

$$\mathrm{aICat}_{\mathbb{E}_1} \to \mathrm{Cat}^\dagger_{\mathbb{E}_1}$$

We first provide Herm on objects of $\mathrm{aICat}_{\mathbb{E}_1}$:

Theorem 2.4.5. *Let $\mathcal{C}$ be a monoidal anti-involutive category. The tensor product of Hermitian pairings given in Definition 2.7 gives* Herm $\mathcal{C}$ *the structure of a monoidal dagger category such that the forgetful map* Herm $\mathcal{C} \to \mathcal{C}$ *is a monoidal equivalence.*

Proof. We start by verifying that $(c_1, h_1) \otimes (c_2, h_2)$ as in Defintion 2.7 defines a Hermitian pairing on $c_1 \otimes c_2$. For this we have to show the diagram

$$\begin{tikzcd}
d(c_1 \otimes c_1) & dc_1 \otimes dc_2 \arrow[l, "\mu_{c_1,c_2}"'] & c_1 \otimes c_2 \arrow[l, "h_1 \otimes h_2"'] \arrow[dl, "\eta_{c_1} \otimes \eta_{c_2}"] \arrow[dd, "\eta_{c_1 \otimes c_2}"] \\
 & d^2 c_1 \otimes d^2 c_2 \arrow[u, "dh_1 \otimes dh_2"] \arrow[dl, "\mu_{dc_1, dc_2}"] & \\
d(dc_1 \otimes dc_2) \arrow[uu, "d(h_1 \otimes h_2)"] & & d^2(c_1 \otimes c_2) \arrow[ll, "d\mu_{c_1,c_2}"]
\end{tikzcd}$$

commutes. This follows because d and η are monoidal and h_1 and h_2 are Hermitian pairings.

Verifying that this makes Herm $\mathcal{C}$ into a monoidal category is now easy given how morphisms are defined in the Hermitian completion. For example, we take the associator $(c_1, h_1) \otimes ((c_2, h_2) \otimes (c_3, h_3)) \to ((c_1, h_1) \otimes (c_2, h_2)) \otimes (c_3, h_3)$ to be the associator $(c_1 \otimes c_2) \otimes c_3 \cong c_1 \otimes (c_2 \otimes c_3)$ and the pentagon identity will be automatically satisfied (for example, at this point we do not even have to check whether the two resulting Hermitian pairings on $c_1 \otimes c_2 \otimes c_3$ are mapped to each other under the associator). Also, since d is monoidal there is an isomorphism $u : d(1) \to 1$. Because η is monoidal, the morphism $\eta(1) : 1 \to d^2(1)$ is equal to $du \circ u^{-1}$. So u is a canonical Hermitian pairing on 1 and so $(1, u) \in \operatorname{Herm} \mathcal{C}$ is a monoidal unit with the unitors equal to those of $\mathcal{C}$.

To show this makes $\operatorname{Herm} \mathcal{C}$ into a monoidal dagger category, we have to prove that $(f_1 \otimes f_2)^\dagger = f_1^\dagger \otimes f_2^\dagger$ and that the unitors and the associator are unitary. Let $f_1 : (c_1, h_1) \to (c_1', h_1')$ and $f_2 : (c_2, h_2) \to (c_2', h_2')$ be morphisms in $\operatorname{Herm} \mathcal{C}$. To obtain $(f_1 \otimes f_2)^\dagger = f_1^\dagger \otimes f_2^\dagger$, we note the following diagram commutes because d is monoidal:

$$\begin{tikzcd}
d(c_1 \otimes c_2) & d(c_1' \otimes c_2') \arrow[l, "d(f_1 \otimes f_2)"'] \\
dc_1 \otimes dc_2 \arrow[u] & dc_1' \otimes dc_2' \arrow[l, "df_1 \otimes df_2"'] \arrow[u] \\
c_1 \otimes c_2 \arrow[u, "h_1 \otimes h_2"] & c_1' \otimes c_2' \arrow[l, "f_1^\dagger \otimes f_2^\dagger"'] \arrow[u, "h_1' \otimes h_2'"]
\end{tikzcd} .$$

For verifying the left unitor $\lambda_c : (1, u) \otimes (c, h) \to (c, h)$ is unitary, we note the following diagram commutes

$$\begin{tikzcd}
d(1 \otimes c) & dc \arrow[l, "d(\lambda_c)"'] \\
d1 \otimes dc \arrow[u] & \\
1 \otimes dc \arrow[u, "u \otimes \mathrm{id}"] \arrow[uur, "\lambda_{d(c)}"'] & \\
1 \otimes c \arrow[u, "\mathrm{id} \otimes h"] \arrow[r, "\lambda_c"] & c \arrow[uuu, "h"']
\end{tikzcd} .$$

This follows by naturality of λ and the unit condition on the functor d. The right unitor is analogous. Showing the associator is unitary is equivalent to showing the following commutes

$$\begin{tikzcd}
d(c_1 \otimes (c_2 \otimes c_3)) \arrow[r, "d\alpha"] & d((c_1 \otimes c_2) \otimes c_3) \\
dc_1 \otimes d(c_2 \otimes c_3) \arrow[u] & d(c_1 \otimes c_2) \otimes dc_3 \arrow[u] \\
dc_1 \otimes (dc_2 \otimes dc_3) \arrow[u] & (dc_1 \otimes dc_2) \otimes dc_3 \arrow[l, "\alpha"'] \arrow[u] \\
c_1 \otimes (c_2 \otimes c_3) \arrow[u, "h_1 \otimes (h_2 \otimes h_3)"] & (c_1 \otimes c_2) \otimes c_3 \arrow[l, "\alpha"'] \arrow[u, "(h_1 \otimes h_2) \otimes h_3"]
\end{tikzcd} .$$

It does by associativity of the monoidal functor d and the associator being natural. $\square$

Next we provide Herm on 1-morphisms of $\mathrm{aICat}_{\mathbb{E}_1}$:

Lemma 2.4.6. *For a monoidal anti-involutive functor,* $\operatorname{Herm} F : \operatorname{Herm} \mathcal{C}_1 \to \operatorname{Herm} \mathcal{C}_2$ *is a monoidal dagger functor.*

Proof. Recall that $u_{\mathcal{C}_i} : 1_{\mathcal{C}_i} \to d1_{\mathcal{C}_i}$ defines a Hermitian pairing on $1_{\mathcal{C}_i}$, which made it into the monoidal unit of $\mathrm{Herm}\,\mathcal{C}_i$. We have to show that $\epsilon : 1_{\mathcal{C}_2} \to \mathrm{Herm}\,F(1, u_{\mathcal{C}_2})$ is unitary. Writing out the definition is exactly the diagram in the definition of a monoidal anti-involutive functor.

Let $(c,h),(c',h') \in \mathrm{Herm}\,\mathcal{C}_1$. We want to show that $\mu_{c,c'} : \mathrm{Herm}\,F(c,h) \otimes \mathrm{Herm}\,F(c',h') \to \mathrm{Herm}\,F((c,h)\otimes(c',h'))$ is a unitary isomorphism. Writing out the Hermitian pairings on the objects involved, this boils down to showing the following diagram commutes.

$$\begin{tikzcd}
F(c)\otimes F(c') \arrow[r, "F(h)\otimes F(h')"] \arrow[d, "\mu_{c,c'}"] & F(dc)\otimes F(dc') \arrow[r, "\phi_c\otimes\phi_{c'}"] \arrow[d, "\mu_{dc,dc'}"] & dF(c)\otimes dF(c') \arrow[r, "\chi_{F(c)\otimes F(c')}"] & d(F(c)\otimes F(c')) \\
F(c\otimes c') \arrow[r, "F(h\otimes h')"] & F(dc\otimes dc') \arrow[r, "F(\chi_{c,c'})"] & F(d(c\otimes c')) \arrow[r, "\phi_{c\otimes c'}"] & dF(c\otimes c') \arrow[u, "d\mu_{c,c'}"']
\end{tikzcd}$$

The left square commutes because F is a monoidal functor. The right rectangle commutes by definition of F being a monoidal anti-involutive functor. □

Recall that monoidal anti-involutive natural transformations are defined as natural transformations that are both anti-involutive and monoidal and similar for monoidal unitary transformations on monoidal dagger categories. Therefore Herm is already defined on 2-morphisms of $\mathrm{aICat}_{\mathbb{E}_1}$. Also we still have $\mathrm{Herm}(F_1 \circ F_2) = \mathrm{Herm}(F_1) \circ \mathrm{Herm}(F_2)$ as monoidal dagger functors, because the monoidal data μ_1, μ_2 of $\mathrm{Herm}(F_1)$ and $\mathrm{Herm}(F_2)$ are simply given by the monoidal data of F_1 and F_2. Therefore we have proven:

Corollary 2.4.7. Herm *is a* 2*-functor*

$$\mathrm{Cat}^{\dagger}_{\mathbb{E}_1} \to \mathrm{aICat}_{\mathbb{E}_1}.$$

To obtain monoidal dagger categories that are not equivalent to Hermitian completions, we need to discuss positivity structures. The only thing to keep in mind, is that a full subcategory of a monoidal category generated by a collection of objects is only monoidal if it is closed under tensor product:

Definition 2.4.8. Let $\mathcal{C}$ be a monoidal anti-involutive category. A positivity structure $P \subseteq \mathrm{obj}\,\mathrm{Herm}\,\mathcal{C}$ is called *monoidal* if it is closed under tensor product.

Example 2.4.9. Let $\mathcal{D}$ be a monoidal dagger category. Then the associated canonical positivity structure in $\mathrm{Herm}\,\mathcal{D}$ of Example 2.3.40 is monoidal. This follows because $\otimes : \mathcal{D}\times\mathcal{D} \to \mathcal{D}$ is a †-functor.

Note that the fact that $\mathrm{Herm}\,\mathcal{C}$ is a monoidal dagger category implies that $\pi_0^U(\mathrm{Herm}\,\mathcal{C})$ is a monoid under tensor product. We therefore obtain:

Lemma 2.4.10. *Let $\mathcal{C}$ be a monoidal anti-involutive category. The dagger category $\mathcal{C}_P$ associated to a subset $P \subseteq \mathrm{Herm}\,\mathcal{C}$ is a monoidal dagger category if and only if P is monoidal. In particular, the dagger category obtained by taking a closed positivity structure with corresponding subset $P \subseteq \pi_0^U(\mathrm{Herm}\,\mathcal{C})$ is a monoidal dagger category if and only if P is a submonoid of $\pi_0^U(\mathrm{Herm}\,\mathcal{C})$.*

Example 2.4.11. For finite-dimensional vector spaces, the monoid structure on $\pi_0(\mathrm{Vect}) = \mathbb{N}$ coming from the monoidal structure on Vect is given by multiplication. The monoid structure on $\pi_0^U(\mathrm{Herm}\,\mathrm{Vect}) \cong \mathbb{N}\times\mathbb{N}$ is given by the formula telling us how the signature of a tensor product looks:

$$(p_1,q_1)(p_2,q_2) = (p_1p_2 + q_1q_2, p_1q_2 + p_2q_1).$$

Even after requiring positivity structures $P \subseteq \pi_0^U(\mathrm{Herm}\,\mathrm{Vect})$ to be monoidal, there are still many positivity structures giving neither $\mathrm{Herm}_{\mathbb{C}}$ nor Hilb. One possibility would be that the positive definite and negative definite line are both contained in P, but their direct sum given by the hyperbolic 2-dimensional space is not. To avoid these more pathological examples, we could require more compatibility conditions. For example, if we require P to be additionally closed under direct sums, then only Hilb and Herm remain as possibilities. Alternatively, we could follow a C^*-category style definition to completely single out Hilb here. In other words, we could require the monoidal dagger category to satisfy the stronger minimality requirement that every weakly positive operator is positive, compare Proposition 2.3.36. Note however that these requirements are very specific about the $\mathbb{C}$-linear situation. For example, we do not know how to apply similar considerations to ask which bordism dagger categories should be considered 'positive' in the sense Hilb might be called a 'positive' dagger category. Namely, bordism categories do not have direct sums and asking every weakly positive morphism to be positive is typically unreasonable.

Remark 2.4.12. If $\mathcal{C}$ is a monoidal anti-involutive category, there can in general be multiple Hermitian pairings on the monoidal unit. Namely, such a Hermitian pairing is equivalent to a 'self-adjoint' isomorphism $h : 1 \to 1$, i.e. such that $dh = h$ (using $d1 = 1$). Two such Hermitian pairings h_1, h_2 are equivalent modulo transfer if and only if they differ by an 'inv-positive', i.e. $h_1 = h_2 \circ df \circ f$, compare Example 2.3.42. For example, for $\mathcal{C} = \mathrm{Vect}$, the monoidal unit $\mathbb{C}$ has $\mathbb{R}^\times$-many self-adjoint automorphisms and $\mathbb{R}_{>0}$ many inv-positive automorphisms. The fact that $\mathbb{R}^\times/\mathbb{R}_{>0} \cong \mathbb{Z}/2$, corresponds to the fact that there are two equivalence classes of Hermitian pairings on $\mathbb{C}$; the positive definite and negative definite Hermitian pairing. However, in a bordism category there exists a unique isomorphism $\emptyset \to \emptyset$ and so $\emptyset$ has a unique Hermitian pairing independent of the chosen monoidal anti-involution. When choosing the trivial Hermitian pairing on 1, the dagger of $f : 1 \to 1$ is simply df. In particular, if d is of the form $\overline{(.)}^*$ for a covariant involution $\overline{(.)}$, we can use the canonical self-duality data $1 \otimes 1 \to 1$ to realize that $f^\dagger = \overline{f}$.

Our goal is to prove the following analogue of Theorem 2.3.15.

Theorem 2.4.13. *The* 2*-category of monoidal dagger categories is equivalent to the* 2*-category of monoidal anti-involutive categories equipped with an equivalence class of monoidal positivity structures:*

$$\mathrm{Cat}^\dagger_{\mathbb{E}_1} \cong (\mathrm{aICat}_{\mathbb{E}_1})_P.$$

Theorem 2.4.14. Herm *is adjoint to the* 2*-functor*

$$\mathrm{Cat}^\dagger_{\mathbb{E}_1} \to \mathrm{aICat}_{\mathbb{E}_1}.$$

It induces an equivalence

$$\mathrm{Cat}^\dagger_{\mathbb{E}_1} \cong (\mathrm{aICat}_{\mathbb{E}_1})_P,$$

between monoidal dagger categories and monoidal anti-involutive categories with positivity structure.

Proof. We first make the unit and the counit of the adjunction for the nonmonoidal case given in Remark 2.3.17 into monoidal functors. Let $(\mathcal{C}, d, \eta, \chi)$ be a monoidal anti-involutive category. Then the anti-involutive functor $K_{\mathcal{C}} : \mathrm{Herm}\,\mathcal{C} \to \mathcal{C}$ has tautological monoidal data, which is associative. The functor intertwines monoidal anti-involutive functors $\mathcal{C}_1 \to \mathcal{C}_2$ with their induced monoidal dagger functors $\mathrm{Herm}\,\mathcal{C}_1 \to \mathrm{Herm}\,\mathcal{C}_2$. The fact that the natural isomorphism $h_\bullet : K_{\mathcal{C}} \circ \dagger \cong d \circ K_{\mathcal{C}}$ is monoidal follows by the formula for the tensor product of two Hermitian pairings.

Now let $\mathcal{D}$ be a monoidal dagger category. We make $U_{\mathcal{D}} : \mathcal{D} \to \operatorname{Herm} \mathcal{D}$ into a monoidal functor with the identity monoidal data which is natural and associative, also see Example 2.4.9. The identity is clearly also a unitary isomorphism

$$(x_1, \mathrm{id}_{x_1}) \otimes (x_2, \mathrm{id}_{x_2}) \cong (x_1 \otimes x_2, \mathrm{id}_{x_1 \otimes x_2})$$

in $\operatorname{Herm} \mathcal{D}$.

Next, we note that U is still natural in $\mathcal{D}$ and K is natural in $\mathcal{C}$, where we used that Herm is a 2-functor, see Corollary 2.4.7. The fact that U and K still define a strict 2-adjunction in this monoidal scenario follows and so we proved the first statement.

For the second statement, we recall from the proof of the main theorem of [68] that in the non-monoidal case U and K restrict to equivalences. More precisely, $U_{\mathcal{D}}$ gives an equivalence between the dagger category $\mathcal{D}$ and the dagger subcategory of $\operatorname{Herm} \mathcal{D}$, given by the positivity structure explained in Example 2.3.40. Since we already proved that $U_{\mathcal{D}}$ is a monoidal dagger functor and $U_{\mathcal{D}}$ restricts to an equivalence of dagger categories which is still a monoidal functor, it restrict to an equivalence of monoidal dagger categories. If P is a monoidal positivity structure on $\mathcal{C}$ it induces a monoidal positivity structure on $\operatorname{Herm} \mathcal{C}$. Moreover, $K_{\mathcal{C}}$ preserves these positivity structures. Therefore if P is monoidal, $K_{\mathcal{C}}$ induces an equivalence of monoidal anti-involutive categories equipped with monoidal positivity structures. □

We finish with the braided and symmetric situations, which are easier because the coherence data is on higher morphisms. A braided/symmetric monoidal dagger category is an $\mathbb{E}_2/\mathbb{E}_\infty$-monoid in dagger categories:

Definition 2.4.15. A *braided monoidal dagger category* is a monoidal dagger category equipped with a unitary braiding. A *symmetric monoidal dagger category* is a braided monoidal dagger category with symmetric braiding. Let $\mathrm{Cat}^{\dagger}_{\mathbb{E}_2}$ denote the 2-category of braided monoidal dagger categories and let $\mathrm{Cat}^{\dagger}_{\mathbb{E}_\infty}$ denote the 2-category of symmetric monoidal dagger categories.

We want to show

Theorem 2.4.16. Herm *extends to a* 2*-functor*

$$\mathrm{aICat}_{\mathbb{E}_2} \to \mathrm{Cat}^{\dagger}_{\mathbb{E}_2}$$

and a 2*-functor*

$$\mathrm{aICat}_{\mathbb{E}_\infty} \to \mathrm{Cat}^{\dagger}_{\mathbb{E}_\infty} .$$

For braided and symmetric monoidal anti-involutive categories $\operatorname{Herm} \mathcal{C}$ is defined as follows.

Lemma 2.4.17. *Suppose the monoidal anti-involution (d, η, u, χ) on the braided monoidal category $\mathcal{C}$ is braided in the sense that $d : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is a braided monoidal functor. Then* $\operatorname{Herm} \mathcal{C}$ *has a unique braiding such that it becomes a braided monoidal dagger category and* $\operatorname{Herm} \mathcal{C} \to \mathcal{C}$ *is a braided monoidal functor. If $\mathcal{C}$ is symmetric, then so is* $\operatorname{Herm} \mathcal{C}$*.*

Proof. As before, because $\operatorname{Herm} \mathcal{C} \to \mathcal{C}$ is an equivalence of monoidal categories onto the objects that admit a Hermitian pairing, the braiding on $\operatorname{Herm} \mathcal{C}$ exists and is uniquely determined by the desired condition. We have to show that $(c_1, h_1) \otimes (c_2, h_2) \cong (c_2, h_2) \otimes (c_1, h_1)$ is a unitary isomorphism. For this we have to show the diagram

$$\begin{tikzcd}
d(c_1 \otimes c_2) & d(c_2 \otimes c_1) \arrow[l, "d(\beta_{c_1,c_2})"'] \\
dc_1 \otimes dc_2 \arrow[u] \arrow[r, "\beta_{dc_1,dc_2}"] & dc_2 \otimes dc_1 \arrow[u] \\
c_1 \otimes c_2 \arrow[u, "h_1 \otimes h_2"] \arrow[r, "\beta_{c_1,c_2}"] & c_2 \otimes c_1 \arrow[u, "h_2 \otimes h_1"']
\end{tikzcd}$$

commutes. The lower part commutes because the braiding is a natural isomorphism and the upper part commutes because d is braided, recalling the braiding defined on $\mathcal{C}^{\mathrm{op}}$ before. If the braiding on $\mathcal{C}$ is symmetric, the induced braiding on $\operatorname{Herm} \mathcal{C}$ is immediately symmetric too. □

Given two braided monoidal categories and and a monoidal functor between them, being braided is a condition. Given a monoidal dagger category that is also braided, being a braided monoidal dagger category is a condition: the braiding is unitary. However, every functor between two braided monoidal dagger categories that is both braided monoidal and dagger is automatically a braided monoidal dagger functor. Given that we have shown that $\operatorname{Herm} \mathcal{C}$ in that case is a braided monoidal dagger category, we automatically get the 2-functor Herm from braided monoidal anti-involutive categories to braided monoidal dagger categories. Symmetric monoidal dagger categories therefore do not any further complications. We have thus proven Theorem 2.4.16. Finally note that the analogues of Theorem 2.3.15 and Theorem 2.4.14 for the braided and symmetric case are now obvious:

Theorem 2.4.18.
- *The 2-category of braided monoidal dagger categories is equivalent to the 2-category of braided monoidal anti-involutive categories equipped with monoidal positivity structure:*
$$\mathrm{Cat}^{\dagger}_{\mathbb{E}_2} \cong (\mathrm{aICat}_{\mathbb{E}_2})_P.$$
- *The 2-category of symmetric monoidal dagger categories is equivalent to the 2-category of symmetric monoidal anti-involutive categories equipped with monoidal positivity structure:*
$$\mathrm{Cat}^{\dagger}_{\mathbb{E}_\infty} \cong (\mathrm{aICat}_{\mathbb{E}_\infty})_P.$$

Proof. Following the line of proof of Theorem 2.4.14 in [68], we see that the only thing left to show is that the unit and counit are braided, which is obvious. □

Remark 2.4.19. The reason that the analogue of Corollary 2.3.26 for monoidal dagger categories does not hold, is that $\pi_0\mathcal{C}$ and $\pi_0^h\mathcal{C}$ will be monoids and the surjective monoid map $\pi_0^h\mathcal{C} \to \pi_0\mathcal{C}$ need not have a splitting. We will see a concrete example of this relevant for unitary topological field theory in Chapter 6.

Example 2.4.20. If $\mathcal{C}$ is a monoidal anti-involutive category with monoidal positivity structure P, then there is a canonical induced monoidal positivity structure P^{op} on $\mathcal{C}^{\circ\mathrm{op}}$ with the anti-involution given in Example 2.2.8 given by the inverses of elements of P. This construction is in agreement with the obvious definition of the opposite of a monoidal dagger category. Also note that there is a canonical induced positivity structure $P^{\otimes\mathrm{op}}$ on $\mathcal{C}^{\otimes\mathrm{op}}$ given by elements of P. It is similarly straightforward to verify that the resulting monoidal dagger category is monoidally unitarily equivalent to the canonical monoidal dagger structure on $(\mathcal{C}_P)^{\otimes\mathrm{op}}$.

2.5 Self-adjoint modifications

It follows by Theorem 2.3.41 that for every anti-involutive category $\mathcal{C}$ in which every object admits a Hermitian pairing, there is a lattice of dagger categories equivalent to $\mathcal{C}$ as an anti-involutive category. This lattice is the poset of positivity structures, i.e. subsets of $\pi_0^h(\mathcal{C})$ that surject to $\pi_0(\mathcal{C})$, see Lemma 2.3.25. This induces a commutative diagram of fully faithful and essentially surjective dagger functors $\mathcal{C}_P \to \mathcal{C}_{P'}$ for $P \subseteq P'$. These functors are unitarily essentially surjective (and hence unitary equivalences) if and only if $P = P'$. It follows by Lemma 2.3.36 that in this lattice the minimal elements are those for which weakly inv-positive and inv-positive agree, while for the unique maximal element (the Hermitian completion) all self-adjoint automorphism are weakly inv-positive. We emphasize that it can still happen that for two different positivity structures P, P' on $\mathcal{C}$ the dagger categories $\mathcal{C}_P$ and $\mathcal{C}_{P'}$ are unitarily equivalent, as long as neither is contained in the other. Note that they can not be equivalent by an equivalence that respects the functor to $\mathcal{C}$ as an anti-involutive category, because the identity anti-involutive functor on $\mathcal{C}$ lifts to a dagger functor $\mathcal{C}_P \to \mathcal{C}_{P'}$ only if $P \subseteq P'$.

However, the equivalence can cover the identity functor on $\mathcal{C}$ with interesting anti-involutivity data ξ, as we will now explore. This is for example the case for the positivity structures on Vect given by the positive and the negative definite inner products. By Theorem 2.3.41, this happens if and only if the identity functor on $\mathcal{C}$ with anti-involutivity data ξ maps P to P'. We will show that the anti-involutive functor $(\mathrm{id}_{\mathcal{C}}, \xi)$ will map P to P_ξ where

$$P_\xi := \{c_2 \xrightarrow{\xi_{c_2}} c_2 \xrightarrow{h} dc_2 : h \in P\},$$

which we will call the *self-adjoint modification of P by ξ*. By fixing P and varying the natural automorphism ξ we thus get a collection of equivalent dagger categories $\mathcal{C}_{P_\xi}$. In case $P_\xi = P$, we obtain an automorphism of $\mathcal{C}_P$. Note that one can think of the data of ξ as a self-adjoint natural automorphism of the identity functor, in the sense that it satisfies $d\xi_x = \xi_{dx}$. We will also see that P_ξ and P are equivalent if and only if ξ_x is positive for all $x \in \mathcal{C}$, in the sense of Corollary 2.5.5. More generally, if $F : \mathcal{C}_P \to \mathcal{D}_Q$ is a dagger functor we will define a notion of self-adjoint natural automorphism ξ of F in Lemma 2.5.1. We can then change the anti-involutivity data of F by ξ to make it into a dagger functor $F_\xi : \mathcal{C}_P \to \mathcal{D}_{Q_\xi}$. We will call this the *self-adjoint modification of F by ξ*. We focus on the symmetric monoidal case, because this will be our main interest for topological field theory. In particular, the main application of this theory will be Section 2.7, where we study self-adjoint modifications of the canonical anti-involutive dual functor on a symmetric monoidal anti-involutive category with duals.

Lemma 2.5.1. *Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a symmetric monoidal anti-involutive functor between symmetric monoidal anti-involutive categories and let ξ be a monoidal natural automorphism of F. Then composing the equivariance data $\phi_x : F(dx) \cong dF(x)$ with $d\xi_x$ gives another symmetric monoidal anti-involutive functor if and only if ξ is self-adjoint natural automorphism of (F, ϕ) in the sense that $\xi_{d^2x}\phi_{dx} = \phi_{dx} d\xi_{dx}$ for all $x \in \mathcal{C}_1$.*

Proof. Since ξ and ϕ are monoidal natural isomorphisms, so is $\xi_{dx} \circ \phi_x : F(dx) \to dF(x)$. We have to show that

$$\begin{tikzcd} F(x) \arrow[r, "F(\eta_x)"] \arrow[d, "\eta_{F(x)}"] & F(d^2x) & F(d^2x) \arrow[l, "\xi_{d^2x}"'] \\ d^2F(x) & dFd(x) \arrow[l, "d\phi_x"] \arrow[u, "\phi_{dx}"] & dFd(x) \arrow[l, "d\xi_{dx}"] \arrow[u, "\phi_{dx}"'] \end{tikzcd}$$

commutes. The left diagram commutes because ϕ makes F into an anti-involutive functor, the right square commutes because ξ is self-adjoint. $\square$

Remark 2.5.2. The terminology of Lemma 2.5.1 is motivated by the case where F preserves d on the nose and $\phi_x = \mathrm{id}_{F(x)}$ for all $x \in \mathcal{C}_1$, for example when F is a †-functor between †-categories. In that case ξ is a self-adjoint natural automorphism of (F, ϕ) if and only if $\xi_{dx} = d\xi_x$ for all $x \in \mathcal{C}_1$. This is more generally true if $\phi_x \neq \mathrm{id}_{F(x)}$ but when it commutes with ξ. As a subexample, we could take a self-adjoint monoidal natural automorphism of $\mathrm{id}_{\mathcal{C}_1}$ and apply F to it to get an self-adjoint monoidal natural automorphism of F. This gives us a way to change the anti-involutive data of a functor by a natural automorphism of the identity functor on the source.

We provide a converse statement. For this, note that monoidal self-adjoint natural automorphisms of $\mathrm{id}_{\mathcal{C}_1}$ form a group. Namely, self-adjoint automorphisms of a fixed object do not form a group, but the product of two commuting self-adjoint automorphisms is again self-adjoint.

Lemma 2.5.3. *Let $\mathcal{C}_1, \mathcal{C}_2$ be symmetric monoidal anti-involutive categories and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a symmetric monoidal equivalence. Suppose $\phi, \phi' : F(dx) \cong dF(x)$ are two data making F into a monoidal anti-involutive functor. Then the composition*

$$\phi(\phi')^{-1}$$

induces an self-adjoint monoidal natural automorphism of $\mathrm{id}_{\mathcal{C}_1}$. In particular, anti-involutivity data of F is a torsor over monoidal self-adjoint natural automorphism of $\mathrm{id}_{\mathcal{C}_1}$.

Proof. Because ϕ and ϕ' are monoidal natural isomorphisms, so is $\phi(\phi')^{-1} : dF \Rightarrow dF$. Since d and F are equivalences, such a natural isomorphism corresponds with a monoidal natural automorphism of $\mathrm{id}_{\mathcal{C}_1}$. This boils down to the fact that the diagram

$$\begin{tikzcd}
dFdx \arrow[r, "\phi_{dx}"] \arrow[d, "d\phi_x"] & F(d^2x) \arrow[r, "\phi'_{dx}"] & F(d^2x) \arrow[d, "d\phi'_x"] \\
d^2F(x) \arrow[rr, bend right, equal] & F(x) \arrow[l, "\eta_{F(x)}"] \arrow[u, "F(\eta_x)"'] \arrow[r, "\eta_{F(x)}"] & d^2F(x)
\end{tikzcd}$$

commutes. Since this construction is inverse to the construction of Lemma 2.5.1, we see that the group of monoidal anti-involutive automorphisms of the identity functor of $\mathcal{C}_1$ acts freely and transitively on anti-involutive symmetric monoidal equivalences with the same underlying symmetric monoidal functor. $\square$

To generalize the above discussion to dagger categories, we include positivity structures into the game. Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category, P a monoidal positivity structure and ξ a monoidal natural automorphism of $\mathrm{id}_{\mathcal{C}_P}$ which is objectwise self-adjoint. By Theorem 2.4.14, ξ corresponds uniquely to a monoidal self-adjoint natural automorphism of the identity functor on the anti-involutive category $\mathcal{C}$ in the sense of Lemma 2.5.1.

Corollary 2.5.4. *Let $F : \mathcal{C}_P \to \mathcal{C}'_Q$ be a symmetric monoidal dagger functor between symmetric monoidal dagger categories induced by monoidal positivity structures P on the symmetric monoidal anti-involutive category $\mathcal{C}$, and Q on the symmetric monoidal anti-involutive category $\mathcal{C}'$. If ξ is a self-adjoint monoidal natural automorphism of $\mathrm{id}_{\mathcal{C}_P}$, then there is an induced symmetric monoidal dagger functor $F_\xi : \mathcal{C}_P \to \mathcal{C}'_{Q_{F(\xi)}}$.*

Proof. By Theorem 2.4.14, F corresponds to an anti-involutive functor which maps P to Q. Therefore by applying Lemma 2.5.1 to the self-adjoint automorphism of F induced by ξ, we get a new anti-involutive functor. It maps to positive pairing $h : c \to dc$ in P to

$$F(c) \xrightarrow{F(h)} F(dc) \xrightarrow{\phi_c} dF(c) \xrightarrow{dF(\xi_c)} dF(c),$$

where ϕ denotes the equivariance data of the original functor F. Because the original functor preserved positive pairings, we have that the composition of the first two morphisms is in Q. Therefore the whole composition is in $Q_{F(\xi)}$. □

Let $F : \mathcal{C}_P \to \mathcal{C}'_Q$ be a symmetric monoidal dagger functor and ξ, ξ' two self-adjoint natural monoidal automorphisms as in Corollary 2.5.4. Note that if $(h : x \to dx) \in P$, then $F(h) \circ F(\xi_x)$ is a transfer of $F(h) \circ F(\xi'_x)$ if and only if $F(\xi_x'^{-1}\xi_x)$ is inv-positive. In that case we therefore have that $Q_{F(\xi)} = Q_{F(\xi')}$.

This requirement is closely related to when two modified anti-involutive functors are unitarily equivalent. Indeed, let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between anti-involutive categories with two anti-involutivity data ϕ_1, ϕ_2. Then (F, ϕ_2) and (F, ϕ_1) are equivalent as anti-involutive functors if and only if there exists a natural automorphism $u : F \Rightarrow F$ such that

$$\begin{tikzcd}
F(dx) \arrow[r, "u_{dx}"] \arrow[d, "(\phi_1)_x"] & F(dx) \arrow[d, "(\phi_2)_x"] \\
F(dx) & F(dx) \arrow[l, "du_x"]
\end{tikzcd}$$

commutes. We obtain the following corollary.

Corollary 2.5.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be an anti-involutive functor and ξ a self-adjoint automorphism of F. Then F is equivalent as an anti-involutive functor to the self-adjoint modification F_ξ if and only if ξ is positive in the sense that*

$$\xi_{dx} = du_x(\phi_2)_x u_{dx}(\phi_2)_x^{-1}$$

Proof. Let ϕ denote the anti-involutivity data of F. We apply the observation above to the case where $\phi_2 = \phi$ and ϕ_1 is the composition of ϕ with ξ_{dx}. The conclusion is that the diagram

$$\begin{tikzcd}
F(dx) \arrow[dd, "\phi_x"] & F(dx) \arrow[l, "u_{dx}"'] \arrow[d, "\phi_x"] \\
 & dF(x) \arrow[d, "\xi_{dx}"] \\
dF(x) \arrow[r, "u_{dx}"] & d(x)
\end{tikzcd}$$

has to commute. □

We will study the above in the special case where $\xi_c^2 = \mathrm{id}_c$ in Sections 2.6 and 2.9.

Remark 2.5.6. In the special case of Corollary 2.5.5 where $F : \mathcal{D} \to \mathcal{D}'$ is a dagger functor between dagger categories, its anti-involutivity data is equal to the identity. Therefore a self-adjoint automorphism of F is a natural automorphism ξ of F such that ξ_x is self-adjoint for all x. Moreover, F

is equivalent as an anti-involutive functor to the modification F_ξ if and only if ξ_x is positive for all x. Note that this is indeed exactly the case that F preserves positivity structures; if P is the canonical positivity structure of $\mathcal{C}$, then $F(P)_\xi$ and $F(P)$ are equivalent positivity structures in this case. In other words, this is consistent with Theorem 2.4.14.

Example 2.5.7. Consider the identity dagger functor on the dagger category $\mathcal{D} = \mathrm{Herm}_{\mathbb{C}}$ of Hermitian vector spaces. We ignore monoidal structures for now. Recall that natural automorphisms of the identity in Vect are given by multiplication by some scalar $\lambda \in \mathbb{C}^\times$. This gives a self-adjoint natural automorphism of the identity on $\mathrm{Herm}_{\mathbb{C}}$ if and only if $\lambda \in \mathbb{R}^\times$. These induce dagger functors on $\mathcal{D}$ that map a Hermitian vector space to the same vector space with its Hermitian form multiplied by λ. Two such functors are unitarily isomorphic if and only if the self-adjoint natural isomorphisms differ by a positive. Note how nonpositive λ is allowed because in this category multiplying a form by a $\lambda \in \mathbb{R}_{<0}$ gives a new allowed form. If we would be working with $\mathcal{D} = \mathrm{Hilb}$, a negative λ would map us to the dagger category where the positivity structure is modified by λ, i.e. Hermitian vector spaces with negative definite inner product. Note that all these self-adjoint natural automorphisms are never monoidal unless they are the identity. In the symmetric monoidal dagger category sHilb there are instead $\mathbb{R}^\times \times \mathbb{R}^\times$-many self-adjoint natural automorphisms of the identity functor.

Remark 2.5.8. From another perspective, the above gives an analysis of functors between dagger categories which are anti-involutive but not necessarily dagger at first glance. Indeed, let $\mathcal{D}_1, \mathcal{D}_2$ be dagger categories and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be an anti-involutive functor. Writing out the definition of an anti-involutive functor, we realize that we are given a collection of automorphisms $\xi_c : F(c) \to F(c)$ for all objects $c \in \mathcal{D}_1$ which do not assemble into a natural automorphism of F, but instead it measures the failure of F to be a dagger functor. More precisely, for all morphisms $f : c \to c'$ the diagram

$$\begin{tikzcd}
F(c^\dagger) \arrow[r, equal] & F(c) \arrow[d, "\xi_c"] & F(c') \arrow[l, "F(f^\dagger)"'] \arrow[r, equal] \arrow[d, "\xi_{c'}"] & F(c'^\dagger) \\
F(c)^\dagger \arrow[r, equal] & F(c) & F(c') \arrow[l, "F(f)^\dagger"'] \arrow[r, equal] & F(c')^\dagger
\end{tikzcd}$$

commutes and $\xi_c = \xi_{c^\dagger} = \xi_c^\dagger$ so that ξ_c is self-adjoint for all $c \in \mathcal{D}_1$. Note that F is a dagger functor if and only if ξ is a natural transformation $F \Rightarrow F$. We can try to change ξ by another self-adjoint natural automorphism ξ' of F, but this will potentially map us out of $\mathcal{D}_2$ into a different positivity structure.

2.6 $B\mathbb{Z}/2$-actions

Most categories of 'fermionic origin' come equipped with an action of the 2-group $B\mathbb{Z}/2$. Here we define a 2-group to be a monoidal category in which all objects and morphisms are invertible under tensor product and all morphisms are invertible under composition. For a review of 2-groups and their actions on categories, see Appendix A.2. If A is an abelian group, we denote by BA the 2-group with 1 object and $\mathrm{Aut}(*) = A$. Note that a BA-action on a category $\mathcal{C}$ consists of a collection of natural automorphisms of the identity $a_c \in \mathrm{Aut}\, c$ for all $a \in A$ that satisfy $a_c \circ a'_c = (aa')_c$ for all $c \in \mathcal{C}$ and $a, a' \in A$.

In this section we will study anti-involutive and dagger categories with $B\mathbb{Z}/2$-action. The main goal is to clarify some sign subtleties in the dagger category of super Hilbert spaces and to see to which extent they generalize to other dagger categories with $B\mathbb{Z}/2$-action. The study of categorical aspects of the spin-statistics theorem will be postponed to Section 2.9 after we have obtained a sufficient understanding of unitary dual functors in Section 2.7.

Definition 2.6.1. Let $(\mathcal{C}, d, \eta)$ be a category with anti-involution. A BA-action $(a, c) \mapsto a_c \in \operatorname{Aut} c$ on $\mathcal{C}$ is anti-involutive if $a_\bullet$ is an involutive natural automorphism for all $a \in A$. In other words, $(a^{-1})_{dc} = da_c$ for all $a \in A$ and $c \in \operatorname{obj} \mathcal{C}$.

Note that a BA-action on a symmetric monoidal category is symmetric monoidal if $a_\bullet$ is a monoidal natural automorphism for all $a \in A$.

Example 2.6.2. sVect has a canonical symmetric monoidal $B\mathbb{Z}/2$-action given by mapping a vector space V to the grading operator $(-1)^F_V : V \to V$. It is anti-involutive for $V \mapsto \overline{V}^*$ because the $B\mathbb{Z}/2$ and $\mathbb{Z}/2$-actions commute.

In analogy with the category of super vector spaces, we typically write a $B\mathbb{Z}/2$-action on a category $\mathcal{C}$ as $c \mapsto (-1)^F_c \in \operatorname{Aut} c$ for $c \in \mathcal{C}$. Note that in the original physics applications there is a distinguished 'fermion number operator' F on V such that $(-1)^F$ is its parity. This suggestive notation has completely lost its meaning and the symbol F does not refer to anything here. Recall that when defining super Hilbert spaces, we had to make an arbitrary choice for vectors of odd parity; we decided to call the vectors with norm a positive multiple of i positive, see Definitions 2.1.5 and 2.1.4. We could have chosen to call a negative multiple of i positive instead in the latter definition. The first thing we make precise is the analogous result for general symmetric monoidal dagger categories with $B\mathbb{Z}/2$-action. Recall that if $\mathcal{C}$ is anti-involutive, P is a positivity structure and $\xi : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$, we denote the modified positivity structure by

$$P_\xi := \{x \xrightarrow{\xi_x} x \xrightarrow{h} dx : h \in P\}.$$

Proposition 2.6.3. *Let $\mathcal{C}$ be a (symmetric monoidal) anti-involutive category with (symmetric monoidal) anti-involutive $B\mathbb{Z}/2$-action and monoidal positivity structure P. Then, there is a symmetric monoidal dagger equivalence*

$$\mathcal{C}_P \cong \mathcal{C}_{P_{(-1)^F}}.$$

Proof. This follows from Corollary 2.5.4 by taking F to be the identity functor and $\xi = (-1)^F$. Indeed, note that because $\xi_x^2 = \mathrm{id}_x$ and ξ_x is anti-involutive, ξ is also self-adjoint. □

Remark 2.6.4. Depending on which description is taken of $\mathcal{C}_{P_{(-1)^F}}$, it can be confusing to distinguish $\mathcal{C}_{P_{(-1)^F}}$ and $\mathcal{C}_P$ at all. For example, let $\mathcal{D}$ be a dagger category. If we consider $\mathcal{D}$ as an anti-involutive category, it has a canonical positivity structure P given by inv-positives in $\mathcal{D}$. Then $\mathcal{D} \cong \mathcal{D}_P \subseteq \operatorname{Herm} \mathcal{D}$ as †-categories, see Example 2.3.40. In this description, $\mathcal{D}_P$ is basically equal to $\mathcal{D}_{P_{(-1)^F}}$. By definition, the former category has objects pairs of an object $x \in \mathcal{D}$ and an automorphism of x of the form $f^\dagger f$ where $f \in \operatorname{Aut} x$. Whereas the latter has objects pairs of an object $x \in \mathcal{D}$ and an automorphism of x of the form $f^\dagger f(-1)^F_x$ where $f \in \operatorname{Aut} x$. Morphisms between such pairs are given by morphisms in $\mathcal{D}$ in both cases. It is straightforward to check using naturality of $(-1)^F$ that the new formula for the dagger on $\mathcal{D}_P$ and $\mathcal{D}_{P_{(-1)^F}}$ is exactly the same.

As an example, consider the case of $\mathcal{C} = \mathrm{sVect}$ with $d = \overline{(.)}^*$ and $\mathcal{C}_P = \mathrm{sHilb}$. Then, $\mathcal{C}_{P_{(-1)^F}}$ is the symmetric monoidal dagger category of super Hermitian vector spaces with positive definite even part and negative definite odd part. The formula for the adjoint of an even linear map is not affected by changing the Hermitian form from positive definite to negative definite on the odd part. However, we want to think of these two dagger categories as 'different but equivalent'; allowing both

P and $P_{(-1)^F}$ as positive pairings would certainly not be a good idea. Thus, we have to arbitrarily pick one over the other and stick to it. This minor difference will play an important role in the spin-statistics theorem, see Section 2.9.

The following discussion generalizes and formalizes the fact that $\mathbb{Z}/2$-graded Hilbert spaces are in one-to-one correspondence with super Hilbert spaces, a result we have seen in Remark 2.1.7. It gives us a way to similarly talk about an 'ungraded symmetric' convention for other anti-involutive categories with $B\mathbb{Z}/2$-actions, such as spin bordism groups.

Definition 2.6.5. Let $(\mathcal{C}, d, \eta)$ be a category with anti-involution and anti-involutive $B\mathbb{Z}/2$-action $c \mapsto (-1)^F_c$. Then, the *opposite anti-involution* of $(\mathcal{C}, d, \eta)$ is the same d, but with η changed by $(-1)^F$.

Note that the opposite anti-involution again defines an anti-involution. Also note that the opposite of the opposite anti-involution is the original anti-involution. A Hermitian pairing for the opposite anti-involution of $(\mathcal{C}, d, \eta)$ is the same as an isomorphism $h : c \to dc$ such that the diagram

$$\begin{tikzcd} c \arrow[r, "h"] \arrow[d, "(-1)^F_c"'] & dc \\ c \arrow[r, "\eta_c"] & d^2c \arrow[u, "dh"'] \end{tikzcd}$$

commutes. When $\mathcal{C} = \mathrm{sVect}$ with the standard anti-involution, this is equivalent to a $\mathbb{Z}/2$-graded Hermitian pairing on a super vector space V, i.e. a nondegenerate sesquilinear even pairing on V such that

$$\langle v, w \rangle = \overline{\langle w, v \rangle} \quad \forall v, w \in V.$$

Therefore, we can think of $(\mathcal{C}, d, \eta \circ (-1)^F)$ as 'switching between the ungraded and the super sign convention' of $(\mathcal{C}, d, \eta)$.

Remark 2.6.6. Given an anti-involutive category $(\mathcal{C}, d, \eta)$ with anti-involutive $B\mathbb{Z}/2$-action $(-1)^F$, we can consider the category of pairs of an object c and an isomorphism $h : c \to dc$ such that

$$\begin{tikzcd} c \arrow[r, "h"] \arrow[d, "(-1)^F_c"'] & dc \\ c \arrow[r, "\eta_c"] & d^2c \arrow[u, "dh"'] \end{tikzcd}$$

commutes. It is still possible to directly construct a dagger for morphisms $f : (c_1, h_1) \to (c_2, h_2)$ by the same formula as in the Hermitian completion. Indeed, the fact that $f^{\dagger\dagger} = f$ follows by naturality of $(-1)^F$. In the special case of sVect this will give the usual adjoint of linear maps between $\mathbb{Z}/2$-graded Hilbert spaces. However, this dagger category is simply the Hermitian completion of $(\mathcal{C}, d, \eta \circ (-1)^F)$, so this construction provides no new insights.

Next we want to show that if there exists a notion of 'i^F' just as in sVect, then the two grading conventions given by $(\mathcal{C}, d, \eta \circ (-1)^F)$ and $(\mathcal{C}, d, \eta)$ are equivalent:

Proposition 2.6.7. *Let $(\mathcal{C}, d, \eta)$ be a category with anti-involution and anti-involutive $B\mathbb{Z}/2$-action $(-1)^F$. Suppose the action refines to an anti-involutive $B\mathbb{Z}/4$-action, which we suggestively write as i^F so that $i^F \circ i^F = (-1)^F$. Let $h : c \to dc$ be a Hermitian pairing for $(\mathcal{C}, d, \eta)$. Then $h \circ i^F_c$ is a Hermitian pairing for the opposite anti-involution $(\mathcal{C}, d, \eta \circ (-1)^F)$.*

Proof. By definition of the action being anti-involutive we have

$$d(i^F_c) = (i^{-1})^F_{dc} = (-1)^F_{dc} \circ i^F_{dc}$$

and so

$$d(h \circ i^F_c) = d(i^F_c) \circ dh = (-1)^F_{dc} \circ i^F_{dc} \circ h \circ \eta_c^{-1}.$$

Using naturality of the $B\mathbb{Z}/2$-action, we obtain the desired. □

Example 2.6.8. The above condition is satisfied for $\mathrm{sVect} = \mathrm{sVect}_{\mathbb{C}}$ with its standard ant-involution and $B\mathbb{Z}/2$-action with i^F defined as in Lemma 2.1.12. Note that this does not work for $\mathrm{sVect}_{\mathbb{R}}$. Namely, the opposite anti-involution is not equivalent to the the canonical involution on $\mathrm{sVect}_{\mathbb{R}}$ given by $V \mapsto V^*$; in one of the two all objects admit Hermitian pairings, but in the other one some do not. To see this concretely, note that an odd-dimensional real vector space in odd degree admits an Hilbert space structure in the ungraded sense, but not in the graded sense.

With the assumptions of the last proposition, let P be a positivity structure on $(\mathcal{C}, d, \eta)$ and P_{i^F} the induced positivity structure on $(\mathcal{C}, d, \eta \circ (-1)^F)$. This notation does not quite agree with the notation P_ξ from before; because i^F is not self-adjoint it does not define a positivity structure on the original anti-involutive category $(\mathcal{C}, d, \eta)$.

Corollary 2.6.9. *The identity functor on $\mathcal{C}$ equipped with anti-involutive data i^F induces an equivalence of dagger categories $(\mathcal{C}, d, \eta, P) \cong (\mathcal{C}, d, \eta \circ (-1)^F, P_{i^F})$.*

Proof. By the correspondence between dagger functors and anti-involutive functors preserving positivity structures, we have to extend the identity functor to $\mathrm{Herm}\,\mathcal{C}$ by $h \mapsto h \circ i^F_c$. It is clearly an equivalence of categories and the last proposition shows it is well-defined. We have to show it is a dagger functor. Let $\ddagger$ denote the new dagger on $\mathcal{C}_{P_{i^F}}$. For $(c_1, h_1), (c_2, h_2)$ objects in $\mathcal{C}_P$, consider the diagram

$$\begin{tikzcd}
c_1 \arrow[d, "i^F_{c_1}"] & c_2 \arrow[l, "f^\ddagger"'] \arrow[d, "i^F_{c_2}"] \\
c_1 \arrow[d, "h_1"] & c_2 \arrow[l, "f^\dagger"'] \arrow[d, "h_2"] \\
dc_1 & dc_2 \arrow[l, "df"']
\end{tikzcd}.$$

The lower square commutes by definition of $f^\dagger$ and the total rectangle commutes by definition of $f^\ddagger$. Then by naturality of i^F we obtain $f^\dagger = f^\ddagger$ and so the identity is a dagger functor. □

Remark 2.6.10. Note that given a $B\mathbb{Z}/4$-action extending the $B\mathbb{Z}/2$-action, there is an inverse $B\mathbb{Z}/4$-action we denote $(i^{-1})^F$. The dagger categories $\mathcal{C}_{P_{(i^{-1})^F}}$ and $\mathcal{C}_{P_{i^F}}$ have their set of Hermitian pairings related by $(-1)^F$ and so can be different if $P \neq P_{(-1)^F}$, even though they are always equivalent as dagger categories by Proposition 2.6.3. For example, whether we choose the comparison map between $\mathbb{Z}/2$-graded Hermitian spaces and super Hermitian vector spaces to be i^F or $(i^{-1})^F$, will change the notion of super Hilbert space by a sign on the odd part.

In the above few results we ignored the monoidal structure, which will make the analysis more subtle. To find out how to deal with this, we return to our main example:

Remark 2.6.11. Recall from Remark 2.1.14 that to build the right monoidal dagger category of $\mathbb{Z}/2$-graded Hilbert spaces from the standard monoidal anti-involution (d, η) on sVect, we need to not only take the opposite anti-involution, but also the monoidal data $d(V \otimes W) \cong dV \otimes dW$ by a sign $v \otimes w \mapsto (-1)^{|v||w|} v \otimes w$. In other words, we should not expect an immediate generalization of Corollary 2.6.9 to the monoidal setting without changing the monoidal data of d. The crux is that i^F is not monoidal since the diagram

$$\begin{tikzcd} \Pi\mathbb{C} \otimes \Pi\mathbb{C} \arrow[r, "\cong"] \arrow[d, "i^F \otimes i^F"] & \mathbb{C} \arrow[d, "\mathrm{id}"] \\ \Pi\mathbb{C} \otimes \Pi\mathbb{C} \arrow[r, "\cong"] & \mathbb{C} \end{tikzcd}$$

does not commute. Instead, the automorphism

$$i^F_{V_1 \otimes V_2} \circ (i^F_{V_1} \otimes i^F_{V_2})^{-1}$$

of $V_1 \otimes V_2$ measuring the failure of i^F being monoidal is exactly given by $v \otimes w \mapsto (-1)^{|v||w|} v \otimes w$. Setting these signs up correctly then results in an equivalence of monoidal dagger categories between the monoidal dagger category of super Hilbert spaces and the monoidal dagger category of $\mathbb{Z}/2$-graded Hilbert spaces, given by changing Hermitian pairings to i^F.

The above remark motivates the following.

Definition 2.6.12. Let $(\mathcal{C}, d, \eta, \chi)$ be a monoidal anti-involutive category with monoidal anti-involutive $B\mathbb{Z}/2$-action and let P be a monoidal positivity structure. Suppose the $B\mathbb{Z}/2$-action admits a refinement to an anti-involutive $B\mathbb{Z}/4$-action i^F, which is not necessarily monoidal. Then the *opposite monoidal anti-involution of* $\mathcal{C}$ is the opposite anti-involution $(\mathcal{C}, d, \eta \circ (-1)^F)$ from before, where the monoidal structure $\chi_{x,y} : d(x \otimes y) \cong dx \otimes dy$ is changed to $\chi^{\mathrm{op}}_{x,y}$ which is the composition

$$i^F_{dx \otimes dy} \circ (i^F_{dx} \otimes i^F_{dy})^{-1} \circ \chi_{x,y}.$$

Proposition 2.6.13. *The functor $F : \mathcal{C}_P \cong \mathcal{C}_{P_{i^F}}$ from Corollary 2.6.9 is a monoidal unitary equivalence between $\mathcal{C}_P$ and the opposite monoidal anti-involution $(\mathcal{C}, d, \eta \circ (-1)^F, \chi^{\mathrm{op}})$ with positivity structure P_{i^F}.*

Proof. The only thing we have to check is that the isomorphism $F(c_1 \otimes c_2) \cong F(c_1) \otimes F(c_2)$ given by the identity on $c_1 \otimes c_2$ is unitary. This follows from the commutativity of the diagram

$$\begin{tikzcd}
c_1 \otimes c_2 \arrow[r, equal] \arrow[d, "i^F_{c_1\otimes c_2}"'] \arrow[dddr, "h_1\otimes h_2"] & c_1 \otimes c_2 \arrow[d, "i^F_{c_1}\otimes i^F_{c_2}"] \\
c_1 \otimes c_2 \arrow[ddd, "h_1\otimes h_2"] & c_1 \otimes c_2 \arrow[d, "h_1\otimes h_2"] \\
 & dc_1 \otimes dc_2 \arrow[d, "(i^F_{dc_1}\otimes i^F_{dc_2})^{-1}"] \\
 & dc_1 \otimes dc_2 \arrow[d, "i^F_{dc_1\otimes dc_2}"] \\
dc_1 \otimes dc_2 \arrow[r, equal] \arrow[d, "\chi_{c_1,c_2}"] & dc_1 \otimes dc_2 \arrow[d, "\chi_{c_1,c_2}"] \\
d(c_1 \otimes c_2) \arrow[r, equal] & d(c_1 \otimes c_2)
\end{tikzcd} .$$

□

Note that under Theorem 2.4.14, we could have equivalently formulated the above using a monoidal dagger category with a monoidal unitary $B\mathbb{Z}/2$-action on $\mathcal{C}$ which extends to a unitary but not necessarily monoidal $B\mathbb{Z}/4$-action. However, the dagger category $\mathcal{C}_{P_{i^F}}$ would be moderately inconvenient to formulate without referring to positivity structures.

Remark 2.6.14. The automorphism

$$i^F_{c_1\otimes c_2} \circ (i^F_{c_1} \otimes i^F_{c_2})^{-1}.$$

of $c_1 \otimes c_2$ that we change the monoidal data of d by is trivial if and only if the $B\mathbb{Z}/4$-action is monoidal. For super vector spaces over $\mathbb{C}$ the above automorphism is given by $v \otimes w \mapsto (-1)^{|v||w|} v \otimes w$. Note that in real super vector spaces this automorphism still exists, even though $(-1)^F$ does not admit a square root.

Remark 2.6.15. In the setting of Corollary 2.6.13, we see that there is an equivalence between monoidal dagger functors into $(\mathcal{C}, d, \eta, \chi, P)$ and monoidal functors into $(\mathcal{C}, d, \eta \circ (-1)^F, \chi^{\mathrm{op}}, P_{i^F})$. In particular, it is irrelevant for dagger functors into super-versions of Hilbert spaces whether we use the ungraded or super notions of Hilbert space, as long as we keep the appropriate translation in mind.

All results in this section until now generalize in a straightforward fashion to symmetric monoidal dagger categories with unitary $B\mathbb{Z}/2$-action (and possibly nonmonoidal unitary $B\mathbb{Z}/4$-action extending it). To state these results, we also reduce the above discussion to the case where the a symmetric monoidal anti-involutive category comes from a symmetric monoidal $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action with duals. By the results of Section 2.2, $dx := \overline{x}^*$ can be made into a symmetric monoidal anti-involution.

Lemma 2.6.16. *Let $\mathcal{C}$ be a symmetric monoidal category with symmetric monoidal $\mathbb{Z}/2 \times BA$-action and duals. Then the BA-action is anti-involutive with respect to $d = \overline{(.)}^*$. If P is a monoidal positivity structure on $(\mathcal{C}, d)$, then the anti-involutive BA-action extends uniquely to a unitary BA-action on $\mathcal{C}_P$.*

Proof. We have to show that $a_{\overline{x}^*} : \overline{x}^* \to \overline{x}^*$ is the inverse of $\overline{a_x}^*$ for all objects $x \in \mathcal{C}$ and $a \in A$. Since the $\mathbb{Z}/2$- and BA-actions assemble into an $\mathbb{Z}/2 \times BA$-action, we have that $a_{\overline{x}} = \overline{a_x}$. Because a_x is a monoidal natural automorphism, the first claim follows by Lemma A.1.9. Since the $a_\bullet$ are anti-involutive monoidal natural automorphisms of the identity functor, the second result follows from the correspondence between dagger categories and anti-involutions of Theorem 2.4.14. □

Corollary 2.6.17. *Let $\mathcal{C}$ be a symmetric monoidal category with $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action and duals. Let P be a monoidal positivity structure for $\overline{(.)}^*$. Suppose the $B\mathbb{Z}/2$-action refines to a non-monoidal anti-involutive $B\mathbb{Z}/4$-action. Then the symmetric monoidal anti-involutive category $(\mathcal{C}, \overline{(.)}^*, \eta \circ (-1)^F, \chi^{op})$ with positivity structure P is equivalent to the opposite monoidal anti-involution $(\mathcal{C}, \overline{(.)}^*, \eta \circ (-1)^F, \chi^{op})$ with positivity structure P_{i^F}.*

Note that changing the monoidal data $d(V \otimes W) \cong dV \otimes dW$ or $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$ by a sign as in the discussions above has no clear analogue in general symmetric monoidal categories with duals and $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action, in particular not for the bordism categories we will consider in Section 5.1. However, changing the monoidal data $d(V \otimes W) \cong dV \otimes dW$ does result in changing $d(V^*) \cong d(V)^*$ by the $B\mathbb{Z}/2$-action. This latter isomorphism will be important in the next section because it determines the data that makes a fixed dual functor anti-involutive. We can still change this anti-involutive data in an ad hoc way by a self-adjoint modification. In other words, once we pick a dual functor, we can decide to modify the canonical equivariance data $d(x^*) \cong d(x)^*$ by the $B\mathbb{Z}/2$-action. This will be an important consideration in Section 2.9.

2.7 Dagger duality

In this section, we will study the interaction between the dagger and the dual functor on symmetric monoidal dagger categories from the perspective of Section 2.3. With this goal in mind, we will first consider dual functors on symmetric monoidal anti-involutive categories $\mathcal{C}$. We will show they come equipped with canonical equivariance data for the anti-involution, so that there is always a canonical dual functor on the Hermitian completion, which will be a symmetric monoidal dagger functor. Suppose now that we are given a symmetric monoidal dagger category presented in the form $\mathcal{C}_P$, for a given monoidal positivity structure P on $\mathcal{C}$. It then becomes a condition whether or not the canonical dual functor on $\operatorname{Herm}\mathcal{C}$ restricts to a symmetric monoidal dagger functor on $\mathcal{C}_P$. This condition is whether the dual functor preserves the positivity data, which concretely means that duals of positive Hermitian pairings are positive.

Unfortunately, it is not always the case for examples relevant in this thesis that this dual functor on an anti-involutive category preserves the positive Hermitian pairings, see for example Proposition 2.1.6. These 'fermionic' examples such as super Hilbert spaces and 'spin-like' bordism categories we will see in Chapter 5, have some abstract features in common; they are all symmetric monoidal categories with duals and $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action. Also, their dagger category structure is most conveniently constructed by choosing Hermitian pairings for the anti-involution $x \mapsto \overline{x}^*$. We will study how to resolve the issue of the canonical dual functor not preserving positivity data by modifying some natural isomorphisms by the $B\mathbb{Z}/2$-action, such as the data $d(x^*) \cong d(x)^*$ specifying how $(.)^*$ is an anti-involutive functor.

We adopt the terminology of [58]:

Definition 2.7.1. Let $\mathcal{D}$ be a rigid monoidal dagger category. A choice of monoidal dual functor $\mathcal{D} \to \mathcal{D}^{\circ\mathrm{op},\otimes\mathrm{op}}$ is called a *unitary dual functor* if it is a monoidal dagger functor.

