A NOTE ON HIGHER ALMOST RING THEORY

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Abstract. We explain a derived version of the basic construction of localisations of module categories by means of idempotent ideals, which lie at the heart of Faltings' almost ring theory.

1. Introduction

Almost ring theory was introduced by Faltings in [Fal88, Fal02], as a way capturing and propagating vanishing phenomena in Galois cohomology, building on initial work of Tate in [Tat67]. The basic set-up of the theory was then reworked by Gabber and Ramero in [GR03, GR04] to simply rely on a commutative ring R, and ideal $I \subseteq R$ satisfying two assumptions:

(1) I is idempotent (that is $I = I^2$), and
(2) I is flat as an R-module.

The most prominent example of this situation is given by the ideal of topologically nilpotent elements I inside the ring of power bounded elements R of a perfectoid field. Gabber and Ramero in fact showed that, without much loss, the second condition can be generalised further by only requiring $I \otimes_R I$ to be flat over R, and eventually that one can do largely without it.

In any case, one says that a morphism of R-modules is an I-*almost isomorphism* if its kernel and cokernel are annihilated by all elements of I. The localisation $\mathrm{aMod}_I(R)$ of $\mathrm{Mod}(R)$ at these maps, the category of I-*almost* R-*modules*, retains many good homological properties: For example the tensor product of R-modules descends to it and its derived category can be described as the localisation of the derived category of R-modules localised at those maps inducing I-almost isomorphisms on homology groups.

In fact, these desirable properties are all consequences of the fact that the multiplication

$$R/I \otimes^{\mathbb{L}}_R R/I \longrightarrow R/I$$

is an equivalence, if I satisfies the properties listed above, making $R \to R/I$ into what is sometimes called a derived localisation. In the case that the flatness assumption is required only for $I \otimes_R I$, the analogous statement is that the commutative differential graded algebra $R /\!\!/ (I \otimes_R I)$ with

$$\mathrm{H}_i(R /\!\!/ (I \otimes_R I)) = \begin{cases} R/I & i = 0 \\ \ker(I \otimes_R I \to I) & i = 1 \\ 0 & \text{else} \end{cases}$$

is a derived localisation of R.

The purpose of the present note is to explain that by passing to derived categories directly one can do away with the flatness hypothesis entirely while still retaining this simple explanation for the good properties of the derived category of I-almost modules. In fact, once attention is restricted to derived categories, one can also replace the ring R by homotopical generalisations. We show:

Theorem A. *Let R be an animated commutative ring and consider the full subcategory LQ_R of $R/\mathrm{AnCRing}$ spanned by the maps $\varphi\colon R \to S$ for which*

(1) the multiplication $S \otimes^{\mathbb{L}}_R S \to S$ is an equivalence, i.e. φ is a derived localisation, and
(2) $\pi_0(\varphi)\colon \pi_0 R \to \pi_0 S$ is surjective.

Then the functor

$$\mathrm{LQ}_R \longrightarrow \{I \subseteq \pi_0 R \mid I^2 = I\}, \quad \varphi \longmapsto \ker(\pi_0 \varphi)$$

is an equivalence of categories, where we regard the target as a poset via the inclusion ordering. The inverse image of some $I \subseteq \pi_0(R)$ is given by the limit of the Amitsur complex for the map $R \to \pi_0(R)/I$.

In fact, there is an entirely analogous result for the $\mathbb{E}_k$-rings of homotopy theory, which in particular gives an existence result for ring structures on certain quotients, something that is neither easy nor generally possible, see e.g. [Bur22]:

Theorem B. *Let A be a connective $\mathbb{E}_k$-ring with $1 \leq k \leq \infty$, respectively. Consider again the full subcategory LQ_R of $A/\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp})$ spanned by the maps $\varphi\colon A \to B$ for which*

(1) the multiplication $B \otimes_A B \to B$ is an equivalence, i.e. φ is a localisation,
(2) B is connective, and
(3) $\pi_0(\varphi)\colon \pi_0 A \to \pi_0 B$ is surjective.

Then the functor

$$\mathrm{LQ}_A \longrightarrow \{I \subseteq \pi_0 A \mid I^2 = I\}, \quad \varphi \longmapsto \ker(\pi_0\varphi)$$

is an equivalence of categories, where we again regard the target as a poset via the inclusion ordering. The inverse image of some $I \subseteq \pi_0(R)$ can be described more directly as A/I^∞, where

$$I^\infty = \lim_{n \in \mathbb{N}^{\mathrm{op}}} J_I^{\otimes_A n}$$

with $J_I \to A$ the fibre of the canonical map $A \to \mathrm{H}(\pi_0(A)/I)$. Furthermore, this inverse system stabilises on π_i for $n > i+1$.

