Finite-to-one equivariant maps and mean dimension

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Abstract

We show that a minimal dynamical system $(X, \mathbb{Z})$ on a compact metric X with $\mathrm{mdim} X = d$ admits for every natural $k > d$ an equivariant map to the shift $([0,1]^k)^{\mathbb{Z}}$ such that each fiber of this map contains at most $[\frac{k}{k-d}]\frac{k}{k-d}$ points.

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1 Introduction

A classical result in Dimension Theory [4, p. 124] says that if a compact metric X is with $\dim X = d$ then for almost every map $f : X \to [0,1]^k$, $k > d$, the fibers of f contain at most $\frac{k}{k-d}$ points. In an attempt to generalize this result to Mean Dimension we prove:

Theorem 1.1 *Let $(X, \mathbb{Z})$ be a minimal dynamical system on a compact metric X with $\mathrm{mdim} X = d$ and let $k > d$ be a natural number. Then almost every map $f : X \to [0,1]^k$ induces the map $f^{\mathbb{Z}} : X \to ([0,1]^k)^{\mathbb{Z}}$ whose fibers contain at most $[\frac{k}{k-d}]\frac{k}{k-d}$ points.*

Here $f^{\mathbb{Z}}$ denotes the equivariant map to the shift $([0,1]^k)^{\mathbb{Z}}$ defined by $f^{\mathbb{Z}}(x) = (f(x+z))_{z\in\mathbb{Z}}$ for $x \in X$. In this note for an additive group G acting on X we use the notation $x + g$ rather than gx to denote the action of $g \in G$ on $x \in X$. This seems to be more consistent with the fact that 0 is the identity element of G.

Although the right estimate for the size of the fibers of $f^{\mathbb{Z}}$ in Theorem 1.1 should probably be $[\frac{k}{k-d}]$, Theorem 1.1 still implies that $f^{\mathbb{Z}}$ is an embedding for $k > 2d$ (the case of a minimal action of the Gutman-Tsukamoto theorem [2]) and $f^{\mathbb{Z}}$ is finite-to-one for $k = [d] + 1$ (the case that motivated this note).

The proof of Theorem 1.1 is self-contained and, basically, an interplay of two beautiful constructions in Topology, namely, Kolmogorov's covers (used in Kolmogorov's proof of the continuous version of Hilbert's 13th problem [3]) and Lindenstrauss' level functions (of Rokhlin-type used in Lindenstrauss' proof of the embedding theorem [5]).

Another essential ingredient of the proof of Theorem 1.1 (that to a certain extent bridges between Kolmogorov's covers and Lindenstrauss' level functions) is Borel's construction, aka the mapping torus in Topological Dynamics, and the following property.

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Theorem 1.2 *For any dynamical system* $(X, \mathbb{Z})$ *on a compact metric* X *one has* $\text{mdim}X \times_{\mathbb{Z}} \mathbb{R} = \text{mdim}X$ *where* $X \times_{\mathbb{Z}} \mathbb{R}$ *is Borel's construction for* $(X, \mathbb{Z})$*.*

The case of a minimal action on X in Theorem 1.2 was obtained by Gutman and Jin [1] based on Lindenstrauss' characterization of mean dimension via metric mean dimension [5]. We prove Theorem 1.2 by a direct argument which does not require any restriction on the action. Theorem 1.2 together with Kolmogorov's covers and Lindenstrauss' level functions seem to provide (at least for minimal actions) an elementary replacement of the signal processing technique used in the proof of Gutman-Tsukamoto's embedding theorem [2].

The note is organized as follows: Section 2 - a description of the notations used in the note, Section 3 - a review of mean dimension, Section 4 - a review of Borel's construction, Section 5 - a proof of Theorem 1.2, Section 6 - a review of Kolmogorov's covers, Section 7 - a review of Lindenstrauss' level functions, Section 8 - a proof of Theorem 1.1.

2 Notations

Everywhere in this note i, j, m, n, l and k stand for natural numbers (=non-negative integers) and z for an integer number (although in most cases it is stated explicitly).

For an additive G group acting on X we use the notation $x+g$ rather than gx to denote the action of $g \in G$ on $x \in X$ and the expression $x + (-g)$ shortens to $x - g$. We also denote $A + g = \{x + g : x \in A\}$ for $A \subset X$, $\mathcal{A} + g = \{A + g : A \in \mathcal{A}\}$ for a collection $\mathcal{A}$ of subsets of X, $A + H = \{x + g : x \in A, g \in H\}$ for $H \subset G$ and $\mathcal{A} + H = \{A + H : A \in \mathcal{A}\}$.

For a function $f : X \to Y$ and collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of X and Y respectively we denote $f(\mathcal{A}) = \{f(A) : A \in \mathcal{A}\}$ and $f^{-1}(\mathcal{B}) = \{f^{-1}(B) : B \in \mathcal{B}\}$.

