Equivariant Poincaré duality for cyclic groups of prime order and the Nielsen realisation problem

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In this companion article to [\[HKK24\]](#page-28-0), we apply the theory of equivariant Poincaré duality developed there in the special case of cyclic groups C_p of prime order to remove, in a special case, a technical condition given by Davis-Lück [\[DL24\]](#page-28-1) in their work on the Nielsen realisation problem for aspherical manifolds. Along the way, we will also give a complete characterisation of C_p –Poincaré spaces as well as introduce a genuine equivariant refinement of the classical notion of virtual Poincaré duality groups which might be of independent interest.

Figure [1](#page-0-0): A wall tiling at the Alhambra, in Granada, Spain, with symmetry group $\degree p3\degree^1$

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1 Introduction

A famous question due to Jakob Nielsen [\[Nie32\]](#page-29-0) in geometric topology is the following: can any finite subgroup $G \subset \pi_0$ h $Aut(\Sigma_g)$ be lifted to an actual continuous group action on Σ_g , for Σ_q a closed oriented surface of genus $g \geq 0$? This turns out to be possible, with Nielsen settling the case of G finite cyclic, and Kerckhoff [\[Ker83\]](#page-28-2) the general case.

In high-dimensions, asking for too direct a generalisation of Nielsen's question inevitably results in wrong statements as there is simply no reason why a general homotopy equivalence $h: M \to M$ should be homotopic to any homeomorphism. However, rigidity phenomena in the theory of closed aspherical manifolds - closed connected manifolds with contractible universal covers - give some hope in generalising Nielsen's and Kerckhoff's results in this direction. This question may thus fairly be called the "generalised Nielsen realisation problem for aspherical manifolds" and has been investigated quite intensively, see for example [\[RS77;](#page-29-1) [DL03;](#page-28-3) [BW08;](#page-28-4) Lüc22; [DL24\]](#page-28-1).^{[2](#page-1-0)}

Unfortunately, even the hypothesis of closed aspherical manifolds is not quite adequate, and we refer to [\[Wei23\]](#page-29-3) for a delightful survey of counterexamples. Nevertheless, it turns out to be quite easy to dodge all potential reasons for counterexamples (for example, the failure for the existence of necessary group extensions due to Raymond–Scott [\[RS77\]](#page-29-1)) by asking a slight variation of the generalised Nielsen problem:

Question 1.1. Let M be an aspherical manifold with fundamental group π and consider an extension of groups $1\to\pi\to\Gamma\to G\to 1$ where G is finite of odd order $^3.$ $^3.$ $^3.$ Does the π -action on the universal cover \widetilde{M} of M extend to a Γ -action such that

$$
\widetilde{M}^H \simeq \begin{cases} * & \text{if } H \leq \Gamma \text{, and } H \text{ is finite;} \\ \emptyset & \text{if } H \leq \Gamma \text{, and } H \text{ is infinite?} \end{cases}
$$

Equivalently, does the π -action on the universal cover of M extend to a Γ -action in a way such that the resulting Γ -space models E_Γ Fin, the universal space for proper Γ -actions?

Provided the answer to Question [1.1](#page-1-2) is yes, one may construct a G -action on M by using the residual action on $\pi \backslash M$. For an account of the relation of Question [1.1](#page-1-2) to the generalized Nielsen realization problem in terms of homomorphisms $G \to \pi_0(hAut(M)) \cong$ $Out(\pi_1(M))$, we refer the reader to the introduction of [\[DL24\]](#page-28-1).

In this article, we give a positive answer to Question [1.1](#page-1-2) in the very special situation when M is high-dimensional, π is hyperbolic, $G = C_p$ for p odd, and if the extension is what we call *pseudofree*, i.e. if each nontrivial finite subgroup $F \subset \Gamma$ satisfies $N_{\Gamma}F = F$. Geometrically, this predicts that any Γ -manifold model M must have discrete fixed points (see Remark [4.1.5\)](#page-18-0), whence the name. One of our main results is the following:

 2 For completeness, we mention here that there is also a large body of work on the Nielsen realisation problem for not necessarily aspherical 4–manifolds, c.f. for instance [\[FL24;](#page-28-5) [BK23;](#page-28-6) [Lee23;](#page-28-7) [Kon24\]](#page-28-8), which has a much more geometric flavour.