To connect to the discussion of almost modules before, recall that for R a commutative animated (e.g. static) ring, the derived category of R depends only on its underlying $\mathbb{E}_1$-ring $\mathrm{H}R$, that is we have $\mathcal{D}(R) \simeq \mathrm{Mod}(\mathrm{H}R)$, and that the animated commutative ring S corresponding to some idempotent $I \subseteq \pi_0(R)$ necessarily satisfies $\mathrm{H}S \simeq (\mathrm{H}R)/I^\infty$ by the uniqueness assertions of the theorems above. We shall therefore denote this ring by R/I^∞ and restrict the discussion to the case of $\mathbb{E}_1$-rings from here on. Recall then that a map of $\mathbb{E}_1$-rings $\varphi\colon A \to B$ with $B \otimes_A B \simeq B$ via the multiplication, gives rise to a stable recollement

$$\mathrm{Mod}(B) \underset{\hom_A(B,-)}{\overset{B\otimes_A -}{\rightleftarrows}} \mathrm{Mod}(A) \underset{\hom_A(\mathrm{fib}(\varphi),-)}{\overset{\mathrm{fib}(\varphi)\otimes_A -}{\rightleftarrows}} \mathrm{Mod}(A)[\varphi\text{-eq's}^{-1}]$$

where the φ-equivalences are those maps of A-modules that become equivalences after tensoring with B, see e.g. [CDH$^+$20, Appendix A.4]; the picture indicates four adjunctions with left adjoints on top, arranged into three horizontal Verdier sequences.

In the case at hand, $B = A/I^\infty$, we show that the image of the fully faithful restriction $\mathrm{Mod}(A/I^\infty) \to \mathrm{Mod}(A)$ consists exactly of those A-modules M with $I \cdot \pi_n(M) = 0$ for all $n \in \mathbb{Z}$, and that a map is a φ-equivalence if and only if it induces an I-almost isomorphism on all homotopy groups. The recollement thus takes the form

$$\mathrm{Mod}(A/I^\infty) \underset{\hom_A(A/I^\infty,-)}{\overset{A/I^\infty\otimes_A -}{\rightleftarrows}} \mathrm{Mod}(A) \underset{\hom_A(I^\infty,-)}{\overset{I^\infty\otimes_A -}{\rightleftarrows}} \mathrm{aMod}_I(A)$$

exhibiting the I-almost A-modules as a split Verdier quotient of $\mathrm{Mod}(A)$, as desired; furthermore, if A is an $\mathbb{E}_k$-algebra, the entire recollement is suitably $\mathbb{E}_{k-1}$-multiplicative, though we shall not further expound that here.

We end this introduction with a typical example, which illustrates the extended range of applicability offered by the removal of the flatness assumption: On the one hand, consider for K a field the commutative ring

$$R_n = K[T_1^{1/2^\infty}, \dots, T_n^{1/2^\infty}] := K[T_{i,j}, i,j \in \mathbb{N}, j \leq n]/(T_{i+1,j}^2 - T_{i,j}, i,j \in \mathbb{N}, j \leq n),$$

together with the ideal

$$I_n = (T_1^{1/2^\infty}, \dots, T_n^{1/2^\infty}) := (T_{i,j}, i,j \in \mathbb{N}, j \leq n).$$

which is evidently idempotent. It is flat only for $n = 1$ but nevertheless $I_n \otimes^{\mathbb{L}}_{R_n} I_n = I_n$ so that

$$R_n/I_n^\infty = R_n/I_n = K$$

is still static. On the other hand, in $\overline{R}_n = R_n/(T_1, \dots, T_n)$ the ideal $\overline{I}_n = I_n/(T_1, \dots, T_n)$ is no longer flat even for $n = 1$ and

$$\pi_*\overline{R}_n/\overline{I}_n^\infty = \Lambda_K[T_1, \dots T_n]$$

is the exterior algebra on n generators in degree 1: The module $\overline{I}_1 \otimes_{\overline{R}_1} \overline{I}_1$ is still flat over $\overline{R}_1$, giving this calculation for $n = 1$ and the general case then follows by multiplicativity. In this example, a well-behaved almost theory still exists at the level of modules for $n = 1$. This fails for $n > 1$, but the derived theory is largely unaffected.

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2. The proof

For the proof recall the category $\mathbb{E}_k$-Mod(A) of $\mathbb{E}_k$-modules over A, as constructed in [Lur17, Sections 3.3 & 3.4], which is again $\mathbb{E}_k$-monoidal under $\otimes_A$. In contrast Mod(A) is only $\mathbb{E}_{k-1}$-monoidal; one has $\mathbb{E}_1$-Mod$(A) = \mathrm{BiMod}(A, A)$ and $\mathbb{E}_\infty$-Mod$(A) = \mathrm{Mod}(A)$.