A map means a continuous function. For a finite open cover $\mathcal{U}$ of a compact metric X the nerve of $\mathcal{U}$ is denoted by $N(\mathcal{U})$ (a finite simplicial complex with $\text{ord}\mathcal{U} = \dim N(\mathcal{U}) + 1$) and a canonical map $f : X \to N(\mathcal{U})$ means a map constructed on the basis of a partition of unity subordinate to $\mathcal{U}$ and an important property of f is that the fibers of f refine $\mathcal{U}$.

3 Mean Dimension

Let $(X, \mathbb{Z})$ be a dynamical system on a compact metric X and d a positive real number. Let us write $\text{mdim}X \leq d^-$ if for every $\epsilon > 0$ there is a finite closed cover $\mathcal{A}$ of X such that $\text{mesh}(\mathcal{A} + z) < \epsilon$ for every integer $z \in \mathbb{Z}$ satisfying $0 \leq zd < \text{ord}\mathcal{A}$. We will write $\text{mdim}X > d^-$ if $\text{mdim}X \leq d^-$ does not hold. Note that slightly enlarging the elements of $\mathcal{A}$ to open sets we may assume that in the above definition that $\mathcal{A}$ is an open cover. We leave to the reader to verify that Gromov's mean dimension $\text{mdim}X$ can be defined as $\text{mdim}X = \inf\{d : \text{mdim}X \leq d^-\}$ (and $\text{mdim}X = \infty$ if the defining set is empty). Note that $1^- < \text{mdim}[0,1]^{\mathbb{Z}} = 1$.

Let $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be covers of X. The notation $\mathcal{A}_0 \vee \cdots \vee \mathcal{A}_n$ is used to denote the cover $\{A_0 \cap \cdots \cap A_n : A_i \in \mathcal{A}_i, 0 \leq i \leq n\}$. Assume that each $\mathcal{A}_i$ is an open finite cover of X and recall that for a canonical map $f_i : X \to K_i = N(\mathcal{A}_i)$ to the simplicial complex K_i which is the nerve of $\mathcal{A}_i$ one has that the fibers of f_i refine the elements of $\mathcal{A}_i$ and

$\dim K_i + 1 = \text{ord}\mathcal{A}_i$. Thus for $f = (f_0, \ldots, f_n) : X \to K_0 \times \cdots \times K_n$ we have that the fibers of f refine $\mathcal{A} = \mathcal{A}_0 \vee \cdots \vee \mathcal{A}_n$ and therefore $\mathcal{A}$ can be refined by an open cover of X of ord $\leq \dim K + 1 = \dim K_0 + \cdots + \dim K_n + 1$.

Also note that if for (not necessarily open) covers $\mathcal{A}_i$ of X we have that $\text{mesh}(\mathcal{A}_i + z) < \epsilon$ for every $0 \leq i \leq n$ and $0 \leq z < m, z \in \mathbb{Z}$, then for the cover $\mathcal{B} = \mathcal{A}_0 \vee (\mathcal{A}_1 - z_1) \vee \cdots \vee (\mathcal{A}_n - z_n)$ where $z_i = im$ we have that $\text{mesh}(\mathcal{B} + z) < \epsilon$ for every $0 \leq z < m(n+1), z \in \mathbb{Z}$.

4 Borel's construction

Let $\mathbb{Z}$ act on a compact metric X. Consider the induced action of $\mathbb{Z}$ on $X \times \mathbb{R}$ defined by $(x,t) + z = (x + z, t + z), z \in \mathbb{Z}$. Borel's construction $X \times_{\mathbb{Z}} \mathbb{R}$ is the orbit space $(X \times \mathbb{R})/\mathbb{Z}$ with the action of $\mathbb{R}$ induced by the action $(x,t) + r = (x, t + r), r \in \mathbb{R}$, on $X \times \mathbb{R}$ and the action of $\mathbb{Z}$ considered as a subgroup of $\mathbb{R}$. We identify X with the subset $X \times_{\mathbb{Z}} \mathbb{R}$ corresponding to $X \times \{0\}$ in $X \times \mathbb{R}$. Note that $X \times_{\mathbb{Z}} \mathbb{R}$ is metric compact, the action of $\mathbb{Z}$ on $X \times_{\mathbb{Z}} \mathbb{R}$ extends the original action of $\mathbb{Z}$ on X, $X \times_{\mathbb{Z}} \mathbb{R} = X + [0,1]$ and for every $r \in \mathbb{R}$ the set $X + r$ is invariant under the action of $\mathbb{Z}$. We denote by π_X, $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{Z}}$ the projections of $X \times \mathbb{R}$ to X, $\mathbb{R}$ and $X \times_{\mathbb{Z}} \mathbb{R}$ respectively and denote by π the map $\pi : X \times_{\mathbb{Z}} \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ induced by $\pi_{\mathbb{R}}$. The space $X \times_{\mathbb{Z}} \mathbb{R}$, also known in Topological Dynamics as the mapping torus of $(X, \mathbb{Z})$, can be represented as the quotient of $X \times [0,1]$ where each point $(x, 0)$ is identified with $(x + 1, 1)$.