Note that it is a condition for a monoidal dual functor on a monoidal dagger category to be a unitary dual functor; we need that $f^{*\dagger} = f^{\dagger *}$ for all morphisms and that the uniqueness of duals isomorphism $(x \otimes y)^* \cong y^* \otimes x^*$ is unitary for all $x, y \in \mathcal{D}$.

Both dagger categories and dual functors are challenging objects of study because it can be hard to figure out exactly what is data and what is a condition. Combining the two makes it even worse. For example, even though two choices of dual functor on a monoidal category are always monoidally naturally isomorphic, there need not be a unitary monoidal natural isomorphism between two unitary dual functors on a monoidal dagger category.

Example 2.7.2. Let $\mathcal{D} = \mathrm{Hilb}$ and assume we have picked a duality $\mathrm{ev}_W : W^* \otimes W \to \mathbb{C}$ on every object W such that the induced monoidal dual functor is a unitary dual functor. Let $z \in \mathbb{C}^\times$ and change the duality only on the specific object V by $\widetilde{\mathrm{ev}}_V := z\,\mathrm{ev}_V$ and $\widetilde{\mathrm{coev}}_V = \mathrm{coev}_V\, z^{-1}$. Suppose $f : V \to W$ is a morphism in Hilb to an object where we did not change the duality. Let us denote by $f^* : W^* \to V^*$ the dual of f under the original duality and $\tilde{f} : W^* \to V^*$ the one where we changed the duality on V and similar for morphisms $g : V \to W$. Then we have that $\tilde{f}$ and f^* will differ by a z while $\tilde{g}$ and g^* will differ by a z^{-1}. We see that $f^{*\dagger} \neq f^{\dagger *}$ if $\overline{z}z \neq 1$ does not lie on the unit circle. This example generalizes to any monoidal dagger category with duals equipped with a nonunitary automorphism of the monoidal unit.

It need not happen that the canonical isomorphism specifying uniqueness of duals is unitary:

Example 2.7.3. Let $\mathcal{D}$ be a monoidal dagger category with duals and let c_1 and c_2 be objects that are not isomorphic by $f : c_1 \to c_2$ where f is not unitary. If ev_{c_1} is a duality on c_1, then f induces a canonical duality on c_2 such that the isomorphism specifying uniqueness of duals is f. As a concrete example consider $\mathcal{D} = \mathrm{Herm}_{\mathbb{C}}$ and take c_1 and c_2 to be a one-dimensional vector space $\mathbb{C}$, one with a positive definite inner product and the other with a negative definite inner product and $f = \mathrm{id}_{\mathbb{C}}$. Then $f^\dagger f = -1$ and so f is not unitary.

Our perspective on dagger categories using Hermitian pairings sheds some light on the above confusions. Therefore we study dual functors on monoidal anti-involutive categories. Because our main interest will be in the symmetric monoidal case in which the story simplifies, we will stick to symmetric monoidal anti-involutive categories from now on. We first prove the analogon of Lemma 2.2.19 for anti-involutions:

Lemma 2.7.4. *Let $\mathcal{C}$ be a symmetric monoidal category with duals and symmetric monoidal anti-involution $(d : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op}}, \eta)$. A choice of dual functor $(.)^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ is canonically anti-involutive.*

Proof. Let $(.)^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ be a choice of monoidal dual functor. Because d is monoidal and, uniqueness of duals provides a canonical monoidal natural isomorphism $d \circ (.)^* \Rightarrow {}^*(.) \circ d$ of monoidal functors $\mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op}}$. Here we in principle had to choose a dual functor ${}^*(.)$ on $\mathcal{C}^{\otimes\mathrm{op}}$, i.e. a right dual functor on $\mathcal{C}$, but we can use the braiding to identify it with $(.)^*$. We then have to show that the diagram

$$\begin{tikzcd} x^* \arrow[d, "\eta_{x^*}"] & (d^2x)^* \arrow[l, "\eta_x^*"'] \\ d^2(x^*) & d(dx^*) \arrow[l] \arrow[u] \end{tikzcd}$$

commutes. This follows from the fact that η is a monoidal natural isomorphism, see Lemma A.1.9. $\square$

Remark 2.7.5. It is an essential ingredient in the above proof that the anti-involutive category is symmetric. Moreover, because the canonical anti-involutive data of the dual functor depends on the braiding, we see that only functors that preserve the braiding preserve this anti-involutive data. This will be important for Proposition 2.7.20.

Remark 2.7.6. We emphasize that just like a choice of dual functor, the choice of anti-involutive dual functor $(.)^*$ as above is provided by duality data on every object alone. In other words, we require no compatibility with d or $\otimes$. Therefore the anti-involutive dual functor of an anti-involutive symmetric monoidal category with duals is completely canonical in the following precise sense. Let $(.)^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}$ be another choice of dual functor and let $\sigma_x : x^* \cong x^\vee$ be the unique monoidal natural isomorphism. Then σ_x is an anti-involutive natural isomorphism. Indeed,

$$\begin{tikzcd} d(c^*) \arrow[r] & d(c)^* \arrow[d, "\sigma_{dc}"] \\ d(c^\vee) \arrow[u, "d\sigma_c"] \arrow[r] & d(c)^\vee \end{tikzcd}$$

commutes because all natural isomorphisms involved are uniqueness of dual isomorphisms of duals of dc, also see Example A.1.7.

Remark 2.7.7. We can apply Lemma 2.7.4 to the case where $\mathcal{D}$ is a symmetric monoidal dagger category, ignoring positivity structures for now. If we fix a dual functor, then the canonical natural isomorphism $\lambda : (.)^{*\dagger} \cong (.)^{\dagger *}$ is given by

$$x^* \xrightarrow{\mathrm{id}_{x^*} \otimes \mathrm{coev}_x} x^* \otimes x \otimes x^* \xrightarrow{\sigma_{x^*,x} \otimes \mathrm{id}_{x^*}} x \otimes x^* \otimes x^* \xrightarrow{\mathrm{coev}_x^\dagger \otimes \mathrm{id}_{x^*}} x^*. \tag{2.8}$$

Because λ expresses uniqueness of duals, it is well-behaved categorically, but in a subtle way. For example, note how the fact that the isomorphism $\lambda : (.)^{*\dagger} \cong (.)^{\dagger *}$ is natural implies that the automorphism (2.8) measures the failure of morphisms $f : x_1 \to x_2$ satisfying $f^{\dagger *} = f^{*\dagger}$. Note that if λ is the identity on all objects, then $(.)^*$ is a unitary dual functor. This follows because natural isomorphisms specifying uniqueness of duals are monoidal. However, the converse is false. Namely, if $(.)^*$ is a unitary dual functor, λ need not be the identity. However, it does follow in that case that λ is a natural automorphism of the functor $(.)^{*\dagger} = (.)^{\dagger *}$. Since $*$ and $\dagger$ are both equivalences, λ therefore corresponds uniquely to a natural automorphism of the identity functor.

By uniqueness of unique dual isomorphisms, λ intertwines the isomorphism $\sigma : x^* \cong x^\vee$ if $x^\vee$ is another choice of dual of x (independently of whether σ is unitary). More precisely, if $\lambda' : (.)^{\vee\dagger} \cong (.)^{\dagger\vee}$ is the natural isomorphism for the other dual functor, then

$$\begin{tikzcd} x^* & x^* \arrow[l, "\lambda_x"'] \arrow[d, "\sigma"] \\ x^\vee \arrow[u, "\sigma^\dagger"] & x^\vee \arrow[l, "\lambda'_x"] \end{tikzcd}$$

commutes. We compare with Example 2.7.2 where we changed a dual functor $(.)^*$ to a dual functor $(.)^\vee$ by modifying ev_V and coev_V on a single Hilbert space V with $z \in \mathbb{C}$. We see that on the object V the automorphisms λ'_V and λ_V differ by multiplication with $z\overline{z} \in \mathbb{C}^\times$. Note in particular that λ_x need not a unitary automorphism of x^* in general. In fact, being the anti-involutivity data of the anti-involutive functor $x \mapsto x^*$, it should be thought of as being self-adjoint instead, compare Section 2.5. In case $(.)^*$ is a unitary dual functor, λ will in fact be a self-adjoint natural automorphism of

$(.)^{*\dagger} = (.)^{\dagger *}$ in the usual sense. In fact, it will be the self-adjoint modification of the anti-involutive dual functor that gives our chosen unitary dual functor. We will see in Lemma 2.7.17 which possible λ arise through this process.

Remark 2.7.8. Monoidality of λ is also saying something subtle, because the monoidality of the functor $x \mapsto x^{*\dagger}$ is potentially different from the monoidality data of $x \mapsto x^{\dagger *}$ if the isomorphism $\rho_{x_1,x_2} : (x_1 \otimes x_2)^* \to x_2^* \otimes x_1^*$ given by uniqueness of duals is not unitary. Monoidality thus means that

$$\begin{tikzcd}
(x_1 \otimes x_2)^* \arrow[r, "\lambda_{x_1 \otimes x_2}"] & (x_1 \otimes x_2)^* \arrow[d, "\rho_{x_1,x_2}"] \\
x_2^* \otimes x_1^* \arrow[u, "\rho^\dagger_{x_1,x_2}"] \arrow[r, "\lambda_{x_1} \otimes \lambda_{x_2}"] & x_2^* \otimes x_1^*
\end{tikzcd}$$

commutes. In other words, the failure of λ being a monoidal natural automorphism of $x \mapsto x^*$ is exactly the failure of the dual functor being a *monoidal* dagger functor.

Corollary 2.7.9. *Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category with duals. Then there is a unitary dual functor on* $\mathrm{Herm}\,\mathcal{C}$ *such that for all objects* $x \in \mathrm{Herm}\,\mathcal{C}$ *the canonical isomorphism* $\lambda_x : x^* \cong x^*$ *expressing the fact that the monoidal functor* $\dagger : \mathrm{Herm}\,\mathcal{C} \to \mathrm{Herm}\,\mathcal{C}^{\circ\mathrm{op}}$ *preserves duals is equal to the identity.*

Proof. A choice of monoidal dual functor is unique up to monoidal natural isomorphism. This dual functor is anti-involutive by Lemma 2.7.4 and the monoidal natural isomorphism is anti-involutive by Remark 2.7.6. By Theorem 2.4.14 and the fact that all Hermitian pairings are allowed in the Hermitian completion, there corresponds a monoidal dagger functor to this

$$(.)^* : \mathrm{Herm}\,\mathcal{C} \to \mathrm{Herm}(\mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}) = \mathrm{Herm}(\mathcal{C})^{\circ\mathrm{op},\otimes\mathrm{op}},$$

which is unique up to monoidal unitary natural isomorphism.

Any other unitary dual functor on $\mathrm{Herm}\,\mathcal{C}$ corresponds up to unitary equivalence with a monoidal anti-involutive dual functor on $\mathcal{C}$ up to monoidal anti-involutive natural isomorphism, but possibly with different anti-involutive data $d(x^*) \cong (dx)^*$. In particular, there is at most one unitary dual functor on $\mathrm{Herm}\,\mathcal{C}$ up to monoidal unitary natural isomorphism with the property that λ_x is the identity for all x. □

Remark 2.7.10. The above unitary dual functor on $\mathrm{Herm}\,\mathcal{C}$ is in a certain sense the unique symmetric monoidal dagger functor up to monoidal unitary natural isomorphism with the property that $\lambda_x = \mathrm{id}_x$. However, we have to be slightly careful; under a unitary monoidal natural isomorphism, λ might be modified by a nontrivial inv-positive monoidal natural automorphism.

We now make some observations on what natural Hermitian pairings there are on the dual c^*, which will make the canonical unitary dual functor on $\mathrm{Herm}\,\mathcal{C}$ more explicit. First note that there is a natural Hermitian pairing on dc, which we do not want to change:

Example 2.7.11. Let $(\mathcal{C}, d)$ be an anti-involutive category and c an object of $\mathcal{C}$. If $h : c \to dc$ is a Hermitian pairing, on c, then $(dh)^{-1} : dc \to d^2c$ is a Hermitian pairing on dc. Moreover, h is a unitary isomorphism between c and dc.

The Hermitian pairing on the dual is somewhat similar in spirit:

Definition 2.7.12. Let $\mathcal{C}$ be a monoidal anti-involutive category equipped with its canonical anti-involutive dual functor and let $h : c \to dc$ be a Hermitian pairing. The *dual Hermitian pairing* on c^* is given by the composition

$$c^* \xrightarrow{h^{*-1}} (dc)^* \cong d(c^*),$$

where the isomorphism used the anti-involutivity data of Lemma 2.7.4. If $\mathcal{C}$ is symmetric and P is a monoidal positivity structure, let P^* denote the positivity structure on $\mathcal{C}$ consisting of all duals of Hermitian pairings in P.

Remark 2.7.13. P^* is most naturally a positivity structure on $\mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}$, but we saw in Example 2.4.20 that there is a natural correspondence between Hermitian pairings on $\mathcal{C}^{\mathrm{op},\otimes\mathrm{op}}$ and $\mathcal{C}$ given by taking the inverse.

In general, there is no reason for c^* to be isomorphic to c. Even if $c^* \cong c$ and h is a Hermitian pairing on c, then c might not be unitarily isomorphic to c^* with the dual Hermitian pairing:

Example 2.7.14. Depending on the context, the dual Hermitian pairing might not be a Hermitian pairing we want to consider on the dual. For example, if (V, h) is a Hermitian super vector space of signature (p_1, p_2, p_3, p_4), then with this Hermitian pairing V^* will have signature (p_1, p_2, p_4, p_3) by Proposition 2.1.6. In other words, for odd degree vector spaces, it will map positive definite Hermitian forms to negative definite Hermitian forms and vice-versa. In particular, V will not be unitarily isomorphic to V^* with this Hermitian pairing. Even worse, the dagger category of super Hilbert spaces is not closed under mapping $V \mapsto V^*$. In other words, with these choices sHilb does not obtain a unitary dual functor. We will deal with this by modifying the canonical dual functor with the self-adjoint natural automorphism $(-1)^F$. Note that this example also shows that there are symmetric monoidal anti-involutive categories $\mathcal{C}$ in which every object admits a self-duality, but the dual (x^*, h') of a Hermitian pairing $h : x \to dx$ is not unitarily isomorphic to the object (x, h) in $\mathrm{Herm}\,\mathcal{C}$. In other words, the canonical dual on $\mathrm{Herm}\,\mathcal{C}$ induces an interesting monoid map $\pi_0^h\mathcal{C} \to \pi_0^h\mathcal{C}$ which covers the identity on $\pi_0\mathcal{C} \to \pi_0\mathcal{C}$.

Example 2.7.15. Consider the ungraded convention sHilb^{ung} for the symmetric monoidal dagger category of super Hilbert spaces in which inner products satisfy

$$\langle v, w\rangle = \overline{\langle w, v\rangle}.$$

We can construct it as a symmetric monoidal anti-involutive category with positivity structure as follows. Take sVect with its canonical symmetric monoidal anti-involution and change both η and χ are changed by a sign. Change the positivity structure we use to make sHilb by i^F, see Section 2.6. It is very tempting to think that in this convention the dual Hermitian pairing does have the same signature as the original. Indeed, it is not hard to check that if $h : V \to dV$ is a Hermitian form in the usual ungraded convention

$$\langle v, w\rangle = \overline{\langle w, v\rangle},$$

then the Hermitian form

$$V^* \xrightarrow{h^{*-1}} (dV)^* \cong d(V^*)$$

has the same signature as h if we take the obvious isomorphism $(dV)^* \cong d(V^*)$. However, because the monoidal data of d is changed by a sign, the canonical isomorphism $d(V^*) \cong (dV)^*$ saying that d preserves duals is changed by $(-1)^F$. Since this isomorphism is used in the Definition 2.7.12, the dual Hermitian pairing also has its signature reversed on the odd part in this convention. This is as expected, because sHilb^{ung} is equivalent as a symmetric monoidal dagger category to the super convention, as follows by Corollary 2.6.17.

Next we want to generalize the discussion from the Hermitian completion to general dagger categories. So our goal will be to lift dual functors on the symmetric monoidal anti-involutive category $(\mathcal{C}, d, \eta)$ to unitary dual functors on $\mathcal{C}_P \subseteq \operatorname{Herm} \mathcal{C}$ given a monoidal positivity structure P. For this note that the canonical unitary dual functor on the Hermitian completion maps an object (c, h) to c^* together with the dual Hermitian pairing of c. Since the dual functor is an equivalence, it induces an anti-involution of monoids $\pi_0^h \mathcal{C} \to \pi_0^h \mathcal{C}$ covering the anti-involution of monoids $\pi_0 \mathcal{C} \to \pi_0 \mathcal{C}$ given by $c \mapsto c^*$. In particular it gives an involution $P \mapsto P^*$ on the collection of positivity structures. Fixed points $P = P^*$ for this involution give canonical unitary dual functors on $\mathcal{C}_P$.

Lemma 2.7.16. *Let $(\mathcal{C}, d, \eta)$ be a symmetric monoidal anti-involutive category with its canonical anti-involutive dual functor $(.)^*$ and let P be a monoidal positivity structure. Then $(.)^*$ induces a symmetric monoidal dagger functor on $\mathcal{C}_P$ which recovers $(.)^*$ as an anti-involutive functor on $\mathcal{C}$ if and only if $P^* = P$.*

Proof. We equip $\mathcal{C}^{\circ\mathrm{op}}$ and $\mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ with the symmetric monoidal anti-involution and monoidal positivity structure introduced in Example 2.4.20. We apply Theorem 2.4.14 to the anti-involutive monoidal functor $(.)^*$ obtained in Lemma 2.7.4. The result is that $(.)^*$ extends to a monoidal dagger functor

$$(.)^* : \mathcal{C}_P \to (\mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}})_{P^{\circ\mathrm{op},\otimes\mathrm{op}}}$$

if and only if it maps elements in P to elements in P^{op}. Writing out Definition 2.3.28 we see that we need $P^* \subseteq P$ which gives the desired. □

Combining Lemma 2.7.16 with Lemma 2.7.4 we can conclude that given a rigid symmetric monoidal dagger category, there is a canonical *candidate* dual functor; we can make the dual functor canonically anti-involutive and then ask whether it preserves the positivity structure. However, it might not be the case. Then we could try to change the canonical equivariance data by some monoidal natural isomorphism $d(x^*) \cong d(x)^*$ to make $(.)^*$ preserve positivity. Equivalently we can first compose the canonical dual functor with the equivalence $\mathcal{C}_P \to \mathcal{C}_{P_\xi}$ induced by a self-adjoint natural automorphism of the dual functor. In other words, we will apply the theory of self-adjoint modifications developed in Section 2.5 to the anti-involutive dual functor. The most important type of self-adjoint automorphisms we will want to modify by is an anti-involutive $B\mathbb{Z}/2$-action $(-1)^F$ in the sense of Section 2.6. Since the canonical dual functor on the Hermitian completion has trivial λ, it is most convenient to change that one by a self-adjoint natural automorphism of the identity functor.

Lemma 2.7.17. *Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category with duals and monoidal positivity structure P. There is a bijection between unitary isomorphism classes of unitary dual functors on $\mathcal{C}_P$ and self-adjoint monoidal natural automorphisms ξ of the identity functor of $\mathcal{C}$ such that*

$$h \in P \Rightarrow (x^* \xrightarrow{\xi_{x^*}} x^* \xrightarrow{h^*} (dx)^* \cong d(x^*)) \in P.$$

modulo inv-positive monoidal automorphisms of $\mathrm{id}_{\mathcal{C}}$.

Proof. A unitary dual functor $(.)^\vee$ up to unitary equivalence is the same as a symmetric monoidal dagger functor $\mathcal{C}_P \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}_{P^{\mathrm{op}}}$ up to unitary equivalence, with the additional property of being equivalent as a symmetric monoidal functor to a chosen dual functor on $\mathcal{C}$. This in turn is equivalent to a symmetric monoidal anti-involutive functor $(.)^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}$ preserving positivity data, which is equivalent as a symmetric monoidal functor to a chosen dual functor on $\mathcal{C}$. Let $(.)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}$ be the canonical anti-involutive dual functor on $\mathcal{C}$ of Corollary 2.7.9, giving us a symmetric monoidal dagger functor $\mathcal{C}_P \to \mathcal{C}_{P^*}^{\circ\mathrm{op},\mathrm{op}}$. By uniqueness of duals, we can assume without loss of generality that $(.)^\vee = (.)^*$ as monoidal functors, but not necessarily as anti-involutive monoidal functors. Since the anti-involutive data of the canonical dual functor on the Hermitian completion is trivial, these two anti-involutivity data differ by a self-adjoint monoidal natural automorphism ξ of $\mathrm{id}_{\mathrm{Herm}\,\mathcal{C}}$, see Remark 2.5.6. The canonical dual functor $(.)^*$ maps P to the positivity data P^* and so $(.)^\vee$ maps P to P^*_ξ. We see that $(.)^\vee$ preserves positivity data if and only if $P = P^*_\xi$. Writing out the definition of preserving positivity data and P^* gives the requirement in the statement. Conversely, a monoidal self-adjoint natural automorphism of $\mathrm{id}_{\mathcal{C}}$ gives a symmetric monoidal anti-involutive dual functor $(.)^\vee = (.)^*_\xi$. This dual functor restrict to a dagger functor $\mathcal{C}_P \to \mathcal{C}_{P^{\mathrm{op}}}^{\circ\mathrm{op},\otimes\mathrm{op}}$ if and only if $P = P^*_\xi$. We have seen that two such functors $(.)^*_\xi$ and $(.)^*_{\xi'}$ are equivalent if and only if they differ by an inv-positive in Section 2.5. □

Example 2.7.18. In the symmetric monoidal dagger category $\mathcal{D} = \mathrm{sHilb}$, there are no nontrivial inv-positive monoidal natural automorphisms of the identity. However, there is a single nontrivial self-adjoint monoidal natural automorphism of the identity given by $(-1)^F$. The condition of Lemma 2.7.17 is satisfied for $\xi = (-1)^F$ but not for $\xi = \mathrm{id}_{\mathrm{id}_{\mathcal{D}}}$, also see Proposition 2.1.6. Therefore, there is just a single unitary dual functor on sHilb. This example also shows that the collection of self-adjoint natural automorphisms ξ of $\mathrm{id}_{\mathcal{D}}$ satisfying the condition in Lemma 2.7.17, is not closed under composition. On $\mathcal{D} = \mathrm{sHerm}_{\mathbb{C}}$, it would also be allowed to take $\xi = \mathrm{id}_{\mathrm{id}_{\mathcal{D}}}$, since the condition on positivity structures is now empty. Taking ξ to be the identity corresponds to the canonical dual functor on the Hermitian completion, which maps a super Hilbert space to a super Hermitian vector space that is negative definite on the odd part.

Remark 2.7.19. Let $\mathcal{D}$ be a symmetric monoidal dagger category and ξ a self-adjoint natural automorphism of the identity. It would be an interesting exercise to spell out the condition on ξ inducing a unitary dual functor on $\mathcal{D}$ coming from Lemma 2.7.17. This can be done without referring to anti-involutive categories and positivity structures, by consulting Example 2.3.40.

There are two different perspectives on the unitary dual functors $(.)^*_\xi$ induced by monoidal self-adjoint automorphisms $\xi : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ from the viewpoint of positivity structures P on anti-involutive categories $\mathcal{C}$. On the one hand we can see $(.)^*_\xi$ as a self-adjoint modification of the canonical dual functor $(.)^*$ on the Hermitian completion. One way to define this is as the composition with the identity functor with anti-involutivity data ξ:

$$\mathcal{C}_P \xrightarrow{(.)^*} \mathcal{C}_{P^*}^{\circ\mathrm{op},\otimes\mathrm{op}} \xrightarrow{\xi} \mathcal{C}_{P^*_\xi}^{\circ\mathrm{op},\otimes\mathrm{op}} = (\mathcal{C}_P)^{\circ\mathrm{op},\otimes\mathrm{op}},$$

where in the last equation we used that $P^* = P_\xi$. On the other hand, we can look for dual functors $(.)^\vee$ on $\mathcal{C}_P$ directly, by starting with the dual functor $(.)^*$ only as a symmetric monoidal functor on $\mathcal{C}$. Note that in this situation, even though for a morphism $f : (c_1, h_1) \to (c_2, h_2)$ in $\mathcal{C}_P$ the morphism $f^* : c_2^* \to c_1^*$ is defined as a morphism in $\mathcal{C}$, the definition of $f^{*\dagger}$ as a morphism $c_1^* \to c_2^*$ depends on the lift $(.)^\vee$. We work ad hoc and propose an arbitrary lift to $\mathcal{C}_P$ which is completely fixed as a symmetric monoidal dagger functor by saying what it does to positive Hermitian pairings:

$$(c, h) \mapsto (c^*, h^\vee) \quad h \in P.$$

Indeed, it will automatically be a functor and comes equipped with canonical monoidal data so that it is simply a condition whether $h \mapsto h^\vee$ defines a unitary dual functor on $\mathcal{C}_P$. This lift is a †-functor (so that $f^{*\dagger} = f^{\dagger *}$) if and only if

$$(c, h) \mapsto (c^* \xrightarrow{h^\vee} d(c^*) \cong (dc)^* \xrightarrow{h^*} c^*) \tag{2.9}$$

is a natural automorphism of the dual functor as an ordinary functor on $\mathcal{C}_P$. Similarly, the dual functor is a monoidal dagger functor if and only if this is a monoidal natural transformation. The choice of the natural automorphism of the identity ξ in Lemma 2.7.17 corresponds to the composition (2.9) which compares the choice of Hermitian pairing on the dual object with the dual of the Hermitian pairing on the original object. We also see concretely that if ξ_x is positive for all x and $h : x \to dx$ is in P then $h^\vee$ and the dual of h will differ by ξ_x and so are equivalent under to transfer. Note that ξ_x^* is the naturality isomorphism $\lambda_x : x^{*\dagger} \cong x^{\dagger *}$ for the unitary dual functor $(.)^*_\xi$ on the Hermitian completion, also see Remark 2.7.7.

Allowing ourselves the room to change the isomorphism $d(x^*) \cong d(x)^*$ makes anti-involutive dual functors no longer unique up to unique anti-involutive monoidal natural isomorphism as in Corollary 2.7.9. Instead they are classified by the collection of self-adjoint automorphisms of $\mathrm{id}_\mathcal{C}$ modulo inv-positive automorphisms. However, we can still make interesting statements about symmetric monoidal dagger functors between such dagger categories with interesting unitary dual functors by comparing it to the canonical dual functor on the Hermitian completion:

Proposition 2.7.20. *Let $\mathcal{C}_1, \mathcal{C}_2$ be symmetric monoidal anti-involutive categories with monoidal positivity structures P_1 and P_2 respectively. Suppose they come equipped with unitary dual functors induced by monoidal self-adjoint natural automorphisms $\xi_1 \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}_1})$ and $\xi_2 \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}_2})$ respectively, satisfying the condition of Lemma 2.7.17. Let $F : (\mathcal{C}_1)_{P_1} \to (\mathcal{C}_2)_{P_2}$ be a symmetric monoidal dagger functor. Then $F((P_1)_{\xi_1}) \subseteq (P_2)_{\xi_2}$.*

Proof. The symmetric monoidal dagger functor corresponds to a symmetric monoidal anti-involutive functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ which maps P_1 to P_2. Such a monoidal functor moreover comes equipped with a canonical natural isomorphism intertwining the dual functors of $\mathcal{C}_1$ and $\mathcal{C}_2$. Since F is symmetric it also preserves the canonical anti-involutive data of the dual functors of $\mathcal{C}_1$ and $\mathcal{C}_2$, see Remark 2.7.5. Therefore there is a canonical natural isomorphism filling the square

$$\begin{tikzcd} \mathrm{Herm}(\mathcal{C}_1) \arrow[r, "F"] \arrow[d, "(.)^*"] & \mathrm{Herm}(\mathcal{C}_2) \arrow[d, "(.)^*"] \\ \mathrm{Herm}(\mathcal{C}_1)^{\circ\mathrm{op},\otimes\mathrm{op}} \arrow[r, dashed, "F"] & \mathrm{Herm}(\mathcal{C}_2)^{\circ\mathrm{op},\otimes\mathrm{op}} \end{tikzcd}$$

where $(.)^*$ denotes the canonical unitary dual functor on the Hermitian completion. If $(.)^*_\xi$ denotes the unitary dual functor on $\mathrm{Herm}\,\mathcal{C}$ obtained by modifying the canonical dual functor with ξ, this gives the commutative square

$$\begin{tikzcd} (\mathcal{C}_1)_{P_1} \arrow[r, "F"] \arrow[d, "(.)^*_{\xi_1}"] & (\mathcal{C}_2)_{P_2} \arrow[d, "(.)^*_{\xi_2}"] \\ (\mathcal{C}_1)^{\circ\mathrm{op},\otimes\mathrm{op}}_{(P_1^{\mathrm{op}})_{\xi_1}} \arrow[r, dashed, "F"] & (\mathcal{C}_2)^{\circ\mathrm{op},\otimes\mathrm{op}}_{(P_2^{\mathrm{op}})_{\xi_2}} \end{tikzcd} .$$

The two vertical maps are equivalences of symmetric monoidal dagger categories and so the dashed arrow exists and has to preserve positivity structures. ☐

Next we relate the discussion above with what are called †-compact categories in the literature using the following lemma:

Lemma 2.7.21. *Let $\mathcal{D}$ be a symmetric monoidal dagger category with duals and let $(.)^* : \mathcal{D} \to \mathcal{D}$ be a monoidal dual functor. Let $\lambda_x : x^* \to x^*$ be the canonical automorphism expressing uniqueness of duals $(.)^{*\dagger} \cong (.)^{\dagger *}$. Then for all objects $x \in \mathcal{D}$ and dualities $\mathrm{ev}_x, \mathrm{coev}_x$ on x the diagram*

$$\begin{tikzcd}
1 & x^* \otimes x \arrow[l, "\mathrm{ev}_x"'] \arrow[d, "\sigma_{x^*,x}"] \\
x \otimes x^* \arrow[u, "\mathrm{coev}_x^\dagger"] \arrow[r, "\mathrm{id}_x \otimes \lambda_x"] & x \otimes x^*
\end{tikzcd}$$

commutes.

Proof. Writing out the explicit expression for λ we see that we have to show the diagram

$$\begin{tikzcd}
x \otimes x^* \arrow[rr, "\mathrm{coev}_x^\dagger"] \arrow[d, "\sigma_{x,x^*}"'] & & 1 \\
x^* \otimes x \arrow[d, "\mathrm{id}_{x^*} \otimes \mathrm{coev}_x \otimes \mathrm{id}_x"'] & x^* \otimes x \arrow[ul, "\sigma_{x^*,x}"] & x^* \otimes x \arrow[u, "\mathrm{ev}_x"] \\
x^* \otimes x \otimes x^* \otimes x \arrow[ur, "\mathrm{id}_{x^* \otimes x} \otimes \mathrm{ev}_x"'] \arrow[rr, "\sigma_{x^*,x} \otimes \mathrm{id}_{x^* \otimes x}"'] & & x \otimes x^* \otimes x^* \otimes x \arrow[u, "\mathrm{coev}_x^\dagger \otimes \mathrm{id}_{x^* \otimes x}"']
\end{tikzcd}$$

commutes. The left part commutes by the triangle identity and the fact that the braiding is symmetric. The right part commutes by the interchange law. □

Remark 2.7.22. Note that even though the automorphism λ_x depends on the duality data chosen, the commutativity of the diagram does not.

Variants of the following definition are well-established [64, 12]. However, in Remark 2.7.28 we will argue the definition is not well-behaved under equivalence.

Definition 2.7.23. Let $\mathcal{D}$ be a symmetric monoidal dagger category equipped with a unitary dual functor $(.)^*$ induced by specific dualities $\mathrm{ev}_x, \mathrm{coev}_x$ on every object x of $\mathcal{D}$. Then $\mathcal{D}$ is called †-*compact (also: dagger compact)* if for every object x the diagram

$$\begin{tikzcd}
1 & x^* \otimes x \arrow[l, "\mathrm{ev}_x"'] \arrow[dl, "\sigma_{x^*,x}"] \\
x \otimes x^* \arrow[u, "\mathrm{coev}_x^\dagger"] &
\end{tikzcd}$$

commutes.

This corollary follows immediately from Lemma 2.7.21.

Corollary 2.7.24. *Let $\mathcal{D}$ be a symmetric monoidal dagger category with unitary dual functor $(.)^*$. Then $(\mathcal{D}, (.)^*)$ is dagger compact if and only if $\lambda_x = \mathrm{id}_{x^*}$ for all $x \in \mathcal{C}$.*

Remark 2.7.25. Definition 2.7.23 is sometimes stated with the dagger on the evaluation map instead. This is equivalent to the above definition, as can be seen by taking the dagger of the above diagram and using that the braiding is unitary.

Example 2.7.26. The canonical unitary dual functor on the Hermitian completion always makes it †-compact. A self-adjoint modification of the unitary dual functor by ξ will change λ by ξ^*. Therefore $(\mathrm{Herm}\,\mathcal{C}, (.)^*_\xi)$ will only be †-compact if ξ is trivial.

Remark 2.7.27. As discussed in Example 2.7.18, sHilb has $(.)^*_{(-1)^F}$ as its only unitary dual functor, given by the modification of the canonical dual functor by $(-1)^F$. It follows by Example 2.7.26 that it does not make the †-category †-compact. At first sight another possible 'fix' to sHilb not being †-compact, would be to change the monoidal data of d by $x \otimes y \mapsto (-1)^{|x||y|} x \otimes y$. Indeed, this will result exactly in the right change in the natural isomorphism $d(x^*) \cong (dx)^*$ by the fermion parity operator, making sure that $P^* = P$. This indeed results in a †-compact symmetric monoidal †-category. However, this manoeuvre will ruin the monoidal structure of sHilb by making the tensor product of odd degree Hilbert spaces negative definite, see Remark 2.1.14. In particular, we need to add some negative definite Hermitian pairings to the positivity structure, making a symmetric monoidal †-category which is not minimal.

Remark 2.7.28. In the above definition and corollary, we required a specific unitary dual functor. The main reason is that Definition 2.7.23 suffers from the fact that the displayed diagram commuting is not stable under varying the duality data. This is because the condition of λ_x being equal to id_{x^*} depends on the choice of dual functor. The problem is two-fold: on the one hand, it can happen that two unitary dual functors are not unitarily equivalent because they are modifications of each other by a nontrivial self-adjoint natural automorphism. On the other hand, a unitary natural isomorphism between two unitary dual functor can still have λ differ by an inv-positive. For example, we have seen in Remark 2.7.7, we can pick an automorphism f of the monoidal unit to change the duality data with and then λ will be changed with the inv-positive 'scalar' $f^\dagger f$.

Lemma 2.7.17 suggests we can do better and obtain a definition of †-compact that is independent of the choice of unitary dual functor as follows. Namely, let $(.)^*_\xi$ be a unitary dual functor on the dagger category $\mathcal{C}_P$. Let $h : x \to dx$ be an object of $\mathcal{C}_P$ and $h^\vee : x^* \to d(x^*)$ a Hermitian pairing on x^* which differs with the dual $h' : x^* \to d(x^*)$ of h by the self-adjoint automorphism ξ_x. For $(.)^*_\xi$ be a unitary dual functor, we need that $h^\vee \in P$ by the assumption of Lemma 2.7.17. If we see coev_x as a morphism $1 \to (x, h) \otimes (x^*, h^\vee)$, then its dagger differs with the dagger of coev_x seen as a morphism $1 \to (x, h) \otimes (x^*, h')$ with the automorphism ξ_x. In particular, we see that if we want Definition 2.7.23 to hold for all allowed dualities, we need $\xi_x = \mathrm{id}_x$, i.e. $(.)^*$ is the canonical unitary dual functor. On the other hand, if we decide on a modification ξ_x that is inv-positive, $(.)^*_\xi$ will be unitarily equivalent to $(.)^*$. Therefore from the perspective of this thesis it would be more reasonable to call a symmetric monoidal dagger category †-compact if it admits a unitary dual functor so that the diagram of Definition 2.7.23 commutes up to an inv-positive automorphism. Alternatively, we could require the stronger condition that every unitary dual functor makes the diagram commute up to an inv-positive automorphism.

The above remark motivates us to modify the definition of †-compact to make it categorically better behaved. There are two natural choices in modifying the definition; allowing only changes by inv-positives and allowing changes also by weakly inv-positives. Up to the problem that unitarily equivalent dual functors might have λ_x change by an inv-positive natural automorphism, this is essentially the difference between requiring the diagram in Definition 2.7.23 to commute either for *all* or for *some* unitary dual functor. The result is two definitions which we will call 'weak' and 'strong' †-compactness. This will also anticipate the definitions in Section 2.9.

Definition 2.7.29. Let $\mathcal{D}$ be a symmetric monoidal dagger category with duals. We say that $\mathcal{D}$ is *weak †-compact* if for every unitary dual functor and $x \in \mathcal{D}$, λ_x is weakly inv-positive. We say that $\mathcal{D}$ is *strong †-compact* if for every unitary dual functor and $x \in \mathcal{D}$, λ_x is inv-positive.

Remark 2.7.30. Clearly every strong †-compact category is weak †-compact. Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category with monoidal positivity structure P. Then $\mathcal{C}_P$ is weak †-compact if and only if $P^* = P$. It is strong †-compact if and only if P is additionally minimal. For example, any Hermitian completion is weak †-compact. We have that $\mathrm{Hilb}_{\mathbb{C}}$ is strong †-compact, while $\mathrm{Herm}_{\mathbb{C}}$ is only weak †-compact. On the other hand, $\mathrm{sHilb}_{\mathbb{C}}$ is not even weak †-compact since $P^* = P_{(-1)^F} \neq P$.

We now briefly restrict to the case where $\mathcal{C}$ is a symmetric monoidal category with symmetric monoidal $\mathbb{Z}/2$-action $c \mapsto \overline{c}$ and let $dc := \overline{c}^*$ be the induced anti-involution as explained in Section 2.2. The canonical anti-involutivity data of $(.)^*$ is the isomorphism $\overline{\overline{x}^*}^* \cong \overline{x}^{**}$ coming from the fact that d is monoidal by

$$\overline{x \otimes y}^* \cong (\overline{x} \otimes \overline{y})^* \cong \overline{y}^* \otimes \overline{x}^* \cong \overline{x}^* \otimes \overline{y}^*.$$

In other words, it uses that $x \mapsto \overline{x}^*$ maps left duals to right duals and then applies the symmetry to make the right dual into a left dual again. We can then not only talk about the dual of a Hermitian pairing, but also about its bar:

Remark 2.7.31. The *bar of Hermitian pairing* $h : c \to \overline{c}^*$ is the Hermitian pairing on $\overline{c}$ given by

$$\overline{c} \xrightarrow{\overline{h}} \overline{\overline{c}^*} \cong \overline{\overline{c}}^*$$

where in the last line we used the data saying that $c \mapsto \overline{c}$ is monoidal to obtain that it preserves duals.

If we are interested in that the dual Hermitian pairing looks like in a concrete example given by a $\mathbb{Z}/2$-action, it can sometimes be more convenient to compute the bar of a Hermitian pairing directly. This suffices because the Hermitian pairing h itself provides a unitary isomorphism between $\overline{c}$ and c^* with the dual Hermitian pairing, compare Example 2.7.11.

Remark 2.7.32. André Henriques and Dave Penneys [30] defined the notion of a bi-involutive monoidal category (for the special case of $\mathbb{C}$-linear tensor categories) as a monoidal dagger category $\mathcal{C}$, together with an op-monoidal involution such that all structure maps of the involution are unitary. This notion is closely related to unitary dual functors, see [58, section 3.5].

2.8 Dagger pivotal structures

In this section, we will give a different perspective on unitary dual functors, which we expect to be especially relevant for generalizations to the non-symmetric case. For this we first review the theory of Dave Penneys [58], who tells us that a unitary dual functor gives a unitary pivotal structure. Recall here that a pivotal structure on a monoidal category $\mathcal{C}$ with chosen dual functor $(.)^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ is a monoidal natural isomorphism $\phi_c : c \to c^{**}$, also see Appendix A.1. Since symmetric monoidal categories already have a canonical pivotal structure given by (A.2), a symmetric monoidal dagger category equipped with a unitary dual functor has two canonical pivotal structures. In all examples considered in this thesis, the construction of the correct dagger dual functor from symmetric monoidal categories with duals and $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action with certain positivity structure, is so that the dagger pivotal structure differs from the pivotal structure coming from the braiding by the $B\mathbb{Z}/2$-action. This property of these two pivotal structures will give us another point of view on the spin-statistics theorem in Section 2.9. The following result is given in [58, Proposition 3.9, Corollary 3.10].

Theorem 2.8.1. *Let $\mathcal{D}$ be a monoidal dagger category together with a monoidal dual functor $(.)^\vee : \mathcal{D} \to \mathcal{D}^{\circ\mathrm{op},\otimes\mathrm{op}}$. Then, $(.)^\vee$ is a unitary dual functor if and only if*

$$\phi_c = (\mathrm{coev}_c^\dagger \otimes \mathrm{id}_{c^{\vee\vee}}) \circ (\mathrm{id}_c \otimes \mathrm{coev}_{c^\vee})$$

defines a pivotal structure. In that case, ϕ_c is unitary.

We call the traces induced by the above dagger-pivotal structure dagger traces. Note that this terminology is abusive, since they depend on a choice of dual functor.

Definition 2.8.2. Let $\mathcal{D}$ be a monoidal dagger category with unitary dual functor and $f : c \to c'$ a morphism. The *left and right dagger trace* $\mathrm{tr}_\dagger\, f$ of f are the left and right trace of f in the pivotal structure induced by the dual functor. Similarly, the *left and right dagger dimension* of an object c are the left and right dagger trace of id_c.

In the case the monoidal dagger category is symmetric, it has a canonical pivotal structure and two pivotal structures compose to give a monoidal natural automorphism of the identity functor. This natural automorphism can be compared to the automorphism $\lambda_x : x^* \to x^*$ giving the canonical anti-involutivity data of the dual functor we obtained in Lemma 2.7.4:

Lemma 2.8.3. *Let $\mathcal{D}$ be a symmetric monoidal dagger category with duals and let $(.)^* : \mathcal{D} \to \mathcal{D}$ be a unitary dual functor. Then, the induced $\dagger$-pivotal structure is given by the composition of the pivotal structure induced by the symmetric monoidal structure and the dual of λ_x.*

Proof. Note that in the figure

$$\begin{tikzcd}
x \arrow[r, "\mathrm{id}_x \otimes \mathrm{coev}_{x^*}"] & x \otimes x^* \otimes x^{**} \arrow[rr, "\mathrm{coev}_x^\dagger \otimes \mathrm{id}_{x^{**}}"] \arrow[d, "\mathrm{id}_x \otimes \lambda_x \otimes \mathrm{id}_{x^*}"'] & & x^{**} \\
& x \otimes x^* \otimes x^{**} \arrow[rr, "\sigma_{x,x^*} \otimes \mathrm{id}_{x^{**}}"] & & x^* \otimes x \otimes x^{**} \arrow[u, "\mathrm{ev}_x \otimes \mathrm{id}_{x^{**}}"']
\end{tikzcd}$$

the square commutes by Lemma 2.7.21. The upper horizontal composition is the dagger pivotal structure on $\mathcal{D}$ induced by the unitary dual functor. We now pass the λ_x through the braiding using its naturality. Then, we can use the universal property of the dual of a morphism to replace λ_x with λ_x^* after passing it through ev_x. Using the explicit expression for the pivotal structure Φ given by the composition (A.2), we obtain that the above composition equals

$$x \xrightarrow{\Phi_x} x^{**} \xrightarrow{\lambda_x^*} x^{**},$$

as desired. □

Corollary 2.8.4. *Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category with duals, let ξ be a natural monoidal self-adjoint automorphism of the identity functor such that $P_\xi = P^*$ and let $(.)^\vee = (.)^*_\xi$ be the corresponding unitary dual functor on $\mathcal{C}_P$. Then, the induced $\dagger$-pivotal structure is given by the composition of ξ with the pivotal structure induced by the symmetric monoidal structure.*

Proof. The canonical unitary dual functor on the Hermitian completion has $\lambda_x = \mathrm{id}_x$. Modifying this dual functor by ξ changes λ to $\lambda_x = \xi_x^*$. □

Remark 2.8.5. By changing the unitary dual functor by a self-adjoint natural automorphism of the identity, we will therefore change dagger traces of morphisms by this natural automorphism. In particular, equivalent unitary dual functors which differ by an inv-positive natural automorphism can still have different dagger traces, but only up to positive automorphisms.

Proposition 2.8.6. *Let $\mathcal{D}, \mathcal{D}'$ be monoidal dagger categories equipped with unitary dual functors and let $F : \mathcal{D} \to \mathcal{D}'$ be a monoidal functor. Then F is pivotal for the †-pivotal structure if and only if $F(x^\vee) \cong F(x)^\vee$ is unitary for all $x \in \mathcal{D}$.*

Proof. The proof is analogous to [58, Proposition 3.40]. □

Remark 2.8.7. Let $\mathcal{C}_1, \mathcal{C}_2$ be symmetric monoidal anti-involutive categories with monoidal positivity structures P_1, P_2 respectively. Suppose $(\mathcal{C}_1)_{P_1}$ and $(\mathcal{C}_2)_{P_2}$ are equipped with unitary dual functors corresponding to natural automorphisms ξ_1 and ξ_2 respectively. Let $F : (\mathcal{C}_1)_{P_1} \to (\mathcal{C}_2)_{P_2}$ be a symmetric monoidal dagger functor. In Proposition 2.7.20 we showed that preservation of duals gives an interesting relationship between ξ_1 and ξ_2. We can analogously discuss whether F is †-pivotal in this context by comparing with the canonical dual functor on the Hermitian completion. We hope this way of comparing dual functors between source and target generalizes well to the non-symmetric case.

2.9 Fermionically dagger compact categories

In Section 2.7 we have seen that given a symmetric monoidal anti-involutive category $\mathcal{C}$ with duals, there is a canonical choice of unitary dual functor on the Hermitian completion $(.)^* : \mathrm{Herm}\,\mathcal{C} \to \mathrm{Herm}\,\mathcal{C}$. Many symmetric monoidal dagger categories in mathematics naturally arise in the form $\mathcal{C}_P$ for some monoidal positivity structure P. Often the unitary dual functor on the Hermitian completion restricts to $\mathcal{C}_P$, which in particular implies $\mathcal{C}_P$ is weak †-compact. Many of these are even strong †-compact. Some examples of weak †-compact categories relevant for topological field theory are the category Hilb and the oriented bordism category $\mathrm{Bord}_{n,n-1}$. However, dagger categories of more 'fermionic nature' such as sHilb and the spin bordism category $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ will not be weak †-compact. Instead, they will be what we will call weak fermionically †-compact, which roughly means that they are weak †-compact up to a $B\mathbb{Z}/2$-action. In terms of the last section, we prefer to think of weak fermionically †-compact categories $\mathcal{C}$ as certain †-categories with duals for which the canonical dual functor on $\mathrm{Herm}\,\mathcal{C}$ does not restrict to $\mathcal{C}$. Instead, it maps a dagger category $\mathcal{C}_P$ to the dagger category $\mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}_{P_{(-1)^F}}$, in which positive pairings are changed by fermion parity.

Similar to what we did in Section 2.7 for †-compact categories, we will introduce two notions of fermionically †-compact that are independent of the choice of unitary dual functor; a strong and a weak one. The weak notion is equivalent to the canonical unitary dual functor on the Hermitian completion mapping P to $P_{(-1)^F}$. Strong fermionically †-compact categories will additionally be minimal, but weak ones are not. For the spin-statistics theorem it is crucial that the bordism category is weak fermionically †-compact and the target †-category sHilb is strong fermionically †-compact.

Lemma 2.9.1. *Let $\mathcal{D}$ be a symmetric monoidal dagger category with duals and unitary monoidal $B\mathbb{Z}/2$-action $(-1)^F$. The following are equivalent:*

1. *There exists a monoidal unitary dual functor* $(.)^* : \mathcal{D} \to \mathcal{D}^{\mathrm{oop},\otimes\mathrm{op}}$*, such that the canonical natural isomorphism* $\lambda_x : (.)^{*\dagger} \cong (.)^{\dagger *}$ *given by the fact that* $\dagger$ *preserves duals is given by an inv-positive multiple of* $x^* \xrightarrow{(-1)^F_{x^*}} x^*$ *on the object* $x \in \mathcal{D}$*.*
2. *If the dagger category* $\mathcal{D} = \mathcal{C}_P$ *is given as a symmetric monoidal anti-involutive category* $\mathcal{C}$ *with positivity structure* P*, then the canonical unitary dual functor on* $\mathrm{Herm}\,\mathcal{C}$ *induces a symmetric monoidal dagger functor*
$$\mathcal{C}_P \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}_{P_{(-1)^F}}.$$
3. *If the dagger category* $\mathcal{D} = \mathcal{C}_P$ *is given as a symmetric monoidal anti-involutive category* $\mathcal{C}$ *with positivity structure* P*, then* $P^* = P_{(-1)^F}$*.*

Proof. Consider a self-adjoint modification $(.)^*_\xi : \mathrm{Herm}\,\mathcal{C} \to \mathrm{Herm}\,\mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}$ of the canonical unitary dual functor on $\mathrm{Herm}\,\mathcal{C}$ for some self-adjoint natural automorphism $\xi : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$. Recall that the corresponding λ is equal to ξ. It descends to a unitary dual functor on $\mathcal{C}_P$ if and only if $P^* = P_\xi$. We see that it λ_x differs from $(-1)^F_x$ with an inv-positive if and only if $P^* = P_{(-1)^F}$. This proves that point 1 and point 3 are equivalent. Point 2 and point 3 are equivalent by using the description of self-adjoint modifications ξ of the identity functor and the corresponding functor $\mathcal{C}_P \to \mathcal{C}_{P_\xi}$. $\square$

Definition 2.9.2. A symmetric monoidal dagger category $\mathcal{D}$ with unitary monoidal $B\mathbb{Z}/2$-action is called *weak fermionically* $\dagger$*-compact* if all objects have duals and the equivalent conditions of Lemma 2.9.1 hold. It is *strong fermionically* $\dagger$*-compact* if for all objects $x \in \mathcal{D}$ and dualities $\mathrm{ev}_x, \mathrm{coev}_x$ on x the diagram

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{ev}_x^\dagger"] \arrow[d, "\mathrm{coev}_x"] & x^* \otimes x \arrow[d, "\sigma_{x^*,x}"] \\ x \otimes x^* \arrow[r, "(-1)^F_x \otimes \mathrm{id}_{x^*}"] & x \otimes x^* \end{tikzcd}$$

commutes up to an inv-positive morphism.

Remark 2.9.3. In case the $B\mathbb{Z}/2$-action on $\mathcal{D}$ is trivial, $\mathcal{D}$ is weak/strong fermionically dagger compact if and only if it is a weak/strong dagger compact category.

Remark 2.9.4. We applied the same caveat as for ordinary $\dagger$-compactness to obtain the definition of strong fermionically $\dagger$-compact. Indeed, recall how changing the duality ev_x might change λ_x by an inv-positive. Therefore, the diagram in Definition 2.9.2 commuting is not stable under changing dual functors. For example, if $\lambda'_x : x^* \to x^*$ corresponds to a different duality with equal underlying object and $\sigma : x^* \to x^*$ is the isomorphism expressing uniqueness of duals, then $\lambda'_x = \sigma^\dagger \lambda_x \sigma$. If we assume the diagram in Definition 2.9.2 commutes with respect to the original duality, then the diagram in Definition 2.9.2 will only commute up to the inv-positive automorphism $\sigma^\dagger \sigma$ of x^*.

More generally, if $f : x^* \to x^\vee$ is any isomorphism, then changing a duality by f will change the diagram by the weakly inv-positive $f^\dagger f$. In a minimal dagger category such as sHilb every weakly inv-positive is inv-positive and so the diagram will still commutes up to an inv-positive. However, in a non-minimal dagger category we can change the duality so that the diagram no longer commutes up to an inv-positive morphism. We conclude that every strong fermionically dagger compact category is minimal. In particular, we can apply this to the case where the dagger category is the Hermitian complete category $\mathrm{sHerm}_{\mathbb{C}}$. Every self-adjoint automorphism is weakly inv-positive and changing a duality with the weakly inv-positive $(-1)^F$ makes clear that $\mathrm{sHerm}_{\mathbb{C}}$ admits unitary dual functors that make it both $\dagger$-compact and fermionically $\dagger$-compact. The canonical dual functor makes it

†-compact and its modification with $(-1)^F$ makes it fermionically †-compact. Only the second dual functor restricts to sHilb, which is why it is not †-compact.

Remark 2.9.5. We could have also decided on a definition of fermionically †-compact that depends on a unitary dual functor. However, a unitary dual functor on a symmetric monoidal dagger category is not canonical. Since a unitary topological field theory will be defined as a symmetric monoidal dagger functor that a priori does not preserve designated unitary dual functors, a definition that only depends on the underlying symmetric monoidal dagger category is more convenient.

We claim that a symmetric monoidal dagger category $\mathcal{C}_P$ is strong fermionically dagger compact if and only if it is both weak fermionically dagger compact and minimal. This essentially follows by Lemma 2.7.21 and the discussion in Remark 2.7.28. As a consequence every strong fermionically †-compact category $\mathcal{D}$ has a unique unitary dual functor, which is the self-adjoint modification of the canonical unitary dual functor on $\operatorname{Herm}\mathcal{D}$ by $(-1)^F$.

Remark 2.9.6. By point 3 of Lemma 2.9.1, $\mathcal{D}$ is weak fermionically dagger compact if and only if $P^* = P_{(-1)^F}$. In terms of anti-involutive categories with positivity structures, this means concretely that for every $h : x \to dx \in P$ and for one and hence every dual functor $(.)^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$, we have that the dual Hermitian pairing composed with the $B\mathbb{Z}/2$-action

$$x \xrightarrow{(-1)^F_x} x \xrightarrow{h^*} (dx)^* \cong d(x^*)$$

is again in P. If $\mathcal{D}$ is a symmetric monoidal dagger category with $B\mathbb{Z}/2$-action it can happen that $P^* = P_{(-1)^F}$ but $\mathcal{D}$ is still not strong fermionically dagger compact. For example, a Hermitian completion always satisfies $P^* = P_{(-1)^F}$, but is almost never strong fermionically †-compact.

Remark 2.9.7. Let $\mathcal{C}$ be a symmetric monoidal anti-involutive category with monoidal anti-involutive $B\mathbb{Z}/2$-action and monoidal positivity structures $P' \subseteq P$. If $\mathcal{C}_{P'}$ is weak fermionically †-compact, then so is $\mathcal{C}_P$. However, the converse is false in general. For example, let $\mathcal{C} = \mathrm{sVect}$ come equipped with the trivial $B\mathbb{Z}/2$-action. Then, sHilb is not †-compact, but $\mathrm{sHerm}_{\mathbb{C}}$ is. Note that $\operatorname{Herm}\mathcal{C}$ is weak fermionically †-compact for any symmetric monoidal anti-involutive category with anti-involutive $B\mathbb{Z}/2$-action.

Lemma 2.9.8. *Suppose $\mathcal{D}$ is strong fermionically dagger compact. Then, the †-pivotal structure for its unique unitary dual functor differs with the pivotal structure induced by the symmetric monoidal structure with the $B\mathbb{Z}/2$-action.*

Proof. This is a direct consequence of Remark 2.8.4 to the case where $\lambda_x = (-1)^F_{x^*}$. □

Remark 2.9.9. It follows by point two of Lemma 2.9.1 and Lemma A.1.18, that the dagger pivotal structure induced by a strong fermionically dagger-compact category $\mathcal{D}$ is spherical. We also have that for all morphisms $f : x_1 \to x_2$ in $\mathcal{C}$, its dagger trace is an 'ungraded trace'

$$\operatorname{tr}_\dagger f = \operatorname{tr}\left(f(-1)^F_{x_1}\right) = \operatorname{tr}\left((-1)^F_{x_2} f\right).$$

In particular,

$$\dim_\dagger x = \operatorname{tr}(-1)^F_x.$$

Remark 2.9.10. It would be interesting to develop a graphical calculus for fermionically dagger compact categories analogous to Selinger [64, Theorem 3.11], also see [65, Theorem 7.9]. Physical intuition would suggest that the operation of a 2π rotation would give the $B\mathbb{Z}/2$-action. However, in the usual graphical calculus, the dual corresponds to a π rotation. The 2π rotation therefore corresponds to the trivialization of the double dual instead.

Proposition 2.9.11. *Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a symmetric monoidal dagger functor between weak fermionically †-compact categories. If $f^\dagger f : F(d_1) \to F(d_1)$ is an inv-positive automorphism of an object in the image, then $f^\dagger f \circ (-1)^F_{F(d_1)} F((-1)^F_{d_1})$ is again positive.*

Proof. Without loss of generality, we can assume that $\mathcal{D}_1 = (\mathcal{C}_1)_{P_1}$, $\mathcal{D}_2 = (\mathcal{C}_2)_{P_2}$ as anti-involutive categories with positivity structures P_1 and P_2 respectively. We can also assume $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an anti-involutive functor that maps P_1 to P_2, by Theorem 2.4.14. For example, we can take $\mathcal{C}_i = \mathcal{D}_i$ as anti-involutive categories with the positivity structure given by the inv-positive morphisms in $\mathcal{D}_i$, see Example 2.3.40. We obtain from Proposition 2.7.20 applied to the case where ξ is the $B\mathbb{Z}/2$-action that $F(P_1 \circ (-1)^F_{\mathcal{C}_1}) \subseteq P_2 \circ (-1)^F_{\mathcal{C}_2}$. So if $h : c_1 \to dc_1$ is in P_1, then we need that

$$F(c_1) \xrightarrow{(-1)^F_{c_1}} F(c_1) \xrightarrow{F((-1)^F_{c_1})} F(c_1) \xrightarrow{F(h)} F(dc_1) \cong dF(c_1)$$

is in P_2. Note that we are sloppy with the order of the $B\mathbb{Z}/2$-action on $\mathcal{C}_2$ and the $B\mathbb{Z}/2$-action on $\mathcal{C}_1$ pushed forward under F, which is justified because they are natural and hence commute. Using the positivity structure associated to the anti-involutive category underlying a dagger category, we obtain the result. □

We now prove an abstract result that will directly imply the spin-statistics theorem for topological field theories, see Section 6.3.

Corollary 2.9.12. *Let $F : \mathcal{D} \to$ sHilb be a symmetric monoidal dagger functor from a weak fermionically †-compact category to the category of finite-dimensional super Hilbert spaces. Then, F is $B\mathbb{Z}/2$-equivariant.*

Proof. Let $d \in \mathcal{D}$ be an object. Since sHilb is weak fermionically †-compact, Proposition 2.9.11 applies to $f = \mathrm{id}_d$. Because $F((-1)^F_d)$ is degree-preserving of order two, we see that $(-1)^F_{F(d)} F((-1)^F_d)$ is an inv-positive automorphism of the super Hilbert space $F(d)$ of order two. This implies in particular that $(-1)^F_{F(d)} F((-1)^F_d)$ is positive as a linear map and so it is diagonalizable. Since it is an involution, it has eigenvalues all equal to ± 1. By positivity, we get $(-1)^F_{F(d)} F((-1)^F_d) = \mathrm{id}_{F(d)}$ and so F is $B\mathbb{Z}/2$-equivariant. □

Remark 2.9.13. Note that the above proof would not work when the target is not sHilb but $\mathrm{sHerm}_{\mathbb{C}}$ instead. Indeed, in $\mathrm{sHerm}_{\mathbb{C}}$ all self-adjoint automorphisms are inv-positive. In particular, since $\mathrm{sHerm}_{\mathbb{C}}$ is weak fermionically †-compact, this gives an example of a symmetric monoidal dagger functor between weak fermionically dagger compact categories that is not $B\mathbb{Z}/2$-equivariant. We claim that it is crucial for the above corollary that sHilb is fermionically †-compact in the strong sense.

Corollary 2.9.14. *The symmetric monoidal dagger category* sHilb *is strong fermionically dagger compact.*

Proof. This follows by Proposition 2.1.16 and the fact that sHilb is minimal. □

Remark 2.9.15. A consequence of this corollary and Remark 2.9.9 is that the dagger dimension of a super Hilbert space V is the usual ungraded dimension of V instead of the super dimension. It is desirable that it disagrees with the dimension with respect to the canonical pivotal structure, because the superdimension can be negative, which is undesirable for reflection-positive topological field theories. Namely, we want 'doubles' $T^\dagger T$ to be positive operators in the usual sense, so that unitary topological field theories will send 'doubles' in the bordism category to positive real numbers.

Remark 2.9.16. The symmetric monoidal dagger category $\mathrm{sHerm}_{\mathbb{C}}$ admits a unitary dual functor that makes the diagram in Definition 2.9.2 commute by the same proof as in Corollary 2.9.14. However, note that the extra room in $\mathrm{sHerm}_{\mathbb{C}}$ allows us to use the more straightforward dual functor that makes the dual of a positive definite odd vector space negative definite. This dual functor does not restrict to sHilb but makes the dagger dimension in $\mathrm{sHerm}_{\mathbb{C}}$ equal to the super dimension. In other words, $\mathrm{sHerm}_{\mathbb{C}}$ is both weak †-compact and weak fermionically †-compact.