Proof of Theorem B. We start with the observation that LQ_A is indeed equivalent to a poset, i.e. its mapping spaces are either empty or contractible, by the characterisation of localisations of an $\mathbb{E}_k$-ring A as $\otimes_A$-idempotent objects in $\mathbb{E}_k$-modules under A (the analogue indeed holds for collections of idempotent objects in any monoidal category). Let us also immediately verify that $\ker(\pi_0\varphi)$ is indeed idempotent for $\varphi\colon A \to B$ a π_0-surjective localisation among connective $\mathbb{E}_k$-rings. Tensoring the fibre sequence $F \to A \to B$ with F gives

$$F \otimes_A F \longrightarrow F \longrightarrow F \otimes_A B$$

and the right hand term vanishes since one has a fibre sequence

$$F \otimes_A B \longrightarrow A \otimes_A B \longrightarrow B \otimes_A B$$

whose right hand map (after identifying $A \otimes_A B \simeq B$) is a section of the multiplication $B \otimes_A B \to B$ and thus an equivalence. But F is connective and the map $\pi_0(F) \to \ker \pi_0\varphi$ surjective by the long exact sequence of φ, whence a chase in the diagram

$$\begin{array}{ccccc}
\ker(\pi_0\varphi) \otimes_{\pi_0 A} \ker(\pi_0\varphi) & & \longrightarrow & & \ker(\pi_0\varphi) \\
\uparrow & & & & \uparrow \\
\pi_0 F \otimes_{\pi_0 A} \pi_0 F & \xrightarrow{\sim} & \pi_0(F \otimes_A F) & \xrightarrow{\sim} & \pi_0 F
\end{array}$$

shows that the multiplication $\ker(\pi_0\varphi) \otimes_{\pi_0 A} \ker(\pi_0\varphi) \to \ker(\pi_0\varphi)$ is surjective as desired.

Next, we verify the last claim from the statement, i.e. that the inverse system $J_I^{\otimes_A n}$ stabilises degreewise. In fact we show slightly more, namely that the cofibre $A/J_I \otimes_A J_I^{\otimes_A n}$ of the canonical map $J_I^{\otimes_A n+1} \to J_I^{\otimes_A n}$ is n-connective. Since $A/J_I = \mathrm{H}(\pi_0(A)/I)$ is an $\mathbb{E}_k$-ring annihilated by I, we immediately deduce that the homotopy groups of this cofibre are annihilated by I (from both sides).

Now, for $n = 0$, the connectivity claim is clear, and if we inductively assume that $A/J_I \otimes_A J_I^{\otimes_A n}$ is n-connective, then

$$A/J_I \otimes_A J_I^{\otimes_A n+1} = \left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_A J_I$$

is clearly also n-connective and its nth homotopy group is $\pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0 A} I$. Since the left hand term is annihilated by I, we compute

$$\pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0 A} I = \pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0(A)/I} \pi_0(A)/I \otimes_{\pi_0 A} I$$
$$= \pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0(A)/I} I/I^2 = 0.$$

As the next step, we show that the tautological map $M = A \otimes_A M \to A/I^\infty \otimes_A M$ is an equivalence whenever the homotopy of M is annihilated by I, or in other words that $I^\infty \otimes_A M \simeq 0$. We start with the simplest case $M = A/J_I$, where the claim is equivalent to the multiplication map

$$\left(\lim_{n\in\mathbb{N}^{\mathrm{op}}} J_I^{\otimes_A n}\right) \otimes_A J_I \longrightarrow \lim_{n\in\mathbb{N}^{\mathrm{op}}} J_I^{\otimes_A n}$$

being an equivalence. But since the limit stabilises degreewise and J_I is connective, we can move the limit out of the tensor product (the cofibre of the interchange map is a limit of terms with growing connectivity), and then the statement follows from finality.

For an arbitrary A-module M concentrated in degree 0 and killed by the action of I, choose a free resolution of $\pi_0 M$ by $\pi_0(A)/I$-modules, which by the Dold-Kan theorem yields a diagram $F\colon \Delta^{\mathrm{op}} \to \mathcal{D}(\pi_0(A)/I)$ with each F_n concentrated in degree 0, $\pi_0(F_n)$ free and $\mathrm{colim}_{\Delta^{\mathrm{op}}} F \simeq (\pi_0 M)[0]$, so that $\mathrm{colim}_{\Delta^{\mathrm{op}}} \iota F \simeq M$, where ι is the composite $\mathcal{D}(\pi_0(A)/I) \simeq \mathrm{Mod}(A/J_I) \to \mathrm{Mod}(A)$. But then

$$I^\infty \otimes_A M \simeq \mathrm{colim}_{k\in\Delta^{\mathrm{op}}} I^\infty \otimes_A \iota F_k \simeq 0$$

since each ιF_k is a direct sum of A/J_I. By exactness of $I^\infty \otimes_A (-)$, the claim then follows for each bounded A-module M whose homotopy is annihilated by I using the Postnikov tower of M. For bounded below M, we have

$$I^\infty \otimes_A M \simeq I^\infty \otimes_A \left(\lim_{k\in\mathbb{N}^{\mathrm{op}}} \tau_{\le k} M\right) \simeq \lim_{k\in\mathbb{N}^{\mathrm{op}}} I^\infty \otimes_A \tau_{\le k} M \simeq 0$$

by commuting the limit out using the same argument as above. Finally, for arbitrary M whose homotopy is killed by I, we find

$$I^\infty \otimes_A M \simeq I^\infty \otimes_A (\mathrm{colim}_{k\in\mathbb{N}} \tau_{\ge -k} M) \simeq \mathrm{colim}_{k\in\mathbb{N}} I^\infty \otimes_A \tau_{\ge -k} M \simeq 0.$$