5 Proof of Theorem 1.2

Clearly we may assume $\text{mdim}X < \infty$. Fix $\epsilon > 0$ and $d > \text{mdim}X$, and take an open cover $\mathcal{A}$ of X such that $\text{mesh}(\mathcal{A} + z) < \epsilon/2$ for every integer $z \in \mathbb{Z}$ such that $0 \leq zd < n = \text{ord}\mathcal{A}$. Consider Borel's construction $X \times_{\mathbb{Z}} \mathbb{R}$. Without loss of generality we may assume that $\text{mesh}(\mathcal{A} + r) < \epsilon/2$ for every $r \in \mathbb{R}$ such that $0 \leq rd < n + 2d$. Take a finite closed cover $\mathcal{B}$ of $X \times [0,1]$ such that $\text{ord}\mathcal{B} \leq \text{ord}\mathcal{A} + 1 = n + 1$ and $\pi_X(\mathcal{B})$ refines $\mathcal{A}$. Clearly $\mathcal{C} = \pi_{\mathbb{Z}}(\mathcal{B})$ is a closed finite cover of $X \times_{\mathbb{Z}} \mathbb{R}$ with $\text{ord}\mathcal{C} \leq 2\text{ord}\mathcal{B} = 2(n+1)$. Recall that X is considered as a subspace $X \times_{\mathbb{Z}} \mathbb{R}$ and note that every point of $X \times_{\mathbb{Z}} \mathbb{R}$ not belonging to X is covered by at most $\text{ord}\mathcal{B}$ elements of $\mathcal{C}$. Also note that we may choose $\mathcal{B}$ so that that $\text{mesh}\pi(\mathcal{C})$ is as small as we wish. Then we may assume that $\text{mesh}(\mathcal{C} + r) < \epsilon$ for every $r \in \mathbb{R}$ such that $0 \leq rd < n + d$.

Recall that $\mathbb{R}$ acts on $X \times_{\mathbb{Z}} \mathbb{R}$, define $\mathcal{D}_i = \mathcal{C} + (i/n)$ for $0 \leq i < n$ and note that $\text{mesh}(\mathcal{D}_i + z) < \epsilon$ for every $z \in \mathbb{Z}$ such that $0 \leq z < [n/d]$. Denote $z_i = [n/d]i$. Then for the cover $\mathcal{D} = \mathcal{D}_0 \vee (\mathcal{D}_1 - z_1) \vee \cdots \vee (\mathcal{D}_{n-1} - z_{n-1})$ of $X \times_{\mathbb{Z}} \mathbb{R}$ we have that $\text{mesh}(\mathcal{D} + z) < \epsilon$ for every $z \in \mathbb{Z}$ such that $0 \leq z < [n/d]n$.

Now consider the set P of the points of $X \times_{\mathbb{Z}} \mathbb{R}$ of ord $> n + 1$ with respect to $\mathcal{C}$. Note that P is a closed subset of $X \subset X \times_{\mathbb{Z}} \mathbb{R}$. Then $P_i = P + (i/n)$ is the set of the points of ord $> n + 1$ with respect to $\mathcal{D}_i$ and $P_i \subset X + (i/n)$. Replace the elements of $\mathcal{D}_i$ by slightly larger open sets to get an open cover $\mathcal{D}'_i$ of $X \times_{\mathbb{Z}} \mathbb{R}$ with $\text{ord}\mathcal{D}'_i = \text{ord}\mathcal{D}_i$, denote by P'_i the set of points of ord $> n + 1$ with respect to $\mathcal{D}'_i$ and consider a canonical map $f_i : X \times_{\mathbb{Z}} \mathbb{R} \to K_i = N(\mathcal{D}'_i)$ to a simplicial complex K_i (the nerve of $\mathcal{D}'_i$) for which the fibers of f_i refine $\mathcal{D}'_i$ and $\dim K_i \leq 2n + 1$. Assuming that the elements of $\mathcal{D}'_i$ are sufficiently close to the elements of $\mathcal{D}_i$ and mesh$\pi(\mathcal{C})$ is small enough we can also assume that there are subcompexes $L_i \subset K_i$ such that $f_i(P'_i + z) \subset L_i$, $\dim(K_i \setminus L_i) \leq n$, and $f_i(f_j^{-1}(L_j) + z) \subset K_i \setminus L_i$ for every i, $j \neq i$ and $z \in \mathbb{Z}$.

