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Quasiclassical approach to the nonlinear Kerr dynamics

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ABSTRACT

We examine the quasiclassical approximation to the self-Kerr nonlinear effect. The corresponding dynamics appears as classical trajectories, with the quantumness of the state included via its Wigner function. We obtain analytical estimates of the optimal squeezing attainable that compare fairly well with the numerical quantum solution. We delimit the range of parameters for which the quasiclassical solution retains relevant quantum features.

1. Introduction

The optical Kerr effect, the intensity-dependent phase shift that light experiences during its propagation through a nonlinear medium, has become a key resource for a variety of areas. Examples include quantum nondemolition measurements [1–7], generation of quantum superpositions [8–15], quantum teleportation [16–19], and quantum gates [20–32], among other. Quadratic Hamiltonians in the photon number have numerous applications in optical lattices [33,34], and most importantly in the context of the Mott insulator transition [35] realized in the laboratory [36].

Special mention deserves the role that this nonlinearity has played in the generation of squeezed states [37,38], particularly using optical fibers [39]. Since the Kerr nonlinearity is very small in silica glass [40], Kerr-based fibers need long propagation distances and high powers. In this limit, one would expect a classical description to be sufficient. Actually, a standard approach based on Maxwell's equations leads to a set of coupled nonlinear Schrödinger equations that describe the behavior of optical fields in nonlinear dispersive media in this classical limit [41]. This has been instrumental in explaining phenomena as diverse as pulse compression [42–44], fiber solitons [45–48], and self-focusing of ultrashort pulses [49–51].

However, there are interesting aspects that cannot be appreciated in this classical approach: at the most basic level, the propagation of light in a Kerr medium is accompanied by unavoidable quantum effects. One might thus rightly ask where is the border between classical and quantum descriptions. We address here this conundrum by exploiting phase-space methods [52–55]. To this end we resort to a quasiclassical approximation [56–59] that employs classical trajectories, whereas the quantumness is taken into account in the initial state via the Wigner function [60,61].

We will focus here on the self-modulation (SPM) effect of the Kerr nonlinearity; that is, the self-induced phase shift experienced by the field during its propagation, which was predicted as early as 1967 [62,63] and has significant implications in the propagation of pulses [64]. Using the quasiclassical approximation and after neglecting higher-order fluctuations, we get an evolution equation for the Wigner function that can be integrated to an analytical form and allows us to estimate different parameters. This is to be compared with the quantum behavior. To this end, we work out a compact analytical solution in the quantum regime that can be efficiently computed for arbitrary number of photons. In this way, we can readily assess the deviations of the quasiclassical solution from the quantum behavior and when purely quantum effects are so mall that can be hardly noticed in practice.

2. Quantum self-Kerr dynamics in phase space

As heralded in the Introduction, the self-Kerr configuration implies that the refractive index seen by a beam is modified by the intensity of that beam. In a quantum description this beam is characterized by a single-mode field of frequency ω and complex amplitude represented by the operator $\hat{a}$, obeying the standard bosonic commutation relation $[\hat{a}, \hat{a}^{\dagger}] = \hat{\mathbb{1}}$, the superscript $\dagger$ standing for the adjoint.

The optical Kerr effect is then modeled by the nonlinear Hamiltonian [65]

$$\hat{H} = \hbar \chi \hat{a}^{\dagger 2} \hat{a}^{2}, \tag{1}$$

where χ is a coupling constant directly related to the third-order nonlinear susceptibility of the medium.

Henceforth, we will consider the field initially be in a coherent state $|\psi\rangle = |\alpha_0\rangle$, as this is the most appropriate way to consider the limit of

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intense fields found in many experimental situations. Throughout this paper we will use the dimensionless time $\tau = \chi t$. The propagator of the Schrödinger equation $\hat{U}(t) = \exp(-i\hat{H}t/\hbar)$ immediately yields the field at time t:

$$|\psi(t)\rangle = \hat{U}(t)|\psi\rangle = \sum_{n=0}^{\infty} \psi_n(t)|n\rangle, \tag{2}$$

where we have projected on the number basis. Here, the time-dependent coefficients $\psi_n(t)$ are given by

$$\psi_n(\tau) = \left(e^{-\bar{n}}\frac{\bar{n}}{n!}\right)^{\frac{1}{2}} e^{i\varphi_0} \exp[-i\tau\, n(n-1)], \tag{3}$$