Now, since $A \to A/J_I = \mathrm{H}(\pi_0(A)/I)$ is a map of $\mathbb{E}_k$-rings, it follows that $J_I \to A$ is an $\mathbb{E}_k$-Smith-ideal in A. It then formally follows that so is $J_I^{\otimes_A n} \to A$, whence $A/J_I^{\otimes_A n}$ and thus A/I^∞ are $\mathbb{E}_k$-rings; for completeness' sake we briefly outline the argument at the end of this section since we are unaware of a reference. Since $\pi_0(A/I^\infty) = \pi_0(A)/I$, all homotopy groups of A/I^∞ are annihilated by I, and so the canonical map $A \to A/I^\infty$ induces an equivalence $A/I^\infty \to A/I^\infty \otimes_A A/I^\infty$, which shows that $A \to A/I^\infty$ is a localisation. Furthermore, it implies that the homotopy of every A/I^∞-module is a $\pi_0(A)/I$-module, so combined with the previous point, we learn that the image of the fully faithful restriction functor $\mathrm{Mod}(A/I^\infty) \to \mathrm{Mod}(A)$ consists exactly of those modules whose homotopy is killed by I, as desired.

Finally, we are ready to verify that the construction $I \mapsto (A \to A/I^\infty)$ induces an inverse to taking kernels. The composition starting with an ideal is clearly the identity. So we are left to show that for every $\varphi\colon A \to B$ in LQ_A the canonical map $\psi\colon A/\ker(\pi_0\varphi)^\infty \to B$, arising from the homotopy of B being annihilated by $\ker(\pi_0\varphi)$, is an equivalence. Per construction it induces an equivalence on π_0. By the lemma below, the functor $\psi_! = B \otimes_{A/\ker(\varphi)^\infty} -\colon \mathrm{Mod}(A/\ker(\pi_0\phi)^\infty) \to \mathrm{Mod}(B)$ is thus conservative when restricted to bounded below modules. But the map

$$B \simeq \psi_!\left(A/\ker(\pi_0\varphi)^\infty\right) \xrightarrow{\psi_!(\varphi)} \psi_!(B) = B \otimes_{A/\ker(\pi_0\varphi)^\infty} B \simeq B \otimes_A B$$

is induced by the unit and thus an equivalence since φ is a localisation. □

2.1. **Lemma** *If $\psi\colon A \to B$ is a map of connective $\mathbb{E}_1$-rings which is an isomorphism on π_0, then*

$$B \otimes_A -\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B)$$

is conservative when restricted to bounded below A-modules.

Proof. If $M \in \mathrm{Mod}(A)$ with $\pi_i(M) = 0$ for $i < n$, then $\pi_n(B \otimes_A M) = \pi_0(B) \otimes_{\pi_0(A)} \pi_n(M) = \pi_n(M)$, so if M is bounded below with $B \otimes_A M \simeq 0$ then also $M \simeq 0$. Considering cofibres of morphisms, this implies the statement. □

Now, the main step in the deduction of Theorem A from Theorem B is to establish the structure of an animated commutative ring R/I^∞ on $(\mathrm{H}R)/I^\infty$, whenever R is itself an animated commutative ring. To this end, recall the Amitsur (or cobar) complex of an algebra B in a monoidal category $(\mathcal{C}, \otimes)$: This is a cosimplicial object in $\mathcal{C}$ with $[n] \mapsto B^{\otimes n+1}$ and face and degeneracy maps induced by the unit and multiplication, respectively, see e.g. [MNN17, Section 2.1]. For a map $R \to S$ of animated commutative rings, we can consider it in $(R/\mathrm{AnCRing}, \otimes^{\mathbb{L}}_R)$ and similarly for a map $A \to B$ of $\mathbb{E}_k$-rings we can consider it in $(A/\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp}), \otimes_A)$ and also in $(\mathbb{E}_k\text{-}\mathrm{Mod}(A), \otimes_A)$; these examples are connected by strong monoidal, limit preserving functors

$$(R/\mathrm{AnCRing}, \otimes^{\mathbb{L}}_R) \longrightarrow (\mathrm{H}R/\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}), \otimes_{\mathrm{H}R}) \quad \text{and} \quad (A/\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp}), \otimes_A) \longrightarrow (\mathbb{E}_k\text{-}\mathrm{Mod}(A), \otimes_A).$$

In [MNN17, Proposition 2.14] it is in particular shown, that the limit of the Amitsur complex for $A \to \mathrm{H}(\pi_0(A)/I)$ formed in $(\mathbb{E}_k\text{-}\mathrm{Mod}(A), \otimes_A)$ agrees with A/I^∞ as an $\mathbb{E}_k$-A-module. The characterisation of the $\mathbb{E}_k$-ring structure on A/I^∞ as arising from being $\otimes_A$-idempotent then shows that this upgrades to an equivalence of $\mathbb{E}_k$-algebras under A.