Consider the map $f : X \times_{\mathbb{Z}} \mathbb{R} \to K = K_0 \times \cdots \times K_{n-1}$ defined by $f(x) = (f_i(x+z_i))_{0 \leq i < n}$. Then the fibers of f refine the cover $\mathcal{D}' = \mathcal{D}'_0 \vee (\mathcal{D}'_1 - z_1) \vee \cdots \vee (\mathcal{D}'_{n-1} - z_{n-1})$ of $X \times_{\mathbb{Z}} \mathbb{R}$ and $f(X \times_{\mathbb{Z}} \mathbb{R}) \subset M = M_0 \cup \cdots \cup M_{n-1}$ where M_i is the product of K_i with all $K_j \setminus L_j$ for $j \neq i$. Note that $\dim M_i \leq 2n + 1 + n(n-1) = n^2 + n + 1$ and hence $\dim M \leq n^2 + n + 1$. Thus $\mathcal{D}'$ can be refined by an open cover of ord $\leq n^2 + n + 2$. Recall that mesh$(\mathcal{D} + z) < \epsilon$ for every $z \in \mathbb{Z}$ such that $0 \leq z < [n/d]n$. Assuming that the elements $\mathcal{D}'$ are sufficiently close to the elements of $\mathcal{D}$ we can assume that the latter property holds for $\mathcal{D}'$ as well and we get that mdim$X \times_{\mathbb{Z}} \mathbb{R} \leq d$ since $\frac{n^2+n+2}{[n/d]n}$ goes to d as n goes to ∞. This implies that mdim$X \times_{\mathbb{Z}} \mathbb{R} =$ mdim$X$.

6 Kolmogorov's covers

For a subset $A \subset \mathbb{R}$ and $r \in \mathbb{R}$ denote $rA = \{ra : a \in A\}$. Fix a natural number n and a real number $\epsilon > 0$ and consider the collections of closed intervals $\mathcal{A}_i = \{\frac{\epsilon}{m+1}[zm + i, zm + i + m] : z \in \mathbb{Z}\}$ for $1 \leq i \leq m + 1$. Note that mesh$\mathcal{A}_i < \epsilon$ and the collections $\mathcal{A}_1, \ldots, \mathcal{A}_{m+1}$ cover $\mathbb{R}$ at least m times. For a natural n denote $\mathcal{A}_i^n = \{A_1 \times \cdots \times A_n : A_1, \ldots, A_n \in \mathcal{A}_i\}$. Then for $n \leq m$ the collections $\mathcal{A}_1^n \ldots \mathcal{A}_{m+1}^n$ cover $\mathbb{R}^n$ at least $m - n + 1$ times. Indeed, for a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ each coordinate of x is covered by at least m collections from $\mathcal{A}_1, \ldots, \mathcal{A}_{m+1}$ and therefore there are at least $m - n + 1$ collections $\mathcal{A}_i$ so that each of them covers all the coordinates of x and and hence for each such collection $\mathcal{A}_i$ the point x is covered by $\mathcal{A}_i^n$. Clearly $\mathcal{A}_1^n, \ldots, \mathcal{A}_{m+1}^n$ are collections of disjoint cubes of diam $< \epsilon$ with respect to the l^∞-norm in $\mathbb{R}^n$. We will refer to $\mathcal{A} = \mathcal{A}_1^n \cup \cdots \cup \mathcal{A}_{m+1}^n$ as Kolmogorov's cover of $\mathbb{R}^n$.

Let $\mathcal{U}$ be a finite open cover of a compact metric X with ord$\mathcal{U} \leq n + 1$ and let m be any integer such that $n \leq m$. Consider a canonical map $f : X \to K = N(\mathcal{U})$ to the nerve of $\mathcal{U}$. Since K is a finite simplicial complex with $\dim K \leq n$ there is a finite-to-one map $g : K \to \mathbb{R}^n$. Then for a sufficiently small $\epsilon > 0$ we have that for Kolmogorov's cover $\mathcal{A}$ the components of the elements of $(g \circ f)^{-1}(\mathcal{A})$ will refine the elements of $\mathcal{U}$ and hence by splitting the elements of $(g \circ f)^{-1}(\mathcal{A}_i^n)$ into finitely many disjoint closed sets and taking into account that $g(K)$ meets only finitely many elements of $\mathcal{A}$ we get from each $(g \circ f)^{-1}(\mathcal{A}_i^n)$ a finite family $\mathcal{F}_i$ of disjoint closed sets such that $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m+1}$ refines $\mathcal{U}$ and covers X at least $m - n + 1$ times. We will refer to $\mathcal{F}$ as a Kolmogorov-Ostrand cover.