 3 Taking G to be of odd order implies that certain UNil-valued obstructions vanish, see [\[DL24,](#page-28-1) Thm. 1.16.] or [\[Wei23,](#page-29-3) Sec. 6.4.].

Theorem A. Consider a group extension

$$
1 \to \pi \to \Gamma \to C_p \to 1 \tag{1}
$$

for an odd prime p. Suppose that

(1) $\pi = \pi_1(M)$ for a closed orientable^{[4](#page-2-0)} aspherical manifold M of dimension at least 5,

(2) π is hyperbolic,

(3) Γ is pseudofree.

Then there exists a cocompact Γ -manifold model for E_{Γ} Fin.

To the best of our knowledge, the most general existence result for manifold models for E_{Γ} Fin that does not refer to specific differential geometric constructions is due to Davis-Lück [$DL24$, Thm. 1.16], whose methods are mainly surgery– and K–theoretic. They prove Theorem [A](#page-1-3) under an additional necessary group homological "Condition (H)" (c.f. Condi-tion [1.3\)](#page-3-0) on Γ which is previously considered mysterious. This Condition (H) was discovered by Lück in [Lüc22] as necessary for the existence of manifold models for E_Γ Fin, but was also used to construct certain models for E_Γ Fin which satisfy some kind of equivariant Poincaré duality. Davis-Lück then show in which situations these can actually be turned into equivariant manifolds. Condition (H), however, seems complicated and hard to verify. Our main contribution to this problem is to show that, in the situation of Theorem [A,](#page-1-3) Condition (H) is actually automatic, and we achieve this by locating it in the more conceptual context of equivariant Poincaré duality as developed in $[HKK24]$. We hope that these techniques will allow us to go beyond the pseudofree situation where Davis-Lück applies, so that in the future we might be able to construct group actions on aspherical manifolds with nondiscrete fixed point sets. As will be clear later, our main input to remove Davis-Lück's Condition (H), Theorem [C,](#page-5-0) does not refer to discrete fixed points at all.

To round off our commentary on Theorem [A,](#page-1-3) it should also be noted that, in principle, the theorem reduces the geometric problem to a purely algebraic one of producing the appropriate group extension under the given hypotheses on p and π . As explained e.g. in [\[DL24,](#page-28-1) p.1], there is an obstruction measuring when a homomorphism $C_p \to \text{Out}(\pi)$ is induced by an extension [\(1\)](#page-2-1). It vanishes for hyperbolic groups as they have trivial center. For the Nielsen realisation problem, Theorem [A](#page-1-3) thus has the following implication.

Corollary 1.2. Let M be a closed orientable aspherical manifold with hyperbolic fundamental group of dimension at least 5, p an odd prime, and $\alpha\colon C_p\to\mathrm{Out}(\pi_1M)$ a homomorphism. Then the Nielsen realisation problem for α admits a solution, provided the associated extension Γ is pseudofree.

Before moving on to elaborate on equivariant Poincaré duality as used in this work, we first state the aforementioned Condition (H) and recall the argument of $[L\ddot{u}c22, L\dot{e}m$ ma 1.9] as to why it is necessary for the conclusion of Theorem [A](#page-1-3) to hold. This shows that Condition

⁴ Orientability is assumed only to simplify the exposition, and can be removed with some care.

(H) is not merely an artefact of the proof strategy of [\[DL24\]](#page-28-1) but is rather a point that must be dealt with in one way or another.

For a pseudofree extension [\(2\)](#page-5-1), a result of Lück–Weiermann [$LW12$] (c.f. Theorem [4.1.4\)](#page-18-1) shows that the subspace $E_{\Gamma} \text{Fin}^{>1}$ of points in $E_{\Gamma} \text{Fin}$ with nontrivial isotropy is discrete, more precisely, $\underline{E_{\Gamma}}{\rm Fin}^{>1}\simeq\coprod_{F\in{\cal M}}\Gamma/F$ where F runs through a set of representatives of conjugacy classes of nontrivial finite subgroups. Writing $H^{\Gamma}_*(X) := H_*(X_{h\Gamma}; \mathbb{Z})$ for the integral Borel homology of a space X with Γ -action, the condition may be stated as:

Condition 1.3 (Condition (H)). For each finite subgroup $F \neq 1$ of Γ , the composite

$$
H_d^{\Gamma}(\underline{E}_{\Gamma}\underline{\mathrm{Fin}}, \underline{E}_{\Gamma}\underline{\mathrm{Fin}}^{>1}) \xrightarrow{\partial} H_{d-1}^{\Gamma}(\underline{E}_{\Gamma}\underline{\mathrm{Fin}}^{>1}) \simeq \bigoplus_{F' \in \mathcal{M}} H_{d-1}(BF') \xrightarrow{\mathrm{proj}_F} H_{d-1}(BF)
$$

is surjective.

To see why condition (H) is necessary for the existence problem, suppose that there exists a d-dimensional cocompact manifold model N for E_Γ Fin. Let us assume for simplicity that N is smooth and that Γ acts smoothly preserving the orientation. As mentioned before, the singular part N^{-1} of N of points with nontrivial isotropy is discrete if the extension is pseudofree. Denote by Q the complement of an equivariant tubular neighbourhood of $N^{>1}$ in N with boundary ∂Q . Then the Γ-action on Q is free and the quotient pair $(\Gamma \backslash Q, \Gamma \backslash \partial Q)$ is a compact d-manifold with boundary. See \S 1 for an illustration. Thus, for every path component L of $\Gamma \backslash \partial Q$, we obtain the commutative diagram

$$
H_d(\Gamma \backslash Q, \Gamma \backslash \partial Q) \longrightarrow H_{d-1}(\Gamma \backslash \partial Q) \xrightarrow{\text{proj}} H_{d-1}(L)
$$

\n
$$
\downarrow \simeq \qquad \qquad \downarrow \qquad \qquad \downarrow
$$

\n
$$
H_d^{\Gamma}(\underline{E_{\Gamma}} \underline{\text{Fin}}, \underline{E_{\Gamma}} \underline{\text{Fin}}^{>1}) \longrightarrow H_{d-1}^{\Gamma}(\underline{E_{\Gamma}} \underline{\text{Fin}}^{>1}) \xrightarrow{\text{proj}} H_{d-1}(BF).
$$

The left vertical arrow is an equivalence using excision and that homology of $(\Gamma \backslash Q, \Gamma \backslash \partial Q)$ agrees with Borel homology of $(Q, \partial Q)$ as the Γ-action is free. The fundamental class of $(\Gamma \backslash Q, \Gamma \backslash \partial Q)$ gets sent to a fundamental class of each boundary component along the upper composite, and so the top composite is surjective. Moreover, note that each component L of $\Gamma \backslash \partial Q$ is obtained as the quotient of a sphere by a free action of the isotropy group F of the corresponding fixed point in $N^{>1}$. Recall that for any free F -action on a $(d-1)$ –sphere S for a finite group F, the map $F \backslash S \to BF$ is is $(d-1)$ –connected and induces a surjection on homology up to degree $d-1$ so the right vertical map is surjective. Together this shows that the bottom composite is surjective in each component.

1.1 Equivariant Poincaré duality

Equivariant Poincare duality is fundamentally about understanding group actions on mani- ´ folds. The notion of a G -equivariant Poincaré complex is designed to satisfy more or less all the homological or cohomological constraints that a smooth G -manifold satisfies. In particular, satisfying equivariant Poincaré duality can *obstruct* the existence of certain group actions

Figure 2: Discs around the singular part for the symmetry group p3

on manifolds. This philosophy is old and has been quite successful, and we exploited it in [\[HKK24\]](#page-28-0) to generalise some classical nonexistence results^{[5](#page-4-1)} using new methods.

In this article, we want to carry across a different point: equivariant Poincaré duality does not only obstruct, but is also quite useful to *construct* group actions on manifolds. The testing ground we chose to demonstrate our claim is Question [1.1.](#page-1-2) Here, we will only employ the theory of equivariant Poincaré duality for the group $G = C_p$, and since this case is much simpler than for general compact Lie groups, we hope that it will demystify the more abstract discussions in [\[HKK24\]](#page-28-0). In particular, we can keep the level of equivariant stable homotopy theory used throughout at a minimum, while still showing some standard manipulations. We hope that readers with an interest in geometric topology and homotopy theory might find this to be a useful first exposition to categories of genuine G -spectra and their uses.