Remark 2.9.17. We briefly compare with similar results by Egger [15]. In [15, Definition 6.2] a notion of an exact Hermitian form on objects of categories with op-monoidal involution $(\mathcal{C}, \overline{(.)})$ is defined. This notion is related to Definition 2.3.4 by Currying and picking the 'involutive object' to be the monoidal unit, at least in the special case where his $*$-autonomous category is a symmetric monoidal category with duality. His notion of a 'Hermitian system' is analogous to our notion of a positivity structure. He also constructs the dagger category associated to a positivity structure in [15, Lemma 6.3]. In [15, Lemma 6.4, 6.5], he provides conditions for the dual functor on $\mathcal{C}$ to lift to a (monoidal) dagger dual functor on the dagger category.

Chapter 3

Fermionic symmetry

In this section we introduce a framework to study symmetries of physical systems with fermions that are allowed to be time-reversing. This allows us to define topological field theories with such symmetries in Section 6.1. The low-energy effective theory of a gapped quantum field theories with symmetry is expected to be a topological field theory with the same symmetry. We claim this makes the setup convenient for the study of topological phases of matter protected by such symmetries.

3.1 Fermionic symmetry groups

When fermions are present, symmetry groups naturally come equipped with the structure of a fermion parity operator and a grading operator, which records which symmetries are time-reversing on space or antiunitary on state spaces. We axiomatize this situation using the following definition which we learned from Peter Teichner, also see [69, Definition 2.1].

Definition 3.1.1. A *fermionic group* K consists of

1. a topological group K;
2. a $\mathbb{Z}/2$-grading on K, i.e. a continuous homomorphism $|.| : K \to \mathbb{Z}/2$;
3. a central element $c \in K$ that squares to one and is even in the grading; $|c| = 0$.

The *underlying bosonic group* K_b of K is the quotient of K by the normal subgroup generated by c. Elements $k \in K$ that are odd in the grading will be called *reversing* and elements that are even are called *preserving*. We write $K = K_{pres} \sqcup K_{rev}$ for the grading decomposition.

Remark 3.1.2. Let K be the spatially internal symmetry group of a physical system, possibly containing fermions. By spatial we mean that K is allowed to contain time-reversal symmetries, which are not internal from a spacetime perspective. We describe the fermionic group structure on K. There is always a special symmetry of order two written $(-1)^F$, which maps a multi-particle state v containing $F \in \mathbb{N}$ fermions and $B \in \mathbb{N}$ bosons to $(-1)^F v$. In other words, if $V = V_B \oplus V_F$ is the one particle state space with its $\mathbb{Z}/2$-grading given by the fermion-boson distinction and

$$\mathcal{F} = \mathrm{Sym}(V) = \mathrm{Sym}(V_B) \otimes \bigwedge V_F$$

is the usual multi-particle Fock space, then $(-1)^F$ is given by the identity on the symmetric factor and the tensor length modulo two on the Grassmann factor. If K preserves bosonic and fermionic

$\mathbb{Z}_2^F$	$\mathbb{Z}_2^T$	$\mathbb{Z}_4^T$	$\mathbb{Z}_2^F \times \mathbb{Z}_2^T$
Spin_1	O_1	Pin_1^-	Pin_1^+

Table 3.1: Some internal fermionic symmetry groups in physics and math notation.

states, then $(-1)^F$ is central. This is for example not the case for supersymmetries, which will not be considered in this thesis.

Usual symmetries in K will act unitarily on V and $\mathcal{F}$, but time-reversal symmetries will act antiunitarily instead. We record this by defining the grading $|.| : K \to \mathbb{Z}/2$ to be nontrivial on an element k if and only if it reverses the direction of time. This motivates why we prefer to call elements with $|k| = 1$ reversing. Calling them odd could cause confusion with objects that are odd in the fermion grading, such as $v \in V_F$ above. In topology reversing symmetries typically manifest themselves as orientation-reversing symmetries.

Remark 3.1.3. We decided to denote the central element by the symbol $c \in K$, instead of the more common physics notation $(-1)^F \in K$. The reason is that we want to reserve the latter symbol for the distinction between fermions and bosons on state space as in the last remark, while $c \in K$ will be more naturally related to the distinction between half-integer spin and integer spin. These two notions are related to the spin-statistics connection, but since there exist non-unitary theories that do not satisfies spin-statistics, we prefer to be more clear about the notation.

It is often reasonable to assume that $c \neq 1$, so that there is a short exact sequence of topological groups

$$1 \to \mathbb{Z}/2 \to K \to K_b \to 1.$$

Because $|c| = 0$, the grading factors through the projection to give us an induced $\mathbb{Z}/2$-grading $|.| : K_b \to \mathbb{Z}/2$. We introduce the following slightly unfortunate terminology.

Definition 3.1.4. A *bosonic group* is a fermionic group with $c = 1$.

Example 3.1.5. Let M be a closed manifold with universal cover $\tilde{M}$. If $\tilde{M}$ is not spin, give $\pi_1(M)$ the bosonic group structure with $c = 1$ and let $|.| : \pi_1(M) \to \mathbb{Z}/2$ be given by the first Stiefel-Whitney class $w_1(M)$. Otherwise, it can be shown that there exists a unique cohomology class in $H^2(B\pi_1; \mathbb{Z}/2)$ that pulls back to $w_2(M)$ under the map $M \to B\pi_1$ classifying the universal cover. This class corresponds to an extension

$$1 \to \mathbb{Z}/2 \to \tilde{\pi}_1 \to \pi_1(M) \to 1.$$

We give $\tilde{\pi}_1$ the structure of a fermionic group with $|.| = w_1$ and c the nontrivial element of the $\mathbb{Z}/2$-subgroup. This example is relevant in the context of Kreck's modified surgery.

Example 3.1.6. Let Cl_{+n} be the real Clifford superalgebra induced by the canonical nondegenerate symmetric bilinear form on $\mathbb{R}^n$, where we take the convention that squares of vectors are positive. Let $\mathrm{Pin}_n^+ \subseteq Cl_{+n}$ be the corresponding pin group, see [46] for basics on Clifford algebras and pin groups. Similarly, we denote the pin group with negative squares by $\mathrm{Pin}_n^- \subseteq Cl_{-n}$. Then Pin_n^+ is a fermionic group with $c = -1 \in Cl_{+n}$ and the grading given by the supergrading of Cl_{+n}. Equivalently the grading is the composition

$$\mathrm{Pin}_n^+ \twoheadrightarrow O_n \xrightarrow{\det} O_1 \cong \mathbb{Z}/2,$$

where the first map is the usual double cover. The spin group Spin_n is the preserving part of Pin_n^+ and so again a fermionic group which has trivial reversing part. More generally, any subgroup of a fermionic group that contains c is naturally a fermionic group. This example works equally well for the Clifford algebra with negative or mixed signature. Given a vector $v \in \mathbb{R}^n$ we will denote the corresponding odd element of Pin_n^+ by v again.

Example 3.1.7. We describe the internal fermionic symmetry group of a class AII topological insulator. Let T denote a time-reversal symmetry with square $T^2 = (-1)^F$ and let Q denote electromagnetic U_1-charge. Since T is antiunitary, it does not commute with the U_1-symmetry generated by Q:

$$e^{iaQ}T = Te^{-iaQ} \quad a \in \mathbb{R}.$$

If $\phi : \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \to \mathrm{Aut}(U_1)$ is the complex conjugation action, then we can describe the fermionic symmetry group that T and Q generate as the semidirect product

$$U_1 \rtimes_\phi \mathbb{Z}/4.$$

The central element is $(1, (-1)^F)$ and the grading is given by $|T| = 1$ and by continuity $|e^{iaQ}| = 0$. It can be reasonable to additionally impose a spin-charge relation saying that a particle has odd charge if and only if it is odd under $(-1)^F$, see [63, section 2.3]. In that case we have to take a diagonal quotient

$$\frac{U_1 \rtimes_\phi \mathbb{Z}/4}{\mathbb{Z}/2}.$$

Definition 3.1.8. A group homomorphism between fermionic groups is called *fermionic* if it intertwines $|.|$ and c.

Example 3.1.9. The fermionic group $U_1 \rtimes_\phi \mathbb{Z}/4$ of example 3.1.7 is isomorphic to Pin_2^- as a fermionic group.

Remark 3.1.10. There are many isomorphisms of groups between fermionic groups that are not fermionic. For example, there are three fermionic group structures on the group $\mathbb{Z}/2$; one for which c is nontrivial, one for which $|.|$ is nontrivial and one for which both are trivial. It is convenient to have some notation for distinguishing different fermionic group structure on the same group. In the physics literature, this is typically done by adding superscripts. For example, the group $\mathbb{Z}/2$ with nontrivial c is denoted $\mathbb{Z}_2^F$ because it is generated by fermion parity $(-1)^F$, while the group $\mathbb{Z}/2$ with nontrivial $|.|$ is denoted $\mathbb{Z}_2^T$, because it is generated by a single time-reversal symmetry with square 1. Mathematically we can think of the first group as being Spin_1 and the second being O_1, compare Table 3.1. In other words, it can be enlightening for a (non-bosonic) fermionic group to denote its canonical $\mathbb{Z}/2$-subgroup generated by c by Spin_1 and the $\mathbb{Z}/2$-group in which its grading is valued as O_1. The third possible fermionic group is the usual bosonic unitary $\mathbb{Z}/2$-symmetry for which both c and $|.|$ are trivial.

Example 3.1.11. Let $A = A_0 \oplus A_1$ be a real superalgebra, i.e. a $\mathbb{Z}/2$-graded algebra. Let $A^\times$ be the group of homogeneous invertible elements. This becomes a fermionic group with $|.|$ given by the grading of A and the central element $c = -1 \in A^\times$. This construction induces a functor $A \mapsto A^\times$ from the category of real superalgebras and even homomorphisms to the category of discrete fermionic groups with fermionic homomorphisms.

This functor admits a left adjoint:

Definition 3.1.12. The *fermionic group algebra* of a discrete fermionic group K is the superalgebra

$$\mathbb{R}^f[K] = \frac{\mathbb{R}[K]}{(c+1)}$$

where $\mathbb{R}[K]$ is the ordinary group algebra graded by $|.|$.

There are analogous constructions when the superalgebra and the fermionic group come equipped with a topology in certain cases.

Example 3.1.13. [69] Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}/2$-graded real C^*-algebra. Let

$$O(A) := \{a \in A_0 \cup A_1 : a^*a = aa^* = 1\} \subseteq A^\times$$

be the topological group of homogeneous orthogonal elements. As for $A^\times$, the grading on A and $c = -1$ give this a fermionic group structure. Therefore $O(A)$ becomes a topological group, which is compact if A is finite-dimensional.

Definition 3.1.14. A *fermionic representation* of the fermionic group K on the complex super vector space V with grading operator $(-1)^F : V \to V$ is a group homomorphism

$$R : K \to GL_{\mathbb{R}}(V)$$

such that

1. $R(k)$ is complex-linear if $|k| = 0$ and complex antilinear if $|k| = 1$;
2. $R(c) = (-1)^F$ is the supergrading.

If $V \in$ sHilb is a super Hilbert space, we call the representation *unitary* if $R(k)$ is unitary for $|k| = 0$ and *antiunitary* for $|k| = 1$:

$$\langle R(k)v, R(k)w\rangle = \begin{cases} \langle v, w\rangle & |k| = 0, \\ (-1)^{|k|}\overline{\langle v, w\rangle} & |k| = 1. \end{cases}$$

Remark 3.1.15. The definition of a fermionic representation is how a physicist would expect a fermionic group to act on the state spaces. We elaborate on the unexpected sign in the definition of an antiunitary operator, which is a consequence of the subtleties in defining the right super Hilbert space structure on the dual of a super Hilbert space, see Proposition 2.1.6. Namely, just like with the dual of a Hilbert space V we have to compose the canonical Hermitian structure on $\overline{V}$ with $(-1)^F_V$ to make it positive definite. This creates extra signs in several formulas. In other words, this choice is necessary if we want to make the Hilbert space structure induce a unitary isomorphism $\overline{V} \to V^*$ as a consequence of Remark 2.7.31. Note that this sign is not directly seen when working with the usual convention for $\mathbb{Z}/2$-graded Hilbert spaces.

Remark 3.1.16. We could have alternatively defined a fermionic representation as a fermionic homomorphism from K to the fermionic group $GL_f(V)$ defined as follows. Preserving elements are even elements of $GL_{\mathbb{C}}(V)$, while reversing elements are even elements of $GL_{\mathbb{R}}(V)$ that are complex antilinear. The element $(-1)^F \in GL_{\mathbb{C}}(V)$ is given by the grading operator $(-1)^F_V$ of V. Analogously, unitary fermionic representations are fermionic homomorphisms into $U_f(V)$ of which the preserving elements are unitary and the reversing elements antiunitary.

We will see in Section 6.2 that the state spaces of a unitary topological field theory with symmetry K often have the natural structure of a unitary fermionic representation.

Remark 3.1.17. In this setting of actions of a finite fermionic group K on quantum state spaces, it can also be natural to consider a 'complex' ungraded group algebra built from K by

$$\frac{\mathbb{R}[K,i]}{(i^2=-1, ki=(-1)^{|k|}ik)}.$$

Namely, an ungraded module over this algebra is equivalent to a fermionic representation and will automatically be graded by the action of c. Note that this algebra is a vector space over $\mathbb{C}$ and contains a canonical subalgebra over $\mathbb{C}$, but if K has reversing elements it is not a complex algebra.

The following definition was given in [69, 2.4] and in [72, proof of theorem 2.2.1.]. It should remind the reader of the graded tensor product of superalgebras.

Definition 3.1.18. Let G, H be fermionic groups. The *fermionic tensor product* $G \otimes H$ is the set $(G \times H)/\langle (c_G, c_H)\rangle$ with the group operation

$$(g_1 \otimes h_1)(g_2 \otimes h_2) = c_G^{|g_2||h_1|} g_1 g_2 \otimes h_1 h_2,$$

the central element $c := 1 \otimes c_H = c_G \otimes 1 \in G \otimes H$ and the grading $|g \otimes h| = |g| + |h|$.

We can think of the fermionic tensor product as the tensor product $G \otimes H = G \otimes_{\mathrm{Spin}_1} H$ over the canonical subgroups $\mathrm{Spin}_1 \subseteq G$ and $\mathrm{Spin}_1 \subseteq H$ generated by c. It follows by a straightforward sign computation using that $|c| = 0$ that the group operation on the fermionic tensor product is associative. It is also easy to show that the fermionic tensor product of three fermionic groups is associative by the obvious associator. Fermionic groups even form a symmetric monoidal category:

Lemma 3.1.19. *There is a natural fermionic isomorphism*

$$K \otimes K' \cong K' \otimes K, \quad k \otimes k' \mapsto c^{|k'||k|} k' \otimes k.$$

Proof. The fact that this is an isomorphism is a straightforward computation. Naturality with respect to fermionic homomorphisms is clear. $\square$

Note that if either G or H has either a trivial grading or trivial c, then

$$G \otimes H \cong \frac{G \times H}{\mathrm{Spin}_1}.$$

Example 3.1.20. Every fermionic group has a canonical fermionic automorphism of order two given by

$$k \mapsto c^{|k|} k.$$

Note that

$$K \otimes K' \cong \frac{K \rtimes K'}{\langle c_K, c_{K'} \rangle},$$

where the action of K' on K is by composing the grading of K' with the map $\mathbb{Z}/2 \to \operatorname{Aut} K$ given by the involution above.

Remark 3.1.21. In the seminal paper [22], Freed and Moore define similar but different symmetry data for symmetries of quantum-mechanical systems, which they coin twisted QM symmetry classes. Since twisted QM symmetry classes contain a canonical U_1-subgroup, we interpret their symmetry groups physically as the charged analogues of our symmetry groups, which are more natural in a neutral fermionic setup. Mathematically, we interpret their groups as a 'Real' analogue of our real symmetry groups in the sense that twisted QM symmetry classes are naturally twists of equivariant KR-theory, while fermionic groups are twists of equivariant KO-theory.

3.2 Spacetime structure groups

The main goal of this section is to define given a fermionic internal symmetry group K and a positive integer n a new fermionic group $G_n(K)$. This group will play an important role to define topological field theories with internal symmetry group K. The reason is that such topological field theories are naturally defined on spacetimes with $G_n(K)$-structure, as we will discuss in Section 6.1, see Section 4.1 for the basics on G-structures. In this section, we will also briefly comment on the connection with the tenfold way in condensed matter. We further strengthen the connection between the theory of fermionic groups and homotopy theory by defining first and second Stiefel-Whitney classes of fermionic groups.

We motivate the precise definition of $G_n(K)$. It is standard to require that a n-dimensional Lorentzian signature quantum field theory comes equipped with a Poincaré group action. Locally, this symmetry reduces to an action of the Lorentz group $SO_{n-1,1}$ or a Spin type extension of it by the fermion parity group Spin_1. In topological field theory we will be working in Euclidean signature in which this Lorentz group becomes Spin_n. Given a quantum field theory with internal fermionic symmetry group K, we will have to combine it with Spin_n using an appropriate construction. The fermionic tensor product turns out to be suitable:

Definition 3.2.1. Let $(K, |.|, c)$ be a fermionic group. The *spacetime structure group* associated to K in spacetime dimension n is

$$G_n(K) := (\mathrm{Pin}_n^+ \otimes K)_{pres}.$$

Also define

$$\hat{G}_n(K) := \mathrm{Pin}_n^+ \otimes K$$

and

$$\overline{G}_n(K) := (\mathrm{Pin}_n^+ \otimes K)_{rev}.$$

Here by the preserving part, we mean the kernel of the diagonal grading

$$|x \otimes k| = |x| + |k|.$$

The group $G_n(K)$ is still fermionic with grading $|x \otimes k| = |x| = |k|$. The reversing part $\overline{G}_n(K)$ is not a group, but it is a $G_n(K)$-torsor.

Let $\hat{\rho} : \hat{G}_n(K) \to O_n$ and $\rho : G_n(K) \to O_n$ denote the homomorphisms induced by projection onto the first factor. There are obvious inclusions between different dimensions and the shape

$$\begin{tikzcd}
 & \hat{G}_n(K) \arrow[rr] \arrow[dd, "\hat{\rho}_n"] & & \hat{G}_{n+1}(K) \arrow[dd, "\hat{\rho}_{n+1}"] \\
G_n(K) \arrow[ur] \arrow[rr] \arrow[dr, "\rho_n"] & & G_{n+1}(K) \arrow[ur] \arrow[dr, "\rho_n"] & \\
 & O_n \arrow[rr] & & O_{n+1}
\end{tikzcd}$$

commutes. The group $\hat{G}_n(K)$ will play an important role in defining orientation-reversal of manifolds with $G_n(K)$-structure, see Section 4.5.

Remark 3.2.2. There is also a noncompact version of the spacetime structure group in which we replace Pin_n^+ by the double cover of $GL_n(\mathbb{R})$ that restricts to Pin_n^+ over O_n. There is also a version with Pin_n^-, which we will denote $\hat{G}_n^-(K) := \mathrm{Pin}_n^- \otimes K$ and $G_n^-(K) := (\mathrm{Pin}_n^- \otimes K)_{pres}$.

Example 3.2.3. Let K be a bosonic symmetry group without time-reversing symmetries. Then $\hat{G}_n(K) = O_n \times K$ and $G_n(K) = SO_n \times K$. More generally if K has time-reversing symmetries, then $\hat{G}_n(K) = O_n \times K$ and $G_n(K)$ is the kernel of the map

$$O_n \times K \to \mathbb{Z}/2,$$

given by comparing the determinant and the grading of K.

Example 3.2.4. Let $K = U_1$ be the circle group with nontrivial $c = -1 \in U_1$. Then $G_n(K) = \mathrm{Spin}^c_n$, also see [63, section 2.3].

Example 3.2.5. Let D be one of the ten superdivision algebras over the real numbers. We equip D with the C^*-algebra structure in which generators e with square 1 have $e^* = e$ and generators f with square -1 have $f^* = -f$. The ten induced fermionic groups $O(D)$ are possible choices of representing internal symmetry groups of the tenfold way in condensed matter. The ten spacetime structure groups associated with $O(D)$ are the ten structure groups given in [20, Tables (9.24) and (9.25)]. For example, for the superdivision algebra $D = Cl_{-2}$, we obtain the spacetime structure group associated to the internal symmetry group of the class AII topological insulator of Example 3.1.7, since $O(Cl_{-2}) = \mathrm{Pin}_2^-$.

We provide another natural formulation of the tenfold way in the setting of fermionic groups:

Theorem 3.2.6. *There exist ten isomorphism classes of Lie groups K that fit in a sequence*

$$1 \to O(D) \to K \to O_1,$$

where D is an ungraded division algebra over $\mathbb{R}$, so that $O(D) = \mathbb{Z}/2, U_1$ or SU_2.

Proof. If the last map is not surjective, we have $K = S(D)$ giving us the three purely even cases. In case the last map is surjective, we consider the three possibilities for D separately:

- $\mathbb{R}$ Since $\mathrm{Aut}\,\mathbb{Z}/2 = 1$, we compute $H^2(\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2$, giving the two possibile extensions $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$;
- $\mathbb{C}$ Segal's cohomology [61] is an invariant of topological groups such that second degree cohomology classifies short exact sequences of topological groups for which the projection homomorphism is a principal bundle. Since $\mathrm{Aut}\,U_1 = \mathbb{Z}/2$ given by complex conjugation, we have to compute the Segal cohomology $H^2_s(\mathbb{Z}/2; U_1)$ with both trivial and nontrivial coefficients. This is isomorphic to $H^3(\mathbb{Z}/2; \mathbb{Z})$ with either the trivial action or the obvious nontwisted action [60, Corollary 97]. In one of the cases we get $\mathbb{Z}/2$ in the other zero, giving a total of three possibilities.
- $\mathbb{H}$ Similarly to [20, Proposition 9.16], we apply [21, Corollary 7.3] to get a diagram of the form

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & SU_2 \arrow[r] & K \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}.$$

  We get that $L = \mathbb{Z}/2 \times \mathbb{Z}/2$ or $L = \mathbb{Z}/4$ and there is an isomorphism

$$K \cong \frac{L \rtimes SU_2}{\mathbb{Z}/2},$$

for some action of L on SU_2. Note that $\operatorname{Aut} SU_2 = SO_3$ only consists of inner automorphisms and therefore we can assume without loss of generality that the action is trivial. We obtain two possibilities depending on L.

□

Example 3.2.7. Let M be a smooth manifold and let K be the (discrete) fermionic group with $K_b = \pi_1(M)$ considered in Example 3.1.5. Define $BG(K)$ as usual by the colimit of $BG_n(K) \to BG_{n+1}(K)$. Then $BG(K) \to BO$ is the tangential 1-type of M, see [72, Theorem 2.1.1 and 2.2.1] for the proof in the normal case. To get the normal 1-type, we replace the tangent bundle of M by the stable normal bundle in Example 3.1.5 which has second Stiefel-Whitney class

$$w_2(\nu(M)) = w_1(M)^2 + w_2(M).$$

We briefly discuss what changes when replacing Pin^+ with Pin^-, starting with an illuminating example.

Example 3.2.8. Consider the fermionic tensor product $E_{p,q}$ of p copies of Pin_1^+ and q copies of Pin_1^-. The result is not isomorphic to $\mathrm{Pin}_{p,q}$ when $p+q > 1$, because $E_{p,q}$ is finite. However, it forms a subgroup called the extraspecial group and its fermionic group algebra

$$\mathbb{R}^f[E_{p,q}] = \frac{\mathbb{R}[E_{p,q}]}{(c+1)} = Cl_{p,q}$$

is the mixed signature Clifford algebra, compare [36, Lemma 2.7]. As a special case, consider the three groups

$$K = \mathrm{Pin}_1^+ \otimes \mathrm{Pin}_1^-, \quad \mathrm{Pin}_1^+ \otimes \mathrm{Pin}_1^+ \quad \text{and} \quad \mathrm{Pin}_1^- \otimes \mathrm{Pin}_1^- .$$

They all sit in a central extension

$$1 \to \mathbb{Z}/2 \to K \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$$

These correspond to four of the eight isomorphism classes of extensions, as we can conclude by computing the group cohomology

$$\begin{aligned} H^2(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2) &\cong \big(H^0(\mathbb{Z}/2;\mathbb{Z}/2) \otimes H^2(\mathbb{Z}/2;\mathbb{Z}/2)\big) \oplus \big(H^1(\mathbb{Z}/2;\mathbb{Z}/2) \otimes H^1(\mathbb{Z}/2;\mathbb{Z}/2)\big) \\ &\oplus \big(H^2(\mathbb{Z}/2;\mathbb{Z}/2) \otimes H^0(\mathbb{Z}/2;\mathbb{Z}/2)\big) \cong (\mathbb{Z}/2)^3. \end{aligned}$$

Indeed the projection to the middle term is nontrivial for all four extensions but the projection on the first and last factor depend on the square of $e_1 \otimes 1$ and $1 \otimes e_1$ respectively. Explicitly, $\mathrm{Pin}_1^- \otimes \mathrm{Pin}_1^-$ is isomorphic to the quaternion group while all other groups are isomorphic to the symmetry group of a square (there are only two nonabelian groups of order eight). Note that even though $\mathrm{Pin}_1^+ \otimes \mathrm{Pin}_1^-$ and $\mathrm{Pin}_1^+ \otimes \mathrm{Pin}_1^+$ are isomorphic as groups, they are not isomorphic as fermionic groups. This is analogous to the fact that Cl_{+2} and $Cl_{1,1}$ are both 2×2 matrix algebras, but they are not isomorphic as superalgebras.

From the example we see that tensoring a symmetry group K with Pin_1^- can give a different group from tensoring with Pin_1^+. The same is true for the preserving part, but only if we 'interchange time-reversal symmetries that square to one with time-reversal symmetries that square to $(-1)^F$'. Because this phenomenon occurs often, we give it a name:

Definition 3.2.9. Let $(K, |.|, c)$ be a fermionic group. Define the *opposite* fermionic group K^{op} to be K as a topological space but with multiplication given by

$$k_1^{\mathrm{op}} k_2^{\mathrm{op}} = c^{|k_1||k_2|}(k_1 k_2)^{\mathrm{op}}$$

and the same $|.|$ and c.

Remark 3.2.10. The name 'opposite' is motivated by the fact that we could have defined the opposite group instead as having opposite multiplication under the symmetric braiding:

$$k_1^{\mathrm{op}} * k_2^{\mathrm{op}} = c^{|k_1||k_2|}(k_2 k_1)^{\mathrm{op}}.$$

This makes the multiplication in the group similar to the multiplication in the opposite of a superalgebra. There is a fermionic isomorphism between this group structure on K and the one of Definition 3.2.9 given by $k \mapsto k^{-1}$.

Example 3.2.11. The identity map on $\mathbb{R}^n$ induces a fermionic isomorphism $(\mathrm{Pin}_n^+)^{\mathrm{op}} \cong \mathrm{Pin}_n^-$.

Example 3.2.12. The obvious bijections

$$(K \otimes \mathrm{Pin}_1^-)_{pres} \cong K \quad (K \otimes \mathrm{Pin}_1^+)_{pres} \cong K^{\mathrm{op}}$$

are fermionic isomorphisms. One might have expected the roles of Pin_1^+ and Pin_1^- to be exchanged in these isomorphisms, but the anti-commuting odd variables cancel the relevant signs. In particular, we have $G_n(\mathrm{Pin}_1^\pm) = \mathrm{Pin}_n^\mp$; a time-reversal symmetry with square $(-1)^F$ gives Pin_n^+ and a time-reversal symmetry with square 1 gives Pin_n^-.

Remark 3.2.13. We define $\mathrm{Pin}_0^+ = \mathrm{Spin}_1$ so that $\hat{G}_0 = K$ and $G_0 = K_{pres}$. This will be somewhat convenient later because it will make

$$\begin{tikzcd} \mathrm{Pin}_1^+ \arrow[r] & O_1 \\ \mathrm{Pin}_0^+ \arrow[r] \arrow[u] & O_0 \arrow[u] \end{tikzcd}$$

into a pullback square in the category of groups.

We now want to compare $\hat{G}(K)$ with $\hat{G}^-(K)$. They are not isomorphic, but their preserving elements are if we replace K by K^{op} on one side:

Lemma 3.2.14. *For fermionic groups G, H, there is a fermionic isomorphism*

$$(G^{\mathrm{op}} \otimes H^{\mathrm{op}})_{pres} \cong (G \otimes H)_{pres}$$

compatible with the maps to G_b and H_b.

Proof. The obvious map $\psi(g^{\mathrm{op}} \otimes h^{\mathrm{op}}) = g \otimes h$ is well-defined, continuous and compatible with the maps to G_b and H_b. We show it is an isomorphism by direct computation:

$$\begin{aligned}(g_1^{\mathrm{op}} \otimes g_2^{\mathrm{op}})(h_1^{\mathrm{op}} \otimes h_2^{\mathrm{op}}) &= c^{|g_2||h_1|} g_1^{\mathrm{op}} h_1^{\mathrm{op}} \otimes g_2^{\mathrm{op}} h_2^{\mathrm{op}} = c^{|g_2||h_1| + |g_1||h_1| + |g_2||h_2|}(g_1 h_1)^{\mathrm{op}} \otimes (g_2 h_2)^{\mathrm{op}} \\ &= c^{|g_2||h_1|}(g_1 h_1)^{\mathrm{op}} \otimes (g_2 h_2)^{\mathrm{op}},\end{aligned}$$

since $|g_1| = |g_2|$ and $|h_1| = |h_2|$. So this element is mapped to

$$(-1)^{|g_2||h_1|} g_1 h_1 \otimes g_2 h_2 = (g_1 \otimes g_2)(h_1 \otimes h_2).$$

The isomorphism clearly is fermionic. □

We can compare with the groups $\hat{G}_n^-(K)$ and $G_n(K)$ from Remark 3.2.2:

Corollary 3.2.15. *For any fermionic group K we have*

$$G_n(K) \cong G_n^-(K^{\mathrm{op}}),$$

compatibly with the map to O_n.

Recall from Example 3.2.8 that $\mathrm{Pin}_1^+ \otimes \mathrm{Pin}_1^+ \ncong \mathrm{Pin}_1^- \otimes \mathrm{Pin}_1^-$, so we really have to restrict to the preserving part in this corollary.

Fermionic groups also have a homotopic interpretation as follows.

Definition 3.2.16. The *first Stiefel-Whitney class* of a fermionic group K is the class $w_1(K) \in H^1(BK_b, O_1)$ given by the grading as an element of

$$H^1(BK_b; O_1) \cong \mathrm{Hom}(\pi_1(BK_b), O_1) \cong \mathrm{Hom}(\pi_0(K_b), O_1)$$

The *second Stiefel-Whitney class* of a fermionic group K with $c \neq 1$ is the class $w_2(K) \in H^2(BK_b; \mathrm{Spin}_1)$ given by the isomorphism class of the extension

$$1 \to \mathrm{Spin}_1 \to K \to K_b \to 1.$$

In the above definition we used the notation Spin_1 or O_1 for the group $\mathbb{Z}/2$ depending on whether it should be interpreted as the canonical subgroup generated by c or the target of the grading of a fermionic group, compare Remark 3.1.10.

Remark 3.2.17. In case $c = 1$ the second Stiefel-Whitney class is not defined. We emphasize that a fermionic group with zero second Stiefel-Whitney class instead corresponds to a split extension

$$1 \to \mathrm{Spin}_1 \to \mathrm{Spin}_1 \times K_b \to K_b \to 1,$$

which in particular has $c \neq 1$.

Remark 3.2.18. The data of a fermionic group structure on K is up to isomorphism equivalent to giving its first and possibly second Stiefel-Whitney class.

Note that $w_1(\mathrm{Pin}_n^-) = w_1(\mathrm{Pin}_n^+) = w_1 \in H^1(BO_n; O_1)$ and $w_2(\mathrm{Pin}_n^+) = w_2 \in H^2(BO_n; \mathrm{Spin}_1)$, but $w_2(\mathrm{Pin}_n^-) = w_2 + w_1^2$. More generally, K^{op} has $w_1(K^{\mathrm{op}}) = w_1(K)$ and $w_2(K^{\mathrm{op}}) = w_2(K) + w_1(K)^2$. The expression for the second Stiefel Whitney class follows because the cup product corresponds to the extension with its product twisted by $c^{|k_1||k_2|}$.

Proposition 3.2.19. *Let K, K' be compact fermionic groups that are not bosonic and consider $K \otimes K'$ as a fermionic group with its diagonal grading. Then*

$$\begin{aligned} w_1(K \otimes K') &= w_1(K) + w_1(K') \in H^1(BK_b \times BK'_b; O_1) \\ w_2(K \otimes K') &= w_2(K) + w_2(K') + w_1(K)w_1(K') \in H^2(BK_b \times BK'_b; \mathrm{Spin}_1) \end{aligned}$$

where we implicitly used the pullbacks along the projections $K_b \times K'_b \to K_b$ and $K_b \times K'_b \to K'_b$.

Proof. Because we want to do algebraic computations, we will have to work with a model of cohomology for topological groups. We choose to work with locally continuous group cohomology $H^\bullet_{loc,c}$ of topological groups, in which group cocycles are assumed to be continuous only in a neighbourhood of the unit [74]. Then $H^2_{loc,c}(G; A)$ classifies extensions of G by A which are locally trivial principal A-bundles. Moreover, if G is compact and A is discrete, then $H^\bullet_{loc,c}(G;A) \cong H^\bullet(BG;A)$ [74, Corollary 4.8, Remark 4.16]. Now let us write the multiplication of K in terms of an explicit representative of the cocycle $w_2(K)$ in locally continuous group cohomology, by picking a locally continuous section of $K \to K_b$. So we view $k \in K$ as $k = (g, \epsilon) \in K_b \times \mathrm{Spin}_1$ with twisted semi-direct product multiplication

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, w_2(K)(g_1, g_2)\epsilon_1\epsilon_2),$$

and similar for K'. Given $k_i = (g_i, \epsilon_i) \in K_i$ and $k'_i \in (g'_i, \epsilon'_i)$, the multiplication in the fermionic group can now be written

$$\begin{aligned}(k_1 \otimes k'_1)(k_2 \otimes k'_2) &= ((g_1, \epsilon_1 c^{|k'_1||k_2|})(g_2, \epsilon_2)) \otimes ((g'_1, \epsilon'_1)(g'_2, \epsilon'_2)) \\ &= (g_1 g_2, c^{|k'_1||k_2|} w_2(K)(g_1, g_2)\epsilon_1\epsilon_2) \otimes (g'_1 g'_2, w_2(K')(g'_1, g'_2)\epsilon'_1\epsilon'_2) \\ &= (g_1 g_2, c^{|k'_1||k_2|} w_2(K)(g_1, g_2) w_2(K')(g'_1, g'_2)\epsilon'_1\epsilon'_2\epsilon_1\epsilon_2) \otimes (g'_1 g'_2, 1).\end{aligned}$$

We see that the w_2-cocycle on $K_b \times K'_b$ equals

$$w_2((K \otimes K')_{pres})(k_1 \otimes k'_1, k_2 \otimes k'_2) = c^{|k'_1||k_2|} w_2(K)(g_1, g_2) w_2(K')(g'_1, g'_2).$$

Note that the $\mathbb{Z}/2$-valued cocycle on $K_b \times K'_b$ given by $c^{|k'_1||k_2|}$ is the cup product of $w_1(K) \in H^1(BK_b; O_1)$ and $w_1(K') \in H^1(BK'_b; O_1)$. The diagonal $\mathbb{Z}/2$-grading on $K \otimes K'$ clearly corresponds to $w_1(K) + w_1(K')$. □

In particular, for $\hat{G}_n(K) = K \otimes \mathrm{Pin}^+_n$, we get $w_2(\hat{G}_n(K)) = w_2(K) + w_2 + w_1(K) \cup w_1$. For $\hat{G}^-_n(K)$ we get $w_2(\hat{G}^-_n(K)) = w_2(K) + w_2 + w_1^2 + w_1(K) \cup w_1$. Both groups have the same first Stiefel-Whitney class given by $w_1(K) + w_1 \in H^1(BK_b \times BO_d; O_1)$.

Remark 3.2.20. We expect Proposition 3.2.19 to hold without compactness assumptions on K and K'.

Remark 3.2.21. Homotopy-theoretically, Proposition 3.2.19 tells us that if we take the fermionic tensor product of fermionic groups with $c \neq 1$ seriously, we should not consider the trivial E_1-structure on the product space $B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ but look for a more interesting one. The right E_1-structure instead deloops to a more interesting space X with only two nontrivial homotopy groups $\pi_2 = \mathbb{Z}/2$ and $\pi_3 = \mathbb{Z}/2$ and a nontrivial k-invariant. We refer the reader to Appendix A.2 for further discussion on cohomology of Eilenberg Maclane spaces, k-invariants and the relationship with 2-groups. This k-invariant corresponds to the nontrivial element of $H^4(B^2\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$ which stabilizes to

$$Sq^2 \in \mathbb{Z}/2 \cong H^2(H\mathbb{Z}/2; \mathbb{Z}/2) \cong H^5(B^3\mathbb{Z}/2; \mathbb{Z}/2) \cong H^4(B^2\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

The resulting two-term spectrum comes up more often, for example in the truncation of the sphere spectrum to π_1 and π_2 and hence also in $M\,\mathrm{Spin}$ and KO-theory. It also corresponds to the naturally induced group structure on $B\,\mathrm{Spin}/BO \cong B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ under direct sums of vector spaces. This latter interpretation also agrees with our intuition that the $\mathbb{Z}/2$-subgroup of a fermionic group deserves the name Spin_1, while the $\mathbb{Z}/2$-grading deserves the name O_1, compare Table 3.1.

Under the homotopy hypothesis, there is also a category-theoretic interpretation. Namely, we can interpret $B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ as the delooping of a 2-group. The resulting monoidal category can be interpreted as the groupoid of orthogonal real $\mathbb{Z}/2$-graded lines. To get the Picard groupoid that corresponds to the 2-term spectrum X described above, we want to take the symmetric braiding on this monoidal category given by the Koszul sign rule $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.

We end the section with several lemmas comparing with the description of the structure groups in [20].[1] For readers interested in comparing with that reference in detail, we note that what is referred to with the symbol K and called the 'internal symmetry group' in [20] is what we would call K_{pres}. In other words, they do not include time-reversal symmetries in their internal symmetry group. We start by comparing the fermionic tensor product with an 'ungraded' tensor product:

Lemma 3.2.22. *If G, H are fermionic groups, then the canonical map*

$$\frac{(G^{\mathrm{op}} \times H)_{pres}}{\mathbb{Z}/2^F} \to (G \otimes H)_{pres}$$

is an isomorphism of fermionic groups.

Proof. Similarly to the definition in the fermionic tensor product, the preserving part of $G^{\mathrm{op}} \times H$ refers to the kernel of the diagonal grading, the quotient is by the subgroup generated by (c_G, c_H). We give the result the fermionic group structure with $c = [c_G, 1] = [1, c_H]$ and $|[g,h]| = |g|_G = |h|_H$. Clearly the map

$$[g^{\mathrm{op}}, h] \mapsto g \otimes h$$

is a well-defined homeomorphism. It follows that this is a group homomorphism by a straightforward sign computation:

$$\begin{aligned}[g_1^{\mathrm{op}}, h_1][g_2^{\mathrm{op}}, h_2] &= [c^{|g_1||g_2|}(g_1 g_2)^{\mathrm{op}}, h_1 h_2] \mapsto c^{|g_1||g_2|}(g_1 g_2)^{\mathrm{op}} \otimes h_1 h_2 \\ &= c^{|h_1||g_2|}(g_1^{\mathrm{op}} \otimes h_1)(g_2^{\mathrm{op}} \otimes h_2)\end{aligned}$$

using that $g_1 \otimes h_1$ is preserving. This group homomorphism is clearly fermionic. □

Note also that for fermionic groups K, K' the 'ungraded tensor product' $\frac{K \times K'}{\mathrm{Spin}_1}$ has second Stiefel-Whitney class $w_2(K) + w_2(K')$, while the second Stiefel-Whitney class of K^{op} is $w_2(K) + w_1(K)^2$. In particular, the second Stiefel-Whitney class of $\frac{K^{\mathrm{op}} \times K'}{\mathrm{Spin}_1}$ typically differs from the second Stiefel-Whitney class of $K \otimes K'$ if $w_1(K)$ and $w_1(K')$ are both nontrivial, by Proposition 3.2.19

By taking $H = \mathrm{Pin}_n^+$ in Lemma 3.2.22, it follows that $G_n(K)$ is isomorphic to the groups H_n of [20, Theorem 2.7] compatibly with the map to O_n by taking $J = K^{\mathrm{op}}$. Also compare with [20, Remark 9.36] where I_n is an arbitrary fermionic group, $\tilde{I}_n = (K \times \mathrm{Pin}_1^+)_{pres}$ and so

$$J = \tilde{I}_n / \mathrm{Spin}_1 \cong I_n.$$

This isomorphism follows for example by Lemma 3.2.22 and the second isomorphism is discussed in Example 3.2.12. We emphasize that physically, $K = J^{\mathrm{op}}$ is the internal symmetry group, which in general is neither fermionically isomorphic to I_n nor J. For example if J has a time-reversal that squares to 1, then K will have a time-reversal that squares to $(-1)^F$.

We state our version of [20, Proposition 3.13].

Lemma 3.2.23. *A choice of norm one vector $v \in \mathbb{R}^n$ induces an isomorphism*

$$G_n(K) \rtimes_{\phi_v} \mathbb{Z}/2 \cong \hat{G}_n(K)$$

where the homomorphism $\mathbb{Z}/2 \to \mathrm{Aut}\, G_n(K)$ maps the generator to

$$\phi_v(x \otimes k) = c^{|k|} v x v^{-1} \otimes k.$$

[1] In the rest of this section we will be referring not to the published version, but to arxiv's v6, which contains a corrected definition of their groups $\hat{H}$.

Proof. The element $v \otimes 1 \in \hat{G}_n(K)$ provides a section of the last map in the short exact sequence

$$1 \to G_n(K) \to \hat{G}_n(K) \to \mathbb{Z}/2 \to 1.$$

Since it squares to one this section is a homomorphism. The homomorphism $\mathbb{Z}/2 \to \operatorname{Aut} G_n(K)$ induced by the splitting is given by conjugation with $v \otimes 1$ which is given by

$$(v \otimes 1)(x \otimes k)(v^{-1} \otimes 1) = c^{|k|} v x v^{-1} \otimes k = \phi_v(x \otimes k)$$

on $x \otimes k \in \hat{G}_n(K)$. $\square$

Note that the above proof does not quite work for $\hat{G}_n^-(K)$ because in that case $(v \otimes 1)^2 = c$, so it does not define a section of the short exact sequence.

Next we compare $\hat{G}_n(K)$ with the group $\hat{H}_n$ introduced in [20, Section 3.3]. The main input involves the following groups.

Definition 3.2.24. Let j be a nonnegative integer, G_{n+j} a topological group, $\rho : G_{n+j} \to O_{n+j}$ a homomorphism and $G_n \to O_n$ its strict pullback in the 1-category of topological groups. Define $G_n^{(j)}$ to be the subgroup of G_{n+j} generated by G_n and those elements for which $\rho(g) \in r_{n+1} \dots r_{n+j} O_n$ where $r_i \in O_n$ denotes reflection in the ith coordinate. In other words, $\rho(g)$ is of the form

$$\begin{pmatrix} A & 0 \\ 0 & -\mathrm{id}_{\mathbb{R}^j} \end{pmatrix},$$

where $A \in O_n$. Define $\rho^{(j)} : G_n^{(j)} \to O_n$ by projection on the A-factor.

$G_n^{(j)}$ is a subgroup of G_{n+j}, because if $\rho(g_1)$ and $\rho(g_2)$ are both in $r_{n+1} \dots r_{n+j} O_n$, then their product is in O_n and so $g_1 g_2 \in G_n$ by the strict pullback property. Given a fermionic group K, note that $G_n^{(j)}(K)$ is the subgroup of $G_{n+j}(K)$ generated by $G_n(K)$ and those elements in $G_{n+j}(K)$ of the form $e_{n+1} \dots e_{n+j} x \otimes k$ for $x \otimes k \in \hat{G}_n(K)$. In particular, $x \otimes k$ is preserving if j is even and reversing when j is odd. Observe that our notation is slightly abusive, because the definition of $G_n^{(j)}$ depends on the choice of $G_{n+j} \to O_{n+j}$ and not just on $G_n \to O_n$:

Example 3.2.25. Take $n = j = 1$ and compare $G_{n+j} = \mathrm{Spin}_2$ with $G_{n+j} = SO_2 \times \mathbb{Z}/2$. In both cases we have that $G_n = \mathbb{Z}/2$ with the trivial map to O_1. In the former case $G_n^{(1)} = \{1, -1, e_1 e_2, -e_1 e_2\}$ is a cyclic group of order four since

$$(e_1 e_2)^2 = -e_1^2 e_2^2 = -1.$$

However, in the latter case $G_n^{(1)} \cong (\mathbb{Z}/2)^2$.

In the notation of [20, section 3.3], the following lemma shows the isomorphism $\hat{G}_n(K) \cong \hat{H}_n$ compatibly with the map to O_n because $G_n^{(3)}(K) \cong \hat{H}_n$, also see [20, Appendix E].

Lemma 3.2.26. 1. *There are fermionic isomorphisms $G_n^{(j)}(K) \cong G_n^{(j+4)}(K)$ that intertwine $\rho^{(j)}$ with $\rho^{(j+4)}$;*

2. *There are fermionic isomorphisms $G_n^{(3)}(K) \cong \hat{G}_n(K)$ and $G_n^{(1)}(K) \cong \hat{G}_n(K)^{\mathrm{op}} \cong \hat{G}^-(K^{\mathrm{op}})$ that intertwine $\hat{\rho}$ with $\rho^{(j)}$;*

Proof. We denote $y_0^{(j)} := e_{n+1} \dots e_{n+j} \in \mathrm{Pin}^+_{n+j}$. We have $y_0^{(j)} v = c^j v y_0^{(j)} \in \mathrm{Pin}^+_{n+j}$ for any vector $v \in \mathbb{R}^n$. In particular, if j is odd, we get

$$(y_0^{(j)} \otimes 1)(x \otimes k) = c^{|x|} x y_0^{(j)} \otimes k = c^{|x|+|k|}(x \otimes k)(y_0^{(j)} \otimes 1)$$

for all $x \otimes k \in \hat{G}_n(K)$. A standard Clifford algebra computation shows that

$$(y_0^{(j)})^2 = \begin{cases} 1 & \text{if } j \equiv 0, 1 \pmod 4 \\ c & \text{if } j \equiv 2, 3 \pmod 4. \end{cases}$$

Given that these relations only depend on j modulo 4 it is clear that

$$y_0^{(j+4)} x \otimes k \mapsto y_0^{(j)} x \otimes k$$

is a fermionic group isomorphism that intertwines $\rho^{(j)}$ with $\rho^{(j+4)}$. Next, we claim that

$$y_0^{(3)} x \otimes k \mapsto x \otimes k$$

for $x \otimes k \in \overline{G}_n(K)$ induces a fermionic isomorphism $G_n^{(3)}(K) \cong \hat{G}_n(K)$. Namely, if $x_1 \otimes k_1 \in G_n(K)$ and $x_2 \otimes k_2 \in \overline{G}_n(K)$, then

$$(x_1 \otimes k_1)(y_0^{(3)} x_2 \otimes k_2) = (y_0^{(3)} x_1 \otimes k_1)(x_2 \otimes k_2).$$

If instead both $x_1 \otimes k_1$ and $x_2 \otimes k_2$ lie in $\overline{G}_n(K)$, then

$$(y_0^{(3)} x_1 \otimes k_1)(y_0^{(3)} x_2 \otimes k_2) = (c(y_0^{(3)})^2 x_1 \otimes k_1)(x_2 \otimes k_2) = (x_1 \otimes k_1)(x_2 \otimes k_2).$$

So the bijection $G_n^{(3)}(K) \to \hat{G}_n(K)$ is indeed a group homomorphism. The same proof works for $j = 1$ except that for the last computation we get

$$(y_0^{(1)} x_1 \otimes k_1)(y_0^{(1)} x_2 \otimes k_2) = c(x_1 \otimes k_1)(x_2 \otimes k_2)$$

so that we get $G_n^{(1)}(K) \cong \hat{G}_n(K)^{\mathrm{op}}$ instead. These isomorphisms clearly intertwine $\hat{\rho}$ with $\rho^{(j)}$. $\square$

Chapter 4

Orientation reversal

The goal of the coming sections is to study the type of categorical structures we studied in Chapter 2 on bordism categories. Since we are interested in topological field theory with fermions and symmetries, we will require our bordisms to come equipped with more geometry, such as a spin structure. This will also give us $B\mathbb{Z}/2$-actions, as discussed in Section 2.6. In the following sections, we will carefully lay out several possible ways to define the orientation reversal of vector bundles equipped with geometric structures and how they can form $\mathbb{Z}/2$-actions. Once we have presented several possibilities for involutions $Y \mapsto \overline{Y}$ on the bordism category, we study the corresponding anti-involutions $Y \mapsto \overline{Y}^*$. It turns out that if we define orientation-reversal appropriately, then there are certain Hermitian pairings on Y that arise naturally in this geometric setting. However, we have to be really careful how the $\mathbb{Z}/2$-actions and Hermitian pairings are defined, in order to get a definition of unitary topological field theory for which physically desirable results hold. This is in particular the case for the spin-statistics theorem, which will be discussed in Section 6.3. A crucial tool will be the theory of fermionically dagger compact dual functors, which we developed in Sections 2.7 and 2.9.

4.1 G-structures

In this section, we provide the notions of geometric structures on vector bundles that will be used to define bordisms with geometric structures in later sections. Even though we will only consider topological theories in this thesis, for which this is not strictly necessary, we work with geometric notions as much as possible, keeping future applications in smooth field theories and index theory in mind. We make the appropriate translations to more homotopy-theoretic descriptions of geometric structures in remarks along the way.

Fix a nonnegative integer n. Let $\rho : G \to GL_n(\mathbb{R})$ be an n-dimensional representation of a topological group G, which we will call a *structure group*. Typically G will be compact Lie and ρ will factor through O_n.

Definition 4.1.1. Let $\eta \to X$ be a rank n real vector bundle over a topological space X. A *G-structure* on η consists of a principal G-bundle $P \to X$ (with its G-action written on the right) and an isomorphism of vector bundles

$$\alpha : \rho_* P := P \times_G \mathbb{R}^n \xrightarrow{\sim} \eta.$$

Here, for a representation V of G we denoted the vector bundle associated to a principal G-bundle P by $P \times_G V$. The above definition was introduced by [11], see [13] for lecture notes on G-structures.

Example 4.1.2. Let $G = O_n$ and $\rho : O_n \hookrightarrow GL_n(\mathbb{R})$ the inclusion. Then a G-structure on η is equivalent to a continuous fiberwise metric on η.

Example 4.1.3. Let $G = SO_n$ and $\rho : SO_n \hookrightarrow GL_n(\mathbb{R})$ the inclusion. Then a G-structure on η is equivalent to a metric and an orientation on η.

Example 4.1.4. Let $G = \mathrm{Spin}_n$ and $\rho : \mathrm{Spin}_n \to GL_n(\mathbb{R})$ the double cover map to SO_n followed by the inclusion. Then a G-structure on η is equivalent to a metric, an orientation and a spin structure on η.

Example 4.1.5. Let $G = GL_n(\mathbb{R}) \times H$ for some Lie group H and let ρ be projection to the first factor. Then a G-structure on $\eta \to X$ is equivalent to a principal H-bundle over X.

Example 4.1.6. Let K be a bosonic internal symmetry group without time reversal symmetries. Then the associated spacetime structure group is $G_n(K) = SO_n \times K$, with ρ the projection to SO_n. We see that a $G_n(K)$-structure on $\eta \to X$ consists of a principal K-bundle over X and a metric and an orientation on η.

If $\sigma : P_1 \to P_2$ is a principal bundle map, we denote the induced map of vector bundles $P_1 \times_G \mathbb{R}^n \to P_2 \times_G \mathbb{R}^n$ again by σ.

Definition 4.1.7. We say two G-structures $(P_1, \alpha_1), (P_2, \alpha_2)$ on the same bundle $\eta \to X$ are *geometrically isomorphic*, if there is an isomorphism $\sigma : P_1 \to P_2$ of principal G-bundles such that the diagram

$$\begin{tikzcd} P_1 \times_G \mathbb{R}^n \arrow[r, "\sigma"] \arrow[d, "\alpha_1"] & P_2 \times_G \mathbb{R}^n \arrow[dl, "\alpha_2"] \\ \eta & \end{tikzcd} \tag{4.1}$$

commutes.

The reason we added the adjective "geometrically" in the above definition, is that there is also a more homotopical notion of when G-structures are equivalent, which is more common in topology:

Definition 4.1.8. A *homotopy* between two G-structures $(P_1, \alpha_1), (P_2, \alpha_2)$ on $\eta \to X$ consists of a map $\sigma : P_1 \to P_2$ such that diagram (4.1) commutes up to homotopy.

Here by a homotopy of vector bundle maps $\eta_1 \to \eta_2$, we mean a homotopy $H_\bullet$ of continuous maps such that H_t is a vector bundle isomorphism for all t. In general we will always assume vector bundle maps are fiberwise isomorphisms (or at least injective in case they are of different rank). This will ensure they correspond one-to-one with principal $GL_n(\mathbb{R})$-bundle maps of the corresponding $GL_n(\mathbb{R})$-frame bundles.

Example 4.1.9. If $\rho : O_n \to GL_n(\mathbb{R})$ is the inclusion and $(P_1, \alpha_1), (P_2, \alpha_2)$ are two O_n-structures on η, a geometric isomorphism between them is equivalent to an isometric bundle isomorphism. However, any two O_n-structures are homotopic because $O_n \hookrightarrow GL_n(\mathbb{R})$ is a homotopy equivalence.

Remark 4.1.10. It would probably be better to include the homotopy of real vector bundle maps filling the diagram (4.1) as part of the data in Definition 4.1.8. To get the resulting data down to a reasonable size, we would then have to quotient out by further homotopies. In other words, a homotopy of G-structures would then be the datum of a homotopy class of homotopies of vector bundle morphisms filling the diagram (4.1). However, we choose not to pursue this here.

Let $G\text{-Str}^{\mathrm{geo}}(\eta), G\text{-Str}^{h}(\eta)$ denote the 1-groupoids of G-structures on a fixed vector bundle η with geometric isomorphism respectively homotopy as its morphisms. We will not use the higher analogues of these groupoids in the bulk of this thesis, but we now include a brief informal discussion of the appropriate generalization, also see [56, 57, 9]. A convenient homotopical way to define an ∞-groupoid $G\text{-Str}^{top}(\eta)$ of G-structures on η of which the fundamental groupoid is $G\text{-Str}^{h}(\eta)$ is as follows. This will be a crucial ingredient for higher-categorical considerations such as for extended topological field theories. There is a classifying space functor $B : \mathrm{TopGp} \to \mathcal{S}$ from the 1-category of topological groups to the $(\infty,1)$-category of ∞-groupoids, which we model by simplicial sets. To avoid usual pathologies with topological groups it is probably more convenient to replace them with simplicial groups in this discussion too. Consider the slice $(\infty,1)$-category $\mathcal{S}_{/BG\to BGL_n(\mathbb{R})}$ of spaces over the map $BG \to BGL_n(\mathbb{R})$. Concretely, objects of this $(\infty,1)$-category are pairs of maps $(\eta : X \to BGL_n(\mathbb{R}), P : X \to BG)$ together with a homotopy h filling the triangle

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "P"] & BGL_n(\mathbb{R}) \\ BG \arrow[ur, "B\rho_n"'] & \end{tikzcd} . \tag{4.2}$$

Our notation in the above diagram suggests that we can identify a principal G-bundle with a map $X \to BG$ and we will soon elaborate on how we do this more systematically. It is often assumed for convenience in model-theoretic settings that $BG \to BGL_n(\mathbb{R})$ is a fibration which allows one to leave out the homotopy in h in the diagram. The 1-morphisms between such triples $(X_1, \eta_1, P_1, h_1), (X_2, \eta_2, P_2, h_2)$ consist of a map $X_1 \to X_2$ and a filling of a tetrahedron

$$\begin{tikzcd} & BG \arrow[ddr, bend left, "B\rho_n"] & \\ & X_2 \arrow[u, dashed, "P_2"] \arrow[dr, dashed, "\eta_2"] & \\ X_1 \arrow[uur, bend left, "P_1"] \arrow[ur, dashed, "f"] \arrow[rr, bend right, "\eta_1"'] & & BGL_n(\mathbb{R}) \end{tikzcd} .$$

Where two of the sides are filled by h_1 and h_2 but the other two sides are extra data. This data corresponds up to homotopy with a vector bundle isomorphism between $f^*\eta_2$ and η_1 and a principal bundle isomorphism between f^*P_2 and P_1. If this 1-morphism covers the identity when restricted to $\mathcal{S}_{/BGL_n(\mathbb{R})}$ only one of the two sides can be nontrivial and so all 1-morphisms are invertible. We thus obtain an ∞-groupoid of G-structures on η, of which the homotopy 1-groupoid is $G\text{-Str}^{h}(\eta)$. Indeed, one can compare the two descriptions of G-structures by using the relationship between homotopy classes of morphisms $P_1 \to P_2$ over X to homotopy classes of homotopies between maps to BG.

We sketch this relationship in more detail, focusing on the case of principal G-bundles instead of G-structures on vector bundles for simplicity. Note that there is a map from $\mathrm{Map}(X, BG)$ to the set of G-Bun(X) principal G-bundles over X given by pulling back the universal bundle $EG \to BG$. We make $\mathrm{Map}(X, BG)$ into a simplicial set in the usual way by considering $\mathrm{Map}(X \times \Delta^n, BG)$ for varying n. We make principal bundles over X into a category enriched in topological spaces by using the topology on the mapping spaces between two principal bundles. This Top-enriched category is an ∞-groupoid, because every map of principal bundles is an isomorphism. We claim that the above map induces an equivalence of ∞-groupoids $\mathrm{Map}(X, BG) \to G\text{-Bun}(X)$. A sketch of proof is as follows. Note that it is an equivalence on connected components by the classical result that homotopy classes of maps from X to BG is in bijection with isomorphism classes of principal G-bundles over X. We can assume without loss of generality that EG has a connection so that given a map $\phi : X \times \Delta^n \to BG$ we get a canonical connection on $\phi^* EG$. Given two maps, $P_1, P_2 : X \to BG$ consider the subspace of the mapping space $\mathrm{Map}(X \times \Delta^1, BG)$ of those maps that restrict to P_1 and P_2 on the respective ends. We have to show that this space is homotopy equivalent to the space of G-bundle maps from P_1 to P_2. The main idea to prove this, would be to realize that a principal bundle over $X \times \Delta^1$ with connection gives a parallel transport map $P_1 \to P_2$. On $X \times \Delta^2$ we get two parallel transport maps by going along either side of the triangle. A choice of homotopy between the paths from $0 \to 1 \to 2$ and $0 \to 2$ in Δ^2 gives a homotopy between the parallel transport maps $P_1 \to P_2$ along the two sides of the triangle. In case $G = GL_n(\mathbb{R})$ we could have worked with vector bundles instead of principal $GL_n(\mathbb{R})$-bundles, this would not change the arguments and gives us a equivalence of ∞-groupoids

$$\mathrm{Map}(X, BGL_n(\mathbb{R})) \to \mathrm{VectBun}(X),$$

where as before we make $\mathrm{VectBun}(X)$ enriched in simplicial sets using homotopies of vector bundle isomorphisms. The two constructions are related by the associated bundle construction and the frame bundle construction.

Also note that composing $P : X \to BG$ with a map $BG \to BH$ induced by a group homomorphism corresponds to induction $P \times_G H$. For the case $H = GL_n(\mathbb{R})$ this will in particular be important to generalize the above discussion from G principal bundles to G-structures.