7 Lindenstrauss' level functions

Let $(X, \mathbb{Z})$ be a non-trivial minimal dynamical system on a compact metric X, U a non-empty open set U in X and $\phi : X \to [0, 1]$ any map such that $\phi(X \setminus U) = 1$ and $\phi^{-1}(0)$ has non-empty interior. Define the random walk on X by stopping at $x \in X$ with the probability $1 - \phi(x)$ and moving from x to $x - 1$ with the probability $\phi(x)$. Then Lindenstrauss' level function $\xi : X \to \mathbb{R}$ is defined at $x \in X$ as the expectation of the number of steps in the random work starting at x. In other words, $\xi(x) =$

$\phi(x)(1-\phi(x-1)) + 2\phi(x)\phi(x-1)(1-\phi(x-2)) + \ldots$ Since the action of $\mathbb{Z}$ is minimal, each random walk will eventually hit $\phi^{-1}(0)$ and stop there, moreover, the number of steps in each random walk is bounded by a number depending only on the set $\phi^{-1}(0)$. Thus ξ is well-defined and continuous. It is easy to see that $\xi(x+1) = \phi(x+1)(\xi(x)+1)$ for every $x \in X$ and therefore given a natural number n we have that $\xi(x+z) = \xi(x) + z$ for every integer $-n \leq z \leq n$ and $x \in X \setminus ((U-n) \cup \cdots \cup (U+n))$. We will refer to $\xi(x)$ as a Lindenstrauss level function determined by U.

8 Proof of Theorem 1.1

Let us first present some auxiliary notations and properties used in the proof.

Let $(Y, \mathbb{R})$ be a dynamical system, A a subset of Y, $\mathcal{A}$ a collection of subsets of Y and $\alpha, \beta \in \mathbb{R}$ positive numbers. The subset A is said to be **(α,β)-small if** $\operatorname{diam}(A+r) < \alpha$ for every $r \in [-\beta, \beta] \subset \mathbb{R}$. The collection $\mathcal{A}$ is said to be **(α,β)-fine** if $\operatorname{mesh}(\mathcal{A}+r) < \alpha$ for every $r \in [0, \beta] \subset \mathbb{R}$. The collection $\mathcal{A}$ is said to be **(α,β)-refined** at a subset $W \subset Y$ if the following two conditions hold: (condition 1) no element of $\mathcal{A}+r$ meets the closures of both $W+r_1$ and $W+r_2$ for every $r, r_1, r_2 \in [-\beta,\beta] \subset \mathbb{R}$ with $|r_1 - r_2| \geq 1$ and (condition 2) if for an element A of $\mathcal{A}$ the set $A + [-\beta,\beta]$ meets the closure of $W + [-\beta, \beta]$ then $\operatorname{diam}(A+r) < \alpha$ for every $r \in [-\beta,\beta] \subset \mathbb{R}$.

Proposition 8.1 *Let $(Y, \mathbb{R})$ be a free dynamical system on a compact metric Y, ω a point in Y and let α and β be positive real numbers. Then there is an open neighborhood W of w and an open cover $\mathcal{V}$ of Y such that* $\operatorname{ord}\mathcal{V} \leq 3$ *and $\mathcal{V}$ is (α,β)-refined at W.*

Proof. Note that $L = w + [-3\beta, 3\beta]$ is an interval in Y and consider a map $g = (g_1, g_2) : Y \to \mathbb{R}^2$ such that g_1 embeds L into $\mathbb{R}$ and $g_2(y)$ is the distance to L from $y \in Y$. Then $g^{-1}(a)$ is a singleton for every $a \in g(L)$ and therefore there is a fine open cover $\mathcal{O}$ of $\mathbb{R}^2$ with $\operatorname{ord}\mathcal{O} = 3$ and a small neighborhood W of w in Y such that the cover $\mathcal{V} = g^{-1}(\mathcal{O})$ of Y is (α,β)-refined at W. Clearly $\operatorname{ord}\mathcal{V} \leq 3$. ∎

Proposition 8.2 *Let $q > 2$ be an integer. Then there is a finite collection $\mathcal{E}$ of disjoint closed intervals in $[0, q) \subset \mathbb{R}$ such that $\mathcal{E}$ splits into the union $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_q$ of q disjoint subcollections having the property that for every $t \in \mathbb{R}$ the set $t + \mathbb{Z} \subset \mathbb{R}$ meets at least $q-2$ subcollections $\mathcal{E}_i$ (a set meets a collection if there is a point of the set that is covered by the collection). Moreover, we may assume that* $\operatorname{mesh}\mathcal{E}$ *is as small as we wish.*

Proof. Let $E_1, \ldots, E_{q-1} \subset [0, q)$ be $q-1$ disjoint closed intervals of length > 1. Clearly for each $t \in \mathbb{R}$ the set $t + \mathbb{Z}$ meets all the intervals E_i. Consider a finite set $A \subset [0, q)$ such that $1 \in A$ and the numbers of A are linearly independent over $\mathbb{Q}$ (the rationals). Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \inf\{|t + z_1 - a_1| + |t + z_2 - a_2| : a_1, a_2 \in A, z_1, z_2 \in \mathbb{Z}, (z_1, a_1) \neq (z_2, a_2)\}.$$