In particular, $(\mathrm{H}R)/I^\infty$ is the limit of the Amitsur complex of $\mathrm{H}R \to \mathrm{H}(\pi_0 R/I)$ formed in $(\mathrm{H}R/\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}), \otimes_{\mathrm{H}R})$, which allows us to lift this structure to that of an animated commutative ring by letting R/I^∞ denote the limit of the Amitsur complex for $R \to \pi_0 R/I$ in $(R/\mathrm{AnCRing}, \otimes^{\mathbb{L}}_R)$. In particular, we then have $\mathrm{H}(R/I^\infty) \simeq (\mathrm{H}R)/I^\infty$ as $\mathbb{E}_\infty$-rings.

Proof of Theorem A. We again observe that LQ_R is a poset, since $- \otimes^{\mathbb{L}}_R S\colon R/\mathrm{AnCRing} \to R/\mathrm{AnCRing}$ is a localisation onto its image for every $\varphi\colon R \to S$ in LQ_R. The construction $R \mapsto R/I^\infty$ in terms of the Amitsur complex thus gives a functor that is evidently right inverse to

$$\mathrm{LQ}_R \longrightarrow \{I \subseteq \pi_0 R \mid I^2 = I\}, \quad \varphi \longmapsto \ker(\pi_0 \varphi).$$

Furthermore, from the case of $\mathbb{E}_\infty$-rings we learn that the natural map $S \simeq R \otimes^{\mathbb{L}}_R S \to R/I^\infty \otimes^{\mathbb{L}}_R S$ is an equivalence if and only if the homotopy groups of S are annihilated by I. In this case we therefore obtain a map $R/I^\infty \to S$ of animated commutative rings under R, and in particular this applies in the case $I = \ker(\varphi)$. But by Theorem B there is only one map $\mathrm{H}R/\ker(\varphi)^\infty \to \mathrm{H}S$ under $\mathrm{H}R$ and this is an equivalence. Since the functor $\mathrm{H}\colon \mathrm{AnCRing} \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ is conservative, we must thus also have $R/\ker(\varphi)^\infty \simeq S$ under R as desired. □

Finally, we briefly sketch the background for our use of Smith-ideals for lack of a reference; see, however, [Hov14] for a treatment of the $\mathbb{E}_1$-version in model categorical language. Consider then an $\mathbb{E}_k$-monoidally cocomplete stable category $\mathcal{C}$, and give $\mathrm{Ar}(\mathcal{C})$ the induced $\mathbb{E}_k$-monoidal Day convolution structure with respect to taking minima on $[1]$, which makes the evaluation functor $t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ strongly $\mathbb{E}_k$-monoidal. A Smith-ideal in an $\mathbb{E}_k$-algebra A in $\mathcal{C}$ is an $\mathbb{E}_k$-algebra $J \to A$ in $\mathrm{Ar}(\mathcal{C})$ lifting the $\mathbb{E}_k$-structure on A. Such objects correspond in a one-to-one fashion to $\mathbb{E}_k$-ring maps out of A by taking (co)fibres: To see this consider $\mathrm{Fun}([1]^2, \mathcal{C})$ equipped with Day convolution with respect to taking minima in the first, and maxima in the second component of $[1]^2$ (note that Day convolution with respect to taking maxima in $[1]$ is just the pointwise monoidal structure on $\mathrm{Ar}(\mathcal{C})$). Now, the full subcategory of $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Fun}([1]^2, \mathcal{C}))$ spanned by the cocartesian squares with lower left corner (i.e. the entry at $(1,0)$) vanishing is on the one hand equivalent to the category of $\mathbb{E}_k$-arrows in $\mathcal{C}$ by taking fibres, and on the other, to the category of $\mathbb{E}_k$-Smith-ideals in $\mathcal{C}$ by taking cofibres; this is clearly true at the level of the underlying categories, and the claim follows from this since such cocartesian squares are closed under the Day convolution on all squares.

Convolving two Smith ideals provides the higher categorical way of taking the sum of ideals, and taking pointwise tensor products generalises taking the product of ideals. Since we used it in the proof above, let us also briefly explain why this pointwise operation is well-defined. Recall that $\mathbb{E}_k$-monoids in the Day convolution structure are identified with lax $\mathbb{E}_k$-monoidal functors.

Consider then the functor $\mathrm{Fun}^{\text{lax-}\mathbb{E}_k}([1],-)\colon \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Cat}) \to \mathrm{Cat}$. Since it preserves products, it lifts to a functor

$$\mathrm{Fun}^{\text{lax-}\mathbb{E}_k}([1],-)\colon \mathrm{Alg}_{\mathbb{E}_{k+l}}(\mathrm{Cat}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_l}(\mathrm{Cat})$$

or in other words, the category of lax $\mathbb{E}_k$-monoidal functors to an $\mathbb{E}_{k+l}$-monoidal category inherits an $\mathbb{E}_l$-monoidal structure, which unwinds to be given by the pointwise tensor product in the target.