In Chapter 5 we will mainly be interested in the case the space X is a manifold M^k of dimension $k \leq n$. A G-structure on M will then always refer to a G-structure on the vector bundle which is a suitable stabilization of the tangent bundle $TM \oplus \underline{\mathbb{R}}^{n-k}$. We will sometimes refer to a manifold with G-structure as a G-manifold, which is unrelated to the notion of a manifold with smooth G-action. If G is a Lie group, we will assume principal bundles and vector bundles are smooth. Note that this changes the groupoid of G-structures up to geometric isomorphism, but not up to homotopy. It is also natural to further restrict to smooth maps of base spaces. The main difference in definitions involving G-structures in this scenario is that a smooth map always induces a map on the tangent bundle. We want to use this map to define maps of G-structures from now on. In particular, we need to consider the subcategories in which only embeddings and diffeomorphisms of the base manifold are allowed to ensure the induced tangent bundle maps are fiberwise isomorphisms:

Definition 4.1.11. Let M_1, M_2 be smooth k-dimensional manifolds equipped with G-structures (P_1, α_1) and (P_2, α_2) respectively. Then a *geometric G-diffeomorphism* from M_1 to M_2 consists of a diffeomorphism $f : M_1 \to M_2$ and a map of principal G-bundles $\sigma : P_1 \to P_2$ covering the map f such that the square

$$\begin{tikzcd} P_1 \times_G \mathbb{R}^n \arrow[r, "\sigma"] \arrow[d, "\alpha_1"] & P_2 \times_G \mathbb{R}^n \arrow[d, "\alpha_2"] \\ TM_1 \oplus \underline{\mathbb{R}}^{n-k} \arrow[r, "df \oplus \mathrm{id}_{\underline{\mathbb{R}}^{n-k}}"] & TM_2 \oplus \underline{\mathbb{R}}^{n-k} \end{tikzcd} \tag{4.3}$$

commutes. Similarly, for a *(homotopical) G-diffeomorphism* the diagram commutes up to homotopy of vector bundle maps. We define a *(homotopical) G-embedding* between manifolds with G-structure of the same dimension analogously when f is an embedding.

Note that the structure of a geometric G-diffeomorphism $(M, P_1, \alpha_1) \to (M, P_2, \alpha_2)$ on the identity diffeomorphism $M \to M$ is the same as a geometric isomorphism of G-structures $(P_1, \alpha_1) \cong (P_2, \alpha_2)$ on $TM \oplus \mathbb{R}^{n-k}$. Similarly making the identity into a homotopic G-diffeomorphism is equivalent to a homotopy of G-structures $(P_1, \alpha_1) \cong (P_2, \alpha_2)$.

Remark 4.1.12. We can also describe the definition of a G-diffeomorphism more homotopically in $\mathcal{S}_{/BG \to BGL_n(\mathbb{R})}$ by realizing that there is a functor from the category of manifolds with diffeomorphisms to $\mathcal{S}_{/BGL_n(\mathbb{R})}$. Concretely, this means the only thing that changes from the previous discussion is that the triangle

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "TM_1"'] & M_2 \arrow[dl, "TM_2"] \\ BGL_n(\mathbb{R}) & \end{tikzcd}$$

has a canonical filling induced by df. A homotopy filling of the square (4.3) corresponds up to homotopy with an equivalence class of 1-morphisms in $\mathcal{S}_{/BG \to BGL_n(\mathbb{R})}$ between the objects induced by (M_1, P_1, α_1) and (M_2, P_2, α_2).

Note that given a diffeomorphism $f : M_1 \to M_2$ and a G-structure (P_2, α_2) on M_2, the pullback principal bundle f^*P_2 with fibers $(f^*P_2)_{m_1} = (P_2)_{f(m_1)}$ becomes a G-structure on TM_1 via

$$f^*\alpha_2 : f^*P_2 \times_G \mathbb{R}^n \xrightarrow{\alpha_2} f^*TM_2 \xrightarrow{df} TM_1.$$

This assembles into a geometric G-diffeomorphism $f : (M_1, f^*P_2, f^*\alpha_2) \to (M_2, P_2, \alpha_2)$. In fact, we sometimes prefer to equivalently think of the data of a G-diffeomorphism on a diffeomorphism $f : M_1 \to M_2$ as an isomorphism $f^*P_2 \cong P_1$ of principal G-bundles satisfying a compaibility condition.

Recall that both the category G-Bun of principal G-bundles over arbitrary spaces as well as the category $\mathrm{VectBun}_n$ of vector bundles of rank n form a stack. Here morphisms are G-equivariant maps and vector bundle isomorphisms respectively, both covering arbitrary continuous maps. We will not use any descent and so we only formulate the prestack data. Our preferred definition of a prestack is a contravariant pseudofunctor (functor between bicategories) Top $\to$ Gpd, where Top is the 1-category of topological spaces and Gpd the 2-category of groupoids. We can then recover the 1-category G-Bun together with its canonical functor to Top as its Grothendieck construction. Explicitly, for every continuous map $f : X_1 \to X_2$, there is an induced covariant functor $f^* : G\text{-Bun}(X_2) \to G\text{-Bun}(X_1)$. Moreover, given two composable morphisms f_1, f_2 there is a canonical isomorphism of principal G-bundles $(f_2 f_1)^*(P) \cong f_1^* f_2^* P$ natural in P. This then satisfies a further compatibility condition for three composable morphisms. Given a homomorphism $G \to H$ we get a natural 2-transformation G-Bun $\Rightarrow$ H-Bun. We also have a natural 2-isomorphism $\mathrm{VectBun}_n \Rightarrow GL_n(\mathbb{R})\text{-Bun}$ given by the frame bundle construction, since the notion of pullback of principal $GL_n(\mathbb{R})$-bundles is compatible with pullback of vector bundles. Therefore, a homomorphism $\rho : G \to GL_n(\mathbb{R})$ induces a natural 2-transformation $G\text{-Bun} \Rightarrow \mathrm{VectBun}_n$.

We define G-Str as the prestack on the category $\mathrm{VectBun}_n$ which assigns $G\text{-Str}^{\mathrm{geo}}(\eta \to X)$ to a vector bundle $\eta \to X$. We want to formulate it on this category, because there is a canonical functor from the category of smooth n-dimensional manifolds $\mathrm{Man}_n^{\mathrm{emb}}$ and embeddings to $\mathrm{VectBun}_n$ and we are interested in $\mathrm{Man}_n^{\mathrm{emb}}$ for their application to (especially non-topological) bordism categories.[1] For topological bordism categories, the map from the groupoid of manifolds and diffeomorphisms to $\mathrm{VectBun}_n$ suffices. For use in later sections, we will now formulate this prestack explicitly. Let $f : X_1 \to X_2$ be covered by a vector bundle isomorphism $\phi : f^*\eta_2 \cong \eta_1$. Then we obtain a functor $f^* : G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_2, \eta_2) \to G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_1, \eta_1)$ given on objects by mapping a G-structure (P_2, α_2) on (X_2, η_2) to the principal bundle $f^*P_2 \to X_1$ and the vector bundle isomorphism

$$f^*P_2 \times_G \mathbb{R}^n \cong f^*(P_2 \times_G \mathbb{R}^n) \xrightarrow{f^*\alpha} f^*\eta_2 \xrightarrow{\phi} \eta_1.$$

On morphisms it maps an isomorphism of G-structures $\sigma : P_2 \to P_2'$ on (X_2, η_2) to the map of principal G-bundles $f^*\sigma : f^*P_2 \to f^*P_2'$ which is a map of G-structures because the diagram

$$\begin{tikzcd}
f^*P_2 \times_G \mathbb{R}^n \arrow[r, "f^*\sigma"] \arrow[d, "f^*\alpha_2"'] & f^*P_2' \times_G \mathbb{R}^n \arrow[dl, "\alpha_2'"] \\
\eta &
\end{tikzcd} \tag{4.4}$$

commutes. This is clearly functorial. Given two composable maps $(f_1, \phi_1) : (X_1, \eta_1) \to (X_2, \eta_2), (f_2, \phi_2) : (X_2, \eta_2) \to (X_3, \eta_3)$, the natural isomorphism filling the triangle

$$\begin{tikzcd}
G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_1, \eta_1) & G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_2, \eta_2) \arrow[l, "f_1^*"'] \\
G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_3, \eta_3) \arrow[u, "(f_2 f_1)^*"] \arrow[ur, "f_2^*"'] &
\end{tikzcd}$$

is given on (P_3, α_3) by the canonical isomorphism $(f_2 f_1)^* P_3 \cong f_1^* f_2^* P_3$ of principal bundles. This is an isomorphism of G-structures because this isomorphism is compatible with the isomorphism $(f_2 f_1)^* \eta_3 \cong f_1^* f_2^* \eta_3$ of vector bundles under ρ. The associativity condition for composing three morphisms follows because it holds for the principal bundle isomorphisms $(f_2 f_1)^* P \cong f_1^* f_2^* P$ and the analogous vector bundle isomorphisms. The above considerations are completely analogous for $G\text{-}\mathrm{Str}^h$ except that (4.4) now commutes up to homotopy.

Note that since every geometric isomorphism between G-structures induces a homotopy, there is a canonical map of prestacks

$$G\text{-}\mathrm{Str}^{\mathrm{geo}} \Rightarrow G\text{-}\mathrm{Str}^h .$$

In general, the essentially surjective functor

$$G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta) \Rightarrow G\text{-}\mathrm{Str}^h(X, \eta)$$

is not full. For example, for $G = O_n$ is gives the inclusion of isometries into all vector bundle maps. Conceptually, $G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta)$ can be thought of as the homotopy fiber over the point $\eta \to X$ of the map from the 1-groupoid of principal G-bundles to the 1-groupoid of vector bundles over X induced by ρ. On the other hand, $G\text{-}\mathrm{Str}^h(X, \eta)$ is the fundamental groupoid of $G\text{-}\mathrm{Str}^{top}(X, \eta)$ which is the the homotopy fiber of the map from the ∞-groupoid of principal G-bundles to the ∞-groupoid of vector bundles over X.

[1] Here we use the word ‘topological’ in the sense of ‘topological’ field theory. In other words, for us topological bordism categories refers to bordism categories that are sensitive to the differential topology but insensitive to the differential geometry of the manifold (such as a metric). This teminology unfortunately clashes with the approach to define $(\infty, 1)$-category of bordisms using topological categories.

If a structure group $\rho : G \to O_n \subseteq GL_n(\mathbb{R})$ lands in orthogonal matrices, a G-structure on η in particular includes an O_n-structure and hence a metric on η. It can be convenient to assume ρ lands in O_n, because of the existence of orthogonal complements. We sometimes prefer to think of these as G-structures (P, α) on n-dimensional vector bundles equipped with a metric, where we require α to preserve the metric using the canonical metric on $P \times_G \mathbb{R}^n$. This metric is well-defined because G acts by orthogonal transformations on $\mathbb{R}^n$. In other words, we sometimes prefer to think of $G\text{-Str}^{\mathrm{geo}}$ as a prestack on the category of vector bundles with metrics and isometries. Note that in particular in that case G-manifolds come equipped with a Riemannian structure and G-embeddings and G-diffeomorphisms are isometric. For topological field theories we will only be interested in G-structures up to homotopy, for which all of this geometric data is irrelevant because the map $BO_n \to BGL_n(\mathbb{R})$ is a homotopy equivalence. In particular, (homotopical) G-diffeomorphisms need not be isometries.

4.2 What on earth is an orientation of a point?

At some point in our mathematical lives we get spoon-fed the idea that the point considered as a zero-dimensional manifold admits two orientations. After some first considerations, this seems to be ridiculous: note that the group of invertible linear maps on a zero-dimensional vector space GL_0 has a single element. By definition, the determinant of a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is the composition

$$\mathbb{R} \cong \bigwedge^n \mathbb{R}^n \xrightarrow{\bigwedge^n T} \bigwedge^n \mathbb{R}^n \cong \mathbb{R}.$$

This definition still works for $n = 0$ since the empty tensor product is a one-dimensional line similar to how the empty product is the number 1. Moreover, the element of GL_0 has determinant one and therefore $SO_0 = 1 = GL_0$. In particular, we see that an SO_0-structure on a GL_0-principal bundle is no data.

This contradicts our understanding that there should be two orientations on the point, like on any other trivial vector bundle. More generally, orientations on η are in canonical bijection with orientations on $\eta \oplus \underline{\mathbb{R}}$. The geometric reason for this in positive dimensions is the fact that the rotation action of SO_{n+1} on S^n is transitive and has stabilizer group SO_n at a choice of pole. The transitivity is the thing that fails for $n = 0$.

In more homotopy-theoretic language, the canonical bijection between SO_n-structures on η and SO_{n+1}-structures on $\eta \oplus \underline{\mathbb{R}}$ is expressed by the fact that

$$\begin{tikzcd} BSO_n \arrow[r] \arrow[d] & BO_n \arrow[d] \\ BSO_{n+1} \arrow[r] & BO_{n+1} \end{tikzcd} \tag{4.5}$$

is a homotopy pullback square. However, note that with the definitions we gave above, this is no longer true if $n = 0$:

$$\begin{tikzcd} \mathrm{pt} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & B\mathbb{Z}/2 \end{tikzcd} .$$

One way to fix it is to instead define '$BSO_0 = \mathbb{Z}/2$', the pointed space consisting of two points. Then the above is a homotopy-pullback:

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & B\mathbb{Z}/2 \end{tikzcd}$$

An orientation on a zero-dimensional vector bundle X then is a continuous map $X \to \mathbb{Z}/2$ (the additional homotopy with the map $X \to BO_0$ is no data). In particular, every zero-dimensional connected manifold has a canonical orientation, which we call the positively oriented point.

Note that definitions of orientations in algebraic topology do make sense for vector bundles of dimension zero. For example, if $V \to X$ is a zero-dimensional vector bundle, then its Thom space is $X \sqcup \{\mathrm{pt}\}$ and so there is a canonical Thom isomorphism

$$H^\bullet(X;\mathbb{Z}) \to \tilde{H}^\bullet(X \sqcup \{\mathrm{pt}\};\mathbb{Z}),$$

corresponding to the positively oriented point in case $X = \{\mathrm{pt}\}$. The negatively oriented point is given by multiplying with the Thom class $u \in \tilde{H}^0(X \sqcup \{\mathrm{pt}\};\mathbb{Z}) \cong H^0(X;\mathbb{Z})$ given by -1.

Another more geometric and equivalent way to define an orientation that does not involve G-structures is as a trivialization of the top exterior power. This also works in zero dimensions as there are two isometric isomorphisms $\bigwedge^0 \mathbb{R}^0 \cong \mathbb{R}$ and the identity gives the preferred positively oriented point. However, since we prefer to work with geometric G-structures, we will from now on always have to be careful in dimension zero or more specifically, when squares of the form (4.5) are not homotopy pullbacks. We will further discuss these issues in Example 4.3.12.

4.3 Suspension, desuspension and reversing suspended directions

The first approach to the orientation-reversal of G-structures we want to discuss is the idea of introducing an extra direction and then reflecting along it. One often considers this notion in the definition of n-dimensional bordisms with structure $G_n \to GL_n(\mathbb{R})$, for which the boundaries are defined as $(n-1)$-dimensional closed manifolds with some related structure $G_{n-1} \to GL_{n-1}(\mathbb{R})$. Then in an informal definition of the bordism category, one distinguishes in- and outgoing manifolds by their 'orientation'. We want to argue that this picture is slightly misleading; in- and outgoing manifolds should be distinguished by the direction of a normal vector, which happens to be equivalent to changing the orientation.

Moreover, if we have access to a structure group $G_{n+1} \to GL_{n+1}(\mathbb{R})$ compatible with $G_n \to GL_n(\mathbb{R})$ in a certain sense, we could similarly define the orientation-reversal of a bordism by reversing the $(n+1)$th direction. This is for example the case when $G_n = G_n(K)$ is the spacetime structure group of a fermionic group. With enough care, defining orientation-reversal for G_n-structures up to homotopy in this way makes sense, but for non-topological field theories most of these notions start to diverge. The reason is that in most cases a G_{n+1}-structure on a vector bundle $\eta \oplus \underline{\mathbb{R}}$ is always homotopic to one that is 'constant in the $\mathbb{R}$-direction', but this is no longer true geometrically. As a solution, we will mostly work with vector bundles equipped with metrics.

What do we mean by 'extra directions'? Suppose throughout this section that we are given a commuting square of topological groups of the following form.

$$\begin{tikzcd} G_n \arrow[r, "\rho_n"] \arrow[d, "f"] & GL_n(\mathbb{R}) \arrow[d] \\ G_{n+1} \arrow[r, "\rho_{n+1}"] & GL_{n+1}(\mathbb{R}) \end{tikzcd} \tag{4.6}$$

Definition 4.3.1. Let (P, α) be a G_n-structure on the bundle $\eta \to X$. Then the *suspension of* (P, α) is the G_{n+1}-structure $(sP, s\alpha)$ on $\eta \oplus \underline{\mathbb{R}}$ where $sP := P \times_{G_n} G_{n+1}$ and $s\alpha : (P \times_{G_n} G_{n+1}) \times_{G_{n+1}} \mathbb{R}^{n+1} \to \eta \oplus \underline{\mathbb{R}}$ is uniquely determined by

$$s\alpha[[p, 1], v] = (\alpha[p, (v_1, \dots, v_n)], v_{n+1}),$$

where $v = (v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$.

For general $g \in G_{n+1}$ we can derive that

$$s\alpha[[p, g], v] = s\alpha[[p, 1], \rho_{n+1}(g)v] = (\alpha[p, (\rho_{n+1}(g)v)_1, \dots (\rho_{n+1}(g)v)_n], (\rho_{n+1}(g)v)_{n+1}).$$

This is well-defined because if $g \in G_n$, then

$$\begin{aligned} s\alpha[[p, f(g)], v] &= (\alpha[p, (\rho_{n+1}(f(g))v)_1, \dots, (\rho_{n+1}(f(g))v)_n], (\rho_{n+1}(f(g))v)_{n+1}) \\ &= (\alpha[p, \rho_n(g)(v_1, \dots, v_{n+1})], v_{n+1}) \\ &= (\alpha([pg, (v_1, \dots, v_n)], v_{n+1}) = s\alpha([pg, 1], v). \end{aligned}$$

Note that $s\alpha$ is an isomorphism because α is. Suspension defines a functor from G_n-structures on η to G_{n+1}-structures on $\eta \oplus \underline{\mathbb{R}}$.

Given a G_{n+1}-structure on $\eta \oplus \underline{\mathbb{R}}$, we can always 'reverse' it by composing with the vector bundle automorphism $\mathrm{id}_\eta \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$ of $\eta \oplus \underline{\mathbb{R}}$. Note that the G_{n+1}-structure does not need to be a suspension to do this, but the vector bundle does need to have one direction trivialized.

Definition 4.3.2. Let (P, α) be a G_{n+1}-structure on $\eta \oplus \underline{\mathbb{R}}$. Then $(P, \alpha)^*$ is defined to be $P^* = P$ and α^* is equal to the composition

$$P \times_{G_{n+1}} \mathbb{R}^{n+1} \xrightarrow{\alpha} \eta \oplus \underline{\mathbb{R}} \xrightarrow{\mathrm{id}_\eta \oplus -\mathrm{id}_{\underline{\mathbb{R}}}} \eta \oplus \underline{\mathbb{R}}.$$

The reason for the notation is that if Y is an n-dimensional closed manifold and (P, α) a G_{n+1}-structure on $TY \oplus \underline{\mathbb{R}}$, then we will see that Y equipped with the G_{n+1}-structure (P^*, α) can be realized naturally as the dual of (Y, P, α) in the category of $(n+1)$-dimensional bordisms with G_{n+1}-structure. Note that $P \mapsto P^*$ becomes a covariant functor on the category of G-structures on $\eta \oplus \underline{\mathbb{R}}$ by defining it to be the identity on morphisms. Moreover, $P^{**} = P$ on the nose, so that this defines a $\mathbb{Z}/2$-action on the category.

Next we want to desuspend G_{n+1}-structures and generalize common statements such as "an orientation on η is equivalent to an orientation on $\eta \oplus \underline{\mathbb{R}}$". For this we have to put more assumptions on the square (4.6). The last section gives us the intuition that for homotopy-theoretic purposes, we should at least assume that the square induces a homotopy pullback square after taking classifying spaces. However, this is not a very geometric condition and so we translate it closer to geometric intuition. Since we want to include metrics later on, we will assume $G_{n+1} \to GL_{n+1}(\mathbb{R})$ lands in

O_{n+1} in the rest of this discussion. Recall that O_{n+1}/O_n is homeomorphic to S^n and so a homotopy pullback square of the desired form

$$\begin{tikzcd} S^n \arrow[r, equal] \arrow[d] & S^n \arrow[d] \\ BG_n \arrow[r, "\rho_n"] \arrow[d, "f"] & BO_n \arrow[d] \\ BG_{n+1} \arrow[r, "\rho_{n+1}"] & BO_{n+1} \end{tikzcd} \tag{4.7}$$

would imply that that the fiber of $BG_n \to BG_{n+1}$ is homotopy equivalent to S^n. For example, if $G_n \to G_{n+1}$ is injective, this implies G_{n+1}/G_n is homotopy equivalent to S^n. In practice, it can be useful to restrict this to be a diffeomorphism.

Definition 4.3.3. We say $\rho_n : G_{n+1} \to O_{n+1}$ is a *strict geometric representation* if acts on S^n transitively through ρ.

If $\rho_n : G_{n+1} \to O_{n+1}$ is a strict geometric representation, we can define a new structure group G_n by a strict pullback

$$\begin{tikzcd} G_n \arrow[r, "\rho_n"] \arrow[d] & O_n \arrow[d] \\ G_{n+1} \arrow[r, "\rho_{n+1}"] & O_{n+1} \end{tikzcd} \tag{4.8}$$

in the 1-category of topological groups, i.e. G_n is the subgroup

$$G_n = \{g \in G_{n+1} : \rho(g) \in O_n\},$$

which is the stabilizer of the action of G_{n+1} on S^n at a pole. We will assume in the rest of this section that ρ_{n+1} is a strict geometric representation and our commutative square (4.6) is a pullback. Note that if G_{n+1} lands in SO_{n+1}, then G_n lands in SO_n and

$$\begin{tikzcd} G_n \arrow[r, "\rho_n"] \arrow[d] & SO_n \arrow[d] \\ G_{n+1} \arrow[r, "\rho_{n+1}"] & SO_{n+1} \end{tikzcd} \tag{4.9}$$

is a strict pullback. Note that for a strict geometric representation we have that G_{n+1}/G_n is homeomorphic to S^n. Note that for $n = 0$ we have that G_{n+1} acts transitively on S^n if and only if it surjects onto O_{n+1}, so in that case a ρ_1 that only hits SO_1 is not a strict geometric representation. We will typically assume generally ρ_{n+1} is either onto SO_{n+1} or O_{n+1}, but the reader is free to substitute these groups by $\operatorname{Im}\rho$. Since surjective group homomorphisms yield fibrations on classifying spaces, this implies either $BG_{n+1} \to BO_{n+1}$ or $BG_{n+1} \to BSO_{n+1}$ is a fibration. Therefore in the first case the strict pullback

$$\begin{tikzcd} BG_n \arrow[r, "B\rho_n"] \arrow[d] & BO_n \arrow[d] \\ BG_{n+1} \arrow[r, "B\rho_{n+1}"] & BO_{n+1} \end{tikzcd} \tag{4.10}$$

is a homotopy pullback. In the latter case, a similar diagram with BSO instead of BO is a homotopy pullback. In that case we can use that

$$\begin{tikzcd} BSO_n \arrow[r, "B\rho_n"] \arrow[d] & BO_n \arrow[d] \\ BSO_{n+1} \arrow[r, "B\rho_{n+1}"] & BO_{n+1} \end{tikzcd} \tag{4.11}$$

is a homotopy pullback for $n > 0$ to conclude the desired homotopy pullback (4.10).

Remark 4.3.4. Note that strict geometric representations certainly does not accommodate for the most general situation. For example, when G_{n+1} is the trivial group it certainly does not act transitively on the circle. We can then still define BG_n by homotopy pullback as a homotopy type. In case this description is too homotopical, we can always represent this space as the classifying space of some topological group G_n, when $n > 0$.

Often we are in the situation where we have a given structure group $G_n \to O_n$ and we are wondering whether it is possible to talk about stabilizing and destabilizing to dimension $n+1$. For this we introduce the following language.

Definition 4.3.5. We say $G_{n+1} \to O_{n+1}$ is a *strict geometric stabilization of* $G_n \to O_n$ if it is a strict geometric representation such that $G_n \to O_n$ is the induced (strict) pullback in the 1-category of topological groups.

Lemma 4.3.6. *Let $n > 0$ and let K be a fermionic group. Then $G_{n+1}(K) \to O_{n+1}$ is a strict geometric stabilization of $G_n(K) \to O_n$. More generally, this holds for $n = 0$ when K has reversing elements.*

Proof. Note that the image of $\rho : G_{n+1}(K) \to O_{n+1}$ always contains SO_{n+1}. Moreover, if K is not purely even, ρ is onto. Therefore the action of $G_{n+1}(K)$ on S^n is transitive for $n > 0$ and for $n = 0$ when K is not purely even. It remains to be shown that

$$G_n(K) = \{x \otimes k \in G_{n+1}(K) : \rho(x \otimes k) \in O_n\},$$

compatibly with the canonical inclusion into $G_{n+1}(K)$. This follows because if $x \in \mathrm{Pin}^+_{n+1}$ has its projection $[x] \in O_{n+1}$ living in the group $[x] \in O_n$, we have that necessarily $x \in \mathrm{Pin}^+_n \subseteq \mathrm{Pin}^+_{n+1}$. Note that in the case $n = 0$ and K has reversing elements, this argument still works with our convention that $\mathrm{Pin}^\pm_0 = \mathrm{Spin}_1$. □

If $G_{n+1} \to O_{n+1}$ is a strict geometric stabilization of $G_n \to O_n$ and we are given a G_n-structure on the real vector bundle η, then the stabilization G_{n+1}-structure on $\eta \oplus \underline{\mathbb{R}}$ equips it with the orthogonal sum metric. If (Q, β) is a G_{n+1}-structure on $\eta \oplus \underline{\mathbb{R}}$, then the G_{n+1}-structure (Q^*, β^*) comes equipped with the same metric; β^* is an isometry since $\mathrm{id}_\eta \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$ is an orthogonal vector bundle automorphism.

Definition 4.3.7. The *desuspension* $(s^{-1}Q, s^{-1}\beta)$ of the G_{n+1}-structure (Q, β) on $\eta \oplus \underline{\mathbb{R}}$ is the G_n-structure on η defined as follows. Let

$$s^{-1}Q := \{q \in Q : \beta[q, e_{n+1}] = (0, 1) \in \eta \oplus \underline{\mathbb{R}}\} \subseteq Q,$$

where $e_{n+1} \in \mathbb{R}^{n+1}$ is the $(n+1)$th standard basis vector. Now $s^{-1}\beta$ is defined as

$$\begin{aligned} s^{-1}\beta :& s^{-1}Q \times_{G_n} \mathbb{R}^n \to \eta \\ s^{-1}\beta[q,v] &= \mathrm{pr}_\eta(\beta[q,(v,0)]) \in \eta, \end{aligned}$$

where $\mathrm{pr}_\eta : \eta \oplus \underline{\mathbb{R}} \to \eta$ is orthogonal projection, $q \in s^{-1}Q$ and $v \in \mathbb{R}^n \subseteq \mathbb{R}^{n+1} \ni (v,0)$.

Be aware that the notation is slightly misleading because $s^{-1}Q$ strongly depends on β.

Lemma 4.3.8. *Let Q a G_{n+1}-structure on the rank $n+1$ vector bundle $\eta \oplus \underline{\mathbb{R}}$. The desuspension $s^{-1}Q$ of Q is a G_n-structure on η.*

Proof. First we will show that $s^{-1}Q$ is a principal G_n-bundle. Let $q \in s^{-1}Q$ and $g \in G_n$. Then $qg \in s^{-1}Q$, since

$$\beta[qg, e_{n+1}] = \beta[q, \rho(g)e_{n+1}] = \beta[q, e_{n+1}] = (0,1).$$

To show the action is free, let $q_1, q_2 \in s^{-1}Q$. We want to show that there exists a unique $g \in G_n$ such that $q_1 g = q_2$. Using the universal property of the pullback G_n, we get a unique $g \in G_{n+1}$ such that $q_1 g = q_2$. We want to show that $g \in G_n$. We know that

$$\beta[q_1, \rho(g)e_{n+1}] = \beta[q_1 g, e_{n+1}] = \beta[q_2, e_{n+1}] = (0,1) = \beta[q_1, e_{n+1}].$$

Because β is bijective, we see that $[q_1, \rho(g)e_{n+1}] = [q_1, e_{n+1}]$ and so $\rho(g)e_{n+1} = e_{n+1}$. Therefore, $\rho(g)$ is a matrix of the form

$$\begin{pmatrix} A & 0 \\ w & 1 \end{pmatrix}$$

where A is an $n \times n$ matrix and $w \in \mathbb{R}^n$. From the fact that $\rho(g)$ is an orthogonal matrix, it follows that $w = 0$. Hence $\rho(g) \in O_n$. By the pullback diagram we derive that $g \in G_n$.

Now note that $s^{-1}\beta[q, \rho(g)v] = s^{-1}\beta[qg, v]$, because the analogous equation holds for β. To show $s^{-1}\beta$ is an isomorphism, it suffices to show that it has no kernel. Let $[q,v] \in \ker s^{-1}\beta$ for some $v \in \mathbb{R}^n$ and $q \in s^{-1}Q$. Then $\beta[q,v] = (0,a)$ for some $a \in \mathbb{R}$. However, we also have $\beta[q, ae_{n+1}] = (0,a)$ by definition of $s^{-1}Q$. Because β is bijective, we see that $v = ae_{n+1}$. But $v_{n+1} = 0$ and hence $v = 0$. $\square$

Note that since the projection pr_η is orthogonal, $s^{-1}\beta$ is an isometry.

Corollary 4.3.9. *Let $\eta \to X$ an n-dimensional vector bundle. Then suspension and desuspension give inverse equivalences of groupoids*

$$\begin{tikzcd} G_{n+1}\text{-}\mathrm{Str}^{\mathrm{geo}}(\eta \oplus \underline{\mathbb{R}}) \arrow[rr, bend right, "s^{-1}"'] & & G_n\text{-}\mathrm{Str}^{\mathrm{geo}}(\eta). \arrow[ll, bend right, "s"'] \end{tikzcd}$$

Proof. We make s^{-1} into a functor as follows. Let $f : (P_1, \alpha_1) \to (P_2, \alpha_2)$ be a morphism of G_{n+1}-structures on $\eta \oplus \underline{\mathbb{R}}$. Then f restricts to a map $s^{-1}P_1 \to s^{-1}P_2$. Indeed, if $p_1 \in P_1$ is in $s^{-1}P_1$ then $\alpha_2[f(p_1), e_{n+1}] = \alpha_1[p_1, e_{n+1}]$ lands in the $\underline{\mathbb{R}}$-summand of $\eta \oplus \underline{\mathbb{R}}$ as desired.

To show that $s^{-1}sP \cong P$, note that by definition

$$s^{-1}sP = \{[p,g] \in P \times_{G_n} G_{n+1} : s\alpha[[p,g], e_{n+1}] = (0,1)\}$$

and
$$s\alpha[[p,g],e_{n+1}] = (\alpha[p,(\rho(g)e_{n+1})_1,\dots,(\rho(g)e_{n+1})_n)],(\rho(g)e_{n+1})_{n+1}).$$
Using the bijectivity of α, this is equivalent to $\rho(g)e_{n+1} = e_{n+1}$. By the same argument as in Lemma 4.3.8, this condition is equivalent to $g \in G_n$ and so $[p,g] = [pg,1]$. We see that the obvious map $P \to s^{-1}sP$, is an isomorphism of principal bundles. It is an isomorphism of G_n-structures because $s^{-1}s\alpha$ agrees with α on elements of $s^{-1}sP$ of the form $[p,1]$ and we have just shown that all elements are of this form.

We now have to show the isomorphism $ss^{-1}P \cong P$ is natural. Let $f : (P_1,\alpha_1) \to (P_2,\alpha_2)$ be a morphism of G_n-structures on η. Then the diagram

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d] & P_2 \arrow[d] \\ s^{-1}sP_1 \arrow[r, "s^{-1}sf"] & s^{-1}sP_2 \end{tikzcd}$$

commutes because on elements of $s^{-1}sP_1$ of the form $[p_1,1]$ we have $s^{-1}sf[p_1,1] = [f(p_1),1]$.

For a G_{n+1}-structure (Q,β) on $\eta \oplus \underline{\mathbb{R}}$, define a G_{n+1}-equivariant map $ss^{-1}Q \to Q$ by $[q,g] \mapsto qg$ for $q \in s^{-1}Q$. This is a map of G_{n+1}-structures by the computation
$$\begin{aligned} ss^{-1}\beta[[q,g],v] &= (s^{-1}\beta[q,\mathrm{pr}_{\mathbb{R}^n\oplus 0}(\rho(g)v)],\mathrm{pr}_{0\oplus\mathbb{R}}(\rho(g)v)) \\ &= (\mathrm{pr}_\eta\,\beta[q,(\mathrm{pr}_{\mathbb{R}^n\oplus 0}(\rho(g)v),0)],\mathrm{pr}_{0\oplus\mathbb{R}}(\rho(g)v)) \\ &= \beta[q,\rho(g)v] \end{aligned}$$
where the last equation followed because $q \in s^{-1}Q$ implies
$$\beta[q,\mathrm{pr}_{0\oplus\mathbb{R}}(\rho(g)v)e_{n+1})] = (0,\mathrm{pr}_{0\oplus\mathbb{R}}(\rho(g)v)).$$
This isomorphism is natural because if $f : Q_1 \to Q_2$ is a morphism of G_{n+1}-structures and $[q,g] \in ss^{-1}Q$, then
$$ss^{-1}f[q,g] = [f(q),g] \mapsto f(q)g = f(qg).$$
□

Note that the above proofs strongly relied on the fact that $\rho(g) \in O_n$. We provide the first way to define orientation-reversal, which works for an arbitrary structure group that admits a strict geometric stabilization:

Definition 4.3.10. If (P,α) is a G_n-structure on η, define $(P',\alpha') := (s^{-1}(sP)^*, s^{-1}(s\alpha)^*)$.

Note that even though $(sP)^* = sP$ as a principal bundle, $s^{-1}(sP)^* \neq s^{-1}sP \cong P$, because the definition of $s^{-1}(sP)^*$ depends on $(s\alpha)^*$. Recall that from a strict geometric representation $G_{n+1} \to O_{n+1}$ we have defined $G_n^{(1)} \to O_n$ in Definition 3.2.24.

Lemma 4.3.11. *Let (P,α) be a G_n-structure on the rank n bundle $\eta \to X$. Explicitly, we have*
$$P' = P \times_{G_n} (G_n^{(1)})_{rev}, \tag{4.12}$$
where $(G_n^{(1)})_{rev} \subseteq G_n^{(1)}$ is the subset of those $g' \in G_{n+1}$ for which $\rho(g')$ is of the form
$$\rho(g') = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \in O_{n+1} \tag{4.13}$$

. For $[p,g'] \in P'$ and $v \in \mathbb{R}^n$ we have

$$\alpha'[[p,g'],v] = \alpha[p, \rho(g')|_{\mathbb{R}^n} v].$$

Proof. We compute using the fact that (4.8) is a pullback square

$$\begin{aligned}
P' &= \{[p,g'] \in P \times_{G_n} G_{n+1} : (s\alpha)^*[[p,g'],e_{n+1}] = (0,1) \in \eta \oplus \underline{\mathbb{R}}\} \\
&= \{[p,g'] \in P \times_{G_n} G_{n+1} : \alpha[p,(\rho(g')e_{n+1})_1, \ldots, (\rho(g')e_{n+1})_n)] = 0 \text{ and } -(\rho(g')e_{n+1})_{n+1} = 1\} \\
&= \{[p,g'] \in P \times_{G_n} G_{n+1} : (\rho(g')e_{n+1})_i = 0 \text{ for } i < n+1 \text{ and } (\rho(g')e_{n+1})_{n+1} = -1\} \\
&= \{[p,g'] \in P \times_{G_n} G_{n+1} : \rho(g')e_{n+1} = -e_{n+1}\} \\
&= \{[p,g'] \in P \times_{G_n} G_{n+1} : \rho(g') = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \in O_{n+1}\}.
\end{aligned}$$

Note that in the formula for α' we restricted $\rho(g') \in O_{n+1}$ to $\mathbb{R}^n \oplus 0 \subseteq \mathbb{R}^{n+1}$, which is in O_n given that $\rho(g')$ satisfies (4.13). Let $v \in \mathbb{R}^n$ and $[p,g'] \in P'$ so that $\rho(g')$ is of the specific form in (4.13). Then

$$\begin{aligned}
\alpha'[[p,g'],v] &= \mathrm{pr}_\eta (s\alpha)^*[[p,g'],(v_1,\ldots,v_n,0)] = \mathrm{pr}_\eta\, s\alpha[[p,g'],(v_1,\ldots,v_n,0)] \\
&= \mathrm{pr}_\eta\, s\alpha[[p,1],(\rho(g)|_{\mathbb{R}^n} v, 0)] = \mathrm{pr}_\eta(\alpha[p,\rho(g)|_{\mathbb{R}^n} v],0) = \alpha[p,\rho(g)|_{\mathbb{R}^n} v],
\end{aligned}$$

as required. □

Example 4.3.12. We review and generalize the discussion in Section 4.2 now that we can talk about suspensions and desuspensions. Let $\rho : G_1 \to O_1$ be a one-dimensional orthogonal representation and $G_0 = \ker \rho \to O_0 = 1$ its strict pullback. If ρ is not surjective, the action of G_1 on S^0 is not transitive and so ρ is not a strict geometric representation. We will now study the consequences of this fact for G_0-structures. A G_0-structure on a zero-dimensional vector bundle is simply a G_0-principal bundle Q. A G_1-structure on X consists of a real line bundle η with metric, a principal G_1-bundle P over X and an isometry $\alpha : P \times_{G_1} \mathbb{R} \cong \eta$. In case ρ is not surjective, there is a canonical isometry $P \times_{G_1} \mathbb{R} \cong \underline{\mathbb{R}}$ and so α is a trivialization of η. But otherwise, η is potentially nontrivial. The suspension of a G_0-structure Q consists of the principal G_1-bundle $P = Q \times_{G_0} G_1$ and $\alpha_x[q,g_1,v] = \rho(g_1)v \in \underline{\mathbb{R}}_x$, where $q \in Q$ lies over $x \in X$.

Note that the trivial G_1-structure $(\underline{\mathbb{R}} \to \{\mathrm{pt}\}, P = G \times \{\mathrm{pt}\}, \alpha = \mathrm{id})$ on the trivial line bundle over a point is the suspension of the only G_0-structure up to isomorphim, which we will call the *positively oriented point*. However, isometric trivializations of a real line bundle over X are an $H^0(X,\mathbb{Z}/2)$-torsor and so we can define the *negatively oriented point* analogously but having $\alpha = -\mathrm{id}$, the nontrivial isometric trivialization of $\underline{\mathbb{R}}$. Note that when ρ is not surjective, the negatively oriented point is not isomorphic to the positively oriented point and so not the suspension of a G_0-structure. However, when ρ is surjective, the positively and negatively oriented point are isomorphic. Indeed, left multiplication by a reversing element $g_1 \notin \ker \rho$ induces an automorphism of trivial principal G_1-bundles P over the point and the diagram

$$\begin{tikzcd}
P \times_G \mathbb{R} \arrow[r, "g_1 \cdot"] \arrow[d, "\alpha"] & P \times_G \mathbb{R} \arrow[dl, "-\alpha"] \\
\underline{\mathbb{R}} &
\end{tikzcd}$$

commutes. Reformulating this slightly: for a G_1-structure (P, α) on a trivial real line bundle we can define a $\mathbb{Z}/2$-grading $P = P_+ \sqcup P_-$ as

$$P_\pm := \{p \in P : \alpha[p, 1] = \pm 1\}.$$

If ρ is surjective, P_+ and P_- are both nonempty and the G_1-action is graded in the sense that reversing elements of G_1 reverse P_+ and P_- while preserving elements preserve them. However, when ρ is not surjective we either have $P_- = \emptyset$ (positively oriented) or $P_+ = \emptyset$ (negatively oriented). In other words, in that case we can think of the negatively oriented point as the 'reversed trivial G-structure'.

If so desired, we could have restricted the above discussion to the case where $G_1 = G_1(K) \cong K^{\mathrm{op}}$ without loss of generality. In that case $G_0 = G_0(K)$ is equal to K_{pres}, keeping our conventions for Pin_0^+ of Remark 3.2.13 in mind.

4.4 The automorphism 2-group of a structure group

This section is strongly inspired by discussions with Stephan Stolz.

If $\phi : G \to G$ is a topological group automorphism and P a principal G-bundle over a space X, let P_ϕ denote the principal G-bundle twisted by ϕ. We denote elements as symbols $p_\phi \in P_\phi$ for $p \in P$ and the action is given by $p_\phi g = (p\phi(g))_\phi$. This gives an action of $\mathrm{Aut}(G)$ on the groupoid of principal G-bundles over X. For example, if $\phi : SO_n \to SO_n$ is conjugation by a reflection r_i with respect to a fixed coordinate direction i, then P_ϕ might be a good model for the 'orientation reversal of the oriented frame bundle P'.

Moreover, if $u \in G$ and $c_u : G \to G$ denotes conjugation by u, then $p_{c_u} \mapsto pu$ defines an isomorphism of principal G-bundles $P_{c_u} \cong P$. For example, if $u = r_i r_j$ is a rotation in the (i, j)-plane, this provides a way to compare the 'orientation reversal in the ith direction' with the 'orientation reversal in the jth direction'. One can work out that these considerations lead to an action of the automorphism 2-group of G on the groupoid of principal G-bundles. For a review on 2-groups and their actions on categories, consider Appendix A.2. Next, we will make the above idea precise in our setting of G-structures on vector bundles.

So suppose $\rho : G \to GL_n(\mathbb{R})$ is a structure group and ϕ covers conjugation with a matrix $A \in GL_n(\mathbb{R})$ and let (P, α) be a G-structure on η. Define a new G-structure (P_ϕ, α_A) by

$$\alpha_A[p_\phi, v] := \alpha[p, Av].$$

This is well-defined, because

$$\alpha_A[p_\phi g, v] = \alpha_A[(p\phi(g))_\phi, v] = \alpha[p\phi(g), Av] = \alpha[p, \rho\phi(g)Av] = \alpha[p, A\rho(g)v] = \alpha_A[p_\phi, \rho(g)v]$$

and α_A is clearly a vector bundle isomorphism. The main goal of this section is to make this construction functorial.

Definition 4.4.1. The *relative automorphism 2-group* of a structure group $G \to GL_n(\mathbb{R})$ is the category $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ with

- objects pairs (ϕ, A) of an automorphism $\phi \in \mathrm{Aut}\, G$ and $A \in GL_n(\mathbb{R})$ a matrix such that $\rho(\phi(g)) = A\rho(g)A^{-1}$;
- morphisms $(\phi_1, A_1) \to (\phi_2, A_2)$ are elements $u \in G$ such that $\phi_2(g) = u\phi_1(g)u^{-1}$ and $\rho(u) = A_2 A_1^{-1}$.

Composition of $u_1 : (\phi_1, A_1) \to (\phi_2, A_2)$ and $u_2 : (\phi_2, A_2) \to (\phi_3, A_3)$ is defined by multiplication.

Composition is well-defined since

$$\rho(u_2u_1) = A_3A_2^{-1}A_2A_1 = A_3A_1.$$

Lemma 4.4.2. *The category* $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ *admits a canonical* 2*-group structure.*

Proof. The tensor product of objects $(\phi, A), (\phi', A')$ is defined by $(\phi \circ \phi', AA')$. This is well-defined, strictly associative and has monoidal unit $(\mathrm{id}_G, \mathrm{id}_{\mathbb{R}^n})$. The tensor product of morphisms $(\phi_1, A_1) \xrightarrow{u} (\phi_2, A_2)$ and $(\phi_1', A_1') \xrightarrow{u'} (\phi_2', A_2')$ is $(\phi_1\phi_1', A_1A_1') \xrightarrow{u\phi_1(u')} (\phi_2\phi_2', A_2A_2')$. This is well-defined because

$$\phi_2\phi_2'(g) = u\phi_1(u'\phi_1'(g)(u')^{-1})u^{-1} = u\phi_1(u')\phi_1\phi_1'(g)\phi_1(u')^{-1}u^{-1}$$

and

$$\rho(u\phi_1(u')) = A_2A_1^{-1}A_1\rho(u')A_1^{-1} = A_2A_2'(A_1')^{-1}A_1^{-1}.$$

It is straightforward to see that all objects are invertible under tensor product and all morphisms are invertible under both tensor product and composition. Finally, to show that tensor product defines a functor, we have to show that given morphisms $u_1 : (\phi_1, A_1) \to (\phi_2, A_2), u_2 : (\phi_2, A_2) \to (\phi_3, A_3), u_1' : (\phi_1', A_1') \to (\phi_2', A_2')$ and $u_2' : (\phi_2', A_2') \to (\phi_3', A_3')$ we have $(u_2 \otimes u_2') \circ (u_1 \otimes u_1') = (u_2 \circ u_1) \otimes (u_2' \circ u_1')$. This follows by the computation

$$\begin{aligned}(u_2 \otimes u_2') \circ (u_1 \otimes u_1') &= u_2\phi_2(u_2')u_1\phi_1(u_1') = u_2u_1\phi_1(u_2')u_1^{-1}u_1\phi_1(u_1') = u_2u_1\phi_1(u_2'u_1') \\ &= (u_2 \circ u_1) \otimes (u_2' \circ u_1').\end{aligned}$$

□

Remark 4.4.3. Let $\mathrm{Group}_2 \subseteq \mathrm{Cat}$ be the 2-category of groups seen as a full subcategory of the 2-category of categories with one object. Then the relative automorphism 2-group of $\rho : G \to GL_n(\mathbb{R})$ can be defined more abstractly as its automorphism 2-group in the slice 2-category $(\mathrm{Group}_2)_{/GL_n(\mathbb{R})}$ of groups over $GL_n(\mathbb{R})$.

Note that given an object (ϕ, A) and an element $u \in G$, there is a canonical morphism $(\phi, A) \to (c_u \circ \phi, \rho(u)A)$, where $c_u : G \to G$ is conjugation with u. Also note that in general it is neither true that ϕ determines A, nor that A determines ϕ. For example, the pair consisting of $\phi = \mathrm{id}_G$ and $A = -\mathrm{id}$ is not the trivial object. On the other hand, G could be of the form $O_n \times K$ with ρ the projection to O_n, $A = \mathrm{id}$ and ϕ induced by a nontrivial automorphism of K.

We now show that the 2-group $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ acts on the groupoid of G-structures on a fixed real vector bundle η, see Appendix A.3 for the definition of an action of a 2-group on a category.

Theorem 4.4.4. *Let* $\eta \to X$ *be a real vector bundle. The construction* $(P, \alpha) \mapsto (P_\phi, \alpha_A)$ *extends to a monoidal functor from* $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ *to the* 2*-group of autoequivalences of the groupoid of* G*-structures* $G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta)$.

Proof. Let $(\phi, A), (\phi', A')$ be two objects of $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$. We clearly have $(P_\phi)_{\phi'} = P_{\phi\phi'}$ and $(\alpha_A)_{A'} = \alpha_{AA'}$, so the construction preserves the tensor product. If $\xi : (P, \alpha) \to (Q, \beta)$ is a map of G-structures on η, then $(\xi)_{A,\phi}(p_\phi) := (\xi(p))_\phi$ defines a map of G-structures and $\xi \mapsto \xi_{\phi,A}$ is functorial. If $(A, \phi) = (1, \mathrm{id}_G)$, then $\xi \mapsto \xi_{\phi,A}$ is the identity functor.

We turn to morphisms $u : (\phi_1, A_1) \to (\phi_2, A_2)$ of $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$. Define the map $\sigma_u(P) : P_{\phi_1} \to P_{\phi_2}$ by $p_{\phi_1} \mapsto (pu^{-1})_{\phi_2}$. This is G-equivariant, because

$$p_{\phi_1} \cdot g = (p\phi_1(g))_{\phi_1} \mapsto (p\phi_1(g)u^{-1})_{\phi_2} = (pu^{-1}\phi_2(g))_{\phi_2} = (pu^{-1})_{\phi_2} \cdot g.$$

It is an isomorphism of G-structures since

$$\alpha_{A_2}[(pu^{-1})_{\phi_2}, v] = \alpha[pu^{-1}, A_2 v] = \alpha[p, \rho(u^{-1})A_2 v] = \alpha[p, A_1 v] = \alpha_{A_1}[p_{\phi_1}, v].$$

This is natural, because

$$\sigma_u(Q)(\xi_{\phi_1}(p_{\phi_1})) = (\xi(p)u^{-1})_{\phi_2} = (\xi(pu^{-1}))_{\phi_2} = \xi_{\phi_2}(\sigma_u(Q)(p_{\phi_1})).$$

Given two composable morphisms u_1, u_2 in $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$, we have $\sigma_{u_2} \circ \sigma_{u_1} = \sigma_{u_2 u_1}$ and σ_u is the identity on $(P, \alpha) \mapsto (P_\phi, \alpha_A)$ if $u = 1$. To get a strict monoidal functor, we still have to check that the trivial monoidal data of the functor is actually natural. For this, we pick morphisms $u : (\phi_1, A_1) \to (\phi_2, A_2), u' : (\phi'_1, A'_1) \to (\phi'_2, A'_2)$ in $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ and compare the tensor product of morphisms $u\phi_1(u')$ with the horizontal composition of the natural transformations σ_u and $\sigma_{u'}$:

$$\begin{tikzcd}
G\text{-}\mathrm{Str}^{\mathrm{geo}}(X,\eta) \arrow[r, bend left, "P\mapsto P_{\phi_1}"] \arrow[r, bend right, "P\mapsto P_{\phi_2}"'] \arrow[r, phantom, "\Downarrow\sigma_u"] & G\text{-}\mathrm{Str}^{\mathrm{geo}}(X,\eta) \arrow[r, bend left, "P\mapsto P_{\phi'_1}"] \arrow[r, bend right, "P\mapsto P_{\phi'_2}"'] \arrow[r, phantom, "\Downarrow\sigma_{u'}"] & G\text{-}\mathrm{Str}^{\mathrm{geo}}(X,\eta)
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
G\text{-}\mathrm{Str}^{\mathrm{geo}}(X,\eta) \arrow[rr, bend left, "P\mapsto P_{\phi_1\phi'_1}"] \arrow[rr, bend right, "P\mapsto P_{\phi_2\phi'_2}"'] \arrow[rr, phantom, "\Downarrow\sigma_{u\phi_1(u')}"] & & G\text{-}\mathrm{Str}^{\mathrm{geo}}(X,\eta)
\end{tikzcd}.$$

This follows from the computation of the map $\sigma_{u\phi_1(u')} : P_{\phi_1\phi'_1} \to P_{\phi_2\phi'_2}$ being

$$\begin{aligned} p_{\phi_1\phi'_1} \mapsto (p(u\phi_1(u'))^{-1})_{\phi_2\phi'_2} &= (p\phi_1(u')^{-1}u^{-1})_{\phi_2\phi'_2} = (pu^{-1}\phi_2(u')^{-1})_{\phi_2\phi'_2} \\ &= ((pu^{-1})_{\phi_2})(u')^{-1})_{\phi'_2}, \end{aligned}$$

which is indeed the horizontal composition of σ_u and $\sigma_{u'}$. Finally, since associators of the monoidal categories and the monoidal data of the functor are trivial, associativity of the monoidal functor is automatic. We have thus defined a monoidal functor from $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ to automorphisms of the groupoid $G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta)$. $\square$

Remark 4.4.5. If G lands in O_n, the action of the 2-group $\mathrm{Aut}(G \to O_n)$ on the 2-group of G-structures is by isometries.

Example 4.4.6. Let $\rho : G \to GL_n(\mathbb{R})$ be a structure group and let $c \in Z(G)$ be a central element in the kernel of ρ. Then c defines an automorphism of the monoidal unit of $\mathrm{Aut}(G \to GL_n(\mathbb{R}))$ and so induces a natural automorphism of the identity functor on $G\text{-}\mathrm{Str}^{\mathrm{geo}}(\eta)$ for any vector bundle $\eta \to X$. In particular, if $G = G_n(K)$ is the spacetime structure group associated to a fermionic group K, then $c = 1 \otimes c = c \otimes 1 \in G$ satisfies this property. We obtain a natural automorphism of the identity functor on $G\text{-}\mathrm{Str}(\eta)$ that squares to one, which we call the spin flip automorphism. In other words, $G\text{-}\mathrm{Str}^{\mathrm{geo}}(\eta)$ has a canonical $B\mathbb{Z}/2$-action, also see Section 2.6. We will use this to define a spin flip $B\mathbb{Z}/2$-action on the bordism category in Chapter 5, see Corollary 5.2.8.

Example 4.4.7. We generalize the discussion at the beginning of the section to orientations on a fixed vector bundle. So let η be a vector bundle and let (P, α) be a $G = SO_n$-structure on η. Let $A = r_i$ be reflection in the ith coordinate and $\phi_i : SO_n \to SO_n$ conjugation with r_i. Then $(P_{\phi_i}, \alpha_{r_i})$ is one way to define the orientation-reversal of an oriented bundle, namely by reflecting in the ith direction. Let $i \neq j$ be two orthogonal directions and let $u = r_i r_j$ be the rotation by 180° in the (i, j)-plane. The proposition above in particular gives a canonical isomorphism between $(P_{\phi_i}, \alpha_{r_i})$ and $(P_{\phi_j}, \alpha_{r_j})$, or equivalently an isomorphism of oriented vector bundles.

Example 4.4.8. We redo the last example for $G = \mathrm{Spin}_n$ and show why we have to be a bit more careful. Let $\phi_i : \mathrm{Spin}_n \to \mathrm{Spin}_n$ be the automorphism induced by conjugation with e_i inside Pin_n^+. Note that we could have also chosen $\mathrm{Spin}_n \subseteq \mathrm{Pin}_n^-$; this would give the same automorphism. We now need to make a choice of an element $u \in \mathrm{Spin}_n$ such that $\rho(u) = r_i r_j$ and there are two possibilities; $u = e_i e_j$ or $u = -e_i e_j = e_j e_i$. In other words, we need to choose a direction to rotate in and rotating by 180° in the other direction will differ by the spin flip $c \in \mathrm{Spin}_n$. These will give two different isomorphisms of G-structures between $(P_{\phi_i}, \alpha_{r_i})$ and $(P_{\phi_j}, \alpha_{r_j})$ and they differ by the spin flip automorphism of Example 4.4.6.

Example 4.4.9. We generalize the last two examples to a general spacetime structure group coming from a fixed fermionic group K as defined in Section 3.2: so let

$$G = G_n(K) = (\mathrm{Pin}_n^+ \otimes K)_{pres} \subseteq \mathrm{Pin}_n^+ \otimes K = \hat{G}_n(K).$$

Conjugation with a reversing element $\hat{g}_1 \in \overline{G}_n(K)$ gives an interesting automorphism $\phi_{\hat{g}_1}$ of G. We can write such an element as $\hat{g}_1 = x_0 \otimes k_0 \in \hat{G}_n(K)$, where x_0 and k_0 have different degree. For example, when $v \in \mathbb{R}^n$ is a vector of norm one, we can take $\hat{g}_1 = v \otimes 1$. Then, the automorphism $\phi_{\hat{g}_1}$ is explicitly given by

$$\phi_v(x \otimes k) = (v \otimes 1)(x \otimes k)(v^{-1} \otimes 1) = (-1)^{|k|} v x v^{-1} \otimes k,$$

which under ρ covers conjugation by the reflection along v in O_n, compare Lemma 3.2.23. If v, w are orthogonal, the element $u := vw \otimes 1 \in G_n$ provides an isomorphism $P_{\phi_v} \cong P_{\phi_w}$. As in the last example, it differs from the isomorphism induced by $u' = wv \otimes 1 \in G_n$ by the spin flip automorphism.

Example 4.4.10. A tiny variation on the above example concerns replacing Pin^+ with Pin^-. We have seen in Corollary 3.2.15 that

$$G_n(K) = (\mathrm{Pin}_n^+ \otimes K)_{pres} \cong (\mathrm{Pin}_n^- \otimes K^{\mathrm{op}})_{pres}, \tag{4.14}$$

but this isomorphism does not lift to an isomorphism on the orientation-graded groups in general. However, for $\hat{G} := (\mathrm{Pin}_n^+)^{\mathrm{op}} \otimes K^{\mathrm{op}}$, taking the lift $\hat{g}_1 = v^{\mathrm{op}} \otimes 1$ does yield the the same automorphism of G as in the last example under the above isomorphism (4.14). This follows by the computation

$$(v^{\mathrm{op}} \otimes 1)(x^{\mathrm{op}} \otimes k^{\mathrm{op}})(-v^{\mathrm{op}} \otimes 1) = -(-1)^{|k|} v^{\mathrm{op}} x^{\mathrm{op}} v^{\mathrm{op}} \otimes k^{\mathrm{op}} = (-1)^{|k|} (vxv)^{\mathrm{op}} \otimes k^{\mathrm{op}},$$

since $(-v^{\mathrm{op}} \otimes 1) = (v^{\mathrm{op}} \otimes 1)^{-1}$ and $(-1)^{|x|}(-1)^{|xv|} = -1$. In particular, this implies that if (P, α) is a $G_n(K)$-structure, the $G_n(K)$-structures $(P_{\phi_v}, \alpha_{r_v})$ obtained from the last example and from this example are equal. However, we will see in the next section, that the corresponding induced $\mathbb{Z}/2$-actions on the groupoid $G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta)$ will in general not be equivalent.

Note that if $\mathcal{C}$ is a 1-category, prestacks form the 2-category $\mathrm{Fun}(\mathcal{C}, \mathrm{Gpd}^{\mathrm{op}})$ and so a prestack F has an automorphism 2-group. Concretely, its objects are natural automorphisms of F and its morphisms are modifications between those. We refer to [54, Appendix B] for a concise summary of bicategories.

Proposition 4.4.11. *Let $G \to O_n$ be a structure group and let $\eta \to X$ be a real vector bundle equipped with a metric. The action of $\mathrm{Aut}(G \to O_n)$ on $G\text{-}\mathrm{Str}^{\mathrm{geo}}(\eta)$ is functorial with respect to maps of spaces, i.e. it induces a 2-group homomorphism λ from $\mathrm{Aut}(G \to O_n)$ to the automorphism 2-group of the prestack*

$$G\text{-}\mathrm{Str}^{\mathrm{geo}} : \mathrm{VectBun}_n \to \mathrm{Gpd}^{\mathrm{op}}.$$

Proof. On objects, the 2-group homomorphism λ maps $(\phi, A) \in \mathrm{Aut}(G \to O_n)$ to a natural 2-transformation $\lambda_{\phi,A} : G\text{-}\mathrm{Str} \Rightarrow G\text{-}\mathrm{Str}$, which on an object (X, η) is the functor

$$F_{\phi,A}(X, \eta) : G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta) \to G\text{-}\mathrm{Str}^{\mathrm{geo}}(X, \eta) \quad (P, \alpha) \mapsto (P_\phi, \alpha_A)$$

defined in Theorem 4.4.4. To make this into a natural 2-transformation, we need to provide for every morphism $f : (X_1, \eta_1) \to (X_2, \eta)$ in $\mathrm{VectBun}_n$, a natural isomorphism filling the square of functors

$$\begin{tikzcd} G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_1, \eta_1) \arrow[r, "F_{\phi,A}(X_1,\eta_1)"] \arrow[dr, Rightarrow] & G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_1, \eta_1) \\ G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_2, \eta_2) \arrow[u, "f^*"] \arrow[r, "F_{\phi,A}(X_2,\eta_2)"'] & G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_2, \eta_2) \arrow[u, "f^*"'] \end{tikzcd}.$$

We take it to be the canonical isomorphism $f^*(P_\phi) \cong f^*(P)_\phi$ of principal G-bundles, which is an isomorphism of G-structures on (X_1, η_1) if (P, α) is a G-structure on (X_2, η_2). This isomorphism is indeed natural with respect to automorphisms of G-structures. If $(X_1, \eta_1) \xrightarrow{f_1} (X_2, \eta_2) \xrightarrow{f_2} (X_3, \eta_3)$ are two morphisms in $\mathrm{VectBun}_n$, the isomorphisms $f_i^*(P_\phi) \cong f_i^*(P)_\phi$ are compatible under composition of f_1 and f_2 using the isomorphisms $(f_2 \circ f_1)^*(P) \cong f_2^* f_1^* P$. This shows $\lambda_{\phi,A}$ is a natural 2-transformation.

Now let $u \in G$ be a morphism from (ϕ_1, A_1) to (ϕ_2, A_2). We want to show that the natural transformation σ_u defines a modification between the natural 2-automorphisms λ_{ϕ_1,A_1} and λ_{ϕ_2,A_2}. This boils down to the fact that the diagram

$$\begin{tikzcd} f^*(P_{\phi_1}) \arrow[r] \arrow[d, "f^*\sigma_u(P)"] & f^*(P)_{\phi_1} \arrow[d, "\sigma_u(f^*P)"] \\ f^*(P_{\phi_2}) \arrow[r] & f^*(P)_{\phi_2} \end{tikzcd} \tag{4.15}$$

commutes for all morphisms $f : (X_1, \eta_1) \to (X_2, \eta_2)$ in $\mathrm{VectBun}_n$ and G-structures (P, α) on (X_2, η_2). Functoriality of λ follows from $\sigma_{u_2} \circ \sigma_{u_1} = \sigma_{u_2 u_1}$.