Equivariant Poincaré duality for the group C_p

Our first theoretical goal is to give a characterisation of C_p -Poincaré duality that is adapted to our application on the Nielsen realisation problem. In [\[HKK24\]](#page-28-0) we showed that if X is a C_p -space, then the following hold:

- (1) If \underline{X} is C_p -Poincaré, then the underlying space X^e and the fixed points X^{C_p} are nonequivariantly Poincaré (c.f. [[HKK24,](#page-28-0) Thm. C]).
- (2) Even if \underline{X} is assumed to be a compact (i.e. equivariantly finitely dominated) C_p -space, requiring X^e and X^{C_p} to be nonequivariantly Poincaré is not sufficient to guarantee that \underline{X} is C_p -Poincaré (c.f. [[HKK24,](#page-28-0) Cor. 5.1.16]).

In this article, we identify the precise additional condition needed to ensure that \underline{X} is C_p Poincaré in the situation of (2) .

Theorem B (c.f. Theorem [3.3.3\)](#page-16-0). Let \underline{X} be a compact C_p -space. Denote by $\varepsilon \colon X^{C_p} \to X^e$ the inclusion of the fixed points, and assume that both X^{C_p} and X^e are nonequivariantly Poincaré. Let $D_{X^{C_p}} \in \text{Fun}(X^{C_p}, \text{Sp}^{BC_p})$ be the dualising sheaf of the fixed point space. Then \underline{X} is C_p -Poincaré if and only if the cofibre of the adjunction unit morphism in $\mathrm{Fun}(X^{C_p}, \mathrm{Sp}^{BC_p})$

$$
D_{X^{C_p}} \to \varepsilon^* \varepsilon_! D_{X^{C_p}}
$$

⁵See also the references therein for more results of this type.

pointwise lies in ${\rm Sp}_{\rm Ind}^{BC_p} \subseteq {\rm Sp}^{BC_p}$, the stable full subcategory generated by the image of $\text{Ind}_{e}^{C_p}: \text{Sp} \to \text{Sp}^{BC_p^{\text{un}}}.$

More details on the terms appearing in the theorem may be found in the material leading up to Theorem [3.3.3.](#page-16-0) In effect, this result gives an interpretation of the genuine equivariant notion of Poincaré duality purely in terms of nonequivariant and Borel equivariant properties. The method of proof is based on various cellular manoeuvres in equivariant stable homotopy theory developed in [§3,](#page-11-0) which might be of independent interest. Armed with this characterisation, we may now return to the modified Nielsen Question [1.1](#page-1-2) as we explain next.

Genunine virtual Poincaré duality groups and the proof of Theorem [A](#page-1-3)

Suppose we are given an extension of groups

$$
1 \to \pi \to \Gamma \to C_p \to 1 \tag{2}
$$

so that there exists a cocompact Γ -manifold N modeling the Γ -homotopy type E_{Γ} Fin. Under reasonable assumptions, we might expect that the C_p -space $\pi \backslash N$ is C_p -Poincaré. Motivated by this expectation, we will call Γ a genuine virtual Poincaré duality group if $\pi\backslash E_{\Gamma}$ Fin is a C_p -Poincaré space. In fact, we define the notion of a genuine virtual Poincaré duality group in a broader context in $\S 4$ and we hope that it can be a useful supplement to the classical theory of virtual Poincaré duality groups that appear for example in $[Bro94]$ $[Bro94]$. In any case, using Theorem [B,](#page-4-2) we will show our main result:

Theorem C (c.f. Theorem [5.1.4\)](#page-23-0). For the extension [\(2\)](#page-5-1), assume that E_Γ Fin is compact. If

- (1) for each nontrivial finite subgroup $F \subset \Gamma$, the group $W_{\Gamma}F$ is a Poincaré duality group;
- (2) the group π is a Poincaré duality group,

then Γ is a genuine virtual Poincaré duality group.