3. Examples and Remarks

(1) A different way of phrasing Theorem B is that there is a one-to-one correspondence between idempotent ideals in $\pi_0(A)$ and idempotent Smith-ideals in A for every connective $\mathbb{E}_k$-algebra A, which makes $I \subseteq \pi_0(A)$ and $I^\infty \to A$ correspond.

(2) As a consequence of the classification of stable recollements, one obtains a cartesian square

$$\begin{array}{ccc} \mathrm{Mod}(A) & \xrightarrow{\hom_A(A/I^\infty,-)\Rightarrow A/I^\infty\otimes_A -} & \mathrm{Ar}(\mathrm{Mod}(A/I^\infty)) \\ \downarrow & & \downarrow \mathrm{cof} \\ \mathrm{aMod}_I(A) & \xrightarrow{A/I^\infty\otimes_A \hom_A(I^\infty,-)} & \mathrm{Mod}(A/I^\infty), \end{array}$$

decomposing the module category of A for every idempotent $I \subseteq \pi_0(A)$, see [CDH+20, Section A.2].

(3) Either directly from the statement of the theorems, or via the construction using the Amitsur complex, one finds that for the exterior sum $I \boxplus_k J$ of two idempotent ideals $I \subseteq \pi_0(A)$ and $J \subseteq \pi_0(A')$ in two k-algebras A and A' (k some $\mathbb{E}_2$-ring), that is the image of

$$(\pi_0 A \otimes_{\pi_0 k} J) \oplus (I \otimes_{\pi_0 k} \pi_0 A') \longrightarrow \pi_0 A \otimes_{\pi_0 k} \pi_0 A' = \pi_0(A \otimes^{\mathbb{L}}_k A'),$$

we have

$$(A \otimes^{\mathbb{L}}_k A')/(I \boxplus_k J)^\infty \simeq A/I^\infty \otimes^{\mathbb{L}}_k A'/J^\infty,$$

or in other words $(I \boxplus_k J)^\infty$ is the exterior sum (over k) of the Smith-ideals I^∞ in A and J^∞ in A'.

This formula evidently also holds for three animated commutative rings in place of k, A and A'.

(4) If R is a static ring with an ideal I that is flat as a left or right R-module and satisfies $I^2 = I$, then $I^{\otimes^{\mathbb{L}}_R n} = I^{\otimes_R n} = I$, so $I^\infty = I$ and $R/I^\infty = R/I$ is static.

As mentioned in the introduction, a commutative ring R together with an idempotent, flat ideal $I \subseteq R$ is indeed one of the standard set-ups for almost mathematics, e.g. in [Bha17, Section 4], and in this case $\mathrm{a}\mathcal{D}_I(R) \simeq \mathrm{aMod}_I(\mathrm{H}R)$ is the derived category of the ordinary category of almost R-modules. Bhatt's notes also cleanly explain that whenever $(K, |\cdot|)$ is a perfectoid field, then $\mathfrak{m} = \{x \in K \mid |x| < 1\}$ is a flat and idempotent ideal in the valuation ring $\mathcal{O} = \{x \in K \mid |x| \leq 1\}$.

(5) Let us also immediately note, that a finitely generated idempotent ideal I in a (static) commutative ring R is necessarily generated by single idempotent element e by Nakayama's lemma and thus, as a direct summand, even projective over R. In this case $R/(e)^\infty = R/(e)$ is simply the factor of R singled out by e, which can also be described as the ordinary localisation $R[(1-e)^{-1}]$ of R.

(6) In fact, for R static the animated ring R/I^∞ is static if and only if $I \otimes^{\mathbb{L}}_R I \simeq I$ via the multiplication: The latter implies the former by the description of $\mathrm{H}R/I^\infty$ in Theorem B, and conversely if $R/I^\infty \simeq R/I$ we learn that $R/I \otimes^{\mathbb{L}}_R R/I \simeq R/I$, which by passing to fibres along the exact sequence $I \to R \to R/I$ first yields $I \otimes^{\mathbb{L}}_R R/I \simeq 0$ and then the claim.

An example, where this occurs without I being flat is the ring $R_n = K[T_1^{1/2^\infty}, \ldots, T_n^{1/2^\infty}]$ from the introduction with $I_n = (T_1^{1/2^\infty}, \ldots, T_n^{1/2^\infty})$. Then as a sequential colimit of principal ideals I_1 is flat over R_1, so the multiplicativity statement for exterior sums of ideals yields

$$R_n/I_n^\infty \simeq (R_1/I_1^\infty)^{\otimes_K n} \simeq (R_1/I_1)^{\otimes_K n} \simeq K.$$