with $|\alpha_0|^2 = \bar{n}$ being the mean number of photons and φ_0 the initial phase of the state. Therefore, each Fock state $|n\rangle$ experiences a nonlinear phase evolution that, for short times, is closely approximated by a rotation of the state with an angle $\tau(\bar{n}+1/2)$. Note that the resulting evolution from (2) is periodic: $|\psi(\tau)\rangle = |\psi(\tau+2\pi)\rangle$; in particular, for $\tau = 2k\pi$, with $k \in \mathbb{Z}$, the initial coherent state is reconstructed. More generally, one can show [12] that m copies of the initial state (with different phases) appear at times $\tau = \ell\pi/m$ (with $\ell, m \in \mathbb{Z}$).

The knowledge of $|\psi(t)\rangle$ requires the evaluation of the series in (2), which is numerically tractable only for few-photon states. We look instead at the evolution in phase space, which we define in terms of dimensionless quadratures x and p, related to the position X and momentum P of the mode by

$$x = \sqrt{\frac{\omega}{\hbar}}X, \qquad p = \sqrt{\hbar\omega}P. \tag{4}$$

We will capitalize on the Wigner function $W(x,p|t)$ at time t, defined as [60]

$$W(x,p|t) = \frac{1}{2\pi\hbar}\int dx'\psi^*\left(x+\tfrac{1}{2}x',t\right)\psi\left(x-\tfrac{1}{2}x',t\right)e^{-ipx'/\hbar}. \tag{5}$$

If we employ the expansion (2) and take into account that the (position) wavefunction of a number state reads [55]

$$\langle x|n\rangle = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(x)\, e^{-\frac{1}{2}x^2}, \tag{6}$$

with $H_n(\cdot)$ a Hermite polynomial [66], the Wigner function can be expressed as

$$W(x,p|\tau) = \frac{1}{2\pi\hbar}e^{-x^2}\sum_{n,m=0}^{\infty}\frac{\psi_n^*(\tau)\psi_m(\tau)}{\sqrt{2^{n+m}n!m!}}\,\mathcal{I}_{nm}(x,p), \tag{7}$$

where

$$\mathcal{I}_{nm}(x,p) = \int dx' e^{-\frac{1}{4}x'^2} H_n\left(x+\tfrac{1}{2}x'\right) H_m\left(x-\tfrac{1}{2}x'\right) e^{-ipx'/\hbar}. \tag{8}$$

This integral (8) can be calculated following the method sketched in Ref. [67]; the result is

$$\mathcal{I}_{nm}(x,p) = 2\sqrt{\pi}e^{-p^2}\begin{cases} 2^m n!(-1)^n L_n^{m-n}\left(2(p^2+x^2)\right)(x+ip)^{m-n}, & m>n, \\ 2^n m!(-1)^m L_m^{n-m}\left(2(p^2+x^2)\right)(x-ip)^{n-m}, & n>m, \end{cases} \tag{9}$$

where $L_n^k(\cdot)$ is the associated Laguerre polynomial [66]. By direct inspection, one can check that $\mathcal{I}_{nm} = \mathcal{I}_{mn}^*$, and the sum (7) can thus be rewritten as

$$W(x,p|\tau) = \frac{1}{\pi}e^{-x^2}\,\mathrm{Re}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n^*(\tau)\psi_{n+k}(\tau)}{\sqrt{2^{2n+k}n!(n+k)!}}\,\mathcal{I}_{nn+k}(x,p), \tag{10}$$

with the proviso that in the sum over k, the term with $k=0$ has to be taken with a coefficient $1/2$. Plugging the explicit form of the state coefficients $\psi_n(\tau)$, we get

$$W(x,p|\tau) = \frac{2}{\pi}\,e^{-2|\alpha|^2-|\alpha_0|^2}\;\mathrm{Re}\sum_{k=0}^{\infty}(2\alpha\alpha_0^*)^k e^{-ik\tau}e^{ik^2\tau}\,\Xi_k(\alpha|\tau), \tag{11}$$

where we have introduced the standard phase-space variable $\alpha = (x+ip)/\sqrt{2}$ (which will also be used to label functions) and

$$\Xi_k(\alpha|\tau) = \sum_{n=0}^{\infty}(-1)^n\frac{\bar{n}^n}{(n+k)!}\;L_n^k(4|\alpha|^2)\,e^{2ikn\tau}. \tag{12}$$

