SOLID LOCALLY ANALYTIC REPRESENTATIONS

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Abstract. We develop the p-adic representation theory of p-adic Lie groups on solid vector spaces over a complete non-archimedean extension of $\mathbf{Q}_p$. More precisely, we define and study categories of solid, solid locally analytic and solid smooth representations. We show that the category of solid locally analytic representations of a compact p-adic Lie group is equivalent to that of quasi-coherent modules over its algebra of locally analytic distributions, generalizing a classical result of Schneider and Teitelbaum. For arbitrary G, we prove an equivalence between solid locally analytic representations and quasi-coherent sheaves over certain locally analytic classifying stack over G. We also extend our previous cohomological comparison results from the case of a compact group defined over $\mathbf{Q}_p$ to the case of an arbitrary group, generalizing results of Lazard and Casselman-Wigner. Finally, we study an application to the locally analytic p-adic Langlands correspondence for GL_1.

arXiv:2305.03162v2 [math.RT] 3 Oct 2023

Contents

1. Introduction

Let p be a prime number, G be a p-adic Lie group defined over a finite extension L of $\mathbf{Q}_p$ and let $\mathcal{K}=(K,K^+)$ be a complete non-archimedean extension of L. The purpose of this article is to give new foundations of the theory of locally analytic representations of G on $\mathcal{K}$-solid vector spaces through the use of condensed mathematics. This generalizes our previous work [RJRC21], where the case G compact and $L=\mathbf{Q}_p$ was studied.

Our first purpose is to provide definitions and study the main properties of categories appearing naturally in the representation theory of a p-adic Lie group. There are at least three of them, namely continuous, smooth and locally analytic representations. Using the formalism of condensed mathematics, we construct and study the (∞-)categories of solid, solid smooth and solid locally analytic representations of G. We denote them, respectively, by $\mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G)$, $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$, $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$. These categories arise as the derived category of a corresponding abelian category of representations. Furthermore, these abelian categories contain fully faithfully all the classical categories of continuous, smooth and locally analytic representations on complete compactly generated locally convex K-vector spaces. One of the main advantages of our approach is that many of the difficulties appearing in fundamental constructions in classical representation theory, such as Hochschild-Serre, Shapiros's lemma, duality, etc., are easily overcome with the use of homological algebra when one works on a solid framework.

Let us now explain our results in more detail. Let G be a p-adic Lie group over L. Let $\mathcal{K}_{\blacksquare}[G]$ be the Iwasawa algebra of G over $\mathcal{K}_{\blacksquare}$, i.e. the free $\mathcal{K}_{\blacksquare}$-vector space generated by G. If $\mathcal{K}=(K,\mathcal{O}_K)$ is a finite extension of $\mathbf{Q}_p$, then $\mathcal{K}_{\blacksquare}[G]$ is the classical Iwasawa algebra of G, i.e. the dual of the space $C(G,K)$ of continuous functions on G. Let $\mathcal{D}^{la}(G,K)$ denote the locally analytic distribution algebra of G, i.e. the dual of the space $C^{la}(G,K)$ of locally analytic functions on G. We denote by $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ and $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{K}_{\blacksquare}[G])$ the (∞-)categories of $\mathcal{D}^{la}(G,K)$ and $\mathcal{K}_{\blacksquare}[G]$-modules on $\mathcal{K}_{\blacksquare}$-vector spaces, respectively. The following result resumes our construction of the category of solid locally analytic representations and its main properties (cf. Propositions 3.2.3, 3.2.5 and 3.2.6).

Theorem A. *There exists a full subcategory* $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)\subset\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ *of solid locally analytic representations of G on $\mathcal{K}_{\blacksquare}$-vector spaces stable under tensor product and colimits, where the inclusion has a right adjoint given by (derived) locally analytic vectors $V\mapsto V^{Rla}$. Moreover, the following properties are satisfied.*

1. *An object $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ is locally analytic if and only if $H^i(V)$ is (non-derived) locally analytic for every $i\in\mathbb{Z}$. In particular, $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ has a natural t-structure.*
2. $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ *is the derived category of its heart.*
3. *The functor of locally analytic vectors satisfies the projection formula, namely, for any $V,W\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$, one has $(V^{Rla}\otimes^L_{\mathcal{K}_{\blacksquare}}W)^{Rla}=V^{Rla}\otimes^L_{\mathcal{K}_{\blacksquare}}W^{Rla}$.*

Remark 1.0.1.

1. Let V be a locally L-analytic representation of G on an LB space in the classical sense. Then point (1) implies that V is an object in $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ that is derived locally analytic. In particular, classical locally analytic representation theory lives naturally in $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$.
2. If G is a p-adic Lie group over $\mathbf{Q}_p$, then $\mathcal{D}^{la}(G,K)$ is an idempotent algebra over the Iwasawa algebra $\mathcal{K}_{\blacksquare}[G]$, namely $\mathcal{D}^{la}(G,K)\otimes^L_{\mathcal{K}_{\blacksquare}[G]}\mathcal{D}^{la}(G,K)=\mathcal{D}^{la}(G,K)$. This implies that the category of $\mathcal{D}^{la}(G,K)$-modules on $\mathcal{K}_{\blacksquare}$-vector spaces embeds fully faithfully in the category of $\mathcal{K}_{\blacksquare}[G]$-modules on $\mathcal{K}_{\blacksquare}$-vector spaces. In particular, $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ is a full subcategory of $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{K}_{\blacksquare}[G])$ and one can also define the locally analytic vectors of $\mathcal{K}_{\blacksquare}[G]$-modules as the right adjoint of this inclusion. For $\mathcal{D}^{la}(G,K)$-modules, this coincides with the construction of Theorem A. Nevertheless, when the group is not defined over $\mathbf{Q}_p$, both constructions of locally analytic vectors differ, c.f. Remark 3.1.6 for a detailed discussion.
3. We also give an analogue of Theorem A for solid smooth representations, cf. §5.2.
4. As a corollary of Theorem A, we obtain a description of $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ and $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$ as quasi-coherent sheaves on the classifying stack $[*/G]$ of G, where G is endowed with the sheaf of locally analytic or smooth functions, cf. Theorem 4.3.3 and Proposition 5.4.2.

We now explain some applications of our theory. The first main result is an equivalence, for G a compact group, between the (derived) category of solid locally analytic representations of G and the category of solid quasi-coherent sheaves over certain non-commutative adic Stein space associated to G. This can be seen as a generalization of a classical anti-equivalence of Schneider and Teitelbaum [ST03], which can be recovered from our equivalence when restricting to the (abelian) subcategory of admissible representations after applying a duality functor. This result can also be seen as a step towards geometrizing the category of solid locally analytic representations. Our second main application is an extension of the cohomological comparison theorems for solid representations from the case where G is compact and defined over $\mathbf{Q}_p$ obtained in [RJRC21] to the general case, extending also the non compact version [CW74] of Lazard's isomorphisms [Laz65] from the case of finite dimensional representations to arbitrary solid representations. The main novelty of our approach to the comparison results is that we deduce them in a completely formal way from adjointness properties between certain functors.

If G is compact the distribution algebra $\mathcal{D}^{la}(G,K)$ is a Fréchet-Stein algebra in the sense of [ST03], and the category of its coadmissible modules can be seen as the category of coherent sheaves over certain (non-commutative) Stein space associated to $\mathcal{D}^{la}(G,K)$. More precisely, for $h\in[0,\infty)$ a parameter depending on some choices, there is a limit sequence of h-analytic distribution algebras $\{\mathcal{D}^h(G,K)\}_{h\geq 0}$ such that $\mathcal{D}^{la}(G,K)=\varprojlim_{h\to\infty}\mathcal{D}^h(G,K)$. For example, if $G=\mathbb{Z}_p$ is the additive group of p-adic integers, by the Amice transform $\mathcal{D}^{la}(\mathbb{Z}_p,K)$ is isomorphic to the global sections of an open unit disc $\mathring{\mathbb{D}}_K$ over K, and the algebras $\mathcal{D}^h(\mathbb{Z}_p,K)$ are overconvergent algebras on closed discs of radius $p^{-\frac{p^{-h}}{p-1}}$. In this way we can think of the sequence $\{\mathcal{D}^h(G,K)\}_{h\geq 0}$ as a family of dagger affinoid algebras defining closed subspaces of a non-commutative Stein space whose global functions are equal to $\mathcal{D}^{la}(G,K)$. We define the category of *solid quasi-coherent* $\mathcal{D}^{la}(G,K)$-modules to be the limit ∞-category

$$\operatorname{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K))=\varprojlim_{h\to\infty}\operatorname{Mod}_{K_\blacksquare}(\mathcal{D}^h(G,K))$$

where the transition maps are given by the K-solid base change $\mathcal{D}^h(G,K)\otimes^L_{\mathcal{D}^{h'}(G,K)}-$ for $h'>h$. Concretely, an object in $\operatorname{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K))$ is a sequence of objects $(\mathcal{F}_h)_{h\geq 0}$ with $\mathcal{F}_h\in\operatorname{Mod}_{K_\blacksquare}(\mathcal{D}^h(G,K))$, together with natural equivalences $\mathcal{D}^h(G,K)\otimes^L_{\mathcal{D}^{h'}(G,K)}\mathcal{F}_{h'}\xrightarrow{\sim}\mathcal{F}_h$ for $h'\geq h$, subject to higher coherences. In the case where $G=\mathbb{Z}_p$, the category $\operatorname{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K))$ is nothing but that of solid quasi-coherent sheaves on $\mathring{\mathbb{D}}_K$. Our second main result is the following.

Theorem B (Theorem 4.1.7)**.** *Let G be a compact p-adic Lie group defined over L. Then there is an equivalence of (stable ∞-)categories*

$$\operatorname{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K))\xrightarrow{\sim}\operatorname{Rep}^{la}_{K_\blacksquare}(G)$$

$$(\mathcal{F}_h)_h\mapsto j_!\mathcal{F}:=(\varprojlim_h\mathcal{F}_h)^{Rla}.$$

Remark 1.0.2.

(1) The functor $j_!$ giving the equivalence of categories can be thought of as taking cohomology with compact support of quasi-coherent sheaves. Indeed, if $G=\mathbb{Z}_p$ the functor $j_!$ is the cohomology with compact supports on $\mathring{\mathbb{D}}_K$ of solid quasi-coherent sheaves as defined in [CS22, Lecture XII] for complex spaces.

(2) The functor $j_!$ of Theorem B does not respect the natural t-structures on both sides and hence does not arise from a functor defined at the level of abelian categories. Indeed, the module $\mathcal{D}^{la}(G,K)$ defines a quasi-coherent sheaf which is given by $\mathcal{F}=(\mathcal{D}^h(G,K))_{h\geq 0}$ and one has that $j_!\mathcal{F}=(\mathcal{D}^{la}(G,K))^{Rla}=C^{la}(G,K)\otimes\chi[-d]$ where d is the dimension of the group G and $\chi=\det(\mathfrak{g})^{-1}$ denotes the determinant of the dual adjoint representation of G on its Lie algebra $\mathfrak{g}$, cf. Corollary 3.1.16.

(3) We also prove an analogous version of Theorem B for solid smooth representations (Proposition 5.2.2), where the category $\operatorname{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{sm}(G,K))$ is defined as $\varprojlim_{H\subset G}\operatorname{Mod}_{K_\blacksquare}(K[G/H])$ for H running through all the open compact subgroups of G.

From Theorem B, we can recover Schneider-Teitelbaum's anti-equivalence as follows.

Proposition C (Proposition 4.2.7). *There is a locally analytic contragradient functor on* $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ *given by*

$$V \mapsto V^{\vee, Rla} = R\underline{\mathrm{Hom}}_K(V,K)^{Rla},$$

and a duality functor $\mathbb{D}$ *on* $\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$*, such that for* $\mathcal{F} \in \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ *one has*

$$j_!(\mathbb{D}(\mathcal{F})) = (j_!\mathcal{F})^{\vee, Rla}.$$

The functor $\mathcal{F} \mapsto j_!\mathbb{D}(W) = (j_!\mathcal{F})^{\vee, Rla}$ *restricts to Schneider and Teitelbaum's classical anti-equivalence between coadmissible* $\mathcal{D}^{la}(G,K)$*-modules and admissible locally analytic representations of* G.

Remark 1.0.3.

(1) In the bigger category $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ of all solid $\mathcal{D}^{la}(G,K)$-modules, the duality functor is given by the formula $\mathbb{D}(V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V, \mathcal{D}^{la}(G,K)\otimes \chi^{-1}[d])$, where χ and d are as before. Note that this functor coincides (up to a twist and a shift in the cohomological degree) with the one defined in [ST03] when G is compact (cf. Corollary 4.2.9 for a discussion of the duality functor in the non-compact case). We refer the reader to Definition 4.1.11 for an explicit definition of $\mathbb{D}$.
(2) Even though the result is stated for a compact group G, one trivially recovers the anti-equivalence of Schneider-Teiltelbaum for non-compact groups, since the classical notions of admissible and coadmissble are local in G, i.e. they only depend on the restriction to an open compact subgroup.
(3) Along the way, the above proposition also answers a question raised in [ST05, p. 26], concerning the extension of the smooth contragradient functor from the category of admissible smooth representations to the cateory of admissible locally analytic representations. We refer the reader to Proposition 5.3.1 for the precise answer to Schneider and Teitelbaum's question.

We now explain our cohomology comparison results. There are natural functors

$$\mathrm{Mod}(\mathcal{K}_{\blacksquare}) \to \mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G) \xrightarrow{F_1} \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_L) \xrightarrow{F_2} \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_{\mathbf{Q}_p}) \xrightarrow{F_3} \mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G),$$

where we denote by $G_{\mathbf{Q}_p}$ the restriction of scalars of G from L to $\mathbf{Q}_p$, and $G_L = G$ to stress that the group is defined over L in order to avoid confusion. All these functors commute with colimits and hence possess right adjoints. The main idea for our comparison results is to reinterpret the cohomological comparison results as formal identities coming from adjunctions and hence reduce them to calculating the right adjoints of the above arrows. Classically, there are many possible cohomology theories associated to G that consider different possible structures of G, e.g., continuous, $\mathbf{Q}_p$ and L-locally analytic, smooth and Lie algebra cohomology.

Definition 1.0.4. We define

- Solid group cohomology $R\Gamma(G,-) : \mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,
- ($\mathbf{Q}_p$-)Locally analytic group cohomology $R\Gamma^{la}(G_{\mathbf{Q}_p},-) : \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_{\mathbf{Q}_p}) \to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,
- (L-)Locally analytic group cohomology $R\Gamma^{la}(G_L,-) : \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_L) \to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,
- Smooth group cohomology $R\Gamma^{sm}(G,-) : \mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$
- Lie algebra cohomology $R\Gamma(\mathfrak{g},-) : \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(U(\mathfrak{g})) \to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,

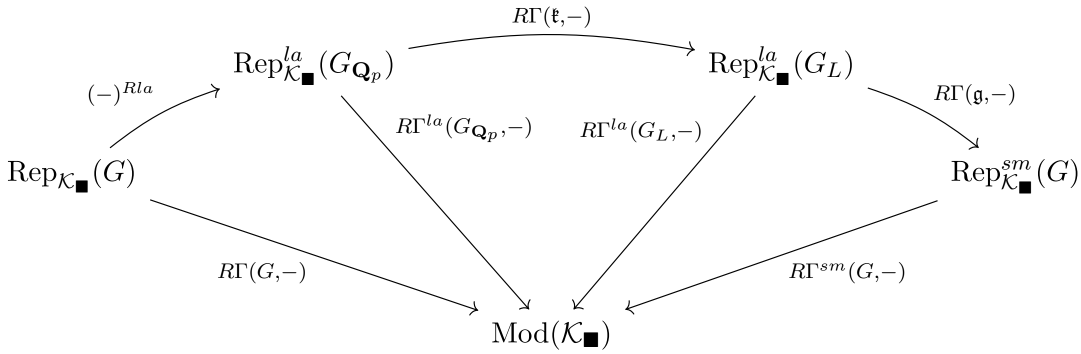
as the right adjoint of the embedding of $\mathrm{Mod}(\mathcal{K}_{\blacksquare})$ in the corresponding category.

One can check (Proposition 6.3.3) that these definitions coincide with the usual definition of cohomology using (continuous, locally analytic, etc...) cochains. Our main key calculation is to show (Proposition 6.2.1) that

(1) The right adjoint of F_1 is given by Lie algebra cohomology $R\Gamma(\mathfrak{g}_L,-) := R\underline{\mathrm{Hom}}_{U(\mathfrak{g}_L)}(K,-)$.
(2) The right adjoint of F_2 is given by $R\Gamma(\mathfrak{k},-) := R\underline{\mathrm{Hom}}_{U(\mathfrak{k})}(K,-)$, where $\mathfrak{k} = \ker(\mathfrak{g}_{\mathbf{Q}_p}\otimes_{\mathbf{Q}_p} L \to \mathfrak{g}_L)$.
(3) The right adjoint of F_3 is given by the functor of locally analytic vectors $(-)^{Rla}$.

Moreover, the right adjoint to the composition of $F_1 \circ \ldots \circ F_j$ $(j = 1,2,3)$ can be interpreted as taking smooth vectors in the corresponding category. Analogously, the right adjoint of $F_2 \circ \ldots F_j$ $(j = 2,3)$ can be interpreted as taking locally L-analytic vectors, and so on. Summarizing this, we obtain our third main result.

Theorem D (Theorem 6.3.4)**.** *We have the following commutative diagram:*



Moreover, since the embedding $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_{\mathbf{Q}_p})$ *in* $\mathrm{Rep}_{\mathcal{K}_\blacksquare}(G)$ *is fully faithful, we have* $R\Gamma(G,V)=R\Gamma(G,V^{Rla})$ *for* $V\in\mathrm{Rep}_{\mathcal{K}_\blacksquare}(G)$*. In particular, if* G *is a* p*-adic Lie group over* $\mathbf{Q}_p$*, we have*

$$R\Gamma(G,V)=R\Gamma(G,V^{Rla})=R\Gamma^{la}(G,V^{Rla})=R\Gamma^{sm}(G,R\Gamma(\mathfrak{g},V^{Rla})).$$

Remark 1.0.5.

(1) When G is compact and V is a finite dimensional representation, the last two equivalences are a classical result of Lazard [Laz65]. When G is given by the $\mathbf{Q}_p$-points of an algebraic group and V is finite dimensional, Casselman-Wigner generalized Lazard's result in [CW74]. For G compact and any solid V, this result was obtained by the authors in [RJRC21].

(2) When G is a p-adic reductive group over $\mathbb{Q}_p$ and V is an admissible Banach representation of G, then $V^{Rla}=V^{la}$ and the isomorphism $R\Gamma(G,V)=R\Gamma(G,V^{la})$ was recently and independently shown by Fust in [Fus23] by reducing the problem to the compact case [RJRC21, Theorem 5.3] via a Bruhat-Tits building argument.

We conclude this introduction with an application of Theorem B to the p-adic Langlands correspondence for GL_1. We heartily thank Eugen Hellman for pointing out this application to us. We let $\mathcal{X}_1$ be the classifying stack of rank 1 (φ,Γ)-modules over the Robba ring on affinoid Tate algebras over $\mathcal{K}=(K,K^+)$, cf. [EGH23, §5]. Since every such (φ,Γ)-module is given, up to a twist by a line bundle on the base, by a continuous (and hence locally analytic) character on $\mathbf{Q}_p^\times=\mathbf{Z}_p^\times\times p^{\mathbb{Z}}$, this stack is represented (cf. [EGH23, §7.1]) by the quotient

$$[(\widetilde{\mathcal{W}}\times\mathbb{G}_m^{an})/\mathbb{G}_m^{an}]$$

with trivial action of $\mathbb{G}_m^{an}$, where $\widetilde{\mathcal{W}}$ is the rigid analytic weight space of $\mathcal{O}_L^\times$ whose points on an affinoid ring A are given by continuous characters $\mathrm{Hom}(\mathcal{O}_L^\times,A)$, and where $\mathbb{G}_m^{an}$ denotes the rigid analytic multiplicative group. Let $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{X}_1)$ be the category of solid quasi-coherent sheaves on $\mathcal{X}_1$. In [EGH23], the authors conjecture that the natural functor

$$\mathfrak{LL}_p^{la}:\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(L^\times_{\mathbb{Q}_p})\to\mathrm{Mod}^{qc}_\blacksquare(\mathcal{X}_1)$$

given by $\mathfrak{LL}_p^{la}(V)=\mathcal{O}_{\mathcal{X}_1}\otimes^L_{\mathcal{D}^{la}(L^\times_{\mathbb{Q}_p},K)}V$ is fully faithful when restricted to a suitable category of "tempered" (or finite slope) locally analytic representations (cf. [EGH23, Equation (7.1.3)]). Here $L^\times_{\mathbb{Q}_p}$ is the restriction of scalars to $\mathbb{Q}_p$ of the p-adic Lie group $L^\times$. On the other hand, for the functor $\mathfrak{LL}_p^{la}$ to be fully faithful without restricting to a smaller subcategory of $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(L^\times_{\mathbb{Q}_p})$, one can also modify the stack $\mathcal{X}_1$, namely, we consider

$$\mathcal{X}_1^{mod}:=[\widetilde{\mathcal{W}}\times\mathbb{G}_m^{alg}/\mathbb{G}_m^{alg}]$$

where $\mathbb{G}_m^{alg}$ is the analytic space, in the sense of [CS20], attached to the ring $(K[T^{\pm1}],K^+)_\blacksquare=\mathcal{K}_\blacksquare\otimes_{\mathbb{Z}}\mathbb{Z}[T^{\pm1}]$.

In order to describe the category of solid quasi-coherent sheaves on the stacks $\mathcal{X}_1^{mod}$ and $\mathcal{X}_1$ in terms of representation theory, we need to introduce some notation. We let $\mathscr{O}(\mathbb{G}_m^{an})=\varprojlim_{n\to\infty}K\langle p^nT,\frac{p^n}{T}\rangle$ and $\ell^{temp}_{\mathbb{Z},K}=\mathscr{O}(\mathbb{G}_m^{an})^\vee$ be the Hopf algebras of functions of the group $\mathbb{G}_m^{an}$ and its dual. We let $\mathbb{Z}^{temp}$ denote the analytic space defined by the algebra $\ell^{temp}_{\mathbb{Z},K}$. We also let $C^{temp}(L^\times_{\mathbb{Q}_p},K)=\mathscr{O}(\widetilde{\mathcal{W}}\times\mathbb{G}_m^{an})^\vee$ be the Hopf algebra of tempered locally analytic functions on $L^{\times}$. Finally, we let $\mathrm{Rep}^{temp}_{\mathcal{K}_{\blacksquare}}(L^{\times}_{\mathbb{Q}_p}) := \mathrm{Mod}^{qc}_{\blacksquare}([*/L^{\times,temp}_{\mathbb{Q}_p}])$ be the category of tempered (locally analytic) representations of $L^{\times}_{\mathbb{Q}_p}$.

Theorem E (Theorem 4.4.4)**.** *There are natural equivalences of stable ∞-categories*

$$\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathbb{Z}/L^{\times,la}_{\mathbb{Q}_p}]) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{X}_1^{mod}), \quad \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathbb{Z}^{temp}/L^{\times,temp}_{\mathbb{Q}_p}]) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{X}_1)$$

Furthermore, the functor $\mathfrak{LL}^{la}_p$ induces equivalences

$$\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(L^{\times}_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\widetilde{\mathcal{W}} \times \mathbb{G}_m^{alg}), \quad \mathrm{Rep}^{temp}_{\mathcal{K}_{\blacksquare}}(L^{\times}_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\widetilde{\mathcal{W}} \times \mathbb{G}_m^{an}).$$

Acknowledgements. We heartily thank Eugen Hellman for pointing out the application of Theorem B to the categorical p-adic Langlands correspondence, and Arthur–César Le Bras for many inspiring conversations that concluded in the stacky interpretation of locally analytic representations. We thank Lucas Mann for several discussions on six functor formalisms and their connection with representation theory. Finally, we thank Johannes Anschütz, Ko Aoki, Yutaro Mikami, Cédric Pepin, Vincent Pilloni, Peter Scholze and Matthias Strauch for their comments and corrections. The first author was supported by the project ANR-19-CE40-0015 COLOSS. The second author thanks the Max Planck Institute for Mathematics for their hospitality during the preparation of this paper.

Notations and auxiliary results. Throughout this paper we use the language of ∞-categories of [Lur09], and the techniques of higher algebra from [Lur17]. We use Clausen and Scholze condensed approach to analytic geometry as presented in the lecture notes [CS19, CS20, CS22]. We refer the work of Mann [Man22b] for complete and rigorous proofs of foundational results on the subject, particularly those regarding the set theoretical subtleties in condensed mathematics. Nevertheless, throughout this paper we will fix an uncountable solid cutoff cardinal κ as in [Man22b, Definition 2.9.11] and work with κ-small condensed sets, it will be clear from the definitions that the functors and adjunctions constructed below are independent of κ, and therefore that they extend naturally to the full condensed categories.

For $\mathcal{C}$ an ∞-category with all small limits and colimits, we let $\mathrm{Cond}(\mathcal{C})$ denote the ∞-category of condensed $\mathcal{C}$-objects, see [Man22b, Definition 2.1.1]. Given $X \in \mathrm{Cond}(\mathcal{C})$ and S a profinite set, we let $\underline{\mathrm{Cont}}(S, X)$ or $C(S, X)$ be the object in $\mathrm{Cond}(\mathcal{C})$ whose values at $S' \in \mathrm{Extdis}$ are $X(S \times S')$. This is still a condensed object by [Man22b, Corollary 2.1.10] under a mild condition on $\mathcal{C}$ (eg. if it is presentable). In particular, we shall write CondSet, CondAb and CondRing for the categories of condensed sets, abelian groups and commutative rings, respectively.

All the analytic rings considered in this document are assume to be animated and complete in the sense of [Man22b, Definition 2.3.10], unless otherwise specified. Given $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M})$ a commutative animated analytic ring we shall write $\mathrm{Mod}_{\mathcal{A}}$ for the symmetric monoidal ∞-category of analytic $\mathcal{A}$-modules and $\mathrm{Mod}^{\heartsuit}_{\mathcal{A}}$ for the heart of its natural t-structure. Given D an $\mathbb{E}_1$-algebra in $\mathrm{Mod}_{\mathcal{A}}$, we let $\mathrm{LMod}_{\mathcal{A}}(D)$ and $\mathrm{RMod}_{\mathcal{A}}(D)$ be the ∞-category of left and right D-modules in $\mathrm{Mod}_{\mathcal{A}}$, if it is clear from the context we will simply write $\mathrm{Mod}_{\mathcal{A}}(D) = \mathrm{LMod}_{\mathcal{A}}(D)$. We say that an analytic ring $\mathcal{A}$ is static if for all extremally disconnected set S, the object $\underline{\mathcal{A}}[S]$ is concentrated in cohomological degree 0. We let $- \otimes^L_{\mathcal{A}} -$ denoted the complete tensor product of $\mathrm{Mod}_{\mathcal{A}}$, and $R\underline{\mathrm{Hom}}_{\mathcal{A}}(-,-)$ the internal Hom space, right adjoint to the tensor. By Warning 7.6 of [CS19], the tensor $- \otimes^L -_{\mathcal{R}}$ is the left derived functor of the tensor $- \otimes_{\mathcal{R}} -$ if $\mathcal{A}[S \times T]$ sits in degree 0 for all extremally disconnected sets. The analytic rings we will consider live over the solid base $\mathbb{Z}_{\blacksquare}$, so this property is always true for them.

Recall that a map $f : N \to M$ of objects in $\mathrm{Mod}_{\mathcal{A}}$ is called trace class ([CS22, Definition 8.1]) if there is a map $\mathcal{A} \to N^{\vee} \otimes M$ with $N^{\vee} = R\underline{\mathrm{Hom}}_{\mathcal{A}}(N, \mathcal{A})$, such that f factors as

$$N \to N \otimes^L_{\mathcal{A}} N^{\vee} \otimes_{\mathcal{A}} M \to M.$$

An object $N \in \mathrm{Mod}_{\mathcal{A}}$ is called nuclear ([CS20, Definition 13.10]) if for all extremally disconnected set S, the natural map

$$\mathcal{A}[S]^{\vee} \otimes^L M(*) \to M(S)$$

is an isomorphism. By [CS20, Proposition 13.14], if $N \in \mathrm{Mod}_{\mathcal{A}}$ is nuclear, then for all S extremally disconnected set and any $M \in \mathrm{Mod}_{\mathcal{A}}$, the natural map

$$(R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], M) \otimes^L_{\mathcal{A}} N)(*) \to (M \otimes^L_{\mathcal{A}} N)(S)$$

is an isomorphism.

We will let $\mathcal{K}=(K,K^+)$ denote a complete non-archimedean extension of $\mathbb{Q}_p$, and let $\mathcal{K}_{\blacksquare}=(K,K^+)_{\blacksquare}$ be the analytic ring attached to the Huber pair as in [And21, §3.3]. Given an algebra D in $\mathrm{Mod}(\mathcal{K}_{\blacksquare})$, we endow D with the induced analytic ring structure from $\mathcal{K}_{\blacksquare}$, and let $-\otimes_D^L-$ (or sometimes $-\otimes^L_{D,\blacksquare}-$) denote the relative tensor product of D-modules in $\mathcal{K}_{\blacksquare}$-vector spaces.

Finally, we address the following proposition that will be used in different parts of the paper.

Proposition 1.0.6. *Let $\mathcal{R}$ be a static commutative analytic ring such that $-\otimes^L_{\mathcal{R}}-$ is the left derived functor of $-\otimes_{\mathcal{R}}-$. Let $\mathcal{A}$ be a static $\mathcal{R}$-Hopf algebra over $\mathcal{R}$ with the induced analytic structure. Suppose that $\mathcal{A}$ is co-commutative and that its antipode is an anti-involution, i.e. $s^2=\mathrm{id}$. Suppose that the self tensor products of analytic rings $\mathcal{A}^{\otimes_{\mathcal{R}}n}$ are static for all $n\in\mathbb{N}$. Then the following assertions hold:*

1. *The tensor product $-\otimes^L_{\mathcal{R}}-$ defines a symmetric monoidal structure on $\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})$ obtained by restriction of scalars along the co-multiplication $\Delta:\mathcal{A}\to\mathcal{A}\otimes_{\mathcal{R}}\mathcal{A}$.*
2. *($\otimes$-$R\underline{\mathrm{Hom}}$ adjunction) The derived internal $\underline{\mathrm{Hom}}$ over $\mathcal{R}$ induces a natural functor*
$$R\underline{\mathrm{Hom}}_{\mathcal{R}}(-,-)_{\star_{1,3}}:\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})\times\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})\to\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})$$
given by precomposing the natural $\mathcal{A}^{op}\otimes_{\mathcal{R}}\mathcal{A}$-module structure with the map $\mathcal{A}\xrightarrow{\Delta}\mathcal{A}\otimes_{\mathcal{R}}\mathcal{A}\xrightarrow{s\otimes 1}\mathcal{A}^{op}\otimes_{\mathcal{R}}\mathcal{A}$, where $s:\mathcal{A}\xrightarrow{\sim}\mathcal{A}^{op}$ is the antipode. Furthermore, $R\underline{\mathrm{Hom}}_{\mathcal{R}}(-,-)_{\star_{1,3}}$ is a right adjoint of the internal tensor product $-\otimes^L_{\mathcal{R}}-$.
3. *(Twisting/untwisting) There are natural equivalences of functors*
$$\Psi:\mathcal{A}\otimes^L_{\mathcal{R}}-\xrightarrow{\sim}\mathcal{A}\otimes^L_{\mathcal{R}}(-)_0$$
$$\Phi:R\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{A},-)_{\star_{1,3}}\xrightarrow{\sim}R\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{A},-)_{\star_1}:=R\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{A},(-)_0),$$
where $(-)_0$ is the trivial $\mathcal{A}$-module structure obtained by restricting scalars along the composition $\mathcal{A}\xrightarrow{\nu}\mathcal{R}\xrightarrow{\mu}\mathcal{A}$.
4. *Let $\iota:\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})\xrightarrow{\sim}\mathrm{RMod}_{\mathcal{R}}(\mathcal{A})$ be the precomposition with the antipode of $\mathcal{A}$. We have natural equivalences of functors*
$$\iota(N)\otimes^L_{\mathcal{A}}M=\underline{\mathcal{R}}\otimes^L_{\mathcal{A}}(N\otimes^L_{\mathcal{R}}M)$$
$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(N,M)=R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{R},R\underline{\mathrm{Hom}}_{\mathcal{R}}(N,M)_{\star_{1,3}})$$
for any $N,M\in\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})$, where $\underline{\mathcal{R}}$ is endowed with the trivial $\mathcal{A}$-structure given by the counit.
5. *Let $\mathcal{B}$ be a static $\mathcal{R}$-Hopf algebra satisfying the same hypothesis as $\mathcal{A}$ and let $\mathcal{A}\to\mathcal{B}$ be a morphism of $\mathcal{R}$-Hopf algebras. Then $\mathcal{B}$ is an idempotent $\mathcal{A}$-algebra if and only if $\mathcal{B}\otimes^L_{\mathcal{A}}\underline{\mathcal{R}}=\underline{\mathcal{R}}$ where $\underline{\mathcal{R}}$ is seen as an $\mathcal{A}$ or $\mathcal{B}$-module via the co-unit.*

Proof. (1) First, let $\mathcal{C}=\mathrm{Mod}_{\mathcal{R}}$ be the symmetric monoidal ∞-category of $\mathcal{R}$-modules, and let $\mathcal{C}^{op}$ be its opposite category. Then, $\mathcal{A}$ defines a commutative Hopf algebra in the symmetric monoidal category $\mathcal{C}^{op}$. Therefore, the category $\mathrm{CoMod}_{\mathcal{A}}(\mathcal{C}^{op})=\varprojlim_{[n]\in\Delta}\mathcal{A}^{\otimes_{\mathcal{R}}n}\text{-}\mathrm{Mod}(\mathcal{C}^{op})$ of (left) comodules of $\mathcal{A}$ over $\mathcal{C}^{op}$ is symmetric monoidal, with symmetric monoidal structure given by $-\otimes^L_{\mathcal{R}}-$ on underlying objects. Part (1) follows since $\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})=(\mathrm{CoMod}_{\mathcal{A}}(\mathcal{C}^{op}))^{op}$, and since the opposite of a symmetric monoidal category is symmetric monoidal.