But I_n is no longer flat for $n \geq 2$: Setting $J_n = (T_1, \dots, T_n) \subseteq R_n$ we for example have

$$\operatorname{Tor}_i^{R_n}(I_n, R_n/J_n) \cong \begin{cases} I_n/J_n \cdot I_n & i = 0 \\ K^{\binom{n}{i+1}} & i \geq 1 \end{cases},$$

which can be read off from the exact sequence

$$I_n \otimes^{\mathbb{L}}_{R_n} R_n/J_n \longrightarrow R_n/J_n \longrightarrow K \otimes^{\mathbb{L}}_{R_n} R_n/J_n$$

in $\mathcal{D}(R_n/J_n)$ together with

$$K \otimes^{\mathbb{L}}_{R_n} R_n/J_n \simeq (K \otimes^{\mathbb{L}}_{R_1} R_1/T_1)^{\otimes_K n} \simeq (\Lambda_K(K[1]))^{\otimes_K n} \simeq \Lambda_K(K^n[1]),$$

which in turn can be read off from the evident free resolution $R_1 \xrightarrow{T_1} R_1$ of R_1/T_1.

(7) Whenever $I \otimes_R I$ is flat over R, one has $R/I^\infty = R /\!\!/ (I \otimes_R I)$ as mentioned in the introduction, where $/\!\!/$ denotes the cofibre in $\mathcal{D}(R)$, modelled by the commutative graded differential algebra with

$$(R /\!\!/ (I \otimes_R I))_i = \begin{cases} R & i = 0 \\ I \otimes_R I & i = 1 \\ 0 & \text{else} \end{cases}$$

so that

$$\pi_i(R/I^\infty) = \begin{cases} R/I & i = 0 \\ \ker(I \otimes_R I \to I) & i = 1 \\ 0 & i \geq 2 \end{cases}$$

in this case: The multiplication map $I^{\otimes^{\mathbb{L}}_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n} \to I^{\otimes^{\mathbb{L}}_R n}$ factors as

$$I^{\otimes^{\mathbb{L}}_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n} \longrightarrow I^{\otimes_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n} \longrightarrow I^{\otimes^{\mathbb{L}}_R n}$$

so the limit computing I^∞ can be replaced by that over the terms $I^{\otimes_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n}$. But this system is constant, as can be seen inductively from the fibre sequence

$$I^{\otimes_R 2} \otimes^{\mathbb{L}}_R I \longrightarrow I^{\otimes_R 2} \longrightarrow I^{\otimes_R 2} \otimes^{\mathbb{L}}_R R/I$$

whose last term is $I^{\otimes_R 2} \otimes_R R/I = I/I^2 \otimes_R I = 0$.

(8) Note that $I^{\otimes_R n} \cong I \otimes_R I$ for all $n \geq 2$ the moment I is idempotent, e.g. by the stability assertion of Theorem B, so that no further flatness hypothesis can sensibly be put on tensor powers of I.

(9) The condition $I \cdot \pi_n M = 0$ of $M \in \operatorname{Mod}(A)$ being almost zero is in fact equivalent to the a priori stronger condition that $I \otimes_{\pi_0 A} \pi_n M = 0$: For the former condition makes $\pi_n M$ into an $\pi_0(A)/I$-module so that

$$I \otimes_{\pi_0 A} \pi_n(M) = I \otimes_{\pi_0 A} \pi_0(A)/I \otimes_{\pi_0(A)/I} \pi_n M = I/I^2 \otimes_{\pi_0(A)/I} \pi_n M = 0.$$

(10) In contrast to this, it need not be true, however, that $I \otimes^{\mathbb{L}}_R M \simeq 0$ for M an I-almost zero R-module: For example, let $R = K[T^{1/2^\infty}]/T$ and $I = (T^{1/2^\infty})$, the ideal generated by all the 2-power roots of T. Then $R/I = K$ is clearly almost 0, but $I \otimes^{\mathbb{L}}_R K \simeq \bigoplus_{i \geq 1} K[2i-1]$ does not vanish: Writing $R(n) = K[T^{1/2^n}]/T$ and $I(n) = (T^{1/2^n})$ for the principal ideal therein, so that $R = \operatorname{colim}_n R(n)$ and $I = \operatorname{colim}_n I(n)$ and consequently $I \otimes^{\mathbb{L}}_R K = \operatorname{colim}_n I(n) \otimes^{\mathbb{L}}_{R(n)} K$, we can freely resolve the inclusion $I(n) \to I(n+1)$ by the periodic

$$\begin{array}{ccccccccc} \dots & \longrightarrow & R(n) & \xrightarrow{\cdot T^{(2^n-1)/2^n}} & R(n) & \xrightarrow{\cdot T^{1/2^n}} & R(n) & \xrightarrow{\cdot T^{(2^n-1)/2^n}} & R(n) \\ & & \downarrow{\scriptstyle \text{incl}} & & \downarrow{\scriptstyle \cdot T^{1/2^{n+1}}} & & \downarrow{\scriptstyle \text{incl}} & & \downarrow{\scriptstyle T^{1/2^{n+1}}} \\ \dots & \longrightarrow & R(n+1) & \xrightarrow{\cdot T^{(2^{n+1}-1)/2^{n+1}}} & R(n+1) & \xrightarrow{\cdot T^{1/2^{n+1}}} & R(n+1) & \xrightarrow{\cdot T^{(2^{n+1}-1)/2^{n+1}}} & R(n+1). \end{array}$$

After tensoring with K each term is K with horizontal maps vanishing and vertical maps alternating between 0 and id_K. This gives the claim upon taking vertical colimits.