In spite of its intimidating appearance, the function (12) admits a closed form:

$$\Xi_k(\alpha|\tau) = \frac{I_k(4|\alpha\alpha_0^*|\,e^{ik\tau})}{(4|\alpha\alpha_0^*|)^{k/2}}\exp\left(-\bar{n}e^{2ik\tau}-ik^2\tau\right), \tag{13}$$

with $I_k(\cdot)$ the modified Bessel function of first kind [66]. Finally, if we use polar coordinates in the complex plane $\alpha = re^{i\varphi}$ (so for the initial state $\alpha_0 = r_0e^{i\varphi_0}$ with $r_0^2 = \bar{n}$), we get

$$W(r,\varphi|\tau) = \frac{2}{\pi}\;e^{-2r^2-r_0^2}\sum_{k=-\infty}^{\infty}e^{ik(\varphi-\varphi_0)}e^{-ik\tau}I_k(4rr_0\,e^{ik\tau})\exp\left(-r_0e^{2ik\tau}\right). \tag{14}$$

This is an exact, compact result that allows one to compute average values. Previous approaches to this problem [68] involve a double summation, which make the evaluation a touchy business. In contradistinction, (14) is numerically efficient, even for relatively intense fields, which is our goal in this paper.

As a check of correctness, let us take $\tau = 0$:

$$W(r,\varphi|0) = \frac{2}{\pi}\;e^{-2(r^2-r_0^2)}\sum_{k=-\infty}^{\infty}e^{ik(\varphi-\varphi_0)}I_k(4rr_0), \tag{15}$$

and performing the summation we have

$$W(\alpha) = \frac{2}{\pi}\;e^{-2|\alpha-\alpha_0|^2}, \tag{16}$$

which coincides with the standard Wigner function for the coherent state $|\alpha_0\rangle$.

One relevant property of the Wigner function is that its marginal distributions are actual quantum probability distributions. For example, for the intensity we have

$$W(r|\tau) = \int d\varphi W(r,\varphi|\tau) = 4\;e^{-2(r^2-r_0^2)}\;I_0(4rr_0), \tag{17}$$

which is independent on time, as the Kerr effect is photon-number preserving.

We can also look at the evolution of phase. The resulting integral can be found in Eq. 2.15.5–4 of Ref. [69], so the final result reads

$$\begin{aligned}\langle e^{-im\varphi}\rangle &= \int d\phi\int dr\,r\;W(r,\phi|\tau)e^{-im\phi} \\ &= \sqrt{\frac{\pi}{2}}r_0e^{-im\varphi_0}e^{-r_0^2}\left[I_{\frac{m-1}{2}}\left(r_0^2\,e^{2im\tau}\right)+I_{\frac{m+1}{2}}\left(r_0^2\,e^{2im\tau}\right)\right].\end{aligned} \tag{18}$$

With this equation we can calculate any relevant phase observable. In particular, for the first moment

$$\langle e^{-i\varphi}\rangle = \sqrt{\frac{\pi}{2}}r_0e^{-i\varphi_0}e^{-r_0^2}\left[I_0\left(r_0^2\,e^{2i\tau}\right)+I_1\left(r_0^2\,e^{2i\tau}\right)\right], \tag{19}$$

which in the limit of high intensities $r \gg 1$ tends to

$$\langle e^{-i\varphi}\rangle \simeq e^{-i\varphi_0}e^{-r_0^2}\exp\left(-i\tau+r_0^2\,e^{2i\tau}\right). \tag{20}$$

3. The quasiclassical limit of self-Kerr dynamics

Instead of the approach in the previous section, we can alternatively look at the time evolution of the density operator ϱ, the von Neumann equation

$$i\hbar\frac{\partial\varrho}{\partial t} = [H,\varrho(t)], \tag{21}$$

and express this evolution in phase space. Since the Wigner function is the symbol of the density operator, we can follow the standard rules of