(2) Given $N,M\in\mathrm{LMod}_{\mathcal{R}}(\mathcal{A})$, we see $R\underline{\mathrm{Hom}}_{\mathcal{R}}(M,N)$ as an $\mathcal{A}$-module via the forgetful functor through the algebra homomorphism $A\xrightarrow{\Delta}A\otimes_{\mathcal{R}}A\xrightarrow{s\otimes 1}A^{op}\otimes_{\mathcal{R}}A$. To prove the $\otimes$-$R\underline{\mathrm{Hom}}$ adjunction, since both functors arise as derived functors of suitable abelian categories with enough projectives and injectives (after fixing the cardinal κ), it suffices to know the non-derived $\otimes$-$\underline{\mathrm{Hom}}$ adjunction of the underlying abelian categories, which is [Sch92, Example 1.2.2 (3)].

(3) Let $\mathcal{C}=\mathrm{Mod}_{\mathcal{R}}$. We have an equivalence of symmetric monoidal categories $\mathrm{Mod}_{\mathcal{R}}(\mathcal{A})=\mathrm{CoMod}_{\mathcal{A}}(\mathcal{C}^{op})^{op}$. Let $f^*:\mathrm{CoMod}_{\mathcal{A}}(\mathcal{C}^{op})\to\mathcal{C}^{op}$ be the forgetful functor taking the underlying object in $\mathcal{C}^{op}$, and let f_* be its right adjoint. In the opposite category $f^{*,op}$ is the forgetful from $\mathcal{A}$ to $\mathcal{R}$-modules, and f_* is its left adjoint $f_*^{op}=\mathcal{A}\otimes^L_{\mathcal{R}}-$. The functor f^* is symmetric monoidal, we then have a natural transformation
$$f_*\mathcal{R}\otimes M\to f_*f^*M$$

for $M \in \mathrm{CoMod}(\mathcal{C}^{op})$. In the opposite category this translates to a natural transformation

$$\mathcal{A} \otimes_{\mathcal{R}} M_0 \to \mathcal{A} \otimes^L_{\mathcal{R}} M.$$

We claim that it is an isomorphism. By writing M as filtered colimits of projective generators, and since $\mathcal{A}[S] = \mathcal{A} \otimes_{\mathcal{R}} \mathcal{R}[S]$, one is reduced to the case when $M = \mathcal{A}_0$. Following the construction, the map of $\mathcal{A}$-modules $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}_0 \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}$ is adjoint to the map

$$\mathcal{A}_0 \xrightarrow{1\otimes \mathrm{id}} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}.$$

An inverse of this map can be given explicitly by the composite

$$\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \xrightarrow{\Delta\otimes\mathrm{id}} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \xrightarrow{\mathrm{id}\otimes s \otimes \mathrm{id}} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \xrightarrow{\mathrm{id}\otimes\nabla} \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}_0,$$

where $m : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \to \mathcal{A}$ is the multiplication map. Finally, the untwisting map Φ for the internal Hom follows from adjunction and the untwisting map Ψ.

(4) The natural transformation for the tensor product is a consequence of the following natural equivalences for $N, M, Y \in \mathrm{Mod}_{\mathcal{R}}(\mathcal{A})$.

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{R} \otimes^L_{\mathcal{A}} (N \otimes_{\mathcal{R}} M), Y) &= R\underline{\mathrm{Hom}}_{\mathcal{A}}(N \otimes_{\mathcal{R}} M, Y) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{A}}(M, R\underline{\mathrm{Hom}}_{\mathcal{R}}(N, Y)_{\star_{1,3}}) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{A}}(M, R\underline{\mathrm{Hom}}_{\mathcal{R}}(\iota(N), Y)) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{R}}(\iota(N) \otimes^L_{\mathcal{A}} M, Y). \end{aligned}$$

The natural equivalence for the internal Hom's follows by the adjunction of point (2).

(5) Suppose that $\mathcal{B}$ is an idempotent $\mathcal{A}$-algebra. Then we have that

$$\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{R} = \mathcal{B} \otimes_{\mathcal{A}} (\mathcal{B} \otimes^L_{\mathcal{B}} \mathcal{R}) = (\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{B}) \otimes^L_{\mathcal{B}} \mathcal{R} = \mathcal{B} \otimes^L_{\mathcal{B}} \mathcal{R} = \mathcal{R}.$$

Conversely, suppose that $\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{R} = \mathcal{R}$, then by (the version for right modules of) part (4) we have

$$\begin{aligned} \mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{B} &= (\mathcal{B} \otimes_{\mathcal{R}} \iota(\mathcal{B})) \otimes^L_{\mathcal{A}} \mathcal{R} \\ &= (\mathcal{B}_0 \otimes_{\mathcal{R}} \iota(\mathcal{B})) \otimes^L_{\mathcal{A}} \mathcal{R} \\ &= \mathcal{B}_0 \otimes_{\mathcal{R}} (\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{R}) \\ &= \mathcal{B}, \end{aligned}$$

in the third equality we used the antipode $s : \mathcal{B}^{op} \xrightarrow{\sim} \mathcal{B}$ to identify the right and left actions of $\mathcal{A}$ on $\mathcal{B}$. An explicit diagram chasing shows that the resulting map $\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ is the multiplication map, proving that $\mathcal{B}$ is an idempotent $\mathcal{A}$-algebra.

□

2. Distribution algebras

We record in this chapter basic properties of the several spaces of functions and algebras of distributions we will be working throughout the text. Most of the results are probably well known but we give statements and proofs for the sake of notation and completeness. Let L be a finite extension of $\mathbb{Q}_p$ and $\varpi \in L$ a pseudo-uniformizer. Let G be a p-adic Lie group over L. We normalize the p-adic absolute value of L such that $|p| = p^{-1}$.

2.1. **Locally analytic distribution algebras.** Let G be a compact p-adic Lie group of dimension d over L. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathcal{L} \subset \mathfrak{g}$ be an $\mathcal{O}_L$-lattice such that $[\mathcal{L}, \mathcal{L}] \subset p\mathcal{L}$. Let $L_{\blacksquare}[G]$ be the Iwasawa algebra of G, i.e., $L_{\blacksquare}[G] = (\varprojlim_{H\subset G} \mathcal{O}_L[G/H])[\frac{1}{p}]$ where H runs over all the compact open subgroups. As it is explained in [Eme17, §5.2], the Lie algebra $\mathcal{L}$ can be integrated to an analytic group $\mathbb{G}_{\mathcal{L}}$ over L whose underlying adic space can be identified with a polydisc of dimension d. More precisely, let $\mathfrak{X}_1, \dots, \mathfrak{X}_d$ be an $\mathcal{O}_L$-basis of $\mathcal{L}$, then the map

$$(T_1, \dots, T_d) \mapsto \exp(T_1\mathfrak{X}_1) \dots \exp(T_d\mathfrak{X}_d)$$

induces an isomorphism of adic spaces between the polydisc $\mathbb{D}^d_L = \mathrm{Spa}(L\langle \underline{T} \rangle, \mathcal{O}_L\langle \underline{T} \rangle)$ and $\mathbb{G}_{\mathcal{L}}$. After shrinking $\mathcal{L}$ if necessary we can assume that $\mathbb{G}_{\mathcal{L}}(L) \subset G$ is a normal compact open subgroup which is moreover a uniform pro-p-group. In the following, we will always assume that $\mathcal{L}$ is small enough such that this holds.

The previous construction can be slightly generalized as follows. Let $\overline{L}$ be an algebraic closure of L, and let $\mathcal{L} \subset \mathfrak{g}_{\overline{L}}$ be a free $\mathcal{O}_{\overline{L}}$-lattice such that $[\mathcal{L}, \mathcal{L}] \subset p\mathcal{L}$. There exists a finite extension F of L such that $\mathcal{L}$ is defined over F, one can define an affinoid group $\mathbb{G}_{\mathcal{L},F}$ over F by integrating $\mathcal{L}$. Furthermore, suppose that the action of Gal_L leaves $\mathcal{L}$ stable, then $\mathbb{G}_{\mathcal{L},F}$ can be obtained as the base change from L of an affinoid group that we denote as $\mathbb{G}_{\mathcal{L}}$. A locally free lattice $\mathcal{L} \subset \mathfrak{g}_{\overline{L}}$ is said *good* if it is Gal_L-stable and $[\mathcal{L}, \mathcal{L}] \subset p\mathcal{L}$, if $\mathcal{L}$ is defined over F we let $\mathcal{L}_F$ denote the Gal_F-invariants of $\mathcal{L}$.

Example 2.1.1. Let us fix a good $\mathcal{O}_L$-lattice $\mathcal{L}_0 \subset \mathfrak{g}$ with group $\mathbb{G}_0$. For $h > 0$ a rational, the lattice $p^h\mathcal{L}_0$ over $\mathfrak{g}_{\overline{L}}$ is good, and it defines an affinoid subgroup $\mathbb{G}_h \subset \mathbb{G}_0$ which is nothing but the polydisc of radius p^{-h}:

$$\mathbb{G}_h = \mathbb{G}_0\left(\frac{T}{p^h}\right).$$

Given a good lattice $\mathcal{L}$ we can also define analytic groups which are Stein spaces, namely, we let $\mathring{\mathbb{G}}_{\mathcal{L}} = \bigcup_{h>0} \mathbb{G}_{p^h\mathcal{L}}$. If $\mathcal{L}$ is already defined over L then $\mathring{\mathbb{G}}_{\mathcal{L}}$ is an open polydisc.

Finally, we can construct affinoid and Stein group neighbourhoods of G by taking finitely many translates of the groups $\mathbb{G}_{\mathcal{L}}$ and $\mathring{\mathbb{G}}_{\mathcal{L}}$. Indeed, since $\mathbb{G}_{\mathcal{L}}(L)$ and $\mathring{\mathbb{G}}_{\mathcal{L}}(L)$ are normal subgroups of G, we can define

$$\mathbb{G}^{(\mathcal{L})} := G\mathbb{G}_{\mathcal{L}} = \bigsqcup_{g \in G/\mathbb{G}_{\mathcal{L}}(L)} g\mathbb{G}_{\mathcal{L}} \text{ and } \mathbb{G}^{(\mathcal{L}^+)} := G\mathring{\mathbb{G}}_{\mathcal{L}} = \bigsqcup_{g \in G/\mathring{\mathbb{G}}_{\mathcal{L}}(L)} g\mathring{\mathbb{G}}_{\mathcal{L}}.$$

If $\mathcal{L}_0$ is a fixed good lattice and $\mathcal{L} = p^h\mathcal{L}_0$ we will simply denote $\mathbb{G}^{(h)} = \mathbb{G}^{(\mathcal{L})}$ and $\mathbb{G}^{(h^+)} = \mathbb{G}^{(\mathcal{L}^+)}$.

With the previous notations we can now define the following distribution algebras and analytic functions.

Definition 2.1.2. Let $\mathcal{L} \subset \mathfrak{g}_{\overline{L}}$ be a good lattice defined over F/L.

(1) Let $\mathbb{G}$ be one of the adic groups $\mathbb{G}_{\mathcal{L}}$, $\mathring{\mathbb{G}}_{\mathcal{L}}$, $\mathbb{G}^{(\mathcal{L})}$ or $\mathbb{G}^{(\mathcal{L}^+)}$. The space of analytic functions of $\mathbb{G}$ with values in L is the space $C(\mathbb{G}, L) = \mathscr{O}(\mathbb{G})$. The algebra of distributions of $\mathbb{G}$ is the dual space $\mathcal{D}(\mathbb{G}, L) = \underline{\mathrm{Hom}}_L(C(\mathbb{G}, L), L)$. If $\mathcal{L}_0$ is fixed as in Example 2.1.1 and $\mathcal{L} = p^h\mathcal{L}_0$, we will simply denote $\mathcal{D}^h(G, L) = \mathcal{D}(\mathbb{G}^{(h^+)}, L)$ and $C^h(G, L) = C(\mathbb{G}^{(h^+)}, L)$.

(2) We let $\widehat{U}(\mathcal{L})^+$ be the $\mathrm{Gal}_{F/L}$-invariants of the p-adic completion of the enveloping algebra of $\mathcal{L}_F$. We also denote $\widehat{U}(\mathcal{L}) = \widehat{U}(\mathcal{L})^+[\frac{1}{p}]$.

(3) Finally, we let $C^{la}(\mathfrak{g}, L) := \varinjlim_{\mathcal{L} \subset \mathfrak{g}} C(\mathbb{G}_{\mathcal{L}}, L)$ and $C^{la}(G, K) = \varinjlim_{\mathcal{L} \subset \mathfrak{g}} C(\mathbb{G}^{(\mathcal{L})}, L)$ be the spaces of locally analytic functions of $\mathfrak{g}$ and G respectively. We let $\mathcal{D}^{la}(\mathfrak{g}, L) = \underline{\mathrm{Hom}}_K(C^{la}(\mathfrak{g}, L), L)$ and $\mathcal{D}^{la}(G, L) = \underline{\mathrm{Hom}}_L(C^{la}(G, L), L)$ be the spaces of locally analytic distributions of $\mathfrak{g}$ and G respectively.

Remark 2.1.3.

(1) We note that, for $\mathbb{G} = \mathbb{G}_{\mathcal{L}}$ or $\mathbb{G}^{(\mathcal{L})}$ (resp. for $\mathbb{G} = \mathring{\mathbb{G}}_{\mathcal{L}}$ or $\mathbb{G}^{(\mathcal{L}^+)}$), the space $C(\mathbb{G}, L)$ is a Banach space (resp. a nuclear Fréchet space), and the distribution algebra $\mathcal{D}(\mathbb{G}, L)$ is a Smith space (resp. an LB-space of compact type), cf. [RJRC21], [ST03] or [Sch02].

(2) The algebra $C^h(G, L)$ is by definition the space of functions of G that are analytic with radius $p^{-h'}$ for any $h' > h$ with respect to the coordinates of $\mathbb{G}_{\mathcal{L}}$. The reason for considering analytic functions on open balls instead of affinoid balls comes from the fact that the algebras $\mathcal{D}^h(G, L)$ are idempotent over $\mathcal{D}^{la}(G, L)$, cf. Corollary 2.1.6 below.

(3) The colimit diagrams $\{C(\mathbb{G}_{\mathcal{L}}, L)\}_{\mathcal{L}}$ and $\{C(\mathring{\mathbb{G}}_{\mathcal{L}}, L)\}_{\mathcal{L}}$ (resp. $\{C(\mathbb{G}^{(\mathcal{L})}, L)\}_{\mathcal{L}}$, $\{C(\mathbb{G}^{(\mathcal{L}^+)}, L)\}_{\mathcal{L}}$ and $\{C^h(G, L)\}_h$) are isomorphic and their colimit is the space of locally analytic functions of $\mathfrak{g}$ (resp. of G). Dually, the limit diagrams of distribution algebras $\{\mathcal{D}(\mathbb{G}_{\mathcal{L}}, L)\}_{\mathcal{L}}$, $\{\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)\}_{\mathcal{L}}$ and $\{\widehat{U}(\mathcal{L})\}_{\mathcal{L}}$ (resp. the limit diagrams $\{\mathcal{D}(\mathbb{G}^{(\mathcal{L})}, L)\}_{\mathcal{L}}$, $\{\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L)\}_{\mathcal{L}}$ and $\{\mathcal{D}^h(G, L)\}_h$) are isomorphic and their limit is equal to $\mathcal{D}^{la}(\mathfrak{g}, L)$ (resp. $\mathcal{D}^{la}(G, L)$). In particular, for $h' > h \geq 0$ we have the inclusions

$$\mathcal{D}(\mathring{\mathbb{G}}_{p^{h'}\mathcal{L}_0}, L) \subset \mathcal{D}(\mathbb{G}_{p^{h'}\mathcal{L}_0}, L) \subset \mathcal{D}(\mathring{\mathbb{G}}_{p^h\mathcal{L}_0}, L),$$

$$\mathcal{D}^{h'}(G, L) \subset \mathcal{D}(\mathbb{G}^{(p^{h'}\mathcal{L}_0)}, L) \subset \mathcal{D}^h(G, L).$$

On the other hand, we have that

$$\mathcal{D}(\mathring{\mathbb{G}}_{p^h\mathcal{L}_0}, L) = \varinjlim_{h'\to (h-\frac{1}{p-1})^+} \widehat{U}(p^{-h'}\mathcal{L}) \tag{2.1}$$

for $h > \frac{1}{p-1}$, see [Eme17, Proposition 5.2.6] and [RJRC21, Corollary 4.18].

(4) Let $\mathcal{L} \subset \mathfrak{g}_{\overline{L}}$ be a good lattice defined over F and let $\mathfrak{X}_1, \ldots, \mathfrak{X}_d$ be a base of $\mathcal{L}_F$ over $\mathcal{O}_F$. One has a power-series description

$$\widehat{U}(\mathcal{L}) \otimes_L F = \widehat{\bigoplus}_{\alpha\in\mathbb{N}^d} F\underline{\mathfrak{X}}^{\alpha}.$$

2.1.1. *Koszul complexes and idempotency.* Let $\mathcal{L} \subset \mathfrak{g}_{\overline{L}}$ be a good lattice, our next goal is to prove that the distribution algebras of Definition 2.1.2 are idempotent algebras over the enveloping algebra $U(\mathfrak{g})$ or the locally analytic distribution algebra $\mathcal{D}^{la}(G, L)$. Since all the algebras involved are co-commutative Hopf algebras, the idea is to show that the co-unit is preserved by base change and to apply Proposition 1.0.6 (5). Without loss of generality let us assume that $\mathcal{L}$ is defined over L.

Proposition 2.1.4. *Let $\mathcal{L} \subset \mathfrak{g}$ be a good lattice, and let $\mathrm{Kos}(\mathfrak{g}, U(\mathfrak{g}))$ be the standard Koszul resolution of L:*

$$0 \to U(\mathfrak{g}) \otimes \bigwedge^d \mathfrak{g} \to \cdots \to U(\mathfrak{g}) \otimes \mathfrak{g} \to U(\mathfrak{g}) \to L \to 0,$$

where the differentials are given by

$$d(v\otimes Z_1\wedge\ldots\wedge Z_k) = \sum_{i=1}^{k}(-1)^{i+1} vZ_i\otimes Z_1\wedge\ldots\wedge\widehat{Z}_i\wedge\ldots\wedge Z_k + \sum_{i<j}(-1)^{i+j} v\otimes[Z_i, Z_j]\wedge\cdots\wedge\widehat{Z}_i\wedge\cdots\wedge\widehat{Z}_j\cdots\wedge Z_k.$$

Let $\mathcal{D}$ denote $\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)$, $\widehat{U}(\mathcal{L})$ or $\mathcal{D}^{la}(\mathfrak{g}, L)$. Then

$$\mathrm{Kos}(\mathfrak{g}, \mathcal{D}) := \mathcal{D} \otimes_{U(\mathfrak{g}),\blacksquare} \mathrm{Kos}(\mathfrak{g}, U(\mathfrak{g}))$$

is a resolution of L as $\mathcal{D}$-module. In particular, $\mathcal{D} \otimes^L_{U(\mathfrak{g}),\blacksquare} L = L$.

Proof. Let $\mathrm{Kos}(\mathcal{L}, U(\mathcal{L})^+)$ be the standard resolution of the trivial representation $\mathcal{O}_L$ and $\varepsilon : \mathrm{Kos}(\mathcal{L}, U(\mathcal{L})^+) \to \mathcal{O}_L$ the augmentation map. There is an $\mathcal{O}_L$-linear homotopy $h_\bullet : U(\mathcal{L})^+ \otimes \bigwedge^\bullet \mathcal{L} \to U(\mathcal{L})^+ \otimes \bigwedge^{\bullet+1} \mathcal{L}$ such that $d_{\bullet+1}h_\bullet + h_{\bullet-1}d_\bullet = \mathrm{id} - \varepsilon$ ([Wei94, Theorem 7.7.2]). Taking a p-adic completion and inverting p, one obtains an homotopy $\widehat{h}_\bullet$ between id and ε for $\mathrm{Kos}(\mathfrak{g}, \widehat{U}(\mathcal{L})^+)$. Inverting p we have an equivalence $\mathrm{Kos}(\mathfrak{g}, \widehat{U}(\mathcal{L})) \xrightarrow{\varepsilon} L$. Taking colimits of the Koszul resolutions for $p^{h'}\mathcal{L}$ as $h' \to (h - \frac{1}{p-1})^+$, one gets an equivalence $\mathrm{Kos}(\mathfrak{g}, \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)) \xrightarrow{\varepsilon} L$. Taking limits of $p^h\mathcal{L}$ as $h \to \infty$, by topological Mittag-Leffler [RJRC21, Lemma 3.27] one gets an equivalence $\mathrm{Kos}(\mathfrak{g}, \mathcal{D}^{la}(\mathfrak{g}, L)) \xrightarrow{\varepsilon} L$. □

Proposition 2.1.5. *Let G be a compact p-adic Lie group over L. There is a Koszul resolution of the trivial $\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L)$-module L*

$$0 \to \mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes \bigwedge^{\dim \mathfrak{g}} \mathfrak{g} \to \cdots \to \mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes \mathfrak{g} \to \mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \to L \to 0$$

obtained as the dual of the de Rham complex of $\mathbb{G}^{(\mathcal{L}^+)}$. Furthermore, the limit along all the lattices $\mathcal{L}$ defines a Koszul resolution of L as $\mathcal{D}^{la}(G, L)$-module. In particular, $\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes^L_{\mathcal{D}^{la}(G,L)} L = L$.

Proof. This is roughly [RJRC21, Proposition 5.12] which is based on the Poincaré Lemma for open polydiscs [Tam15, Lemma 26]. Let $\mathbb{G}^{(\mathcal{L}^+)}$ be the Stein group defined by $\mathcal{L}$, since $\mathcal{L}$ is defined over L, $\mathbb{G}^{(\mathcal{L}^+)}$ is a finite disjoint union of open polydiscs. We can then consider the de Rham complex $DR(\mathbb{G}^{(\mathcal{L}^+)})$ of $\mathbb{G}^{(\mathcal{L}^+)}$. Taking a basis of the tangent space by right invariant vector fields, the de Rham complex is written explicitly as a left G-equivariant complex

$$[C(\mathbb{G}^{(\mathcal{L}^+)}, L) \to \cdots \to C(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes \bigwedge^i \mathfrak{g}^\vee \to \cdots \to C(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes \bigwedge^{\dim \mathfrak{g}} \mathfrak{g}^\vee]$$

and differentials induced by right derivations. By the Poincaré Lemma, the natural map $L \to DR(\mathbb{G}^{(\mathcal{L}^+)})$ is an equivalence. Taking duals, one gets a resolution of $\mathcal{D}(\mathcal{L}^+(G), L)$-modules

$$\mathrm{Kos}(\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}), L) := DR(\mathbb{G}^{(\mathcal{L}^+)})^\vee \to L$$

which is noting but finitely many translates of the Koszul resolution of Proposition 2.1.4. Taking limits along all lattices $\mathcal{L}$, one gets an equivalence of $\mathcal{D}^{la}(G, L)$-modules

$$\mathrm{Kos}(\mathcal{D}^{la}(G, L)) \to L.$$

One clearly has

$$\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes^L_{\mathcal{D}^{la}(G,L)} L = \mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes^L_{\mathcal{D}^{la}(G,L)} \mathrm{Kos}(\mathcal{D}^{la}(G, L)) = \mathrm{Kos}(\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}), L) = L.$$

□

Corollary 2.1.6. (1) *Let $\mathcal{L} \subset \mathfrak{g}$ be a good lattice and let $\mathcal{D}$ denote $\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)$, $\widehat{U}(\mathcal{L})$ or $\mathcal{D}^{la}(\mathfrak{g}, L)$. Then $\mathcal{D} \otimes^L_{U(\mathfrak{g}),\blacksquare} \mathcal{D} = \mathcal{D}$. In particular, the ∞-category $\mathrm{Mod}_{L_\blacksquare}(\mathcal{D})$ of solid $\mathcal{D}$-modules is a full subcategory of $\mathrm{Mod}_{L_\blacksquare}(U(\mathfrak{g}))$.*

(2) *We have $\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) \otimes^L_{\mathcal{D}^{la}(G,L),\blacksquare} \mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) = \mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L)$. In particular $\mathrm{Mod}_{L_\blacksquare}(\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L))$ is a full subcategory of $\mathrm{Mod}_{L_\blacksquare}(\mathcal{D}^{la}(G, L))$.*

Proof. This follows from the Koszul resolutions of Propositions 2.1.4, 2.1.5, and the idempotent algebra criterion for co-commutative Hopf algebras of Proposition 1.0.6 (5). □

Next, we want to relate the distribution algebras associated to an immersion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$.

Proposition 2.1.7. *Let $\mathfrak{g}$ be a Lie algebra over L and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Let $\mathcal{L} \subset \mathfrak{g}$ be a good lattice and let $\mathcal{L}_{\mathfrak{h}} = \mathcal{L} \cap \mathfrak{h}$. Let $\mathcal{D}(\mathcal{L})$ denote $\widehat{U}(\mathcal{L})$, $\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)$ or $\mathcal{D}^{la}(\mathfrak{g}, L)$ (resp. for $\mathcal{L}_{\mathfrak{h}}$), and let $\mathcal{D}(\mathcal{L}/\mathcal{L}_{\mathfrak{h}}) := \mathcal{D}(\mathcal{L}) \otimes_{\mathcal{D}(\mathcal{L}_{\mathfrak{h}})} L$.*

(1) *Let $\mathcal{T} \subset \mathcal{L}$ be a free complement of $\mathcal{L}_{\mathfrak{h}}$ in $\mathcal{L}$ with basis $\mathfrak{Y}_1, \ldots, \mathfrak{Y}_s$ and $\mathfrak{t} = \mathcal{T}[\frac{1}{p}]$. Let $\mathbb{G}_T \subset \mathbb{G}_{\mathcal{L}}$ be the image by the exponential of the ordered basis $\mathfrak{Y}_1, \ldots, \mathfrak{Y}_s$, and let $\mathring{\mathbb{G}}_{\mathcal{T}} = \bigcup_{h>0} \mathbb{G}_{p^h\mathcal{T}}$ be the open polydisc. Then we have isomorphisms of solid L-vector spaces*

$$\mathcal{D}(\mathcal{L}/\mathcal{L}_{\mathfrak{h}}) \cong \mathcal{D}(\mathcal{T})$$

where

$$\mathcal{D}(\mathcal{T}) = \begin{cases} \widehat{U}(\mathcal{T}) := \widehat{\bigoplus}_{\alpha \in \mathbb{N}^s} L\underline{\mathfrak{Y}}^\alpha, \\ \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{T}}, L) := \underline{\mathrm{Hom}}_L(\mathscr{O}_{\mathring{\mathbb{G}}_{\mathcal{T}}}, L) \\ \mathcal{D}^{la}(\mathfrak{t}, L) := \varprojlim_{h\to\infty} \widehat{U}(p^h\mathcal{T}). \end{cases}$$

(2) *We have an isomorphism of right $\mathcal{D}(\mathcal{L}_{\mathfrak{h}})$-modules*

$$\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{T}) \otimes^L_{L_\blacksquare} \mathcal{D}(\mathcal{L}_{\mathfrak{h}}).$$

Furthermore, we have an equivalence of left $\mathcal{D}(\mathcal{L})$-modules $\mathrm{Kos}(\mathfrak{h}, \mathcal{D}(\mathcal{L})) \xrightarrow{\varepsilon} \mathcal{D}(\mathcal{L}/\mathcal{L}_{\mathfrak{h}})$ where $\mathrm{Kos}(\mathfrak{h}, \mathcal{D}(\mathcal{L}))$ is the Koszul complex

$$\mathrm{Kos}(\mathfrak{h}, \mathcal{D}(\mathcal{L})) = [0 \to \mathcal{D}(\mathcal{L}) \otimes_L \bigwedge^{\dim \mathfrak{h}} \mathfrak{h} \to \cdots \mathcal{D}(\mathcal{L}) \otimes_L \mathfrak{h} \to \mathcal{D}(\mathcal{L})].$$

In particular, $\mathcal{D}(\mathcal{L}/\mathcal{L}_h) = \mathcal{D}(\mathcal{L}) \otimes^L_{\mathcal{D}(\mathcal{L}_{\mathfrak{h}}),\blacksquare} L$, and taking $\mathfrak{h} = \mathfrak{g}$, one recovers Proposition 2.1.4.

Proof. The proof of the proposition follows the same lines as those of Propositions 2.1.4 and 2.1.5. Since $\mathcal{L} = \mathcal{L}_{\mathfrak{h}} \oplus \mathcal{T}$, we can write $\mathring{\mathbb{G}}_{\mathcal{L}} = \mathring{\mathbb{G}}_{\mathcal{T}} \times \mathring{\mathbb{G}}_{\mathcal{L}_{\mathfrak{h}}}$. Taking global sections one finds that $\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L) = \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{T}}, L) \otimes^L_{L_\blacksquare} \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}_{\mathfrak{h}}}, L)$. We can then take the relative de Rham complex of the map

$$\mathring{\mathbb{G}}_{\mathcal{L}} \to \mathring{\mathbb{G}}_{\mathcal{L}}/\mathring{\mathbb{G}}_{\mathcal{L}_{\mathfrak{h}}} \cong \mathring{\mathbb{G}}_{\mathcal{T}},$$

and by taking duals we find the Koszul complex $\mathrm{Kos}(\mathfrak{h}, \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}), L)$ which is quasi-isomorphic to $\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)$ by the Poincaré Lemma. The case for $\mathcal{D}^{la}(\mathfrak{g}, L)$ and $\mathcal{D}^{la}(\mathfrak{h}, L)$ is obtained by taking limits along all lattices $\mathcal{L}$ in the previous construction.

Finally, for $\widehat{U}(\mathcal{L})$ and $\widehat{U}(\mathcal{L}_{\mathfrak{h}})$, consider the Koszul resolution of $U(\mathcal{L})$ as $\mathcal{L}_{\mathfrak{h}}$-module. Since $U(\mathcal{L}) = U(\mathcal{L}_{\mathfrak{h}}) \otimes_{\mathcal{O}_L} U(\mathcal{T})$ where $U(\mathcal{T}) = \bigoplus_{\alpha} \underline{\mathfrak{Y}}^{\alpha}$, the same argument of [Wei94, Theorem 7.7.2] provides an homotopies between id and the augmentation map

$$\mathrm{Kos}(\mathcal{L}_{\mathfrak{h}}, U(\mathcal{L}) \xrightarrow{\varepsilon} U(\mathcal{T}).$$

Taking p-adic completions and inverting p one gets the Koszul complex for the $\widehat{U}$-algebras, and the equality $\widehat{U}(\mathcal{L}) = \widehat{U}(\mathcal{T}) \otimes^L_{L_{\blacksquare}} \widehat{U}(\mathcal{L}_{\mathfrak{h}})$. □

The following particular case will be of special interest: Let $\tilde{\mathfrak{g}}$ be the Lie algebra $\mathfrak{g}$ seen as a Lie algebra over $\mathbb{Q}_p$, similarly we let $\widetilde{G}$ be the restriction of G to $\mathbb{Q}_p$. Take $\mathfrak{k} = \ker(\widetilde{\mathfrak{g}} \otimes_{\mathbb{Q}_p} L \to \mathfrak{g})$. Let $\mathcal{L} \subset \mathfrak{g}_{\overline{L}}$ be a good lattice and let $\widetilde{\mathcal{L}}$ be its restriction to $\mathbb{Q}_p$, i.e. the lattice obtained by its $\mathrm{Gal}_{\mathbb{Q}_p} / \mathrm{Gal}_L$-translates in

$$\widetilde{\mathfrak{g}}_{\overline{L}} = \mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{L} = \prod_{\sigma : L \to \overline{L}} \mathfrak{g}_{\sigma, \overline{L}}.$$

Corollary 2.1.8. *Let $\mathcal{D}$ denote one of the algebras $\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L)$, $\widehat{U}(\mathcal{L})$, $\mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)$, $\mathcal{D}^{la}(\mathfrak{g}, L)$ or $\mathcal{D}^{la}(G, L)$. Let $\widetilde{\mathcal{D}}$ be the analogue algebra associated to $\widetilde{G}$ and $\widetilde{\mathcal{L}}$. Then there is a natural equivalence of right $\mathcal{D}^{la}(\mathfrak{k}, L)$-modules*

$$\widetilde{\mathcal{D}} \otimes_{\mathbb{Q}_p} L = \mathcal{D} \otimes^L_{L_{\blacksquare}} \mathcal{D}^{la}(\mathfrak{k}, L).$$

In particular, $(\widetilde{\mathcal{D}} \otimes^L_{\mathbb{Q}_p} L) \otimes_{\mathcal{D}^{la}(\mathfrak{k}, L)} L = \mathcal{D}$.

Proof. This follows from Proposition 2.1.7 once we note that $\mathcal{D}(\mathbb{G}^{(\mathcal{L}^+)}, L) = \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L) \otimes_L L[G/\mathring{\mathbb{G}}_{\mathcal{L}}(L)] = L[G/\mathring{\mathbb{G}}_{\mathcal{L}}(L)] \otimes_L \mathcal{D}(\mathring{\mathbb{G}}_{\mathcal{L}}, L)$. The case of $\mathcal{D}^{la}(G, L)$ follows by taking limits of the $\mathbb{G}^{(\mathcal{L}^+)}$-cases. □

2.1.2. *Locally analytic functions and distributions.* We now define locally analytic functions on G taking values in a solid vector space V. Recall from [RJRC21] that we have defined analytic rings $C(\mathbb{G}^{(h)}, L)_{\blacksquare} = (C(\mathbb{G}^{(h)}, L), C(\mathbb{G}^{(h)}, \mathcal{O}_L))_{\blacksquare}$ in order to define h-analytic and locally analytic vectors of a solid representation. The following Lemma says basically that, in the limit, the analytic structure becomes trivial.

Lemma 2.1.9. *Let $h' > h$, we have natural maps of analytic rings $(\mathscr{O}(\mathbb{G}^{(h)}), \mathscr{O}^+(\mathbb{G}^{(h)}))_{\blacksquare} \to (\mathscr{O}(\mathbb{G}^{(h')}), \mathcal{O}_L)_{\blacksquare} \to (\mathscr{O}(\mathbb{G}^{(h')}), \mathscr{O}^+(\mathbb{G}^{(h')}))_{\blacksquare}$. In particular for $V \in \mathrm{Mod}(L_{\blacksquare})$ we have maps*

$$C(\mathbb{G}^{(h)}, L)_{\blacksquare} \otimes^L_{L_{\blacksquare}} V \to C(\mathbb{G}^{(h')}, L) \otimes^L_{L_{\blacksquare}} V \to C(\mathbb{G}^{(h')}, L)_{\blacksquare} \otimes^L_{L_{\blacksquare}} V.$$

Proof. By [And21, Lemma 3.31] one has that for an affinoid ring (A, A^+), $(A, \mathcal{O}_L)_{\blacksquare} = (A, A^{min,+})_{\blacksquare}$ where $A^{min,+}$ is the integral closure of $\mathcal{O}_L + A^{00}$. The lemma follows from [And21, Proposition 3.34] and the fact that we have morphisms of Huber pairs $(\mathscr{O}(\mathbb{G}^{(h)}), \mathscr{O}^+(\mathbb{G}^{(h)})) \to (\mathscr{O}(\mathbb{G}^{(h')}), \mathscr{O}(\mathbb{G}^{(h')})^{min,+}) \to (\mathscr{O}(\mathbb{G}^{(h')}), \mathscr{O}^+(\mathbb{G}^{(h')}))$. Indeed, if $\frac{T}{p^h}$ denotes a variable of the group $\mathbb{G}^{(h)}$, one can write $\frac{T}{p^h} = p^{h'-h} \frac{T}{p^{h'}}$, proving that the image of $\frac{T}{p^h}$ in $\mathscr{O}(\mathbb{G}^{(h')})$ is topologically nilpotent. □

Definition 2.1.10. Let $V \in \mathrm{Mod}(L_{\blacksquare})$, we define the following spaces of functions with values in V.