Note that ϕ is continuous and periodic with period 1 and $\phi(t) > 0$ for every t. Then $\sigma = \inf\{\phi(t) : t \in \mathbb{R}\} > 0$. Set Ω to be the open $\sigma/3$-neighborhood of A and note that Ω

contains at most one point of $t+\mathbb{Z}$ for every $t \in \mathbb{R}$. Also note that for every $1 \leq i \leq q-1$ the set $E_i \setminus \Omega$ splits into a finite collection of disjoint closed intervals and denote this collection by $\mathcal{E}_i$. Thus we have that for every $t \in \mathbb{R}$ the set $t+\mathbb{Z}$ meets at least $q-2$ collections from $\mathcal{E}_1, \dots, \mathcal{E}_{q-1}$. Assuming that A is sufficiently dense in $[0, q)$ we can get mesh$\mathcal{E}_i$ as small as we wish for all i. Setting $\mathcal{E}_q$ to be empty or any finite collection of disjoint small closed intervals in $[0, q)$ that don't meet $\mathcal{E}_1, \dots, \mathcal{E}_{q-1}$ we get the collection $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_q$ with the required properties. ■

Proof of Theorem 1.1. Let $f = (f_1, \dots, f_k) : X \to [0,1]^k$ be any map and let $\epsilon > 0$ and $\delta > 0$ be such that under each f_i the image of every subset of X of diam $< 3\epsilon$ is of diam $< \delta$ in $[0, 1]$. Our goal is to approximate f by a δ-close map ψ such that the fibers of $\psi^{\mathbb{Z}}$ contain at most $\gamma = \gamma(d)$ points with pairwise distances larger than 3ϵ where $\gamma(t) = [k/(k-t]k/(k-t), t < k$.

Since $[\gamma] = [\gamma(t)]$ for every $t > d$ sufficiently close to d we can replace d by a slightly larger real number and assume that mdim$X < d$. Take natural numbers n and q such that mdim$X < n/q < d$ and set $m = qk$. Then $n < qd < qk = m$ and $m/(m-n) \leq k/(k-d)$. By Theorem 1.2 we have mdim$X \times_{\mathbb{Z}} \mathbb{R} =$ mdim$X$. Then, assuming that n is large enough, there is an open cover $\mathcal{U}$ of $X \times_{\mathbb{Z}} \mathbb{R}$ such that ord$\mathcal{U} \leq n-2$ and $\mathcal{U}$ is (ϵ, q)-fine. Clearly we may assume that $q > 2$.

Since the theorem obviously holds if X is a singleton, we may assume that $(X, \mathbb{Z})$ is non-trivial. Fix a point $w \in X$ and let $l > q$ be a (sufficiently large) positive integer which will be defined later and will depend only on q. By Proposition 8.1 there is an open cover $\mathcal{V}$ of $X \times_{\mathbb{Z}} \mathbb{R}$ and a neighborhood W of w in $X \times_{\mathbb{Z}} \mathbb{R}$ such that $\mathcal{V}$ is $(\epsilon, 2l)$-refined at W.

Now replacing $\mathcal{U}$ by an open cover of ord $\leq n$ refining $\mathcal{U} \vee \mathcal{V}$ we can assume that ord$\mathcal{U} \leq n$, $\mathcal{U}$ is (ϵ, q)-fine and $\mathcal{U}$ is $(\epsilon, 2l)$-refined at W. Clearly we can replace W by any smaller neighborhood of w and assume that W is $(\epsilon, 3l)$-small and the elements of $\mathcal{D}_W$ are disjoint where $\mathcal{D}_W$ is the collection of the closures of $W + z$ for the integers $z \in [-2l, 2l]$.

Refine $\mathcal{U}$ by a Kolmogorov-Ostrand cover $\mathcal{F}$ of $X \times_{\mathbb{Z}} \mathbb{R}$ such that $\mathcal{F}$ covers $X \times_{\mathbb{Z}} \mathbb{R}$ at least $m-n$ times and $\mathcal{F}$ splits into $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m$ the union of finite families of disjoint closed sets $\mathcal{F}_i$. Note that $\mathcal{F}$ is (ϵ, q)-fine and $(\epsilon, 2l)$-refined at W.

Let $\xi : X \to \mathbb{R}$ be a Lindenstrauss level function determined by W restricted to X. Denote $W^+ = W + \mathbb{Z} \cap [-l, l]$ and $X^- = X \setminus W^+$. Recall that $\xi(x+z) = \xi(x) + z$ for every $x \in X^-$ and an integer $-l \leq z \leq l$.

We need an additional auxiliary notation. Let $\mathcal{A}$ be a collection of subsets of $X \times_{\mathbb{Z}} \mathbb{R}$, $\mathcal{B}$ a collection of intervals in $\mathbb{R}$. For $B \in \mathcal{B}$ and $z \in \mathbb{Z}$ consider the collection $\mathcal{A} + B$ restricted to $\xi^{-1}(B + qz)$ and denote by $\mathcal{A} \oplus_\xi B$ the union of such collections for all $z \in \mathbb{Z}$. Now denote by $\mathcal{A} \oplus_\xi \mathcal{B}$ the union of the collections $\mathcal{A} \oplus_\xi B$ for all $B \in \mathcal{B}$. Note that $\mathcal{A} \oplus_\xi \mathcal{B}$ is a collection of subsets of X.