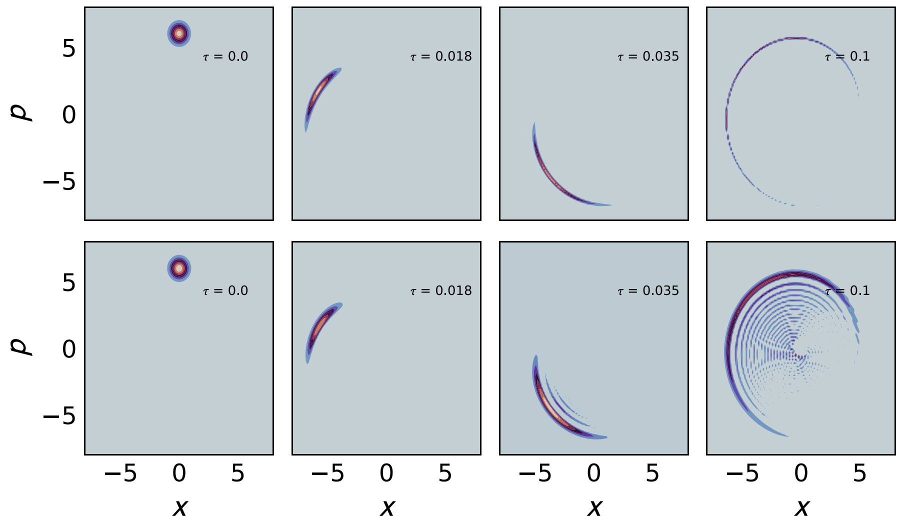

Fig. 1. Density plots of the Wigner function for an initial coherent state with $\bar{n} = 36$ at times (from left to right) $\tau = 0$, 0.018, 0.035, and 0.1. In this case, the optimal squeezing time $\tau_{\min}$ is 0.032. The upper panel corresponds to the quasiclassical approximation, whereas the lower panel is the exact quantum solution.

the Wigner–Weyl isomorphism to map operators onto functions on the phase space [52,53], to obtain

$$i\frac{\partial W}{\partial t} = 2\chi(|\alpha|^2 - 1)\left(\alpha^* \frac{\partial W}{\partial \alpha^*} - \alpha\frac{\partial W}{\partial \alpha}\right) - \frac{\chi}{2}\left(\alpha^* \frac{\partial^3 W}{\partial \alpha^{*2}\partial\alpha} - \alpha\frac{\partial^3 W}{\partial\alpha^2\partial\alpha^*}\right), \tag{22}$$

or, using again polar coordinates

$$\frac{\partial W}{\partial t} = 2\chi(r^2-1)\frac{\partial W}{\partial\varphi} - \frac{\chi}{8}\left(\frac{1}{r^2}\frac{\partial^3}{\partial^3\varphi} + \frac{\partial^3}{\partial r^2\partial\varphi} + \frac{1}{r}\frac{\partial^3}{\partial\varphi^2\partial r}\right)W. \tag{23}$$

In general, there are no analytical solutions to this equation and one should solve it numerically. However, as worked out in the previous Section, we are interested in the limit $r \gg 1$. If, in addition, we consider short times, such that $\tau \ll 1$, the evolution can be well approximated by

$$\frac{\partial W}{\partial t} = 2\chi(r^2-1)\frac{\partial W}{\partial\varphi}. \tag{24}$$

The physical interpretation of this equation is direct: as $\partial/\partial\varphi$ generates rotations in phase space, (24) reflects that the mode amplitude experiences rotations with angles proportional to the intensity. The neglected terms in (23) correspond to quantum fluctuations, which hopefully are not playing a major role for highly-excited fields. Notice that the evolution is specified only by classical trajectories, much in the spirit of the quasiclassical approximation.

Eq. (24) can be immediately solved:

$$W(r,\varphi|\tau) = W(r, \varphi = \varphi_0 - \tau + 2r^2\tau). \tag{25}$$

Moreover, since we are considering an initial coherent state $|\alpha_0\rangle$, the Wigner function takes the form

$$W(r,\varphi|\tau) = \frac{2}{\pi}\exp\left(-2|re^{i\varphi_0 - i\tau + i2r^2\tau} - \alpha_0|^2\right). \tag{26}$$

From this quasiclassical solution, we can calculate any observable of the system. For example, let us consider the rotated quadrature

$$x_\theta = \frac{1}{\sqrt{2}}(ae^{i\theta} + e^{-i\theta}a^\dagger), \tag{27}$$

with minimal fluctuations

$$S = \min_\theta \mathrm{Var}(X_\theta) = 1 + 2\left(\langle a^\dagger a\rangle - |\langle a\rangle|^2\right) - 2|\langle a^2\rangle - \langle a\rangle^2|. \tag{28}$$

Using the previous solution, we find

$$S = 1 + 2w^2 - 2w^3\tau + 2w\tau - 2w\sqrt{1+w^2} + 2w^2\tau\frac{2+3w^2}{\sqrt{1+w^2}}, \tag{29}$$

where $w = 2r^2\tau$. Retaining only the leading terms, this gives the simple expression

$$S \simeq \frac{1}{4w^2} + 4w^3\tau, \tag{30}$$

which gives us an optimal squeezing (and the corresponding time)

$$S_{\min} \simeq \tfrac{3}{2}(2r)^{-2/3}, \qquad \tau_{\min} \simeq (2r)^{-5/3}, \tag{31}$$

which coincides with previous estimates obtained from a very different method [70].