(11) The algebra $R = K[T^{1/2^\infty}]/T$ and ideal $I = (T^{1/2^\infty})/T$ from the previous point form a typical example for which $I \otimes_R I$, but not I itself, is flat: We already used above that over $K[T^{1/2^\infty}]$ the ideal $(T^{1/2^\infty})$ is flat, and one easily checks that

$$I \otimes_R I \simeq R \otimes_{K[T^{1/2^\infty}]} (T^{1/2^\infty}) \otimes_{K[T^{1/2^\infty}]} (T^{1/2^\infty}),$$

whereas I itself is not the base change of $(T^{1/2^\infty})$. From the fibre sequence $I \otimes^{\mathbb{L}}_R I \to I \to I \otimes^{\mathbb{L}}_R K$ and the calculation in the previous point, one then reads off that $\ker(I \otimes_R I \to I) \cong K$ which gives

$$\pi_*(R/I^\infty) = \Lambda_K(K[1]),$$

an exterior algebra on one generator in degree 1.

(12) For

$$R_n = K[T_1^{1/2^\infty}, \dots, T_n^{1/2^\infty}]/(T_1, \dots, T_n)$$

and $I_n = (T_1^{1/2^\infty}, \dots, T_n^{1/2^\infty})$, we then have $R_n/I_n^\infty = (R_1/I_1^\infty)^{\otimes^{\mathbb{L}}_K n}$, so

$$\pi_*(R_n/I_n^\infty) = \Lambda_K(K^n[1]),$$

by the previous point. This in particular shows that even for static rings R, the animated rings R/I^∞ can have arbitrarily high non-trivial homotopy in the absence of any flatness assumption on I.

(13) Theorem A shows that an animated commutative ring structure on R induces a unique compatible one on R/I^∞; using the desciption of animated commutative rings as algebras over the monad of derived symmetric powers [Rak20, Section 4.2], this implies that $\mathcal{D}(R)$ and $\mathcal{D}(R/I^\infty)$ are equipped with derived functors of Sym^n, compatible under extension of scalars. It follows that also the category $\mathrm{a}\mathcal{D}_I(R)$ of derived almost modules carries such operations compatible with the left adjoint to the localisation $\mathcal{D}(R) \to \mathrm{a}\mathcal{D}_I(R)$. In most examples, these derived symmetric powers in fact simply descend from $\mathcal{D}(R)$ to $\mathrm{a}\mathcal{D}_I(R)$: This is true precisely if for every $k \geq 2$ the ideal I is generated by the k-th powers of its elements and in particular it happens in all the examples above and always when $I \otimes_R I$ is flat; to see this combine [GR03, Proposition 2.1.7 (ii)], [GR04, Theorem 14.1.57], [GR04, Example 14.1.60] and recall that $\mathbb{L}\mathrm{Sym}^n_R(M[2]) \cong \mathbb{L}\Gamma^n_R(M)[2n]$, so that ruling out descent for derived symmetric powers is the same as ruling it out for derived divided powers. For an explicit example where it fails, let S denote the set of finite strings of 0's and 1's and take

$$R = \mathbb{F}_p[T_s \mid s \in S]/(T_s - T_{s*0} \cdot T_{s*1}, T_s^p \mid s \in S)$$

with I generated by all the variables; then the ideal generated by the p-th powers of elements of I is trivial.

This is the only important structural result we are aware of, that actually requires such a flatness assumption.

(14) Incidentally, as the algebra R/I^∞ from the previous item pulls back to R under Frobenius, i.e. $R/I^\infty \otimes^{\mathbb{L}}_R \mathrm{Fr}^* R \simeq R$ (by Theorem A this can be checked on π_0 after all), it also yields an example, where the Frobenius endomorphism of R does not induce the identity, or even an isomorphism, on the locale of derived localisations of R, i.e. the smashing spectrum of $\mathcal{D}(R)$, answering a question of Efimov.

(15) Finally, let us remark that the categories $\mathrm{aMod}_I(A)$, alongside categories of sheaves on locally compact Hausdorff spaces, and categories of nuclear modules in condensed mathematics, are typical examples of compactly assembled categories that need not be compactly generated, e.g. for $I = (T^{1/2^\infty}) \subset K[T^{1/2^\infty}] = R$ this is due to Keller [Kel94]. In long anticipated work Efimov [Efi24] recently defined a version of algebraic K-theory for such categories, and there results a fibre sequence

$$\mathrm{K}(\mathrm{aMod}_I(A)) \longrightarrow \mathrm{K}(A) \longrightarrow K(A/I^\infty)$$

of spectra for every idempotent $I \subseteq \pi_0 A$. It is this connection to algebraic K-theory that originally sparked the present note.

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