(1) For G compact the space of $\mathbb{G}^{(h)}$-analytic functions

$$C(\mathbb{G}^{(h)}, V) := C(\mathbb{G}^{(h)}, L)_{\blacksquare} \otimes^L_{L_{\blacksquare}} V.$$

(2) For G compact the space of $\mathbb{G}^{(h^+)}$-analytic functions

$$C^h(G, V) = R \varprojlim_{h' > h} C(\mathbb{G}^{(h')}, V) = R \varprojlim_{h' > h} (C(\mathbb{G}^{(h')}, L) \otimes^L_{L_{\blacksquare}} V)$$

where the second equality holds by Lemma 2.1.9.

(3) For G arbitrary the space of locally analytic functions

$$C^{la}(G, V) := \prod_{g \in G/G_0} (C^{la}(gG_0, L) \otimes^L_{L_{\blacksquare}} V)$$

with $G_0 \subset G$ an open compact subgroup.

(4) For G arbitrary we define the algebra of locally analytic distributions of G as

$$\mathcal{D}^{la}(G, L) = \underline{\mathrm{Hom}}_L(C^{la}(G, L), L).$$

Remark 2.1.11. Let G be a compact p-adic Lie group and $V \in \mathrm{Mod}(L_{\blacksquare})$. Then we have that

$$C^{la}(G,V) = \varinjlim_h C(\mathbb{G}^{(h)}, L) \otimes^L_{L_{\blacksquare}} V = \varinjlim_h C(\mathbb{G}^{(h)}, V) = \varinjlim_h C^h(G,V) = \varinjlim_h C^h(G,L) \otimes^L_{L_{\blacksquare}} V,$$

where the first equality is by definition, the others follow by Lemma 2.1.9 and the cofinality of the algebras in Remark 2.1.3 (3).

Lemma 2.1.12. *We have*

$$\mathcal{D}^{la}(G,L) = L_{\blacksquare}[G] \otimes_{L_{\blacksquare}[G_0]} \mathcal{D}^{la}(G_0,L) = \bigoplus_{g \in G/G_0} \mathcal{D}^{la}(gG_0, L)$$

for any G_0 compact open subgroup of G. Moreover, $\mathcal{D}^{la}(G,L)^{\vee} = C^{la}(G,L)$.

Proof. The first claim is trivial in the compact case, and the duality between the space of distributions and locally analytic functions follows by the duality of nuclear Fréchet and LB spaces of compact type, see [RJRC21, Theorem 3.40]. Let us prove the general case.

Provided the first formula is proved, then we immediately have:

$$\mathcal{D}^{la}(G,L)^{\vee} = \prod_{g\in G/G_0} (\mathcal{D}^{la}(gG_0,L))^{\vee} = \prod_{g \in G/G_0} C^{la}(gG_0, L) = C^{la}(G,L),$$

which proves the last statement.

We now prove the first formula of the statement. We write

$$\begin{aligned} C^{la}(G,L) &= \prod_{g\in G/G_0} C^{la}(gG_0,L) \\ &= \prod_{g\in G/G_0} \varinjlim_{h>0} C^h(G_0,L) \\ &= \varinjlim_{\substack{g\in G/G_0 \\ h_g>0}} \prod_{g\in G/G_0} C^{h_g}(G,L). \end{aligned}$$

Therefore,

$$C^{la}(G,L)^{\vee} = \varprojlim_{\substack{g\in G/G_0 \\ h_g>0}} \bigoplus_{g\in G/G_0} \mathcal{D}^{h_g}(G,L). \tag{2.2}$$

We want to prove that the RHS is equal to $\bigoplus_{g\in G/G_0} \mathcal{D}^{la}(gG_0,L)$. Notice that the RHS injects into $\varprojlim_{\substack{g\in G/G_0 \\ h_g>0}} \prod_{g\in G/G_0} \mathcal{D}^{h_g}(G_0,L) = \prod_{g\in G/G_0} \mathcal{D}^{la}(gG_0,L)$. let $(a_g)_{g\in G/G_0}$ be a sequence in the RHS of (2.2), it suffices to prove that all but finitely many elements $a_g \in \mathcal{D}^{la}(gG_0,L)$ vanish. Suppose the opposite, then we can find infinitely many $g \in G/G_0$, and numbers $h_g > 0$, such that the image of a_g in $\mathcal{D}^{h_g}(gG_0,L)$ is non-zero, but this contradicts the fact that (a_g) defines an element in $\bigoplus_{g\in G/G_0} \mathcal{D}^{h_g}(gG_0,L)$. Moreover, the same applies when evaluating at an arbitrary profinite set S. The proposition follows. □

The same proof of the above lemma implies the analogous duality between continuous functions and the Iwasawa algebra.

Lemma 2.1.13. *Let G be an arbitrary p-adic Lie group, then $C(G,L)^{\vee} = L_{\blacksquare}[G]$.*

Proof. This follows by the duality between Fréchet and LS spaces [RJRC21, Theorem 3.40] and the last argument of Lemma 2.1.12 to commute products with sums. Namely,

$$C(G,L)^{\vee} = \left(\prod_{g\in G/G_0} C(gG_0,L) \right)^{\vee} = \bigoplus_{g\in G/G_0} L_{\blacksquare}[gG_0] = L_{\blacksquare}[G].$$

□

2.2. **Smooth functions and distribution algebras.** Let L be a finite extension of $\mathbf{Q}_p$, G a p-adic Lie group over L. For defining smooth functions and distributions on G one only needs G to be a locally profinite group. However, since we will be only interested in p-adic Lie groups we prefer to stay in this situation.

Definition 2.2.1 ([Man22b, Definition 3.4.7])**.** Let S be a profinite set and $V \in \mathrm{Mod}(L_{\blacksquare})$, the space of smooth functions from S to V is the solid $L_{\blacksquare}$-vector space given by

$$C^{sm}(S,V) = \underline{\mathrm{Cont}}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} V.$$

In particular, since $\mathbb{Z}$ is discrete, we have $C^{sm}(S,V) = \varinjlim_i \underline{\mathrm{Cont}}(S_i,V)$ where $S = \varprojlim_i S_i$ is written as a limit of finite subsets.

Lemma 2.2.2 ([Man22b, Lemma 3.4.8])**.** *Let S be a profinite set and $V \in \mathrm{Mod}^{\heartsuit}(L_{\blacksquare})$. The following hold*

(1) *The values of $C^{sm}(S,V)$ at a profinite set T are given by*

$$C^{sm}(S,V)(T) := \mathrm{Cont}(S, V(T)),$$

where $V(T)$ is discrete.

(2) *The natural map $C^{sm}(S,V) \to \underline{\mathrm{Cont}}(S,V)$ is injective.*

Definition 2.2.3 ([Man22b, Definition 3.4.9])**.** Let G be a locally profinite group and $V \in \mathrm{Mod}(L_{\blacksquare})$ a solid L-vector space. We define the space of smooth functions of G with values in V to be the solid L-vector space with values at a profinite T given by

$$C^{sm}(G,V)(T) = \mathrm{Cont}(G, V(T)).$$

Equivalently, if $H \subset G$ is an open compact subgroup, we have that

$$C^{sm}(G,V) = \prod_{g \in G/H} C^{sm}(gH, V).$$

Definition 2.2.4. The algebra of smooth L-valued distributions of G is defined as

$$\mathcal{D}^{sm}(G,L) := \mathcal{D}^{la}(G,L)/(\mathfrak{g}).$$

Proposition 2.2.5. *The following assertions hold.*

(1) *If G is compact, then*

$$\mathcal{D}^{sm}(G,L) = \varprojlim_{H \subset G} L_{\blacksquare}[G/H] = C^{sm}(G,L)^{\vee}.$$

Furthermore, $C^{sm}(G,L) = \mathcal{D}^{sm}(G,L)^{\vee}$.

(2) *For arbitrary G and any open compact subgroup $G_0 \subset G$, we have*

$$\mathcal{D}^{sm}(G,L) = L_{\blacksquare}[G] \otimes_{L_{\blacksquare}[G_0]} \mathcal{D}^{sm}(G_0,L) = \mathcal{D}^{sm}(G_0,L) \otimes_{L_{\blacksquare}[G_0]} L_{\blacksquare}[G].$$

Furthermore, we have $\mathcal{D}^{sm}(G,L) = C^{sm}(G,L)^{\vee}$ and $C^{sm}(G,L) = \mathcal{D}^{sm}(G,L)^{\vee}$.

(3) *For G arbitrary, there is an isomorphism of left $\mathcal{D}^{la}(\mathfrak{g},L)$-modules*

$$\mathcal{D}^{la}(G,L) = \mathcal{D}^{la}(\mathfrak{g},L) \otimes_{L_{\blacksquare}} \mathcal{D}^{sm}(G,L),$$

resp. for right $\mathcal{D}^{la}(\mathfrak{g},L)$-modules.

(4) *For G arbitrary, there is an isomorphism*

$$\mathcal{D}^{sm}(G,L) = L \otimes^{L}_{\mathcal{D}^{la}(\mathfrak{g},L),\blacksquare} \mathcal{D}^{la}(G,L).$$

Proof. We first prove (1), (2) and (3) at the same time. For any open compact subgroup H of G we have an $\mathcal{D}^{la}(H,L)$-equivariant isomorphism

$$\mathcal{D}^{la}(G,L) = \mathcal{D}^{la}(H,L) \otimes_{L_{\blacksquare}} L_{\blacksquare}[G/H].$$

Taking limits for $H \subset G$ we get a $\mathcal{D}^{la}(\mathfrak{g},L)$-equivariant isomorphism

$$\mathcal{D}^{la}(G,L) = \mathcal{D}^{la}(\mathfrak{g},L) \otimes_{L_{\blacksquare}} (\varprojlim_{H \subset G} L_{\blacksquare}[G/H])$$

so that $\mathcal{D}^{sm}(G,L) = L\otimes^L_{\mathcal{D}^{la}(\mathfrak{g},L),\blacksquare}\mathcal{D}^{la}(G,L) = \varprojlim_{H\subset G} L_{\blacksquare}[G/H]$. The duality between $\mathcal{D}^{sm}(G,L)$ and $C^{sm}(G,L)$ follows from the previous computation and the duality between Fréchet and LS spaces of [RJRC21, Theorem 3.40].

Finally we prove (4), it follows the same lines of the proof of Lemma 2.1.12. By *loc. cit.* we have $\mathcal{D}^{la}(G,L) = L_{\blacksquare}[G]\otimes_{L_{\blacksquare}[G_0]}\mathcal{D}^{la}(G_0,L) = \mathcal{D}^{la}(G_0,L)\otimes_{L_{\blacksquare}[G_0]} L_{\blacksquare}[G]$ for any compact open subgroup G_0. The tensor product formula of (4) follows by (3). Finally, we prove the duality between $\mathcal{D}^{sm}(G,L)$ and $C^{sm}(G,L)$ in the non-compact case. We can write

$$\mathcal{D}^{sm}(G,L) = \bigoplus_{g\in G/G_0}\mathcal{D}^{sm}(gG_0,L) \text{ and } C^{sm}(G,L) = \prod_{g\in G/G_0} C^{sm}(gG_0,L).$$

Then

$$\mathcal{D}^{sm}(G,L)^{\vee} = \prod_{g\in G/G_0}(\mathcal{D}^{sm}(gG_0,L))^{\vee} = \prod_{g\in G/G_0} C^{sm}(gG_0,L) = C^{sm}(G,L).$$

For the other duality, we write

$$\begin{aligned} C^{sm}(G,L) &= \prod_{g\in G/G_0} C^{sm}(gG_0,L) \\ &= \prod_{g\in G/G_0}\varinjlim_{H\subset G_0} C^{sm}(gG_0/H,L) \\ &= \varinjlim_{\substack{g\in G/G_0\\ H_g\subset G_0}}\prod_{g\in G/G_0} C^{sm}(gG_0/H_g,L). \end{aligned}$$

Therefore,

$$C^{sm}(G,L)^{\vee} = \varprojlim_{\substack{g\in G/G_0\\ H_g\subset G_0}}\bigoplus_{g\in G/G_0} L_{\blacksquare}[gG_0/H_g]. \tag{2.3}$$

We want to prove that the RHS is equal to $\bigoplus_{g\in G/G_0}\mathcal{D}^{sm}(gG_0,L)$. Notice that the RHS injects into $\varprojlim_{\substack{g\in G/G_0\\ H_g\subset G_0}}\prod_{g\in G/G_0} L_{\blacksquare}[gG_0/H_g] = \prod_{g\in G/G_0} D^{sm}(gG_0,L)$. let $(a_g)_{g\in G/G_0}$ be a sequence in the RHS of (2.3). It suffices to prove that all but finitely many elements $a_g\in\mathcal{D}^{sm}(gG_0,L)$ vanish. Suppose the opposite, then we can find infinitely many $g\in G/G_0$, and open subgroups H_g, such that the image of a_{g_i} in $L_{\blacksquare}[gG_0/H_{g_i}]$ is non-zero, but this contradicts the fact that (a_g) defines an element in $\bigoplus_{g\in G/G_0} L_{\blacksquare}[G/H_g]$. Moreover, the previous holds when evaluating at any profinite set S. The proposition follows. ☐

Corollary 2.2.6. *Let G be a compact p-adic Lie group. Then $\mathcal{D}^{sm}(G,L) = \prod_{\rho}\rho\otimes\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,L)}(\rho,\mathcal{D}^{sm}(G,L))$ where ρ runs over all the irreducible finite dimensional smooth representations of G. In particular:*

1. *The functor $\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,L)}(\rho,-)$ is an exact functor in the abelian category of $\mathcal{D}^{sm}(G,L)$-modules.*
2. *$\mathcal{D}^{sm}(G,L)$ is self-injective (algebraically).*
3. *$L_{\blacksquare}[G/H]$ is a idempotent $\mathcal{D}^{sm}(G,L)$-algebra for all $H\subset G$ normal open subgroup.*

Proof. Any group algebra of a finite group G_0 over a field of characteristic zero is isomorphic to the product of $\rho\otimes\underline{\mathrm{Hom}}_{L_{\blacksquare}[G_0]}(\rho, L_{\blacksquare}[G_0])$ where ρ runs over all irreducible representations of G_0. Since $\mathcal{D}^{sm}(G,L) = \varprojlim_H L_{\blacksquare}[G/H]$ if G is compact, the first part of the corollary follows. The second statement is clear since ρ is a direct summand of $\mathcal{D}^{sm}(G,L)$, so a projective module. The second assertion follows since any direct product of division algebras is self-injective, cf. [Lam99, Corollary 1.33B]. For the last claim, notice that $L_{\blacksquare}[G/H]$ is a direct summand of $\mathcal{D}^{sm}(G,L)$, namely, the projection $\mathcal{D}^{sm}(G,L)\to L_{\blacksquare}[G/H]$ has a section given by the Haar measure of H. Writing $\mathcal{D}^{sm}(G,L) = L_{\blacksquare}[G/H]\oplus M$ as $\mathcal{D}^{sm}(G,L)$-modules, tensoring with $L_{\blacksquare}[G/H]$ gives

$$L_{\blacksquare}[G/H] = L_{\blacksquare}[G/H]\otimes^L_{\mathcal{D}^{sm}(G,L),\blacksquare} L_{\blacksquare}[G/H]\oplus M\otimes^L_{\mathcal{D}^{sm}(G,L),\blacksquare} L_{\blacksquare}[G/H],$$

but the image of M in $L_{\blacksquare}[G/H]$ is zero, this implies that $L_{\blacksquare}[G/H] = L_{\blacksquare}[G/H]\otimes^L_{\mathcal{D}^{sm}(G,L),\blacksquare} L_{\blacksquare}[G/H]$ proving the corollary. ☐

Lemma 2.2.7. *Let G be compact. Then*

$$\mathcal{D}^h(G,L)\otimes^L_{\mathcal{D}^{la}(G,L),\blacksquare}\mathcal{D}^{sm}(G,L)=L_\blacksquare[G/G_{h^+}],$$

where $G_{h^+}:=\mathbb{G}_{p^h\mathcal{L}_0^+}(L)$.

Proof. By (3) of 2.2.5, we have $\mathcal{D}^h(G,L)\otimes_{\mathcal{D}^{la}(G,L),\blacksquare}\mathcal{D}^{sm}(G,L)=\mathcal{D}^h(G,L)\otimes^L_{\mathcal{D}^{la}(\mathfrak{g},L),\blacksquare}L$. On the other hand, we have $\mathcal{D}^h(G,L)=L_\blacksquare[G/G_{h^+}]\otimes_{L_\blacksquare}\mathcal{D}(\mathring{\mathbb{G}}_{p^h\mathcal{L}_0},L)$. Hence we reduce to showing that $\mathcal{D}(\mathring{\mathbb{G}}_{p^h\mathcal{L}_0},L)\otimes^L_{\mathcal{D}^{la}(\mathfrak{g},L),\blacksquare}L=L$, which follows from Corollary 2.1.6. □

Corollary 2.2.8. *Let G be an arbitrary p-adic Lie group over L and let $G_{\mathbb{Q}_p}$ be G considered as a p-adic Lie group over $\mathbb{Q}_p$. Let $\mathfrak{k}=\ker(\mathfrak{g}_{\mathbb{Q}_p}\otimes L\to\mathfrak{g})$. Then*

$$\mathcal{D}^{la}(G,L)=L\otimes^L_{U(\mathfrak{k})}\mathcal{D}^{la}(G_{\mathbb{Q}_p},L)=\mathcal{D}^{la}(\mathfrak{g},L)\otimes^L_{\mathcal{D}^{la}(\mathfrak{g}_{\mathbb{Q}_p},L)}\mathcal{D}^{la}(G_{\mathbb{Q}_p},L).$$

Proof. We have

$$L\otimes^L_{U(\mathfrak{k})}\mathcal{D}^{la}(G_{\mathbb{Q}_p},L)=L\otimes^L_{U(\mathfrak{k})}\mathcal{D}^{la}(\mathfrak{g}_{\mathbf{Q}_p},L)\otimes^L_{L_\blacksquare}\mathcal{D}^{sm}(G,L)=\mathcal{D}^{la}(\mathfrak{g},L)\otimes^L_{L_\blacksquare}\mathcal{D}^{sm}(G,L)=\mathcal{D}^{la}(G,L),$$

where in the first and last equality we used Proposition 2.2.5 (3) and the middle equality follows from Corollary 2.1.8. This shows the first assertion. The last identity is proven in a similar way:

$$L\otimes^L_{U(\mathfrak{k})}\mathcal{D}^{la}(G_{\mathbb{Q}_p},L)=L\otimes^L_{U(\mathfrak{k})}\mathcal{D}^{la}(\mathfrak{g}_{\mathbf{Q}_p},L)\otimes^L_{\mathcal{D}^{la}(\mathfrak{g}_{\mathbf{Q}_p},L)}\mathcal{D}^{la}(G_{\mathbf{Q}_p},L)=\mathcal{D}^{la}(\mathfrak{g},L)\otimes^L_{\mathcal{D}^{la}(\mathfrak{g}_{\mathbf{Q}_p},L)}\mathcal{D}^{la}(G_{\mathbf{Q}_p},L),$$

where the second equality follows also from Corollary 2.1.8. □

3. Solid locally analytic representations

In [RJRC21] the authors introduced the concept of a solid locally analytic representation for compact p-adic Lie groups over $\mathbb{Q}_p$. The goal of this first section is to extend the main results of *loc. cit.* to the case where G is a locally profinite p-adic Lie group defined over a finite extension of $\mathbb{Q}_p$.

Let L be a finite extension of $\mathbb{Q}_p$ and $\varpi\in L$ a pseudo-uniformizer. Let (K,K^+) be a complete non-archimedean field extension of L. Let G be a p-adic Lie group over L. In §3.1, motivated from the main theorems of [RJRC21], we define the derived L-analytic vectors of a solid representation of G. We will show that they can be recovered as the $\mathbb{Q}_p$-locally analytic vectors which are killed by some "Cauchy-Riemann equations". In §3.2 we define the ∞-category of locally analytic representations of G, which will be a full subcategory of the category of solid $\mathcal{D}^{la}(G,K)$-modules, where $\mathcal{D}^{la}(G,K)$ is the locally analytic distribution algebra of G. If in addition G is defined over $\mathbb{Q}_p$, the ∞-category of locally analytic representations is itself a full subcategory of the solid G-representations. Finally, in §3.3. we give sufficient conditions for a solid representation to be locally analytic.

3.1. Locally analytic vectors.

Let G be a p-adic Lie group over a finite extension L of $\mathbf{Q}_p$ and let $\mathcal{K}=(K,K^+)$ be a complete non-archimedean extension of L. We denote $\mathcal{K}_\blacksquare$ the analytic ring associated to $\mathcal{K}$. In the following we review the definition of locally L-analytic vectors of solid G-modules on $\mathcal{K}_\blacksquare$-vector spaces. We shall fix a good lattice $\mathcal{L}_0\subset\mathfrak{g}$ defined over L, and for $h>0$ we let $\mathbb{G}^{(h)}$ and $\mathbb{G}^{(h^+)}$ denote the analytic groups $\mathbb{G}^{(p^h\mathcal{L}_0)}$ and $\mathbb{G}^{(p^h\mathcal{L}_0^+)}$ containing G (resp. we let $\mathbb{G}_h$ and $\mathbb{G}_{h^+}$ denote $\mathbb{G}_{p^h\mathcal{L}_0}$ and $\mathring{\mathbb{G}}_{p^h\mathcal{L}_0}$).

Remark 3.1.1. When G is compact and $L=\mathbb{Q}_p$, the notation of [RJRC21] and the one presented in this paper agree for the spaces of functions, i.e. $C(\mathbb{G}^{(h)},K)$ and $C(\mathbb{G}^{(h^+)},K)$. Notice however that the distribution algebras $\mathcal{D}(\mathbb{G}^{(h)},K)$ and $\mathcal{D}(\mathbb{G}^{(h^+)},K)$ are written, respectively, as $\mathcal{D}^{(h)}(G,K)$ and $\mathcal{D}^{(h^+)}(G,K)$ in *loc. cit.*. In the current paper we are writing $\mathcal{D}^h(G,K)=\mathcal{D}(\mathbb{G}^{(h^+)},K)$ and $C^h(G,K)=C(\mathbb{G}^{(h^+)},K)$ instead since these are the spaces that we use more often, we apologise for the discrepancy in the notations.

Lemma 3.1.2.

(1) *Let G be a compact group, then the functors $V\mapsto C(\mathbb{G}^{(h)},V)$ and $V\mapsto C^h(G,V)$ for $V\in\mathrm{Mod}(\mathcal{K}_\blacksquare)$ are naturally promoted to exact functors*

$$\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,L))\to\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G^3,L)).$$

(2) *Let G be arbitrary, then the functor $V \mapsto C^{la}(G,V)$ for $V \in \mathrm{Mod}(\mathcal{K}_{\blacksquare})$ is naturally promoted to an exact functor*

$$\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G^3,K)).$$

Moreover, the functors $V \mapsto C(\mathbb{G}^{(h)},V)$ and $V \mapsto C^{la}(G,V)$ are exact in the abelian categories.

Proof. For the compact case it suffices to prove the lemma for $C(\mathbb{G}^{(h)},-)$, namely the other functors are constructed as limits or colimits of this. But then by [RJRC21, Corollary 2.19] we have

$$C(\mathbb{G}^{(h)},V) = R\underline{\mathrm{Hom}}_L(\mathcal{D}(\mathbb{G}^{(h)},L),V),$$

as $\mathcal{D}(\mathbb{G}^{(h)},L)$ is a $\mathcal{D}^{la}(G,L)$-algebra one has the desired left and right natural actions of $\mathcal{D}^{la}(G\times G,K) = \mathcal{D}^{la}(G,L)\otimes^L_{L_{\blacksquare}}\mathcal{D}^{la}(G,L)\otimes^L_{L_{\blacksquare}}\mathcal{K}_{\blacksquare}$ on $C(\mathbb{G}^{(h)},V)$. On the other hand, $\mathcal{D}(\mathbb{G}^{(h)},L)$ is a Smith space, so projective as $L_{\blacksquare}$-vector space by [RJRC21, Lemma 3.8 (2)], this implies that $V\mapsto C(\mathbb{G}^{(h)},V)$ is exact in the abelian category. If in addition V is a $\mathcal{D}^{la}(G,K)$-module then one has the full action of $\mathcal{D}^{la}(G^3,K)$ as wanted.

In the non-compact case, note that we have natural equivalences

$$C^{la}(G,V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^{la}(G,K),C^{la}(G_0,V))$$

for both the left or right regular action of $\mathcal{D}^{la}(G,K)$ on $C^{la}(G_0,V)$ and any compact open subgroup $G_0\subset G$. This endows $C^{la}(G,V)$ with commuting left and right regular action of $\mathcal{D}^{la}(G,K)$, if in addition V is a $\mathcal{D}^{la}(G,K)$-module then we have the compatible action of

$$\mathcal{D}^{la}(G^3,K) = \mathcal{D}^{la}(G,K)\otimes^L_{\mathcal{K}_{\blacksquare}}\mathcal{D}^{la}(G,K)\otimes^L_{\mathcal{K}_{\blacksquare}}\mathcal{D}^{la}(G,K)$$

as desired. Finally, since $C^{la}(G,V) = \varinjlim_h C(\mathbb{G}^{(h)},V)$, the functor $V\mapsto C^{la}(G,V)$ is exact in the abelian category. □

Remark 3.1.3. The action of G^3 on a function f in any of the three cases is heuristically given by $((g_1,g_2,g_3)\star f)(h) = g_3\cdot f(g_1^{-1}hg_2)$. If V arises as the solid vector space attached to a locally convex vector space then the action of $G\times G\times G$ is given precisely by these formulas.

Given $I\subset\{1,2,3\}$ a non-empty subset and $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G^3,K))$ we let $V_{\star_I}\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ be the restriction of V to the I-diagonal of $\mathcal{D}^{la}(G^3,K)$, i.e. V equipped with he $\mathcal{D}^{la}(G,K)$-module structure induced by the embedding $\iota_I: G\to G^3$, $\iota_I(g)_j = g$ if $j\in I$ and $\iota_I(g)_j = e_G$ if $j\notin I$, where $e_G\in G$ denotes the identity element.

Definition 3.1.4. Let G be a p-adic Lie group over L.

(1) For G compact the functor of (derived) $\mathbb{G}^{(h)}$-analytic vectors $(-)^{Rh-an}:\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))\to\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ is defined as

$$V^{Rh-an} := R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K,(C(\mathbb{G}^{(h)},V)_{\star_{1,3}}),$$

where the action of $\mathcal{D}^{la}(G,K)$ on V^{Rh-an} is induced by the $\star_2$-action (the right regular action). Similarly, the (derived) $\mathbb{G}^{(h^+)}$-analytic vectors is the functor on solid $\mathcal{D}^{la}(G,K)$-modules given by

$$V^{Rh^+-an} := R\varprojlim_{h'>h} V^{Rh'-an} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K,C^h(G,V)_{\star_{1,3}}).$$

If $V\in\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ we let V^{h-an} and V^{h^+-an} denote the H^0 of their derived analytic vectors.

(2) For G compact, we say that an object $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ is h-analytic (resp. h^+-analytic) if the natural arrow $V^{Rh-an}\to V$ (resp. $V^{Rh^+-an}\to V$) is an equivalence. If $V\in\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$, we say that V is non-derived h-analytic if the map $V^{h-an}\to V$ is an equivalence (resp. for h^+).

(3) For G arbitrary we define the functor of locally analytic vectors $(-)^{Rla}:\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))\to\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ as

$$V^{Rla} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K,C^{la}(G,V)_{\star_{1,3}})$$

where we see V^{Rla} endowed with the $\star_2$-action of $\mathcal{D}^{la}(G)$.

(4) For G arbitrary, we say that an object $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ is locally analytic if the natural arrow $V^{Rla} \to V$ is an equivalence. If $V \in \mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ we write $V^{la} := H^0(V^{Rla})$. If $V \in \mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, we say that V is non-derived locally analytic if $V^{la} \to V$ is an isomorphism.

Remark 3.1.5. The distinction between derived and non derived locally analytic representations might look subtle at the beginning, we will see in Proposition 3.2.5 that there is no actual difference.

Remark 3.1.6. The definition of locally analytic vectors might seem slightly strange since we are taking as an input a module over the distribution algebra instead of a solid representation of G as it is usual. Note that, for any $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$ one can define the analytic vectors of V as

$$V^{Rla} := R\underline{\mathrm{Hom}}_{\mathcal{K}_\blacksquare[G]}(K, C^{la}(G,V)_{\star_{1,3}}).$$

If $G = G_{\mathbb{Q}_p}$ is a defined over $\mathbb{Q}_p$, then $\mathcal{D}^{la}(G,K)$ is an idempotent algebra over $\mathcal{K}_\blacksquare[G]$ and the inclusion of $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ into $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$ is fully faithful. Then, for any $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ one has

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{K}_\blacksquare[G]}(K, C^{la}(G,V)_{\star_{1,3}}) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{K}_\blacksquare[G]}(\mathcal{D}^{la}(G,K), C^{la}(G,V)_{\star_{1,3}})) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^{la}(G,K) \otimes^L_{\mathcal{K}_\blacksquare[G]} \mathcal{D}^{la}(G,K), C^{la}(G,V)_{\star_{1,3}})) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, C^{la}(G,V)_{\star_{1,3}}) \end{aligned}$$

proving that both definitions agree. However, if G is defined over $L \neq \mathbb{Q}_p$ and V is a $\mathcal{D}^{la}(G,K)$-module, then the locally analytic vectors of V considered as a solid G-representation are given by $V^{Rla} \otimes R\Gamma(\mathfrak{k}, L)$, where V^{Rla} are the locally analytic vectors as $\mathcal{D}^{la}(G,K)$-module and $R\Gamma(\mathfrak{k}, L)$ is the Lie algebra cohomology of $\mathfrak{k} = \ker(\mathfrak{g} \otimes_{\mathbb{Q}_p} L \to \mathfrak{g})$, see Theorem 6.3.4. This shows that there are different versions of "locally analytic vectors", depending on the category we start with.

Let us prove some basic properties of the functor of locally analytic vectors.

Proposition 3.1.7. *The following assertions hold.*

(1) *Let $G_0 \subset G$ be any open subgroup and $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, there is a natural equivalence $V^{Rla}|_{G_0} = (V|_{G_0})^{Rla}$ between the restriction to G_0 of the G-locally analytic vectors of V and the G_0-locally analytic vectors of $V|_{G_0}$.*

(2) *The functor $(-)^{Rla} : \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ is the right derived functor of $W \mapsto W^{la}$ on the abelian category $\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$.*

(3) *The functor $(-)^{Rla} : \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ preserves small colimits. The same holds for $(-)^{Rh-an}$ and G-compact.*

(4) *If G is compact, then $V^{Rla} = \varinjlim_h V^{Rh-an} = \varinjlim_h V^{Rh^+-an}$.*

Proof.

(1) By construction one has that

$$C^{la}(G,V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^{la}(G,K), C^{la}(G_0,V))$$

where the $\mathcal{D}^{la}(G_0,K)$ acts by left multiplication on $\mathcal{D}^{la}(G,K)$ and by the left regular action on $C^{la}(G_0,V)$. One finds that

$$\begin{aligned} V^{Rla} &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, (C^{la}(G,V))_{\star_{1,3}}) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^{la}(G,K), C^{la}(G_0,V)_{\star_{1,3}})) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(K, C^{la}(G_0,V)_{\star_{1,3}}) \\ &= (V|_{G_0})^{Rla}. \end{aligned}$$

(2) By Lemma 3.1.2 the functor $V \mapsto C^{la}(G,V)$ is exact in the abelian category of solid $\mathcal{D}^{la}(G,K)$-modules. Then, one has that

$$V^{Rla} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, C^{la}(G,V)_{\star 1,3})$$

is a derived $\underline{\mathrm{Hom}}$-functor, which implies that it is the right derived functor of the invariants $V^{la} = C^{la}(G,K)^{G_{\star_{1,3}}}$.

(3) By (1), we can assume that G is compact. By definition of $(-)^{Rla}$ and $(-)^{Rh-an}$, since $V \mapsto C^{la}(G,V) = C^{la}(G,K) \otimes^L_{\mathcal{K}_\blacksquare} V$ and $V \mapsto C(\mathbb{G}^{(h)},V) = C(\mathbb{G}^{(h)},K)_\blacksquare \otimes^L_{\mathcal{K}_\blacksquare} V$ commute with colimits, it suffices to show that K is compact as $\mathcal{D}^{la}(G,K)$-module, this follows from Proposition 2.1.5.
(4) Since taking locally analytic vectors commutes with colimits by (3), and since one can assume G to be compact by (1), this is as consequence of the compacity of K as $\mathcal{D}^{la}(G,K)$-module and Remark 2.1.11.

□

The following proposition relates the functor of analytic vectors with the distribution algebras.

Proposition 3.1.8. *Let G be a compact p-adic Lie group over L, and let $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. Then*

$$V^{Rh-an} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}(\mathbb{G}^{(h)},K),V)$$
$$V^{Rh^+-an} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^h(G,K),V).$$

In particular, an object $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ is h^+-analytic if and only if it is a module over the idempotent $\mathcal{D}^{la}(G,K)$-algebra $\mathcal{D}^h(G,K)$.

Proof. This follows from the same proof of Theorem 4.36 of [RJRC21] using Corollary 2.19 of *loc. cit.* □

Remark 3.1.9. It should be not true that the distribution algebra $\mathcal{D}(\mathbb{G}^{(h)},K)$ is an idempotent $\mathcal{D}^{la}(G,K)$-algebra for general G. For example, if $G = \mathbb{Z}_p$, then $\mathcal{D}(\mathbb{G}^{(h)},\mathbb{Q}_p)$ can be described as the generic fiber of the formal complete PD-envelope of X of a polynomial algebra $\mathbb{Z}_p[X]$, which is not an idempotent $\mathbb{Z}_p[X]$-algebra.