4. Discussion

Fig. 1 presents plots of the Wigner function for an initial coherent state with $r_0^2 = \bar{n} = 36$, at four different times. The lower panel corresponds to the exact quantum solution, whereas the upper panel is the quasiclassical solution. In both cases and due to the fact that the number operator is a constant of motion, the probability distributions during their evolution remain located around a circle of radius $r = 6$. As discussed before, beginning from a circle, the Wigner function turns into an ellipse and squeezing appears in an appropriate direction. In the semiclassical approximation, only rotation and stretching effects are present, corresponding to the linear terms in the evolution equation: the intensity fluctuations modulate the nonlinear refractive index and this in turn modulates the phase of traveling light. Photons with stronger amplitude will acquire phase faster than the photons with smaller amplitude. In the quantum solution, the ellipse changes into a banana shape and a "tail" of the interference fringes appears, squeezing increases and the state becomes non-Gaussian. These oscillations in phase space decorating the classical trajectories are distinctive of the quantumness and have been studied in detail [71,72]. We can take as the characteristic time for the evolution the value $\tau_{\min}$ given in (31), which in this case is $\tau_{\min} \simeq 0.0318$. For times lesser than $\tau_{\min}$, both solutions look quite similar, but they differ significantly for times much larger than $\tau_{\min}$.

To better appreciate the differences between the quantum and quasiclassical behaviors, in Fig. 2 we present plots of the Wigner function at times $\tau_{\min}$ for different initial average photon numbers $\bar{n}$, namely, 8, 16, 64, and 128. The snapshots look very similar for all values of $\bar{n}$, which means that at this characteristic time the predicted optimal squeezing in the quasiclassical approximation can be taken as exact.

A more quantitative way to assess how much close the quantum and the quasiclassical solutions are is via the corresponding fidelity [73]. For pure states this measure reduces to the overlap of the solutions

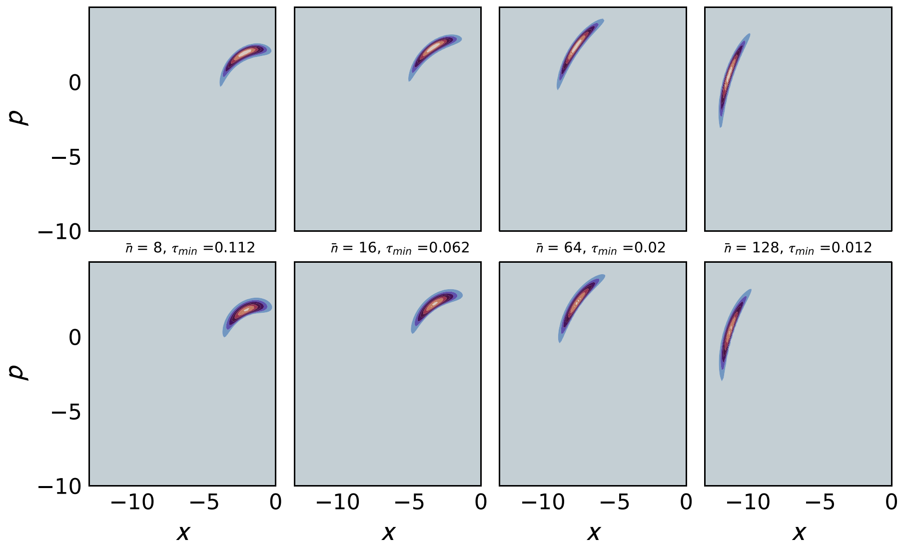

Fig. 2. Density plots of the Wigner function at the optimal squeezing times given $\tau_{\min}$ as in Eq. (31) for an initial coherent state with average number of photons $\bar{n}$ (from left to right) 8, 16, 64, and 128. As in the previous figure, the upper panel corresponds to the quasiclassical approximation, whereas the lower panel is the exact quantum solution.