Consider a finite collection $\mathcal{E}$ of disjoint closed intervals in $[0, q) \subset \mathbb{R}$ satisfying the conclusions of Proposition 8.2. For $1 \leq i \leq k$ define the collection $\mathcal{D}_i$ of subsets of X as the union of the collections $\mathcal{F}_i \oplus_\xi \mathcal{E}_1$, $\mathcal{F}_{i+k} \oplus_\xi \mathcal{E}_2$, $\dots$, $\mathcal{F}_{i+(q-1)k} \oplus_\xi \mathcal{E}_q$. Note that assuming that mesh$\mathcal{E}$ is small enough we may also assume that $\mathcal{F}_i^+ = \mathcal{F}_i + [-\text{mesh}\mathcal{E}, \text{mesh}\mathcal{E}]$ is a collection of disjoint sets and the collection $\mathcal{F}^+ = \mathcal{F} + [-\text{mesh}\mathcal{E}, \text{mesh}\mathcal{E}]$ is (ϵ, q)-fine and $(\epsilon, 2l)$-refined at W and, as a result, we get that $\mathcal{D}_i$ is a collection of disjoint closed sets of X of diam $< \epsilon$ and each element of $\mathcal{D}_i$ meets at most one element of $\mathcal{D}_W$.

Define a map $\psi = (\psi_1, \dots, \psi_k) : X \to [0,1]^k$ so that for each i the map ψ_i is δ-close to f_i, ψ_i sends the elements of $\mathcal{D}_W$ restricted to X and the elements of $\mathcal{D}_i$ to singletons in $[0,1]$ and ψ_i separates the elements of $\mathcal{D}_W$ restricted to X together with the elements of $\mathcal{D}_i$ not meeting $\mathcal{D}_W$. We will show that the fibers of $\psi^{\mathbb{Z}}$ contain at most γ points with pairwise distances larger than 3ϵ.

Denote by S the set of all the pairs of integers (i,j) with $0 \le i \le l-1$ and $1 \le j \le k$. We say that a point $x \in X$ is marked by a pair $(i,j) \in S$ if $x+i$ is covered by $\mathcal{D}_j$ and denote by $S_x \subset S$ the set of the pairs by which x is marked. Let us compute the size of S_x for a point $x \in X^-$. Consider a non-negative integer z such that $\xi(x) \le zq < (z+1)q < \xi(x)+l$. Recall that $x+(zq-\xi(x))$ is covered by at least $m-n$ collections from the family $\mathcal{F}_1, \dots, \mathcal{F}_m$ and $\xi(x)+\mathbb{Z}$ meets at least $q-2$ collections from $\mathcal{E}_1, \dots, \mathcal{E}_q$. Then (†) the point x is marked by at least $m-n-2k$ pairs (i,j) of S with $zq \le \xi(x)+i < (z+1)q$.

Indeed, for every $\mathcal{E}_p$ that meets $\xi(x)+\mathbb{Z}$ pick up $z_p \in \mathbb{Z}$ such that $\xi(x)+z_p$ is covered by $\mathcal{E}_p$. Denote $i_p = z_p + zq$ and note that $zq \le \xi(x)+i_p < (z+1)q$ and, hence, $0 \le i_p < l$. Also note that different p define different i_p and for every $1 \le j \le k$ such that $\mathcal{F}_{j+(p-1)k}$ covers the point $x+(zq-\xi(x))$ we have that the collection $\mathcal{D}_j$ covers $x+i_p$, and therefore x is marked by the pair (i_p, j) in S. Thus if $\xi(x)+\mathbb{Z}$ meets all the collections $\mathcal{E}_1, \dots, \mathcal{E}_q$ the number of pairs $(i,j) \in S$ with $zq \le \xi(x)+i < (z+1)q$ marking x will be at least the number of times $x+(zq-\xi(x))$ is covered by the collections $\mathcal{F}_1, \dots, \mathcal{F}_m$, which is at least $m-n$. Each time $\xi(x)+\mathbb{Z}$ misses a collection from $\mathcal{E}_1, \dots, \mathcal{E}_q$ reduces the above estimate by at most k. Since $\xi(x)+\mathbb{Z}$ can miss at most two collections from $\mathcal{E}_1, \dots, \mathcal{E}_q$ we arrive at the required estimate $m-n-2k$ in (†).

Thus, by (†), the point x is marked by at least $(\frac{l}{q}-3)(m-n-2k)$ pairs of S. Since $m-n \ge m(k-d)/k = q(k-d)$ we have $|S_x| \ge (\frac{l}{q}-3)(m-n-2k) \ge ((l/q)-3)(q(k-d)-2k) = l(k-d)\Delta$ where $\Delta = (1-\frac{3q}{l})(1-\frac{2k}{(k-d)q})$. Note that we may take q sufficiently large and then l sufficiently large with respect to q and assume that $\Delta < 1$ and Δ is as close to 1 as we wish.