The significance of this result is that we relate the new notion of genuine virtual Poincaré duality groups, which enjoys good conceptual properties, with the classical notion of Poincaré duality groups, which is easier to check. It will turn out that, in the situation of Theorem [A,](#page-1-3) the group Γ satisfies the conditions of Theorem [C,](#page-5-0) and so we see that $\pi \backslash E_{\Gamma}$ Fin is C_p -Poincaré. In particular, this opens up the way for an equivariant fundamental class analysis (c.f. [\[HKK24,](#page-28-0) §4.5]) on the problem at hand, yielding the following:

Theorem D. Let Γ be as in Theorem [A.](#page-1-3) Then Γ satisfies Condition (H).

A proof of this, and the more general Proposition [5.2.3,](#page-25-0) will be given in [§5.2.](#page-24-0) Taking this for granted for the moment, we may now provide the proof of Theorem [A.](#page-1-3)

Proof of Theorem [A.](#page-1-3) This is now a direct consequence of [\[DL24,](#page-28-1) Theorem 1.4]. The group theoretic assumptions therein are satisfied by Lemma [5.1.1,](#page-20-0) whereas Condition (H) is shown to hold in Theorem [D.](#page-5-2) \Box

Structural overview

We recall in \S 2 some notions and constructions from the theory of equivariant Poincaré duality [\[HKK24\]](#page-28-0) that will be pertinent to our purposes. Next, in [§3,](#page-11-0) we work towards proving Theorem [B,](#page-4-2) and for this, it will be necessary to develop some theory on compact objects in C_p –genuine spectra. This we do in §[§3.1a](#page-12-0)nd [3.2,](#page-14-0) which might be of independent interest. Having set up the requisite basic theory, we return to the problem at hand and define the notion of genuine virtual Poincaré duality groups in $\S4$, refining the classical notion of virtual Poincaré duality groups. Therein, we will also prove a characterisation result tailored to our needs. Finally, we put together all the elements and prove Theorems C and D in [§5.](#page-20-1)

Conventions

This paper is written in the language of ∞ –categories as set down in [\[Lur09;](#page-29-5) [Lur17\]](#page-29-6), and so by a *category* we will always mean an ∞ –category unless stated otherwise.

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2 Recollections

There will be two types of equivariance in this paper, each playing a distinct role. The first kind will be defined for an arbitrary Lie group, which is covered in [§2.1;](#page-6-1) the second kind, covered in [§2.2,](#page-8-0) will be defined only for finite groups (in fact, it is defined more generally for compact Lie groups as in [\[HKK24\]](#page-28-0)) and is the one that supports stable homotopy theory and the theory of equivariant Poincaré duality in \S 2.3. More details on the materials in §[§2.2a](#page-8-0)nd [2.3](#page-10-0) together with references to the original sources may be found in [\[HKK24\]](#page-28-0).

2.1 Equivariant spaces

Throughout, let Γ be a Lie group.

Notation 2.1.1. Let $\mathcal{O}(\Gamma)$ be the topological category of homogeneous Γ –spaces, the full topological subcategory of the category of topological Γ-spaces on objects isomorphic to Γ/H for closed subgroups $H \leq \Gamma$.

Definition 2.1.2. The category S_{Γ} of Γ -spaces is defined as the category of presheaves $\mathrm{Psh}(\mathcal{O}(\Gamma)) \coloneqq \mathrm{Fun}(\mathcal{O}(\Gamma)^{\mathrm{op}}, \mathcal{S}).$

Construction 2.1.3 (Fundamental adjunctions). Genuine equivariant spaces participate in many adjunctions, the fundamental one that we will need being the following: let $\alpha: K \to \Gamma$ be a continuous homomorphism of Lie groups. By left Kan extension and restriction along the (opposite) induction functor ${\rm Ind}_{\alpha}^{\cal O} \colon {\cal O}(K)^{\rm op} \to {\cal O}(\Gamma)^{\rm op},$ we obtain the adjunction

$$
\text{Ind}_{\alpha} \coloneqq (\text{Ind}_{\alpha}^{\mathcal{O}})_{!} \colon \mathcal{S}_{K} \xrightarrow{\sim} \mathcal{S}_{\Gamma} : \text{Res}_{\alpha} \coloneqq (\text{Ind}_{\alpha}^{\mathcal{O}})^{*}.
$$