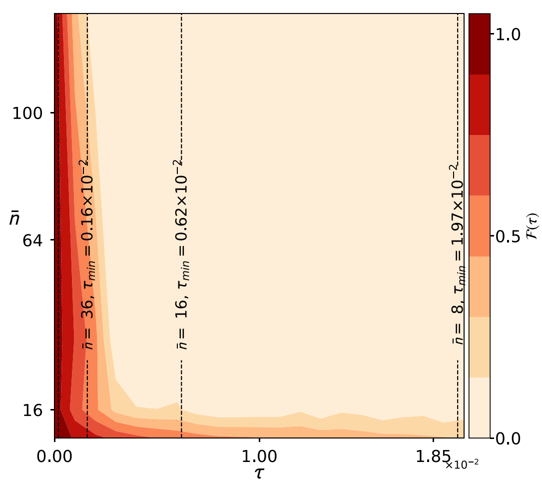

Fig. 3. Density plot of the fidelity $\mathcal{F}(\tau)$ between the quantum and the quasiclassical solutions, as a function of τ and the average photon number of the initial coherent state $\bar{n}$. We have also included the lines corresponding to the optimal squeezing times $\tau_{\min}$ for the average photon numbers indicated.

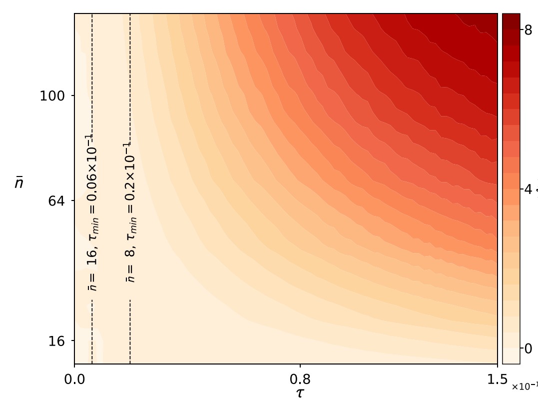

Fig. 4. Density plot of the negativity $\mathcal{N}(\tau)$, as a function of τ and the average photon number of the initial coherent state $\bar{n}$. We have also included the lines corresponding to the optimal squeezing times $\tau_{\min}$ for the average photon numbers indicated.

$\mathcal{F}(\tau) = |\langle \psi_{\mathrm{q}}(\tau)|\psi_{\mathrm{cl}}(\tau)\rangle|^2$, where the subscript q refers to quantum and cl to quasiclassical. Actually, this concept can be elegantly translated into phase space, so that we have

$$\mathcal{F}(\tau) = \mathrm{Tr}[\varrho_{\mathrm{q}}(\tau)\varrho_{\mathrm{cl}}(\tau)] = \int d\alpha W_{\mathrm{q}}(\alpha|\tau) W_{\mathrm{cl}}(\alpha|\tau). \qquad (32)$$

Here, W_{q} corresponds to the solution (14) and W_{cl} to the semiclassical one (26). This quantity is plotted in Fig. 3. We have also included the values of $\tau_{\min}$ for a few values of the average photon number $\bar{n}$. Matching the physical intuition, the fidelity remains larger for low-intensity fields.

Another interesting question is how much quantum is the exact solution (14). We shall use as an indicator of quantumness the quantity

$$\mathcal{N}(\tau) = \int d\alpha\, |W(\alpha|\tau)| - 1, \qquad (33)$$

which is the doubled volume of the integrated negative part of the Wigner function [74]. Similar quantities related to the volume of the negative part of the Wigner function were used [75,76] to describe the interference effects that determine the departure from classical behavior.

This negativity $\mathcal{N}(\tau)$ has been considered as a resource for quantum tasks [77]. The function is $\mathcal{N}(\tau)$ presented in Fig. 4. As expected, $\mathcal{N}(\tau)$ is more pronounced for small number of photons and tends to disappear for intense fields, where the Wigner function is purely Gaussian and, as so, is strictly positive.

5. Concluding remarks

In summary, we have presented a simple quasiclassical approximation to the light propagation in a self-Kerr medium. By neglecting quantum effects, the quasiclassical approximation reduces the computational complexity and enables more efficient simulations, making it easier to study and optimize the performance of Kerr-based systems. Interestingly, even if the states considered are bright and we neglect quantum correlations, we still observe nonclassical effects, that have been tested with a new exact expression for the full quantum dynamics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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