Now we will show that there is no set contained in a fiber of $\psi^{\mathbb{Z}}$ and containing more than γ points with pairwise distances larger than 3ϵ. Aiming at a contradiction assume that such a set $\Gamma \subset X$ exists with $|\Gamma| = [\gamma]+1$. First note that Γ can contain at most one point in $X \setminus X^-$ since each ψ_i separates $\mathcal{D}_W$ restricted to X and $\mathrm{mesh}(\mathcal{D}_W) \le \epsilon$.

Now assume that Γ contains a point $x \in X \setminus X^-$ and a point $y \in X^-$. Take any pair $(i,j) \in S_y$ and an element $D \in \mathcal{D}_j$ that contains $y+i$. Since $\psi_j^{\mathbb{Z}}$ does not separate x and y we get that D meets the element of $\mathcal{D}_W$ containing $x+i$, and (††) the latter is impossible because $\mathcal{F}^+$ is $(\epsilon, 2l)$-refined at W and W is $(\epsilon, 3l)$-small that implies that x and y are 2ϵ-close.

Indeed, D is contained in an element of $\mathcal{F}^+ + t$ for some $0 \le t < q$. Then y is covered by an element of $\mathcal{F}^+ + t - i$ that intersects the element of $\mathcal{D}_W$ containing x, and the facts that $(\epsilon, 2l)$-refined at W and W is $(\epsilon, 3l)$-small yield the conclusion of (††).

Thus, by (††), we get that $\Gamma \subset X^-$. Now we can assume that $\Delta > \gamma/|\Gamma|$ and get that $\sum_{x\in\Gamma} |S_x| > |\Gamma| l(k-d)(\gamma/|\Gamma|) = lk[k/(k-d)]$. Then, since $|S| = lk$, there is a pair (i_*, j_*) in S and $\Gamma_* \subset \Gamma$ such that $|\Gamma_*| = [k/(k-d)]+1$ and every point in Γ_* is marked by (i_*, j_*). Thus $\Gamma_* + i_*$ is covered by $\mathcal{D}_{j_*}$. Then (†††) there is an element D of $\mathcal{D}_{j_*}$ containing $\Gamma_* + i_*$.

Indeed, if $\Gamma_* + i_*$ meets two elements of $\mathcal{D}_{j_*}$ then those elements cannot touch the same element of $\mathcal{D}_W$ (because the points of Γ_* are 3ϵ-distant and $\mathcal{F}^+$ is $(\epsilon, 2l)$-refined at W) and, hence will be separated by $\psi_{j_*}^{\mathbb{Z}}$. Thus (†††) holds.

Then, by (†††), we have that $\xi(\Gamma_*)$ is contained in an element (interval) of $\mathcal{E}+qz$ for some non-negative integer z. Since mesh$\mathcal{E}$ is small there is an integer $0 \leq q_* \leq q-1$ such that $\xi(\Gamma_* - q_*) \subset [qz, qz+2]$. Denote by S_* the set of pairs (i,j) such that $0 \leq i \leq q-3, 1 \leq j \leq k$, and say that a point $x \in \Gamma_* - q_*$ is marked by $(i,j) \in S_*$ if $x+i$ is covered by $\mathcal{D}_j$. Since $\mathcal{F}$ covers X at least $m-n$ times we get by an argument similar to the one applied above that each point of $\Gamma_* - q_*$ is marked by at least $m-n-6k \geq q(k-d) - 6k = q(k-d)\Delta_*$ pairs in S_* where $\Delta_* = 1 - \frac{6k}{q(k-d)}$. Note that $(k/(k-d))/|\Gamma_*| < 1$ and assume that q is such that $(k/(k-d))/|\Gamma_*| < \Delta_*$. Then, since $|S_*| = k(q-2)$, there are two distinct points x and y of $\Gamma_* - q_*$ marked by the same pair (i,j) of S_*. Again by the above reasoning $x+i$ and $y+i$ cannot lie in different elements of $\mathcal{D}_j$. Thus $x+i$ and $y+i$ lie in the same element D of $\mathcal{D}_j$. Then, since $\mathcal{F}^+$ is (ϵ, q)-fine, this implies that the points $x+q_*$ and $y+q_*$ of Γ_* are ϵ-close.

Indeed, D is contained in an element of $\mathcal{F}^+ + t + i$ with $t \in [0,2)$. Then x and y are contained in an element of $\mathcal{F}^+ + t$, and $x+q_*$ and $y+q_*$ are contained in an element of $\mathcal{F}^+ + t + q_*$. Hence $x+q_*$ and $y+q_*$ are ϵ-close since $0 \leq t + q_* < q$ and $\mathcal{F}^+$ is (ϵ, q)-fine.

Thus the assumption $|\Gamma| > \gamma$ leads to a contradiction and hence ψ is the desired approximation of f. Then the theorem follows by a standard Baire category argument. ■

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