To show that λ is monoidal, we have to show that the monoidal data from Theorem 4.4.4 defines natural modifications

$$\lambda_{(\phi_1,A_1),(\phi_2,A_2)} : \lambda_{\phi_2,A_2} \circ \lambda_{\phi_1,A_1} \Rrightarrow \lambda_{\phi_2 \circ \phi_1, A_2 \circ A_1}.$$

for all objects $(\phi_1, A_1), (\phi_2, A_2) \in \mathrm{Aut}(G \to O_n)$. This natural modification is given by the identity $((P_{\phi_1})_{\phi_2}, (\alpha_{A_1})_{A_2}) = (P_{\phi_1 \circ \phi_2}, \alpha_{A_1 A_2})$ on a fixed object (X, η) of $\mathrm{VectBun}_n$. To show that $\lambda_{(\phi_1,A_1),(\phi_2,A_2)}$ is indeed a modification, we only need for a morphism $f : (X_1, \eta_1) \to (X_2, \eta_2)$ that the isomorphism $f^*(P_\phi) \cong f^*(P)_\phi$ of G-structures is compatible with the equality $(P_{\phi_1})_{\phi_2} = P_{\phi_1\phi_2}$ which is easy to see.

Next, we have to show that the monoidal data is natural in morphisms $u_1 : (\phi_1, A_1) \to (\phi_1', A_1')$ and $u_2 : (\phi_2, A_2) \to (\phi_2', A_2')$, i.e. that the diagram

$$\begin{tikzcd}
\lambda_{\phi_2,A_2} \circ \lambda_{\phi_1,A_1} \arrow[r, "\lambda_{(\phi_1,A_1),(\phi_2,A_2)}"] \arrow[d, "\sigma_{u_2} * \sigma_{u_1}"] & \lambda_{\phi_2\circ\phi_1, A_2A_1} \arrow[d, "\sigma_{u_2u_1}"] \\
\lambda_{\phi_2',A_2'} \circ \lambda_{\phi_1',A_1'} \arrow[r, "\lambda_{(\phi_1',A_1'),(\phi_2',A_2')}"] & \lambda_{\phi_2'\circ\phi_1', A_2'A_1'}
\end{tikzcd}$$

commutes. This follows from the naturality of the tensor product, as discussed in Theorem 4.4.4. Finally, we have to show that the monoidal data is associative, which again was discussed in Theorem 4.4.4. $\square$

Remark 4.4.12. The action of $\mathrm{Aut}(G \to O_n)$ on $G\text{-Str}^{\mathrm{geo}}$ induces an action of $\mathrm{Aut}(G \to O_n)$ on homotopy G-structures $G\text{-Str}^h$. This can be shown using naturality with respect to inclusions

$$X \times \{0\} \hookrightarrow X \times [0,1] \hookleftarrow X \times \{1\}.$$

More generally, these considerations generalize from intervals to arbitrary simplices. Therefore, there is also an action on the simplicial set $G\text{-Str}^{top}(X)$ of G-structures. This could be useful for understanding orientation-reversal on $(\infty,1)$-bordism categories. In particular, it is most likely relevant to define unitarity for topological field theories that are 'extended upwards', i.e. when the target is an $(\infty, 1)$-category such as the category of chain complexes of complex vector spaces. However, we will not prove any of these statements. The main reason is that homotopies of G-structures will not suffice; we will also be interested in (homotopical) G-diffeomorphisms in Chapter 5, which are more general.

4.5 Orientation-graded G-structures

In this section, we define the most general notion of orientation-reversal considered in this thesis. To ensure we have a metric, we will work with G-structures that land in O_n, but this assumption is not strictly necessary in this section. The starting point is a choice of $\mathbb{Z}/2$-graded group $\hat{G}$ of which the even part is G together with an extension of ρ:

Definition 4.5.1. An *orientation-graded structure group* consists of a structure group $G \to O_n$ together with a $\mathbb{Z}/2$-extension $\hat{G}$ of G and a group homomorphism $\hat{\rho} : \hat{G} \to O_n$ compatible with ρ:

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] \arrow[d, "\rho"] & \hat{G} \arrow[r] \arrow[dl, "\hat{\rho}"] & \mathbb{Z}/2 \arrow[r] & 1 \\
& O_n & & &
\end{tikzcd} \tag{4.16}$$

with the triangle commuting. We will call $\hat{G} \to \mathbb{Z}/2$ the orientation $\mathbb{Z}/2$-grading of $\hat{G}$ and write the reversing part in this grading $\overline{G} := \hat{G} \setminus G$.

The main example to keep in mind are the spacetime structure groups $\hat{G}(K)$ introduced in Section 3.2, such as $G = SO_n$ and $\hat{G} = O_n$ or $\hat{G} = \mathrm{Pin}_n^+$ and $G = \mathrm{Spin}_n$. In our notation for $\mathbb{Z}/2$-subgroups as explained in Remark 3.1.10, we should think of the orientation-grading as valued in O_1, not in Spin_1. For the rest of this section we fix an orientation-graded structure group $G \to \hat{G} \xrightarrow{\hat{\rho}} O_n$. The following is analogous to [20, Definition 4.1]

Definition 4.5.2. Let (P, α) be a G-structure on $\eta \to X$. The associated *orientation-graded* $\hat{G}$-structure is the principal bundle $\hat{P} := P \times_G \hat{G}$ together with the isometry $\hat{\alpha} : \hat{P} \times_G \hat{G} \cong \eta$ given by

$$\hat{\alpha}[[p, \hat{g}], v] := \alpha[p, \hat{\rho}(\hat{g})v].$$

The property that $\hat{\rho}$ and ρ are compatible implies that $\hat{\alpha}$ is well-defined. Note that since $\overline{G}$ is a G-torsor, $\hat{P}$ is the disjoint union of two principal G-bundles $\hat{P} = P \sqcup \overline{P}$, where $\overline{P} = P \times_G \overline{G}$ and we identified $P \times_G G \cong P$.

Definition 4.5.3. The *bar* of the G-structure (P, α) is $(\overline{P}, \overline{\alpha})$, where $\overline{\alpha}$ is the restriction of $\hat{\alpha}$ to $\overline{P} \times_G \mathbb{R}^n$.

Lemma 4.5.4. *Given a fixed real vector bundle η with metric, $\overline{(.)}$ induces a $\mathbb{Z}/2$-action on the groupoid of G-structures on η.*

Proof. For $\phi : (P, \alpha) \to (Q, \beta)$ a map of G-structures, define $\overline{\phi} : \overline{(P, \alpha)} \to \overline{(Q, \beta)}$ as

$$\overline{\phi}[p, \hat{g}] = [\phi(p), \hat{g}]$$

for $\hat{g} \in \overline{G}$ and $p \in P$. This is well-defined since

$$[\phi(pg^{-1}), g\hat{g}] = [\phi(p), \hat{g}]$$

for every $g \in G$. It is a principal bundle map since

$$\overline{\phi}([p, \hat{g}] \cdot g) = [\phi(p), \hat{g}g] = [\phi(p), \hat{g}] \cdot g.$$

It is a map of G-structures since

$$\overline{\beta}([[\phi(p), \hat{g}], v]) = \beta([\phi(p), \hat{\rho}(\hat{g})v]) = \alpha([p, \hat{\rho}(\hat{g})v]) = \overline{\alpha}([[\phi(p), \hat{g}], v]).$$

Composition is clearly functorial, so we have defined $\overline{(.)}$ as a functor.

Define a map $\mu_P : \overline{\overline{P}} \to P$ by

$$[[p, \hat{g}_1], \hat{g}_2] \mapsto p \cdot (\hat{g}_1\hat{g}_2)$$

The map is well-defined and G-equivariant because $\hat{g}_1\hat{g}_2 \in G$. This is a map of G-structures by the computation

$$\overline{\overline{\alpha}}[[[p, \hat{g}_1], \hat{g}_2], v] = \alpha[p, \hat{\rho}(\hat{g}_1)\hat{\rho}(\hat{g}_2)v] = \alpha[p, \hat{\rho}(\hat{g}_1\hat{g}_2)v] = \alpha[p \cdot (\hat{g}_1\hat{g}_2), v],$$

so it is a geometric isomorphism of G-structures. It is a natural transformation because if $\phi : (P, \alpha) \to (Q, \beta)$ is a map of G-structures, then

$$\overline{\overline{\phi}}[[p, \hat{g}_1], \hat{g}_2] = [[\phi(p), \hat{g}_1], \hat{g}_2] \mapsto \phi(p) \cdot (\hat{g}_1\hat{g}_2) = \phi(p \cdot (\hat{g}_1\hat{g}_2)).$$

Finally note that $\overline{\mu_P} = \mu_{\overline{P}}$ by the computation

$$\begin{aligned}\overline{\mu_P}[[[p,\hat{g}_1],\hat{g}_2],\hat{g}_3] &= [\mu_P([[p,\hat{g}_1],\hat{g}_2]),\hat{g}_3] = [p\cdot(\hat{g}_1\hat{g}_2),\hat{g}_3] = [p,(\hat{g}_1\hat{g}_2)\hat{g}_3] = [p,\hat{g}_1](\hat{g}_2\hat{g}_3)\\ &= \mu_{\overline{P}}[[[p,\hat{g}_1],\hat{g}_2],\hat{g}_3].\end{aligned}$$

□

Remark 4.5.5. From a more homotopical perspective, the orientation grading on $\hat{G}$ gives a fibration

$$BG \to B\hat{G} \to B\mathbb{Z}/2$$

or in other words a homotopy $\mathbb{Z}/2$-action $\mathbb{Z}/2 \to \mathrm{Aut}_h(BG)$. This induces a homotopy $\mathbb{Z}/2$-action on principal G-bundles. Moreover, the choice of $\hat{\rho}$ trivializes the pushforward homotopy $\mathbb{Z}/2$-action on O_n:

$$\begin{tikzcd} BG \arrow[r] \arrow[d, "B\rho"] & B\hat{G} \arrow[r, "Bp"] \arrow[d, "(B\hat{\rho}, Bp)"] & B\mathbb{Z}/2 \arrow[d, equal] \\ BO_n \arrow[r] & BO_n \times B\mathbb{Z}/2 \arrow[r] & B\mathbb{Z}/2 \end{tikzcd} \tag{4.17}$$

This gives a homotopy $\mathbb{Z}/2$-action on homotopy G-structures. For the orientation-grading on the spacetime structure group corresponding to an internal fermionic symmetry group, the homotopy $\mathbb{Z}/2$-action on $BG_n(K)$ defined by $\hat{G}_n(K)$ corresponds to the fibration

$$BG_d(K) \to BK_b \times BO_d \xrightarrow{(w_1(K)+w_1, w_2(K)+w_2+w_1(K)w_1)} B\mathbb{Z}/2 \times B^2\mathbb{Z}/2,$$

where $w_i = w_i(\mathrm{Pin}_d^+) \in H^i(BO_d;\mathbb{Z}/2)$ are the usual Stiefel-Whitney classes. For $\hat{G}_n^-(K^{\mathrm{op}})$ we would instead get the fibration

$$BG_d^-(K) \to BK_b \times BO_d \xrightarrow{(w_1(K)+w_1, w_2(K)+w_1(K)^2+w_2+w_1^2+w_1(K)w_1)} B\mathbb{Z}/2 \times B^2\mathbb{Z}/2.$$

Since we are taking the fiber of the map $w_1(K)+w_1$, the two maps $w_1(K)^2$ and w_1^2 are homotopic, so that $BG_d(K) \cong BG_d^-(K)$ compatibly with the maps to $BK_b \times BO_d$ in agreement with Corollary 3.2.15. However, the two fibrations are not isomorphic in general. Indeed, assume a homotopy equivalence $BK_b \times BO_d \cong BK_b \times BO_d$ would be compatible with the respective maps to $B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$. Taking the homotopy fibers of the maps to $B^2\mathbb{Z}/2$ we obtain the classifying space of the group classifying the corresponding $\mathbb{Z}/2$-extensions. These two groups are $\hat{G}_d(K) = K \otimes \mathrm{Pin}_d^+$ and $\hat{G}_d^-(K) = K \otimes \mathrm{Pin}_d^-$ and we have seen that these are not isomorphic in general, see Example 3.2.8. Therefore the two $\mathbb{Z}/2 \times B\mathbb{Z}/2$-actions can be different. We will soon see in a less homotopy-theoretic way how they differ.

To get a more explicit algebraic understanding of $\overline{(.)}$ and connect to previous sections, let us translate the short exact sequence to a cohomological description. In other words, we would like to pick a nonabelian 2-cocycle representing a cohomology class in some nonabelian topological group cohomology $H^2(\mathbb{Z}/2;G)$ classifying the extension $\hat{G}$ of G. We will not specify what type of nonabelian cohomology we mean here but instead spell out the result explicitly. Fix a non-homomorphic splitting of the short exact sequence (4.16), i.e. an arbitrary reversing element $\hat{g}_1 \in \overline{G}$. Let $\phi_{\hat{g}_1} : G \to G$ denote the induced automorphism $\phi_{\hat{g}_1}(g) = \hat{g}_1 g \hat{g}_1^{-1}$. This automorphism could be inner, but it need not be. For example, when $G = SO_n$ and $\hat{G} = O_n$, we can take $\hat{g}_1$ to be reflection in the nth coordinate and conjugation with a reflection is not inner in SO_n. However, by construction $\phi_{\hat{g}_1}$

covers an inner automorphism of O_n given by conjugation with $\hat\rho(\hat g_1)$. In particular $(\phi, \hat\rho(g_1))$ is an object of $\operatorname{Aut}(G \to O_n)$. Also $(\phi \circ \phi)(g) = \hat g_1^2 g \hat g_1^{-2}$ is canonically inner and so we obtain an associated morphism $(1,1) \to (\phi^2, \rho(\hat g_1^2))$ in $\operatorname{Aut}(G \to O_n)$. Finally note that $\phi(\hat g_1^2) = \hat g_1^2$.

Example 4.5.6. [20, Remark 4.2] We consider the case where the extension is split. So let $G \to O_n$ be a structure group and $\phi \in \operatorname{Aut} G$ an involution covering conjugation with $A \in O_n$. Define $\hat G = G \rtimes_\phi \mathbb{Z}/2$. We can define $\hat\rho$ by ρ on the first factor and by A on the second factor. This is a group homomorphism by the assumption that ϕ covers conjugation with A. A canonical splitting is given by $\hat g_1 = (1, -1) \in \overline{G}$ and we have $\phi_{\hat g_1} = \phi$.

Example 4.5.7. Suppose we are given a strict geometric representation $\rho : G_{n+1} \to O_{n+1}$. Recall that

$$G_n^{(1)} := \{g \in G_{n+1} : \rho_{n+1}(g) \in O_n \text{ or } \rho_{n+1}(g) \in r_{n+1} O_n\}$$

and $\rho^{(1)} : G_n^{(1)} \to O_n$ was defined to be $\rho_{n+1}(g)|_{\mathbb{R}^n} \in O_n$. We obtain an orientation-graded group

$$1 \to G_n \to G_n^{(1)} \to \mathbb{Z}/2 \to 1.$$

Given a G_n-structure (P, α), its associated bar is by definition what we obtained in Lemma 4.3.11:

$$P' = \{[p, g] \in sP : \rho(g) = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \in O_n\}$$

with

$$\alpha'[[p, g], v] = \alpha[p, \rho(g)|_{\mathbb{R}^n}(v)].$$

Example 4.5.8. We restrict the last example to the case where $G_{n+1} = G_{n+1}(K)$ and we recall that $\hat G_n^-(K^{\mathrm{op}}) \cong G_n^{(1)}(K)$ by Lemma 3.2.26. Consider the splitting $\tilde g_1 = v e_{n+1} \otimes 1 \in G_n^{(1)}(K)$, where $v \in \mathbb{R}^n$ is a vector of norm one. Note that $\tilde g_1$ is indeed reversing even though it is a preserving element in G_{n+1}. Also note that $\tilde g_1^2 = -1$, so this choice does not split the extension homomorphically. The automorphism of G_n induced by the splitting is

$$x \otimes k \mapsto v e_{n+1} x e_{n+1} v \otimes k = (-1)^{|x|} v x v \otimes k.$$

This is the same automorphism as the automorphism of G_n induced by the splitting $\hat g_1 = 1 \otimes v$ of the extension $\hat G_n$ as explained in Example 4.4.9.

Proposition 4.5.9. *Let $\eta \to X$ be a real vector bundle. Fix a splitting $\hat g_1 \in \hat G$ of the short exact sequence* (4.16)*, which does not necessarily square to one. Let $\phi(g) = \hat g_1 g \hat g_1^{-1}$ be the induced automorphism of G and set $A := \hat\rho(\hat g_1)$ so that (ϕ, A) is an object of* $\operatorname{Aut}(G \to O_n)$*. Then $P \mapsto P_\phi$ together with the isomorphism $\sigma_{\hat g_1^2} : (P_\phi)_\phi = P_{\phi^2} \cong P$ induced by the morphism $\hat g_1^2 : (1,1) \to (\phi^2, A^2)$ in* $\operatorname{Aut}(G \to O_n)$ *defines a $\mathbb{Z}/2$-action. There is a natural isomorphism*

$$T_P : (P_\phi, \alpha_A) \cong (\overline{P}, \overline{\alpha})$$

given by

$$p_\phi \mapsto [p, \hat g_1]$$

which is an isomorphism of $\mathbb{Z}/2$-actions.

Proof. We already know $P \mapsto P_\phi$ is a functor and $\sigma_{\hat{g}_1^2}$ a natural transformation. The two morphisms $P_{\phi^3} \to P_\phi$ are given by $p_{\phi^3} \mapsto (p\hat{g}_1^{-2})_\phi$ and $p_{\phi^3} \mapsto (p)_\phi \hat{g}_1^{-2}$, which are equal because $\phi(\hat{g}_1^{-2}) = \hat{g}_1^{-2}$. So this defines a $\mathbb{Z}/2$-action.

For the equivalence of $\mathbb{Z}/2$-actions, note that the proposed map is G-equivariant:

$$p_\phi \cdot g = (p\phi(g))_\phi \mapsto [p\phi(g), \hat{g}_1] = [p, \phi(g)\hat{g}_1] = [p\hat{g}_1, g\hat{g}_1^{-1}\hat{g}_1] = [p, \hat{g}_1]g.$$

It is a map of G-structures:

$$\overline{\alpha}[[p, \hat{g}_1], v] = \alpha[p, \hat{\rho}(\hat{g}_1)v] = \alpha_A[p, v].$$

For naturality, let $\xi : (P, \alpha) \to (Q, \beta)$ be a map of G-structures. Then

$$\begin{tikzcd}
P_\phi \arrow[r] \arrow[d, "\xi_\phi"] & \overline{P} \arrow[d, "\overline{\xi}"] \\
Q_\phi \arrow[r] & \overline{Q}
\end{tikzcd}$$

commutes since both ways to go through the diagram map p_ϕ to the element $[\xi(p), \hat{g}_1] \in \overline{Q}$. To check that this is an isomorphism of $\mathbb{Z}/2$-actions we have to show commutativity of the diagram

$$\begin{tikzcd}
P_{\phi^2} \arrow[r, "T_{P_\phi}"] \arrow[d, "\sigma_{\hat{g}_1^2}"] & \overline{P_\phi} \arrow[r, "T_{\overline{P}}"] & \overline{\overline{P}} \arrow[d, "\mu_P"] \\
P \arrow[rr, equal] & & P
\end{tikzcd}.$$

Moving in the diagram through the upper right corner gives

$$p_{\phi^2} \mapsto [p_\phi, \hat{g}_1] \mapsto [[p, \hat{g}_1], \hat{g}_1] \mapsto p \cdot (\hat{g}_1)^2$$

which is indeed $\sigma_{\hat{g}_1^2}$. □

Proposition 4.5.10. *Let K be a fermionic group and (P, α) a $G_n(K)$-structure for $n > 0$. Let*

$$\overline{(P,\alpha)}^{\hat{G}}, \overline{(P,\alpha)}^{\hat{G}'}, \overline{(P,\alpha)}^{G^{(1)}}, \overline{(P,\alpha)}^{G^{(3)}}$$

denote the different notions of orientation reversal corresponding to the different extensions by $G_n(K)$ we defined. They are all isomorphic as $G_n(K)$-structures. The $\mathbb{Z}/2$-actions induced by $\hat{G}$ and $G^{(3)}$ are isomorphic and those induced by $\hat{G}^-$ and $G^{(1)}$ are isomorphic. However, the $\mathbb{Z}/2$-actions induced by $\hat{G}$ and $\hat{G}^-$ have their isomorphisms $\overline{\overline{P}} \cong P$ different by a spin flip and so are potentially nonequivalent when $c \neq 1$.

Proof. It follows by Lemma 3.2.26 that $\hat{G}_n(K) \cong G_n^{(3)}(K)$ and $\hat{G}_n^-(K) \cong G_n^{(1)}(K)$ compatibly with the maps to O_n. Therefore the notions of orientation reversal they induce for $G_n(K)$-structures is identical and so are the induced $\mathbb{Z}/2$-actions.

To show that all four groups $\overline{(P, \alpha)}$ are naturally isomorphic, note that all four extensions admit a splitting for which

$$\phi(x \otimes k) = (-1)^{|x|} e_n x e_n \otimes k$$

is the same involution on $G_n(K)$ lifting conjugation by r_n in O_n. Therefore by the previous proposition, all these notions of orientation reversal are naturally isomorphic to $(P, \alpha) \mapsto (P_\phi, \alpha_{r_n})$.

To show that these are not isomorphic as $\mathbb{Z}/2$-actions, note that the isomorphism $\overline{\overline{P}} \cong P$ is induced by $\hat{g}_1^2$ for a given splitting $\hat{g}_1$. For $\hat{G}_n(K)$ the splitting $e_n \otimes 1$ squares to 1 in $G_n(K)$, but it squares to c in $\hat{G}_n^-(K)$ and so the actions induced by $\hat{G}_n(K)$ and $\hat{G}_n^-(K)$ differ by c on the isomorphism $\overline{\overline{P}} \cong P$. □

Corollary 4.5.11. *Let K be a fermionic group and $\hat{G}(K)$ the usual orientation-graded structure group induced by K. Then there is a $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action on the prestack G-$\mathrm{Str}^{\mathrm{geo}}$ given by orientation-reversal and the spin flip.*

Proof. We have seen that given a vector bundle $\eta \to X$, the orientation-grading induces a canonical $\mathbb{Z}/2$-action on G-$\mathrm{Str}^{\mathrm{geo}}(\eta)$. This $\mathbb{Z}/2$-action is isomorphic to the $\mathbb{Z}/2$-action $P \mapsto P_\phi$ induced by a choice of splitting $\hat{g}_1 \in \hat{G}$ such as $\phi(x \otimes k) = (-1)^{|k|} vxv \otimes k$ for a fixed $v \in \mathbb{R}^n$. We have also seen that the latter gives a $\mathbb{Z}/2$-action on the prestack G-$\mathrm{Str}^{\mathrm{geo}}$ in Proposition 4.4.11. The central element c induces a $B\mathbb{Z}/2$-action on G-$\mathrm{Str}^{\mathrm{geo}}$ as explained in Example 4.4.6. A category equipped with both a $\mathbb{Z}/2$-action $\overline{\cdot}$ and a $B\mathbb{Z}/2$-action $x \mapsto x_c \in \operatorname{Aut} x$ extends to a $\mathbb{Z}/2 \times B\mathbb{Z}/2$-action if $x_{\overline{c}} = \overline{x_c}$. This follows in our case because c is fixed by ϕ. □

Functoriality of the $\mathbb{Z}/2$-action induced by an orientation-graded group $\hat{G}$ with respect to vector bundle maps now follows by combining Proposition 4.5.9 with Proposition 4.4.11. We spell out explicitly what this means. Let $f : X_1 \to X_2$ be covered by a vector bundle isometry $\mu : \eta_1 \to \eta_2$. Then the pullback functor

$$G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_2, \eta_2) \to G\text{-}\mathrm{Str}^{\mathrm{geo}}(X_1, \eta_1)$$

comes equipped with canonical $\mathbb{Z}/2$-equivariance data as follows. If (P, α) is a G-structure on (X_2, η_2) there is a canonical principal bundle isomorphism

$$\overline{f^*P} = f^*P \times_G \overline{G} \cong f^*(P \times_G \overline{G}) \cong f^*\overline{P}.$$

This is a map of G-structures because the triangle

$$\begin{tikzcd}
(f^*P) \times_G \overline{G} \times_G \mathbb{R}^n \arrow[r] \arrow[d] & f^*(P \times_G \overline{G}) \times_G \mathbb{R}^n \arrow[dl] \\
f^*P \times_G \mathbb{R}^n &
\end{tikzcd}$$

induced by $\hat{\rho}$ commutes. Note that

$$\begin{tikzcd}
f^*(P \times_G \overline{G} \times_G \overline{G}) \arrow[d] & f^*(P \times_G \overline{G}) \times_G \overline{G} \arrow[l] \\
f^*P & f^*P \times_G \overline{G} \times_G \overline{G} \arrow[l] \arrow[u]
\end{tikzcd}$$

commutes. If $\sigma : (P_1, \alpha_1) \to (P_2, \alpha_2)$ is an isomorphism of G-structures, then

$$\begin{tikzcd}
\overline{f^*P_1} \arrow[r] \arrow[d, "\overline{f^*\sigma}"] & f^*\overline{P}_1 \arrow[d, "f^*\overline{\sigma}"] \\
\overline{f^*P_2} \arrow[r] & f^*\overline{P_2}
\end{tikzcd}$$

commutes.

4.6 Hermitian pairings on G-structures

In the previous section we have defined a $\mathbb{Z}/2$-action on the G-structures on a vector bundle coming from an orientation-graded group

$$G \to \hat{G} \xrightarrow{\hat{\rho}} O_n.$$

In Chapter 5 we will extend this $\mathbb{Z}/2$-action to a $\mathbb{Z}/2$-action on the category of n-dimensional bordisms with G-structure and consider the induced anti-involution in the sense of Section 2.2. After establishing this, we will be interested in understanding Hermitian pairings on $(n-1)$-dimensional closed manifolds with G-structure with respect to this anti-involution. The goal of this section is to establish the existence of certain isomorphisms between G-structures on vector bundles that will induce such Hermitian pairings in Section 5.2.

The general approach in this section is as follows. We fix a dimension $n \geq 0$ and a strict geometric representation $G_{n+1} \to O_{n+1}$. We take the orientation grading on the pullback G_n to be $\hat{G} = G_n^{(1)}$ so that the induced $\mathbb{Z}/2$-action on G_n-structures is $P \mapsto P'$ as in Definition 4.3.10. Our goal is to construct certain isomorphisms of G_n-structures $h : (P, \alpha)'^* \cong (P, \alpha)$ on vector bundles of the form $(X, \eta \oplus \underline{\mathbb{R}})$, where the star is as in Definition 4.3.2. We warn the reader that even though $P \mapsto P'$ is defined in terms of the star operation too, this is a reflection in one dimension higher so that one can think of P'^* as being 'reflected in the top two dimensions n and $n+1$'. Our construction of h in Proposition 4.6.2 unfortunately requires (P, α) to be a stabilization of a G_{n-1}-structure, where $G_{n-1} \to O_{n-1}$ is the further pullback. To assume this without loss of generality we will need to assume that the pullback $G_n \to O_n$ of $G_{n+1} \to O_{n+1}$ is again a strict geometric representation. Namely, in that case the distinction between G_{n-1}-structures on η and G_n-structures on $\eta \oplus \underline{\mathbb{R}}$ does not matter.

Another problem we will face is that the isomorphisms h will turn out not to be a Hermitian pairing on Y for this $\mathbb{Z}/2$-action $Y \mapsto Y'$ in general, see Lemma 4.6.14. For structure groups coming from a fermionic group however, the hermiticity equation relating h and dh will still hold up to a spin flip as we show in Corollary 4.6.15. As a consequence, h_Y will define a Hermitian pairing if we decide to modify the isomorphism $Y'' \cong Y$ by a spin flip by modifying the $\mathbb{Z}/2$-action. By Proposition 4.5.10, the most important consequence will be a definition of Hermitian pairings for the case where the orientation-grading on $G_n(K)$ is given by $\hat{G} = \hat{G}_n(K)$. However, when K is a fermionic group, we have to be a bit careful when $n = 1$ when $G_n(K)$ is not a strict geometric representation and so not every $G_n(K)$-structure is a suspension. We provide an ad hoc fix for this problem in Lemma 4.6.11. This results in in Corollary 4.6.16, which provides Hermitian pairings for the $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$ that work in all dimensions. It is not clear to us how the analogous construction of Hermitian pairings in [20, proposition 4.8] circumvents this issue in dimension $n = 1$.

We finish this section by comparing the Hermitian pairing on Y^* with the Hermitian pairing on Y. This will be a prerequisite for showing that the bordism category we will construct in Section 5.2 is weak fermionically †-compact. This in turn is an essential ingredient in the proof of the spin-statistics theorem in Section 6.3.

We switch to the precise definitions of Hermitian pairings on vector bundles with G-structures. Let $\hat{G}$ be an orientation-graded group and let $\overline{(.)}$ denote the induced orientation-reversal $\mathbb{Z}/2$-action on the prestack of geometric G-structures. Note that the identity map induces an isomorphism $\overline{P}^* \cong \overline{P^*}$ of G-structures because

$$(\overline{\alpha})^*[[p, \hat{g}], v] = r_n \alpha[p, \hat{\rho}(\hat{g})v] = \overline{\alpha^*}[[p, \hat{g}], v].$$

This isomorphism is also clearly natural. We will later show that the induced isomorphism in the bordism category is the one induced by the monoidality of the functor $P \mapsto \overline{P}$. We now define the notion of Hermitian pairing on G-structures, which is very similar to the notion we gave in Definition 2.3.4. The main difference is an inverse on the starred h coming from the fact that $P \mapsto P^*$ is a covariant involution on G-structures. This will be reconciled when pushing the Hermitian pairing to the bordism category.

Definition 4.6.1. A *Hermitian pairing* on a G-structure (P, α) is an isomorphism $h : P \to \overline{P}^*$ such that the composition

$$P \xrightarrow{h} \overline{P}^* \xrightarrow{\overline{h^*}} \overline{(\overline{P}^*)}^* \cong \overline{\overline{P}}^{**} \cong \overline{\overline{P}} \cong P \tag{4.18}$$

is the identity.

Note that a map $h : P_1 \to \overline{P_2}$ of G-structures uniquely extends to a map of G-structures $\hat{h} : \hat{P}_1 \to \hat{P}_2$ via $\hat{h}[p, \hat{g}] = h(p)\hat{g}$. For a Hermitian pairing $h : P \to \overline{P}^*$ we take $P_1 = P$ and $P_2 = P^*$, which is equal to P as a principal G-bundle. In this description, the condition that h has to satisfy simply reduces to $\hat{h} \circ \hat{h} = \mathrm{id}_{\hat{P}}$. Conversely any map $\hat{h} : \hat{P} \to \hat{P}$ satisfying $\hat{h}^2 = \mathrm{id}_{\hat{P}}$ that is 'odd in the orientation grading of $\hat{P}$' defines a Hermitian pairing when restricted to P.

For now we work with general structure groups and start with an approach assuming the existence of sufficiently many suspensions to compare P and P'^*. The idea will be to pick a lift of a rotation in G to rotate between P and P'^*. It will be an application of the following general result.

Proposition 4.6.2. *Let $G_{n+k} \to O_{n+k}$ be a structure group for some $k \geq 1, n \geq 0$ and $G_{n+i} \to O_{n+i}(\mathbb{R})$ for $0 \leq i < k$ the structure groups obtained by strict pullback. Fix an element $u \in G_{n+k}$ covering the matrix $\mathrm{id}_{\mathbb{R}^n} \oplus A \in O_{n+k}(\mathbb{R})$ for some $A \in O_{n+k}$. Let (Q, β) be a G_n-structure on the vector bundle $\eta \to X$ and let (P, α) denote its k-fold suspension. Let (P, α_A) be the G_{n+k}-structure on $\eta \oplus \underline{\mathbb{R}}^k$ obtained by composing α with $\mathrm{id}_\eta \oplus A$. If $u \in G_{n+k}$ centralizes G_n:*

$$ug' = g'u \quad \forall g' \in G_n$$

then the G_{n+k}-equivariant extension of the multiplication by u map

$$\tilde{h}_u[q, 1] = [q, u] \in s^k Q$$

is an isomorphism of G_{n+k}-structures $(P, \alpha_A) \cong (P, \alpha)$. Moreover, this construction is functorial in G_n-structure maps on (X, η, Q).

Proof. The G_{n+k}-equivariant extension of $\tilde{h}_u$ is given by

$$\tilde{h}_u[q, g] = [q, ug] \quad g \in G_{n+k}.$$

This is well-defined because if $g' \in G_n$, then using the centralizing property

$$\tilde{h}_u[p, g'g] = [p, ug'g] = [pg', ug] = \tilde{h}_u[pg', g].$$

To show that it is a map of G_{n+k}-structures, it is convenient to separate the first n from the last k coordinates

$$\rho(g) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

If $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^k$ then $\rho(ug) = \rho(u)\rho(g) = (\mathrm{id}_{\mathbb{R}^n} \oplus A)\rho(g)$ implies

$$\rho(ug) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} B_{11}v_1 + B_{21}v_2 \\ A(B_{12}v_1 + B_{22}v_2) \end{pmatrix}.$$

Using such linear algebra we show $\tilde{h}_u$ is a map of G_{n+k}-structures:

$$\begin{aligned} s^k\alpha[[q, ug], v] &= (\alpha[q, \mathrm{pr}_1((\mathrm{id}_{\mathbb{R}^n} \oplus A)\rho(g)v)], \mathrm{pr}_2((\mathrm{id}_{\mathbb{R}^n} \oplus A)\rho(g)v)) \\ &= (\alpha[q, B_{11}v_1 + B_{21}v_2], A(B_{12}v_1 + B_{22})v_2) \\ &= (\mathrm{id}_\eta \oplus A)(s^k\alpha[[q, g], v]), \end{aligned}$$

where $\mathrm{pr}_1 : \mathbb{R}^{n+k} \to \mathbb{R}^n$ and $\mathrm{pr}_2 : \mathbb{R}^{n+k} \to \mathbb{R}^k$ are orthogonal projections. To show functoriality, first note that the construction $(P, \alpha) \mapsto (P, \alpha_A)$ is functorial. Now let $f : (X_1, \eta_1, Q_1) \to (X_2, \eta_2, Q_2)$ be an isomorphism of G-structures potentially covering a nontrivial morphism $(X_1, \eta_1) \to (X_2, \eta_2)$. It follows by direct inspection that the diagram of principal bundles

$$\begin{tikzcd} s^kQ_1 \arrow[r, "f"] \arrow[d, "\tilde{h}_u"] & s^kQ_2 \arrow[d, "\tilde{h}_u"] \\ s^kQ_1 \arrow[r, "f"] & s^kQ_2 \end{tikzcd}$$

commutes. □

Remark 4.6.3. Note that the construction $(P, \alpha) \mapsto (P, \alpha_A)$ used in the above proposition is not the same as the construction considered in Section 4.4. Namely, we need the vector bundle to be partially trivialized and we do not twist the principal bundle by an automorphism of G.

Remark 4.6.4. Proposition 4.6.2 still holds true in case when ρ_{n+k} only lands in $GL_{n+k}(\mathbb{R})$. Note also that we assumed $A \in O_{n+k}$ in the proof to make sure α_A is an isometry.

Remark 4.6.5. Note that in general $\tilde{h}_u[q, g] = [q, ug] \neq [q, gu]$. Intuitively one can think of h_u as given by 'multiplication with u on the left'. Namely, right multiplication with a noncentral element u of G does not provide an automorphism of a principal G-bundle. However, left multiplication with u on a trivial G-bundle does provide an automorphism. Since in the above proposition the G-bundle is 'trivial in the last k directions' and u 'commutes with the other n directions', h_u is well-defined.

Example 4.6.6. Let K be a fermionic group, $k \geq 2$ and pick $u = vw \otimes 1 \in G_{n+k}(K)$, where $v, w \in \mathbb{R}^k \subseteq \mathbb{R}^{n+k}$ are orthonormal. Since $|vw| = 0$, u centralizes $G_n(K)$. If (Q, α) is a $G_n(K)$-structure on $\eta \oplus \underline{\mathbb{R}}^k$, then $\tilde{h}_u[q, g] = [q, (vw \otimes 1)g]$ is an isomorphism $(s^kQ, s^k\alpha) \cong (s^kQ, (s^k\alpha)_A)$

Consider a pullback arising from a strict geometric representation

$$\begin{tikzcd} G_n \arrow[r, "\rho_n"] \arrow[d] & O_n \arrow[d] \\ G_{n+1} \arrow[r, "\rho_{n+1}"] & O_{n+1} \end{tikzcd} \tag{4.19}$$

for $n \geq 1$. We also consider the further pullback to G_{n-1} but will not assume $G_n \to O_n$ is a strict geometric representation. Given a G_n-structure (P, α) on $\eta \oplus \underline{\mathbb{R}}$ we can talk about $(P, \alpha)^* = (P, \alpha_{r_n})$ and about $(P, \alpha)' = (s^{-1}(sP)^*, s^{-1}(s\alpha)^*) = (s^{-1}(sP)^*, s^{-1}(s\alpha)_{r_{n+1}})$. We emphasize again that we are now using the star in the 'two directions' n and $n + 1$.

Corollary 4.6.7. *Let $u \in G_{n+1}$ be a lift of $A = r_nr_{n+1} \in O_n$ that centralizes G_{n-1} and let (Q, β) be a G_{n-1}-structure on η. Then the desususpension of $\tilde{h}_u$ gives an isomorphism $h_u : (sQ, s\beta) \cong (sQ, s\beta)'^*$.*

Proof. We only have to show that $s^{-1}(s^2Q,(s^2\beta)_A)$ gives us $(sQ,s\beta)'^*$. Then we can define $h_u := s^{-1}(\tilde{h}_u)^{-1}$ which finishes the proof. For this note that by a similar computation as in the proof of Lemma 4.3.11 the principal G_{n+1}-bundle is

$$\{[q,g]\in s^2Q : r_n r_{n+1}(s^2\beta[q,g]) = (0,1)\in(\eta\oplus\underline{\mathbb{R}})\oplus\underline{\mathbb{R}}\} = (sQ)'$$

since the reflection r_n is irrelevant in the argument. As principal G_n-bundles $(sQ)' = (sQ)'^*$, so this is the correct desuspension. The G_n-structures also agree by a computation

$$\begin{aligned} s^{-1}(s\beta)_{r_n r_{n+1}}[[q,g],v] &= \mathrm{pr}_{\eta\oplus\underline{\mathbb{R}}}\left((s\beta)_{r_n r_{n+1}}[[q,g],(v,0)]\right) = r_{n+1}\,\mathrm{pr}_{\eta\oplus\underline{\mathbb{R}}}\left(s\beta[[q,g],(v,0)]\right)\\ &= r_{n+1}(\beta')[[q,g],v] = (\beta')^*[[q,g],v]. \end{aligned}$$

□

If $[q,g]\in sQ$ the explicit definition of $h_u : sQ\to(sQ)'^*$ is given by $h[q,g]=[q,ug]$. Note that indeed $\rho(ug) = r_n r_{n+1}\rho(g)\in r_{n+1}O_n$ so that $ug\in G_n'$.

Remark 4.6.8. In the proof, it is crucial that the G_n-structure on $\eta\oplus\underline{\mathbb{R}}$ is a suspension. When working in a purely homotopical setting this distinction is irrelevant, but in geometric settings it depends whether G_n-structures on $\eta\oplus\underline{\mathbb{R}}$ are always suspensions. This need not be the case if $G_n\to O_n$ is not a strict geometric representation.

From now on we sometimes abusively write v for elements of $\hat{G}_n(K)$ of the form $v\otimes 1$.

Corollary 4.6.9. *Let K be a fermionic group, $n>0$, $G_n := G_n(K)$ and let (Q,β) a G_{n-1}-structure on η. Then there is an isomorphism $h_u : sQ\to(sQ)'^*$ given by $h[q,g]=[q,ce_{n+1}e_ng]$.*

Proof. This follows by Corollary 4.6.7 because $u = ce_{n+1}e_n$ stabilizes $G_{n-1}(K)$. □

Remark 4.6.10. Not that in the above corollary we could have chosen the other lift $u' = e_ne_{n+1}\otimes 1 = -e_{n+1}e_n\otimes 1\in G_{n+1}(K)$ of r_nr_{n+1}. Given the order in which the prime and the star occur in P'^* it does seem to be more natural to use e_ne_{n+1}. To get an isomorphism $P\to(P^*)'$ it would be tempting to then use the other lift $e_{n+1}e_n$. However, then the diagram

$$\begin{tikzcd} P \arrow[r, "h_u"] \arrow[d, "h_{u'}"] & P'^* \\ (P^*)' \arrow[ur] & \end{tikzcd}$$

will not commute; the two directions differ by the spin flip automorphism. So we really have to make an arbitrary choice between e_ne_{n+1} and $e_{n+1}e_n$ in the construction of h_u once and for all. This choice will however not affect anything by the discussion at the beginning of Section 2.6 and we prefer this sign convention because then the isomorphism $\overline{P}^*\cong P$ for the orientation-grading coming from a fermionic group will not have a sign.

The proof of Corollary 4.6.7 also works for $n=0$, but still we restricted to $n>0$ in Corollary 4.6.9. The reason is that an oriented structure group $G_1\to SO_1$ is not a strict geometric representation and so there is no desuspension in general. As a consequence, a negatively oriented point will not fit in the above framework and so we need to define h_u in a different way, also see the discussion at Example 4.3.12. We will refrain from defining h_u for general structure groups in dimension one, instead focussing on $G_n = G_n(K)$ coming from a fermionic group and $u = e_1e_2\otimes 1\in G_2(K)$.

To motivate the definition, let us first assume K has reversing elements so that all $G_1(K) = K^{\mathrm{op}}$-structures on trivial bundles are suspensions of some $G_0(K)$-structure and so h_u is already defined by Corollary 4.6.7. Let Q be a $G_0(K) = K_{pres}$-structure so that $sQ = Q \times_{K_{pres}} K^{\mathrm{op}} = (sQ)_+ \sqcup (sQ)_-$ where $P_\pm$ are defined as in Example 4.3.12. Then by definition $h_u[q, 1] = [q, u]$ when $[q, 1] \in (sQ)_+$ but when $[q, g_1] \in (sQ)_-$ where $g_1 \in G_1(K)$ is reversing, then $h_u[q, g_1] = [q, ug_1] = [q, cg_1u]$. Indeed, it follows by a short computation that $g_1 = e_1 \otimes k \in G_1(K)$ anticommutes with u:

$$(-e_1e_2 \otimes 1)(e_1 \otimes k) = -e_1e_2e_1 \otimes k = e_1e_1e_2 \otimes k = (e_1 \otimes k)(e_1e_2 \otimes 1). \tag{4.20}$$

We use the same formula to define h_u in case $K = K_{pres}$:

Lemma 4.6.11. *Let (P, α) be a $G_1(K)$-structure on $\underline{\mathbb{R}} \to X$ and define*

$$P_\pm := \{p \in P : \alpha[p, 1] = \pm 1\}.$$

Let $u = -e_1e_2 \otimes 1 \in G_2(K)$. Then $h_u : P \to P'^*$ given by

$$h_u(p) = \begin{cases} [p, u] = [p, -e_1e_2] & p \in P_+ \\ [p, cu] = [p, e_1e_2] & p \in P_- \end{cases}$$

defines an isomorphism of $G_1(K)$-structures. Moreover, in case P is a suspension, the h_u agrees with the isomorphism obtained from Corollary 4.6.7.

Proof. We have $ug = c^{|g|}gu$ for all $g \in G_1(K)$. This is clear for preserving g and follows for reversing g by the computation (4.20) above. Therefore h_u is $G_1(K)$-equivariant. To show it is an isomorphism of $G_1(K)$-structures we compute for $v \in \mathbb{R}$ and $p_- \in P_-$ that

$$\alpha'^*[p, cu, v] = \alpha[p, -\rho(cu)|_{\mathbb{R}\oplus 0}(v)] = \alpha[p, v].$$

The same computation without c applies when $p_+ \in P_+$. In case $P = sQ$ this agrees with the formula of Corollary 4.6.7 by the discussion directly above this proposition. ☐

Remark 4.6.12. By the same proof, the map defined by $\tilde{h}_u(p) = [p, u]$ for all $p \in P$ would also give an isomorphism $P \to P'^*$. However, this formula is not only inconsistent with Corollary 4.6.7, but it will also lead to the wrong notion of unitary topological field theory. Namely in the case where $K = \mathrm{Spin}_1$, we see that $\tilde{h}_u$ is the same as h_u on a positively oriented point, but different from h_u by a spin flip on the negatively oriented point. As a consequence the spin bordism dagger category will be weak dagger compact, instead of the desired weak fermionically dagger compact. In particular, it will make the double of a half-circle the periodic spin circle, which is not bounding. This is a problem for the spin-statistics theorem, as discussed in the introduction. Our intuition for this is that in spacetime dimension one there is not enough room to encode the fact that 'rotation by 2π gives c'.

Remark 4.6.13. Let (P, α) be a $G_1(K)$-structure on $\underline{\mathbb{R}} \to X$. Then $\hat{P}$ always has a nontrivial $+$ and $-$ part. For example note how

$$\begin{aligned}(\hat{P})_+ &= \{[p, k \otimes e_1] : p \in P_-\} \cup \{[p, k \otimes 1] : p \in P_+\} \\ &\subseteq P \times_{G_1(K)} \hat{G}_1(K)\end{aligned}$$

independent of whether $k \in K$ is time-reversing.

We now come to the question of whether the isomorphism $h_u : P \to P'^*$ defines a Hermitian pairing on P. For bosonic symmetry groups, it turns out that it does, but for general structure groups it does not. Essentially, the error term to h_u being a Hermitian pairing is $u^2 \in G_{n+1}$ not being equal to 1, i.e. whether a 2π rotation lifts to a trivial element in G_{n+1}. In particular, for fermionic symmetry groups, $\hat{h}_u$ will square to a spin flip.

Lemma 4.6.14. *Let $h_u : P \to P'^*$ denote either the isomorphism constructed in Corollary 4.6.7 or the one of Lemma 4.6.11 for the special case where $G_{n+1} = G_{n+1}(K)$. Then the composition*

$$P \xrightarrow{h} P'^* \xrightarrow{h'^*} (P'^*)'^* \cong P''^{**} \cong P'' \cong P \tag{4.21}$$

is given by h_{u^2}.

Proof. For $n > 0$, let (Q, β) be a G_{n-1}-structure on η and $P = sQ$ its suspension. Given our assumptions $G_n(K) \to O_n$ is a strict geometric representation and so the choice (Q, β) of desuspension is unique. Note that we have $\rho(u^2) \in O_n$ and so by the strict pullback property $u^2 \in G_n$. This element centralizes G_{n-1} so that $h'_{u^2} : sQ \to sQ$ is defined. We obtain

$$P' = Q \times_{G_{n-1}} G_n \times_{G_n} (G_n^{(1)})^{rev} \cong Q \times_{G_{n-1}} (G_n^{(1)})^{rev}$$

and multiplication in $(G_n^{(1)})^{rev}$ gave the isomorphism

$$P'' = Q \times_{G_{n-1}} (G_n^{(1)})^{rev} \times_{G_n} (G_n^{(1)})^{rev} \to Q \times_{G_{n-1}} G_n = P.$$

The stars do not change anything for the principal bundle morphisms and so the desired composition is

$$[q, g] \mapsto [q, ug] \mapsto [q, u^2 g].$$

Noting that $h_u(p) \in P_\pm$ if and only if $p \in P_\pm$, a similar computation applies to the formula of Lemma 4.6.11. □

Corollary 4.6.15. *Let K be a fermionic group and $h_u : P \to P'^*$ the isomorphism induced by $u = e_n e_{n+1} \otimes 1$. Then the composition* (4.18) *is given by the spin flip automorphism of P.*

Proof. We have that $u^2 = c$ is central and so $h_{u^2}[p, g] = [p, u^2 g] = [p, g] \cdot c$ as desired. □

We conclude that if K is bosonic, $G_n(K)$-manifolds have canonical Hermitian pairings with respect to the $\mathbb{Z}/2$-action $P \mapsto P'$. However, when $c \neq 1$ the canonical candidate Hermitian pairings square to c instead. Note that it follows from the fact that there exists some $h : P \to P'^*$ such that the composition (4.18) is given by the spin flip that in general there is no Hermitian pairing on P. It follows by the discussion in Section 2.6 that in case the spin flip automorphism admits a square root, then a Hermitian pairing for $P \mapsto \overline{P}$ induces a Hermitian pairing for $P \mapsto P'$. However, a square root of the spin flip automorphism does not always exist. The case where $G(K) = \mathrm{Spin}$ already gives a counterexample. In fact, we will see in Example 5.2.19 that the $\mathbb{Z}/2$-action on the spin bordism category induced by $P \mapsto P'$ does not have Hermitian pairings on all objects.

We can solve this problem by changing the isomorphism $P'' \cong P$ with a spin flip. This will make the $\mathbb{Z}/2$-action isomorphic to the $\mathbb{Z}/2$-action $P \mapsto \overline{P}$ induced by the orientation-grading $G(K) \hookrightarrow \hat{G}(K)$ by Proposition 4.5.10:

Corollary 4.6.16. *Let K be a fermionic group and let $P \mapsto \overline{P}$ be the $\mathbb{Z}/2$-action induced by the orientation-grading $\hat{G}_n(K)$ on $G_n(K)$. If (P, α) is a $G_n(K)$-structure on $\eta \oplus \underline{\mathbb{R}}$ it has a Hermitian pairing h such that under Proposition 4.5.10 it corresponds to $h_u : P \to P'^*$. Explicitly it is given by*

$$h[q, g] = [q, e_n g] \quad [q, g] \in Q' = Q \times_{G_{n-1}(K)} G_n^{(1)}(K)$$

if $n > 1$ and Q is a desuspension of P and

$$h(p) = \begin{cases} [p, e_n] & p \in P_+, \\ [p, -e_n] & p \in P_-, \end{cases}$$

if $n = 1$.

Proof. The proof is immediate, but we derive an explicit formula for h. We start by writing out the isomorphism $\overline{P} \cong P'$ more explicitly. Recall that $\hat{G}_n^-(K) \cong G_n^{(1)}(K)$ via

$$x \otimes k \mapsto \begin{cases} x \otimes k & |x| = |k|, \\ e_{n+1} x \otimes k & |x| \neq |k|. \end{cases}$$

This defines the natural isomorphism $f_P : \overline{P} \cong P'$ as

$$f_P[p, \hat{g}] = f_P[p, e_{n+1}\hat{g}] \quad \hat{g} \in \overline{G}.$$

If $x \otimes k \in G_n(K)$ we obtain in particular

$$e_n x \otimes k \mapsto e_{n+1} e_n x \otimes k = -e_n e_{n+1} x \otimes k \in G_n^{(1)}(K).$$

Now suppose $n > 1$ so that $G_n(K)$ is a strict geometric representation and we can assume without loss of generality that $P = sQ$ is a suspension. We then obtain the formula

$$h[q, g] = [q, e_n g]$$

For $n = 1$ we use Lemma 4.6.11 to obtain for $p \in P$ a G_1-structure on $\eta = \underline{\mathbb{R}}$ which is not necessarily a suspension the isomorphism $P \to \overline{P}^*$ given by

$$h(p) = \begin{cases} [p, e_n] & p \in P_+, \\ [p, -e_n] & p \in P_-. \end{cases}$$

$\square$

Remark 4.6.17. To check that f_P indeed defines an isomorphism of $\mathbb{Z}/2$-actions the reader is invited to write out explicitly the formulas that show the diagrams

$$\begin{tikzcd} P \arrow[r, "c"] & P & P'' \arrow[l] \\ \overline{\overline{P}} \arrow[u] \arrow[r, "\overline{f_P}"] & \overline{P'} \arrow[ur, "f_{P'}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} \overline{\overline{P}} \arrow[r, "\overline{f_P}"] \arrow[d, "f_{\overline{P}}"] & \overline{P'} \arrow[d, "f_{P'}"] \\ \overline{P}' \arrow[r, "f'_P"] & P'' \end{tikzcd}$$

commute.

Remark 4.6.18. Recall the discussion in Remark 4.6.10 about picking the other lift $-u$ instead of u. This will eventually change what we call a positive pairing on the bordism category from $h : x \to dx$ to $h : x \xrightarrow{c} x \to dx$. This is analogous to the choice in super Hilbert spaces between calling odd vectors with norm $+i$ positive versus calling $-i$ positive. Since the spin flip will give the $B\mathbb{Z}/2$-action on the bordism category, this operation will map the positivity structure P on the anti-involutive bordism category to the modified positivity structure $P_{(-1)^F}$ in the sense of Section 2.5. It follows by modifying the identity functor by the self-adjoint natural automorphism $(-1)^F$ that picking the other lift will give an equivalent symmetric monoidal dagger category, also see Proposition 2.6.3.

Remark 4.6.19. It follows by the naturality discussed in Proposition 4.6.2 that the Hermitian pairings in Corollary 4.6.16 are natural in case P is the suspension of a $G_{n-1}(K)$-structure Q. More precisely, if $f : (X_1, \eta_1) \to (X_2, \eta_2)$ is a morphism in $\mathrm{VectBun}_{n-1}$ which is covered by a morphism $(Q_1, \beta_1) \to (Q_2, \beta_2)$, then it intertwines the Hermitian pairings on the suspensions sQ_1 and sQ_2.

Example 4.6.20. For r a prime, let $G_2 := \mathrm{Spin}_2^r$ be the r-fold connected cover of SO_2 so that $\mathrm{Spin}_2^2 = \mathrm{Spin}_2$. Then any $u \in G_2$ lifting a rotation by $180°$ satisfies that u^2 is central of order r and so generates a $B\mathbb{Z}/r$-action on G-Str by Section 4.4. The composition (4.18) is the generator of this action. Note that when $r \neq 2$ we cannot perform the same trick of changing the $\mathbb{Z}/2$-action $P \mapsto P'$ by the generator of this action, because if u^2 does not square to 1 the isomorphism $\mu_P : P \cong \overline{\overline{P}}$ will not satisfy the desired $\overline{\mu_P} = \mu_{\overline{P}}$. We also remark that if $r > 2$, then G_2 is an example of a structure group for which a stabilization G_3 does not exist even homotopically in the sense that

$$\begin{array}{ccc} BG_3 & \longrightarrow & BO_3 \\ \uparrow & & \uparrow \\ BG_2 & \longrightarrow & BO_2 \end{array}$$

can never be a homotopy pullback. This follows by a computation using the grid of long exact sequences associated to the above diagram by using that parallel maps in a homotopy pullback have homtopy equivalent fibers.

We have successfully constructed Hermitian pairings on $G_n(K)$-structures (P, α) on vector bundles of the form $\eta \oplus \underline{\mathbb{R}}$ for the $\mathbb{Z}/2$-action $P \mapsto \overline{P}$ induced by the orientation-grading $G_n(K) \to \hat{G}_n(K)$ The final goal of this section is to ask how the Hermitian pairings on P and P^* are related. This is an essential ingredient in proving that the bordism category is weak fermionically dagger-compact, which in turn is essential to prove the spin-statistics theorem, see Corollary 2.9.12. Looking at the formulas, it is very tempting to think that h_{P^*} will become the dual Hermitian pairing of h_P in the bordism category in the sense of Definition 2.7.12. However, a subtlety in the definition of the Hermitian pairing on (P, α) is that the definition depended on a choice of desuspension (Q, β) of (P, α). We already know of a desuspension of $(P, \alpha)^*$; it is the prime in the nth direction. The fact that the Hermitian pairing on P^* will differ with the dual Hermitian pairing with a spin flip is now a consequence of the following computation.

Lemma 4.6.21. *Let $g \in (G_{n-1}^{(1)})_{rev}$, i.e. an element of $G_n(K)$ satisfying*

$$\rho(g) = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \in O_n.$$

Then g anticommutes with $1 \otimes e_n$.

Proof. Write $g = k \otimes xe_n$ with $x \in \mathrm{Pin}^+_{n-1}$. Compute

$$g(1 \otimes e_n) = k \otimes xe_n^2 = (-1)^{|x|} k \otimes e_n x e_n = (-1)^{|x|+|k|}(1 \otimes e_n)(k \otimes xe_n)$$

Since $|x| = |k| + 1$ this finishes the proof. $\square$

Corollary 4.6.22. *For $n > 0$ let (Q, β) be a $G_{n-1}(K)$-structure on (X, η) and let (P, α) be its suspension. The diagram*

$$\begin{tikzcd} P^* \arrow[r, "h_{P^*}"] \arrow[d, "c_{P^*}"] & \overline{P^*}^* \arrow[d] \\ P^* \arrow[r, "h_P^*"] & \overline{P}^{**} \end{tikzcd}$$

commutes.

Proof. Recall that $(Q, \beta)'$ is the desuspension of $(P, \alpha)^*$ and $Q' = Q \times_{G_{n-1}} (G^{(1)}_{n-1})_{rev}$ by Lemma 4.3.11. By the previous lemma we have for $[q, g] \in Q'$ that

$$h_{P^*}[q, g] = [[q, g], e_n] = [[q, c], e_n g].$$

This differs from h_P^* with multiplication by $c \in G_n(K)$. $\square$

The above statement is similarly true for $n = 0$ when we take h_P to be the map of Lemma 4.6.11. However, the proof is completely different; it follows from the fact that $(P_\pm)^* = (P^*)_\mp$.

Remark 4.6.23. In work in progress [44], Kreck, Stolz and Teichner have constructed an isomorphism $P \cong \overline{P}^*$ of $G_n(K)$-structures of which the definition does not depend on a choice of desuspension of the $G_n(K)$-structure. In particular, it does not require a workaround for $n = 1$ and $K = K_{pres}$ as in Corollary 4.6.16, but agrees with our formula in all dimensions.

Chapter 5

Fermionic bordism

5.1 The bordism category

In Chapter 4 we studied orientation-reversal of G-structures on vector bundles over topological spaces. We will now restrict to the case the space is a manifold and the vector bundle is a suitable stabilization of the tangent bundle. To make a bordism category we will then also be interested in bordisms between such manifolds with G-structures. Since we will only consider the classical nonextended notion of topological field theory in this thesis, many of the subtleties in defining the bordism category are not relevant, but we will point out some of the relevant modifications towards the end of this section. We decide to work with the following definition of the bordism category, which will be technically convenient for topolological field theories.

Definition 5.1.1. Let $G \to O_n$ be a structure group and let $(Y_0^{n-1}, P_0, \alpha_0), (Y_1^{n-1}, P_1, \alpha_1)$ be closed $(n-1)$-manifolds with G-structures. Then a *G-bordism*[1] $X : Y_0 \rightsquigarrow Y_1$ consists of a compact manifold X with boundary and a G-structure (Q, β) together with a partition of the boundary into two parts

$$\partial X = (\partial X)_{in} \sqcup (\partial X)_{out}$$

and G-diffeomorphisms

$$(Y_0^{n-1}, P_0, \alpha_0)^* \cong (X, Q, \beta)|_{(\partial X)_{in}}, \quad (Y_1^{n-1}, P_1, \alpha_1) \cong (X, Q, \beta)|_{(\partial X)_{out}}.$$

We provide some explanation on how to think of the two parts Y_0 and Y_1 of the boundary pictorially and in particular why we take the starred G-structure Y_0^* in the definition, also see Figure 5.1. The main idea is that we want the normal vector field to point in the direction in which time flows. Recall that implicitly, a G-structure on an $(n-1)$-dimensional manifold Y is a G-structure on the once stabilized tangent bundle. We prefer to think of the single stabilization of TY as a choice of vector normal to the space Y inside spacetime in say the positive time direction, which is also the direction in which we will compose bordisms. More precisely, given a compact Riemannian manifold X, we get two isomorphisms $TX|_{\partial X} \cong T\partial X \oplus \underline{\mathbb{R}}$ given by choosing either the normal inward or outward pointing vector field. These two isomorphisms differ by the vector bundle automorphism $\mathrm{id}_{T\partial X} \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$. We once and for all decide to trivialize normals to boundaries of manifolds with their outgoing arrows. Note that to get the canonical isomorphism $TX|_{\partial X} \cong T\partial X \oplus \underline{\mathbb{R}}$, it was crucial for our structure group G to land in O_n.

[1]Some authors refer to bordisms as 'cobordisms'. For us, these two terms are synonyms.