For general groups, we have the following immediate consequence.

Corollary 3.1.10. *Let G be a p-adic Lie group over L and let $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. Then*

$$V^{Rla} = \varinjlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^h(G_0,K),V) = \varinjlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}(\mathbb{G}_0^{(h)},K),V),$$

where G_0 is any open compact subgroup of G.

The following result verifies that taking locally analytic vectors defines an idempotent functor.

Proposition 3.1.11. *Suppose that G is compact. Let $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, then $V^{Rh-an} = (V^{Rla})^{Rh-an}$ and $V^{Rh^+-an} = (V^{Rla})^{Rh^+-an}$. In particular, for any group G, $(V^{Rla})^{Rla} = V^{Rla}$, and the locally analytic vectors of a $\mathcal{D}^{la}(G,K)$-module is a locally analytic representation of G.*

Proof. It suffices to prove that $V^{Rh-an} = (V^{Rla})^{Rh-an}$, the other cases follow from this after taking limits or colimits.

$$\begin{aligned}
(V^{Rla})^{Rh-an} &= \varinjlim_{h_1} (V^{Rh_1{}^+-an})^{Rh-an} \\
&= \varinjlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}(\mathbb{G}^{(h)},K), R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^{h_1}(G,K),V)) \\
&= \varinjlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^{h_1}(G,K) \otimes^L_{\mathcal{D}^{la}(G,K)} \mathcal{D}(\mathbb{G}^{(h)},K),V) \\
&= \varinjlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}(\mathbb{G}^{(h)},K),V) \\
&= V^{Rh-an},
\end{aligned}$$

where the first equality follows from Proposition 3.1.7 (3), the second equality follows from Proposition 3.1.8, the third equality is a $\otimes$-$\underline{\mathrm{Hom}}$ adjunction, the fourth equality follows from the fact that $\mathcal{D}^{h_1}(G,K)$ is an idempotent $\mathcal{D}^{la}(G,K)$-algebra and that $\mathcal{D}(\mathbb{G}^{(h)},K)$ is a $\mathcal{D}^{h_1}(G,K)$-module for all h_1 big enough, and the last equality is Proposition 3.1.8 again. □

The following proposition provides a different way to compute locally analytic vectors as a relative tensor product of $\mathcal{D}^{la}(G,K)$-modules.

Proposition 3.1.12. *Let G be a compact p-adic Lie group. The following assertions hold.*

(1) *Let $V, W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$. Let $V \otimes^L_{\mathcal{K}_\blacksquare} W$ be endowed with the diagonal action. Then there is a natural equivalence*
$$R\underline{\mathrm{Hom}}_{\mathcal{K}_\blacksquare[G]}(K, V \otimes^L_{\mathcal{K}_\blacksquare} W) = (K(\chi_{\mathbb{Q}_p}) \otimes^L_{\mathcal{K}_\blacksquare} \iota(V)) \otimes^L_{\mathcal{K}_\blacksquare[G]} W[-d]$$
where $\iota(V)$ is the right G-module induced by V under the natural involution $\iota : \mathcal{K}_\blacksquare[G] \to \mathcal{K}_\blacksquare[G]$, $\chi_{\mathbb{Q}_p} = \det(\mathfrak{g}_{\mathbb{Q}_p})^{-1}$, and $d = \dim_{\mathbb{Q}_p} G$.

(2) *Let $\mathcal{D}$ denote $\mathcal{D}^{la}(G, K)$ or $\mathcal{D}^{la}(\mathfrak{g}, K)$. Let $V, W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D})$, then there is a natural equivalence*
$$R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, V \otimes^L_{\mathcal{K}_\blacksquare} W) = (K(\chi) \otimes^L_{\mathcal{K}_\blacksquare} \iota(V)) \otimes^L_{\mathcal{D}} W[-d]$$
where $\iota(V)$ is the right $\mathcal{D}$-module obtained by the involution of $\mathcal{D}$, $\chi = \det(\mathfrak{g})^{-1}$ and $d = \dim_L G$.

Proof. Without lose of generality we can take K to be a finite extension of $\mathbb{Q}_p$, the general case is deduced by taking a base change. By Theorem 5.19 of [RJRC21] one has that
$$R\underline{\mathrm{Hom}}_{\mathcal{K}_\blacksquare[G]}(K, V \otimes^L_{\mathcal{K}_\blacksquare} W) = K(\chi_{\mathbb{Q}_p}) \otimes^L_{\mathcal{K}_\blacksquare[G]} (V \otimes^L_{\mathcal{K}_\blacksquare} W)[-d].$$
where we see $K(\chi_{\mathbb{Q}_p})$ as a right representation. By Proposition 1.0.6 (4), we have natural equivalences
$$K(\chi_{\mathbb{Q}_p}) \otimes^L_{\mathcal{K}_\blacksquare[G]} (V \otimes^L_{\mathcal{K}_\blacksquare} W)[-d] = 1 \otimes^L_{\mathcal{K}_\blacksquare[G]} (\iota(K(\chi_{\mathbb{Q}_p})) \otimes^L_{\mathcal{K}_\blacksquare} V \otimes^L_{\mathcal{K}_\blacksquare} W)[-d] = (K(\chi_{\mathbb{Q}_p}) \otimes^L_{\mathcal{K}_\blacksquare} \iota(V)) \otimes^L_{\mathcal{K}_\blacksquare[G]} W[-d],$$
this shows (1). By Propositions 2.1.4 and 2.1.5, the trivial representation is a perfect $\mathcal{D}$-module, in particular dualizable, this implies that the natural functor
$$R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, \mathcal{D}) \otimes^L_{\mathcal{D}} W \to R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, W)$$
for any $W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D})$ is an equivalence. Then, for $V, W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D})$, by Proposition 1.0.6 (4) we have natural equivalences
$$R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, V \otimes^L_{\mathcal{K}} W) = R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, \mathcal{D}) \otimes^L_{\mathcal{D}} (V \otimes^L_{\mathcal{K}_\blacksquare} W) = (R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, \mathcal{D}) \otimes^L_{\mathcal{K}_\blacksquare} \iota(V)) \otimes^L_{\mathcal{D}} W.$$
We are left to compute $R\underline{\mathrm{Hom}}_{\mathcal{D}}(K, \mathcal{D}) = K(\chi)$. For $\mathcal{D} = \mathcal{D}^{la}(\mathfrak{g}, K)$ this follows by an explicit computation using the Koszul resolution of Proposition 2.1.4. For $\mathcal{D} = \mathcal{D}^{la}(G, K)$ one argues as follows: K is a $\mathcal{D}^{sm}(G, K)$-module and $\mathcal{D}^{sm}(G, K) = K \otimes^L_{\mathcal{D}^{la}(\mathfrak{g}, K)} \mathcal{D}^{la}(G, K)$ by Proposition 2.2.5 (3). Then
$$\begin{aligned}
R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, \mathcal{D}^{la}(G, K)) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^{sm}(G, K) \otimes^L_{\mathcal{D}^{sm}(G,K)} K, \mathcal{D}^{la}(G, K)) \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^{sm}(G, K), \mathcal{D}^{la}(G, K))) \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(\mathfrak{g},K)}(K, \mathcal{D}^{la}(G, K))) \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K, K(\chi) \otimes^L_{\mathcal{D}^{la}(\mathfrak{g},K)} \mathcal{D}^{la}(G, K)) \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K, K(\chi) \otimes_{\mathcal{K}_\blacksquare} \mathcal{D}^{sm}(G, K)) \\
&= K(\chi).
\end{aligned}$$
□

Remark 3.1.13. The last calculation in the proof is a special case of our cohomological comparison isomorphisms that will be shown in §6.

Remark 3.1.14. In Proposition 3.1.12 we see χ as a right $\mathcal{D}^{la}(G, K)$-module. It arises as the determinant of the right action of G on $\mathfrak{g}^\vee$ given by
$$(H \cdot g)(v) = H(gvg^{-1})$$
for $H \in \mathfrak{g}^\vee$, $v \in \mathfrak{g}$ and $g \in G$. We will often consider χ as a left representation as well, in this case, it arises as the determinant of the contragradient representation $\mathfrak{g}^\vee$ with action
$$(g \cdot H)(v) = H(g^{-1}vg).$$

Corollary 3.1.15. *Let G be an arbitrary p-adic Lie group over L of dimension d. The following assertions hold.*

(1) *Let $V \in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$, then for any open compact sugroup $G_0 \subset G$ one has*

$$V^{Rla} = (\iota(C^{la}(G_0,K)_{\star_1}) \otimes K(\chi)[-d]) \otimes^L_{\mathcal{D}^{la}(G_0,K)} V.$$

In particular, the functor $(-)^{Rla}$ has cohomological dimension d.

(2) *Suppose that G is defined over $\mathbb{Q}_p$ and let $V \in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{K}_{\blacksquare}[G])$. Then for any open compact sugroup $G_0 \subset G$ one has*

$$V^{Rla} = (\iota(C^{la}(G_0,K)_{\star_1}) \otimes K(\chi)[-d]) \otimes^L_{\mathcal{K}_{\blacksquare}[G_0]} V$$

where the locally analytic vectors are as in Remark 3.1.6.

(3) *Let $V, W \in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$, there is a natural equivalence*

$$(V \otimes^L_{\mathcal{K}_{\blacksquare}} W^{Rla})^{Rla} = V^{Rla} \otimes_{\mathcal{K}_{\blacksquare}} W^{Rla}.$$

Proof. (1) By Proposition 3.1.7 the locally analytic vectors are independent of $G_0 \subset G$ compact open, so we can assume without loss of generality that G is compact. Then, by Proposition 3.1.12 (2) one has

$$\begin{aligned} V^{Rla} &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G)}(K, C^{la}(G,K)_{\star_1} \otimes^L_{\mathcal{K}_{\blacksquare}} V) \\ &= (\iota(C^{la}(G,K)_{\star_1}) \otimes K(\chi)) \otimes^L_{\mathcal{D}^{la}(G,K)} V. \end{aligned}$$

(2) This follows from the same argument of the previous point using Proposition 3.1.12 (1) instead.

(3) We can assume that G is compact. The orbit map $\mathscr{O}_W : W^{Rla} \to C^{la}(G,K) \otimes^L_{\mathcal{K}_{\blacksquare}} W^{Rla}$ induces a natural equivalence

$$C^{la}(G, W^{Rla})_{\star_1} \xrightarrow{\sim} C^{la}(G, W^{Rla})_{\star_{1,3}}, \tag{3.1}$$

at the level of functions this maps sends $f : G \to W$ to the function $\widetilde{f} : G \to W$ given by $\widetilde{f}(g) = g \cdot f(g)$. Then, one computes

$$\begin{aligned} (V \otimes^L_{\mathcal{K}} W^{Rla})^{Rla} &= (\iota(C^{la}(G,K)_{\star_1}) \otimes K(\chi)) \otimes^L_{\mathcal{D}^{la}(G,K)} (V \otimes^L_{\mathcal{K}_{\blacksquare}} W^{Rla}) \\ &= (\iota(C^{la}(G,K)_{\star_1} \otimes^L_{\mathcal{K}_{\blacksquare}} W^{Rla}) \otimes K(\chi)) \otimes^L_{\mathcal{D}^{la}(G,K)} V \\ &= (\iota(C^{la}(G,W^{Rla})_{\star_{1,3}}) \otimes K(\chi)) \otimes^L_{\mathcal{D}^{la}(G,K)} V \\ &= (\iota(C^{la}(G,W^{Rla})_{\star_1}) \otimes K(\chi)) \otimes^L_{\mathcal{D}^{la}(G,K)} V \\ &= \left((\iota(C^{la}(G,K)_{\star_1}) \otimes K(\chi)) \otimes^L_{\mathcal{D}^{la}(G,K)} V \right) \otimes^L_{\mathcal{K}_{\blacksquare}} W^{Rla} \\ &= V^{Rla} \otimes^L_{\mathcal{K}_{\blacksquare}} W^{Rla}. \end{aligned}$$

In the first equality we use part (1). In the second equality we move W to the left part of the tensor using Proposition 1.0.6 (4). The third equality is the definition $C^{la}(G,W) = C^{la}(G,K) \otimes^L_{\mathcal{K}_{\blacksquare}} W$. The fourth equality uses the natural equivalence (3.1). In the fifth equality we take out the tensor with W^{Rla} since $\mathcal{D}^{la}(G,K)$ is acting trivially on it. In the last equality we use part (1) again. □

The previous computation implies that there are representations with higher locally analytic vectors.

Corollary 3.1.16. *Let G be a p-adic Lie group over L of dimension d. Then for any profinite set S we have*

$$(\mathcal{D}^{la}(G,K) \otimes_{\mathcal{K}_{\blacksquare}} \mathcal{K}_{\blacksquare}[S])^{Rla} = (C^{la}_c(G,K) \otimes K(\chi)[-d]) \otimes_{\mathcal{K}_{\blacksquare}} \mathcal{K}_{\blacksquare}[S]$$

where $C^{la}_c(G,K) = \mathcal{D}^{la}(G,K) \otimes_{\mathcal{D}^{la}(G_0,K)} C^{la}(G_0,K)$ is the space of compactly supported locally analytic functions of G. If G is defined over $\mathbb{Q}_p$ we also have

$$(\mathcal{K}_{\blacksquare}[G \times S])^{Rla} = (C^{la}_c(G,K) \otimes K(\chi)[-d]) \otimes_{\mathcal{K}_{\blacksquare}} \mathcal{K}_{\blacksquare}[S].$$

Proof. By Corollary 3.1.15 (1) we have that

$$\begin{aligned}(\mathcal{D}^{la}(G,K)\otimes_{\mathcal{K}_{\blacksquare}}\mathcal{K}_{\blacksquare}[S])^{Rla} &= (\iota(C^{la}(G_0,K))_{\star 1}\otimes K(\chi)[-d])\otimes_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^{la}(G,K)\otimes_{\mathcal{K}_{\blacksquare}}\mathcal{K}_{\blacksquare}[S]) \\ &= \Big((\iota(C^{la}(G_0,K))_{\star 1}\otimes K(\chi)[-d])\otimes_{\mathcal{D}^{la}(G_0,K)}\mathcal{D}^{la}(G,K)\Big)\otimes_{\mathcal{K}_{\blacksquare}}\mathcal{K}_{\blacksquare}[S] \\ &= (C^{la}_c(G,K)\otimes K(\chi)[-d])\otimes_{\mathcal{K}_{\blacksquare}}\mathcal{K}_{\blacksquare}[S].\end{aligned}$$

The second claim follows by the same argument using Corollary 3.1.15 (2) instead. □

3.2. **The category of locally analytic representations.** Let L be a finite extension of $\mathbb{Q}_p$. Our next goal is to define the ∞-category of locally analytic representations and discuss some general properties of it.

Definition 3.2.1. We define the ∞-category of locally analytic representations, denoted as $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$, to be the full subcategory of $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ whose objects are locally analytic representations of G. In other words, $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ is the full subcategory of solid $\mathcal{D}^{la}(G,K)$-modules whose objects are the V such that $V^{Rla}=V$.

Our next task is to show that the derived category of locally analytic representations has a natural t-structure and that it is the derived category of its heart.

Lemma 3.2.2. *Given $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$, one has that*

$$V^{Rla}=\varinjlim_{b\to+\infty}\varprojlim_{a\to-\infty}(\tau^{[a,b]}V)^{Rla},$$

in the homotopy category, where $a,b\in\mathbb{Z}$ with $a\leq b$ and $\tau^{[a,b]}=\tau^{\geq a}\circ\tau^{\leq b}$ is the canonical truncation in the interval $[a,b]$ in cohomological notation.

Proof. This follows from the fact that $(-)^{Rla}$ has finite cohomological dimension, see Corollary 3.1.15 (1). □

We now prove some basic and fundamental properties of the category of solid locally analytic representations.

Proposition 3.2.3. $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ *is stable under all small colimits of* $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ *and tensor products over* $\mathcal{K}_{\blacksquare}$.

Proof. This follows from the fact that taking locally analytic vectors preserves colimits, cf. Proposition 3.1.7, and the projection formula of Corollary 3.1.15 (2). □

Lemma 3.2.4. *Let G be compact. Then* $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ *is the full subcategory of* $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ *stable under all small colimits containing the categories* $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{h}(G,K))$ *for all $h\geq 0$.*

Proof. This follows from Corollary 3.1.10. □

Proposition 3.2.5. *An object $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ is locally analytic if and only if $H^i(V)$ is non-derived locally analytic for all $i\in\mathbb{Z}$. In particular, the t-structure of $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ induces a t-structure on $\mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G)$.*

Proof. We can assume without lose of generality that G is compact. If V is locally analytic then $V^{Rla}=\varinjlim_h V^{Rh^+-an}$ and $H^i(V)=\varinjlim_h H^i(V^{Rh^+-an})$, but V^{Rh^+-an} is a $\mathcal{D}^h(G,K)$-module. This shows that the cohomology groups $H^i(V)$ are colimits of $\mathcal{D}^h(G,K)$-modules for $h\to\infty$ and is locally analytic by Lemma 3.2.4 (so afortriori non-derived locally analytic). Conversely, suppose that $H^i(V)$ is non-derived locally analytic for all $i\in\mathbb{Z}$. We want to show that V is locally analytic. By Lemma 3.2.2 we can assume that V is bounded with support in cohomological degrees $[0,k]$. By an inductive argument one the lenght of the support of V, we can assume that $\tau^{\geq 1}V$ is locally analytic, then V is an extension

$$H^0(V)\to V\to\tau^{\geq 1}V\to H^0(V)[1].$$

Since $H^0(V)$ is non-derived locally analytic, it can be written as a filtered colimit of $\mathcal{D}^h(G,L)$-modules, then it is actually locally analytic by Lemma 3.2.4. This exhibits V as the fiber of $\tau^{\geq 1}V\to H^0(V)[1]$ which is a locally analytic representation by Proposition 3.2.3. □

Proposition 3.2.6. *The category* $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}[G]}$ *is a Grothendieck abelian category. Moreover* $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ *is the* ∞*-derived category of* $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}[G]}$.

Proof. To show that $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$ is a Grothendieck category, by the above results, it is enough to see that it has a small family of generators. Let $G_0 \subset G$ be a compact open subgroup. Since we are working with κ-small condensed sets, by Lemma 3.2.4 the category $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ is generated by $\{\mathcal{D}^{la}(G,K) \otimes_{\mathcal{D}^{la}(G_0,K)} \mathcal{D}^h(G_0,K) \otimes_{\mathcal{K}_{\blacksquare}} K_{\blacksquare}[S]\}_{h,S}$ where $h>0$ and S runs over all the κ-small profinite sets.

Let us first prove that the right adjoint of the fully faithful inclusion $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Mod}^{\heartsuit}(\mathcal{D}^{la}(G,K))$ of abelian categories is given by the (non-derived) locally analytic vectors. Let $V \in \mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$ and $W \in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$, we want to prove that the natural map

$$\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V, W^{la}) \to \underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V,W)$$

is an equivalence. Then, it suffices to take $V = \mathcal{D}^{la}(G,K) \otimes_{\mathcal{D}^{la}(G_0,K)} \mathcal{D}^h(G_0,K) \otimes_{\mathcal{K}_{\blacksquare}} \mathcal{K}_{\blacksquare}[S]$ and show that the natural map

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V, W^{Rla}) \to R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V,W)$$

is an equivalence. Indeed, one can find a resolution $P^{\bullet} \to V$ of V where each term is a direct sum of elements in $\{\mathcal{D}^{la}(G,K) \otimes_{\mathcal{D}^{la}(G_0,K)} \mathcal{D}^h(G,K) \otimes_{\mathcal{K}_{\blacksquare}} K_{\blacksquare}[S]\}_{h,S}$ and calculate $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V,W)$ in terms of this resolution. Let $V = \mathcal{D}^{la}(G,K) \otimes_{\mathcal{D}^{la}(G_0,K)} \mathcal{D}^h(G_0,K) \otimes_{\mathcal{K}_{\blacksquare}} K_{\blacksquare}[S]$, since we are taking internal $\underline{\mathrm{Hom}}$, we can assume that $S = *$. By Proposition 3.1.11 we have that

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V,W) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^h(G_0,K), W) \\ &= W^{Rh^+-an} \\ &= (W^{Rla})^{Rh^+-an} \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V, W^{Rla}), \end{aligned}$$

proving the claim.

Now, let I be a κ-small injective $\mathcal{D}^{la}(G,K)$-module. By [Sta22, Tag 015Z] $I^{la} = I^{Rla}$ is an injective object in $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$. Moreover, we have that

$$\begin{aligned} I^{Rla} &= \varinjlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^h(G_0,K), I) \\ &= \varinjlim_h \underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^h(G_0,K), I) \\ &= I^{la}. \end{aligned}$$

Then, if $V \in \mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$ and $I^{\bullet}$ is an injective resolution of V as $\mathcal{D}^{la}(G,K)$-module, we have

$$V = V^{Rla} = I^{\bullet, Rla} = I^{\bullet, la},$$

so that $I^{\bullet,la}$ is an injective resolution of V in $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$. The previous implies that the $R\underline{\mathrm{Hom}}$ in the derived category of $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$ can be computed as the $R\underline{\mathrm{Hom}}$ in $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$. Since $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$ is left complete by Lemma 3.2.2, one deduces that it is the ∞-derived category of $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G)$. □

As a byproduct of the proof of Proposition 3.2.6, we have the following result.

Corollary 3.2.7. *The fully faithful inclusion* $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K))$ *has for right adjoint the functor of locally analytic vectors* $V \mapsto V^{Rla}$.

We end this section briefly discussing some functorial properties of the categories of locally analytic representations. Let $H \to G$ be a morphism of p-adic Lie groups over L and denote by $\mathfrak{h} \to \mathfrak{g}$ the corresponding map between their Lie algebras. We have a natural morphism of projective systems of good lattices $\{\mathcal{M}\}_{\mathcal{M}\subset \mathfrak{h}_{\mathcal{L}}} \to \{\mathcal{L}\}_{\mathcal{L}\subset\mathfrak{g}_{\mathcal{L}}}$. In particular, if $\mathcal{M}$ maps to $\mathcal{L}$, we have a morphism of distribution algebras $\widehat{U}(\mathcal{M}) \to \widehat{U}(\mathcal{L})$. On the other hand, the forgetful functor $F : \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(H,K))$

restricts to a forgetful functor $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) \to \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(H)$. It has a right adjoint which is given by the locally analytic induction

$$F : \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) \rightleftharpoons \mathrm{Mod}^{la}_\blacksquare(H, K) : \text{la-}\mathrm{Ind}_H^G(-)$$

where

$$\text{la-}\mathrm{Ind}_H^G(V) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(H,K)}(\mathcal{D}^{la}(G,K), V)^{RH-la}.$$

If $H \subset G$ is an open subgroup, then the forgetful functor commutes with limits in the category of locally analytic representations (computed as the locally analytic vectors of the limit in $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$). Then it has a left adjoint called the *compactly supported induction* and is given by

$$\text{la-cInd}_H^G(V) = \mathcal{D}^{la}(G,K) \otimes^L_{\mathcal{D}^{la}(H,K)} V.$$

3.3. Detecting locally analyticity.

We finish this section with some additional results that can come in handy when proving that a solid representation is locally analytic. In the following we let G be an uniform pro-p-group over $\mathbb{Q}_p$ of dimension d and let $\log : G \to \mathfrak{g} = \mathrm{Lie}\, G$ be its logarithm.

Lemma 3.3.1. *Let $g_1, \ldots, g_s \in G$ and let $\Gamma_1, \ldots, \Gamma_s \subset G$ be their generated pro-p-groups. Suppose that the smallest Lie subalgebra of $\mathfrak{g}$ containing $\mathrm{Lie}\,\Gamma_1 + \cdots + \mathrm{Lie}\,\Gamma_s$ is $\mathfrak{g}$ itself. Then there exists a tuple $(i_1, \ldots, i_r)$ with $i_j \in \{1, \ldots, s\}$ such that the multiplication map*

$$m : \Gamma_{i_1} \times \cdots \times \Gamma_{i_r} \to G$$

has open image around $1 \in G$. Furthermore, the map m admits a section of p-adic manifolds locally around $1 \in G$.

Proof. Let $\mathfrak{X}_{\mathrm{i}} = \log(g_i) \in \mathrm{Lie}\,\Gamma_i$. Let $\mathscr{S} \subset G$ be the subset of elements that can be written as a product of powers of $g_i's$. We claim that $\{h\mathfrak{X}_i h^{-1} : 1 \leq i \leq s, \quad h \in \mathscr{S}\}$ contains a basis of $\mathfrak{g}$. Indeed, by density of $\mathscr{S}$ in an open subset of G, the $\mathbb{Q}_p$-spam V of the objects $h\mathfrak{X}_i h^{-1}$ is a G-stable subspace of $\mathfrak{g}$ which in addition is stable under the Lie bracket. By hypothesis $V = \mathfrak{g}$ and we can find such a base. Let us take $h_1, \ldots, h_d \in \mathscr{S}$, and $\kappa_1, \ldots, \kappa_d \in \{1, \ldots, s\}$ such that $\{\mathfrak{Y}_i := h_i \mathfrak{X}_{\kappa_i} h_i^{-1}\}_{i=1}^d$ is a basis of $\mathfrak{g}$, and set $\widetilde{g}_i = h_i g_{\kappa_i} h_i^{-1}$. Given a tuple $\underline{\iota} = (\iota_1, \ldots, \iota_r)$ let us write $\Gamma_{\underline{\iota}} = \Gamma_{\iota_1} \times \cdots \Gamma_{\iota_r}$ and let $m(\Gamma_{\underline{\iota}}) \subset G$ be its image under the multiplication map. Let us take $\underline{\iota}$ big enough so that $h_i, h_i^{-1} \in m(\Gamma_{\underline{\iota}})$ for all $i = 1, \ldots, d$. We claim that

$$m : (\Gamma_{\underline{\iota}} \times \Gamma_{\kappa_1} \times \Gamma_{\underline{\iota}}) \times \cdots \times (\Gamma_{\underline{\iota}} \times \Gamma_{\kappa_d} \times \Gamma_{\underline{\iota}}) \to G$$

has open image around $1 \in G$. Indeed, denoting $\widetilde{\Gamma}_i = \widetilde{g}_i^{\mathbb{Z}_p}$, we have inclusions $\widetilde{\Gamma}_i \to (\Gamma_{\underline{\iota}} \times \Gamma_{\kappa_i} \times \Gamma_{\underline{\iota}})$ induced by $\widetilde{\gamma}_i^a = h_i g_{\kappa_i}^a h_i^{-1}$, but the multiplication map $\widetilde{m} : \widetilde{\Gamma}_1 \times \cdots \times \widetilde{\Gamma}_d \to G$ has open image around $1 \in G$ since it is an isomorphism at the tangent space of the identity. Finally, $\widetilde{m}$ is a local isomorphism and we can find a local section $s : G \to \widetilde{\Gamma}_1 \times \cdots \times \widetilde{\Gamma}_d \subset (\Gamma_{\underline{\iota}} \times \Gamma_{\kappa_1} \times \Gamma_{\underline{\iota}}) \times \cdots \times (\Gamma_{\underline{\iota}} \times \Gamma_{\kappa_d} \times \Gamma_{\underline{\iota}})$ as wanted. □

Proposition 3.3.2. *Keep the hypothesis of Lemma 3.3.1. An object $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$ is locally analytic if and only if its restriction to Γ_i is locally analytic for all $i = 1, \ldots, s$. Furthermore, if the restriction to each Γ_i is h-analytic for some h, then V is itself h-analytic representation of G for some (maybe different) h.*

Proof. If V is an analytic representation of G it is obviously a analytic representation of Γ_i for all i. Let us show the converse. By Proposition 3.2.5 (2) we can assume that $V \in \mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$ is concentrated in degree 0. By Lemma 3.3.1, there is a tuple $\underline{\iota} = (\iota_1, \ldots, \iota_r)$ of elements in $\{1, \ldots, s\}$ such that the multiplication map

$$m : \Gamma_{\underline{\iota}} := \Gamma_{\iota_1} \times \cdots \times \Gamma_{\iota_r} \to G$$

is surjective, and such that we can find a section $s : G_0 \to \Gamma_{\underline{\iota}}$ from an open compact subgroup.

For each $i \in \{1, \ldots, s\}$ we let $\mathscr{O}_i : V \to \underline{\mathrm{Cont}}(\Gamma_i, V)$ and $\mathscr{O}_i^{la} : V \to C^{la}(\Gamma_i, V)$ denote the orbit maps. Then, composing orbit maps we obtain a commutative diagram

$$\begin{array}{ccccccc} & & \underline{\mathrm{Cont}}(\Gamma_{\iota_r}, V) & \xrightarrow{\mathscr{O}_{\iota_{r-1}}} & \underline{\mathrm{Cont}}(\Gamma_{\iota_{r-1}} \times \Gamma_{\iota_r}, V) & \longrightarrow \cdots \xrightarrow{\mathscr{O}_{\iota_1}} & \underline{\mathrm{Cont}}(\Gamma_{\underline{\iota}}, V) \\ & \nearrow^{\mathscr{O}_{\iota_r}} & \uparrow & & \uparrow & & \uparrow \\ V & & & & & & \\ & \searrow_{\mathscr{O}_{\iota_r}^{la}} & & & & & \\ & & C^{la}(\Gamma_{\iota_r}, V) & \xrightarrow{\mathscr{O}^{la}_{\iota_{r-1}}} & C^{la}(\Gamma_{\iota_{r-1}} \times \Gamma_{\iota_r}, V) & \longrightarrow \cdots \xrightarrow{\mathscr{O}^{la}_{\iota_1}} & C^{la}(\Gamma_{\underline{\iota}}, V). \end{array}$$

Then, taking pullbacks along s, we have a commutative diagram

$$\begin{array}{cccc} & \underline{\mathrm{Cont}}(\Gamma_{\underline{\iota}}, V) & \xrightarrow{s^*} & \underline{\mathrm{Cont}}(G_0, V) \\ \nearrow^{\mathscr{O}_{\underline{\iota}}} & \uparrow & & \uparrow \\ V & & & \\ \searrow_{\mathscr{O}^{la}_{\underline{\iota}}} & & & \\ & C^{la}(\Gamma_{\underline{\iota}}, V) & \xrightarrow{s^*} & C^{la}(G_0, V) \end{array}$$

such that $s^* \circ \mathscr{O}_{\underline{\iota}} = \mathscr{O}_{G_0}$ is the orbit map of G_0. It is easy to see that $s^* \circ \mathscr{O}^{la}_{\underline{\iota}}$ lands in V^{la}, namely, we have to see that $s^* \circ \mathscr{O}^{la}_{\underline{\iota}}(V)$ lands in the $\star_{1,3}$-invariant vectors of $C^{la}(G_0, V)$, but this can be easily prove using the fact that each $\mathscr{O}_i^{la}$ lands in the $\star_{1,3}$-invariant vectors of $C^{la}(\Gamma_i, V)$ for all i. Thus, we find a G_0-equivariant map $V \to V^{la}$ whose composition with the natural map $V^{la} \to V$ is the identity. One deduces that V is a direct summand of V^{la} and afortriori a locally analytic representation of G.

The proof for the h-analyticity follows the same lines as above, knowing that the product $m\Gamma_{\underline{\iota}} \to G$ and the section $s : G_0 \to \Gamma_{\underline{\iota}}$ is locally on the Γ_i and G given by some **analytic** power series. ☐

The following proposition tells us when the generic fiber of a p-adically complete G-representation is locally analytic.

Proposition 3.3.3. *Let $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare, \geq 0}(\mathcal{K}_\blacksquare[G])$ a connective solid $\mathcal{K}_\blacksquare[G]$-module. Suppose that the following holds:*

1. *There exists a p-adically complete object $V^+ \in \mathrm{Mod}_{K^+_\blacksquare, \geq 0}(K^+_\blacksquare[G])$ with $V^+ \otimes^L_{K^+_\blacksquare} K = V$.*
2. *The action of G on V^+/p factors through a finite quotient, i.e. there exists an open subgroup $G_0 \subset G$ such that the restriction of V^+/p to G_0 belongs to the image of $\mathrm{Mod}((K^+/p)_\blacksquare)$ into $\mathrm{Mod}_{K^+_\blacksquare}(K^+_\blacksquare[G_0])$ via the trivial representation.*

Then V is a h-analytic representation for some $h > 0$.

Proof. We can assume without loss of generality that K is a finite extension of $\mathbb{Q}_p$. Let $g_1, \ldots, g_d \in G$ be a local basis of G, and set $\Gamma_i = g_i^{\mathbb{Z}_p}$. By Proposition 3.3.2, it is enough to show that the restriction of V to Γ_i is h-analytic for $i = 1, \ldots, d$. Thus, we can assume that $G = \mathbb{Z}_p$. In this case we have $\mathcal{O}_{K,\blacksquare}[\mathbb{Z}_p] = \mathcal{O}_K[[X]]$ with $X = [1] - 1$, and being h-analytic for some $h > 0$ is equivalent to the existence of $\varepsilon > 0$ such that V is a $K\langle \frac{X}{p^\varepsilon} \rangle$-module, or equivalently, that $K\langle \frac{X}{p^\varepsilon} \rangle \otimes^L_{\mathcal{O}_K[[X]]} V = V$.

The tensor product $K\langle \frac{X}{p^\varepsilon} \rangle \otimes^L_{\mathcal{O}_K[[X]]} V$ is represented by the cone

$$\mathrm{cone}[V \otimes^L_{K,\blacksquare} K\langle T \rangle \xrightarrow{p^\varepsilon T - X} V \otimes^L_{K,\blacksquare} K\langle T \rangle]. \tag{3.2}$$

By taking the g_i small enough, we can assume that the multiplication by X is homotopic to 0 on V^+/p. Then, the multiplication map $X : V^+ \to V^+$ factors through a map

$$\begin{array}{ccc} V^+ & \xrightarrow{\widetilde{X}} & V^+ \\ & \searrow_{X} & \downarrow{\scriptstyle p} \\ & & V^+. \end{array}$$

Take $\varepsilon = 1$, the cone (3.2) is equivalent to the cone

$$\mathrm{cone}[V \otimes^L_{K,\blacksquare} K\langle T\rangle \xrightarrow{T-\widetilde{X}} V \otimes^L_{K_\blacksquare} K\langle T\rangle]$$

which is the generic fiber of

$$C = \mathrm{cone}[V^+ \otimes^L_{\mathcal{O}_K,\blacksquare} \mathcal{O}_K\langle T\rangle \xrightarrow{T-\widetilde{X}} V^+ \otimes^L_{\mathcal{O}_K,\blacksquare} \mathcal{O}_K\langle T\rangle].$$

On the other hand, we have a natural equivalence

$$\mathrm{cone}[V^+ \otimes^L_{\mathcal{O}_K} \mathcal{O}_K[T] \xrightarrow{T-\widetilde{X}} V^+ \otimes^L_{\mathcal{O}_K} \mathcal{O}_K[T]] = V^+.$$

Taking derived p-completions and using [Man22b, Proposition 2.12.10] one gets an equivalence

$$\mathrm{cone}[V^+ \otimes^L_{\mathcal{O}_K,\blacksquare} \mathcal{O}_K\langle T\rangle \xrightarrow{T-\widetilde{X}} V^+ \otimes^L_{\mathcal{O}_K,\blacksquare} \mathcal{O}_K\langle T\rangle] = V^+.$$

By inverting p one deduces that (3.2) is equivalent to V finishing the proof. □

Remark 3.3.4. The same proof of Proposition 3.3.3 holds for a quotient V^+/p^ε for any $\varepsilon > 0$, namely, it is enough to suppose that V^+/p^ε arises as a trivial G_0-representation.

4. Geometric interpretation of locally analytic representations

Let G be a p-adic Lie group over a finite extension L of $\mathbb{Q}_p$. The purpose of this section is to identify the category of locally analytic representations inside the category $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. If G is compact, the algebra $\mathcal{D}^{la}(G,K)$ can be though as the global sections of a non-commutative Stein space. Global sections of sheaves over this space will give objects of $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, and we will prove that the functor of "global sections with compact support" induces an equivalence of stable ∞-categories between quasi-coherent sheaves of this space and $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G)$.

In a second interpretation, for general G, we will show that the category of solid locally analytic representations of G can be described as the derived category of co-modules of the coalgebra $C^{la}(G,K)$ of L-analytic functions. Heuristically, if G^{la} denotes the "analytic spectrum of $C^{la}(G,K)$", the previous description provides a natural equivalence between $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G)$ and solid quasi-coherent sheaves of the classifying stack $[*/G^{la}]$.

4.1. Locally analytic representations as quasi-coherent $\mathcal{D}^{la}(G,K)$-modules.

Definition 4.1.1. Let us write $\mathcal{D}^{la}(G,K) = \varprojlim_{h\to\infty} \mathcal{D}^h(G,K)$ as a limit of h-analytic distribution algebras. We define the category $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ of solid quasi-coherent modules over $\mathcal{D}^{la}(G,K)$ as the ∞-category

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) := \varprojlim_{h>0} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^h(G,K)),$$

where the transition maps in the limit are given by base change.

Objects in the category $\mathcal{C} = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ are sequences of modules $(V_h)_h$ with $V_h \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^h(G,K))$ such that for $h' > h$ one has $\mathcal{D}^h(G,K) \otimes^L_{\mathcal{D}^{h'}(G,K)} V_{h'} = V_h$. Given two objects $(V_h)_h$ and $(W_h)_h$ in $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, the spectra of morphisms is given by

$$R\mathrm{Hom}_{\mathcal{C}}((V_h)_h, (W_h)_h) = \varprojlim_h R\mathrm{Hom}_{\mathcal{D}^h(G,K)}(V_h, W_h).$$

The following lemma will give a sufficient condition for a morphism of objects in $\mathcal{C}$ to be an equivalence.

Lemma 4.1.2. *Let $(R_n)_{n\in\mathbb{N}}$ be a limit sequence of $\mathbb{E}_1$-$\mathcal{K}_\blacksquare$-algebras and let $\mathcal{C} = \varprojlim_n \mathrm{Mod}_{\mathcal{K}_\blacksquare}(R_n)$ be the limit category along base change. Let $f_\bullet : (X_n)_n \to (Y_n)_n$ be a morphism of objects in $\mathcal{C}$, and suppose that there are arrows $h_{n+1} : Y_{n+1} \to X_n$ of R_{n+1}-modules making the following diagram commutative*

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_n \\ \downarrow{\scriptstyle f_{n+1}} & \nearrow{\scriptstyle h_{n+1}} & \downarrow{\scriptstyle f_n} \\ Y_{n+1} & \longrightarrow & Y_n. \end{array}$$

Then $f_\bullet$ is an equivalence in $\mathcal{C}$.

Proof. We have to prove that each $f_{n+1} : X_{n+1} \to Y_{n+1}$ is an equivalence. We have a commutative diagram by extension of scalars

$$\begin{array}{ccccc} X_{n+1} & \longrightarrow & R_n \otimes^L_{R_{n+1}} X_{n+1} & \xrightarrow{\sim} & X_n \\ \Big\downarrow{\scriptstyle f_{n+1}} & & {\scriptstyle 1\otimes f_{n+1}}\Big\downarrow \quad \nearrow{\scriptstyle 1\otimes h_{n+1}} & & \Big\downarrow{\scriptstyle f_n} \\ Y_{n+1} & \longrightarrow & R_n \otimes^L_{R_{n+1}} Y_{n+1} & \xrightarrow{\sim} & Y_n. \end{array}$$

A diagram chasing shows that the map $Y_n \xrightarrow{\sim} R_n \otimes^L_{R_{n+1}} Y_{n+1} \xrightarrow{1\otimes h_{n+1}} X_n$ defines a homotopy inverse of f_n proving that $f_\bullet$ is an equivalence. □

Next, we define natural functors between the category of modules over $\mathcal{D}^{la}(G,K)$ and $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$.

Lemma 4.1.3. *Let $j^* : \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ be the localization functor sending a $\mathcal{D}^{la}(G,K)$-module V to the sequence $(V_h)_h$ with $V_h = \mathcal{D}^h(G,K) \otimes^L_{\mathcal{D}^{la}(G,K)} V$. Then j^* has a right adjoint j_* given by*

$$j_*(V_h)_h = R\varprojlim_h V_h.$$

Proof. Let us denote $\mathcal{C} = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, let $V = (V_h) \in \mathcal{C}$ and $W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. We have a natural map $W \to R\varprojlim_h(\mathcal{D}^h(G,K) \otimes^L_{\mathcal{D}^{la}(G,K)} W)$, and by construction we have

$$\begin{aligned} R\mathrm{Hom}_{\mathcal{C}}(j^*W, V) &= R\varprojlim_h R\mathrm{Hom}_{\mathcal{D}^h(G,K)}(\mathcal{D}^h(G,K) \otimes^L_{\mathcal{D}^{la}(G,K)} W, V_h) \\ &= R\varprojlim_h R\mathrm{Hom}_{\mathcal{D}^{la}(G,K)}(W, V_h) \\ &= R\mathrm{Hom}_{\mathcal{D}^{la}(G,K)}(W, R\varprojlim_h V_h), \end{aligned}$$

proving that the right adjoint of j^* is j_* as wanted. □

Our next goal is to construct a left adjoint $j_!$ for the localization functor $j^* : \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. We shall exploit the fact that the maps $\mathcal{D}^{h'}(G,K) \to \mathcal{D}^h(G,K)$ and $C^h(G,K) \to C^{h'}(G,K)$ are of trace class for $h' > h$. Moreover, they factor through $\mathcal{D}^{la}$-modules

$$\mathcal{D}^{h'}(G,K) \to \overline{\mathcal{D}}^{h'}(G,K) \to \mathcal{D}^h(G,K)$$

and

$$C^h(G,K) \to \overline{C}^h(G,K) \to C^{h'}(G,K)$$

with $\overline{\mathcal{D}}^{h'}(G,K)$ and $\overline{C}^h(G,K)$ being compact projective as $\mathcal{K}_\blacksquare$-vector spaces. We will write $C^{h',B}(G,K)$ and $\mathcal{D}^{h,B}(G,K)$ for the duals of $\overline{\mathcal{D}}^{h'}(G,K)$ and $\overline{C}^h(G,K)$ respectively, these are K-Banach spaces.

Lemma 4.1.4. *Let $f : V \to W$ be a trace class map of $\mathcal{K}_\blacksquare$-vector spaces. There is a morphism $R\underline{\mathrm{Hom}}_K(W,-) \to V^\vee \otimes^L_{\mathcal{K}_\blacksquare} -$ making the following diagram commutative*

$$\begin{array}{ccc} W^\vee \otimes^L_{\mathcal{K}_\blacksquare} - & \longrightarrow & R\underline{\mathrm{Hom}}_K(W,-) \\ \Big\downarrow & \swarrow & \Big\downarrow \\ V^\vee \otimes^L_{\mathcal{K}_\blacksquare} - & \longrightarrow & R\underline{\mathrm{Hom}}_K(V,-) \end{array}$$

Proof. This is analogue to [CS22, Lemma 8.2]. By definition the map f arises from a morphism $K \to V^\vee \otimes^L_{\mathcal{K}_\blacksquare} W$. Let $P \in \mathrm{Mod}(\mathcal{K}_\blacksquare)$ we have morphisms functorial in P

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{K}}(W, P) &\to R\underline{\mathrm{Hom}}_{\mathcal{K}}(V^\vee \otimes_{\mathcal{K}_\blacksquare} W, V^\vee \otimes_{\mathcal{K}_\blacksquare} P) \\ &\to R\underline{\mathrm{Hom}}_{\mathcal{K}}(K, V^\vee \otimes^L_{\mathcal{K}_\blacksquare} P) \\ &= V^\vee \otimes_{\mathcal{K}_\blacksquare} P \\ &\to R\underline{\mathrm{Hom}}_K(V, P). \end{aligned}$$

□

Corollary 4.1.5. *Let $f : V \to W$ be a morphism of $\mathcal{D}^{la}(G, K)$-modules which is trace class as $\mathcal{K}_\blacksquare$-vector spaces such that the morphism $\widetilde{f} : K \to V^\vee \otimes^L_{\mathcal{K}_\blacksquare} W$ defining f is $\mathcal{D}^{la}(G, K)$-equivariant. Then there is a map $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(W \otimes \chi[-d], -) \to V^\vee \otimes^L_{\mathcal{D}^{la}(G,K)} -$ (depending on $\widetilde{f}$) making the following diagram commutative*

$$\begin{array}{ccc} W^\vee \otimes^L_{\mathcal{D}^{la}(G,K)} - & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(W \otimes \chi[-d], -) \\ \downarrow & \swarrow & \downarrow \\ V^\vee \otimes^L_{\mathcal{D}^{la}(G,K)} - & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(V \otimes \chi[-d], -) \end{array}$$

where $V^\vee = R\underline{\mathrm{Hom}}_K(V, K)$, $\chi = (\det \mathfrak{g})^{-1}$ and $d = \dim_L(G)$.

Proof. By Lemma 4.1.4 we have morphism functorial on P

$$R\underline{\mathrm{Hom}}_K(W \otimes \chi[-d], P) \to (V \otimes \chi[-d])^\vee \otimes^L_{\mathcal{K}_\blacksquare} P \to R\underline{\mathrm{Hom}}_K(V \otimes \chi[-d], P),$$

since the map $K \to (V)^\vee \otimes_{\mathcal{K}_\blacksquare} W$ is $\mathcal{D}^{la}(G, K)$-equivariant, then the previous are morphisms of $\mathcal{D}^{la}(G, K)$-modules. Taking invariants and using Proposition 3.1.12 (2) one finds the desired commutative diagram. □

Lemma 4.1.6. *Let $h > h'$, the trace class maps $\mathcal{D}^{h'}(G, K) \to \mathcal{D}^h(G, K)$ and $C^h(G, K) \to C^{h'}(G, K)$ arise from a natural $\mathcal{D}^{la}(G, K)$-equivariant arrow $K \to C^{h'}(G, K) \otimes_{\mathcal{K}_\blacksquare} \mathcal{D}^h(G, K)$.*

Proof. Without lose of generality we can assume that K is a finite extension of $\mathbb{Q}_p$. We can factor $\mathcal{D}^{h'}(G, K) \to \overline{\mathcal{D}}^{h'}(G, K) \to \mathcal{D}^h(G, K)$ where $\overline{\mathcal{D}}^{h'}(G, K)$ is a distribution algebra whose underlying $\mathcal{K}_\blacksquare$-vector space is compact projective with dual $C^{h,B}(G, K)$. Then the morphism $\mathcal{D}^{h'}(G, K) \to \mathcal{D}^h(G, K)$ comes from the map

$$K \to \underline{\mathrm{Hom}}_K(\overline{\mathcal{D}}^{h'}(G, K), \mathcal{D}^h(G, K)) = C^{h',B}(G, K) \otimes_{\mathcal{K}_\blacksquare} \mathcal{D}^h(G, K) \to C^{h'}(G, K) \otimes_{\mathcal{K}_\blacksquare} \mathcal{D}^h(G, K),$$

which is $\mathcal{D}^{la}(G, K)$-equivariant by construction. Similarly, the map $C^h(G, K) \to C^{h'}(G, K)$ factors through a smith space $\overline{C}^h(G, K)$ with dual $\mathcal{D}^{h,B}(G, K)$. Thus, $C^h(G, K) \to C^{h'}(G, K)$ arises from the map

$$K \to R\underline{\mathrm{Hom}}_K(\overline{C}^h(G, K), C^{h'}(G, K)) = \mathcal{D}^{h,B}(G, K) \otimes_{\mathcal{K}_\blacksquare} C^{h'}(G, K) \to \mathcal{D}^h(G, K) \otimes_{\mathcal{K}_\blacksquare} C^{h'}(G, K).$$

□

Theorem 4.1.7. *The map j^* has a left adjoint $j_!$ given by*

$$j_! : \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G, K)) \to \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G, K))$$

$$j_!(V_h)_h = (R \varprojlim_h V_h)^{Rla}.$$

The functor $j_!$ is fully faithful, and $j_! j^* W = W^{Rla}$ for all $W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G, K))$ so that the essential image of $j_!$ is the category $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G)$. In particular, it induces an equivalence of (stable ∞)-categories

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G, K)) \xrightarrow{\sim} \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G).$$

Proof. The lines of the proof are as follows. We will first prove that there is a natural equivalence $j^*j_!V \xrightarrow{\sim} V$ for $V \in \mathrm{Mod}^{qc}(\mathcal{D}^{la}(G,K))$. Then, we show that for $W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, the map $W \to j_*j^*W$ gives rise a natural equivalence $W^{Rla} \xrightarrow{\sim} j_!j^*W$. Taking inverses, these define a unit $V \xrightarrow{\sim} j^*j_!V$ and a counit $j_!j^*W \xrightarrow{\sim} W^{Rla} \to W$ which will give automatically an adjunction such that $j_!$ is fully faithful with essential image the category of locally analytic representations. To lighten notations, we will denote $\mathcal{D}^{la} = \mathcal{D}^{la}(G,K)$, $\mathcal{D}^h = \mathcal{D}^h(G,K)$ and $C^h = C^h(G,K)$ for any $h > 0$, and we omit the decoration for derived limits and tensor products. We will also use Corollary 3.1.10 to write the locally analytic vectors as colimits of $\underline{\mathrm{Hom}}$'s spaces from distribution algebras.

Step 1. We first show that there is a natural equivalence $j^*j_!(V_h)_h \to (V_h)_h$. Unravelling the definitions, we have

$$j^*j_!(V_h)_h = \big(\mathcal{D}^{h_3} \otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1})\big)_{h_3}.$$

In the above description, observe that we can assume that $h_1 \geq h_2 \geq h_3$. Observe that the map $\mathcal{D}^{la} \to \mathcal{D}^{h_2}$ induces a map

$$\mathcal{D}^{h_3} \otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}) \to V_{h_3}. \tag{4.1}$$

Indeed, this follows since

$$\mathcal{D}^{h_3} \otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{la}, \varprojlim_{h_1} V_{h_1}) = \mathcal{D}^{h_3} \otimes_{\mathcal{D}^{la}} \varprojlim_{h_1} V_{h_1} \to V_{h_3}.$$

This provides a natural morphism $j^*j_!(V_h)_h \to (V_h)_h$ for $(V_h)_h \in \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la})$. We want to prove that this map is an equivalence, for this we will use Lemma 4.1.2. The key idea to will be to successively use that, for $h > h'$ the restriction maps $C^h \to C^{h'}$ are trace class maps and use Corollary 4.1.5 to move from one sequencial diagram to the other.

Consider, for any $h_2 \geq h_3 > h_3'$ the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{D}^{h_3} \otimes_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}) & \longrightarrow & \mathcal{D}^{h_3'} \otimes_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}) \\
\downarrow & & \downarrow \\
\varprojlim_{h_1} \big(\mathcal{D}^{h_3} \otimes_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, V_{h_1})\big) & \longrightarrow & \varprojlim_{h_1} \big(\mathcal{D}^{h_3'} \otimes_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, V_{h_1})\big) \\
\downarrow & & \downarrow \\
R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3} \otimes \chi[-d], R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1})) & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3'} \otimes \chi[-d], R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}))
\end{array}$$

The horizontal maps are the obvious maps, and the first vertical maps are the natural maps. The only maps needing explanation are the last vertical ones and the dotted diagonal arrows. The last vertical arrows are constructed as follows: we have $C^{h_3} = \varprojlim_{h'<h_3} C^{h'}$ and $\mathcal{D}^{h_3} = \varinjlim_{h'<h_3} \mathcal{D}^{h'} = \varinjlim_{h'<h_3} R\underline{\mathrm{Hom}}_K(C^{h'}, K)$ with trace class transition maps. The second equality for the distribution algebras follows since the maps $\mathcal{C}^h \to \mathcal{C}^{h'}$ factor through the compact projective $\mathcal{K}_\blacksquare$-vector space $\overline{C}^h$, so in the colimit the derived or non-derived $\underline{\mathrm{Hom}}$'s are equal. Then, the second vertical arrows arise from the natural maps

$$\varinjlim_i V_i^\vee \otimes (-) \to \varinjlim_i R\underline{\mathrm{Hom}}(V_i, -) \to R\underline{\mathrm{Hom}}(R\varprojlim_i V_i, -).$$

The dashed arrows are given by applying Corollary 4.1.5 to the restriction map $f: V = C^{h_3} \to W = C^{h_3'}$ which is trace class and evaluating it at $R\mathrm{Hom}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, V_{h_1})$ for each h_1 and passing to the limit. Furthermore, evaluating Corollary 4.1.5 at the object $R\mathrm{Hom}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, X)$ with $X = V_{h_1}$ and $\varprojlim_{h_1} V_{h_1}$ gives us a map

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3} \otimes \chi[-d], R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, X)) \to \mathcal{D}^{h_3'} \otimes_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, X).$$

Corollary 4.1.5 also implies that the previous functors are natural on X and that the dashed arrows in the diagram above are compatible. We note that by adjunction

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3}\otimes\chi[-d], R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1})) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3}\otimes\chi[-d]\otimes_{\mathcal{D}^{la}}\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1})) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3}\otimes\chi[-d], \varprojlim_{h_1} V_{h_1})), \end{aligned}$$

where the last equality follows since C^{h_3} is already a D^{h_2}-module, as $h_2 \geq h_3$. The same holds for the analogous term with h_3'.

On the other hand, we have another commutative diagram

$$\begin{array}{ccc} \varprojlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3}\otimes\chi[-d], V_{h_1}) & \longrightarrow & \varprojlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3'}\otimes\chi[-d], V_{h_1}) \\ \uparrow & & \uparrow \\ \varprojlim_{h_1}\left(\mathcal{D}^{h_3}\otimes^L_{\mathcal{D}^{la}} V_{h_1}\right) & \longrightarrow & \varprojlim_{h_1}\left(\mathcal{D}^{h_3'}\otimes^L_{\mathcal{D}^{la}} V_{h_1}\right) \\ \wr\uparrow & & \wr\uparrow \\ V_{h_3} & \longrightarrow & V_{h_3'} \end{array}$$

Summarizing, joining both diagrams and taking colimits as $h_2 \to \infty$ we get a commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{h_3}\otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}) & \longrightarrow & \mathcal{D}^{h_3'}\otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_1) \\ \downarrow & \nearrow\text{(dashed)} & \downarrow \\ \varprojlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3}\otimes\chi[-d], V_{h_1}) & \longrightarrow & \varprojlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^{h_3'}\otimes\chi[-d], V_{h_1}) \\ \uparrow & & \uparrow \\ V_{h_3} & \longrightarrow & V_{h_3'}. \end{array}$$

Finally, a diagram chasing shows that the vertical maps commute with the morphism (4.1), obtaining a (final!) commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{h_3}\otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}) & \longrightarrow & \mathcal{D}^{h_3'}\otimes_{\mathcal{D}^{la}} \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1} V_{h_1}) \\ \downarrow & \nearrow\text{(dashed)} & \downarrow \\ V_{h_3} & \longrightarrow & V_{h_3'}. \end{array}$$

Now Lemma 4.1.2 concludes the proof of Step 1.

Step 2. Next, we will prove that for $W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ the unit map $W \to j_*j^*W$ induces an equivalence on locally analytic vectors $W^{Rla} \xrightarrow{\sim} (j_*j^*W)^{Rla} = j_!j^*W$. Composing the inverse of this map together with the natural arrow $W^{Rla} \to W$ one obtains a counit $j_!j^*W \to W$. To prove the equivalence on locally analytic vectors note

$$\begin{aligned}(j_*j^*W)^{Rla} &= \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \varprojlim_{h_1}(\mathcal{D}^{h_1}\otimes^L_{\mathcal{D}^{la}} W))\\ &= \varinjlim_{h_2}\varprojlim_{h_1} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, \mathcal{D}^{h_1}\otimes^L_{\mathcal{D}^{la}} W)\\ &= \varinjlim_{h_2}\varprojlim_{h_1}((C^{h_2}\otimes\chi[-d])\otimes_{\mathcal{D}^{la}}\mathcal{D}^{h_1}\otimes^L_{\mathcal{D}^{la}} W)\\ &= \varinjlim_{h_2}(C^{h_2}\otimes\chi[-d])\otimes^L_{\mathcal{D}^{la}} W\\ &= \varinjlim_{h_2} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{h_2}, W)\\ &= W^{Rla},\end{aligned}$$

where the first equality is just the definition, the second one is obvious, in the third and fifth equalities we use Corollary 4.1.5, and the fourth follows since C^{h_2} is already a $\mathcal{D}^{h_1}$-module since one can assume $h_1 \geq h_2$ in the limit.

Step 3. We now show the adjunction using the first two steps. Indeed, let $V \in \mathrm{Mod}^{qc}(\mathcal{D}^{la}(G,K))$ and $W \in \mathrm{Mod}(\mathcal{D}^{la}(G,K))$. We have

$$\begin{aligned}R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(j_!V, W) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(j_!V, W^{Rla})\\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(j_!V, (j_*j^*W)^{Rla})\\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(j_!V, j_*j^*W)\\ &= R\underline{\mathrm{Hom}}_{\mathcal{C}}(j^*j_!V, j^*W)\\ &= R\underline{\mathrm{Hom}}_{\mathcal{C}}(V, j^*W),\end{aligned}$$

where in the first and third equalities we used the adjunction of Corollary 3.2.7 since $j_!V$ is locally analytic. The second equality follows from *Step 2*, the fourth equality follows from Lemma 4.1.3, and the last equality follows from *Step 1*.

Step 4. Finally, the last thing to check is that the essential image of $j_!$ are the locally analytic functions. But this follows immediately from *Step 2* since $W^{Rla} = j_!j^*W$ for any $W \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. This concludes the proof of the theorem. ☐

Corollary 4.1.8. *Let $V \in \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, then the counit map $j^*j_*V \to V$ is an equivalence. In particular, j_* also defines a fully faithfull embedding from $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ into $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ with essential image those $\mathcal{D}^{la}(G,K)$-modules W such that $W = j_*j^*W$.*

Proof. By definition one has $j_*V = R\varprojlim_h V_h$. By Theorem 4.1.7 j^* is a right adjoint, in particular it commutes with limits, one deduces that $j^*j_*V = R\varprojlim_h j^*V_h$, by definition this object is the sequence

$$((R\varprojlim_h j^*V_h)_{h'})_{h'} = (\varprojlim_h(\mathcal{D}^{h'}\otimes_{\mathcal{D}^{la}} V_h))_{h'} = (V_{h'})_{h'}$$

which proves the corollary. ☐

We now give some examples showing how this equivalence behaves. In particular, it does not preserve the natural t-structures on both sides and hence does not induce at all an equivalence of abelian categories.

Example 4.1.9. We have

(1) $j^*\mathcal{D}^{la}(G,K) = (\mathcal{D}^h(G,K))_h$.
(2) $j_!j^*\mathcal{D}^{la}(G,K) = C^{la}(G,K)\otimes\chi[-d]$.
(3) $j^*C^{la}(G,K) = (\mathcal{D}^h(G,K)\otimes\chi^{-1}[d])_h$.
(4) If V is a $\mathcal{D}^h(G,K)$-module then the sequence $(V)_{h'>h}$ defines an element in $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ and one has $j_!(V)_h = j_*(V)_h = V$. In particular, for each $h > 0$, Theorem 4.1.7 restricts to the equivalences of [RJRC21, Theorem 4.36].

Proof. The first point follows by definition. Part (2) follows from (1) and Corollary 3.1.16. Indeed, we have

$$j_!j^*\mathcal{D}^{la}(G,K) = (\varprojlim_h \mathcal{D}^h(G,K))^{Rla} = \mathcal{D}^{la}(G,K)^{Rla}.$$

Applying j^* to the second example, we obtain

$$j^*C^{la}(G,K) = j^*j_!j^*\mathcal{D}^{la}(G,K)\otimes\chi^{-1}[d] = j^*\mathcal{D}^{la}(G,K)\otimes\chi^{-1}[d] = (\mathcal{D}^h(G,K)\otimes\chi^{-1}[d])_h,$$

where for the second equality we used the equivalence of $j^*j_! \to \mathrm{id}$ of Theorem 4.1.7. The last point follows directly from the definitions. Indeed, if $V\in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ is in fact a $\mathcal{D}^h(G,K)$-module, then $j^*V = (\mathcal{D}^{h'}(G,K)\otimes_{\mathcal{D}^{la}(G,K)} V)_{h'} = (V)_{h'\geq h}$, which is a constant sequence, and we have $j_*(V)_h = \varprojlim_h V = V$ and $j_!(V)_h = (j_*V)^{Rla} = V^{Rla} = V$. □

Example 4.1.10. As the notation suggests, the functors j^*, j_* and $j^!$ should come from a 6-functor formalism of "non-commutative spaces" which at the moment is not available. When $G=\mathbb{Z}_p$, nevertheless, the functors j^*, j_* and $j_!$ can be interpreted as part of the six functors of the open rigid ball of radius one.

Definition 4.1.11. We define a duality functor on $\mathcal{C} = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ by mapping an object $V=(V_h)_h$ to

$$\mathbb{D}(V) := j^*\big(\varprojlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V_h,\mathcal{D}^h(G,K)\otimes\chi^{-1}[d])\big) = \varprojlim_h j^*R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V_h,\mathcal{D}^h(G,K)\otimes\chi^{-1}[d]).$$

Lemma 4.1.12. *Let $V\in\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, then*

$$j^*R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^{la}(G,K)) = j^*(\varprojlim_h(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V_h,\mathcal{D}^h(G,K))).$$

Proof. We compute

$$j^*R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^{la}) = (\mathcal{D}^h\otimes^L_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^{la}))_h.$$

By Corollary 4.1.5, this system is cofinal with the system

$$(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d], R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^{la}))_h.$$

But

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d], R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^{la})) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d]\otimes_{\mathcal{D}^{la}} V,\mathcal{D}^{la})\\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d],\mathcal{D}^{la})),\end{aligned}$$

and hence, applying again Corollary 4.1.5, we get that the Pro-system $(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d],\mathcal{D}^{la}))_h$ is equivalent to the Pro-system $(\mathcal{D}^h\otimes_{\mathcal{D}^{la}}\mathcal{D}^{la})_h = (\mathcal{D}^h)_h$. We deduce from Corollary 4.1.8 a natural equivalence of Pro-systems $j^*R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^{la}) = (R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V_h,\mathcal{D}^h)_h$ which proves the lemma. □

Proposition 4.1.13. *Let $V\in\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, then*

$$j^*R\underline{\mathrm{Hom}}_K(V,K) = \mathbb{D}(j^*V)$$

where we use the involution of $\mathcal{D}^{la}(G,K)$ to see both modules as left $\mathcal{D}^{la}(G,K)$-modules. In other words, the duality functors as K-vector space or $\mathcal{D}^{la}(G,K)$-module become the same (modulo a twist) in the category of quasi-coherent $\mathcal{D}^{la}(G,K)$, e.q. in the category of solid locally analytic representations.

Proof. By definition we have that $j^*R\underline{\mathrm{Hom}}_K(V,K) = (\mathcal{D}^h\otimes^L_{\mathcal{D}^{la}} R\underline{\mathrm{Hom}}_K(K,V))_h$. By Corollary 4.1.5 the Pro-system $j^*(R\underline{\mathrm{Hom}}_K(V,K))$ is cofinal with the Pro-system

$$(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d], R\underline{\mathrm{Hom}}_K(V,K)))_h.$$

We also have that

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(C^h\otimes\chi[-d], R\underline{\mathrm{Hom}}_K(V,K)) &= R\underline{\mathrm{Hom}}_K(C^h\otimes\chi[-d]\otimes^L_{\mathcal{D}^{la}} V,K)\\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V, R\underline{\mathrm{Hom}}_K(C^h\otimes\chi[-d],K)).\end{aligned}$$

Using Lemma 4.1.4 we see that the Pro-system $(R\underline{\mathrm{Hom}}_K(C^h\otimes\chi[-d],K))_h$ is cofinal with $(\mathcal{D}^h\otimes\chi^{-1}[d])_h$. One deduces that $j^*(R\underline{\mathrm{Hom}}_K(V,K))$ is cofinal with the Pro-system $(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(V,\mathcal{D}^h\otimes\chi^{-1}[d]))_h$. One concludes by Lemma 4.1.12. □

4.2. **Admissible and coadmissible representations.**

Definition 4.2.1. We define the derived category of perfect $\mathcal{D}^{la}(G,K)$-modules to be the inverse limit $\mathrm{Mod}_K^{perf}(\mathcal{D}^{la}(G)) = \varprojlim_h \mathrm{Mod}_K^{perf}(\mathcal{D}^h(G,K))$ of perfect $\mathcal{D}^h(G,K)$-modules. Under the fully faithful embedding $j_* : \mathrm{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K)) \to \mathrm{Mod}_{K_\blacksquare}(\mathcal{D}^{la}(G,K))$, we denote by $\mathrm{Mod}_K^{coad}(\mathcal{D}^{la}(G))$ the essential image of $\mathrm{Mod}_K^{perf}(\mathcal{D}^{la}(G))$ and call it the derived category of coadmissible $\mathcal{D}^{la}(G,K)$-modules. Analogously, under the equivalence $j_! : \mathrm{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K)) \to \mathrm{Rep}^{la}_{K_\blacksquare}(G)$, we denote by $\mathrm{Rep}^{ad}_K(G)$ the essential image of $\mathrm{Mod}_K^{coad}(\mathcal{D}^{la}(G))$ and call it the derived category of admissible locally L-analytic representations of G.

Let us relate $\mathrm{Rep}^{ad}_K(G)$ with a more classical definition of the category of admissible representations. We first need to recall some properties of the distribution algebras.

Proposition 4.2.2 ([ST03])**.**

(1) *There are Banach distribution algebras $\mathcal{D}^{(h)}(G,K)$ with dense and trace class transition maps $\mathcal{D}^{(h')}(G,K) \to \mathcal{D}^{(h)}(G,K)$ for $h' > h$, such that $\mathcal{D}^{la}(G,K) = \varprojlim_h \mathcal{D}^{(h)}(G,L)$ is presented as a Fréchet-Stein algebra. In particular the rings $\mathcal{D}^{(h)}(G,K)$ are noetherian so any finite $\mathcal{D}^{(h)}(G,K)$-module is naturally a Banach space, and the morphisms of algebras $\mathcal{D}^{la}(G,K) \to \mathcal{D}^{(h)}(G,K)$ and $\mathcal{D}^{(h')}(G,K) \to \mathcal{D}^{(h)}(G,K)$ for $h' > h$ are flat.*
(2) *The rings $\mathcal{D}^{(h)}(G,K)$ are Auslander of dimension $d = \dim_L G$. In particular, any $\mathcal{D}^{(h)}(G,K)$-module of finite type has a finite projective resolution of length at most d.*

Remark 4.2.3. The algebras $\mathcal{D}^{(h)}(G,K)$ used by Schneider and Teitelbaum (denoted by $D_r(G,K)$ in *loc. cit.*) are different from those $\mathcal{D}^h(G,K)$ used in this paper. It should be true that that algebras $\mathcal{D}^h(G,K)$ are noetherian and Auslander of dimension d, and that the transition maps $\mathcal{D}^{la}(G,K) \to \mathcal{D}^h(G,K)$ and $\mathcal{D}^{h'}(G,K) \to \mathcal{D}^h(G,K)$ are flat for $h' > h$, see [CS22, Theorem 10.5]. On the other hand, the systems $(\mathcal{D}^{(h)}(G,K))_h$ and $(\mathcal{D}^h(G,K))_h$ are cofinal, this implies that we can also write

$$\mathrm{Mod}^{qc}_{K_\blacksquare}(\mathcal{D}^{la}(G,K)) = \varprojlim_h \mathrm{Mod}_{K_\blacksquare}(\mathcal{D}^{(h)}(G,K)).$$

Corollary 4.2.4. *The category $\mathrm{Mod}_K^{coad}(\mathcal{D}^{la}(G,K))$ has a natural t-structure with heart given by the abelian category of coadmissible $\mathcal{D}^{la}(G,K)$-modules, i.e. $\mathcal{D}^{la}(G,K)$-modules of the form $V = \varprojlim_h (V_h)_h$, where the V_h's are $\mathcal{D}^{(h)}(G,K)$-modules of finite type such that $\mathcal{D}^{(h)}(G,K) \otimes^L_{\mathcal{D}^{(h')}} V_{h'} = V_h$ for $h' > h$.*

Proof. The flatness of the rings of distribution algebras implies that the t-structures on the categories $\mathrm{Mod}_K^{perf}(\mathcal{D}^{(h)}(G,K))$ are preserved under base change, this shows that $\mathrm{Mod}_K^{coad}(\mathcal{D}^{la}(G,K))$ has a natural t-structure and that the heart is, by definition, the abelian category of coadmissible $\mathcal{D}^{la}(G,K)$-modules of [ST03]. □

Remark 4.2.5. One can ask for the relation of the (triangulated) bounded derived category of the abelian category of coadmissible $\mathcal{D}^{la}(G,K)$-modules and the homotopy category of the bounded objects in $\mathrm{Mod}_K^{coad}(\mathcal{D}^{la}(G,K))$. We do not have an answer to this question, however the first could be poorly behaved as the abelian category of coadmissible $\mathcal{D}^{la}(G,K)$-modules might not have enough injectives or projectives.