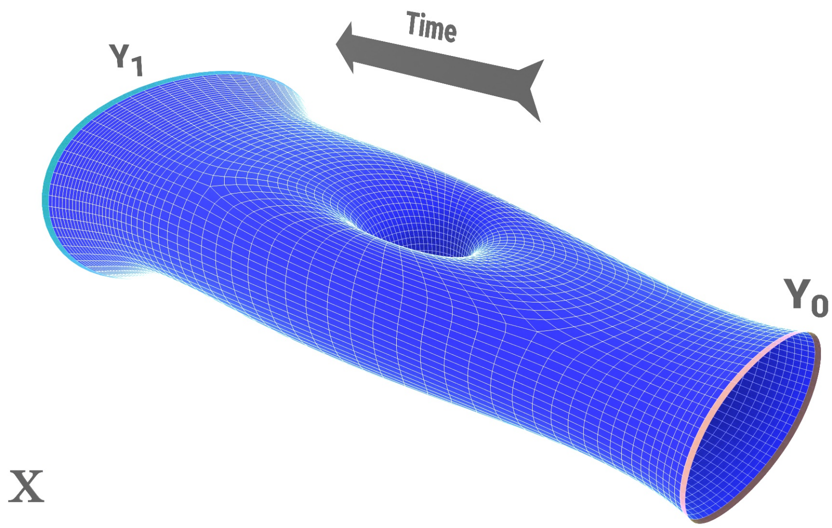

Figure 5.1: An n-dimensional bordism X from the $(n-1)$-dimensional manifold Y_0 to the $(n-1)$-dimensional manifold Y_1.

Now assume X has its boundary decomposed as $\partial X = (\partial X)_{in} \sqcup (\partial X)_{out}$ which we think of as a spacetime going from $(\partial X)_{in}$ to $(\partial X)_{out}$. Then we want to use the opposite normal vector for $(\partial X)_{in}$ and $(\partial X)_{out}$ so that the vector points into the spacetime for $(\partial X)_{in}$ and out of the spacetime for $(\partial X)_{out}$. This is the reason we have to take the star of only $(\partial X)_{in}$; the star performs the operation of composing the G-structure datum α by $\mathrm{id}_{TY} \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$. We will see later that this reversal will define a canonical categorical dual of objects in our definition of the bordism category. This also motivates our convention for trivializing a boundary with its outgoing arrow; we want to think of a bordism from Y_0 to Y_1 as equivalent to a manifold with boundary $Y_1 \sqcup Y_0^*$ similarly to how we can think of a linear map from V_0 to V_1 as an element of $V_1 \otimes V_0^*$.

Remark 5.1.2. In the definition of a G-bordism we allow Y_0 and Y_1 to be empty. For example, closed n-dimensional G-manifolds are endomorphisms of the empty set.

Definition 5.1.3. The *(topological) bordism category* $\mathrm{Bord}^G_{n,n-1}$ associated to a structure group $G \to O_n$ has objects closed $(n-1)$-manifolds with G-structure and morphisms G-bordisms modulo G-diffeomorphism relative boundary. Composition of bordisms $X : Y_0 \rightsquigarrow Y_1$ and $X' : Y_1 \rightsquigarrow Y_2$ is given by gluing along Y_1. In order to make the result smooth, we first pick G-diffeomorphisms of X and X' relative boundary with bordisms $\tilde{X}$ and $\tilde{X}'$ which are of the form $Y_1 \times [0,1]$ near the outgoing respectively incoming boundary.

Here a G-diffeomorphism relative boundary between (X, Q, β) and (X', Q', β') from (Y_0, P_0, α_0) to (Y_1, P_1, α_1) is a G-diffeomorphism which commutes with the two G-diffeomorphisms at both boundaries, e.g. $Y_0 \cong (\partial X)_{in} \to (\partial X')_{in} \cong Y_0$ is the identity. Recall that G-diffeomorphism refers to the weak homotopical notion of Definition 4.1.11, not the strict notion of a geometric G-diffeomorphism. This single choice of definition is what makes 5.1.3 define a *topological* bordism category in the sense of *topological* field theory; it cannot distinguish different metrics.[2] For example, even though objects of the bordism category come equipped with a Riemannian metric, morphisms are equivalence classes of Riemannian manifolds with extra structure and the Riemannian metric does not preserve this equivalence. If two objects are G-diffeomorphic, they will be isomorphic in the bordism category by a mapping cylinder construction, see Corollary 5.1.20. Since G-diffeomorphisms in general do not preserve the metric, this in particular will imply that diffeomorphic but nonisometric $(n-1)$-dimensional closed manifolds become isomorphic in the $G = O_n$-bordism category.

Remark 5.1.4. Physically we think of the bordisms as *spacetimes*. In particular, since objects of the bordism category represent space at a specific time we sometimes refer to them as *time slices*.

To define the composition in the bordism category, we used the existence of smooth collars. For this it is essential that we chose to work with G-structures up to the homotopical notion of G-diffeomorphism. For example, two Riemannian manifolds with isometric boundaries need not glue together to a well-defined metric on the composition. Note also that composition is well-defined; the G-diffeomorphism class relative boundary of the composition is independent of the choices of $\tilde{X}$ and $\tilde{X}'$.

The cylinder $Y \times [0,1]$ on the closed $(n-1)$-dimensional G-manifold Y with G-structure constant in the time direction and the decomposition of the boundary into the two copies of Y defines the identity on Y. For this it is again crucial that we take the homotopical notion of G-diffeomorphism; this tells us that enlarging a bordism X from Y_1 to Y_2 with G-structure constant in the time direction around Y_1 with $Y_1 \times [0,1]$ around the source does not change the G-diffeomorphism type relative boundary of X. The bordism category becomes monoidal under disjoint union and symmetric using the canonical diffeomorphisms $Y_1 \sqcup Y_0 \cong Y_0 \sqcup Y_1$.

Remark 5.1.5. Given a bordism X from Y_0 to Y_1, there is a canonical way to view it as a bordism from $Y_0 \sqcup Y_1^*$ to $\emptyset$ or as a bordism from $\emptyset$ to $Y_0^* \sqcup Y_1$. Applying this to the identity on Y we get canonical morphisms $\mathrm{ev}_Y : Y^* \sqcup Y \rightsquigarrow \emptyset$ and $\mathrm{coev}_Y : \emptyset \rightsquigarrow Y^* \sqcup Y$ that realize Y^* as the dual of Y.

Remark 5.1.6. We could have also defined a bordism from Y_{in} to Y_{out} to be a G-manifold X with its boundary decomposed into two parts $\partial X = (\partial X)_{in} \sqcup (\partial X)_{out}$ with the property that $(\partial X)_{in} = Y_{in}^*$ and $(\partial X)_{out} = Y_{out}$ as G-manifolds. This defines an equivalent category and is the definition of the bordism category used in [45]. Forgetting the boundary parametrization conveniently decreases the amount of data one has to carry around. However, it would make the construction of the mapping cylinder functor from the groupoid of G-diffeomorphisms to the bordism category more opaque because it would force the gluing by the G-diffeomorphism to be at some arbitrary slice in the interior of the bordism. It also does not generalize well to more geometric bordism categories.

We make a couple of remarks that are relevant for generalizing the notion of bordism to higher-categorical and more geometric situations, also see [70, 71].

Remark 5.1.7. If we would want to define an $(\infty,1)$-category, we would not quotient out by G-diffeomorphisms relative boundary. Instead we would make a moduli space of G-bordisms between two fixed closed G-manifolds of which the connected components are the morphism sets of the bordism category of Definition 5.1.3. For this it is convenient to require an embedding of all bordisms into $\mathbb{R}^\infty$. This space has enough room to avoid knotting and embedding spaces have a natural compact-open topology. We get a space of bordisms between two fixed embedded manifolds, so this allows for topologically enriched categories which are models of $(\infty,1)$-categories. $(\infty,1)$-bordism categories

[2] Unfortunately the terminology is such that topological field theories do require smooth (not just topological) manifolds. A further unfortunate clash of terminology concerns the study of topological categories of bordisms. This is one popular way to define bordism $(\infty,1)$-categories.

are relevant for topological field theories with suitably derived targets, such as the $(\infty, 1)$-category of chain complexes over the complex numbers. This target could be relevant for other applications in mathematical physics, such as cohomological field theories and the BV-BRST formalism.

It can also be convenient to fix the boundaries to fixed hyperplanes in $\mathbb{R}^\infty$ to define composition. This approach is also useful to build an (∞, n)-bordism category, see [17] for an introduction to this topic. For a definition of the (∞, n)-bordism category as an n-fold complete Segal space, it can furthermore be convenient to describe (higher) bordisms as manifolds equipped with certain cuts along submanifolds. We refer to [10, 24] for details.

We provide a discussion on how to change the objects of the bordism category appropriately in a fully geometric setting. The first naive guess will be to not identify bordisms if they are homotopically G-diffeomorphic relative boundary, but only when they are geometrically G-diffeomorphic relative boundary. However, we have seen in that case that composition of bordisms will not be well-defined because the G-structure might not glue well along the boundary. To fix this this, we realize that in the topological setting we can equivalently think of bordisms as manifolds X in which the 'boundaries' sit inside it with a small cylindrical collar $Y \times (-\epsilon, \epsilon)$, which allowed us to glue. For clarity, we provide some precise elementary comparison results with the previous formulation. With further applications to higher-dimensional bordisms in mind, we work in a more general setting where we replace the interval with an arbitrary contractible patch.

Lemma 5.1.8. *Let Y be an n-dimensional manifold, $G \to GL_{n+k}(\mathbb{R})$ a structure group and (P_Y, β_Y) a G-structure on $TY \oplus \underline{\mathbb{R}}^k$. There is a canonical functor*

$$\{G - \textit{structures on } TY \oplus \underline{\mathbb{R}}^k\} \to \{G - \textit{structures on } T(Y \times \mathbb{R}^k)\}. \tag{5.1}$$

induced by $P_{Y \times \mathbb{R}^k} := P_Y \times \mathbb{R}^k \to Y \times \mathbb{R}^k$ for a G_{n+k}-structure on $TY \oplus \underline{\mathbb{R}}^k$.

Proof. The statement follows immediately from the prestack structure on G-structures applied to the projection map $\mathrm{pr}_1 : Y \times \mathbb{R}^k \to Y$, since the bundle $TY \oplus \underline{\mathbb{R}}^k$ over Y pulls back to the tangent bundle of $Y \times \mathbb{R}^k$ using the specified direction in $\mathbb{R}^k$:

$$T(Y \times \mathbb{R}^k) \cong TY \times T\mathbb{R}^k \cong (TY \to Y) \boxplus (\underline{\mathbb{R}}^k \to \mathbb{R}^k) = (TY \oplus \underline{\mathbb{R}}^k) \times \mathbb{R}^k.$$

We write out concretely what this means for the G-structure: make $P_{Y \times \mathbb{R}^k}$ into a principal G-bundle over $Y \times \mathbb{R}^k$ by equipping it with the G-action only acting on P_Y. Now $P_{Y \times \mathbb{R}^k}$ becomes a G-structure on $T(Y \times \mathbb{R}^k)$ via the map

$$\beta_{Y \times \mathbb{R}^k}[(p, t), v] = (\beta_Y[p, v], t) \in (TY \oplus \underline{\mathbb{R}}^k) \times \mathbb{R}^k.$$

□

In particular for $k = 1$ we see that $Y \times (-\epsilon, \epsilon)$ has a canonical induced G-structure for every ϵ. Note that taking two different $\epsilon_1 < \epsilon_2$ will give the same G-structures when comparing under the G-embedding $Y \times (-\epsilon_1, \epsilon_1) \hookrightarrow Y \times (-\epsilon_2, \epsilon_2)$. Without including metrics, the functor of Lemma 5.1.8 is not essentially surjective for geometric G-structures: it maps to G-structures that are constant in the $\mathbb{R}^k$-direction. In particular for $k = 1$ these are only the G-structures that are constant in the time direction. Now suppose G maps to O_{n+k} so that Y has a metric, then the functor

$$\{G - \text{structures on } TY \oplus \underline{\mathbb{R}}^k\} \to \{G - \text{structures on } T(Y \times \mathbb{R}^k)\}$$

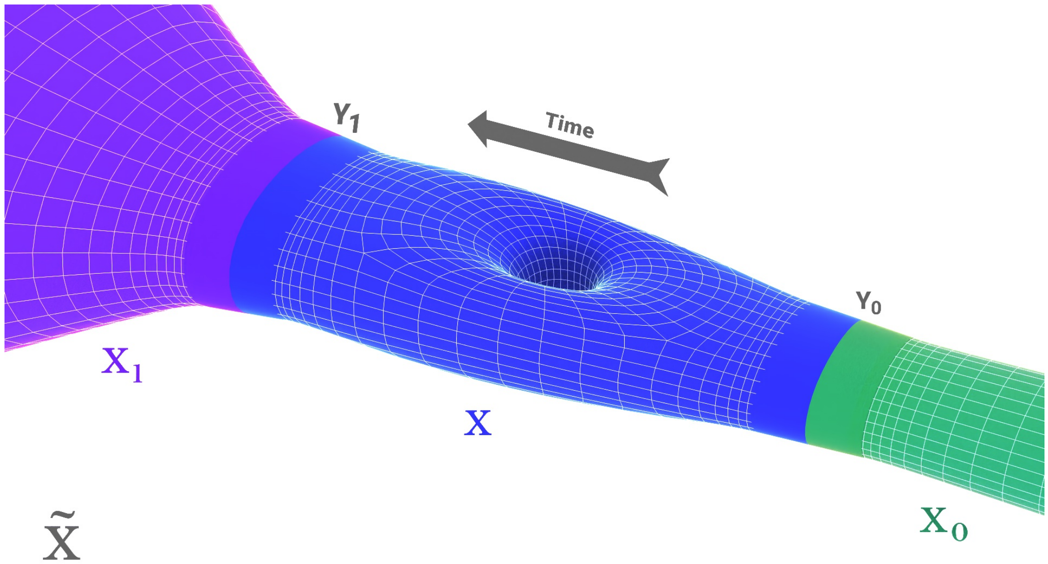

Figure 5.2: A picture of a bordism $\tilde{X}$ in a definition that is more suitable for generalizations to non-topological functorial quantum field theories, following Stolz-Teichner [70, 71]. The middle blue region X is called the core and should be thought of as the 'actual bordism'. The incoming green region X_0 and outgoing purple region X_1 stretch out to infinity and should be thought of as 'redundant data'. The bordism goes from the object Y_0, which comes equipped with little collar $Y_0 \times (-\epsilon_0, \epsilon_0)$, to the collar $Y_1 \times (-\epsilon_1, \epsilon_1)$. These collars lie partially inside the core and partially outside as pictured.

equips $Y \times \mathbb{R}^k$ with the product with the Euclidean metric. Furthermore, if G is a strict geometric representation, the groupoid of G_{n+k}-structures on $TY \oplus \underline{\mathbb{R}}$ is equivalent to the groupoid of G_{n+k-1}-structures on TY via the suspension functor. If the pullbacks G_{n-i} continue to satisfy the strictness hypothesis, this is similarly true for G_n-structures TY and G_{n+k}-structures on $TY \oplus \underline{\mathbb{R}}^k$. Since we will be working with G-structures homotopically in Definition 5.1.3, the data of the metrics in this discussion will be irrelevant.

We can now formulate bordisms without ever referring to $(n-1)$-dimensional manifolds in the first place as follows, we refer to [70, 71] for details. Even though the following formulation is only necessary for non-topological bordism categories, we still think of it is as being pictorially convenient for topological field theory, especially for confusions concerning orientations and in- versus outgoing boundaries. Therefore we sometimes prefer this picture in proofs. However, it is more technical and elaborate to formulate. Let (P_0, α_0) and (P_1, α_1) be G-structures on $Y_0 \times (-\epsilon_0, \epsilon_0)$ and $Y_1 \times (-\epsilon_1, \epsilon_1)$ respectively. A geometric G-bordism from (Y_0, P_0, α_0) to (Y_1, P_1, α_1) is a not necessarily compact G-manifold $\tilde{X}$ without boundary, together with a decomposition of $\tilde{X} = X_0 \cup_{(\partial X)_0} X \cup_{(\partial X)_1} X_1$ into three parts, see Figure 5.2. Here X is a compact manifold with boundary called the core, which we think of as the 'actual bordism' and agrees with what we called X before. The subsets X_0 and X_1 are respectively what happened in the past before Y_0 happened and what will happen in the future after Y_1 happened. We then require the G-manifold $\tilde{X}$ to come equipped with G-embeddings of $Y_0 \times (-\epsilon_0, \epsilon_0)$ and $Y_1 \times (-\epsilon_1, \epsilon_1)$ so that the collars get embedded in the correct subsets. More precisely, one requires that $Y_0 \times (0, \epsilon_0)$ maps into X_0, $Y_1 \times (\epsilon_1, 0)$ maps into X_1 while $Y_0 \times (\epsilon_0, 0]$ and $Y_1 \times [0, \epsilon_1)$ map into X. Here we adopted the convention that time flows in the negative direction or in other words: bordisms go from right to left. We prefer this convention because then composition corresponds to the usual order of composition for functions. This also explains our convention for the directions of the diffeomorphisms from the boundaries in Definition 5.1.3 and the direction of the normal vector is the direction time flows.

We will want to identify two such bordisms $\tilde{X}$ and $\tilde{X}'$ if there exists a G-diffeomorphism between their cores $X \cong X'$ that extends to some neighborhood inside $\tilde{X}$. However, it does not have to extend to all of $\tilde{X}$; we want to ignore everything that happened before Y_0 and after Y_1. This formulation makes precise the idea of having a bordism together with a small collar which we can use to make gluing of bordisms well-defined.

One issue of the above definition is that the boundaries of the bordism depend on ϵ and on the G-structure on all of the area surrounding Y. Instead we want to identify objects with different ϵ by looking at G-embeddings for two different ϵ. We then need to take germs of collars around Y in the sense that we will identify $Y \times (-\epsilon_1, \epsilon_1)$ and $Y \times (-\epsilon_2, \epsilon_2)$ if there exists $Y \times (-\epsilon_3, \epsilon_3)$ for some ϵ_3 such that $\epsilon_3 < \epsilon_2$ and $\epsilon_3 < \epsilon_1$ and G-embeddings

$$Y \times (-\epsilon_3, \epsilon_3) \hookrightarrow Y \times (-\epsilon_1, \epsilon_1) \quad Y \times (-\epsilon_3, \epsilon_3) \hookrightarrow Y \times (-\epsilon_2, \epsilon_2).$$

Note that for $G = O_n$ if we do not quotient by the homotopical notion of G-diffeomorphism, the cylinders $Y \times [0, t]$ of different sizes give potentially non-identity bordisms. These will correspond to evolving time on the space Y by the amount t, i.e. for a functorial field theory they will be mapped to the time evolution operator e^{-tH} where H is the Hamiltonian. Moreover, Y will no longer be categorically dualizable, so that functorial field theories can assign infinite-dimensional state spaces to time slices.

Remark 5.1.9. In the above discussion there was no need to impose any restrictions on how the G-structure of a collar $Y \times (-\epsilon, \epsilon)$ varies in the time-direction. We already discussed the option that it is constant in the time direction, but another interesting but weaker assumption is to require it to be symmetric under reflection over the center $Y \times 0$. In other words, the G-structure might be stable under pullback by the diffeomorphism

$$\begin{aligned} S : & Y \times (-\epsilon, \epsilon) \to Y \times (-\epsilon, \epsilon) \\ & (y, t) \mapsto (y, -t). \end{aligned}$$

This is a necessary assumption for the collar to have the chance to admit something like a Hermitian pairing. Indeed, the constructions we saw in Section 4.6 all require a reflection. This connects with the original idea of reflection positivity that pairs of fields that are reflected towards each other by time-reversal to obtain a nondegenerate positive pairing, also compare with the discussion in [43] in the 'unitarity' subsection. We can only perform such an operation if the collar is preserved under this reflection. As a consequence, the state spaces of unitary non-topological functorial field theories associated to such symmetric time slices are expected to be Hilbert spaces, but general time slices are not. The starring operation $Y \mapsto Y^*$ is also closely related to pulling back along S. This will be made precise now in Corollary 5.1.11.

In order to make the above discussion about Hermitian pairings more precise and relate the operation of reflecting the direction of time of a collar $Y \times (-\epsilon, \epsilon)$ to the star operation, we first prove a general lemma.

Lemma 5.1.10. *Let Y be an n-dimensional manifold, $G \to GL_{n+k}(\mathbb{R})$ a structure group and (P_Y, β_Y) a G-structure on $TY \oplus \underline{\mathbb{R}}^k$. Let $A \in GL_k(\mathbb{R})$ induce the diffeomorphism $S_A : (y, v) \mapsto (y, Av)$ of $Y \times \mathbb{R}^k$. Then S_A induces a canonical G-diffeomorphism between the images of (P_Y, β_Y) and $(P_Y, (\beta_Y)_A)$ under the functor* (5.1)*. If additionally, $G \to O_{n+k}$ and $A \in O_k$, then S_A is an isometric G-diffeomorphism.*

Proof. The map

$$S_* : P_Y \times \mathbb{R}^k \to P_Y \times \mathbb{R}^k$$

given by the identity on P_Y and A on $\mathbb{R}^k$ is a map of G-principal bundles covering S_A. Under the isomorphism $T(Y \times \mathbb{R}^k) \cong (TY \oplus \underline{\mathbb{R}}^k) \times \mathbb{R}^k$ the differential of S_A is given by the identity on TY and $(x, v) \mapsto (Ax, Av)$ on $(x, v) \in T\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$. We see that the desired diagram

$$\begin{tikzcd}
(P_Y \times \mathbb{R}^k) \times_G \mathbb{R}^{n+k} \arrow[r, "S_*"] \arrow[d, "\beta_Y"] & (P_Y \times \mathbb{R}^k) \times_G \mathbb{R}^{n+k} \arrow[d, "(\mathrm{id}_{TY} \oplus A) \circ \beta_Y"] \\
T(Y \times \mathbb{R}^k) \arrow[r, "dS"] & T(Y \times \mathbb{R}^k)
\end{tikzcd}$$

commutes. Since S_A is an isometry if $A \in O_k$, the last statement follows as well. □

We can thus represent $(P, \alpha)^*$ as the collar $Y \times (-\epsilon, \epsilon)$ reflected along the middle of the interval:

Corollary 5.1.11. *Let $k = 1, A = -1 \in O_1$ so that*

$$S_{r_{n+1}} : Y \times (-\epsilon, \epsilon) \to Y \times (-\epsilon, \epsilon)$$

is the reflection of a collar along Y. Then (P_Y, β_Y) and $(P_Y, \beta_Y)^*$ become isometrically G_{n+1}-diffeomorphic under S after applying the functor (5.1)*.*

Next, we discuss the consequences of the above lemma for Hermitian pairings between bordisms.

Corollary 5.1.12. *Let $k = 2$ and let $A = \mathrm{rot}_t \in SO_2$ denote anti-clockwise rotation by $t \in \mathbb{R}/2\pi\mathbb{Z} \cong S^1$. We obtain an S^1-family of isometric diffeomorphisms of $Y \times \mathbb{R}^2$, which for a fixed $t \in \mathbb{R}/2\pi\mathbb{Z}$ gives a G_{n+1}-diffeomorphism between images of (P_Y, β_Y) and $(P_Y, (\beta_Y)_{\mathrm{rot}_t})$ under the functor 5.1. For $t = \pi$ in case (P_Y, β_Y) is the suspension of Q, then $(P_Y, (\beta_Y)_{\mathrm{rot}_t})$ is the suspension of Q'^*.*

The above corollary gives us a more geometric viewpoint of what Hermitian pairings are about; the operation $Q \mapsto Q^*$ reflects along the $(n+1)$th coordinate, while $Q \mapsto Q'$ reflects along the nth coordinate so that $Q \mapsto Q'^*$ is a rotation by π in the $(n, n+1)$-plane. The Hermitian pairing allows one to rotate back. We claim that this perspective is useful to show that the double of a manifold with boundary is nullbordant, also see Figure 0.1.

Remark 5.1.13. Recall from Example 2.3.24 that the category of cospans $\mathrm{CoSpan}\,\mathcal{C}$ on a category $\mathcal{C}$ with pushouts is a dagger category. Bordisms are at least on a first glance defined very similarly as certain cospans of manifolds. For example in more geometric definitions of bordisms such as the one discussed above, bordisms are examples of cospans in the category $\mathrm{Man}_n^{\mathrm{emb},G}$ of n-dimensional G-manifolds with G-embeddings. Also the symmetric monoidal product on $\mathrm{Bord}_{n,n-1}^G$ is defined using the coproduct in $\mathrm{Man}_n^{\mathrm{emb},G}$, just as in $\mathrm{CoSpan}\,\mathcal{C}$. As an aside: this puts a problem we had with the definition of composition of geometric bordisms in a categorical perspective; the category of G-manifolds and G-embeddings does not admit all pushouts, which causes composition of cospans to not always exist. One important condition on the embeddings into the bordism is the compatibility of the embeddings of the boundaries of the bordisms with the time direction; we cannot simply flip a time-slice Y in the time direction and expect that it defines a bijection between bordisms from Y and bordisms to Y, because this will no longer give a G-embedding. As a consequence, objects in the bordism category (unlike the category of spans) do not admit a canonical self-duality, only a canonical duality between Y and its time-reversal Y^*. Therefore, unlike sometimes assumed in the literature [4], bordism categories are not canonically dagger categories. This justifies the necessity of the elaborate constructions in Section 4.6 in order to construct Hermitian pairings on the bordism category.

Remark 5.1.14. Our discussion of the subtleties in defining geometric bordism categories is not exhaustive. Some relevant topics we did not discuss include

1. The option of requiring only one side of the collar as part of the data. This will for example make sure that in one dimension the macaroni of length $t = 0$ only exists on one side. We want the evaluation to exist because it will give the inner product on state space. But we do not want the coevaluation to exist because it will force the identity operator on state space to be Hilbert-Schmidt, which can only happen when state-space is finite-dimensional. This approach might additionally be useful to enforce time slices to be time symmetric for the purposes of Remark 5.1.9.

2. Including smoothness in the definition of a functorial field theory by defining families of bordisms. For this we need to make both the bordism category and the target category fibred over the site of manifolds and require the functor to respect this structure.

3. In certain definitions of bordisms in a Riemmannian setting there is no natural morphism $Y_1 \rightsquigarrow Y_2$ in the geometric bordism category associated to a diffeomorphism $Y_1 \to Y_2$ or any attempt to make such a morphism by a mapping cylinder construction will not result in an isomorphism. For this it can be convenient to separately include diffeomorphisms as invertible 'thin bordisms' in the definition of the geometric bordism category.

4. It is probably reasonable from the perspective of physics to require geometric bordism categories to come equipped with connections for the relevant principal bundles. More precisely, if $\rho : G \to O_n$ is a structure group we require the principal G-bundles over spacetime to come equipped with a connection with the property that it pushes forward to the Levi-Civita connection for the Riemannian metric under ρ.

We come back to our preferred Definition 5.1.3 of the bordism category without collars suitable for topological field theory. Note that Y^* is the dual of Y in $\mathrm{Bord}^G_{n,n-1}$:

Lemma 5.1.15. *Let Y^{n-1} be a manifold with G-structure (P, α). Then the time slice induced by $(P, \alpha)^*$ is the dual of the time slice induced by (P, α) in the bordism category.*

Proof. The construction of Lemma 5.1.8 for $k = 1$ gives the cylinder $X = Y \times [0, 1]$ a G-structure constant in the time direction. By definition of the bordism category, we can view it not just canonically as a bordism $Y \rightsquigarrow Y$ but also as a bordism $\mathrm{ev}_Y : Y^* \sqcup Y \rightsquigarrow \emptyset$ or as a bordism $\mathrm{coev}_Y : \emptyset \rightsquigarrow Y \sqcup Y^*$. It is now straightforward to show that the snake identities hold. □

Recall that in a symmetric monoidal category $\mathcal{C}$ with duals, uniqueness of duals supplies a canonical trivialization of the double dual $x^{**} \cong x$ for $x \in \mathcal{C}$.

Lemma 5.1.16. *Let (Y, P, α) be a closed $(n-1)$-dimensional manifold with G-structure on $TY \oplus \underline{\mathbb{R}}$. Then the isomorphism $Y \cong Y^{**}$ in the bordism category induced by the equality $(P, \alpha)^{**} = (P, \alpha)$ agrees with the isomorphism induced by the symmetric braiding.*

Proof. Recall that if c_1, c_2 are two duals of c, then any isomorphism $c_1 \to c_2$ intertwining the evaluation maps is equal to the isomorphism expressing uniqueness of duals. Therefore, it suffices to show that the diagram

$$\begin{tikzcd} Y \sqcup Y^* \arrow[r, equal] \arrow[d, "\beta_{Y,Y^*}"] & Y^{**} \sqcup Y^* \arrow[d, "\mathrm{ev}_{Y^*}"] \\ Y^* \sqcup Y \arrow[r, "\mathrm{ev}_Y"] & \emptyset \end{tikzcd}$$

commutes. This follows by direct inspection. □

Definition 5.1.17. Let $(Y_1, P_1, \alpha_1), (Y_2, P_2, \alpha_2)$ be G-manifolds and let $\phi : Y_1 \to Y_2$ a G-diffeomorphism. Then the *mapping cylinder of* ϕ is the bordism as a manifold with boundary is given by the cylinder on Y_2 with its G-structure constant in the time direction :

$$C(\phi) := (Y_2)_\phi := (Y_2 \times [0,1], Q, \beta) : Y_1 \rightsquigarrow Y_2.$$

Here (Q, β) is $P_2 \times [0,1]$ with its G-structure induced by α_2, in other words it is the pullback of the G-structure under the projection $Y_2 \times [0,1] \to Y_2$. However, as a bordism its incoming boundary diffeomorphism is changed by ϕ:

$$(Y_2, P_2, \alpha_2) \to (Y_2 \times [0,1], Q, \beta)|_{Y_2 \times \{0\}} \quad (Y_1, P_1, \alpha_1) \xrightarrow{\phi} (Y_2, P_2, \alpha_2) = (Y_2 \times [0,1], Q, \beta)|_{Y_2 \times \{1\}}.$$

Lemma 5.1.18. *Let $\mathcal{G}$ be the groupoid of $(n-1)$-dimensional G-manifolds and G-diffeomorphisms. The mapping cylinder construction gives a functor $C : \mathcal{G} \to \mathrm{Bord}^G_{n,n-1}$.*

Proof. The functor C is tautological on objects and given by Definition 5.1.17 on morphisms. We have to show C is a functor. It clearly preserves identities. Now take G-manifolds (Y_1, P_1, α_1), (Y_2, P_2, α_2), (Y_3, P_3, α_3) and G-diffeomorphisms $\phi_1 : Y_1 \to Y_2$ and $\phi_2 : Y_2 \to Y_3$. We have to show that $(Y_3)_{\phi_2} \cup_{\phi_2} (Y_2)_{\phi_1}$ is G-diffeomorphic relative boundary to $(Y_3)_{\phi_2 \circ \phi_1}$. After making sure both cylinders are of the same length, we take the G-diffeomorphism given by the identity on the Y_3 part of $(Y_3)_{\phi_2} \cup_{\phi_2} (Y_2)_{\phi_1}$ and by ϕ_2 on the Y_2 part of $(Y_3)_{\phi_2} \cup_{\phi_2} (Y_2)_{\phi_1}$. This is a smooth map given that $Y_2 \times [0,1]$ and $Y_3 \times [0,1]$ are glued together using ϕ_2. □

Remark 5.1.19. We could have also made the construction by putting the diffeomorphism ϕ on the other side of the bordism. This would give the same functor because the two bordisms are diffeomorphic relative boundary by ϕ. However, given our convention of the direction of the embedding from the time slices into the bordism, we would need to put an inverse on ϕ to make it functorial.

Restricting the functor C to a single object of $\mathcal{G}$ and only allowing G-diffeomorphisms that are the identity as ordinary diffeomorphisms gives:

Corollary 5.1.20. *Let Y be a $(n-1)$-dimensional manifold. Then there is a functor $C : G\text{-}\mathrm{Str}(Y) \to \mathrm{Bord}^G_{n,n-1}$ which maps a G-structure on Y to its corresponding object in the bordism category.*

We define an *isotopy* of G-diffeomorphisms $\phi_0, \phi_1 : (Y_1, P_1, \alpha_1) \to (Y_2, P_2, \alpha_2)$ to be a homotopy between them that stays within the space of G-diffeomorphisms. In other words, it consists of a family of diffeomorphisms $\phi_\bullet : Y_1 \times [0,1] \to Y_2$ together with an isomorphism of G-structures $\pi^*(P_1, \alpha) \cong \phi_\bullet^*(P_2, \alpha_2)$ which is compatible with the isomorphisms $(P_1, \alpha) \cong \phi_0^*(P_2, \alpha_2)$ and $(P_1, \alpha) \cong \phi_1^*(P_2, \alpha_2)$ on either side. Here $\pi : Y_2 \times [0,1] \to Y_2$ denotes the projection.

Lemma 5.1.21. *Let ϕ_0 and ϕ_1 be isotopic G-diffeomorphisms $Y_1 \to Y_2$. Then $C(\phi_0) = C(\phi_1)$.*

Proof. We have to show that the bordisms $(Y_2)_{\phi_0}$ and $(Y_2)_{\phi_1}$ are G-diffeomorphic relative boundary. Let ϕ_t be an isotopy between ϕ_0 and ϕ_1. Define a G-diffeomorphism of $Y_2 \times [0, 1/2]$ by

$$\psi(y_2, t) = \phi_{1-t} \circ \phi_t^{-1}(y_2).$$

Note that $\psi_{1/2} = \mathrm{id}_{Y_2}$ and $\psi_0 = \phi_1 \circ \phi_0^{-1}$ and so ψ is a G-diffeomorphism relative boundary between the two mapping cylinders $(Y_2)_{\phi_0}$ and $(Y_2)_{\phi_1}$. □

On the groupoid of G-structures the star operation was covariant, but a dual functor is contravariant. However, they are still mapped to each other in the appropriate sense:

Lemma 5.1.22. *Let Y be a closed $(n-1)$-dimensional manifold and $f : (P_1, \alpha_1) \to (P_2, \alpha_2)$ an isomorphism of G-structures on Y. Then*

$$C(f^{-1*}) = C(f)^\vee$$

where on the left hand side the superscript star means the covariant functor $Y \mapsto Y^*$ on G-structures and on the right hand side $\vee$ means the dual in the bordism category.

Proof. It is easy to show that

$$\begin{array}{ccc} Y_1^* \sqcup Y_1 & \xrightarrow{\mathrm{ev}_{Y_1}} & \emptyset \\ {\scriptstyle C(f^{-1*})}\uparrow & & \uparrow{\scriptstyle \mathrm{ev}_{Y_2}} \\ Y_2^* \sqcup Y_1 & \xrightarrow{C(f)} & Y_2^* \sqcup Y_2 \end{array}$$

commutes. □

5.2 Involutions and Hermiticity

In Chapter 4 we studied orientation-reversal $\mathbb{Z}/2$-actions on the groupoid of G-structures on a fixed vector bundle. We also studied what we call Hermitian pairings on such G-structures. Here we leverage our knowledge from this groupoid to the bordism category.

We start with some abstract lemmas showing that the 2-group action of Section 4.4 lifts to a symmetric monoidal action on the bordism category. Proposition 4.5.9 then gives an orientation $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$ on $\mathrm{Bord}^G_{n,n-1}$ for an orientation-graded structure group $G \to \hat{G} \xrightarrow{\hat{\rho}} O_n$ in the sense of Section 4.5. The first step is to construct functors between bordism categories induced by morphisms between prestacks $G_1\text{-Str} \Rightarrow G_2\text{-Str}$. Recall that here $G\text{-Str}$ refers to the prestack $\mathrm{VectBun}_n \to \mathrm{Gpd}$ on the category of vector bundles and vector bundle isomorphisms.

Lemma 5.2.1. *Let $G_1 \to O_n$ and $G_2 \to O_n$ be structure groups and $\phi : G_1\text{-Str} \Rightarrow G_2\text{-Str}$ a morphism of prestacks. Then there is an induced symmetric monoidal functor $\zeta(\phi) : \mathrm{Bord}^{G_1}_{n,n-1} \to \mathrm{Bord}^{G_2}_{n,n-1}$.*

Proof. Let $\phi : G_1\text{-Str} \Rightarrow G_2\text{-Str}$ be a natural 2-transformation. Explicitly this means that for every vector bundle $\eta \to X$ we get a functor

$$\phi_{(X,\eta)} : G_1\text{-Str}(X, \eta) \to G_2\text{-Str}(X, \eta)$$

and for every morphism of vector bundles $f : (X_1, \eta_1) \to (X_2, \eta_2)$ a natural transformation filling the square

$$\begin{tikzcd} G_1\text{-Str}(X_1,\eta_1) \arrow[r, "\phi_{(X_1,\eta_1)}"] & G_2\text{-Str}(X_1,\eta_1) \\ G_1\text{-Str}(X_2,\eta_2) \arrow[u, "f^*"] \arrow[r, "\phi_{(X_2,\eta_2)}"'] \arrow[ur, phantom, "\Leftarrow \phi_f"] & G_2\text{-Str}(X_2,\eta_2) \arrow[u, "f^*"'] \end{tikzcd} \tag{5.2}$$

satisfying some further conditions. Note that if $\eta \to Y$ is a rank $n-1$ vector bundle and $f : (Y, \eta \oplus \underline{\mathbb{R}}) \to (Y, \eta \oplus \underline{\mathbb{R}})$ is the morphism in $\mathrm{VectBun}_n$ given by the identity on Y and $f : \mathrm{id}_\eta \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$ on the vector bundle, then $G\text{-Str}(f)(P,\alpha) = (P,\alpha)^*$. Therefore given a G_1-structure (P,α) on $(Y, \eta \oplus \underline{\mathbb{R}})$, ϕ_f provides an isomorphism of G_2-structures on $(Y, \eta \oplus \underline{\mathbb{R}})$ between $\phi_{(Y,\eta\oplus\underline{\mathbb{R}})}((P,\alpha)^*)$ and $(\phi_{(Y,\eta\oplus\underline{\mathbb{R}})}(P,\alpha))^*$.

Now let

$$(Y_2, \eta_2 = TY_2 \oplus \underline{\mathbb{R}}, P_2, \alpha_2) \hookrightarrow (X, \xi = TX, Q, \beta) \hookleftarrow (Y_1, \eta_1 = TY_1 \oplus \underline{\mathbb{R}}, P_1, \alpha_1)$$

be a morphism in the bordism category. We apply ϕ to this diagram to get a new morphism as follows. Let j_1 denote the morphism in $\mathrm{VectBun}_n$ given by the inclusion $(Y_1, TY_1 \oplus \eta_1) \to (X, TX)$ and let i_1 denote its enhancement to a G_1-embedding, i.e. an isomorphism $i_1 : j_1^*(Q,\beta) \to (P_1,\alpha_1)^*$ of G_1-structures on Y_1. This gives an isomorphism of G_2-structures $\phi_{Y_1}(i_1) : \phi_{Y_1}(j_1^*(Q,\beta)) \to \phi_{Y_1}((P_1,\alpha_1)^*)$ on X. To show how $\phi_{Y_1}(i_1)$ can make j_1 into a G_2-embedding for the new G_2-structures, we have to provide compatibility data with the G_2-structure $\phi_X(Q,\beta)$ on X. For this we apply the diagram (5.2) to $f = j_1$ to obtain the isomorphisms

$$j_1^*\phi_X(Q,\beta) \xrightarrow{\phi_{j_1}(Q,\beta)} \phi_{Y_1} j_1^*(Q,\beta) \xrightarrow{\phi_{Y_1}(i_1)} \phi_{Y_1}((P_1,\alpha_1)^*) \cong (\phi_{Y_1}(P_1,\alpha_1))^*$$

of G_2-structures on Y_1. This defines the G_2-embedding of $\phi_{Y_1}(P_1,\alpha_1)$ into $\phi_X(Q,\beta)$. Applying the same procedure to Y_2 but without the star we obtain a G_2-bordism $\zeta(\phi)(X) = \phi_X(Q,\beta)$ from $\zeta(\phi)(Y_1) = \phi_{Y_1}(P_1,\alpha_1)$ to $\zeta(\phi)(Y_2) = \phi_{Y_2}(P_2,\alpha_2)$. This is obviously compatible with the composition of bordisms. It also sends the identity bordism on (Y,P,α) to the identity bordism on $(Y, \phi_Y(P,\alpha))$

We have to show that $\zeta(X)$ is well-defined up to G_2-diffeomorphism relative boundary. If $f : X \to X'$ is a diffeomorphism between bordisms, it is useful to consider the diagram

$$\begin{tikzcd} & G_1\text{-Str}(Y_1) \arrow[r, "\phi_{Y_1}"] & G_2\text{-Str}(Y_1) & \\ {} \arrow[r, phantom, "\Leftarrow"] & G_1\text{-Str}(X) \arrow[u, "j_1^*"] \arrow[r, "\phi_X"] \arrow[ur, phantom, "\Leftarrow \phi_{j_1}"] & G_2\text{-Str}(X) \arrow[u, "j_1^*"'] \arrow[r, phantom, "\Rightarrow"] & {} \\ & G_1\text{-Str}(X') \arrow[u, "f^*"] \arrow[r, "\phi_{X'}"] \arrow[ur, phantom, "\Leftarrow \phi_f"] \arrow[uu, bend left=80, "j_1'^*"] & G_2\text{-Str}(X') \arrow[u, "f^*"'] \arrow[uu, bend right=80, "j_1'^*"'] & \end{tikzcd} .$$

The diagram becomes sphere-shaped if we further fill the outside by $\phi_{j_1'}$ and the result commutes because ϕ is natural. Now let $f : X \to X'$ be a G_1-diffeomorphism of bordisms so that $f \circ j_1 = j_1'$ and we are given an isomorphism of G_1-structures $\lambda : f^*(Q',\beta') \cong (Q,\beta)$ such that $i_1' : j_1'^*(Q',\beta') \to (P_1,\alpha_1)$ agrees with the composition

$$j_1'^*(Q',\beta') = (f \circ j_1)^*(Q',\beta') \cong j_1^* f^*(Q',\beta) \xrightarrow{j_1^*(\lambda)} j_1^*(Q,\beta) \xrightarrow{i_1} (P_1,\alpha_1)^*.$$

We want to show that the same diffeomorphism lifts to a G_2-diffeomorphism relative boundary given by an isomorphism of G_2-structures $\mu : f^*\phi_{X'}(Q',\beta') \cong \phi_X(Q,\beta)$ such that the diagram

$$\begin{tikzcd}
\phi_{Y_1}((P_1,\alpha_1)^*) & \phi_{Y_1} j_1^*(Q,\beta) \arrow[l, "\phi_{Y_1}(i_1)"'] \\
\phi_{Y_1} j_1'^*(Q',\beta') \arrow[u, "\phi_{Y_1}(i_1')"] & j_1^*\phi_X(Q,\beta) \arrow[u, "\phi_{j_1}(Q,\beta)"'] \\
j_1'^*\phi_{X'}(Q',\beta') \arrow[u, "\phi_{j_1'}(Q',\beta')"] & j_1^* f^*\phi_{X'}(Q',\beta')) \arrow[l] \arrow[u, "j_1^*(\mu)"']
\end{tikzcd}$$

commutes. Just as we did for the embeddings $j_1 : Y_1 \hookrightarrow X$, we take μ to be given by the composition

$$f^*\phi_{X'}(Q',\beta') \xrightarrow{\phi_f} \phi_X f^*(Q',\beta') \xrightarrow{\phi_X(\lambda)} \phi_X(Q',\beta').$$

It then follows by naturality of $j_1^* f^* \cong j_1'^*$, the fact that ϕ_{j_1} is a natural transformation and the condition on λ that the above diagram commutes.

Next we need to make $\zeta(\phi)$ into a symmetric monoidal functor. If $(Y_1,\eta_1),(Y_2,\eta_2)$ are rank n vector bundles, then $\eta_1 \sqcup \eta_2 \to Y_1 \sqcup Y_2$ makes the inclusions of the two factors into morphisms in $\mathrm{VectBun}_n$ and this forms the coproduct in this category. If $(Y_1,\eta_1,P_1,\alpha_1),(Y_2,\eta_2,P_2,\alpha_2)$ are two G-structures, then $P_1 \sqcup P_2$ is a G-structure on $\eta_1 \sqcup \eta_2$ and this defines an equivalence of groupoids $G\text{-Str}(Y_1 \sqcup Y_2) \cong G\text{-Str}(Y_1) \times G\text{-Str}(Y_2)$. The projections to the two factors are moreover compatible with G-Str applied to the inclusions of Y_1 and Y_2, i.e. G-Str sends coproducts to products. We obtain a diagram

$$\begin{tikzcd}
 & G_1\text{-Str}(Y_1) \times G_1\text{-Str}(Y_2) \arrow[dl] \arrow[d, "\sim"] \arrow[dr] & \\
G_1\text{-Str}(Y_1) \arrow[d, "\phi_{Y_1}"] & G_1\text{-Str}(Y_1 \sqcup Y_2) \arrow[l] \arrow[r] \arrow[d, "\phi_{Y_1 \sqcup Y_2}"] & G_1\text{-Str}(Y_2) \arrow[d, "\phi_{Y_2}"] \\
G_2\text{-Str}(Y_1) & G_2\text{-Str}(Y_1 \sqcup Y_2) \arrow[l] \arrow[r] & G_2\text{-Str}(Y_2) \\
 & G_2\text{-Str}(Y_1) \times G_2\text{-Str}(Y_2) \arrow[ul] \arrow[u, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

which tells us that $\phi_{Y_1 \hookrightarrow Y_1 \sqcup Y_2}$ and $\phi_{Y_2 \hookrightarrow Y_1 \sqcup Y_2}$ provide a natural isomorphism between the diagonal matrix $\phi_{Y_1} \times \phi_{Y_2}$ and the functor $\phi_{Y_1 \sqcup Y_2}$ considered as a functors $G_1\text{-Str}(Y_1) \times G_1\text{-Str}(Y_2) \to G_2\text{-Str}(Y_1) \times G_2\text{-Str}(Y_2)$. Moreover, given a disjoint union of three objects Y_1, Y_2, Y_3 in $\mathrm{VectBun}_n$ the two ways to embed, say, Y_1 as

$$Y_1 \hookrightarrow Y_1 \sqcup Y_2 \hookrightarrow Y_1 \sqcup Y_2 \sqcup Y_3 \quad \text{and} \quad Y_1 \hookrightarrow Y_1 \sqcup Y_3 \hookrightarrow Y_1 \sqcup Y_2 \sqcup Y_3$$

are equal. This gives a relation between the composition of natural transformations obtained by applying ϕ to this diagram saying that

$$\begin{tikzcd}
& \phi_{Y_1} \times \phi_{Y_2 \sqcup Y_3} \arrow[dl, Rightarrow] & \\
\phi_{Y_1 \sqcup Y_2 \sqcup Y_3} & & \phi_{Y_1} \sqcup \phi_{Y_2} \sqcup \phi_{Y_3} \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
& \phi_{Y_1 \sqcup Y_2} \times \phi_{Y_3} \arrow[ul, Rightarrow] &
\end{tikzcd} \tag{5.3}$$

commutes.

In particular if Y_1, Y_2 and Y_3 are $(n-1)$-dimensional closed manifolds with G-structures $(P_1, \alpha_1), (P_2, \alpha_2)$ and (P_3, α_3), then $\phi_{Y_1 \sqcup Y_2}$ and $\phi_{Y_1} \times \phi_{Y_2} \cong \phi_{Y_1 \sqcup Y_2}$ provide a natural isomorphism of G-structures

$$\zeta_{Y_1,Y_2} : \phi(P_1, \alpha_1) \sqcup \phi(P_2, \alpha_2) \cong \phi((P_1, \alpha_1) \sqcup (P_2, \alpha_2))$$

which induces an isomorphism in the bordism category by the mapping cylinder functor C of Corollary 5.1.20. It follows by the diagram (5.3) that this monoidal structure on the functor $\zeta(\phi)$ is associative in $(P_1, \alpha_1), (P_2, \alpha_2)$ and (P_3, α_3).

We now show that the monoidal data is natural. Let (X_1, Q_1, β_1) be a bordism from (Y_1, P_1, α_1) to (Y_1', P_1', α_1') and (X_2, Q_2, β_2) a bordism from (Y_2, P_2, α_2) to (Y_2', P_2', α_2'). Then the fact that under the equivalences ζ_{Y_1,Y_2} the bordism $\zeta(X_1 \sqcup X_2)$ becomes equal to $\zeta(X_1) \sqcup \zeta(X_2)$ follows by several applications of naturality of ϕ_f. The monoidal functor is clearly symmetric and so the proof is finished. $\square$

The above lemma also allows us to compare different bordism categories.

Corollary 5.2.2. *Let $G_1 \to O_n$ and $G_2 \to O_n$ be structure groups and $\rho : G_1 \to G_2$ a homomorphism commuting with the structure maps to O_n. Then there is an induced symmetric monoidal functor* $\mathrm{Bord}_{n,n-1}^{G_1} \to \mathrm{Bord}_{n,n-1}^{G_2}$.

Proof. It is straightforward to check that

$$P_1 \mapsto P_1 \times_\rho G_2$$

defines a morphism of prestacks $G_1\text{-Str} \Rightarrow G_2\text{-Str}$. $\square$

Lemma 5.2.3. *Let $\mathbf{Aut}(G\text{-Str})$ be the 2-group of automorphisms of the prestack $G\text{-Str}$ and $\mathbf{Aut}(\mathrm{Bord}_{n,n-1}^G)$ the 2-group of symmetric monoidal automorphisms of the bordism category. There is a 2-group homomorphism $\zeta : \mathbf{Aut}(G\text{-Str}) \to \mathbf{Aut}(\mathrm{Bord}_{n,n-1}^G)$ in agreement with the construction of Lemma 5.2.1.*

Proof. By taking $G_1 = G_2 = G$ in the previous lemma we have defined ζ on objects. Let $\sigma : \phi \Rrightarrow \psi$ be a natural modification between two natural 2-transformations $\phi : G\text{-Str} \Rightarrow G\text{-Str}$ and $\psi : G\text{-Str} \Rightarrow G\text{-Str}$. Explicitly this means that for all rank n vector bundles $\eta \to X$ we have a natural transformation

$$\begin{tikzcd}
G\text{-Str}(X, \eta) \arrow[r, bend left, "\phi_{(X,\eta)}"] \arrow[r, bend right, "\psi_{(X,\eta)}"'] \arrow[r, phantom, "\Downarrow \sigma_{(X,\eta)}"] & G\text{-Str}(X, \eta)
\end{tikzcd}$$

such that for all vector bundle isomorphisms $f : (X_1, \eta_1) \to (X_2, \eta_2)$ and all G-structures (P_2, α_2) on (X_2, η_2) the diagram

$$\begin{tikzcd} f^*\phi_{(X_2,\eta_2)}(P_2,\alpha_2) \arrow[d, "f^*\sigma_{(X_2,\eta_2)}(P_2,\alpha_2)"'] & \phi_{(X_1,\eta_1)}f^*(P_2,\alpha_2) \arrow[l, "\phi_f(P_2,\alpha_2)"'] \arrow[d, "\sigma_{f^*(X_2,\eta_2)}(P_2,\alpha_2)"] \\ f^*\psi_{(X_2,\eta_2)}(P_2,\alpha_2) & \psi_{(X_1,\eta_1)}f^*(P_2,\alpha_2) \arrow[l, "\psi_f(P_2,\alpha_2)"'] \end{tikzcd} \tag{5.4}$$

commutes. We have to define a monoidal natural transformation $\zeta(\sigma) : \zeta(\phi) \Rightarrow \zeta(\psi)$ between symmetric monoidal functors $\mathrm{Bord}^G_{n,n-1} \to \mathrm{Bord}^G_{n,n-1}$. If (Y_1, P_1, α_1) is an object of the bordism category, we apply the mapping cylinder construction of Corollary 5.1.20 to the morphism $\phi_{Y_1}(P_1, \alpha) \xrightarrow{\sigma_{Y_1}(P_1,\alpha_1)} \psi_{Y_1}(P_1, \alpha)$ of G-structures to obtain a morphism in the bordism category. It similarly follows by that corollary that this is natural in (Y_1, P_1, α_1).

Monoidality of $\zeta(\sigma)$ already follows on the level of G-structures as follows. Let $(Y_1, \eta_1, P_1, \alpha_1)$, $(Y_2, \eta_2, P_2, \alpha_2)$ be G-structures. We apply naturality of σ_Y in Y to get a compatibility with disjoint union. The result is that under the natural isomorphism $\phi_{Y_1 \sqcup Y_2} \cong \phi_{Y_1} \times \phi_{Y_2}$ we get that $\sigma_{Y_1 \sqcup Y_2}$ is mapped to $\sigma_{Y_1} \times \sigma_{Y_2}$. We have thus defined the morphism $\zeta(\sigma) : \zeta(\phi) \Rightarrow \zeta(\psi)$ between symmetric monoidal functors.

To make ζ into a monoidal functor, we have to give compatibility data for the composition of two natural transformations $\phi : G\text{-Str} \Rightarrow G\text{-Str}$ and $\phi' : G\text{-Str} \Rightarrow G\text{-Str}$ on the bordism category. If (Y_1, P_1, α_1) is an object of the bordism category then $\zeta(\phi') \circ \zeta(\phi)(Y_1, P_1, \alpha_1) = \zeta(\phi' \circ \phi)(Y_1, P_1, \alpha_1)$ are already equal. This is moreover clearly compatible with respect to $\sigma : \phi \Rrightarrow \psi$ and $\sigma' : \phi' \Rrightarrow \psi'$ and associative, already on the level of G-structures. To show that this defines a natural isomorphism $\zeta(\phi') \circ \zeta(\phi) \cong \zeta(\phi' \circ \phi)$ we still have to show compatibility with bordisms. For this we check compatibility between the composition

$$(\phi' \circ \phi)_{Y_2}(P_2, \alpha_2) \xrightarrow{(\phi'\circ\phi)_{Y_2}(i_2)} (\phi' \circ \phi)_{Y_2} j_2^*(Q, \beta) \xrightarrow{(\phi'\circ\phi)_{j_2}} (\phi' \circ \phi)_X(Q, \beta)$$

and the functor $\zeta(\phi')$ applied to the inclusion of the boundary $\zeta(\phi)(Y)$ given by

$$\phi_{Y_2}(P_2, \alpha_2) \xrightarrow{\phi_{Y_2}(i_2)} \phi_{Y_2} j_2^*(Q, \beta) \xrightarrow{\phi_{j_2}} \phi_X(Q, \beta)$$

which is clear and goes analogously for the incoming boundary. To show $\zeta(\phi') \circ \zeta(\phi) \cong \zeta(\phi' \circ \phi)$ is a monoidal natural transformation we have to show compatibility with disjoint union. This can again be done on the level of G-structures, where it is clear. □

Lemma 5.2.4. *The homomorphism ζ is compatible under the mapping cylinder construction with the action of $\mathbf{Aut}(G\text{-Str})$ on $G\text{-Str}(Y)$ for Y an $(n-1)$-dimensional manifold.*

Proof. Let $\phi \in \mathbf{Aut}(G\text{-Str})$ and let $\mathcal{G}$ denote the groupoid of $(n-1)$-dimensional G-manifolds and G-diffeomorphisms. We have to provide a natural transformation filling the square

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\phi"] \arrow[d, "C"] & \mathcal{G} \arrow[d, "C"] \\ \mathrm{Bord}^G_{n,n-1} \arrow[r, "\zeta(\phi)"] & \mathrm{Bord}^G_{n,n-1} \end{tikzcd} .$$

Note that on objects the diagram commutes and so it we will try and show the diagram strictly commute by computing it on morphisms.

Let $f : Y_1 \to Y_2$ be a diffeomorphism together with the datum $\lambda : f^*(P_2, \alpha_2) \to (P_1, \alpha_1)$ making it into a G-diffeomorphism. Let $\pi : Y_2 \times [0,1] \to Y_2$ denote the projection and $i_1, i_2 : Y_2 \to Y_2 \times [0,1]$ the inclusions of the two boundaries as morphisms in $\mathrm{VectBun}_n$. Note that the mapping cylinder of f is the G-structure $\pi^*(P_2, \alpha_2)$ on $Y_2 \times [0,1]$ with the following boundary G-embeddings. Using that $\pi i_2 = \mathrm{id}_{Y_2}$, we get a canonical identification $i_2^* \pi^*(P_2, \alpha_2) \cong (P_2, \alpha)$. On the incoming side we similarly use $\pi i_1 = \mathrm{id}_{Y_2}$ and then compose with λ to get the further identification with (P_1, α_1).

Using the above description of the mapping torus, we see that going through the upper right part of the diagram gives the manifold with boundary $\pi^* \phi_{Y_2}(P_2, \alpha_2)$ while the lower left part of the diagram gives $\phi_{Y_2 \times [0,1]} \pi^*(P_2, \alpha)$. These two manifolds are related by the G-diffeomorphism which is the identity as a diffeomorphism and equal to $\phi_\pi(P_2, \alpha_2)$ on the G-structure level. We have to show this commutes with the boundary identifications using the definition of $\zeta(\phi)$ applied to a bordism given in the proof of Lemma 5.2.1. On the incoming boundary this follows from the fact that

$$\begin{tikzcd}
j_1^* \pi^* \phi_{Y_2}(P_2, \alpha_2) \arrow[rr] \arrow[d, "j_1^* \phi_\pi(P_2, \alpha_2)"] & & \phi_{Y_2}(P_2, \alpha_2) \\
j_1^* \phi_{Y_2 \times [0,1]} \pi^*(P_2, \alpha_2) \arrow[rr, "\phi_{j_1}(\pi^*(P_2, \alpha_2))"] & & \phi_{Y_2} j_1^* \pi^*(P_2, \alpha_2) \arrow[u]
\end{tikzcd}$$

commutes using that ϕ is a natural 2-transformation. The proof is similar for the outgoing boundary.

If $\sigma : \phi \Rrightarrow \psi$ is a natural modification then $\zeta(\sigma)$ is already defined through the mapping cylinder construction. The composition data induced by the composition of two natural transformations $\phi : G\text{-Str} \Rightarrow G\text{-Str}$ and $\phi' : G\text{-Str} \Rightarrow G\text{-Str}$ is the identity on objects and therefore also compatible with the mapping cylinder construction. □

From now on we abuse notation and denote the automorphisms $\zeta(\phi)$ of the bordism category induced by $\phi : G\text{-Str} \Rightarrow G\text{-Str}$ again by ϕ.

Remark 5.2.5. The proof of the above lemmas is insensitive to the origin of the morphism $(Y_1, TY_1 \oplus \eta_1) \to (X, TX)$ in $\mathrm{VectBun}_n$. In particular, the same proof would work in a bordism category defined using embeddings of collars of time slices.

We obtain using Proposition 4.4.11 and Lemma 5.2.3

Lemma 5.2.6. *The automorphism* 2*-group* $\mathbf{Aut}(G \to O_n)$ *of the prestack of* G*-structures acts symmetric monoidally on the bordism category.*

Using Proposition 4.5.9, Lemma 5.2.3 and Lemma 5.2.4 we get:

Corollary 5.2.7. *Let* $G \to \hat{G} \to O_n$ *be an orientation-graded structure group. Then the* $\mathbb{Z}/2$*-action of Lemma 4.5.4 induces a symmetric monoidal* $\mathbb{Z}/2$*-action on* $\mathrm{Bord}^G_{n,n-1}$. *If* $f : Y_1 \to Y_2$ *is* G*-diffeomorphism of* $(n-1)$*-dimensional* G*-manifolds, then* $C(\overline{f}) = \overline{C(f)}$.

Moreover, using Corollary 4.5.11 we get:

Corollary 5.2.8. *Let* K *be a fermionic group. Then the orientation-reversal and spin flip give a symmetric monoidal* $\mathbb{Z}/2 \times B\mathbb{Z}/2$*-action on* $\mathrm{Bord}^{G_n(K)}_{n,n-1}$.

Note that by construction of the action from Lemma 5.2.6, the $B\mathbb{Z}/2$-action in the above corollary maps a time slice (Y^{n-1}, P, α) to the mapping torus Y_c of the spin flip action on P. More precisely, Y_c is $Y \times [0,1]$ as a manifold with G-structure, but the incoming boundary is modified with the spin diffeomorphism of (Y^{n-1}, P, α) which is the identity on the manifold Y but given by multiplication with $c \in G_n(K)$ on P. This $B\mathbb{Z}/2$-action is an essential ingredient in defining what it means for a topological field theory to have a connection between spin and statistics, see Section 6.3.

Example 5.2.9. Let K be a bosonic internal symmetry group. Then the induced $B\mathbb{Z}/2$-action on $\mathrm{Bord}_{n,n-1}^{G_n(K)}$ is trivial.