Lemma 4.2.6. *Let $V \in \mathrm{Mod}^{perf,\heartsuit}_{K_\blacksquare}(\mathcal{D}^{la}(G))$ be a perfect $\mathcal{D}^{la}(G,K)$-module in the heart. Then $(j_*V)^{\vee,Rla}$ is a locally analytic representation concentrated in degree 0.*

Proof. Let $V = (V_h)_h$ be a perfect $\mathcal{D}^{la}(G,K)$ module. By definition we have

$$(j_*V)^{\vee,Rla} = \varinjlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}}(\mathcal{D}^{(h)}, R\underline{\mathrm{Hom}}_K(j_*V,K)) = \varinjlim_h R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h)} \otimes^L_{\mathcal{D}^{la}} j_*V, K).$$

By Corollary 4.1.8 we have $j^*j_*V = V$, so that $\mathcal{D}^{(h)} \otimes_{\mathcal{D}^{la}} j_*V = V_h$. Therefore

$$(j_*V)^{\vee,Rla} = \varinjlim_h R\underline{\mathrm{Hom}}_K(V_h,K),$$

but $V_{h'}$ is a $\mathcal{D}^{(h')}$-module of finite presentation, and $V_h = \mathcal{D}^{(h)} \otimes_{\mathcal{D}^{(h')}} V_{h'}$. One deduces that $V_{h'} \to V_h$ is a trace class map, defined by a trace map $K \to H^0(V_{h'}^\vee) \otimes_{K_\blacksquare} V_{h'}$. Let $W_{h'} := H^0(V_{h'}^\vee)$, one then has a factorization

$$\begin{aligned} V_h^\vee &\to R\underline{\mathrm{Hom}}_K(W_{h'} \otimes^L_{\mathcal{K}_\blacksquare} V_h, W_{h'}) \\ &\to W_{h'} \\ &\to V_{h'}^\vee \end{aligned}$$

where the first map is the obvious one, the second follows from the trace map $K \to H^0(V_{h'}^\vee) \otimes_{\mathcal{K}_\blacksquare} V_{h'}$, and the last from the natural map $W_{h'} = H^0(V_{h'}^\vee) \to V_{h'}^\vee$. One concludes that

$$\varinjlim_h V_h^\vee = \varinjlim_h W_h$$

sits in degree 0 which proves the lemma. □

The reader might ask about the relation between the equivalence provided by Theorem 4.1.7 and the classical anti-equivalence of categories [ST03, Theorem 6.3] of Schneider and Teitelbaum. In [RJRC21, Proposition 4.42] we have shown how one can recover this result from our previous work. The following result, which is a summary of many of the previous results of this section, shows how Schneider and Teitelbaum's equivalence sits inside the equivalence of Theorem 4.1.7, proving that our theorem can be seen as a refinement of [ST03, Theorem 6.3].

Proposition 4.2.7. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) & \xrightarrow{j^*} & \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) \\ \downarrow{\scriptstyle (-)^{\vee, Rla}} & & \downarrow{\scriptstyle \mathbb{D}(-)} \\ \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) & \xleftarrow{j_!} & \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)), \end{array}$$

where the right vertical arrow is given by the dualizing functor of Definition 4.1.11. Moreover, when restricted to the abelian category of coadmissible $\mathcal{D}^{la}(G,K)$-modules, the composition $j_! \circ \mathbb{D} \circ j^*$ restricts to the anti-equivalence of [ST03, Theorem 6.3].

Proof. We first prove that the diagram is commutative. By Proposition 4.1.13, we know that $\mathbb{D} \circ j^* V = j^* R\underline{\mathrm{Hom}}_K(V,K)$, so that

$$j_! \circ \mathbb{D} \circ j^* = (j_! j^* V^\vee) = (V^\vee)^{Rla}$$

by the second step of the proof of Theorem 4.1.7. Lemma 4.2.6 shows that, when we restrict to the subcategory $\mathrm{Mod}^{coad,\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$, this composition of functors is concentrated in degree 0 and hence coincides with $V \mapsto \underline{\mathrm{Hom}}_K(V,K)$ which is an admissible locally analytic representation. □

Proposition 4.2.8. *Let $V \in \mathrm{Rep}^{ad}_K(G)$ be an admissible locally analytic representation. Then, letting $V^\vee := R\underline{\mathrm{Hom}}_K(V,K)$, we have*

$$\mathbb{D}(j^* V^\vee) = j^* V$$

Proof. Since V is admissible one has that $V = j_! W$ for $W \in \mathrm{Mod}^{perf}_K(\mathcal{D}^{la}(G,K))$, in particular $j^* V = W$. The object W is reflexive for the functor $\mathbb{D}(-)$ being a limit diagram of perfect $\mathcal{D}^h(G,K)$-modules, one deduces that $W = \mathbb{D}(\mathbb{D}(W))$. On the other hand, Proposition 4.1.13 says that $\mathbb{D}(W) = j^*(V^\vee)$, one deduces that $j^* V = \mathbb{D}(\mathbb{D}(W)) = \mathbb{D}(j^* V^\vee)$ proving the proposition. □

We conclude by studying the dualizing functor in the non-compact case. Let G be a locally profinite p-adic Lie group over L and $G_0 \subset G$ an open compact subgroup. We denote

$$\overline{\mathcal{D}}^{la}(G,K) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(\mathcal{D}^{la}(G,K), \mathcal{D}^{la}(G_0,K)) = \prod_{g \in G/G_0} \mathcal{D}^{la}(G_0,K),$$

one easily verifies that this is the dual space of the locally analytic functions of G with compact support. We define a duality functor in $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ by $\mathbb{D}(W) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(W, \overline{\mathcal{D}}^{la}(G,K) \otimes \chi^{-1}[d])$. Notice that by adjunction

$$\mathbb{D}(W) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_0,K)}(W, \mathcal{D}^{la}(G_0,K) \otimes \chi^{-1}[d])$$

so that it is the natural induction of the duality functors from compact p-adic Lie groups. Observe that the duality functor just defined is compatible with the duality functor on $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K))$ of Definition 4.1.11, namely if $W\in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K))$, then by Lemma 4.1.12 one has $\mathbb{D}(j^*W)=j^*\mathbb{D}(W)$. We have the following proposition.

Corollary 4.2.9. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) & \xleftarrow{(-)^{Rla}} & \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G)) \\ {\scriptstyle (-)^{\vee,Rla}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}(-)} \\ \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare(G)} & \xleftarrow[(-)^{Rla}]{} & \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G)). \end{array}$$

In other words, the duality functor $\mathbb{D}$ is compatible with the duality functor $(-)^{\vee,Rla}$ of $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G)$.

Proof. Observe that, if $G_0\subset G$ is an open compact subgroup, since for any $W\in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$ we have $j^*(W)=j^*j_!j^*(W)=j^*(W^{Rla})$ by Theorem 4.1.7, the diagram of Proposition 4.2.7 can be written as

$$\begin{array}{ccc} \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_0) & \xleftarrow{j_!} & \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K)) \\ {\scriptstyle (-)^{\vee,Rla}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}(-)} \\ \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_0) & \xleftarrow[j_!]{} & \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K)). \end{array}$$

The corollary follows since $\mathbb{D}(W)=R\underline{\mathrm{Hom}}_{G_0}(W,\mathcal{D}^{la}(G_0,K)\otimes\chi^{-1}[d])$ is the duality functor for G_0, and the duality functor $\mathbb{D}:\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K))\to \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K))$ is the pullback by j^* of the duality functor on $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G_0,K))$ by Lemma 4.1.12. □

4.3. **Locally analytic representations as comodules of $C^{la}(G,K)$.** Let G be a p-adic Lie group over L. In this section we show that the category of locally L-analytic representations of G can be undestood as the derived category of quasi-coherent sheaves over a suitable "classifying stack" $[*/G^{la}]$ of G. Throughout this paper we will only see this stack as a formal object for which the category of quasi-coherent sheaves can be defined by hand as a limit of a cosimplicial diagram, an honest definition as a stack will require a notion of stack on analytic rings that we will not explore in this work.

Definition 4.3.1.

(1) Let G be a group acting on a space X. We define the simplicial diagram $(G^n\times X)_{[n]\in\Delta^{op}}$ with boundary maps $d_n^i: G^n\times X\to G^{n-1}\times X$ for $0\leq i\leq n$ defined by

$$d_n^i(g_n,\cdots,g_1,x)=\begin{cases}(g_n,\ldots,g_2,g_1x) & \text{if } i=0\\ (g_n,\ldots,g_{i+1}g_i,\ldots,g_1,x) & \text{if } 0<i<n\\ (g_{n-1},\ldots,g_1,x) & \text{if } i=n\end{cases}$$

and degeneracy maps $s_n^i: G^n\times X\to G^{n+1}\times X$ for $0\leq i\leq n$ given by sending the tuple $(g_n,\ldots,g_1,x)$ to $(g_n,\ldots,1,\ldots,g_1,x)$ with 1 in the $i+1$-th coordinate.

(2) Let $G_0\subset G$ be an open compact subgroup. We define the category of quasi-coherent sheaves on G^{la} to be

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(G^{la}):=\prod_{g\in G/G_0}\mathrm{Mod}_{\mathcal{K}_\blacksquare}(C^{la}(gG_0,K)).$$

(3) We define the category of quasi-coherent sheaves on the classifying stack $[*/G^{la}]$ to be the limit

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{la}])=\varprojlim_{[n]\in\Delta}\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(G^{n,la}).$$

Remark 4.3.2. The definition of $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(G^{la})$ is made in such a way that for G compact we can see G^{la} as the analytic spectrum of $C^{la}(G,K)$, and that for G arbitrary $G^{la}=\bigsqcup_{g\in G/G_0}gG_0^{la}$. Then, the definition of $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{la}])$ follows the intuition that $[*/G^{la}]$ is the geometric realization of the simplicial space $(G^{n,la})_{n\in\Delta^{op}}$.

Theorem 4.3.3. *There is a natural equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{la}]),$$

where the tensor product in the LHS is the tensor product over $\mathcal{K}_\blacksquare$.

We need a lemma.

Lemma 4.3.4. *There is a natural symmetric monoidal equivalence between the abelian category $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$ of locally analytic representations of G, and the abelian category of comodules of the functor $C^{la}(G,-)$ mapping $V \in \mathrm{Mod}^{\heartsuit}(\mathcal{K}_\blacksquare)$ to $C^{la}(G,V) = \prod_{g\in G/G_0}(C^{la}(gG_0,K)\otimes_{\mathcal{K}_\blacksquare} V)$.*

Proof. Given a map $\mathscr{O} : V \to C^{la}(G,V)$ we have a morphism $V \to C^{la}(G,V) \to \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K),V)$ which by adjunction gives rise a map $\rho : \mathcal{D}^{la}(G,K)\otimes_K V \to V$. If $\mathscr{O}$ is a comodule then ρ is a module structure and V defines an object in $\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. Restricting the co-module structure to G_0 one finds that the morphism $\mathscr{O}|_{G_0} : V \to C^{la}(G_0,K)\otimes_{\mathcal{K}_\blacksquare} V$ lands in the invariants of the $\star_{1,3}$-action of $\mathcal{D}^{la}(G,K)$ in right term. Thus, by taking invariants one finds that V is a direct summand of V^{la} which implies that V is locally analytic itself, i.e. $V \in \mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$. Conversely, given $V \in \mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$ one has an orbit map $\mathscr{O} : V \to C^{la}(G,V)$ which is clearly a comodule for the functor $C^{la}(G,-)$. It is easy to check that these constructions are inverse each other. ☐

Proof of Theorem 4.3.3. By [Man22b, Proposition A.1.2] the category $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{la}])$ is the derived category of descent datum of $*$ over G^{la} via the trivial action, which is the same as the abelian category of comodules $V \to C^{la}(G,V)$. By Lemma 4.3.4 this abelian category is naturally isomorphic to $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$ as symmetric monoidal categories, taking derived categories one has an equivalence

$$\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{la}])$$

as symmetric monoidal stable ∞-categories. ☐

Corollary 4.3.5. *Let G be a compact p-adic Lie group over L, then we have natural equivalences of stable ∞-categories*

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K)) = \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{la}]).$$

4.4. **Classifying stack of rank one (φ,Γ)-modules and locally analytic representations of GL_1.** In this section, we explore an interesting application of Theorem 4.1.7 for the group $\mathcal{O}_L^\times$ to the locally analytic categorical p-adic Langlands correspondence for GL_1 as formulated in [EGH23].

We let $\mathcal{X}_1$ be the classifying stack of rank 1 (φ,Γ)-modules over the Robba ring on affinoid Tate algebras over $\mathcal{K} = (K,K^+)$, cf. [EGH23, §5]. This stack is represented (cf. [EGH23, §7.1]) by the quotient

$$[(\widetilde{\mathcal{W}} \times \mathbb{G}_m^{an})/\mathbb{G}_m^{an}]$$

with trivial action of $\mathbb{G}_m^{an}$, where $\widetilde{\mathcal{W}}$ is the rigid analytic weight space of $\mathcal{O}_L^\times$ whose points on an affinoid ring A are given by continuous (eq. $\mathbb{Q}_p$-locally analytic) characters $\mathrm{Hom}(\mathcal{O}_L^\times, A)$, and where $\mathbb{G}_m^{an}$ denotes the rigid analytic multiplicative group. Let $L^\times_{\mathbb{Q}_p}$ be the restriction of scalars of $L^\times$ from L to $\mathbb{Q}_p$. In [EGH23], the authors conjecture that the natural functor

$$\mathfrak{LL}_p^{la} : \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(L^\times_{\mathbb{Q}_p}) \to \mathrm{Mod}^{qc}_\blacksquare(\mathcal{X}_1) \tag{4.2}$$

given by $\mathfrak{LL}_p^{la}(\pi) = \mathcal{O}_{\mathcal{X}_1} \otimes^L_{\mathcal{D}^{la}(L^\times_{\mathbb{Q}_p},K)} \pi$ (cf. [EGH23, Equation (7.1.3)]) is fully faithful when restricted to a suitable category of "tempered" (or finite slope) locally analytic representations.

On the other hand, for the functor $\mathfrak{LL}_p^{la}$ to be fully faithful without restricting to a smaller subcategory of $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(L^\times_{\mathbb{Q}_p})$, one can also modify the stack $\mathcal{X}_1$, namely, we consider

$$\mathcal{X}_1^{mod} := [\widetilde{\mathcal{W}} \times \mathbb{G}_m^{alg}/\mathbb{G}_m^{alg}]$$

where $\mathbb{G}_m^{alg}$ is the analytic space attached to the ring $(K[T^{\pm 1}],K^+)_\blacksquare = \mathcal{K}_\blacksquare \otimes_{\mathbb{Z}} \mathbb{Z}[T^{\pm 1}]$. To lighten notation we will use the version of $\mathcal{X}_1$ and $\mathcal{X}_1^{mod}$ involving the space $\mathcal{W} \subset \widetilde{\mathcal{W}}$ of L-locally analytic characters, and the group $L^\times$ instead. The same arguments will hold for the spaces defined over $\mathbb{Q}_p$.

To describe the category of solid quasi-coherent sheaves of the original stack $\mathcal{X}_1$ in terms of representation theory we need to introduce a certain algebra of "tempered sequences" on $\mathbb{Z}$.

Definition 4.4.1.

(1) We let $\ell^{temp}_{\mathbb{Z},K} \subset \prod_{\mathbb{Z}} K$ be the subalgebra with respect to the pointwise multiplication consisting on sequences $(a_n)_{n\in\mathbb{Z}}$ such that there exists $r>0$ such that $\sup_{n\in\mathbb{Z}}\{|a_n|p^{-r|n|}\}<\infty$. Equivalently, Let $\mathscr{O}(\mathbb{G}_m^{an}) = \varprojlim_{n\to\infty} K\langle p^nT, \frac{p^n}{T}\rangle$, then $\ell^{temp}_{\mathbb{Z},K} = \mathscr{O}(\mathbb{G}_m^{an})^{\vee}$. We let $\mathbb{Z}^{temp}$ denote the analytic space defined by the algebra $\ell^{temp}_{\mathbb{Z},K}$.

(2) We let $L^{\times,temp}$ be the analytic space associated to the algebra $C^{temp}(L^{\times},K) := C^{la}(\mathcal{O}_L^{\times},L)\otimes^L_{\mathcal{K}_{\blacksquare}} \ell^{temp}_{\mathbb{Z},K}$ of tempered locally analytic functions on $L^{\times}$. Equivalently, we have

$$C^{temp}(L^{\times},K) = \mathscr{O}(\mathcal{W}\times\mathbb{G}_m^{an})^{\vee}.$$

(3) We let $\mathrm{Rep}^{temp}_{\mathcal{K}_{\blacksquare}}(L^{\times}) := \mathrm{Mod}^{qc}_{\blacksquare}([*/L^{\times,temp}])$ be the category of tempered (locally analytic) representations of $L^{\times}$.

Remark 4.4.2. In [CS20, Definition 13.5] Clausen and Scholze have introduced a notion of analytic space as certain sheaves in the category of analytic rings with respect to steady localizations. The analytic spaces $\mathbb{Z}^{temp}$ and $L^{\times,temp}$ can be considered in this category, or equivalently, as the presheaves on analytic rings corepresented by the corresponding algebra.

Lemma 4.4.3. *The spaces $\mathbb{Z}^{temp}$ and $L^{\times,temp}$ have unique commutative group structures compatible with the natural maps $\mathbb{Z}\to\mathbb{Z}^{temp}$ and $L^{\times,la}\to L^{\times,temp}$.*

Proof. A commutative group structure on $\mathbb{Z}^{temp}$ and $L^{\times,temp}$ is the same as a commutative Hopf algebra structure on their spaces of functions. But by definition $\ell^{temp}_{\mathbb{Z},K}$ and $C^{temp}(L^{\times},K)$ are the duals of the global sections of $\mathbb{G}_m^{an}$ and $\mathcal{W}\times\mathbb{G}_m^{an}$ which are themselves commutative groups, proving that $\ell^{temp}_{\mathbb{Z},K}$ and $C^{temp}(L^{\times},K)$ have a natural structure of commutative Hopf algebras. □

Theorem 4.4.4. *There are natural equivalences of stable ∞-categories*

$$\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathbb{Z}/L^{\times,la}]) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{X}_1^{mod}), \quad \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathbb{Z}^{temp}/L^{\times,temp}]) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{X}_1) \tag{4.3}$$

Furthermore, the functor $\mathfrak{LL}_p^{la}$ defined in (4.2) *induces equivalences*

$$\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(L^{\times}) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{W}\times\mathbb{G}_m^{alg}), \quad \mathrm{Rep}^{temp}_{\mathcal{K}_{\blacksquare}}(L^{\times}) \xrightarrow{\sim} \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{W}\times\mathbb{G}_m^{an}). \tag{4.4}$$

Remark 4.4.5. The equivalences of Theorem 4.4.4 (1) should follow from a Cartier duality theory for quasi-coherent sheaves in analytic spaces, this would imply that the natural symmetric monoidal structures are transformed in the convolution products via the Fourier-Moukai transform. In the cases of the theorem, we will roughly prove that modules over the Hopf algebras of the groups are equivalent to comodules of the dual Hopf algebras.

Proposition 4.4.6. *Let A be a flat solid $\mathcal{K}_{\blacksquare}$-algebra. Then there are natural equivalences*

$$\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathrm{AnSpec}\,A/\mathbb{G}_m^{alg}]) = \mathrm{Func}(\mathbb{Z},\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(A))$$

and

$$\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathrm{AnSpec}\,A\times\mathbb{G}_m^{alg}) = \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathrm{AnSpec}\,A/\mathbb{Z}])$$

functorial with respect to base change $B\otimes^L_{A,\blacksquare}-$. In particular, the same statement hold for analytic spaces glued from flat $\mathcal{K}$-algebras.

Proof. By [Man22b, Proposition A.1.2], the ∞-category $\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}([\mathrm{AnSpec}\,A/\mathbb{G}_m^{alg}])$ is the derived category of $A[T^{\pm1}]$-comodules over A. The data of a $A[T^{\pm1}]$-comodule is the same as the data of a $\mathbb{Z}$-graded A-algebra, namely, given M an $A[T^{\pm1}]$-comodule and $\mathscr{O}: M\to M\otimes_A A[\pm1]$ the co-module map, one has a graduation $M=\bigoplus_i M(i)$ by defining $M(i) = \mathscr{O}^{-1}(M\otimes T^{-i})$. Conversely, if $M=\bigoplus_{i\in\mathbb{Z}} M(i)$ one defines the co-module structure $\mathscr{O}: M\to M\otimes A[T^{\pm1}]$ by mapping $\mathscr{O}: M(i)\xrightarrow{\sim} M(i)\otimes T^{-i}$. We have constructed a natural equivalence

$$\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathrm{AnSpec}\,A/\mathbb{G}_m) \xrightarrow{\sim} \mathrm{Func}(\mathbb{Z},\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(A)),$$

taking derived categories we get the first equivalence.

For the second one, the category $\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathrm{AnSpec}\, A \times \mathbb{G}_m^{alg})$ is by definition the derived category of $\mathcal{K}_\blacksquare$-solid $A[T^{\pm 1}] = A[\mathbb{Z}]$-modules, i.e. $\mathbb{Z}$-representations on solid A-modules. This gives a natural equivalence

$$\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(A[T^{\pm 1}]) = \mathrm{Mod}^{qc,\heartsuit}_{\mathcal{K}_\blacksquare}([\mathrm{AnSpec}\, A/\mathbb{Z}]),$$

taking derived categories one obtains the second equivalence of the lemma. □

Remark 4.4.7. In the proof of the following lemma we are going to use some facts coming from a 6-functor formalism for solid quasi-coherent sheaves of analytic stacks over $\mathbb{Q}_{p,\blacksquare}$ in the $\mathscr{D}$-topology as in [Sch23, Definition 4.14]. This theory has been partially constructed in [CS19] and [CS22] for schemes or complex analytic spaces, and the methods of [Man22b, Appendix A.5], [Man22a, §5-9] and [Sch23] are enough to give proper foundations. In particular, we assume that:

(1) The family E of morphisms in the 6-functor formalism (see [Man22b, Definition A.5.7]) contains all maps $f : X \to Y$ of rigid spaces. In particular, we have shriek functors $f_!$ and $f^!$ satisfying proper base change and projection formula, and compatible under compositions.
(2) Let $f : \mathcal{A} \to \mathcal{B}$ be a map of analytic rings that defines a map of analytic spectra $f : \mathrm{AnSpec}\, \mathcal{B} \to \mathrm{AnSpec}\, \mathcal{A}$. If the pullback $f^* : \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{B}}$ is an open immersion in the sense of [CS22, Proposition 6.5], then $f \in E$ and $f_!$ is the left adjoint of f^*. Similarly, if $\mathcal{B} = \mathcal{B}_{\mathcal{A}/}$ has the induced analytic ring structure, then $f \in E$ and $f_! = f_*$ is the right adjoint of f^*.
(3) Smooth morphisms of rigid spaces are cohomologically smooth (cf. [Sch23, Definition 5.1]). For partially proper smooth rigid spaces over a point this follows from the proof of [CS22, Proposition 13.1] for complex analytic spaces. Moreover, given $f : X \to Y$ a smooth map of rigid spaces, we have that $f^! = f^! \mathscr{O}_Y \otimes f^*$ and we have a natural isomorphism $f^! \mathscr{O}_Y = \Omega^{\dim X - \dim Y}_{X/Y}[\dim X - \dim Y]$, the last equality can be proven via the same argument of [CS19, Theorem 11.6].
(4) Being cohomologically smooth is local in the target for the $\mathscr{D}$-topology (see [Sch23, Definition 4.18 (2)]), this follows from arguments analogue to those of [Man22a, Lemma 8.7 (ii)]. In particular, if $\mathbb{G}$ is a smooth rigid group over $\mathcal{K} = (K, K^+)$, and $* = \mathrm{AnSpec}\, \mathcal{K}_\blacksquare$, then $f : * \to [*/\mathbb{G}]$ is cohomologically smooth. Indeed, by definition $[*/\mathbb{G}]$ is the geometric realization of the Čech nerve $\{\mathbb{G}^n\}_{n \in \Delta^{op}}$, so that the map $* \to [*/\mathbb{G}]$ is a $\mathscr{D}$-cover and $* \times_{[*/\mathbb{G}]} * = \mathbb{G}$ which is cohomologically smooth over $*$ by (3).
(5) Being cohomologically proper is local in the target for the $\mathscr{D}$-topology, this follows from the same arguments of [Man22a, Lemma 9.8 (iii)]. In particular, if $\mathbb{G} = \mathrm{AnSpec}\, A$ is the analytic affinoid group associated to a $\mathcal{K}_\blacksquare$-algebra with the induced analytic structure, then the map $* \to [*/\mathbb{G}]$ is cohomologically proper.

In this section we do not pretend to give proper foundations of the theory of analytic stacks or the 6-functor formalism of solid quasi-coherent sheaves. Instead, we only give an example of the power of these abstract tools, and their relation with our Theorem 4.1.7 and categorical Langlands for GL_1. This section is completely independent of the rest of the paper.

Before stating the next proposition, we explain how the formalism of categorified locales of [CS22] allows us to see $\mathbb{G}_m^{an}$ and $\mathbb{G}_m^{alg}$ in the same footing. let $\mathbb{P}^{1,an}_K$ be the projective space over K with coordinates $[x, y]$ seen as a rigid space, let $0 = [0 : 1]$ and $\infty = [1 : 0]$ be marked points. Then $\mathbb{P}^{1,an}_K$ can be given a structure of categorified local as in [CS22, Definition 11.14]. We can identify $\mathbb{G}_m^{an}$ as the complement of $\{0, \infty\}$ in $\mathbb{P}^{1,an}_K$ as rigid analytic spaces. We can embed

$$j : \mathbb{G}_m^{an} \subset \mathbb{G}_m^{alg}$$

as the open subspace in the sense of categorified locales whose complement is the idempotent $K[T^{\pm 1}]$-algebra

$$C = K\{T\}[T^{-1}] \oplus K\{T^{-1}\}[T]$$

where $K\{U\} = \varinjlim_{r \to \infty} K\langle \frac{U}{p^r} \rangle$ is the algebra of germs of functions of $\mathbb{A}^{1,an}_K$ at 0, and unit map $K[T^{\pm 1}] \to C$ given by $(1, -1)$. Indeed, by [CS22, Proposition 5.3 (4)] the idempotent algebra defined by $\{0, \infty\}$ in $\mathbb{P}^{1,an}_K$ is equal to

$$D = K\{T\} \oplus K\{T^{-1}\}$$

with $T = x/y$, namely, we can write $\{0, \infty\}$ as the intersection of the union of two discs centered in 0 and ∞ and radius going to 0. By [CS22, Theorem 6.10] we have a natural isomorphism of analytic spaces $\mathbb{P}^{1,an}_K = \mathbb{P}^{1,alg}_K$ between the rigid analytic and the schematic projective spaces (in the notation of *loc. cit.* the rigid analytic and the schematic projective space correspond to $C(X, X)$ and $C(X)$ respectively). Taking pullbacks of D through the map $\mathbb{G}_m^{alg} \to \mathbb{P}^{1,alg}_K$ one obtains that $C = K\{T\}[T^{-1}] \oplus K\{T^{-1}\}[T]$ is the complement idempotent algebra of $\mathbb{G}_m^{an}$ in $\mathbb{G}_m^{alg}$ as claimed.

Proposition 4.4.8. *Let A be an animated solid $\mathcal{K}_\blacksquare$-algebra. There are natural equivalences*

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathrm{AnSpec}\, A/\mathbb{G}_m^{an}]) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathrm{AnSpec}(A \otimes_{\mathcal{K}_\blacksquare} \ell^{temp}_{\mathbb{Z},K}))$$

and

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathrm{AnSpec}\, A \times \mathbb{G}_m^{an}) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathrm{AnSpec}(A)/\mathbb{Z}^{temp}])$$

natural with respect to base change $B \otimes^L_A -$. In particular, the same statement holds for analytic spaces glued from animated $\mathcal{K}_\blacksquare$-algebras.

Proof. To simplify notation we will assume that $A = K$, the same arguments hold for general A. Let $* = \mathrm{AnSpec}\, \mathcal{K}_\blacksquare$. We start with the proof of the first equivalence. Consider the map $f : * \to [*/\mathbb{G}_m^{an}]$ of stacks obtained as the geometric realization of the morphism of simplicial analytic spaces

$$f_\bullet : (\mathbb{G}_m^{an,n+1})_{n \in \Delta^{op}} \to (\mathbb{G}_m^{an,n})_{n \in \Delta^{op}}, \tag{4.5}$$

where the map $f_n : \mathbb{G}_m^{an,n+1} \to \mathbb{G}_m^{an,n}$ is the projection towards the first n components. In particular, as $\mathbb{G}_m^{an}$ is cohomologically smooth, the map f is cohomologicaly smooth. This implies that $f^! \cong f^* \otimes f^! 1$ and $f^! 1$ invertible, which shows that f^* has a left adjoint given by $f_\natural = f_!(- \otimes f^! 1)$ (the homology). Then, f^* is a conservative functor that preserves limits and colimits and, by Barr-Beck-Lurie theorem [Lur17, Theorem 4.7.3.5], we have a natural equivalence

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/\mathbb{G}_m^{an}]) = \mathrm{Mod}_{f^* f_\natural}(\mathrm{Mod}(\mathcal{K}_\blacksquare)).$$

By the projection formula, $f^* f_\natural$ is a $\mathrm{Mod}(\mathcal{K}_\blacksquare)$-linear functor, this shows that $\mathrm{Mod}_{f^* f_\natural}(\mathrm{Mod}(\mathcal{K}_\blacksquare)) = \mathrm{Mod}_{\mathcal{K}_\blacksquare}(f^* f_\natural(K))$ by [Lur17, Theorem 4.8.4.1]. By Lemma 4.4.9 below we have that the object $f^* f_\natural(K)$ is naturally isomorphic to $\ell^{temp}_{\mathbb{Z},K}$ as Hopf algebras, and hence we obtain

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/\mathbb{G}_m^{an}]) = \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\ell^{temp}_{\mathbb{Z},K})$$

which shows the first part of the lemma.

For the second part, we consider the projection map $q : \mathbb{G}_m^{an} \to *$ and let $g : \mathbb{G}_m^{alg} \to *$ so that $q = g \circ j$, we also write $h : * \to [*/\mathbb{Z}^{temp}]$. It suffices to prove that the adjunction $q_\natural \rightleftharpoons q^*$ is co-monadic. Indeed, assuming this, by Barr-Beck-Lurie one has

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathbb{G}_m^{an}) = \mathrm{CoMod}_{q_\natural q^*}(\mathrm{Mod}(\mathcal{K}_\blacksquare)).$$

The projection formula implies that the functor $q_\natural q^*$ is $\mathrm{Mod}(\mathcal{K}_\blacksquare)$-linear so that by Lemma 4.4.9 we have

$$\mathrm{CoMod}_{q_\natural q^*}(\mathrm{Mod}(\mathcal{K}_\blacksquare)) = \mathrm{CoMod}_{\ell^{temp}_{\mathbb{Z},K}}(\mathrm{Mod}(\mathcal{K}_\blacksquare)).$$

Finally, by [Lur17, Theorem 4.7.5.2] (3) we have a natural equivalence

$$\mathrm{CoMod}_{h^* h_*}(\mathrm{Mod}(\mathcal{K}_\blacksquare)) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/\mathbb{Z}^{temp}]).$$

Indeed, the left adjointable condition is a consequence of proper base change as h is a proper map (cf. Remark 4.4.7 (5)). By projection formula and proper base change $h^* h_*$ is $\mathrm{Mod}(\mathcal{K}_\blacksquare)$-linear and one has $\mathrm{CoMod}_{h^* h_*}(\mathrm{Mod}(\mathcal{K}_\blacksquare)) = \mathrm{CoMod}_{h^* h_*(K)}(\mathrm{Mod}(\mathcal{K}_\blacksquare))$, but [Lur17, Theorem 4.7.5.2] (2) implies that $h^* h_*(K) = \ell^{temp}_{\mathbb{Z},K}$ as co-algebra, proving what we wanted.

We are left to prove co-monadicity of the adjunction $q_\natural \rightleftharpoons q^*$:

- The functor $q_\natural$ is conservative: it is (modulo a twist) the composition of the forgetful functor $g_* : \mathrm{Mod}_{\mathcal{K}_\blacksquare}(K[T^{\pm 1}]) \to \mathrm{Mod}(\mathcal{K}_\blacksquare)$ and the fully faithful inclusion $j_! : \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathbb{G}_m^{an}) \to \mathrm{Mod}_{\mathcal{K}_\blacksquare}(K[T^{\pm 1}])$.

- The functor $q_\natural$ preserves $q_\natural$-split totalizations. Since $q_\natural = q_!(- \otimes \Omega^1_{\mathbb{G}_m^{an}}[1])$, it suffices to see that $q_!$ preserves $q_!$-split totalizations. Let $M \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(K[T^{\pm 1}])$, we can write
$$q_!(j^*M) = g_*(j_!j^*M) = [K[T^{\pm 1}] \to C] \otimes^L_{K[T^{\pm 1}],\blacksquare} M,$$
and since $K[T^{\pm 1}]$ is a Hopf-algebra we have that
$$[K[T^{\pm 1}] \to C] \otimes^L_{K[T^{\pm 1}],\blacksquare} M = ([K[T^{\pm 1}] \to C] \otimes^L_{\mathcal{K}_\blacksquare} M) \otimes^L_{K[T^{\pm 1}]} K \tag{4.6}$$
where $K[T^{\pm 1}]$ acts antidiagonally in $[K[T^{\pm 1}] \to C] \otimes^L_{\mathcal{K}_\blacksquare} M$ and K is the trivial representation of $\mathbb{G}_m^{alg}$. Observe that K is a perfect $K[T^{\pm 1}]$-module by the exact sequence
$$0 \to K[T^{\pm 1}] \xrightarrow{T-1} K[T^{\pm 1}] \to K \to 0$$
and hence the functor $- \otimes^L_{K[T^{\pm 1}]} K$ commutes with limits. Let $(M_n)_{[n]\in\Delta}$ be a cosimplicial diagram in $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(K[T^{\pm 1}])$ such that $(j^*M_n)_{[n]\in\Delta}$ is $q_!$-split. Then we have
$$\begin{aligned} q_!(\varprojlim_{[n]\in\Delta} j^*M_n) &= q_!(\varprojlim_{[n]\in\Delta} j^*j_!j^*M_n) \\ &= q_!(j^* \varprojlim_{[n]\in\Delta} j_!j^*M_n) \\ &= ([K[T^{\pm 1}] \to C] \otimes^L_{\mathcal{K}_\blacksquare} \varprojlim_{[n]\in\Delta} q_!j^*M_n) \otimes^L_{K[T^{\pm 1}]} K \\ &= (\varprojlim_{[n]\in\Delta} ([K[T^{\pm 1}] \to C] \otimes^L_{\mathcal{K}_\blacksquare} q_!j^*M_n)) \otimes^L_{K[T^{\pm 1}]} K \\ &= \varprojlim_{[n]\in\Delta} ((([K[T^{\pm 1} \to C] \otimes^L_{\mathcal{K}_\blacksquare} q_!j^*M_n) \otimes^L_{K[T^{\pm 1}]} K) \\ &= \varprojlim_{[n]\in\Delta} q_!j^*M_n. \end{aligned}$$
In the first equivalence we used that $j^*j_!$ is the identity. In the second equivalence we used that j^* commute with limits being the pullback of an open immersion. In the third equality we use (4.6). The fourth equivalence follows by [Mat16, Examples 3.11 and 3.13]. The fifth follows since the functor $- \otimes^L_{K[T^{\pm 1}]} K$ commutes with limits. The last equality follows from (4.6) again.