We also record the following immediate corollary of Proposition 4.5.10:

Corollary 5.2.10. *Let K be a fermionic group. Then of the four orientation-gradings $\hat{G}_n(K)$, $\hat{G}_n^-(K)$, $G_n^{(1)}(K)$ and $G_n^{(3)}(K)$, we have that*

1. *$\hat{G}_n(K)$ and $G_n^{(3)}(K)$ induce equivalent $\mathbb{Z}/2$-actions on $\mathrm{Bord}_{n,n-1}^{G_n(K)}$;*
2. *$\hat{G}_n^-(K)$ and $G_n^{(1)}(K)$ induce equivalent $\mathbb{Z}/2$-actions on $\mathrm{Bord}_{n,n-1}^{G_n(K)}$;*
3. *the $\mathbb{Z}/2$-actions induced by $\hat{G}_n(K)$ and $G_n^{(1)}(K)$ differ by the $B\mathbb{Z}/2$-action on the identification $Y \cong \overline{\overline{Y}}$.*

Lemma 5.2.11. *Let $\phi : G\text{-Str} \Rightarrow G\text{-Str}$ be a prestack automorphism and (Y, P, α) a closed $(n-1)$-dimensional manifold with G-structure. Then the isomorphism $\phi_Y((P,\alpha)^*) \cong (\phi_Y(P,\alpha))^*$ induced by the vector bundle isomorphism $\mathrm{id}_{TY} \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$ on $TY \oplus \underline{\mathbb{R}} \to Y$ induces the unique isomorphism in the bordism category expressing the fact that the monoidal functor $\phi : \mathrm{Bord}_{n,n-1}^G \to \mathrm{Bord}_{n,n-1}^G$ preserves duals.*

Proof. We have to show that the diagram

$$\begin{tikzcd}
\phi(Y^*) \sqcup \phi(Y) \arrow[r, "\sim"] & \phi(Y^* \sqcup Y) \arrow[d, "\phi(\mathrm{ev}_Y)"] \\
\phi(Y)^* \sqcup \phi(Y) \arrow[u, "\sim"] \arrow[r, "\mathrm{ev}_{\phi(Y)}"] & \emptyset
\end{tikzcd}$$

commutes, which follows by direct inspection. □

Corollary 5.2.12. *Let $G \to \hat{G} \to O_n$ be an orientation-graded structure group and (Y, P, α) a closed $(n-1)$-dimensional manifold with G-structure. Recall that the identity map on P induces an isomorphism $\overline{Y}^* \cong \overline{Y^*}$ of G-structures. This isomorphism induces the unique isomorphism in the bordism category expressing the fact that the monoidal functor $Y \mapsto \overline{Y}$ preserves duals.*

Proof. It follows because the identity map on P that induces the isomorphism $\overline{Y}^* \cong \overline{Y^*}$ is induced by the vector bundle isomorphism $\mathrm{id}_{TY} \oplus -\mathrm{id}_{\underline{\mathbb{R}}}$, see the end of Section 4.5 for the functoriality of $\overline{(.)}$ in vector bundle maps. □

Let $G \to \hat{G} \to O_n$ be an orientation-graded structure group and consider $\mathrm{Bord}_{n,n-1}^G$ to come equipped with the induced symmetric monoidal anti-involution $Y \mapsto \overline{Y}^*$ through the process described in Section 2.2. Then the definition of a Hermitian pairing on an object of the category $\mathrm{Bord}_{n,n-1}^G$ we considered in Definition 2.3.4 reduces to the following concrete definition, also see the discussion around the diagram (2.3).

Definition 5.2.13. Let (Y, P, α) be a $(n-1)$-dimensional manifold with G-structure on $TY \oplus \underline{\mathbb{R}}$. Then a *Hermitian pairing on* Y is an isomorphism $h : Y \cong \overline{Y}^*$ in the bordism category such that the composition

$$Y \xrightarrow{h} \overline{Y}^* \xrightarrow{\overline{h}^{*-1}} \overline{(\overline{Y}^*)}^* \cong Y \tag{5.5}$$

is equal to the identity on Y.

Here the isomorphism $\overline{(\overline{Y}^*)}^* \cong Y$ is given by the inverse of η_Y for the involution $d = \overline{(.)}^*$. We recall from Section 2.2 that it is given by the composition

$$\overline{(\overline{Y}^*)}^* \cong \overline{\overline{Y}}^{**} \cong \overline{\overline{Y}} \cong Y,$$

where the first isomorphism follows from the fact that the monoidal functor $Y \mapsto \overline{Y}$ preserves duals, the second follows by uniqueness of duals, and the last follows by the data of being a $\mathbb{Z}/2$-action. For the G-structure (P, α) on Y, this isomorphism is explicitly given by the mapping cylinder of the G-structure isomorphism induced by

$$\overline{\overline{P}} = P \times_G \overline{G} \times_G \overline{G} \to P \quad [p, g_1, g_2] \mapsto p \cdot (g_1 \cdot g_2) \quad g_1, g_2 \in \overline{G}$$

as follows by Corollary 5.2.12, Lemma 5.1.16 and Lemma 4.5.4.

Now let h be a Hermitian pairing on the G-structure (P, α) in the sense of Definition 4.6.1. By Lemma 5.1.22 and the fact that the mapping cylinder construction preserves the bar, it follows that h induces a Hermitian pairing on Y in the bordism category.

Remark 5.2.14. In theory there could be many choices of Hermitian pairings on objects of the bordism category that do not come from constructions on the level of G-structures of a fixed manifold. Such Hermitian pairings would involve nontrivial diffeomorphisms or more general isomorphisms in the bordism category, such as h-cobordisms.

Remark 5.2.15. Let K be a fermionic group and consider $\mathrm{Bord}_{n,n-1}^{G_n(K)}$ as a symmetric monoidal anti-involutive category induced by one of the four involutions of Corollary 5.2.10. It follows from the fact that all these involutions assemble into $\mathbb{Z}/2 \times B\mathbb{Z}/2$-actions and Lemma 2.6.16 that the spin flip natural automorphism is anti-involutive. We see that the orientation-gradings $\hat{G}_n(K)$ and $\hat{G}_n^-(K)$ give opposite anti-involutions in the sense of Definition 2.6.5.

Example 5.2.16. Suppose $G_{n+1} \to O_{n+1}$ is a strict geometric representation and $\hat{G} = G^{(1)}$ is the induced orientation-grading on G_n giving us the $\mathbb{Z}/2$-action $Y \mapsto Y'$ on $\mathrm{Bord}_{n,n-1}^G$. Let $u \in G_{n+1}$ be a lift of a rotation in the plane spanned by e_n and e_{n+1}, which commutes with all elements of the strict pullback G_{n-1}. Then, Corollary 4.6.7 implies that there is an isomorphism $h_Y : Y \cong Y'^*$ in $\mathrm{Bord}_{n,n-1}^G$, but it need not be a Hermitian pairing in general. However, if we take $\hat{G} = \hat{G}_n(K)$ to be the orientation-graded structure group from a fermionic group and $u = 1 \otimes e_n e_{n+1}$, the isomorphism $h_Y : Y \cong \overline{Y}^*$ defined in Corollary 4.6.16 does induce a Hermitian pairing. Since the structure group $G_n(K)$ is the most relevant from the perspective of physics as well, we will mostly focus on this setting. In that case we will focus on the notion of positivity on $\mathrm{Bord}_{n,n-1}^G$ given by the positive Hermitian pairings of Definition 5.2.17.

Now let $(\mathrm{Bord}_{n,n-1}^G)_P \subseteq \mathrm{Herm}(\mathrm{Bord}_{n,n-1}^G)$ denote the bordism dagger category obtained from some monoidal positivity structure P on the symmetric monoidal anti-involutive category $\mathrm{Bord}_{n,n-1}^G$. Recall what this means explicitly: P is a collection of 'positive' Hermitian pairings $Y \to \overline{Y}^*$ we allow. The category $(\mathrm{Bord}_{n,n-1}^G)_P$ has objects pairs $(Y, h : Y \to \overline{Y}^*)$ where $h \in P$ and morphisms are given by bordisms, see Definitions 2.3.20 and 2.4.8. For simplicity, we will further assume that the positivity structure is closed under transfer.

The dagger of a bordism $X : Y_1 \rightsquigarrow Y_2$ seen as a morphism from (Y_1, h_1) to (Y_2, h_2) is given by the composition

$$Y_1 \xrightarrow{h_1} \overline{Y}_1^* \xrightarrow{\overline{X}^*} \overline{Y}_2^* \xrightarrow{h_2^{-1}} Y_2$$

in the bordism category. Note that in case for every object Y of the bordism category there is a unique positive Hermitian pairing $(h_Y : Y \to \overline{Y}) \in P$, the ordinary symmetric monoidal category underlying $(\mathrm{Bord}^G_{n,n-1})_P$ is equal to $\mathrm{Bord}^G_{n,n-1}$ on the nose. This is for example the case for the orientation-graded spacetime structure groups associated to fermionic groups: This will not change the resulting symmetric monoidal dagger bordism category up to symmetric monoidal unitary equivalence.

Definition 5.2.17. Let K be a fermionic group and $\hat{G} = \hat{G}_n(K)$ for $n \geq 1$ the orientation-grading of $G_n(K)$ from Definition 3.2.1. Let (Y, P, α) a $(n-1)$-dimensional manifold with G_n-structure on $TY \oplus \underline{\mathbb{R}}$ and let $\overline{Y}$ denote the orientation reversal of Y in the orientation-grading $\hat{G}$. The *positive Hermitian pairing* of Y is the isomorphism $Y \cong \overline{Y}^*$ in the bordism category induced by the isomorphism of G-structures of Corollary 4.6.16 under the mapping cylinder construction of Corollary 5.1.20.

Definition 5.2.18. Let K be a fermionic group. Equip $\mathrm{Bord}^{G_n(K)}_{n,n-1}$ with the $\mathbb{Z}/2$-action induced by the orientation-grading $\hat{G} = \hat{G}_n(K)$ for $n \geq 1$ and the induced symmetric monoidal anti-involution. The *bordism dagger category* $(\mathrm{Bord}^{G_n(K)}_{n,n-1})_P$ *with internal symmetry* K is the symmetric monoidal dagger category obtained by equipping the anti-involutive category $\mathrm{Bord}^{G_n(K)}_{n,n-1}$ with the positivity structure of Definition 5.2.17.

Note that this positivity structure of Definition 5.2.17 is clearly monoidal, so this indeed makes $\mathrm{Bord}^{G_n(K)}_{n,n-1}$ into a symmetric monoidal dagger category.

Example 5.2.19. If $\mathcal{C} = \mathrm{Bord}^{\mathrm{Spin}_n}_{n,n-1}$ is the spin bordism category, there does not exist as there is no bordism $X : M \to M$ such that $X \circ X$ is diffeomorphic to the spin flip in general. It also does not hold for Pin^+_n but it does work for Pin^-_n. In particular, Corollary 2.6.17 does not apply to this symmetric monoidal dagger category. Heuristically this means that unlike for super Hilbert spaces, there is no 'equivalent ungraded convention' for the spin bordism category in the sense of Definition 2.6.5. In fact, changing the isomorphism $\eta_Y : Y \cong d^2 Y$ with a spin flip will often have as a result that no Hermitian pairings exist at all on the $(n-1)$-dimensional closed spin manifold Y. For example, suppose $n = 1$ and Y is the positively oriented point. In the bordism category it has only one nontrivial automorphism which is given by the spin flip. Therefore there are only two possible isomorphisms $h : Y \to dY$; the positive Hermitian pairing of Y from Definition 5.2.17 and its composition with Y_c. The latter is another Hermitian pairing for the same η and so neither defines a Hermitian pairing for the spin-flipped η. Looking at Proposition 4.5.10, this also implies that the $\mathbb{Z}/2$-action $Y \mapsto Y'$ on the bordism category does not admit Hermitian pairings in general.

A dagger structure on the bordism category allows us to define what the double of a G-manifold with boundary is, compare [20, section 4.4]:

Definition 5.2.20. Let $X : \emptyset \rightsquigarrow Y$ be a morphism in the bordism dagger category $(\mathrm{Bord}^{G_n(K)}_{n,n-1})_P$. Then its *double* is the closed n-dimensional manifold with $G_n(K)$-structure $X^\dagger X : \emptyset \rightsquigarrow \emptyset$.

Note that doubles are exactly the weakly positive endomorphisms of the empty set in $(\mathrm{Bord}_{n,n-1}^{G_n(K)})_P$ in the sense of Definition 2.3.31.

Remark 5.2.21. Let $\hat{G}$ be an orientation-graded group and consider the induced symmetric monoidal anti-involutive category $\mathrm{Bord}_{n,n-1}^{G}$. Then the definition of a double of a morphism $X : \emptyset \rightsquigarrow Y$ makes sense after choosing a Hermitian pairing h on Y. The result strongly depends on h. For example, suppose we take X to be the macaroni $\mathrm{ev}_Y : \emptyset \rightsquigarrow Y \sqcup Y^*$ in the spin bordism dagger category obtained from Definition 5.2.18. Then changing the Hermitian pairing $Y \sqcup Y^*$ by a spin flip on only one of the two factors will change $X^\dagger \circ X$ from $Y \times S^1_{ap}$ to $Y \times S^1_{per}$.

Remark 5.2.22. We have defined the positivity structure on the bordism category by picking a single Hermitian pairing on every object. However, we still do not know whether the bordism category is minimal in general, because it could happen that the transfer of the positive Hermitian pairing h_{Y_1} on one manifold Y_1 under an isomorphism $Y_1 \cong Y_2$ in the bordism category results in a Hermitian pairing not equal to h_{Y_2}. We provide an example of a groupoid completion of a bordism category that is not minimal in dimension one in Example 6.2.13. However, in such low dimensions every isomorphism in the bordism category comes from a G-diffeomorphism. We will show in Proposition 5.2.24 that mapping cylinders of G-diffeomorphisms are always unitary. This shows bordism dagger categories with internal symmetry K for which every isomorphism in the $G_n(K)$-bordism category is G-diffeomorphic relative boundary to a mapping cylinder are all minimal.

Remark 5.2.23. Let K be a fermionic group. It follows by Remark 4.6.19 that Hermitian pairings are natural in G-diffeomorphisms in the following sense. Let Y_1^{n-1} and Y_2^{n-1} be closed $G_{n-1}(K)$-manifolds, which we consider as objects of $\mathrm{Bord}_{n,n-1}^{G_n(K)}$ using the induced $G_n(K)$-structures on $TY_1 \oplus \underline{\mathbb{R}}$ and $TY_2 \oplus \underline{\mathbb{R}}$. Note that if $n = 0$ not all objects of the bordism category arise this way, since $G_n(K) \to O_n$ need not be a strict geometric representation If $f : Y_1 \to Y_2$ is a $G_{n-1}(K)$-diffeomorphism it intertwines the Hermitian pairings on Y_1 and Y_2. A corollary of this is Proposition 5.2.24, which also applies in case $n = 1$.

Proposition 5.2.24. *Let K be a fermionic group and $G = G_n(K)$. If $\phi : Y_1 \to Y_2$ is a G-diffeomorphism, then $(Y_2)_\phi$ is unitary in the n-dimensional bordism dagger category.*

Proof. First suppose that $n > 1$ so that Hermitian pairings are defined by destabilizations. Remark 4.6.19 implies that if (Q_1, β_1) is a $G_{n-1}(K)$-structure on Y_1, (Q_2, β_2) a $G_{n-1}(K)$-structure on Y_2 and $\phi : Y_1 \to Y_2$ a $G_{n-1}(K)$-diffeomorphism, we get the commutative diagram of principal G_{n-1}-bundles

$$\begin{tikzcd} s^2 Q_1 \arrow[r, "\tilde{h}_u"] \arrow[d, "s^2 f"] & s^2 Q_1 \arrow[d, "s^2 f"] \\ s^2 Q_2 \arrow[r, "\tilde{h}_u"] & s^2 Q_2 \end{tikzcd}.$$

By desuspending this diagram, we see that Hermitian pairings are functorial in G-structures. We now apply the mapping cylinder functor and use $C(f^*) = C(f)^{\vee -1}$ and $C(\overline{f}) = \overline{C(f)}$ from Corollary 5.2.7. We get that

$$\begin{tikzcd} Y_1 \arrow[r, "h_{Y_1}"] \arrow[d, "C(f)"'] & \overline{Y}_1^* \\ Y_2 \arrow[r, "h_{Y_2}"] & \overline{Y}_2^* \arrow[u, "\overline{C(f)}^*"'] \end{tikzcd} \tag{5.6}$$

commutes. This is the definition of a unitary morphism in a dagger category defined from Hermitian pairings.

When $n = 1$, the formula for h_Y is different by a spin flip on $P_\pm$. However, if $f : P_1 \to P_2$ is a morphism of G-structures it maps $(P_1)_\pm$ to $(P_2)_\pm$. We see that when passing through the diagram (5.6) either both or none of the two Hermitian structions will give an extra spin flip multiplication and so $C(f)$ is still unitary. $\square$

In the symmetric monoidal dagger category $\mathrm{sHerm}_{\mathbb{C}}$ we saw that the canonical dual V^* of an object V is isomorphic to V but not unitarily. In $\mathrm{Herm}\,\mathrm{Bord}_{n,n-1}^{G_n(K)}$ the dual Y^* of a time slice Y is in general already non-unitarily not always isomorphic to Y unless K has time-reversal symmetry. However, the positivity structure P of Definition 5.2.17 still satisfies the similar requirement that the positive Hermitian pairing on the dual object Y^* differs from the dual of the positive Hermitian pairing on Y by the $B\mathbb{Z}/2$-action:

Theorem 5.2.25. *Let K be a fermionic group. The bordism dagger category $(\mathrm{Bord}^{G_n(K)})_P$ with internal symmetry K is weak fermionically dagger compact.*

Proof. We have shown that $\mathrm{Bord}^{G_n(K)}$ is a symmetric monoidal category with duals. The symmetric monoidal involution $Y \mapsto \overline{Y}$ induced by the orientation-grading through Corollary 5.2.7 gives a symmetric monoidal anti-involution $Y \mapsto \overline{Y}^*$ by the procedure described in Section 2.2. Only allowing the positive Hermitian pairings P on objects of $\mathrm{Bord}^{G_n(K)}$ in the sense of Definition 5.2.17 gives us a symmetric monoidal dagger category $(\mathrm{Bord}^{G_n(K)})_P \subseteq \mathrm{Herm}(\mathrm{Bord}^{G_n(K)})$ as described in Sections 2.3 and 2.4. To show that this symmetric monoidal dagger category is weak fermionically dagger compact, it suffices to show that it admits a dual functor, so that for every object Y the Hermitian pairing on Y^* differs from the dual Hermitian pairing of Y by the $B\mathbb{Z}/2$-action, compare Remark 2.9.6 This follows by applying the mapping cylinder construction to the result of Corollary 4.6.22. $\square$

Remark 5.2.26. In general, it is not a big achievement to show that a bordism dagger category is weak fermionically dagger compact. For example, the Hermitian completion of an anti-involutive bordism category with anti-involutive $B\mathbb{Z}/2$-action is weak fermionically dagger compact. However, the positivity structure P is relatively 'small', in the sense that we are not aware of any objects that are isomorphic in $(\mathrm{Bord}^{G_n(K)})_P$ but not unitarily. In other words, we do not know whether $(\mathrm{Bord}^{G_n(K)})_P$ is strong fermionically †-compact in general.

Example 5.2.27. The bordism categories corresponding to the fermionic group $K = \mathrm{Spin}_1$ and the bosonic group $K = \mathbb{Z}/2$ are $\mathrm{Bord}_{n,n-1}^{\mathrm{Spin}_n}$ and $\mathrm{Bord}_{n,n-1}^{SO_n\times\mathbb{Z}/2}$ respectively. These two categories are the same in spacetime dimension $n = 1$ but different for $n > 1$. However, the $B\mathbb{Z}/2$-actions are different already for $n = 1$; on the first category it is given by the spin flip, while for the latter the action is trivial. The $\mathbb{Z}/2$-actions on these categories are equivalent however, and so the symmetric monoidal †-categories $\mathrm{Herm}(\mathrm{Bord}_{1,0}^{SO_n\times\mathbb{Z}/2})$ and $\mathrm{Herm}(\mathrm{Bord}_{1,0}^{\mathrm{Spin}_n})$ are equivalent as symmetric monoidal †-categories. Since they are Hermitian completions, they are both weak †-compact and weak fermionically †-compact. However, they are neither strong †-compact nor strong fermionically †-compact since they are not minimal †-categories; the positively oriented point admits two equivalence classes of Hermitian pairings which are related by composing with the $B\mathbb{Z}/2$-action automorphism. However, it follows by Theorem 5.2.25 that we can do better and pick smaller positivity structures P so that $(\mathrm{Bord}_{1,0}^{SO_1\times\mathbb{Z}/2})_P$ is still weak †-compact, while $(\mathrm{Bord}_{1,0}^{\mathrm{Spin}_1})_P$ is weak fermionically †-compact. This is a consequence of the interesting positivity structures we introduced to deal with the subtleties in dimension one discussed in Section 4.6. It turns out that $(\mathrm{Bord}_{1,0}^{SO_1\times\mathbb{Z}/2})_P$ is even strong †-compact while $(\mathrm{Bord}_{1,0}^{\mathrm{Spin}_1})_P$ is strong fermionically †-compact. Indeed, these dagger categories are minimal as in this low dimension all isomorphisms in the bordism category are unitary, see the discussion at Remark 5.2.22. More precisely, in $(\mathrm{Bord}_{1,0}^{SO_1\times\mathbb{Z}/2})_P$ the Hermitian structure on the negatively oriented point $+^*$ has the dual of the Hermitian structure on the positively orientated point $+$ in the sense of Definition 2.7.12. On the other hand, in $(\mathrm{Bord}_{1,0}^{\mathrm{Spin}_1})_P$ has its the Hermitian structure on the negatively oriented point $+^*$ changed by the spin flip $+_c$. This will cause the double of ev_+ to be the periodic circle in the former dagger category and the anti-periodic circle in the latter dagger category.

We now elaborate further on why we call $(\mathrm{Bord}^{G_n(K)})_P$ a dagger category with internal symmetry K. This is motivated by the physical intuition that state spaces of a quantum field theory with symmetry K should equipped with a representation of K. More specifically, if $G = G_n(K)$ for K an internal fermionic symmetry group we expect the Hilbert space attached to a time slice to come equipped with a representation of K so that time-preserving elements act linearly and time-reversing elements anti-linearly. We now make the first steps in this direction by working purely on the bordism category. We will restrict to manifolds with trivialized G-structures $P = Y \times G$. Note that this implies that Y is framed, but this is not sufficient.

Corollary 5.2.28. *Let Y be a $(n-1)$-dimensional manifold. Let $\mathcal{G}_Y$ be the full subgroupoid of $G\text{-}\mathrm{Str}(Y)$ in which Y has a G-structure (P,α) so that $P = Y \times G$ is trivial as a principal G-bundle. There are isomorphisms $(Y,P,\alpha) \cong (Y,P,\alpha_{\rho(g)})$ in the bordism category functorial in $g \in G$.*

Proof. By Corollary 5.1.20 it suffices to work in $G\text{-}\mathrm{Str}(Y)$ and then apply the mapping cylinder construction. Let (P,α) be a G-structure so that $P = Y \times G$ is trivial as a principal G-bundle. Given $g \in G$ define $R_g : P \to P$ by left multiplication $R_g(y,g') = (y,gg')$. Note that this is a G-equivariant map and $\alpha[y,R_g(g')] = \alpha[y\rho(g),g']$. Therefore this defines a map of G-structures $R_g : (P,\alpha) \to (P,\alpha_{\rho(g)})$ which is clearly functorial in g. □

Remark 5.2.29. If $g(t) \in G$ is a smooth path, then there is a homotopy of G-diffeomorphisms between $R_{g(0)}$ and $R_{g(1)}$. By Lemma 5.1.21 we see that in the case $g(0) = g(1)$, the two isomorphisms $(Y,P,\alpha) \cong (Y,P,\alpha_{\rho(g)})$ in the above corollary are equal. The fact that we can 'only see $\pi_0(G)$' is a consequence of the fact that we are working 1-categorically. The $(\infty,1)$-bordism category would remember more than just the connected components of the topological group of G-diffeomorphisms of a time slice.

The following corollary tells us that time-preserving symmetries in K gives us canonical automorphisms of every spatial slice, but for time-reversing elements we have to change the orientation of space.

Corollary 5.2.30. *Let K be a fermionic group and let Y be a $(n-1)$-dimensional manifold with a $G_n = G_n(K)$-structure (P,α) so that $P = Y \times G$ is trivial as a principal G-bundle. Let $\hat{K}$ be the groupoid with two objects $+$ and $-$ in which objects have automorphisms K_{pres} while $\mathrm{Hom}(+,-) = K_{rev}$ with the obvious composition. Then there is a homomorphism of groupoids from $\hat{K}$ to the subgroupoid of $\mathrm{Bord}_{n,n-1}^G$ on the two objects Y and $\overline{Y}$. This homomorphism factors through $\pi_0(K)$.*

Proof. By the last corollary, it suffices to show that there is a groupoid homomorphism from K to the subgroupoid of $G\text{-}\mathrm{Str}(Y)$ on the two manifolds with G-structure Y and $\overline{Y}$ and then apply the mapping cylinder construction. If $|k| = 0$ is time-preserving then $1 \otimes k \in G$ gets mapped to $1 \in O_n$

under ρ and so the mapping cylinder gives automorphisms of (Y, P, α) in the bordism category functorial in $k \in K_{pres}$. If $|k| = 1$ is time-reversing then $e_n \otimes k \in G$ instead gives an isomorphism $(Y, P, \alpha) \cong (Y, P, \alpha_{e_n}) = (Y, P, \alpha)^*$. We use the Hermitian pairing on Y to get the corresponding isomorphism $f_k : Y \to \overline{Y}$. Assume now that $n > 1$ so that we can use the first formula in Corollary 4.6.16 for the Hermitian pairing, using the obvious desuspension $Y \times G_{n-1}$ of P. Since the Hermitian pairing is then explicitly given by multiplication with e_n this isomorphism is simply given by

$$(y, g) \mapsto (y, (1 \otimes k)g) \quad P = Y \times G \to \overline{P} = Y \times \overline{G}.$$

Note that this is functorial in $k \in K$. For example, if $k_1, k_2 \in K_{rev}$ are both time-reversing, then clearly

$$\overline{f_{k_1}} f_{k_2} = f_{k_1 k_2}.$$

In case $n = 1$, the only thing that changes is that the formula of the Hermitian pairing changes by a spin flip for P_-. However, somewhat confusingly, time-reversing symmetries will still map $P_\pm$ to $P_\pm$, compare Remark 4.6.13. Therefore functoriality still holds. The fact that $f_k = f_{k'}$ is k and k' are in the same path component is a consequence of Remark 5.2.29 together with a fact that a path in K induces a path in $\hat{G}$ mapping to $1 \in O_n$. □

Remark 5.2.31. Recall from Example 3.2.12 that the subgroup of $G_n(K)$ consisting of elements of the form $1 \otimes k$ for $k \in K_{pres}$ and $e_n \otimes k$ for $k \in K_{rev}$ is isomorphic to K^{op}. This follows because $k \in K_{rev}$ and e_n anti-commute. We thus see in Corollary 5.2.30 that when assuming time-reversing elements act by duals, it is K^{op} that acts on objects of the bordism category, but if we want them to act by mapping to the orientation-reversal $\overline{Y}$ it is the internal symmetry group K itself that acts. This confusing sign change will ensure that in a unitary topological field theory the internal symmetry group K will act on state spaces as is expected from physics. We will discuss this in more detail at the end of Chapter 6.

Remark 5.2.32. If K is a bosonic group with no time-reversal symmetries so that $G_n(K) = SO_n \times K$, a $G_n(K)$-structure (P, α) on Y^{n-1} is given by an orientation and a principal K-bundle over Y. In particular, if the principal K-bundle Q is trivial, we can define ρ_k for $k \in K$ purely on Q without needing the assumption that Y is framed to give us an action of K on the object (Y, P, α) in the bordism category.

Chapter 6

Hermitian and unitary topological field theory

6.1 Fermionic topological field theory

In Atiyah-Segal style topological field theory we axiomatize n-dimensional quantum field theory by recording the assignment of the vector space of states to an $(n-1)$-dimensional space and the time evolution operator to an n-dimensional spacetime bordism. If a quantum field theory has a symmetry K, we can couple it to a *background* gauge field to get a quantum field theory defined on spacetimes equipped with a principal K-bundle with connection. Here we allow nontrivial principal bundles, which could correspond physically to a nontrivial instanton sectors. In topological field theories we do not include gauge fields because they form a contractible space, but the principal K-bundle is still relevant data.

When fermions are present, some things change: to deal with spinors we need spin structures and we have to record the $\mathbb{Z}/2$-grading $(-1)^F$ on state spaces. If our topological field theory has the symmetry of a fermionic group (possibly with time-reversal) K as discussed in Definition 3.1.1, we can still couple it to background gauge fields in a certain sense. However, the interesting interplay between time-reversal symmetries in the fermionic group and orientation-reversing Lorentz symmetries, as well as between $c \in K$ and the special element $c \in \mathrm{Spin}_n$, makes it more complicated. Following [20], we argue that the spacetimes on which the topological field theories are defined should come equipped with a $G_n(K)$-structure as in Definition 3.2.1, also see [25, section 2.1]. In particular, it is to be expected that

1. the IR limit of a gapped n-dimensional QFT with fermionic symmetry K is a topological field theory with $G_n(K)$-structure;
2. for an n-dimensional QFT with anomalous fermionic symmetry K, the anomaly is classified by an $(n+1)$-dimensional invertible topological field theory with $G_n(K)$-structure.

Definition 6.1.1. Let K be a fermionic group and let $G = G_n(K)$. A *fermionic topological field theory (TFT) with internal fermionic symmetry group K in spactime dimension n* is a symmetric monoidal functor $Z : \mathrm{Bord}_n^G \to \mathrm{sVect}$.

Example 6.1.2. When $K = \mathrm{Spin}_1 = \mathbb{Z}_2^F$ is the internal symmetry group only containing fermion parity, we have $G_n(K) = \mathrm{Spin}_n$. We see that a fermionic TFT with internal fermionic symmetry group K in spactime dimension n is a symmetric monoidal functor $Z : \mathrm{Bord}_{n,n-1}^{\mathrm{Spin}_n} \to \mathrm{sVect}$. In case where $K = \mathrm{Pin}_1^-$ is the symmetry group corresponding to a time-reversal with square $(-1)^F$, we have $G_n(K) = \mathrm{Pin}_n^+$, and so we get a functor $Z : \mathrm{Bord}_{n,n-1}^{\mathrm{Pin}_n^+} \to \mathrm{sVect}$. For $K = \mathrm{Pin}_1^+$ a time-reversal with square 1, we would get Pin_n^- instead.

Definition 6.1.3. Let Z be a fermionic TFT and let X be a closed G-manifold seen as a bordism from the empty manifold to itself, see Remark 5.1.2. Then the complex number corresponding to the linear map $Z(X) : \mathbb{C} = Z(\emptyset) \to Z(\emptyset) = \mathbb{C}$ is called the *partition function on the spacetime* X.

Remark 6.1.4. In physics, the partition function is often evaluated specifically on a spacetime of the form $Y \times S^1_\beta$, where Y is a time slice and S^1_β is a circle of length β. In this case β could either refer to inverse temperature (in statistical mechanics) or to a time parameter. For example, for a one-dimensional quantum field theory we can think of this partition function as being the trace of the time evolution operator e^{-tH} where H is the Hamiltonian. For a fermionic TFT however

$$Z(Y \times S^1_\beta) = \operatorname{sdim} Z(Y)$$

will be independent of β. Intuitively, this is a consequence of the Hamiltonian being zero.

Example 6.1.5. A fermionic TFT for the case where $K = 1$ is the trivial bosonic group, results in a symmetric monoidal functor

$$\mathrm{Bord}_{n,n-1}^{SO_n} \to \mathrm{sVect}\,.$$

More generally, for a bosonic group K without time-reversal, the functor

$$\mathrm{Bord}_{n,n-1}^{SO_n \times K} \to \mathrm{sVect}$$

requires as input oriented Riemannian space(time)s with principal K-bundle. Note the curious fact that even though the symmetry group is bosonic, we do allow nontrivial $\mathbb{Z}/2$-gradings on state spaces. However, it will be a consequence of the spin-statistics theorem that in case such a TFT is unitary, it will automatically land in Vect, see Corollary 6.3.6. Not every TFT satisfies spin-statistics though. For example, in spacetime dimension $n = 1$, there is a theory that assigns the odd line to a point, which has partition function -1 on a circle. This theory should be thought of as 'integer spin but fermionic', compare [34, Appendix E.1].

Example 6.1.6. A one-dimensional fermionic spin TFT

$$Z : \mathrm{Bord}_{1,0}^{\mathrm{Spin}_1} \to \mathrm{sVect}$$

is classified by the finite-dimensional super vector space $Z(+)$ assigned to the positively oriented point $+$, together with the even involution $Z(+_c) : Z(+) \to Z(+)$ induced by the spin flip. In particular, the two involutions $Z(+_c)$ and $(-1)^F_{Z(+)}$ have no reason to be related in general. This follows for example from the cobordism hypothesis (which is a theorem in dimension one, see [29]), together with the fact that the structure group Spin_1 is equal to $\mathbb{Z}/2$ with the trivial map to O_1. We will see in Section 6.3 that $Z(+_c) = (-1)^F_{Z(+)}$ if and only if this theory has a connection between spin and statistics. We thus see again that there is no reason for a spin-statistics connection to exist in a general non-unitary TFT.

There are two spin circles; the periodic circle S^1_{per} with the disconnected double cover and the antiperiodic circle S^1_{ap} with the disconnected double cover. The trace induced by the symmetric monoidal structure on the bordism category gives

$$\dim(+) = S^1_{per}.$$

Since traces are mapped to traces by symmetric monoidal functors and the trace in sVect is the supertrace, this will be mapped to sdim $Z(+)$. Similarly S^1_{ap} is mapped to the supertrace of $Z(+_c)$.

Remark 6.1.7. Note that even though in this thesis sVect denotes finite-dimensional super vector spaces, the existence of duals in the bordism category will automatically enforce finite-dimensional state spaces even when the target allows for infinite dimensions. At first sight it seems to be depressing from a physical perspective that TFTs cannot have infinite-dimensional state-spaces, but after some more thought on the interpretation of TFTs this does seem reasonable; TFTs can be thought of as the zero energy part of a gapped quantum field theory that can be evaluated on compact spacetimes. It is not a very strong assumption for such quantum field theories that the ground state is only finitely degenerate.

Remark 6.1.8. From a mathematical perspective, there seems to be no reason why including 'spinors' on the bordism side would require one to set the target to allow 'fermions'. In other words, one could just as well study functors

$$\mathrm{Bord}_{n,n-1}^{\mathrm{Spin}_n} \to \mathrm{Vect}\,.$$

However, we decide to follow current physics wisdom and study TFTs that both have fermions and spinors in this thesis.

There are examples of structure groups $G \to O_n$ for which no dagger structure on $\mathrm{Bord}^G_{n,n-1}$ can exist:

Remark 6.1.9. For $G = 1$, the category $\mathrm{Bord}^G_{n,n-1}$ of bordisms with unstable framing does not admit the structure of a dagger category in general. For example, suppose for $n = 2$ that $\dagger : \mathrm{Bord}^G_{n,n-1} \to (\mathrm{Bord}^G_{n,n-1})^{\mathrm{op}}$ is a dagger structure. Recall that a closed genus g surface admits a framing if and only if $g = 1$. However, if we puncture a genus g surface to get a manifold with boundary S^1, then it does admit a framing. Now consider such a surface Σ with genus $g > 1$ as an element of $\mathrm{Hom}_{\mathrm{Bord}^G_{n,n-1}}(\emptyset, (S^1, \tau))$, where the S^1 comes equipped with the restricted framing τ induced by the framing of the bulk. Then $\Sigma^\dagger$ is some element of $\mathrm{Hom}_{\mathrm{Bord}^G_{n,n-1}}((S^1, \tau), \emptyset)$. But note that the composition $\Sigma^\dagger\Sigma$ is a closed framed manifold of genus at least $g > 1$, which leads to a contradiction.

Example 6.1.10. Two-dimensional spin TFTs have a classification, similar to the classification of two-dimensional oriented TFTs by commutative Frobenius algebras [51, Section 2.6].

6.2 Unitary topological field theory

After all the preparatory work on studying orientation-reversing involutions on the bordism category in Section 5.2, we are fully equipped to define unitary TFTs. We start with some general discussion on the relationship between two definitions of unitary topological field theory; one through dagger functors and the other through $\mathbb{Z}/2$-equivariant functors. The fact that these two approaches are equivalent will be a consequence of the more general theory developed in Section 2.2.

Euclidean quantum field theories are typically obtained by Wick-rotating a genuine Lorentzian signature quantum field theory. The analogous property such a Euclidean theory obtains when the Lorentzian theory is unitary, is typically called 'reflection positivity', see [20, section 3.2] and references therein. Since Atiyah's axioms for TFTs are naturally in Euclidean signature, it is thus more appropriate to call unitary TFTs reflection positive instead. We decide to follow the terminology of [20] and call a $\mathbb{Z}/2$-equivariance structure on a TFT Z a reflection structure, and we call it reflection positive if it satisfies an extra positivity condition. Under the relationship between dagger and $\mathbb{Z}/2$-equivariant functors, we will call the corresponding TFTs Hermitian, respectively unitary. In other words, a Hermitian TFT will be a dagger functor from a bordism category to Hermitian vector spaces, while a unitary TFT will be a dagger functor to Hilbert spaces. It follows by the theory developed in Section 2.2 and Section 2.4, that Hermitian TFTs are equivalent to TFTs with reflection structure and unitary TFTs are equivalent to reflection positive TFTs. We start working with slightly more general structure groups than in the last section, allowing arbitrary orientation-graded structure groups in the sense of Section 4.5.

Definition 6.2.1. Let $G \hookrightarrow \hat{G} \xrightarrow{\hat{\rho}} O_n$ be an orientation-graded structure group giving the bordism category $\mathrm{Bord}^G_{n,n-1}$ a $\mathbb{Z}/2$-action. A *TFT with reflection structure* is a $\mathbb{Z}/2$-equivariant functor $Z : \mathrm{Bord}^G_{n,n-1} \to \mathrm{sVect}$ for the $\mathbb{Z}/2$-actions $\overline{(.)}$ given by orientation-reversal on $\mathrm{Bord}^G_{n,n-1}$ and the standard complex-conjugation action on sVect.

If $Z : \mathrm{Bord}^{G_n(K)}_{n,n-1} \to \mathrm{sVect}$ is a fermionic TFT with internal fermionic symmetry group K, we can use the orientation-grading $\hat{G}_n(K)$ to talk about a reflection structure on it. Explicitly writing out Definition A.3.6 in the appendix (for the more general symmetric monoidal situation), tells us that $\mathbb{Z}/2$-equivariance data is a collection of isomorphisms $Z(\overline{Y}) \cong \overline{Z(Y)}$ that are natural with respect to bordisms, such that

$$\begin{array}{ccccc}
Z(\overline{Y_1 \sqcup Y_2}) & \longleftarrow & Z(\overline{Y}_1 \sqcup \overline{Y}_2) & \longleftarrow & Z(\overline{Y}_1) \otimes Z(\overline{Y}_2) \\
\downarrow & & & & \downarrow \\
\overline{Z(Y_1 \sqcup Y_2)} & \longleftarrow & \overline{Z(Y_1) \otimes Z(Y_2)} & \longleftarrow & \overline{Z(Y_1)} \otimes \overline{Z(Y_2)}
\end{array} \tag{6.1}$$

and

$$\begin{array}{ccc}
Z(Y) & \longrightarrow & Z(\overline{\overline{Y}}) \\
\downarrow & & \downarrow \\
\overline{\overline{Z(Y)}} & \longleftarrow & \overline{Z(\overline{Y})}
\end{array} \tag{6.2}$$

commute.

Remark 6.2.2. If Z is an n-dimensional TFT with reflection structure and X is a closed manifold, then $Z(\overline{X}) = \overline{Z(X)}$. In other words, the partition function of the orientation-reversed spacetime is the complex conjugate of the original spacetime.

If $G \subseteq \hat{G} \xrightarrow{\hat{\rho}} O_n$ is an orientation-graded structure group, then $\mathrm{Bord}^G_{n,n-1}$ comes equipped with the anti-involution $Y \mapsto \overline{Y}^*$ induced by the involution $Y \mapsto \overline{Y}$ through the procedure described in Section 2.2. Let P be a monoidal positivity structure on $\mathrm{Bord}^G_{n,n-1}$ in the sense of Definition 5.2.13 and let $h \in P$. In particular, we will thus assume that every object of $\mathrm{Bord}^G_{n,n-1}$ admits at least one Hermitian pairing. If Z is a TFT with reflection structure, the condition (6.2) will make sure the composition

$$Z(Y) \xrightarrow{Z(h)} Z(\overline{Y}^*) \to Z(\overline{Y})^* \to \overline{Z(Y)}^* \tag{6.3}$$

defines a Hermitian pairing on the super vector space $Z(Y)$ in the sense of Section 2.1. The condition (6.1) ensures compatibility between the Hermitian pairings on the disjoint union of two time slices and the tensor product of the Hermitian pairings on their state spaces.

Definition 6.2.3. A TFT $Z : \mathrm{Bord}^G_{n,n-1} \to \mathrm{sVect}$ with reflection structure is called *reflection positive with respect to* P if for all $h \in P$, the Hermitian pairings of the composition (6.3) are positive definite in the sense of Definition 2.1.4.

Next, we will provide a different definition of reflection positive TFTs using dagger functors. For this, we need to assume that the bordism category is a symmetric monoidal dagger category. We will take it to be the †-category $(\mathrm{Bord}^G_{n,n-1})_P$ associated to the positivity structure P.

Definition 6.2.4. A *Hermitian TFT* is a symmetric monoidal dagger functor

$$(\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHerm}_{\mathbb{C}}\,.$$

A *unitary TFT* is a symmetric monoidal dagger functor

$$(\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHilb}\,.$$

Theorem 6.2.5. *There is an equivalence between the category of TFTs with reflection structure and the category of Hermitian TFTs.*

Proof. By Theorem 2.2.24, there is an equivalence of categories between TFTs equipped with $\mathbb{Z}/2$-equivariance data and the category of symmetric monoidal anti-involutive functors $\mathrm{Bord}^G_{n,n-1} \to \mathrm{sVect}$, where both sides are equipped with the $d = \overline{(.)}^*$ anti-involution. As a symmetric monoidal anti-involutive category, $(\mathrm{Bord}^G_{n,n-1})_P$ is equivalent to $\mathrm{Bord}^G_{n,n-1}$. So by the universal property of the Hermitian completion of Corollary 2.3.16 and its generalization to the symmetric monoidal case, we get an equivalence of categories between symmetric monoidal anti-involutive functors $\mathrm{Bord}^G_{n,n-1} \to \mathrm{sVect}$ and dagger functors $(\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{Herm}(\mathrm{sVect})$. Since the Hermitian completion of sVect is the symmetric monoidal dagger category $\mathrm{sHerm}_{\mathbb{C}}$, the theorem follows. □

Remark 6.2.6. Note that the theorem is true independent of the Hermitian pairing P we chose on objects of the bordism category. Namely, the category of dagger functors $(\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHerm}_{\mathbb{C}}$ only depends on the choice of $\mathbb{Z}/2$-action on the bordism category. In contrast, the category of dagger functors $(\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHilb}$ will strongly depend on the Hermitian pairings we choose. The more Hermitian pairings on the bordism category we call positive, the less such dagger functors exist.

Theorem 6.2.7. *There is an equivalence between the category of reflection positive TFTs and the category of unitary TFTs.*

Proof. This is a direct consequence of Theorem 6.2.5 and the symmetric monoidal generalization of Theorem 2.3.41. □

Next, we observe some simple consequences of these definitions.

Remark 6.2.8. In quantum mechanics (one-dimensional quantum field theory), the time evolution operator e^{-itH} – which is what the theory assigns to a bordism of (Lorentzian) length t – is a unitary operator if the quantum field theory is unitary.[1] For a unitary (or Hermitian) TFT there is similarly a condition on the adjoint $Z(X)^\dagger$ induced by a bordism $X : (Y_1, h_1) \to (Y_2, h_2)$. However, $Z(X)$ will

[1] For non-Hermitian systems, the Hamiltonian H need not be self-adjoint and time evolution loses information.

typically not be a unitary operator. Instead we have only that $Z(X)^\dagger = Z(X^\dagger)$, where $X^\dagger$ is the bordism

$$Y_2 \xrightarrow{h_2} \overline{Y_2}^* \xrightarrow{\overline{X}^*} \overline{Y_1}^* \xrightarrow{h_1^{-1}} Y_1,$$

which can heuristically be described as X with opposite orientation, viewed as a bordism in the other direction. The caveat however, is that the result depends on the choices of $h_1.h_2 \in P$.

Remark 6.2.9. If $X : (Y_1, h_1) \to (Y_2, h_2)$ is a morphism in $(\mathrm{Bord}^G_{n,n-1})_P$ and $Z : (\mathrm{Bord}^G_{n,n-1})_P \to$ sHilb is a unitary TFT, then $Z(X^\dagger X) = Z(X)^\dagger Z(X)$ is a positive semi-definite endomorphism of $Z(Y_1)$. Note that this would not necessarily be true for a Hermitian TFT because in $\mathrm{sHerm}_{\mathbb{C}}$ there are weakly positive operators that do not give positive semi-definite matrices.

Remark 6.2.10. If Z is a Hermitian TFT and Y is a time slice, then the state space $Z(Y)$ is a super Hermitian vector space so that in particular the even and odd vectors are orthogonal. This implies that the operator $(-1)^F_{Z(Y)}$ is always unitary.

Remark 6.2.11. In our definition of unitary TFT, we used the convention that the inner products of super Hilbert spaces V satisfy

$$\langle v, w\rangle = (-1)^{|v||w|}\overline{\langle w, v\rangle}$$

and odd vectors $v \in V$ satisfy $\langle v, v\rangle \in i\mathbb{R}_{\geq 0}$. We briefly spell out what to modify, when the reader prefers to work with the more common convention for $\mathbb{Z}/2$-graded Hilbert spaces, for which the Koszul sign is omitted in the above formula, compare Remarks 2.1.14, 2.1.14 and 2.1.15. For the symmetric monoidal $\mathbb{Z}/2$-action $V \mapsto \overline{V}$, we do not change the isomorphism $V \cong \overline{\overline{V}}$, but we change its monoidal data $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$ by the sign $\overline{v \otimes w} \mapsto (-1)^{|v||w|}\overline{v} \otimes \overline{w}$. For the induced symmetric monoidal anti-involution $dV = \overline{V}^*$, this results in changing the isomorphism $V \cong d^2V$ with a $(-1)^F_V$ and the isomorphism $d(V \otimes W) \cong dV \otimes dW$ with the same mixed sign. Note that this also changes the isomorphisms $\overline{V^*} \cong \overline{V}^*$ and $d(V^*) \cong (dV)^*$ by $(-1)^F$. In particular, the notion of dual Hermitian pairing of Definition 2.7.12 is changed by a sign on the odd part. With these conventions, all results in this chapter hold, because Corollary 2.6.17 tells us that this symmetric monoidal dagger category of $\mathbb{Z}/2$-graded Hilbert spaces is equivalent to sHilb. Any other choices of convention will result in the tensor square of an odd line having a negative definite inner product.

Example 6.2.12. Consider again the case of a trivial bosonic symmetry group $K = 1$. Since the $B\mathbb{Z}/2$-action is trivial, the oriented n-dimensional dagger bordism category is $\dagger$-compact in the usual sense. Let $Z : \mathrm{Bord}^{SO_n}_{n,n-1} \to \mathrm{sHerm}_{\mathbb{C}}$ be a Hermitian TFT in which we allow graded vector spaces and let Y^{n-1} be a time slice. Then $Z(Y)$ is a Hermitian vector space of some signature (p_1, p_2, p_3, p_4). There is a canonical Hermitian pairing on the dual which has signature (p_1, p_2, p_4, p_3). For example, if $Z(Y)$ is purely odd, then $Z(Y)$ has a positive definite Hilbert space structure if and only if the induced Hermitian pairing on $Z(Y)^*$ is negative definite. In particular, if we want both $Z(Y)$ and $Z(Y)^*$ to be a Hilbert space, we need $Z(Y)$ to be a purely even vector space. This is the spin-statistics theorem in the bosonic case. We used the bosonic nature through the fact that the $B\mathbb{Z}/2$-action given by the spin flip is trivial. In general, we will need to modify the Hermitian pairing on $Z(Y)^*$ by this spin flip which changes the signature further.

We study the example of one-dimensional Hermitian and unitary Pin^+-theories. This example is interesting for several reasons:

1. It gives an example of a(n invertible) TFT of which the partition function satisfies $Z(\overline{X}) = \overline{Z(X)}$ for all closed manifolds, but it cannot be made into a Hermitian (let alone unitary) TFT;

2. it reveals some of the subtleties of the interplay between time-reversal symmetries and fermion parity.

Example 6.2.13. Consider a one-dimensional TFT with internal symmetry group $K = \mathrm{Pin}_1^-$ consisting of a time-reversal symmetry with square $(-1)^F$. The associated spacetime structure group is $G_n(K) = \mathrm{Pin}_n^+$ and so we consider symmetric monoidal functors.

$$Z : \mathrm{Bord}_{1,0}^{\mathrm{Pin}_1^+} \to \mathrm{sVect}\,.$$

The theory is determined by a finite-dimensional super vector space $V := Z(*) = V_0 \oplus V_1$, a graded-symmetric nondegenerate bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C},$$

such that V_0 and V_1 are orthogonal, and an even involution $Z(*_c) : V \to V$, such that

$$\langle Z(*_c)v, w\rangle = \langle v, Z(*_c)w\rangle \quad \forall v, w \in V.$$

One way to prove the above classification is using the one-dimensional cobordism hypothesis (c.f. [29]). Geometrically, the bilinear form is what the theory assigns to the semicircle bordism from $*\sqcup*$ to $\emptyset$. The previous equation tells us that we can freely move the spin flip around on this semicircle. Just like for spin, there are two circles. The periodic circle is the mapping torus of the identity on a point and turns out to bound a Möbius strip. For the antiperiodic circle we need to insert an extra spin flip and this circle bounds the disk. The former circle is the trace of the point in the pivotal structure induced by the braiding, and since Z is pivotal it will map $Z(S^1_{per})$ to the super dimension of V. Similarly, it maps S^1_{ap} to the supertrace of $Z(*_c)$.

Now let us assume $\mathrm{Bord}_{1,0}^{\mathrm{Pin}_1^+}$ comes equipped with a symmetric monoidal dagger structure and a dual functor such that it becomes strong fermionically dagger compact. For example, we can take the standard positive Hermitian pairings of Definition 5.2.17. Suppose Z is a theory with reflection structure, so that our preferred Hermitian pairing on the point makes V into a super Hermitian vector space of some signature (p_1, p_2, p_3, p_4). Since $Z(*_c)$ and $(-1)^F_V$ are both unitary and commute, we can let $\{e_i\}$ be an orthonormal basis for which these operators are diagonal. Write the bilinear form in the basis as

$$a = \sum_{i,j} a_{ij} e^i e^j \quad a_{ij} = (-1)^{|e_i||e_j|} a_{ji}.$$

By the assumption that the bordism category is strong fermionically †-compact, the double of a semicircle is the antiperiodic circle which is mapped to the supertrace of $Z(*_c)$. Note that we really need strongness here. For example, if we take all Hermitian structures on this bordism category to be positive, the double of the semicircle is either periodic or antiperiodic, depending on which Hermitian pairing we chose on $* \sqcup *$. On the other hand, the above expresseion should agree with $a^\dagger a$, which can be computed using Proposition 2.1.9 to be

$$\sum_{i,j} p^{ij} \overline{a_{ij}} a_{ij} = \sum_{i,j} (-1)^{|e_i||e_j|} p^{ij} \overline{a_{ji}} a_{ij},$$

where $p^{ij} \in \{\pm 1, \pm i\}$ can be determined as follows. We have that p^{ij} is imaginary if $|e_i| + |e_j| \neq 0$, but this will not be relevant because $a_{ij} = 0$ if e_i and e_j are of different degree. The sign of p^{ij} will be negative if and only if (e_i, e_i) and (e_j, e_j) are of different sign.

This gives a constraint for Pin^+ theories with reflection structure, even if they are not reflection positive. For example, if a_{ij} is diagonal in this basis, we obtain

$$\sum_i (-1)^{|e_i|}|a_{ii}|^2 = \operatorname{str} Z(*_c) = \sum_i (-1)^{|e_i|}(-1)^{s(e_i)/2},$$

where $(-1)^{s(e_i)/2}$ is ± 1 depending on the spin of e_i, i.e. it is the eigenvalue of the eigenvector e_i for $Z(*_c)$. We see that in that case $Z(*_c) = \mathrm{id}_V$. One interesting consequence of this observation is that only the trivial invertible one-dimensional Pin^+-TFT has a reflection structure. Indeed, note that the graded symmetry condition on $\langle\cdot,\cdot\rangle$ requires the odd part of V to be even dimensional, so all invertible one-dimensional Pin^+-TFTs have $V = \mathbb{C}$ the even line. There is then still a nontrivial invertible TFT with c acting by -1. Note that this theory is not unitary. One can see this either by the fact that this theory does not satisfy spin-statistics (see Section 6.3) or by using the classification of unitary invertible TFTs by bordism groups and the vanishing of the bordism group $\Omega_1^{\mathrm{Pin}^+} = 0$. However, many non-unitary theories are still Hermitian, but this one is not.

Now suppose Z is a unitary TFT. Then, $p^{ij} = 1$ and so we get

$$\sum_{i,j} |a_{ij}|^2 = \sum_i (-1)^{|e_i|}(-1)^{s(e_i)/2},$$

which by strong fermionic †-compactness is equal to the ungraded dimension of V.

Remark 6.2.14. We give another perspective on the above example of a one-dimensional invertible TFT that does not lift to a Hermitian TFT. We look through the lens of Picard groupoids, being sketchy about the details. For invertible TFTs, we can use the universal property of groupoidification to equivalently consider them as functors of Picard groupoids out of the groupoidification of the bordism category to the Picard groupoid of superlines, see [20]. The groupoidification $\widehat{\mathrm{Bord}}^G_{n,n-1}$ of $\mathrm{Bord}^G_{n,n-1}$ has isomorphism classes of objects equal to $\pi_0(\widehat{\mathrm{Bord}}^G_{n,n-1}) = \Omega^G_{n-1}$. For example for $n = 1$ and $G = \mathrm{Pin}_n^+$, this gives $\Omega_0^{\mathrm{Pin}^+} = \mathbb{Z}/2$. In good cases (as outlined in [45]), we have that $\pi_1(\widehat{\mathrm{Bord}}^G_{n,n-1}) = SKK^G_n$ is given by the SKK-group [40], which can be computed in simple examples [59]. In this case, $SKK_1^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by the bounding circle. In particular, the disjoint union of two points is isomorphic to the empty set in the groupoidification, because the semicircle is a morphism between the two. The dagger structure on the bordism category induces a dagger structure on the Picard groupoid. Now note that the disjoint union of two points is not unitarily isomorphic to the empty manifold. For example, the double of the semicircle is bounding and so always gives a nontrivial element in

$$\mathrm{Aut}_{\widehat{\mathrm{Bord}}^{\mathrm{Pin}_1^+}_{1,0}}(\emptyset) \cong SKK_1^{\mathrm{Pin}^+}.$$

We conclude that $\pi_0^U(\widehat{\mathrm{Bord}}^{\mathrm{Pin}_1^+}_{1,0}) \cong \mathbb{Z}/4$ and so the surjective group homomorphism

$$\pi_0^U\left(\widehat{\mathrm{Bord}}^{\mathrm{Pin}_1^+}_{1,0}\right) \to \pi_0\left(\widehat{\mathrm{Bord}}^{\mathrm{Pin}_1^+}_{1,0}\right) \cong \mathbb{Z}/2$$

does not have a splitting. We see that this is an example of a symmetric monoidal dagger category that is not only not minimal, but also has no monoidal dagger subcategory that is minimal, compare

Remark 2.4.19. Note that it is still possible to lift the nontrivial homomorphism

$$\mathbb{Z}/2 \to \pi_0\left(\widehat{\mathrm{Bord}}_{1,0}^{\mathrm{Pin}_1^+}\right) \to \pi_0(\mathrm{sLine}) \cong \mathbb{Z}/2$$

to a homomorphism

$$\mathbb{Z}/4 \cong \pi_0^U\left(\widehat{\mathrm{Bord}}_{1,0}^{\mathrm{Pin}_1^+}\right) \to \pi_0^U(\mathrm{sHermLine}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2,$$

where sHermLine is the dagger category of one-dimensional Hermitian vector spaces. Indeed, we can either assign the negative definite or the positive definite inner product space to the odd line. The problem here is that the theory that assigns the odd line to the point needs to have partition function -1 on the antiperiodic circle. Since this circle is a double $X^\dagger X$, it evaluates to $-1 = Z(X)^\dagger Z(X) \in \mathbb{R}^\times$ in the target category. But an isomorphism $f : L \to \mathbb{C}_+$ from a super Hermitian line to the positive definite even line can only satisfy $f^\dagger f = -1$ if L has a negative definite inner product. In this case however $L = Z(+ \sqcup +) \cong Z(+) \otimes Z(+)$ and so Z has to assign the even line with negative definite Hermitian pairing to the disjoint union of two points. This is impossible, because the odd line does not have a Hermitian pairing of which its square is a negative define pairing on the even line. In the language of [44], this is a consequence of the fact that this invertible field theory does not intertwine the 'dagger k-invariants' of the source and target Picard dagger categories.

6.3 The spin-statistics theorem

The spin-statistic theorem is a cornerstone of quantum field theory. It states that in a unitary quantum field theory a particle is a fermion if and only if it has half-integer spin. We now proceed to make this precise in the context of unitary topological field theory.

The statistics of a particle is determined by how it braids with other particles; exchanging two fermions should give an extra minus sign, while exchanging bosons with fermions or bosons with each other does not. This is made precise in the world of TFTs as follows. Recall that we equip the state spaces V with a grading operator $(-1)^F_V : V \to V$ corresponding to fermion parity to make them into super vector spaces $V = V_0 \oplus V_1 \in \mathrm{sVect}$. As in Section 2.1, we then require the symmetric braiding β on sVect to come with a Koszul sign:

$$\beta_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

For example, suppose $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$ is a one particle Hilbert space, which is $\mathbb{Z}/2$-graded by the distinction between fermions and bosons. Since we will be working with quantum field theories in which the one particle Hilbert space has been second quantized, we will need to consider the associated Fock space $V = \mathrm{Sym}\,\mathcal{H}$ of *graded*-symmetric tensor-powers of V. So if $\mathcal{H}$ is a fermion, we put it in odd super degree resulting in a Grassmann algebra, while if it is a boson, we put it in even degree so that we obtain a symmetric algebra.

Now suppose $(\mathcal{H}, R)$ is a finite-dimensional irreducible representation of Spin_n which we think of as a single type of spinful particles. Note that since $c \in \mathrm{Spin}_n$ is central, $R(c) = \pm\,\mathrm{id}_{\mathcal{H}}$ and $\mathcal{H}$ factors through SO_n if and only if $R(c) = \mathrm{id}_{\mathcal{H}}$. We say $(\mathcal{H}, R)$ has integer spin when $R(c) = \mathrm{id}_{\mathcal{H}}$ and half-integer spin otherwise. Note that this terminology is consistent with the usual classification of irreducible representations of Spin_3 in terms of spin in the case $n = 3$. The corresponding Fock space V has an induced operator $R(c)$ measuring the total spin of the multiparticle state modulo

1. More generally, we could have started with more particle content corresponding to a reducible representation of Spin_n with nontrivial super grading respected by the Spin_n-representation.

In the situation where V is not just a super vector space but also a representation R of Spin_n, we can compare the $\mathbb{Z}/2$-grading $R(c)$ given by integer vs half-integer spin with the $\mathbb{Z}/2$-grading given by fermion parity $(-1)^F_V$. If they are the same, then the spin-statistics theorem is satisfied and otherwise not. We will refer to the connection between spin and statistics as the 'spin-statistics connection'. By the 'spin-statistics theorem' we will instead mean that every unitary quantum field theory has a spin-statistics connection. This clarifies the distinction between these two concepts.

For fermionic TFTs we do not quite have an action of Spin_n on state spaces in general, but we can still formulate spin-statistics as follows. Let K be a fermionic internal symmetry group and $G_n(K) \to O_n$ the associated spacetime structure group. The case where $K = \mathrm{Spin}_1 = \mathbb{Z}_2^F$ only contains fermion parity recovers $G_n(K) = \mathrm{Spin}_n$ in this discussion. Let $Z : \mathrm{Bord}_{n,n-1}^{G_n(K)} \to \mathrm{sVect}$ be a fermionic TFT with internal symmetry K and let Y^{n-1} be a time slice in the $G_n(K)$-bordism category. Then $c \in G_n(K)$ is central and squares to one and so induces an involution on Y given by multiplication on the principal $G_n(K)$-bundle by c inside the groupoid G-$\mathrm{Str}(Y)$ of G-structures on Y. This induces a mapping cylinder bordism $Y_c : Y \to Y$, which as a manifold with boundary is simply $Y \times [0,1]$, but the identification with Y is twisted on one side with the above involution, compare Corollary 5.2.8.

Definition 6.3.1. Let $Z : \mathrm{Bord}_{n,n-1}^{G_n(K)} \to \mathrm{sVect}$ be a fermionic TFT with internal symmetry K. We say Z has a *spin-statistics connection* if for all time slices Y, the spin flip automorphism is mapped to the super grading:

$$Z(Y_c) = (-1)^F_V.$$

Remark 6.3.2. Often, fermionic QFTs are required to have a spin-statistics connection from the get-go, for example in [27, 71]. In our opinion however, this should be seen only as a consequence of the spin-statistics theorem.

Remark 6.3.3. In Section 5.2 we have seen that there is a symmetric monoidal $B\mathbb{Z}/2$-action on $\mathrm{Bord}_{n,n-1}^{G_n(K)}$ given by mapping Y to the automorphism $Y_c : Y \to Y$. There is also a symmetric monoidal $B\mathbb{Z}/2$-action on sVect given by mapping V to $(-1)^F_V$. Note that Z has a spin-statistics connection if and only if it is $B\mathbb{Z}/2$-equivariant. In this formulation it is clear that for a once- or higher extended TFTs a spin-statistics connection becomes data.

Example 6.3.4. We illustrate the spin-statistics theorem for a one-dimensional oriented TFT with target sVect. The structure group SO_n is of the form $G_n(K)$ for the internal bosonic symmetry group $K = 1$. In particular, since $c = 1$, the induced $B\mathbb{Z}/2$-action on the bordism category $\mathrm{Bord}_{1,0}^{SO_1}$ is trivial. Such a theory

$$Z : \mathrm{Bord}_{1,0}^{SO_1} \to \mathrm{sVect}$$

is classified by a single finite-dimensional super vector space $Z(*) = V$. The partition function $Z(S^1) = \mathrm{sdim}\, V$ is given by the trace of the identity in the category sVect. We will now show it has a spin-statistics connection if and only if V is purely even. Now consider the semicircle bordism $\mathrm{coev}_+ : \emptyset \to + \sqcup +$, where $+$ denotes the positively oriented point and $-$ the negatively oriented point. Then, $\mathrm{coev}^\dagger \circ \mathrm{coev} = S^1$ and so a unitary TFT should map S^1 to

$$Z(\mathrm{coev})^\dagger Z(\mathrm{coev}) = \dim_{ungr} V,$$

where the equation follows because sHilb is strong fermionically dagger compact, see Corollary 2.9.14. We conclude that for a unitary TFT we need $\dim_{ungr} V = \mathrm{sdim}\, V$, which can only happen if V is purely even. This argument essentially generalizes to a proof of the spin-statistics theorem in general, see Remark 6.3.8.

The following theorem is the main result of this thesis, the proof of the spin-statistics theorem for unitary fermionic topological field theories.

Theorem 6.3.5. *Let K be a fermionic group and let*

$$Z : (\mathrm{Bord}_{n,n-1}^{G_n(K)})_P \to \mathrm{sHilb}$$

be a unitary TFT with internal symmetry K. Then, Z has a spin-statistics connection.

Proof. This is an immediate consequence of Theorem 5.2.25 and Corollary 2.9.12. □

Corollary 6.3.6. *If K is a bosonic symmetry group, then any unitary TFT lands in purely even complex vector spaces.*

Remark 6.3.7. The theorem immediately generalizes to any other weak fermionically †-compact bordism category. In particular, a Hermitian completion $\mathrm{Herm}(\mathrm{Bord}_{n,n-1}^{G_n(K)})$ is always weak fermionically †-compact. Hence, dagger functors

$$Z : \mathrm{Herm}(\mathrm{Bord}_{n,n-1}^{G_n(K)}) \to \mathrm{sHilb}$$

always satisfy the spin-statistics theorem too. Note that every dagger functor Z induces a unitary TFT in the sense of Definition 6.2.4 by composing with the inclusion $(\mathrm{Bord}_{n,n-1}^{G_n(K)})_P \subseteq \mathrm{Herm}(\mathrm{Bord}_{n,n-1}^{G_n(K)})$. However, using the smaller positivity structure P has the advantage of having many more interesting unitary TFTs, see Remark 6.3.11.

Remark 6.3.8. Since the above proof is based on the very abstract and categorical considerations of Section 2.9, we spell out the complete logic of the proof. So let Y be an object of $(\mathrm{Bord}_{n,n-1}^{G})_P$ and let $Z : (\mathrm{Bord}_{n,n-1}^{G})_P \to \mathrm{sHilb}$ be a unitary TFT with internal symmetry K. We want to show that the spin flip on Y gets mapped to the fermion parity operator of $Z(Y)$ under Z. Let $\mathrm{ev}_Y : Y^* \sqcup Y \rightsquigarrow \emptyset$, $\mathrm{coev}_Y : \emptyset \rightsquigarrow Y \sqcup Y^*$ be the duality pairings making $(\mathrm{Bord}_{n,n-1}^{G})_P$ dagger compact. Topologically they are the macaroni bordisms given by the cylinder on Y, in which we see both boundaries as incoming, respectively outgoing. We can express the mapping torus of the identity on Y as

$$Y \times S^1_{per} = \mathrm{ev}_Y \circ \sigma_{Y,Y^*} \circ \mathrm{coev}_Y =: \mathrm{tr}\,\mathrm{id}_Y = \dim Y,$$

where for the trace we used the canonical pivotal structure of a symmetric monoidal category. Here, the G-structure on the periodic circle S^1_{per} is defined as the mapping torus of the identity on the positively oriented point. Since Z is a symmetric monoidal functor, it is pivotal for the pivotal structures induced by the symmetric monoidal structures. Therefore it preserves the trace and so

$$Z(Y \times S^1_{per}) = Z(\dim Y) = \dim_s Z(Y)$$

since the canonical trace in the category sVect is the supertrace. On the other hand, the dagger dimensions are also mapped to each other, which results in the computation

$$\dim_\dagger Z(Y) = Z(\mathrm{coev}_Y)^\dagger Z(\mathrm{coev}_Y) = Z(\mathrm{coev}_Y^\dagger \,\mathrm{coev}_Y) = Z(\mathrm{tr}\, Y_c) = \mathrm{tr}_s\, Z(Y_c). \tag{6.4}$$

Here, we implicitly used that the functor Z is symmetric to ensure that the respective anti-involutive dual functors are mapped to each other, see Proposition 2.7.20 and Remark 2.7.5. The G-manifold

$\operatorname{tr} Y_c$ appearing here deserves the name $S^1_{ap} \times Y$, as it generalizes the analogous spin manifold. Since the dagger dimension is given by the ungraded dimension, and $Z(Y_c)$ only has eigenvalues ± 1, we conclude from the computation (6.4) that $Z(Y_c)$ must be the grading of $Z(Y)$. This difference between these two types of dimensions is a consequence of the fact that the Hermitian pairing on Y^* is the dual of the Hermitian pairing on Y composed with Y_c, see Corollary 4.6.22.

Example 6.3.9. Let $Z : \mathrm{Bord}^{SO_2}_{2,1} \to \mathrm{sVect}$ be a unitary TFT with $K = 1$ so that Z lands in Vect. If we forget about the structure of Hermitian pairings on state spaces, Z is classified by a finite-dimensional commutative complex Frobenius algebra A given by $(Z(S^1), Z(D^2) : Z(S^1) \to \mathbb{C})$ [42]. The bilinear pairing $A \cong A^*$ induced by the Frobenius structure and the positive definite Hermitian pairing $A \cong \overline{A}^*$ together give an algebra isomorphism $A \cong \overline{A}$ making A into a commutative C^*-algebra. In particular, A is semisimple and so a direct sum of copies of $\mathbb{C}$, also see [78].

Remark 6.3.10. We have provided TFTs in spacetime dimension one that do not have a spin-statistics connection and such examples can be constructed in arbitrary odd dimensions. However, we are not aware of any TFTs in even dimension that do not have a spin-statistics connection. It can be shown by an ad-hoc computation using the cobordism hypothesis that two-dimensional extended TFTs with target the 2-category sAlg of superalgebras always admit the structure of a spin-statistics connection. We however do not have a conceptual proof for this fact and are agnostic about whether it should generalize to higher dimensions.

Remark 6.3.11. Since dagger functors correspond to anti-involutive functors preserving positivity structures, we have that every symmetric monoidal dagger functor

$$Z : \mathrm{Herm}\left(\mathrm{Bord}^{G_n(K)}\right) \to \mathrm{sHilb}$$

induces a unitary TFT

$$(\mathrm{Bord}^{G_n(K)})_P \to \mathrm{sHilb}\,.$$

In particular, Z satisfies spin-statistics. However, many TFTs are not of this form. In other words, requiring all Hermitian pairings to be positive in the bordism category is way too restrictive. For example, if h_Y is a Hermitian pairing on the time slice Y and we require both h_Y and $h_Y \circ Y_c$ to be positive, then this will enforce $Z(Y_c)$ to be a positive operator. By the spin-statistics theorem it then follows that $Z(Y)$ is purely even. More generally, suppose that we modify a positive Hermitian pairing h_Y by a self-adjoint automorphism $X : Y \rightsquigarrow Y$ in the usual bordism dagger category of Definition 5.2.18. Then, making a new bordism dagger category $(\mathrm{Bord}^{G_n(K)})_{P'}$ in which h_Y and $h_Y \circ X$ are both positive will enforce $Z(X)$ to be a positive operator for all dagger functors

$$(\mathrm{Bord}^{G_n(K)})_{P'} \to \mathrm{sHilb}\,.$$

It is well-established that time-preserving symmetries act unitarily on state spaces while time-reversing symmetries act antiunitarily. We now make this precise for unitary fermionic TFTs. This is essentially a consequence of Corollary 5.2.30 and Proposition 5.2.24. We briefly review the setup of those results. For simplicity, we will assume the spacetime dimension is at least two in this discussion, but the results also hold in spacetime dimension one with the appropriate adaptations related to how Hermitian pairings are defined for G-structures without suspensions. Let K be a fermionic group and let P denote the standard positivity structure on $\mathrm{Bord}^G_{n,n-1}$ for $G = G_n(K)$ given in Definition 5.2.17. Let $Z : \mathrm{Bord}^G_{n,n-1} \to \mathrm{sVect}$ be a TFT with reflection structure and Y^{n-1} a time slice such that the underlying principal G-bundle $Q = Y \times G$ is trivialized. Recall that left multiplication by $k \in K \subseteq \hat{G}$ induces $\hat{G}(K)$-structure automorphisms of $\hat{Q} = Y \times \hat{G}$. Seeing Z as a

$\mathbb{Z}/2$-equivariant functor, we get a representation of K on $Z(Y)$ such that time-preserving elements of K act complex-linearly and time-reversing elements act anti-linearly. The action of the group is moreover by unitary/antiunitary operators:

Lemma 6.3.12. *Let K be a fermionic group and $Z : (\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHilb}$ a unitary fermionic TFT with $G = G_n(K)$. Let Y^{n-1} be a G-manifold and let $f : Y \to \overline{Y}$ be a G-diffeomorphism. Then, there is a canonically induced antiunitary map*

$$Z(Y) \xrightarrow{f} Z(\overline{Y}) \cong \overline{Z(Y)}.$$

Proof. Note that since Z is a symmetric monoidal dagger functor, it comes equipped with canonical $\mathbb{Z}/2$-equivariance data by Theorem 6.2.7. In other words, from the fact that it preserves duals and the anti-involution, we obtain the isomorphism $Z(\overline{Y}) \cong \overline{Z(Y)}$. Because the bordism category is weak fermionically dagger compact, the Hermitian pairing $h_{\overline{Y}}$ on $\overline{Y}$ and $\overline{h_Y}$ differ by a spin flip, also see Remark 2.7.31. We therefore obtain that the induced Hermitian pairing on $\overline{Z(Y)}$ differs from the bar of the Hermitian pairing on $Z(Y)$ by $Z(Y_c)$. By the spin-statistics theorem, we obtain that the Hermitian pairing on $\overline{Z(Y)}$ is the canonical one up to a spin-flip and so a unitary map $Z(Y) \to \overline{Z(Y)}$ is the same as an antiunitary map. By Proposition 5.2.24 it follows that the mapping cylinder of f defines a unitary isomorphism $Y \to \overline{Y}$ in $(\mathrm{Bord}^G_{n,n-1})_P$ and so an antiunitary map $Z(Y) \to \overline{Z(Y)}$. □

Proposition 6.3.13. *Let K be a fermionic group and $Z : (\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHilb}$ a unitary fermionic TFT with $G = G_n(K)$. Let Y^{n-1} be a time slice such that the principal G-bundle $Q = Y \times G$ is trivialized. Then, the Hilbert space $Z(Y)$ has the canonical structure of a representation $\rho : K \to GL_{\mathbb{R}}(Z(Y))$ of K such that $\rho(k)$ is unitary when k is time-preserving and antiunitary when it is time-reversing.*

Proof. We have seen in Corollary 5.2.30 that K induces a collection of G-diffeomorphisms $Y \to Y$ for $k \in K_{pres}$ and $Y \to \overline{Y}$ for $k \in K_{rev}$. These G-diffeomorphisms give bordisms by the mapping cylinder construction and are compatible under composition. It follows by Lemma 6.3.12 that the time-reversing elements k give antiunitary operators $\rho(k)$ of Y and similarly time-preserving elements give unitary operators. □

Remark 6.3.14. The same proof tells us that if Z is a Hermitian TFT with internal symmetry K, then all state spaces $Z(Y)$ come equipped with automorphisms $\rho(k) : Z(Y) \to Z(Y)$ such that

$$\rho(k_1)\rho(k_2) = (-1)^F_{Z(Y)}\rho(c)\rho(k_1k_2)$$

if $k_1, k_2 \in K$ are both time-reversing, and similar equations when only one or neither is time-reversing. Note that indeed the action $\rho(c)$ of $c \in K$ on $Z(Y)$ agrees with the spin flip $B\mathbb{Z}/2$-action $Z(Y_c)$ under Z. This also gives another perspective on Example 6.2.13: a one-dimensional Hermitian Pin^+-TFT has internal symmetry group Pin^-_1 consisting of a single time-reversal symmetry with square c. This gives an antiunitary operator T on the super Hermitian vector space assigned to the point satisfying $T^2 = (-1)^F_{Z(Y)}$. Since quaternionic structures only exist on even-dimensional complex vector spaces, we see that the odd part of $Z(Y)$ has to be even dimensional. This in particular recovers the observation that the invertible Pin^+-TFT which assigns the odd line to a point has no reflection structure.

Remark 6.3.15. Physically we like to think of the mapping cylinder of left multiplication by $k \in K$ as putting the corresponding codimension one symmetry defect orthogonal to the direction of time. From this perspective, it is not surprising that forcing a state in a TFT through this symmetry defect gives a representation of the symmetry group on the state spaces. We expect this perspective to generalize well to higher (form) symmetry groups and extended TFTs, in which higher symmetries could be realized as bordisms between bordisms.

Remark 6.3.16. Note that the representation obtained in Proposition 6.3.13 factors through $\pi_0(K)$. It would be interesting to study appropriate smooth versions in TFTs in the sense of fibred categories over the site of manifolds [70, section 2.7] and see whether it it possible to recover smooth representations of K. Another possible direction to generalize in would be to look at higher bordism categories which could potentially see higher homotopy groups of K.

It would also be interesting to study in how far the above proposition generalizes to the case where Q is nontrivial. Recall that in the case where K is bosonic and time-preserving, Q is the data of a principal K-bundle and the tangent bundle with its metric. If the tangent bundle is nontrivial but the principal K-bundle is trivial, there will still be an induced K-action by left multiplication on Q and hence a unitary representation on the state spaces of a unitary TFT. For a general fermionic group K, spacetime does not come equipped with a principal K-bundle because fermion parity mixes with the spin group, but there is a K_b-principal bundle. We therefore conjecture

Conjecture 6.3.17. *Let K be a fermionic group and $Z : (\mathrm{Bord}^G_{n,n-1})_P \to \mathrm{sHilb}$ a unitary fermionic TFT with $G = G_n(K)$. Let (Y^{n-1}, Q, β) be a time slice equipped with a trivialization of the principal K_b-bundle $Q \times_G K_b$ obtained by pushing along the projection homomorphism $G \to K_b$. Then, the Hilbert space $Z(Y)$ has the structure of a representation ρ of K such that $\rho(k)$ is unitary when k is time-preserving and antiunitary when it is time-reversing. The representation agrees with the representation of Proposition 6.3.13 in case Q is trivial.*

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Appendix A

Category theory

A.1 Monoidal categories and duality

Let $(\mathcal{C}, \otimes, 1, \alpha)$ be a monoidal category. We refer to [73, Chapter I] and [49, Chapters VII and XI] for an introduction to monoidal, braided monoidal and symmetric monoidal categories. Using Mac Lane's coherence theorem for monoidal categories [49, section VII.2], we will typically assume $\mathcal{C}$ is strictly unital and associative without loss of generality. We also often assume functors are strictly unital. We denote by $\mathcal{C}^{\otimes \mathrm{op}}$ the category $\mathcal{C}$ with reversed tensor product and by $\mathcal{C}^{\circ \mathrm{op}}$ the category with reversed composition but nonreversed tensor product.

Definition A.1.1. A *duality pairing* between $x, y \in \mathcal{C}$ consists of two morphisms $\mathrm{ev}_{x,y} : x \otimes y \to 1$ and $\mathrm{coev}_{x,y} : 1 \to y \otimes x$ satisfying the triangle identities.

Given objects $x, y \in \mathcal{C}$, there might or might not exist a duality pairing between them. Even if some duality pairing does exist, an arbitrary map $\mathrm{ev}_{x,y} : x \otimes y \to 1$ might or might not extend to a duality pairing between them. However, if $\mathrm{ev}_{x,y}$ extends to a duality pairing, then $\mathrm{coev}_{x,y}$ is uniquely determined. A (left) dual of an object $x \in \mathcal{C}$ is an object $x^* \in \mathcal{C}$, equipped with a duality pairing $\mathrm{ev}_x : x^* \otimes x \to 1$, $\mathrm{coev}_x : 1 \to x \otimes x^*$ between them. In that case we also call x a right dual of x^*. Duals are unique in the sense that for a different choice $\mathrm{ev}'_x : x' \otimes x \to 1$ of dual, there is a unique isomorphism $x' \cong x^*$ commuting with the evaluation maps, or equivalently with the coevaluation maps. This isomorphism is explicitly given by

$$x^* \xrightarrow{\mathrm{id}_{x^*} \otimes \mathrm{coev}'_x} x^* \otimes x \otimes x' \xrightarrow{\mathrm{ev}_x \otimes \mathrm{id}_{x'}} x'. \tag{A.1}$$

The dual of a morphism $f : c_1 \to c_2$ is the unique morphism $f^* : c_2^* \to c_1^*$ making the diagram

$$\begin{tikzcd}
c_2^* \otimes c_1 \arrow[r, "\mathrm{id}_{c_2^*} \otimes f"] \arrow[d, "f^* \otimes \mathrm{id}_{c_2}"'] & c_2^* \otimes c_2 \arrow[d, "\mathrm{ev}_{c_2}"] \\
c_1^* \otimes c_1 \arrow[r, "\mathrm{ev}_{c_1}"] & 1
\end{tikzcd}$$

commute. Equivalently, we could have required the above diagram for the coevaluation instead. Concretely, we can express the dual as the composition

$$c_2^* \xrightarrow{\mathrm{id}_{c_2^*} \otimes \mathrm{coev}_{c_1}} c_2^* \otimes c_1 \otimes c_1^* \xrightarrow{\mathrm{id}_{c_2^*} \otimes f \otimes \mathrm{id}_{c_1^*}} c_2^* \otimes c_2 \otimes c_1^* \xrightarrow{\mathrm{ev}_{c_2}} c_1^*.$$

It is straightforward to check using the triangle identities that the diagram

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{ev}_x^*"] \arrow[d, "\mathrm{coev}_{x^*}"] & (x^* \otimes x)^* \\ x^* \otimes x^{**} \arrow[ur] & \end{tikzcd}$$

commutes. Note that $1^* = 1$ can be realized as a dual of itself by the isomorphism $1 \otimes 1 \to 1$. Duality data equips $\mathcal{C}$ with a dual functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. The dual functor is even monoidal in a certain sense:

Example A.1.2. Let $c_1, c_2 \in \mathcal{C}$ come equipped with duality data. Then, $c_2^* \otimes c_1^*$ is canonically a dual of $c_1 \otimes c_2$ via

$$c_2^* \otimes c_1^* \otimes c_1 \otimes c_2 \xrightarrow{\mathrm{id}_{c_2^*} \otimes \mathrm{ev}_{c_1} \otimes \mathrm{id}_{c_2}} c_2^* \otimes c_2 \xrightarrow{\mathrm{ev}_{c_2}} 1.$$

This defines a monoidal natural isomorphism between the two monoidal functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}^{\mathrm{oop}}$ given by the tensor product followed by the dual functor and the tensor product of the dual functor. It follows that if we choose duality data on all objects, then the induced dual functor is canonically a monoidal functor

$$(.)^* : \mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op},\mathrm{oop}}.$$

The following two lemmas are a convenient setting to make dual functors and uniqueness of duals functorial, generalizing some of the discussion above. These results were communicated to us by Jan Steinebrunner.

Lemma A.1.3. *Suppose $G\colon \mathcal{C} \to \mathcal{D}$ is a monoidal functor and we are given any collection of duality pairings*

$$\mathrm{ev}_x^G \colon (Gx)' \otimes G(x) \to 1,$$

indexed by $x \in \mathcal{C}$ satisfying no further conditions. Then, there is a well-defined and unique monoidal functor $F\colon \mathcal{C} \to \mathcal{D}^{\mathrm{oop},\otimes\mathrm{op}}$ such that $(Gx)' = F(x)$ and $\{\mathrm{ev}_\bullet^G\}$ defines a duality pairing between F and G.

Example A.1.4. Take $G = \mathrm{id}$ and suppose we are given dualities $\mathrm{ev}_x : x^* \otimes x \to 1$ on every object. Then, the functor F obtained in Lemma A.1.3 is the monoidal dual functor associated to the duality pairings $\mathrm{ev}_x : F(x) \otimes x \to 1$ in the sense of Example A.1.2.

Lemma A.1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor and let G and G' be monoidal functors $\mathcal{C} \to \mathcal{D}^{\otimes\mathrm{op},\mathrm{oop}}$. Suppose we have chosen for every $x \in \mathcal{C}$ a duality pairing between $F(x)$ and $G(x)$*

$$\mathrm{ev}_x^{G,F} \colon G(x) \otimes F(x) \to 1,$$

such that the diagrams

$$\begin{tikzcd} G(x) \otimes F(y) \arrow[r, "Gf \otimes \mathrm{id}_{F(y)}"] \arrow[d, "\mathrm{id}_{G(x)} \otimes Ff"'] & G(y) \otimes F(y) \arrow[d, "\mathrm{ev}_y^{G,F}"] \\ G(x) \otimes F(x) \arrow[r, "\mathrm{ev}_x^{G,F}"] & 1 \end{tikzcd}$$

$$\begin{tikzcd} G(y) \otimes G(x) \otimes F(x) \otimes F(y) \arrow[r, "\mathrm{id}_{G(y)} \otimes \mathrm{ev}_x^{G,F} \otimes \mathrm{id}_{F(y)}"] \arrow[d, "\cong"] & G(y) \otimes F(y) \arrow[d, "\mathrm{ev}_y^{G,F}"] \\ G(x \otimes y) \otimes F(x \otimes y) \arrow[r, "\mathrm{ev}_{x \otimes y}^{G,F}"] & 1 \end{tikzcd}$$

commute for all $x, y \in \mathcal{C}$ and $f: y \to x$ and similarly for G'. Then there is a unique monoidal natural isomorphism $\eta\colon G \cong G'$ such that the diagram

$$\begin{tikzcd} & G'(x) \otimes F(x) \arrow[dr, "\mathrm{ev}_x^{G',F}"] & \\ G(x) \otimes F(x) \arrow[ur, "\eta_x \otimes \mathrm{id}_{F(x)}"] \arrow[rr, "\mathrm{ev}_x^{G,F}"] & & 1 \end{tikzcd}$$

commutes for all $x \in \mathcal{C}$.

Example A.1.6. If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor and $\mathrm{ev}_c : c^* \otimes c \to 1$ a duality, then

$$F(x^*) \otimes F(x) \to F(x^* \otimes x) \xrightarrow{F(\mathrm{ev}_c)} F(1) \cong 1$$

is a duality. This gives a canonical monoidal natural isomorphism μ_F between the two monoidal functors $\mathcal{C} \to \mathcal{D}^{\circ\mathrm{op},\otimes\mathrm{op}}$ given by $F \circ (.)^*$ and $(.)^* \circ F$.

Example A.1.7. More generally, if $F\colon \mathcal{C} \to \mathcal{D}^{\otimes\mathrm{op},\circ\mathrm{op}}$ and $G\colon \mathcal{C} \to \mathcal{D}$ are functors with a duality pairing $\{\mathrm{ev}_\bullet^{F,G}\}$ between them and $K\colon \mathcal{D} \to \mathcal{E}$ is a monoidal functor, then

$$\mathrm{ev}_x^{KF,KG} := \left(KF(x) \otimes KG(x) \cong K(F(x) \otimes G(x)) \xrightarrow{K(\mathrm{ev}_x)} 1\right)$$

is a duality paring between $KF\colon \mathcal{C} \to \mathcal{E}$ and $KG\colon \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}} \to \mathcal{E}$.

Similarly, if $L\colon \mathcal{B} \to \mathcal{C}$ is a monoidal functor then

$$\mathrm{ev}_x^{FL,GL} := \mathrm{ev}_{L(x)}$$

is a duality paring between $FL\colon \mathcal{B} \to \mathcal{D}^{\circ\mathrm{op}\otimes\mathrm{op}}$ and $GL\colon \mathcal{B} \to \mathcal{D}$. Suppose now $G'\colon \mathcal{C} \to \mathcal{D}$ is another functor equipped with a duality pairing to F and denote the induced monoidal transformation by $\eta\colon G \Rightarrow G'$. Then the canonical transformations $K \circ G \Rightarrow K \circ G'$ and $G \circ K \Rightarrow G' \circ K$ induced by the pairings of above, are given by $x \mapsto K(\eta_x)$ and $x \mapsto \eta_{K(x)}$ respectively.

Example A.1.8. Let $(.)^* : \mathcal{C} \to \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}}$ be a (left) dual functor. Then the induced functor $(.)^* : \mathcal{C}^{\circ\mathrm{op}} \to \mathcal{C}^{\otimes\mathrm{op}}$ is a right dual functor for the duality in which ev_c and coev_c are exchanged. Similarly, $(.)^* : \mathcal{C}^{\otimes\mathrm{op}} \to \mathcal{C}^{\circ\mathrm{op}}$ is a right dual functor and $(.)^* : \mathcal{C}^{\circ\mathrm{op},\otimes\mathrm{op}} \to \mathcal{C}$ is a left dual functor.

We provide a few lemmas on duality that will be useful in the main text.

Lemma A.1.9. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be monoidal functors with a monoidal natural isomorphism $\phi : F \Rightarrow G$. Then, the following diagram commutes*

$$\begin{tikzcd} F(x^*) \arrow[r, "\phi_{x^*}"] \arrow[d, "\mu_F"] & G(x^*) \arrow[d, "\mu_G"] \\ F(x)^* & G(x)^* \arrow[l, "\phi_x^*"] \end{tikzcd} .$$

Proof. It suffices to show that the composite isomorphism

$$F(x^*) \xrightarrow{\phi_{x^*}} G(x^*) \xrightarrow{\mu_G} G(x)^* \xrightarrow{\phi_x^*} F(x)^*$$

satisfies the property that characterises $\mu_F \colon F(x^*) \cong F(x)^*$. This means we have to show the diagram

$$\begin{tikzcd}
F(x^*) \otimes F(x) \arrow[r, "\phi_{x^*} \otimes \mathrm{id}_{F(x)}"] \arrow[d, "\mathrm{id}_{F(x^*)} \otimes \phi_x"] \arrow[dd, bend right=60] & G(x^*) \otimes F(x) \arrow[r, "\mu_G \otimes \mathrm{id}_{F(x)}"] \arrow[d, "\mathrm{id}_{G(x^*)} \otimes \phi_x"] & G(x)^* \otimes F(x) \arrow[r, "\phi_x^* \otimes \mathrm{id}_{F(x)}"] \arrow[d, "\mathrm{id}_{G(x)^*} \otimes \phi_x"] & F(x)^* \otimes F(x) \arrow[d, "\mathrm{ev}_{F(x)}"] \\
F(x^*) \otimes G(x) \arrow[r, "\phi_{x^*} \otimes \mathrm{id}_{G(x)}"] & G(x^*) \otimes G(x) \arrow[r, "\mu_G \otimes \mathrm{id}_{G(x)}"] \arrow[d] & G(x)^* \otimes G(x) \arrow[r, "\mathrm{ev}_{G(x)}"] & 1 \arrow[d] \arrow[dl] \\
F(x^* \otimes x) \arrow[r, "\phi_{x^* \otimes x}"] \arrow[rrr, bend right, "F(\mathrm{ev}_x)"'] & G(x^* \otimes x) \arrow[r, "G(\mathrm{ev}_x)"] & G(1) & F(1) \arrow[l, "\phi_1"]
\end{tikzcd}$$

commutes. The two leftmost upper squares commute by an interchange and the right upper square commutes by the universal property of the dual of a morphism. The lower left figure and the triangle commute because ϕ is monoidal. The lowest figure commutes because ϕ is natural and the remaining square commutes because μ_G expresses uniqueness of duals. $\square$

Lemma A.1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be monoidal functors between monoidal categories equipped with dual functors. Let $\phi_x : F(x)^* \to F(x^*)$ and $\gamma_x : G(x)^* \to G(x^*)$ be the natural isomorphisms from Example A.1.6. The diagram*

$$\begin{tikzcd}
F(G(x)^{**}) & F(G(x^*)^*) \arrow[l, "F(\gamma_x^*)"'] \\
F(G(x)^*)^* \arrow[u, "\phi_{G(x)^*}"] & F(G(x^*))^* \arrow[l, "F(\gamma_x)^*"] \arrow[u, "\phi_{G(x^*)}"']
\end{tikzcd}$$

commutes.

Proof. This follows from the fact that ϕ is a natural transformation. $\square$

We now want to show that if $\mathcal{C}$ is a braided monoidal category with duals, then a dual functor $\mathcal{C} \to \mathcal{C}^{\circ\mathrm{op}, \otimes\mathrm{op}}$ is braided monoidal. For this we first need the following lemma:

Lemma A.1.11. *The diagram*

$$\begin{tikzcd}
c_1 \otimes c_2 \arrow[r, "\sigma_{c_1, c_2}"] \arrow[d, "\mathrm{coev}_{c_2}"] & c_2 \otimes c_1 \\
c_2 \otimes c_2^* \otimes c_1 \otimes c_2 \arrow[r, "\sigma^{-1}_{c_1, c_2^*}"] & c_2 \otimes c_1 \otimes c_2^* \otimes c_2 \arrow[u, "\mathrm{ev}_{c_2}"']
\end{tikzcd}$$

commutes.

Proof. We consider the following diagram:

$$\begin{tikzcd}
c_1 \otimes c_2 \arrow[rr, "\sigma_{c_1,c_2}"] \arrow[d, "\mathrm{coev}_{c_2}"'] & & c_2 \otimes c_1 \arrow[dl, "\mathrm{coev}_{c_2}"'] \\
& c_2 \otimes c_2^* \otimes c_2 \otimes c_1 \arrow[ur, bend right, "\mathrm{ev}_{c_2}"'] & \\
c_2 \otimes c_2^* \otimes c_1 \otimes c_2 \arrow[ur, "\sigma_{c_1,c_2}"] \arrow[rr, "\sigma^{-1}_{c_1,c_2^*}"'] & & c_2 \otimes c_1 \otimes c_2^* \otimes c_2 \arrow[ul, "\sigma_{c_1,c_2^*\otimes c_2}"'] \arrow[uu, "\mathrm{ev}_{c_2}"']
\end{tikzcd}.$$

The upper left part commutes by an interchange. The lower triangle commutes by a hexagon identity of the braiding. The right part commutes by naturality of the braiding in the second tensor factor applied to the morphism ev_{c_2}. We are done by applying the snake identity. $\square$

The lemma above also holds on the other side, the proof being analogous.

Lemma A.1.12. *If σ is a braiding on the rigid monoidal category $\mathcal{C}$ and $c_1, c_2 \in \mathcal{C}$ then*

$$\begin{tikzcd}
c_1^* \otimes c_2^* \arrow[r, "\sigma_{c_1^*,c_2^*}"] \arrow[d, "\sim"] & c_2^* \otimes c_1^* \arrow[d, "\sim"] \\
(c_2 \otimes c_1)^* \arrow[r, "\sigma^*_{c_1,c_2}"] & (c_1 \otimes c_2)^*
\end{tikzcd}$$

commutes.

Proof. We provide a string diagram proof in Figure A.1 using the previous lemma. $\square$

The double dual need not be isomorphic to the original object in general.

Definition A.1.13. Let $\mathcal{C}$ be a rigid category equipped with a monoidal dual functor $(.)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{oop},\otimes\mathrm{op}}$. A *pivotal* structure ϕ is a monoidal natural isomorphism from the identity functor on $\mathcal{C}$ to

$$(.)^{**} : \mathcal{C} \to \mathcal{C}.$$

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between pivotal categories is called *pivotal*, if it preserves the pivotal structure with respect to the canonical isomorphisms $F(c^*) \cong F(c)^*$ for all objects $c \in \mathcal{C}$. If $f : c \to c$ is an endomorphism, then its *left trace* with respect to ϕ is the endomorphism of 1 given by

$$1 \xrightarrow{\mathrm{coev}_c} c \otimes c^* \xrightarrow{f\otimes \mathrm{id}_{c^*}} c \otimes c^* \xrightarrow{\phi_c \otimes \mathrm{id}_{c^*}} c^{**} \otimes c^* \xrightarrow{\mathrm{ev}_{c^*}} 1.$$

The *right trace* is given by

$$1 \xrightarrow{\mathrm{coev}_{c^*}} c^* \otimes c^{**} \xrightarrow{\mathrm{id}_{c^*} \otimes \phi_c^{-1}} c^* \otimes c \xrightarrow{\mathrm{id}_{c^*} \otimes f} c^* \otimes c \xrightarrow{\mathrm{ev}_c} 1.$$

A pivotal structure ϕ is called *spherical* if left and right traces agree.

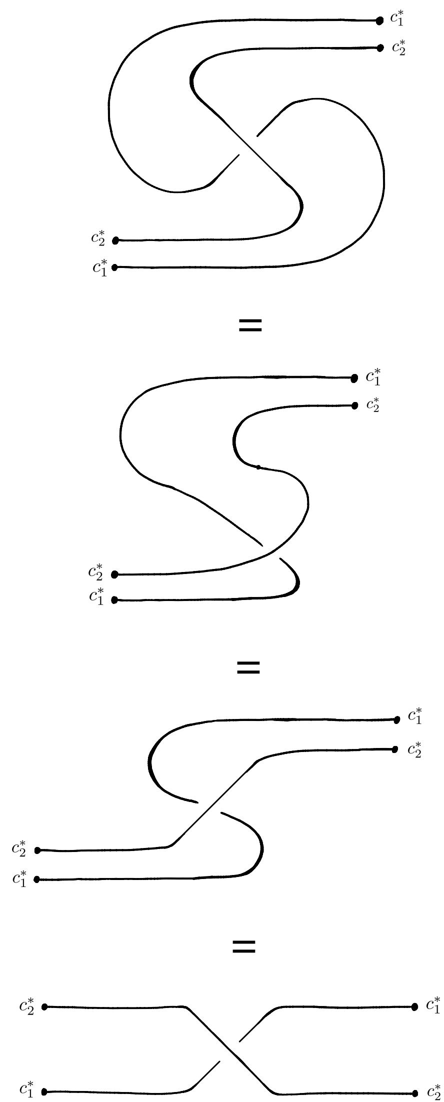

Figure A.1: A string diagram proof of Lemma A.1.12, which we read from right to left. In the first line, we use Lemma A.1.11 to change the order of the crossing, for which we had to introduce an extra coevaluation in c_2 above the crossing. In the second line, we use the snake identity to cancel this new coevaluation with the evaluation above it. In the last line, we apply the last lemma again on the other side.

If $\mathcal{C}$ has a symmetric braiding σ, then there is a canonical pivotal structure Φ coming from the fact that c^* is also a dual of c^{**} under

$$c^* \otimes c^{**} \xrightarrow{\sigma_{c^*,c^{**}}} c^{**} \otimes c^* \xrightarrow{\mathrm{ev}_{c^*}} 1.$$

Explicitly, this means that Φ_c is the composition

$$c \xrightarrow{\mathrm{coev}_{c^*} \otimes \mathrm{id}_c} c^* \otimes c^{**} \otimes c \xrightarrow{\mathrm{id}_{c^*} \otimes \sigma_{c^{**},c}} c^* \otimes c \otimes c^{**} \xrightarrow{\mathrm{ev}_c \otimes \mathrm{id}_{c^{**}}} c^{**}. \tag{A.2}$$

This pivotal structure is moreover spherical [73, Lemma 3.5, Corollary 3.6]. Note that the isomorphism Φ_c can also be defined when $\mathcal{C}$ is only braided but not symmetric. However, in that case Φ will not be a monoidal natural transformation and so $\mathcal{C}$ is not canonically pivotal.

Lemma A.1.14. *[73, Exercise 3.3.5] Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal categories with duals and let $\Phi_x : x \to x^{**}$ denote the canonical pivotal structure on $\mathcal{C}$ and $\mathcal{D}$. Then F is pivotal.*

Proof. Let $\phi_x : F(x^*) \cong F(x)^*$ denote the natural isomorphism specifying uniqueness of duals. We have to show that the diagram

$$\begin{tikzcd}
F(x) \arrow[r, "\Phi_{F(x)}"] \arrow[d, "F(\Phi_x)"] & F(x)^{**} \\
F(x^{**}) & F(x^*)^* \arrow[l, "\phi_{x^*}"] \arrow[u, "\phi_x^*"']
\end{tikzcd}$$

commutes. For this it suffices to show that going from the upper left to the upper right corner through the downward path is an isomorphism expressing uniqueness of duals. So we need to show that the composition intertwines the relevant evaluation maps as in the following diagram, in which we omitted tensors with identities on arrows for reasons of space.

$$\begin{tikzcd}
 & F(x) \otimes F(x)^* \arrow[rr, "\sigma"] \arrow[dl, "F(\Phi_x)"'] & & F(x)^* \otimes F(x) \arrow[dll, "F(\Phi_x)"'] \arrow[dl, "\phi_x"] \arrow[dd, bend left, "\mathrm{ev}_{F(x)}"] \\
F(x^{**}) \otimes F(x)^* \arrow[r, "\sigma"] \arrow[ddr, "\phi_x"] & F(x)^* \otimes F(x^{**}) \arrow[d, "\phi_x"] & F(x^*) \otimes F(x) \arrow[r] \arrow[dl, "F(\Phi_x)"] & F(x^* \otimes x) \arrow[d, "F(\mathrm{ev}_x)"'] \\
 & F(x^*) \otimes F(x^{**}) \arrow[d, "\sigma"] & & 1 \\
F(x^*)^* \otimes F(x)^* \arrow[uu, "\phi_{x^*}"] \arrow[drr, "\phi_x"] \arrow[dr, bend right, "\phi_x^*"'] & F(x^{**}) \otimes F(x^*) \arrow[r] & F(x^{**} \otimes x^*) \arrow[ur, bend left, "F(\mathrm{ev}_{x^*})"] & \\
 & F(x)^{**} \otimes F(x)^* \arrow[uurr, bend right, "\mathrm{ev}_{F(x)^*}"'] & F(x^*)^* \otimes F(x^*) \arrow[ul, "\phi_{x^*}"'] \arrow[uur, "\mathrm{ev}_{F(x^*)}"'] &
\end{tikzcd}$$

To show this diagram commutes we need to use the naturality of the braiding twice. We also use that ϕ_x^* is the dual of ϕ_x, the fact that F preserves the braiding, the fact that ϕ_{x^*}, ϕ_x and Φ_x express uniqueness of duals. $\square$

Lemma A.1.15. *If $\mathcal{C}$ is a symmetric monoidal category with duals and $\Phi_x : x \to x^{**}$ denotes the canonical pivotal structure, then Φ_{x^*} and Φ_x^* are inverses:*

$$\begin{tikzcd} x^* \arrow[r, bend left, "\Phi_{x^*}"] & x^{***}. \arrow[l, bend left, "\Phi_x^*"] \end{tikzcd}$$

Proof. It suffices to show that Φ_x^* is the isomorphism specifying uniqueness of duals of x^{**}. We show that it preserves the appropriate coevaluation maps. Consider the diagram

$$\begin{tikzcd} x^{**} \otimes x^{***} \arrow[d] \arrow[drr, "\sigma_{x^{**},x^{***}}"] & & \\ (x^{**} \otimes x^*)^* \arrow[r, "\sigma^*_{x^*,x^{**}}"] & (x^* \otimes x^{**})^* \arrow[d, "(\mathrm{id}_{x^*} \otimes \Phi_x)^*"] & x^{***} \otimes x^{**} \arrow[l] \arrow[d, "\Phi_x^* \otimes \mathrm{id}_{x^{**}}"] \\ 1 \arrow[u, "\mathrm{ev}^*_{x^*}"] \arrow[uu, bend left=60, "\mathrm{coev}_{x^{**}}"] \arrow[r, "\mathrm{ev}_x^*"] & (x^* \otimes x)^* & x^* \otimes x^{**} \arrow[l] \end{tikzcd}$$

The upper triangle commutes by Lemma A.1.12. The left square commutes by taking the dual of the diagram

$$\begin{tikzcd} x^{**} \otimes x^* \arrow[d, "\mathrm{ev}_{x^*}"] & x^* \otimes x^{**} \arrow[l, "\sigma_{x^*,x^{**}}"'] \\ 1 \arrow[r, bend right, "\mathrm{coev}_{x^*}"'] & x^* \otimes x \arrow[l, "\mathrm{ev}_x"] \arrow[u, "\mathrm{id}_{x^*} \otimes \Phi_x"'] \end{tikzcd} ,$$

defining Φ_x as a unique dual isomorphism. ☐

Remark A.1.16. The above lemma is actually true for any pivotal structure [65, Lemma 4.11]. Note how this implies in particular that the left trace of f is the right trace of f^*.

Lemma A.1.17. *[58, Lemma 2.14] A pivotal functor preserves traces.*

The following lemma shows †-traces on fermionically †-compact categories are spherical.

Lemma A.1.18. *Let $\phi_x : x \to x^{**}$ be a pivotal structure and $\xi_x : x \to x$ a monoidal natural automorphism of $\mathrm{id}_{\mathcal{C}}$. Then $\phi \circ \xi$ is a pivotal structure. If ϕ is spherical, then $\phi \circ \xi$ is spherical if ξ generates a $B\mathbb{Z}/2$-action.*

Proof. The first statement follows from the fact that that composition preserves monoidal natural isomorphisms. Note that the right trace of $f : c \to c$ with respect to $\phi \circ \xi$ is given by the right trace of $f \circ \xi^{-1}$ with respect to ϕ. The left trace of $f : c \to c$ with respect to $\phi \circ \xi$ is given by the left trace of $\xi \circ f$ with respect to ξ. By cyclicity of the trace, the latter is also equal to the left trace of $f \circ \xi$. We see that if ϕ is spherical and $\xi_x = \xi_x^{-1}$, then $\phi \circ \xi$ is spherical. ☐

A.2 2-groups

Homotopy-theoretically, n-groups are defined as homotopy $(n-1)$-types with a multiplication that is associative up to coherent homotopy.[1] More precisely, they are monoids in the infinity category of spaces such that the underlying space is a homotopy $(n-1)$-type. Equivalently, by May's recognition theorem, we can consider the classifying space of this monoid, which is a connected homotopy n-type. Note that for $n = 1$, a 1-group G is an ordinary group and under the classifying space construction, this is equivalent to a connected homotopy 1-type $K(G, 1)$.

We want a similar down-to-earth algebraic description for 2-groups. Luckily, monoidal categories and bicategories are well-understood. Invoking the homotopy hypothesis, we could therefore define (the classifying space of) a 2-group alternatively as a 2-groupoid with a single object, i.e. a bicategory with a single object in which all 1- and 2-morphisms are invertible under composition. Equivalently, we could define a 2-group as a grouplike monoid in the 2-category of groupoids. A popular model for strict 2-groups are crossed modules and every 2-group is equivalent to a strict 2-group, see [5] for more details on different models of 2-groups. From now on we will work with the latter:

Definition A.2.1. A 2-*group* is a monoidal category $(\mathcal{G}, \otimes, 1, \alpha)$ such that all objects and morphisms are invertible under tensor product and composition.

Example A.2.2. If G is a group, it is a 2-group with object set G, tensor product given by multiplication and only identity morphisms.

Example A.2.3. If A is an abelian group, BA denotes the 2-group with a single object and A as its automorphisms. Note that the automorphisms of the identity object have two commuting multiplications given by composition and tensor product. Therefore this construction only works when A is abelian, by Eckmann-Hilton.

Example A.2.4. Let $\mathcal{B}$ be a bicategory and $b \in \mathcal{B}$ an object. The monoidal category $\mathrm{Aut}(b)$ of invertible automorphisms of b is then a 2-group called its automorphism 2-group. For example, let $\mathcal{B} = \mathrm{Group}_2$ be the 2-category of groups considered as a full subcategory of the 2-category of categories with one object and only invertible morphisms. The automorphism 2-group of an object G of this 2-category has as its objects group automorphisms $f : G \to G$. A morphism $f_1 \to f_2$ is an element $g \in G$ such that $f_2(g') = gf_1(g')g^{-1}$ for all $g' \in G$. The tensor product is given by composition of automorphisms and the composition of 1-morphisms is given by the product in G.

Recall that we will typically assume that monoidal functors are strictly unital:

Definition A.2.5. A 2-*group homomorphism or functor* between 2-groups is a strongly monoidal strictly unital functor.

We review some of the general theory and classification of 2-groups, see [35, section 2] for another review. This will not be used extensively in the main text, but will be essential for the understanding of invertible topological field theories Let $(\mathcal{G}, 1, \otimes, \alpha)$ be a 2-group, where crucially we will not assume that the associator is trivial. If an object c of a monoidal category is invertible, it is possible to choose an inverse c^{-1} and isomorphisms $c \otimes c^{-1} \cong 1 \cong c^{-1} \otimes c$, which satisfy the triangle identities. Therefore, a choice of inverse is equivalent to a choice of dual object with the property that the evaluation and coevaluation maps are isomorphisms. Since dual functors are unique up to unique natural isomorphism, inverses are unique in the same sense. We will from now on assume a chosen

[1] This definition is unrelated to the definition of a p-group for p a prime as a group having only elements of order a power of p.

inverse c^{-1} for every object c, with chosen isomorphisms $c\otimes c^{-1}\cong 1\cong c^{-1}\otimes c$ satisfying the triangle identities, i.e. a dual functor on $\mathcal{G}$. Let $\pi_0 := \pi_0(\mathcal{G})$ denote the set of objects modulo isomorphisms and $\pi_1 := \pi_1(\mathcal{G}) = \operatorname{Aut} 1$. Then π_0 is a group under tensor product. Similarly, π_1 is a group under $\otimes$, but it also has another compatible group structure given by composition. By an Eckmann-Hilton argument it follows that the tensor product and composition are equal on $\operatorname{Aut} 1$ and π_1 is abelian. There is a left action $\rho(c)[f]$ of $c\in\pi_0$ on $f\in\pi_1$ given by conjugation $\mathrm{id}_c\otimes f\otimes \mathrm{id}_{c^{-1}}$. We translate automorphisms of $c\in\pi_0$ to automorphisms of the monoidal unit by tensor product on the right with $\mathrm{id}_{c^{-1}}$. This is a group isomorphism $\pi_1\cong \operatorname{Aut} c$ since

$$(\gamma_1\otimes \mathrm{id}_{c^{-1}})\circ(\gamma_2\otimes \mathrm{id}_{c^{-1}}) = (\gamma_1\gamma_2)\otimes \mathrm{id}_{c^{-1}} \quad \forall \gamma_1,\gamma_2\in \operatorname{Aut} c.$$

The tensor product $\operatorname{Aut} c_1\otimes \operatorname{Aut} c_2 \to \operatorname{Aut} c_1c_2$ when translated to a map $\pi_1\times\pi_1\to\pi_1$ in this convention becomes $(f_1,f_2)\mapsto f_1\rho(c_1)[f_2]$ since

$$\begin{aligned}\gamma\otimes\delta\otimes \mathrm{id}_{(c_1c_2)^{-1}} &= (\gamma\otimes \mathrm{id}_{c_1^{-1}})\otimes(\mathrm{id}_{c_1}\otimes\delta\otimes \mathrm{id}_{c_2^{-1}}\otimes \mathrm{id}_{c_1^{-1}})\\ &= (\gamma\otimes \mathrm{id}_{c_1^{-1}})\otimes \rho(c_1)[\delta\otimes \mathrm{id}_{c_2^{-1}}] \quad \forall\gamma: c_1\to c_1, \delta: c_2\to c_2.\end{aligned}$$

In particular, we have that if $\gamma: c'\to c'$ and $f=\gamma\otimes \mathrm{id}_{(c')^{-1}}$ is the corresponding element of π_1, then the morphism $\gamma\otimes \mathrm{id}_c$ also corresponds to f, but $\mathrm{id}_c\otimes\gamma$ corresponds to $\rho(c)[f]$. The associator $\alpha: (c_1\otimes c_2)\otimes c_3\cong c_1\otimes(c_2\otimes c_3)$ becomes a map $\alpha:\pi_0\times\pi_0\times\pi_0\to\pi_1$. With this convention, the pentagon identity is equivalent to $\alpha\in Z^3(\pi_0;(\pi_1)_\rho)$ being a 3-cocycle with values in the π_0-module (π_1,ρ).

Remark A.2.6. If we would have translated automorphisms of c to automorphisms of 1 by tensoring with $\mathrm{id}_{c^{-1}}$ from the left, we would instead get as the tensor product $(f_1,f_2)\mapsto \rho(c_2^{-1})[f_1]f_2$ and the associator would satisfy the cocycle condition for the right action of π_0 on π_1 given by $f\cdot c=\rho(c^{-1})[f]$.

Definition A.2.7. Given a 2-group $\mathcal{G}$, the quadruple $(\pi_0,\pi_1,\rho,\alpha\in H^3(\pi_0;(\pi_1)_\rho))$ is called its *skeletal data*.

Given a quadruple $(\pi_0,\pi_1,\rho,\alpha\in Z^3(\pi_0;(\pi_1)_\rho))$, where π_0 is a group, π_1 is an abelian group and $\rho:\pi_0\to \operatorname{Aut}\pi_1$ a homomorphism, we can construct a 2-group $\mathcal{G}'$ with object set π_0, morphism sets $\operatorname{Hom}_{\mathcal{G}'}(c,c)=\pi_1$ and $\operatorname{Hom}_{\mathcal{G}'}(c_1,c_2)=\emptyset$ if $c_1\neq c_2$. The composition is given by the product in π_1, the tensor product on objects is given by the product in π_0, the tensor product on morphisms is uniquely determined by requiring $\mathrm{id}_c\otimes f\otimes \mathrm{id}_{c^{-1}}$ to be $\rho(c)[f]$ and $f\otimes f'=f\circ f'$ for $f,f'\in\pi_1$. The associator is induced by α. The 2-group $\mathcal{G}'$ made from the skeletal data of $\mathcal{G}$ is called its *skeletal model* and is equivalent to $\mathcal{G}$. In fact, the skeletal data is a complete invariant.

Theorem A.2.8. *Two 2-groups are equivalent if and only if they have the same skeletal data. Given a quadruple $(\pi_0,\pi_1,\rho,\alpha\in H^3(\pi_0;(\pi_1)_\rho)$, where π_0 is a group, π_1 is an abelian group and $\rho:\pi_0\to \operatorname{Aut}\pi_1$ a homomorphism, there exists a 2-group with that skeletal data.*

It can sometimes be useful to use the homotopy hypothesis to compare 2-groups with connected homotopy 2-types by including them in ∞-groups, which are equivalent to arbitrary homotopy types. We can make the relationship with connected homotopy 2-types and their Postnikov towers explicit as follows. A homotopy 1-type X is classified by the group $\pi_1(X)$, the abelian group $\pi_2(X)$,[2] the

[2]The shift by one in the indices is because X is the classifying space of $\mathcal{G}$.

action of $\pi_1(X)$ on $\pi_2(X)$ and the k-invariant (or Postnikov invariant) classifying the fibration

$$\begin{tikzcd} B^2\pi_2(X) \arrow[d] \\ X \arrow[d] \\ B\pi_1(X) \end{tikzcd} .$$

Such a k-invariant is given by a map $B\pi_1(X) \to B\,\mathrm{Aut}_h\, B^2\pi_2(X)$, where Aut_h denotes the grouplike E_1-space of homotopy automorphisms of a space. Now it is a fact that [16]

$$\mathrm{Aut}_h\, B^n A \cong \mathrm{Aut}\, A \rtimes B^n A.$$

In particular, the k-invariant is a pair of a group homomorphism $\rho : \pi_1(X) \to \mathrm{Aut}(\pi_2(X))$ and a class in $H^2(\pi_1; (\pi_2)_\rho)$.

Example A.2.9. Let $\mathcal{B}$ be the Morita bicategory of algebras over a field k. The automorphism 2-group of an algebra A is the monoidal category of invertible (A, A)-bimodules and invertible bimodule maps between them. It has $\pi_0 = \mathrm{Pic}\, A$ given by the Picard group of A and $\pi_1 = Z(A)^\times$ is the group of automorphisms of the (A, A)-bimodule A.

A.3 2-group actions on categories

An action of a 2-group $(\mathcal{G}, \otimes, 1, \alpha)$ on a category $\mathcal{C}$ is a monoidal functor

$$\mathcal{G} \to (\mathrm{End}\,\mathcal{C}, \circ),$$

where $\mathrm{End}\,\mathcal{C}$ is the category of endofunctors of $\mathcal{C}$ with monoidal product given by composition of functors. Equivalently, it is a pseudofunctor

$$B\mathcal{G} \to \mathrm{Cat}_1,$$

from the bicategory with one object $*$ and $\mathrm{Hom}_{B\mathcal{G}}(*, *) = \mathcal{G}$ to the 2-category of categories with the property that $* \mapsto \mathcal{C}$. Spelling out such a definition, this means that

Definition A.3.1. An *action of a 2-group $\mathcal{G}$ on a category $\mathcal{C}$* consists of

1. for every object $g \in \mathcal{G}$, a functor $\rho(g) : \mathcal{C} \to \mathcal{C}$;

2. for every morphism $\gamma : g_1 \to g_2 \in \mathcal{G}$, a natural isomorphism $\rho(\gamma) : \rho(g_1) \Rightarrow \rho(g_2)$;

3. for every two objects $g, g' \in \mathcal{G}$, a natural isomorphism $R_{g,g'} : \rho(g) \circ \rho(g') \Rightarrow \rho(g \otimes g')$;

such that

1. for three objects $g, g', g'' \in \mathcal{G}$, the diagram

$$\begin{tikzcd}[column sep=small] & \rho(g'') \circ \rho(g') \circ \rho(g) & \\ \rho(g'') \circ \rho(g' \otimes g) \arrow[ur, Rightarrow, "\mathrm{id}_{\rho(g'')} \bullet R_{g',g}"] & & \rho(g'' \otimes g') \circ \rho(g) \arrow[ul, Rightarrow, "R_{g'',g'} \bullet \mathrm{id}_{\rho(g)}"'] \\ \rho(g'' \otimes (g' \otimes g)) \arrow[u, Rightarrow, "R_{g'',g' \otimes g}"] \arrow[rr, Rightarrow, "\rho(\alpha(g'',g',g))"'] & & \rho((g'' \otimes g') \otimes g) \arrow[u, Rightarrow, "R_{g'' \otimes g',g}"'] \end{tikzcd} \tag{A.3}$$

commutes, where $\bullet$ denotes horizontal composition of natural transformations;

2. for two composable morphisms $\gamma_1 : g_1 \to g_2, \gamma_2 : g_2 \to g_3 \in \mathcal{G}$, we have $\rho(\gamma_2) \circ \rho(\gamma_1) = \rho(\gamma_2 \circ \gamma_1)$;

3. for any two morphisms $\gamma : g_1 \to g_2, \gamma' : g_1' \to g_2' \in \mathcal{G}$, we have the commutative diagram

$$\begin{tikzcd} \rho(g_1' \otimes g_1) \arrow[r, Rightarrow, "R_{g_1',g_1}"] \arrow[d, Rightarrow, "\rho(\gamma' \otimes \gamma)"'] & \rho(g_1') \circ \rho(g_1) \arrow[d, Rightarrow, "\rho(\gamma') \bullet \rho(\gamma)"] \\ \rho(g_2' \otimes g_2) \arrow[r, Rightarrow, "R_{g_2',g_2}"'] & \rho(g_2') \circ \rho(g_2) \end{tikzcd};$$

4. $\rho(1) = \mathrm{id}_{\mathcal{C}}$;

5. $\rho(\mathrm{id}_g) = \mathrm{id}_{\rho(g)}$ for all objects g.

Remark A.3.2. We assumed that the action is strictly unital, which can be done without loss of generality.

Remark A.3.3. The condition $\rho(\gamma_2) \circ \rho(\gamma_1) = \rho(\gamma_2 \circ \gamma_1)$ is redundant, as it follows by writing the composition of two morphisms as a tensor product and then applying the preservation of tensor products of morphisms. This would not be the case if we replace $\mathcal{G}$ by an arbitrary monoidal category.

Example A.3.4. An action of an ordinary group G on a category is an action of the 2-group, of which the objects are G and all morphisms are identities.

Example A.3.5. Let G, H be two groups. Consider H as a 2-group with only trivial morphisms and let $\mathcal{C} = BG$ be the category with one object and morphisms G. Then, an action of H on $\mathcal{C}$ is exactly the same data as an H-valued nonabelian 2-cocycle on G, i.e. a collection of automorphisms $\rho(g) \in \operatorname{Aut} H$ and elements $\tau(g_1, g_2) \in H$ for $g_1, g_2 \in G$, such that

$$\tau(g_1, g_2) \cdot \rho(g_1 g_2)[h] = (\rho(g_1) \circ \rho(g_2))[h] \cdot \tau(g_1, g_2)$$

satisfying a twisted cocycle condition. Up to isomorphism, this is equivalent to giving an extension

$$1 \to G \to K \to H \to 1,$$

which we can recover as the (homotopy) colimit of the functor $\rho : BH \to \mathrm{Gpd}$ between $(2,1)$-categories.

Definition A.3.6. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between categories with $\mathcal{G}$-action. Then, F is said to be $\mathcal{G}$-*equivariant* if it comes equipped with a collection of natural transformations $\lambda_g : F \circ \rho(g) \cong \rho(g) \circ F$ for every object $g \in \mathcal{G}$ such that

- The diagram
$$\begin{tikzcd}
F \circ \rho(g'g) \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & F \circ \rho(g') \circ \rho(g) \arrow[d, Rightarrow] \\
& \rho(g') \circ F \circ \rho(g) \arrow[d, Rightarrow] \\
\rho(g'g) \circ F \arrow[r, Rightarrow] & \rho(g') \circ \rho(g) \circ F
\end{tikzcd}$$
commutes for all objects $g, g' \in \mathcal{G}$;

- the diagram
$$\begin{tikzcd}
F \circ \rho(g_1) \arrow[r, Rightarrow, "F\rho(\gamma)"] \arrow[d, Rightarrow] & F \circ \rho(g_2) \arrow[d, Rightarrow] \\
\rho(g_1) \circ F \arrow[r, Rightarrow, "\rho(\gamma)F"] & \rho(g_2) \circ F
\end{tikzcd}$$
commutes for all morphisms $\gamma : g_1 \to g_2$ in $\mathcal{G}$;

- $\lambda_1 = \mathrm{id}_F$.

A natural transformation $F \Rightarrow F'$ between equivariant functors is *equivariant* if the diagram

$$\begin{tikzcd}
F \circ \rho(g) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F' \circ \rho(g) \arrow[d, Rightarrow] \\
\rho(g) \circ F \arrow[r, Rightarrow] & F' \circ \rho(g)
\end{tikzcd}$$

commutes for all objects $g \in \mathcal{G}$.

All definitions in this appendix generalize to monoidal $\mathcal{G}$-actions by requiring all functors and natural transformations occurring in the definition of an action to be monoidal. Similarly, if the category is braided or symmetric, we require all functors to preserve the braiding.