□

Lemma 4.4.9. *Consider the cartesian square*
$$\begin{array}{ccc} \mathbb{G}_m^{an} & \xrightarrow{q} & * \\ \downarrow q & & \downarrow f \\ * & \xrightarrow{f} & [*/\mathbb{G}_m^{an}]. \end{array}$$
Then $f^*f_\natural K = q_\natural q^* K$ is canonically isomorphic to $\ell^{temp}_{\mathbb{Z},K}$ as Hopf algebras.

Proof. Let $j : \mathbb{G}_m^{an} \subset \mathbb{G}_m^{alg}$ and $g : \mathbb{G}_m^{alg} \to *$. We have that
$$\begin{aligned} f^*f_\natural(K) &= q_\natural q^*(K) \\ &= q_!(\Omega^1_{\mathbb{G}_m^{an}}[1]) \\ &\cong q_!(\mathscr{O}_{\mathbb{G}_m^{an}}[1]) \\ &= g_*(j_!(\mathscr{O}_{\mathbb{G}_m^{an}}))[1] \\ &= [K[T^{\pm 1}] \to C][1] \\ &\cong \ell^{temp}_{\mathbb{Z},K}. \end{aligned}$$
The first equality follows from proper base change. The second one follows from the identity $q_\natural = q_!(- \otimes (q^!K))$ and Remark 4.4.7 (3). The third one follows since $\Omega^1_{\mathbb{G}_m^{an}} \cong \mathscr{O}_{\mathbb{G}_m^{an}}$ by taking the differential dT/T as

a basis. The fourth one follows since $f = g \circ j$, j is an open immersion, and $\mathbb{G}_m^{alg} = \mathrm{AnSpec}\,\mathcal{K}[T^{\pm 1}]$ has the induced analytic structure from $\mathcal{K}_{\blacksquare}$, see Remark 4.4.7 (2). The fifth one follows from the formula for $j_!$ for an open immersion given in [CS22, Lecture V]. In the last isomorphism we write $[K[T^{\pm 1}] \to C][1] = TK\{T\} \bigoplus K \bigoplus T^{-1}K\{T^{-1}\}$ to identify it with $\ell^{temp}_{\mathbb{Z},K}$. This shows that $f^* f_\natural(K)$ is a solid $\mathcal{K}_{\blacksquare}$-vector space that is abstractly isomorphic to $\ell^{temp}_{\mathbb{Z},K}$, which is an LB space of compact type.

We claim moreover that they are actually naturally isomorphic. For this, by the duality of LB and Fréchet spaces of compact type (see [RJRC21, Theorem 3.40]) it suffices to see that their duals are naturally isomorphic. Indeed

$$R\underline{\mathrm{Hom}}_K(f^* f_\natural K, K) = R\underline{\mathrm{Hom}}_K(K, f^* f_* K)$$

and $f^* f_* K$ is naturally isomomorphic to $q_* q^* K = \mathscr{O}(\mathbb{G}_m^{an})$ by smooth base change [Man22a, Proposition 8.5 (ii.b)]. This shows that $f^* f_\natural K$ is naturally isomorphic to the (abelian) dual of $\mathscr{O}(\mathbb{G}_m^{an})$ which by definition is $\ell^{temp}_{\mathbb{Z},K}$.

It is left to see that the Hopf algebra structure of $f^* f_\natural K = q_\natural q^* K$ is identified with the Hopf algebra structure of $\ell^{temp}_{\mathbb{Z},K}$. The proof of this fact is probably standard but we include it for completeness. Let us start with the algebra structure. Let us write $\mathbb{G} = \mathbb{G}_m^{an}$, and consider the Čech nerve $\mathbb{G}^{\bullet}$ and $\mathbb{G}^{\bullet+1}$ of the maps $* \to [*/\mathbb{G}]$ and $\mathbb{G} \to [\mathbb{G}/\mathbb{G}]$ with respect to the left multiplication map, see Definition 4.3.1. Let $f_{\bullet} : \mathbb{G}^{\bullet+1} \to \mathbb{G}^{\bullet}$ be the natural map of simplicial spaces corresponding to the $\mathbb{G}$-equivariant map $\mathbb{G} \to *$. The boundary map $d^0_n : [n-1] \to [n]$ defines a functor $d^0_{\bullet} : \Delta^{op} \to \Delta^{op}$. Let $\mathrm{Mod}_{\blacksquare}(\mathbb{G}^{\bullet})$ be the category of quasi-coherent sheaves of the simplicial analytic space $\mathbb{G}^{\bullet}$. The pullback of $\mathbb{G}^{\bullet}$ along $d^0_{\bullet}$ is the simplicial space $\mathbb{G}^{\bullet+1}$ and the associated map $d^0_{\bullet} : \mathbb{G}^{\bullet+1} \to \mathbb{G}^{\bullet}$ is equal to $f_{\bullet}$. This shows that the co-unit $f_{\bullet,\natural} f^*_{\bullet} \to 1$ is computed in a co-cartesian section $(M_n)_{[n]\in\Delta} \in \mathrm{Mod}_{\blacksquare}(\mathbb{G}^{\bullet})$ as the co-unit

$$d^0_{\bullet,\natural} d^{0,*}_{\bullet} M_{\bullet} \to M_{\bullet}.$$

This map is adjoint to the orbit or co-multiplication map $M_{\bullet} \to d^0_{\bullet,*} d^{0,*}_{\bullet} M$. This proves that the algebra structure of $f^* f_\natural(K)$ is the dual of the coalgebra structure of $\mathscr{O}(\mathbb{G})$, which by definition is the algebra structure of $\ell^{temp}_{\mathbb{Z},K}$. We now prove that the natural isomorphism $f^* f_\natural K = \ell^{temp}_{\mathbb{Z},K} = q_\natural q^* K$ is as coalgebras, namely, it arises from the diagonal map $\mathbb{G}_m^{an} \to \mathbb{G}_m^{an} \times \mathbb{G}_m^{an}$ (equivalently, from the comonad $q_\natural q^*$), and this map is dual to the multiplication map $\mathscr{O}(\mathbb{G}_m^{an}) \otimes \mathscr{O}(\mathbb{G}_m^{an}) \to \mathscr{O}(\mathbb{G}_m^{an})$, proving what we wanted. ☐

We now show the analogue of Proposition 4.4.8 for the weight space $\mathcal{W}$.

Proposition 4.4.10. *Let A be an animated $\mathcal{K}_{\blacksquare}$-algebra. Then there are natural equivalences*

$$\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathrm{AnSpec}\,A \times \mathcal{W}) = \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}([\mathrm{AnSpec}\,A/\mathcal{O}_L^{\times,la}])$$

and

$$\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathrm{AnSpec}\,A \times \mathcal{O}_L^{\times,la}) = \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}([\mathrm{AnSpec}\,A/\mathcal{W}])$$

natural with respect to base change $B \otimes^L_A -$. In particular, the same statement hold for analytic spaces glued from animated $\mathcal{K}_{\blacksquare}$-algebras.

Proof. We just mention how to modify the main points of the proof of Proposition 4.4.8. Since $\mathcal{W}$ is a smooth group over K, the only difference with the case of $\mathbb{G}_m^{an}$ is to find a replacement for $\mathbb{G}_m^{an} \subset \mathbb{G}_m^{alg}$. Let $D = \mathcal{D}^{la}(\mathcal{O}_L^{\times}, K)$ be the distribution algebra over K. We claim that the pullback map $j^* : \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(D) \to \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{W})$ is an open localization as in [CS22, Proposition 6.5]. Indeed, by Theorems 4.1.7 and 4.3.3 j^* has a fully faithful left adjoint

$$j_! : \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{W}) \to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(D)$$

such that

$$j_! j^* M = R\underline{\mathrm{Hom}}_D(K, C^{la}(\mathcal{O}_L^{\times}) \otimes^L_{\mathcal{K}_{\blacksquare}} M) = C^{la}(\mathcal{O}_L^{\times}, K) \otimes \omega^{-1} \otimes^L_{D,\blacksquare} M. \tag{4.7}$$

where ω is a suitable dualizing sheaf. This implies that $j_!$ satisfies the projection formula and that j^* is indeed an open localization. Then, replacing $K[T^{\pm 1}]$ with D and (4.6) with (4.7) the same proof of Proposition 4.4.8 holds in this situation. ☐

Remark 4.4.11. The first equivalence of Proposition 4.4.10 for $A = K$ is Theorem 4.1.7 for $G = \mathcal{O}_L^{\times}$. It should be possible to give a proof of Theorem 4.1.7 using the more categorical approach of the Proposition 4.4.10.

Remark 4.4.12. The flatness assumption in Proposition 4.4.6 can be dropped by using the same arguments as in Proposition 4.4.8. Indeed, one considers the Cartesian square

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1} & * \\ \downarrow q & & \downarrow f \\ * & \xrightarrow{f} & [*/\mathbb{Z}], \end{array}$$

then one computes that the Hopf algebra $f^* f_\natural K$ is naturally isomorphic to $K[T^{\pm 1}]$, and that the conditions of the (co)monadicity theorem are satisfied.

We can finally move to the proof of the main result of this section.

Proof of Theorem 4.4.4. We start with the proof of the first equivalence. By Propositions 4.4.6 and 4.4.10 we have natural equivalences

$$\begin{aligned} \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathcal{W} \times \mathbb{G}_m^{alg}/\mathbb{G}_m^{alg}]) &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{W} \times \mathbb{G}_m^{alg} \times \mathbb{Z}) \\ &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathcal{W} \times \mathbb{Z}/\mathbb{Z}]) \\ &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathbb{Z}/\mathcal{O}_L^{\times, la} \times \mathbb{Z}]) \\ &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathbb{Z}/L^{\times, la}]). \end{aligned}$$

Observe that, in the third equivalence, we used that

$$\begin{aligned} \mathrm{Mod}_{\mathcal{K}_\blacksquare}([\mathcal{W} \times \mathbb{Z}/\mathbb{Z}]) &= \varprojlim_{[n]\in\Delta} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{W} \times \mathbb{Z} \times \mathbb{Z}^n) \\ &= \varprojlim_{[n]\in\Delta} \mathrm{Mod}_{\mathcal{K}_\blacksquare}([\mathbb{Z} \times \mathbb{Z}^n/\mathcal{O}_L^{\times, la}]) \\ &= \varprojlim_{[n]\in\Delta} \varprojlim_{[m]\in\Delta} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathbb{Z} \times \mathbb{Z}^n \times (\mathcal{O}_L^{\times, la})^m) \\ &= \varprojlim_{[n]\in\Delta} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathbb{Z} \times \mathbb{Z}^n \times (\mathcal{O}_L^{\times, la})^n) \\ &= \mathrm{Mod}_{\mathcal{K}_\blacksquare}([\mathbb{Z}/\mathcal{O}_L^{\times, la} \times \mathbb{Z}]). \end{aligned}$$

Analogously, Propositions 4.4.8 and 4.4.10 show that

$$\begin{aligned} \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathcal{W} \times \mathbb{G}_m^{an}/\mathbb{G}_m^{an}]) &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{W} \times \mathbb{G}_m^{an} \times \mathbb{Z}^{temp}) \\ &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathcal{W} \times \mathbb{Z}^{temp}/\mathbb{Z}^{temp}]) \\ &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathbb{Z}^{temp}/\mathcal{O}_L^{\times, la} \times \mathbb{Z}^{temp}]) \\ &= \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([\mathbb{Z}^{temp}/L^{\times, temp}]). \end{aligned}$$

This finishes the proof of the first equivalences. The second equivalences follow from the exact same arguments and Theorem 4.3.3, the fact that the functor defining the equivalence if given by $\mathfrak{LL}_p^{la}$ follows from construction and the adjunction of $j_!$ and j^* in Theorem 4.1.7. □

5. Solid smooth representations

Let G be a p-adic Lie group over a finite extension L of $\mathbb{Q}_p$ and let $\mathcal{K} = (K, K^+)$ be a complete non-archimedean field extension of L. In this section we construct the ∞-category of smooth representations of G on $\mathcal{K}_\blacksquare$-vector spaces and study its main properties.

5.1. **Solid smooth representations.** Let G be a p-adic Lie group, and let $\mathrm{Mod}(\mathcal{D}^{sm}_{\mathcal{K}_{\blacksquare}}(G,\mathcal{K}))$ be the derived (∞-)category of $\mathcal{D}^{sm}(G,K)$-modules on $\mathcal{K}_{\blacksquare}$-vector spaces. In this paragraph we will define the category of smooth representations of G on $\mathcal{K}_{\blacksquare}$-vector spaces as a suitable full subcategory of $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,\mathcal{K}))$.

Definition 5.1.1.

(1) Let $V\in \mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$, the smooth vectors of V are defined by

$$V^{sm}=\varinjlim_{H\subset G} V^H=\varinjlim_{H\subset G}\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],V)$$

where H runs over all the open compact subgroups of G. We say that V is a smooth representation of G is the natural map $V^{sm}\to V$ is an isomorphism.

(2) We let $(-)^{Rsm}:\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ be the functor of derived smooth vectors

$$V^{Rsm}=\varinjlim_{H\subset G} V^{RH}=\varinjlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],V).$$

We say that an object in $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ is smooth if the natural arrow $V^{Rsm}\to V$ is an equivalence. We let $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)\subset \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ be the full subcategory consisting of smooth objects.

Remark 5.1.2. In (1) of the previous definition we defined smooth vectors for a module over the smooth distribution algebra. One can of course give a similar definition for a solid G representation, namely, if $V\in \mathrm{Mod}^{\heartsuit}(\mathcal{K}_{\blacksquare}[G])$ one defines

$$V^{sm}=\varinjlim_{H\subset G} V^H=\varinjlim_{H\subset G}\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{K}_{\blacksquare}[G/H],V).$$

If V is in addition a $\mathcal{D}^{sm}(G,K)$ then both definitions are the same. However, at derived level it turns out that the smooth distribution algebra is better suited to define derived smooth representations, e.g., the derived smooth representations will embed fully faithfully into $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$, but not into $\mathrm{Mod}(\mathcal{K}_{\blacksquare}[G])$, see §6 for a more concrete explanation of this fact.

We start by proving some basic facts on smooth representations.

Lemma 5.1.3. *Let $V\in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$, then*

$$V^{Rsm}=\varinjlim_{H\subseteq G'} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G',K)}(\mathcal{K}_{\blacksquare}[G'/H],V)$$

for any open subgroup $G'\subseteq G$. Moreover, we have $(V^{Rsm})^{Rsm}=V^{Rsm}$. In particular, the derived category $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)\subset \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ is stable under all colimits.

Proof. Observe that, for $V\in \mathrm{Mod}(\mathcal{D}^{sm}(G,K))$, $G'\subseteq G$ an open subgroup, $G_0\subseteq G'$ open compact and $H\subseteq G_0$ an open compact, since $\mathcal{K}_{\blacksquare}[G'/H]=\mathcal{D}^{sm}(G',K)\otimes^L_{\mathcal{D}^{sm}(G_0,K)}\mathcal{K}_{\blacksquare}[G_0/H]$, by a base change we have

$$\varinjlim_{H\subset G'} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G',K)}(\mathcal{K}_{\blacksquare}[G'/H],V)=\varinjlim_{H\subset G_0} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{K}_{\blacksquare}[G_0/H],V).$$

This shows the first claim. For the second one, let $G_0\subseteq G$ be a compact open subgroup. Then we have

$$\begin{aligned}(V^{Rsm})^{Rsm}&=\varinjlim_{H\subset G_0} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0)}(\mathcal{K}_{\blacksquare}[G_0/H],\varinjlim_{H'\subset G_0} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{K}_{\blacksquare}[G_0/H],V))\\ &=\varinjlim_{H\subset G_0}\varinjlim_{H'\subset G_0} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0)}(\mathcal{K}_{\blacksquare}[G_0/H],R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{K}_{\blacksquare}[G_0/H],V))\\ &=\varinjlim_{H\subset G_0} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0)}(\mathcal{K}_{\blacksquare}[G_0/H]\otimes^L_{\mathcal{D}^{sm}(G_0,K)}\mathcal{K}_{\blacksquare}[G_0/H],V)\\ &=\varinjlim_{H\subset G_0} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0)}(\mathcal{K}_{\blacksquare}[G_0/H],V)\\ &=V^{Rsm},\end{aligned}$$

where the first and last equalities follow from definition, the second one from the fact that $\mathcal{K}_{\blacksquare}[G_0/H]$ is a compact $\mathcal{D}^{sm}(G_0,K)$-module and the third one follows since $\mathcal{K}_{\blacksquare}[G_0/H]$ is idempotent over $\mathcal{D}^{sm}(G_0,K)$ (cf. Corollary 2.2.6).

Finally, for the last statement, let $\{V_i\}_{i\in I}$ be a colimit diagram of smooth representations, to check that $\varinjlim_i V_i$ is smooth we can restrict to G compact, in this case we have that

$$\begin{aligned}(\varinjlim_i V_i)^{Rsm} &= \varinjlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],\varinjlim_i V_i)\\ &= \varinjlim_i \varinjlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],V_i)\\ &= \varinjlim_i V_i^{Rsm} = \varinjlim_i V_i,\end{aligned}$$

where in the second equality we used again the compacity of the $\mathcal{D}^{sm}(G_0,K)$-module $\mathcal{K}_{\blacksquare}[G_0/H]$ □

The following two lemmas describe the smooth vectors in a similar way as we have previously defined continuous and loclaly analytic vectors (cf. [RJRC21]).

Lemma 5.1.4. *The functor $V\mapsto C^{sm}(G,V)$ of smooth functions induces an exact functor of derived categories*

$$C^{sm}(G,-):\mathrm{Mod}(\mathcal{K}_{\blacksquare}[G])\to \mathrm{Mod}(\mathcal{K}_{\blacksquare}[G^3])$$

and

$$C^{sm}(G,-):\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G^3,K))$$

where (g_1,g_2,g_3) acts on a function $f:G\to V$ by $((g_1,g_2,g_2)\cdot f)(h)=g_3f(g_1^{-1}hg_2)$.

Proof. Let $V\in \mathrm{Mod}^{\heartsuit}(\mathcal{K}_{\blacksquare})$. If G_0 is compact we have that $C^{sm}(G_0,V)=\varinjlim_{H\subset G_0}\mathrm{Hom}_K(\mathcal{K}_{\blacksquare}[G_0/H],V)$. One deduces that the functor $V\mapsto C^{sm}(G_0,V)$ is exact and that it is a $\mathcal{D}^{sm}(G,K)$-module for the left and right regular actions. This implies the lemma for $G=G_0$ compact. For general G and $V\in\mathrm{Mod}^{\heartsuit}(\mathcal{K}_{\blacksquare})$, by definition we have that $C^{sm}(G,V)=\prod_{g\in G/G_0}C^{sm}(gG_0,V)=\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{D}^{sm}(G,K),C^{sm}(G_0,V))$ for both the left or right regular action of G_0 on $C^{sm}(G_0,V)$. Therefore the functor $V\mapsto C^{sm}(G,V)$ is exact and the left and right regular actions of G are upgraded to left and right regular actions of $\mathcal{D}^{sm}(G,K)$, proving the lemma. □

Lemma 5.1.5. *Let $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$. Then, for any open subgroup $G'\subseteq G$ we have*

$$V^{Rsm}=R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G',K)}(K,C^{sm}(G',V)_{\star 1,3}).$$

Proof. We start by proving the result for a compact subgroup. Let now $G_0\subset G$ be a compact open and let $V\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G_0,K))$. We recall that we have

$$C^{sm}(G_0,V)=\varinjlim_{H\subset G_0}C(G_0/H,V)=\varinjlim_{H\subset G_0}R\underline{\mathrm{Hom}}_K(\mathcal{K}_{\blacksquare}[G_0/H],V) \tag{5.1}$$

where H runs over all the normal open compact subgroups. Notice that the $\star_{1,3}$ on the LHS translates to the contragradient action of the RHS (heuristically we have $g\cdot f(x)=gf(g^{-1}x)$ for $f\in R\underline{\mathrm{Hom}}_{\mathcal{K}}(\mathcal{K}_{\blacksquare}[G_0/H],V)$ and $x\in\mathcal{K}_{\blacksquare}[G_0/H]$). Taking G_0-invariants in Equation (5.1) (cf. Proposition 1.0.6 (4)) and since K is a direct summand of $\mathcal{D}^{sm}(G_0,K)$, we obtain

$$\begin{aligned}R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(K,C^{sm}(G_0,V)_{\star1,3}) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(K,\varinjlim_{H\subset G_0}R\underline{\mathrm{Hom}}_{\mathcal{K}}(\mathcal{K}_{\blacksquare}[G_0/H],V))\\ &= \varinjlim_{H\subset G_0}R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(K,R\underline{\mathrm{Hom}}_{\mathcal{K}}(\mathcal{K}_{\blacksquare}[G_0/H],V))\\ &= \varinjlim_{H\subset G_0}R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{K}_{\blacksquare}[G_0/H],V)\\ &= V^{Rsm}.\end{aligned}$$

Let now $G'\subseteq G$ be an open subgroup and $G_0\subset G$ be an open compact subgroup. Without loss of generality we can assume $G'=G$. First observe that for $V\in\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ we have a natural isomorphism

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{D}^{sm}(G,K),C^{sm}(G_0,V)_{\star_{1,3}}) &= \underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\bigoplus_{g\in G_0\backslash G}\mathcal{D}^{sm}(G_0,K)\cdot g, C^{sm}(G_0,V)) \\ &= \prod_{g\in G_0\backslash G}\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{D}^{sm}(G_0,K)\cdot g, C^{sm}(G_0,V)) \\ &= \prod_{g\in G_0\backslash G} C^{sm}(G_0 g, V) = C^{sm}(G,V)_{\star_{1,3}}\end{aligned}$$

where the G-action on the first term is induced by the right action on $\mathcal{D}^{sm}(G,K)$. The inverse $C^{sm}(G,V)_{\star_{1,3}} \to \underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{D}^{sm}(G,K),C^{sm}(G_0,V)_{\star_{1,3}})$ is given by sending a smooth function $f: G\to V$ to the map $\tilde{f}: G\to C^{sm}(G_0,V)$ given by $\tilde{f}(g) = (g\star_{1,3} f)|_{G_0}$. We deduce a natural equivalence

$$C^{sm}(G,V)_{\star_{1,3}} \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{D}^{sm}(G,K),C^{sm}(G_0,V)_{\star_{1,3}})$$

for all $V\in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$, so that

$$\begin{aligned}R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K,C^{sm}(G,V)_{\star_{1,3}}) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K, R\mathrm{Hom}_{\mathcal{D}^{sm}(G_0,K)}(\mathcal{D}^{sm}(G,K),C^{sm}(G_0,V)_{\star_{1,3}})) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_0,K)}(K,C^{sm}(G_0,V)_{\star_{1,3}})\end{aligned}$$

proving the statement. ☐

Lemma 5.1.6. *Let $V\in\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$, then $H^i(V)^{sm} = H^i(V^{Rsm})$ for all $i\in\mathbb{Z}$, i.e., taking smooth vectors is exact in the abelian category of solid $\mathcal{D}^{sm}(G,K)$-modules.*

Proof. Taking smooth vectors is independent of the open subgroup of G, so we can assume that G is compact. In this case we can write $V^{Rsm} = \varinjlim_{H\subset G} R\underline{\mathrm{Hom}}_G(\mathcal{K}_\blacksquare[G/H],V)$ where H runs over all the normal open compact subgroups of G, but $\mathcal{K}_\blacksquare[G/H]$ is a projective $\mathcal{D}^{sm}(G,K)$-algebra, the lemma follows since taking filtered colimits is exact. ☐

Proposition 5.1.7. *An object $V\in\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$ is smooth if and only if $H^i(V)$ is smooth for all $i\in\mathbb{Z}$. Therefore, the natural t-structure of $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$ induces a t-structure on $\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G)$. Moreover, $\mathrm{Rep}^{sm,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$ is a Grothendieck abelian category and $\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G)$ is the derived category of its heart.*

Proof. An object $V\in\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$ is smooth if and only if the natural map $V^{Rsm}\to V$ is an equivalence if and only if $H^i(V)^{sm} = H^i(V^{Rsm}) = H^i(V)$ for all $i\in\mathbb{Z}$. the fact that the category $\mathrm{Mod}^{sm,\heartsuit}(G,\mathcal{K}_\blacksquare)$ is an abelian Grothendieck is clear, cf. [Man22b, Lemma 3.4.10]. Note that a system of generators of the category is given by the objects $\mathcal{K}_\blacksquare[G/H]\otimes_{\mathcal{K}_\blacksquare}\mathcal{K}_\blacksquare[S]$ where H runs over the open compact subgroups of G and S over the (κ-small) profinite sets. Let $\mathscr{C}$ be the derived category of $Rep^{sm,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$. By [Lur17, Proposition 1.3.3.7] we have a natural morphism $\mathscr{C}\to\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G)$. To prove that this is an equivalence it suffices to show that for $V,W\in\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$ smooth representations we have that

$$R\mathrm{Hom}_{\mathcal{C}}(V,W) = R\mathrm{Hom}_{\mathcal{D}^{sm}(G,K)}(V,W).$$

Let $I^\bullet$ be an injective resolution of W as $\mathcal{D}^{sm}(G,K)$-modules, then $I^{\bullet,Rsm} = I^{\bullet,sm}$ is an injective resolution of W in $\mathscr{C}^\heartsuit = \mathrm{Rep}^{sm,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$. We have that

$$\begin{aligned}R\mathrm{Hom}_{\mathcal{D}^{sm}(G,K)}(V,W) &= \mathrm{Hom}_{\mathcal{D}^{sm}(G,K)}(V,I^\bullet) \\ &= \mathrm{Hom}_{\mathcal{D}^{sm}(G,K)}(V,I^{\bullet,sm}) \\ &= \mathrm{Hom}_{\mathcal{C}^\heartsuit}(V,I^{\bullet,sm}) \\ &= R\mathrm{Hom}_{\mathcal{C}}(V,W),\end{aligned}$$

finishing the proof of the result. ☐

Proposition 5.1.8. *The inclusion $\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G)\to\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K))$ has a right adjoint given by the smooth vectors functor $V\mapsto V^{Rsm}$.*

Proof. Assume first that G is compact. Let V be a smooth representation of G and $W \in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$, then

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V,W) &= \varprojlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V^{RH},W) \\ &= \varprojlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H]\otimes^L_{\mathcal{D}^{sm}(G,K)} V^{RH},W) \\ &= \varprojlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V^{RH},R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],W)) \\ &= \varprojlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V^{RH},W^{RH}) \\ &= \varprojlim_{H\subset G} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V^{RH},(W^{Rsm})^{RH}) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V,W^{Rsm}), \end{aligned}$$

where in the first equality we used the fact that V is (derived) smooth, the second follows from the fact that V^{RH} is a $\mathcal{K}[G/H]$-module and that $\mathcal{K}_{\blacksquare}[G/H]$ is idempotent over $\mathcal{D}^{sm}(G,K)$ (cf. Corollary 2.1.6), the third equality follows by adjunction, the fourth by definition, the fifth one is obvious, and the last one follows by applying all the first four equalities in a reverse order with W replaced by W^{Rsm}.

Let G be a general p-adic Lie group and let $V \in \mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$ and $W \in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$. It suffices to show the adjunction at the level of abelian categories (cf. [Sta22, Tag 0FNC]), so we can assume both V and W to be in degree 0. Moreover, since by Proposition 5.1.7 the abelian category of smooth representations is generated by $\mathcal{K}_{\blacksquare}[G/H]\otimes_{\mathcal{K}_{\blacksquare}}\mathcal{K}_{\blacksquare}[S]$ for $H\subset G$ open compact and S profinite, we can assume $V=\mathcal{K}_{\blacksquare}[G/H]\otimes_{\mathcal{K}_{\blacksquare}}\mathcal{K}_{\blacksquare}[S]$. Moreover, since we are computing the internal $\underline{\mathrm{Hom}}$ we can even assume that $V=\mathcal{K}_{\blacksquare}[G/H]$. But then we have that $R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],W)=W^{RH}=W^H$ are the H-invariant vectors which coincide with the H-invariant vectors of W^{sm}, i.e.

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],W)=R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],W^{sm})$$

proving what we wanted. □

5.2. **Smooth representations as quasi-coherent $\mathcal{D}^{sm}(G,K)$-modules.** In this section we will give two alternative descriptions of the categories of solid smooth representations which are the analogue of those appearing in Corollary 4.3.5.

Definition 5.2.1. Let G be a compact p-adic Lie group, we define the category of solid quasi-coherent modules over $\mathcal{D}^{sm}(G,K)$ as

$$\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))=\varprojlim_{H\subset G}\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G/H,K)),$$

where H runs over all the normal open subgroups and the transision maps are base changes. We let $j^*:\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))\to \mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ be the pullback functor $j^*W=(\mathcal{D}^{sm}(G/H)\otimes^L_{\mathcal{D}^{sm}(G)}W)_H$.

Proposition 5.2.2. *The pullback functor $j^*:\mathrm{Mod}_{\mathcal{K}}(\mathcal{D}^{sm}(G,K))\to\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ has a right adjoint $j_*(V_H)_H=R\varprojlim_H V_H$ and a left adjoint $j_!V_H=(j_*V)^{Rsm}$. Furthermore, $j^*j_*V=j^*j_!V=V$ for $V\in\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ and $j_!j^*W=W^{Rsm}$ for $W\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$. The functor is a fully faithful embedding with essential image $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$.*

Proof. Let $V=(V_H)_H\in\mathrm{Mod}^{qc}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$ and $W\in\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$. One has

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(W,j_*V) &= R\varprojlim_H R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(W,V_H) \\ &= R\varprojlim_H R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K[G/H]\otimes^L_{\mathcal{D}^{sm}(G,K)}W,V_H) \end{aligned}$$

where H runs over open compact subgroups of G, proving that j_*V is the right adjoint of j^*. The other statements of the proposition follow easily by unraveling the definitions, $\otimes$-$\underline{\mathrm{Hom}}$ adjunction and using the fact that $K[G/H]$ is a direct summand of $\mathcal{D}^{sm}(G,K)$, so in particular compact and dualizable. □

5.3. **Smooth dualizing functors.** The following result answers a question raised by Schneider and Teitelbaum in [ST05, p. 26] on the extension of the contragradient functor for smooth representation to the category of locally analytic representations.

Proposition 5.3.1. *Let $V \in \mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G)$. Then*

$$(V^\vee)^{Rsm} = (V^\vee)^{Rla}.$$

In other words, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G) & \longrightarrow & \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G) \\ \downarrow{\scriptstyle ((-)^\vee)^{Rsm}} & & \downarrow{\scriptstyle ((-)^\vee)^{Rla}} \\ \mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G) & \longrightarrow & \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G). \end{array}$$

Proof. This is a consequence of Corollary 4.2.9 and the analogous calculation for smooth representations, which follow from [ST05, Corollary 3.7]. Indeed these statements assert that both functors are given by the same duality functor in the category $\mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{la}(G,K))$. But we give a direct proof. We can and do assume that G is compact, or even a uniform pro-p-group. We have

$$\begin{aligned}(V^\vee)^{Rla} &= \varinjlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(\mathcal{D}^h(G,K), R\underline{\mathrm{Hom}}_K(V,K)) \\ &= \varinjlim_h R\underline{\mathrm{Hom}}_K(\mathcal{D}^h(G,K)\otimes_{\mathcal{D}^{la}(G,K)}\mathcal{D}^{sm}(G,K)\otimes_{\mathcal{D}^{sm}(G,K)} V, K) \\ &= \varinjlim_h R\underline{\mathrm{Hom}}_K(K[G/\mathbb{G}^{(h^+)}(L)]\otimes_{\mathcal{D}^{sm}(G,K)} V, K) \\ &= \varinjlim_h R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K[G/\mathbb{G}^{(h^+)}(L)], V^\vee) \\ &= (V^\vee)^{Rsm},\end{aligned}$$

where the first, second and fourth equalities follow from definition and adjunction, and the third one follows from the equality $\mathcal{D}^h(G,K)\otimes_{\mathcal{D}^{la}(G,K)}\mathcal{D}^{sm}(G,K) = K[G/\mathbb{G}^{(h^+)}(L)]$ of Lemma 2.2.7. The fifth one follows since the groups $\mathbb{G}^{(h^+)}(L)$ form a cofinal system of open neighbourhoods of the identity in G. □

5.4. **Smooth representations as comodules over $C^{sm}(G,K)$.** We now explain the analogue equivalence of Theorem 4.3.3 for smooth representations.

Definition 5.4.1. Let G be a p-adic Lie group and $G_0 \subset G$ an open compact subgroup. We let

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(G^{sm}) = \prod_{g\in G/G_0} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(C^{sm}(gG_0,K)).$$

We define the quasi-coherent modules of $[*/G^{sm}]$ to be

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{sm}]) = R\varprojlim_{n\in\Delta} \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(G^{n,sm}).$$

Proposition 5.4.2. *There is a natural equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{sm}]).$$

In particular, if G is compact, we have natural equivalences of stable ∞-categories

$$\mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}(\mathcal{D}^{sm}(G,K)) = \mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}^{qc}_{\mathcal{K}_\blacksquare}([*/G^{sm}]).$$

Proof. This follows by the same proof of Theorem 4.3.3, the only thing to verify is that the abelian category of smooth representations is naturally equivalent to the abelian category of comodules $V \to C^{sm}(G,V)$, which is obvious. □

5.5. **Locally algebraic representations of reductive groups.** In this last section we introduce a category of solid locally algebraic representations for the L-points of a reductive group $\mathbf{G}/L$. Let $C^{\mathrm{alg}}(\mathbf{G},K)$ be the ring of algebraic functions of $\mathbf{G}$, i.e., the global sections of the affine group scheme $\mathbf{G}_K$. For $G_0\subset \mathbf{G}(L)$ a compact open subgroup we define the space of locally algebraic functions of G_0 (relative to $\mathbf{G}$) to be

$$C^{\mathrm{lalg}}(G_0,K)=C^{sm}(G_0,K)\otimes_K C^{\mathrm{alg}}(\mathbf{G},K).$$

We let $\mathcal{D}^{\mathrm{lalg}}(G_0,K)=\underline{\mathrm{Hom}}_K(C^{\mathrm{lalg}}(G_0,K),K)$ be the locally algebraic distribution algebra of G_0 and for any $G_0\subset G\subset \mathbf{G}(L)$ an open subgroup we denote

$$\mathcal{D}^{\mathrm{lalg}}(G,K):=\mathcal{K}_{\blacksquare}[G]\otimes_{\mathcal{K}_{\blacksquare}[G_0]}\mathcal{D}^{\mathrm{lalg}}(G_0,K)$$

the locally algebraic distribution algebra functions of G.

Definition 5.5.1. Let $V\in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$.

1. We let $C^{\mathrm{lalg}}(G,V):=\prod_{g\in G/G_0}(C^{\mathrm{lalg}}(gG_0,K)\otimes^L_{\mathcal{K}_{\blacksquare}}V)$ be the space of locally algebraic functions of G with values in V. The space $C^{\mathrm{lalg}}(G,V)$ has three commuting actions of $\mathcal{D}^{\mathrm{lalg}}(G,K)$ given by the left $\star_1$ and right $\star_2$ regular actions, and the action $\star_3$ on V.
2. Define the functor of locally algebraic vectors $(-)^{R\,\mathrm{lalg}}:\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$ to be
$$V^{R\,\mathrm{lalg}}:=R\underline{\mathrm{Hom}}_{\mathcal{D}^{\mathrm{lalg}}(G,K)}(K,C^{\mathrm{lalg}}(G,V)_{\star_{1,3}})$$
endowed with the $\star_2$-action of $\mathcal{D}^{\mathrm{lalg}}(G,K)$.
3. We say that an object $V\in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$ is locally algebraic if the natural map $V^{R\,\mathrm{lalg}}\to V$ is an equivalence. We let $\mathrm{Rep}^{\mathrm{lalg}}_{\mathcal{K}_{\blacksquare}}(G)\subset \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$ be the full subcategory of locally algebraic functions.

Lemma 5.5.2. *Let G be a compact open subgroup of $\mathbf{G}(L)$. We have natural isomorphisms of $\mathcal{D}^{\mathrm{lalg}}(G^2,K)$-modules (for the actions $\star_1$ and $\star_2$)*

$$C^{\mathrm{lalg}}(G,K)=\bigoplus_{\pi,\lambda}(\pi\otimes V^{\lambda})_{\star_1}\otimes(\pi\otimes V^{\lambda})^{\vee}_{\star_2}$$

and

$$\mathcal{D}^{\mathrm{lalg}}(G,K):=\prod_{\pi,\lambda}(\pi\otimes V^{\lambda})^{\vee}_{\star_1}\otimes(\pi\otimes V^{\lambda})_{\star_2}$$

where π runs over all the smooth irreducible representations of G, and V^{λ} over all the irreducible representations of $\mathbf{G}$.

Proof. This follows from Lemma 2.2.6 and [GW09, Theorem 4.2.7] describing the algebra of functions of $\mathbf{G}$ in terms of irreducible representations. □

Proposition 5.5.3. *The following assertions hold.*

1. *Let $V\in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$, the natural map $(V^{R\,\mathrm{lalg}})^{R\,\mathrm{lalg}}\to V^{R\,\mathrm{lalg}}$ is an equivalence.*
2. *The functor $(-)^{R\,\mathrm{lalg}}$ commute with colimits.*
3. *Let $V,W\in \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$, then $(V^{R\,\mathrm{lalg}}\otimes^L_{\mathcal{K}_{\blacksquare}}W)^{R\,\mathrm{lalg}}=V^{R\,\mathrm{lalg}}\otimes^L_{\mathcal{K}_{\blacksquare}}W^{R\,\mathrm{lalg}}$. In particular, $\mathrm{Rep}^{\mathrm{lalg}}_{\mathcal{K}_{\blacksquare}}(G)$ has a natural symmetric monoidal structure.*
4. *The functor $(-)^{R\,\mathrm{lalg}}$ is the right adjoint of the inclusion $\mathrm{Rep}^{\mathrm{lalg}}_{\mathcal{K}_{\blacksquare}}(G)\subset \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$.*
5. *The functor $(-)^{R\,\mathrm{lalg}}$ is exact in the abelian category $\mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{\mathrm{lalg}}(G,K))$. In particular, $\mathrm{Rep}^{\mathrm{lalg}}_{\mathcal{K}_{\blacksquare}}(G)$ has a natural t-structure.*
6. *The ∞-category $\mathrm{Rep}^{\mathrm{lalg}}_{\mathcal{K}_{\blacksquare}}(G,K)$ is the derived category of its heart.*

Proof. This follows the same arguments of Propositions 3.2.3, 3.2.5 and 3.2.6 in the locally analytic case, or the Propositions 5.1.7 and 5.1.8 in the smooth case. We give a sketch for completeness. Let $G_0\subset G$ be an open compact subgroup, by adjunction we have that

$$W^{R\,\mathrm{lalg}}=R\underline{\mathrm{Hom}}_{\mathcal{D}^{\mathrm{lalg}}(G,K)}(K,C^{\mathrm{lalg}}(G,W))=R\underline{\mathrm{Hom}}_{\mathcal{D}^{\mathrm{lalg}}(G_0,K)}(K,C^{\mathrm{lalg}}(G_0,W)),$$

then for (1)-(3) and (5) we can assume that G is compact. By Lemma 5.5.2 any finite dimensional representation of G is a direct summand of $\mathcal{D}^{\mathrm{lalg}}(G,K)$, in particular they are projective. This implies that

$(-)^{R\,\mathrm{lalg}}$ is an exact functor in the abelian category and that it commutes with colimits. Moreover, we have that

$$\begin{aligned} W^{R\,\mathrm{lalg}} &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{\mathrm{lalg}}(G,K)}(K, (C^{\mathrm{lalg}}(G,K) \otimes^L_{\mathcal{K}_\blacksquare} W)_{\star 1,3}) \\ &= \varinjlim_{\pi,\lambda} R\underline{\mathrm{Hom}}_{\mathcal{D}^{\mathrm{lalg}}(G,K)}(K, (\pi \otimes V^\lambda) \otimes ((\pi \otimes V^\lambda)^\vee \otimes W)_{\star 1,3}) \\ &= \varinjlim_{\pi,\lambda} R\underline{\mathrm{Hom}}_{\mathcal{D}^{\mathrm{lalg}}(G,K)}((\pi \otimes V^\lambda)^\vee, W) \otimes (\pi \otimes V^\lambda)^\vee. \end{aligned}$$

Then, to prove that the functor $(-)^{\mathrm{lalg}}$ is idempotent it suffices to prove it for the representations of the form $W = (\pi \otimes V^\lambda)^\vee$, which follows from the previous formula and the irreducibility and projectiveness of $\pi \otimes V^\lambda$ as $\mathcal{D}^{\mathrm{lalg}}(G,K)$-modules. So far we have proven parts (1), (2) and (5). For part (3) we can assume that $W = C^{\mathrm{lalg}}(G,K)$ in which case we can untwist the diagonal action of $C^{\mathrm{lalg}}(G,K) \otimes C^{\mathrm{lalg}}(G,K) \otimes V$ to a representation where $\mathcal{D}^{\mathrm{lalg}}(G,K)$ acts trivially on the first factor. Taking invariants by $\mathcal{D}^{\mathrm{lalg}}(G,K)$ one gets that

$$(C^{\mathrm{lalg}}(G,K) \otimes W)^{R\,\mathrm{lalg}} = C^{\mathrm{lalg}}(G,K) \otimes W^{R\,\mathrm{lalg}}.$$

Parts (4) and (6) follow the same lines of their analogues for smooth representations, see Propositions 5.1.7 and 5.1.8. □

6. Adjunctions and cohomology

In this final section, we show how the cohomology comparison theorems of [RJRC21, §5.2] are explained in terms of adjunctions.

6.1. **Geometric solid representations.** Following the interpretation of the categories of locally analytic and smooth representations as quasi-coherent sheaves of "classifying stacks of G^{la} and G^{sm}", one can introduce a different category of "continuous geometric" representations where now G is the analytic space defined by the algebra of its continuous functions.

Definition 6.1.1. Let $G_0 \subset G$ be an open compact subgroup.

(1) Let $V \in \mathrm{Mod}(\mathcal{K}_\blacksquare)$, we define the space of "geometric continuous" functions of G on V to be
$$C^{geo}(G,V) := \prod_{g \in G/G_0} (C(gG_0,K) \otimes^L_{\mathcal{K}_\blacksquare} V).$$
(2) We define the category of quasi-coherent sheaves of the underlying profinite group G^{prof} to be $\mathrm{Mod}^{qc}(G^{prof}) = \prod_{g\in G/G_0} \mathrm{Mod}_{\mathcal{K}_\blacksquare}(C(gG_0,K))$.
(3) We define the category of "continuous geometric" representations of G to be the simplicial limit
$$\mathrm{Rep}^{geo}_{\mathcal{K}_\blacksquare}(G) := \mathrm{Mod}^{qc}([*/G^{prof}]) := \varprojlim_{n\in\Delta} \mathrm{Mod}^{qc}(G^{prof,n}).$$

Lemma 6.1.2. *Let $V \in \mathrm{Mod}^\heartsuit(\mathcal{K}_\blacksquare)$ and S a profinite set. Then the natural map $C(S,K) \otimes_{\mathcal{K}_\blacksquare} V \to C(S,V)$ is an injection.*

Proof. It is enough to take $K = \mathbb{Q}_p$. Since any solid $\mathbb{Q}_p$-vector space is a colimit of quotients of compact projective $\mathbb{Q}_{p,\blacksquare}$-vector spaces, we can assume that V fits in a short exact sequence $0 \to \mathbb{Q}_{p,\blacksquare}[S] \to \mathbb{Q}_{p,\blacksquare}[S'] \to V \to 0$. Taking lattices $0 \to \mathbb{Z}_p[S] \to \mathbb{Z}_p[S'] \to Q \to 0$ (after rescaling if necessary), it suffices to show that the map
$$C(S,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} Q \to C(S,Q)$$
is injective. But both objects are p-adically complete, so it suffices to show that their reduction modulo p^n are injective, i.e. that we have monomorphisms
$$C^{sm}(S,Q/p^n) \to C(S,Q/p^n).$$
This is Lemma 3.4.8 (iii) of [Man22b]. □

Lemma 6.1.3. *Let $\mathcal{A}$ be the category of comodules $V \to C^{geo}(G,V)$ with $V \in \mathrm{Mod}^\heartsuit(\mathcal{K}_\blacksquare)$. Then $\mathcal{A}$ is a Grothendieck abelian full subcategory of $\mathrm{Mod}^\heartsuit_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$ with derived ∞-category naturally equivalent to $\mathrm{Rep}^{geo}_{\mathcal{K}_\blacksquare}(G)$.*

Proof. The fact that $\mathcal{A}$ is an abelian category follows from the fact that $V \mapsto C^{geo}(V,K)$ is an exact functor. We have a natural functor $\mathcal{A} \to \mathrm{Mod}^{\heartsuit}_{\mathcal{K}_{\blacksquare}}(\mathcal{K}_{\blacksquare}[G])$ sending the comodule V to the representation defined by the orbit map $V \to C^{geo}(G,V) \to C(G,V)$. It is clear that for $V,W \in \mathcal{A}$ one has $\mathrm{Hom}_{\mathcal{A}}(V,W) \subset \mathrm{Hom}_{\mathcal{K}_{\blacksquare}[G]}(V,W)$. Conversely, let $f: V \to W$ be a morphism of $\mathcal{K}_{\blacksquare}[G]$-modules. We have a diagram whose lower square is commutative

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \downarrow & & \downarrow \\ C^{geo}(G,V) & \longrightarrow & C^{geo}(G,V) \\ \downarrow & & \downarrow \\ C(G,V) & \longrightarrow & C(G,W) \end{array}$$

and the such that lower vertical arrows are injective by Lemma 6.1.2, then the upper square must be commutative proving that $\mathrm{Hom}_{\mathcal{A}}(V,W) = \mathrm{Hom}_{\mathcal{K}_{\blacksquare}[G]}(V,W)$.

To prove that $\mathcal{A}$ is a Grothendieck abelian category, it is left to show that $\mathcal{A}$ has enough compact generators. Using [Man22b, Proposition A.1.2], one deduces that $\mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}(G)$ is the derived category of $\mathcal{A}$. Let $V \in \mathcal{A}$, the orbit map gives a G-equivariant injection

$$V \to C^{geo}(G,V)$$

for the $\star_2$-action. Then, writing V as a colimit of quotients $Q = \mathrm{coker}(\mathcal{K}_{\blacksquare}[S] \to \mathcal{K}_{\blacksquare}[S'])$ of compact projective generators, one sees that a family of generators are the subobjects of $C^{geo}(G,Q)$ for Q as before. $\square$

We have a natural morphism of coalgebras $C^{la}(G,K) \to C(G,K)$ which heuristically should induce a group homomorphism $G^{prof} \to G^{la}$ and as consequence a morphism of their classifying stacks $f: [*/G^{prof}] \to [*/G^{la}]$. We can define a pullback functor $f^*: \mathrm{Mod}^{qc}([*/G^{la}]) \to \mathrm{Mod}^{qc}([*/G^{prof}])$ which corresponds to a forgetful functor $F: \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}(G)$ sending the co-module $V \to C^{la}(G,V)$ to the co-module $V \to C^{la}(G,V) \to C^{geo}(G,V)$. The functor f^* preserves colimits, so it admits a right adjoint that we can call the pushforward $f_*: \mathrm{Mod}^{qc}([*/G^{prof}]) \to \mathrm{Mod}^{qc}([*/G^{la}])$. At the level of representations we can think of f_* as a locally analytic vectors functor $(-)^{Rla}: \mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$.

Definition 6.1.4. We define the "continuous geometric" cohomology $R\Gamma^{geo}(G,-): \mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}(G) \to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$ to be the right adjoint of the trivial representation functor $\mathrm{Mod}(\mathcal{K}_{\blacksquare}) \to \mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}(G)$.

We have the following proposition.

Proposition 6.1.5. *The forgetful functor $f^*: \mathrm{Mod}^{qc}([*/G^{la}]) \to \mathrm{Mod}^{qc}([*/G^{prof}])$ is fully faithful. The right adjoint of f^* on a geometric representation V can be computed as*

$$f_*V = R\Gamma^{geo}(G, C^{la}(G,V)_{\star_{1,3}}).$$

Proof. By Lemma 4.3.4 the category $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G)$ is the derived category of comodules of the functor $C^{la}(G,-)$. Similarly, by Lemma 6.1.3 the category $\mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}(G)$ is the derived category of the abelian category of co-modules of $C^{geo}(G,-)$. Moreover, we have fully faithful inclusion of abelian categories $\mathrm{Rep}^{la,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G) \subset \mathrm{Rep}^{geo,\heartsuit}_{\mathcal{K}_{\blacksquare}}(G) \subset \mathrm{Mod}^{\heartsuit}(\mathcal{K}_{\blacksquare}[G])$. This implies that the right adjoint of the first inclusion is given by the locally anlaytic vectors functor that can be computed as $C^{la}(G,V)^{G_{\star_{1,3}}}$. Taking right derived functors we see that $f_*V = R\Gamma^{geo}(G, C^{la}(G,V))$ for any $V \in \mathrm{Rep}^{geo}_{\mathcal{K}_{\blacksquare}}[G]$.

It is left to show that the unit map $1 \to f_*f^*$ is an equivalence. Let $G_0 \subset G$ be a compact open subgroup, we have a commutative diagram of morphisms of stacks

$$\begin{array}{ccc} [*/G_0^{prof}] & \xrightarrow{\tilde{f}} & [*/G_0^{la}] \\ \big\downarrow{\scriptstyle \tilde{g}} & & \big\downarrow{\scriptstyle g} \\ [*/G^{prof}] & \xrightarrow{f} & [*/G^{la}] \end{array}$$

The pullback functors correspond to forgetful functors, and the vertical pushforward functions are given by inductions. Indeed, we can check this at the level of abelian categories where the right adjoint of a forgetful functor is clearly an induction. As a consequence one deduces that

$$R\Gamma^{geo}(G, C^{la}(G,V)) = R\Gamma^{geo}(G_0, C^{la}(G_0, V)).$$

Thus, we can assume without loss of generality that G is compact. In this case $C^{la}(G,V) = C^{la}(G,K) \otimes^L_{\mathcal{K}} V$ and $C^{geo}(G,V) = C(G,K) \otimes^L_{\mathcal{K}} V$.

Notice that for $V \in \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G)$ we have a natural equivalence of representations $C^{la}(G,V)_{\star_{1,3}} \xrightarrow{\sim} C^{la}(G,V)_{\star_2}$. Thus, it suffices to show that for a trivial representation V one has $R\Gamma^{geo}(G, C^{la}(G,V)_{\star_2}) = V$, equivalently, that $R\Gamma^{geo}(G, C^{la}(G,V)_{\star_1}) = V$. Writing V as limit of canonical and stupid truncations we can assume that V is a solid $\mathcal{K}_\blacksquare$-vector space in degree 0. But by Proposition 6.3.3 down below one can compute this geometric cohomology using geometric cochains, i.e. $R\Gamma^{geo}(G, C^{la}(G,V)_{\star_1})$ is represented by the bar complex of geometric cochains

$$[C^{la}(G,V) \to C^{geo}(G, C^{la}(G,V)) \to C^{geo}(G^2, C^{la}(G,V)) \to \cdots],$$

which is the same as the tensor product of the bar complex

$$[C^{la}(G,K) \to C(G,K) \otimes^L_{\mathcal{K}} C^{la}(G,K) \to \cdots] \otimes^L_{\mathcal{K}_\blacksquare} V.$$

But $C^{la}(G,K)$ is a nuclear $\mathcal{K}_\blacksquare$-vector space, so that the geometric bar complex of $C^{la}(G,K)$ is equal to the solid bar complex which computes $R\mathrm{Hom}_{\mathcal{K}_\blacksquare[G]}(K, C^{la}(G,K)) = K$ proving what we wanted. □

Remark 6.1.6. Under the hypothesis of a formalism of six functor for analytic stacks, the previous proof simplifies a lot. Let $f : [*/G^{geo}] \to [*/G^{la}]$ be the natural map of stacks, it suffices to prove that the natural map $\mathrm{id} \to f_* f^*$ is an equivalence. The map f is going to be a cohomologically proper map as the fibers are isomorphic to $[G_0^{la}/G_0^{geo}]$ for G_0 any compact open subgroup, so $f_* = f_!$ and by projection formula we only need to prove that $1_{[*/G^{la}]} \to f_* 1_{[*/G^{geo}]}$ is an equivalence, this follows from the explicit computation using the bar complexes and the description of f_*.

6.2. **Adjunctions.** Let G be as always a p-adic Lie group defined over a finite extension L of $\mathbf{Q}_p$, $\mathcal{K}_\blacksquare = (K, K^+)$ a complete non-achimedean extension of L and $\mathcal{K}_\blacksquare = (K,K^+)_\blacksquare$. To avoid any confusion, when talking about locally analytic representations, in this section we will note $G_L = G$ to stress that we see the group G defined over L and we denote by $G_{\mathbf{Q}_p}$ the p-adic Lie group G viewed over $\mathbf{Q}_p$. For continuous and smooth representations this disctinction is unnecessary since their definition is independent of the Lie group structures, and we will simply use the notation G. We have the following diagram of categories.

$$\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G) \xrightarrow{F_1} \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_L) \xrightarrow{F_2} \mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_{\mathbf{Q}_p}) \xrightarrow{F_3} \mathrm{Rep}_{\mathcal{K}_\blacksquare}(G), \tag{6.1}$$

where $\mathrm{Rep}_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}_{\mathcal{K}_\blacksquare}(\mathcal{K}_\blacksquare[G])$ denotes the category of solid representations of G, and where the natural functors F_i are just the forgetful functors. Since all these functors commute with colimits, they all have right adjoints and the purpose of this first section is to calculate each of them.

Proposition 6.2.1.

(1) *The right adjoint of* F_1 *is given by Lie algebra cohomology* $R\Gamma(\mathfrak{g}_L, -) := R\underline{\mathrm{Hom}}_{U(\mathfrak{g}_L)}(K, -)$.
(2) *The right adjoint of* F_2 *is given by* $R\Gamma(\mathfrak{k}, -) := R\underline{\mathrm{Hom}}_{U(\mathfrak{k})}(K,-)$, *where* $\mathfrak{k} = \ker(\mathfrak{g}_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} L \to \mathfrak{g}_L)$.
(3) *The right adjoint of* F_3 *is given by the functor of locally analytic vectors* $(-)^{Rla}$.

Proof. Let $V \in \mathrm{Rep}^{la}(G_L)$ and $W \in \mathrm{Rep}^{la}(G_{\mathbf{Q}_p})$. Then

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_{\mathbf{Q}_p},K)}(V,W) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_{\mathbf{Q}_p},K)}(\mathcal{D}^{la}(G_L,K) \otimes^L_{\mathcal{D}^{la}(G_L,K)} V, W) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_L,K)}(V, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_{\mathbf{Q}_p},K)}(\mathcal{D}^{la}(G_L,K), W) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_L,K)}(V, R\underline{\mathrm{Hom}}_{U(\mathfrak{k})}(K,W)), \end{aligned}$$

where the first two equalities are trivial and the last one follows from adjunction via Lemma 2.2.8. This proves (2).

Recall from Proposition 2.2.5 that $\mathcal{D}^{sm}(G_L,K)=K\otimes^L_{\mathcal{D}^{la}(\mathfrak{g}_L,K)}\mathcal{D}^{la}(G_L,K)$. Then, using the exact same argument as in the proof of (2), we have, for $V\in \mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$ and $W\in \mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_L)$,

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_L,K)}(V,W)=R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G_L,K)}(V,R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(\mathfrak{g}_L,K)}(K,W)),$$

proving (1).

By point (1) of Proposition 3.2.5, the right adjoint to the fully faithful inclusion $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_{\mathbb{Q}_p})\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G_{\mathbb{Q}_p},L))$ is given by the functor $(-)^{Rla}$. Since the (fully faithful) inclusion $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G_{\mathbb{Q}_p},K))\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{K}_{\blacksquare}[G])$ has a right adjoint given by $R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{D}^{la}(G_{\mathbb{Q}_p},K),-)$, the third assertion follows since we know that $\big(R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{D}^{la}(G_{\mathbb{Q}_p},K),W)\big)^{Rla}=W^{Rla}$. ☐

Remark 6.2.2. Consider the following sequence of adjunctions

$$\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)\underset{R\Gamma(\mathfrak{g}_L,-)}{\overset{F_1}{\rightleftarrows}}\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_L)\underset{R\Gamma(\mathfrak{k},-)}{\overset{F_2}{\rightleftarrows}}\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_{\mathbf{Q}_p})\underset{(-)^{Rla}}{\overset{F_3}{\rightleftarrows}}\mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G).$$

One can define functors of smooth or locally analytic vectors from different categories of representations as right adjoint of forgetful functors. For example, let F be the composite forgetful functor $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)\to \mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G)$, then its right adjoint can be computed as the composite of the right adjoints of the forgetful functors

$$\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))\to \mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{K}_{\blacksquare}[G])=\mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G).$$

This can be computed by applying simple adjunctions as follows: if $V\in \mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$ and $W\in \mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G)$, then

$$\begin{aligned}R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(V,W)&=R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V,R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{D}^{sm}(G,K),W))\\&=R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(V,\big(R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{D}^{sm}(G,K),W)\big)^{Rsm})\end{aligned}$$

where the first equality follows using the fact that V is a $\mathcal{D}^{sm}(G,K)$-module and adjunction, the second one by Proposition 5.1.8. Thus, the right adjoint of F is

$$\begin{aligned}W^{Rsm}&:=\big(R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{D}^{sm}(G,K),W)\big)^{Rsm}\\&=\varinjlim_{H\subset G}R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(\mathcal{K}_{\blacksquare}[G/H],R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{D}^{sm}(G,K),W))\\&=\varinjlim_{H\subset G}R\underline{\mathrm{Hom}}_{\mathcal{K}_{\blacksquare}[G]}(\mathcal{K}_{\blacksquare}[G/H],W).\end{aligned}$$

6.3. **Cohomology and comparison theorems.** We now introduce all the cohomology theories we are interested in, namely, Lie algebra, smooth, locally L and $\mathbf{Q}_p$-analytic, and solid group cohomologies. We will first define them and show that these definitions recover the usual ones at abelian level. Finally, we will show how they compare to each other by some formal adjunctions.

There is a natural map from the category $\mathrm{Mod}(\mathcal{K}_{\blacksquare})$ to each of the categories appearing in (6.1) given by trivial representations.

Definition 6.3.1. We define

- Solid group cohomology $R\Gamma(G,-):\mathrm{Rep}_{\mathcal{K}_{\blacksquare}}(G)\to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,
- ($\mathbf{Q}_p$-)Locally analytic group cohomology $R\Gamma^{la}(G_{\mathbf{Q}_p},-):\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_{\mathbf{Q}_p})\to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,
- (L-)Locally analytic group cohomology $R\Gamma^{la}(G_L,-):\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_L)\to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,
- Smooth group cohomology $R\Gamma^{sm}(G,-):\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)\to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$
- Lie algebra cohomology $R\Gamma(\mathfrak{g},-):\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(U(\mathfrak{g}))\to \mathrm{Mod}(\mathcal{K}_{\blacksquare})$,

as the right adjoint functor of the map from $\mathrm{Mod}(\mathcal{K}_{\blacksquare})$ to the corresponding category.

Remark 6.3.2. As the categories $\mathrm{Rep}^{sm}_{\mathcal{K}_{\blacksquare}}(G)$, $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_L)$ and $\mathrm{Rep}^{la}_{\mathcal{K}_{\blacksquare}}(G_{\mathbf{Q}_p})$ embed fully faithfully, respectively, in the categories $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{sm}(G,K))$, $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G_L,K))$ and $\mathrm{Mod}_{\mathcal{K}_{\blacksquare}}(\mathcal{D}^{la}(G_{\mathbf{Q}_p},K))$, we also have that

$$R\Gamma^{la}(G_{\mathbf{Q}_p}, V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_{\mathbf{Q}_p},K)}(K, V),$$
$$R\Gamma^{la}(G_L, V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G_L,K)}(K, V),$$
$$R\Gamma^{sm}(G, V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{sm}(G,K)}(K, V).$$

Moreover, since the categories $\mathrm{Rep}^{sm}_{\mathcal{K}_\blacksquare}(G)$ and $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_L)$ are the derived categories of their heart, the smooth and locally analytic cohomology functors can be computed as the right derived functors of the G-invariants of their respective representation categories.

By [Man22b, Corollary 3.4.17], smooth cohomology can be computed using smooth cochains. We prove the same for geometric, solid and locally analytic representations.

Proposition 6.3.3. *Let $\mathrm{Rep}^{?}_{\mathcal{K}_\blacksquare}(G)$ denote the category of smooth, L-locally analytic, geometric or solid representations of G, and let $R\Gamma^?(G,-)$ denote their corresponding cohomology functor. Let $V \in \mathrm{Rep}^{?,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$ be a representation in degree 0 and let $[C^?(G^\bullet, V), d^n]$ be the bar complex in $\mathrm{Mod}(\mathcal{K}_\blacksquare)$ with n-th term $C^?(G^n, V)$ and n-th boundary map*

$$d^n(f)(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

Then there is a natural equivalence

$$R\Gamma^?(G, V) = [C^?(G^\bullet, V), d^\bullet].$$

Proof. We follow the same proof of [Man22b, Lemma 3.4.15]. Let $? - \mathrm{Ind} : \mathrm{Mod}(\mathcal{K}_\blacksquare) \to \mathrm{Rep}^?_{\mathcal{K}_\blacksquare}(G)$ be the right adjoint of the forgetful functor, and let r denote the composition of the forgetful functor of $\mathrm{Rep}^?_{\mathcal{K}_\blacksquare}(G)$ with $? - \mathrm{Ind}$, for $n \geq 0$ we let $r^n(-)$ denote the application of n-times r. By adjunction, we have natural transformations $r^n(-) \to r^{n+1}(-)$ for all $n \geq 0$. For $M \in \mathrm{Rep}^{?,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$, we claim that the complex

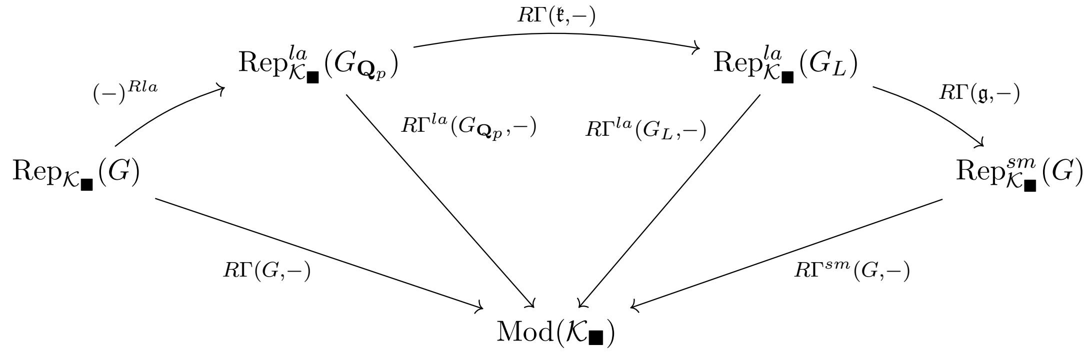
$$0 \to M \to r(M) \to r^2(M) \to \cdots \tag{6.2}$$

is exact and that $r^n(M) = C^?(G^n, M)$. First, we claim that for any $W \in \mathrm{Mod}(\mathcal{K}_\blacksquare)$ one has $? - \mathrm{Ind}(W) = C^?(G, W)$. It suffices to take $W \in \mathrm{Mod}^\heartsuit(\mathcal{K}_\blacksquare)$, in which case we need to compute the right adjoint of the forgetful functor of abelian categories $\mathrm{Rep}^{?,\heartsuit}_{\mathcal{K}_\blacksquare}(G) \to \mathrm{Mod}^\heartsuit(\mathcal{K}_\blacksquare)$. For ? solid one has $\mathrm{Rep}^?_{\mathcal{K}_\blacksquare}(G) = \mathrm{Mod}^\heartsuit(\mathcal{K}_\blacksquare[G])$ and the induction is just $C(G, V)$. For ? being smooth, locally analytic or geometric, the category $\mathrm{Rep}^{?,\heartsuit}_{\mathcal{K}_\blacksquare}(G)$ is the category of comodules of the exact functor $C^?(G, -)$, and one easily checks that the right adjoint of the forgetful functor is simply $V \mapsto C^?(G, V)$ proving the claim.

Now, unraveling the definitions, one has that the sequence (6.2) is given by the usual bar complex of the respective representation category, which is an exact complex as they are constructed functorially from the augmented cosimplicial object $(G^{n+1})_{n \in \Delta^{op}} \xrightarrow{\varepsilon} *$. To conclude the proof we need to show that $R\Gamma^?(G, ? - \mathrm{Ind}(W)) = W$ for any $W \in \mathrm{Mod}(\mathcal{K}_\blacksquare)$, but the functor $R\Gamma^?(G, ? - \mathrm{Ind}(-))$ is the right adjoint of the composite of the trivial representation and the forgetful functor which is the identity on $\mathrm{Mod}(\mathcal{K}_\blacksquare)$, so it is equivalent to the identity. This finishes the proof. □

All our comparison results are subsumed in the following statement, which generalizes in particular our main results [RJRC21, Theorem 5.3 and Theorem 5.5] from the case of a compact p-adic Lie group defined over $\mathbf{Q}_p$ to that of a (non-necessarily compact) p-adic Lie group defined over a finite extension of $\mathbf{Q}_p$.

Theorem 6.3.4. *We have the following commutative diagram:*



Moreover, since the embedding $\mathrm{Rep}^{la}_{\mathcal{K}_\blacksquare}(G_{\mathbf{Q}_p})$ *in* $\mathrm{Rep}_{\mathcal{K}_\blacksquare}(G)$ *is fully faithful then, for any* $V \in \mathrm{Rep}_{\mathcal{K}_\blacksquare}(G)$*, we have* $R\Gamma(G,V) = R\Gamma(G,V^{Rla})$*. In particular, if* G *is defined over* $\mathbf{Q}_p$*, we have*

$$R\Gamma(G,V) = R\Gamma(G,V^{Rla}) = R\Gamma^{la}(G,V^{Rla}) = R\Gamma^{sm}(G,R\Gamma(\mathfrak{g},V^{Rla})).$$

Proof. It follows by the adjunctions of Proposition 6.2.1. □

6.4. **Homology and duality.** We conclude with some applications to duality between cohomology and homology. The following result is the infinitesimal analoge of [RJRC21, Theorem 5.19].

Proposition 6.4.1. *Let* $V \in \mathrm{Mod}_{\mathcal{K}_\blacksquare}(U(\mathfrak{g}))$*. Then we have*

$$R\Gamma(\mathfrak{g},V) = K(\chi)[-d] \otimes^L_{U(\mathfrak{g})} V.$$

In particular, if $V \in \mathrm{Rep}_{\mathcal{K}_\blacksquare}(G)$*, then*

$$R\Gamma(G,V) = R\Gamma^{sm}(G, K(\chi)[-d] \otimes^L_{U(\mathfrak{g})} V^{Rla}).$$

Proof. This follows exactly the same argument as in [RJRC21, Theorem 5.19] replacing the Lazard-Serre resolution by Chevalley-Eilenberg resolution to calculate cohomology. The last assertion follows from the first one and Theorem 6.3.4. □

The problem for showing a global result when G is not compact is that the trivial object K might not be a perfect $\mathcal{K}_\blacksquare[G]$-module. Nevertheless, this is indeed the case when either $G = \mathbb{G}(\mathbf{Q}_p)$ arises as the $\mathbf{Q}_p$-points of a connected reductive group over $\mathbf{Q}_p$ by [Koh11, Theorem 6.6] or G is solvable by [Koh11, Theorem 6.5]. From these facts, one immediately deduces the following:

Corollary 6.4.2. *Let* G *be either given by the* $\mathbf{Q}_p$*-point of a connected reductive group* $\mathbb{G}$ *defined over* $\mathbf{Q}_p$ *or solvable and let* $V \in \mathrm{Rep}_{\mathcal{K}_\blacksquare}(G)$*. Then*

$$R\Gamma(G,V) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,K)}(K, \mathcal{D}^{la}(G,K)) \otimes^L_{\mathcal{D}^{la}(G,K)} V^{Rla}.$$

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