Specialising to the two cases of $\alpha = \iota: H \rightarrowtail \Gamma$ being a closed subgroup and $\alpha = \theta: \Gamma \rightarrowtail Q$ being a continuous surjection of Lie groups with kernel N , the adjunction above yields the following two adjunctions which we have given special notations

$$
\operatorname{Ind}_{H}^{\Gamma} := \operatorname{Ind}_{\iota} : \mathcal{S}_{H} \xrightarrow{\longrightarrow} \mathcal{S}_{\Gamma} : \operatorname{Res}_{H}^{\Gamma} := \operatorname{Res}_{\iota}
$$

$$
N \setminus (-) := \operatorname{Ind}_{\theta} : \mathcal{S}_{\Gamma} \xrightarrow{\longleftarrow} \mathcal{S}_{Q} : \operatorname{infl}_{\Gamma}^{Q} := \operatorname{Res}_{\theta}.
$$

Importantly, in the special case of a continuous surjection $\theta \colon \Gamma \to Q$, we have an adjunction $\operatorname{Res}_{\theta}^{\mathcal O}\colon \mathcal O(Q)^\mathrm{op}\rightleftharpoons \mathcal O(\Gamma)^\mathrm{op}:\operatorname{Ind}_{\theta}^{\mathcal O}$, and so $\operatorname{infl}_{\Gamma}^{\mathcal Q}=(\operatorname{Ind}_{\theta}^{\mathcal O})^*\simeq (\operatorname{Res}_{\theta}^{\mathcal O})_!.$

Observation 2.1.4. In particular, suppose we have homomorphisms of Lie groups $\iota: N \rightarrow \Gamma$ and $\theta \colon \Gamma \to Q$ which are injective and surjective, respectively, and such that the composite $\theta \circ \iota \colon N \to Q$ is also surjective. Writing π for the kernel of θ , we thus see that ker $(\theta \circ \iota)$ = N ∩ π. Since for composable homomorphisms of Lie groups α and β we have Ind_α∘β \simeq ${\rm Ind}_\alpha\circ{\rm Ind}_\beta,$ we see that in this case, there is a natural equivalence of functors $\mathcal S_N\to\mathcal S_Q$

$$
\pi \setminus \operatorname{Ind}_N^{\Gamma}(-) \simeq (N \cap \pi) \setminus (-).
$$

Construction 2.1.5 (Singular parts). Denote by $s: \mathcal{O}^{\text{sing}}(\Gamma) \subseteq \mathcal{O}(\Gamma)$ the full subcategory on the orbits Γ/H with H nontrivial. We then get the adjunction

$$
\operatorname{Psh}(\mathcal{O}^{\operatorname{sing}}(\Gamma)) \overset{-s_!}{\underset{\leftarrow}{\leftarrow}} \overset{s_!}{\longrightarrow} \operatorname{Psh}(\mathcal{O}(\Gamma)) = \mathcal{S}_{\Gamma}
$$

by restricting and left Kan extending along s . We abbreviate $(-)^{>1}$ = $s_!s^*$, writing $\varepsilon: \underline{X}^{>1} \to \underline{X}$ for the adjunction counit. Note that for an orbit $\underline{\Gamma/H} \in S_{\Gamma}$ we have $\Gamma/H^{>1} \simeq \emptyset$ if $H = e$ and $\varepsilon: \Gamma/H^{>1} \to \Gamma/H$ is an equivalence otherwise. We refer to $\underline{X}^{>1}$ as the *singular part of* \underline{X} and think of $\varepsilon\colon \underline{X}^{>1}\to \underline{X}$ as the inclusion of all points with nontrivial isotropy.

Notation 2.1.6. In the special case of $\Gamma = C_p$, we have $\mathcal{O}^{\text{sing}}(C_p) = \{C_p/C_p\} \simeq *,$ so that ${\rm Psh}(\mathcal{O}^{\rm sing}(C_p))\simeq \mathcal{S}.$ In this case, one can work out that $s_!$ just assigns a space to the constant diagram as an object in $\mathcal{S}_{C_p}.$ Moreover, $s^*\underline{X}\,=\,X^{C_p},$ and so we will also write $\underline{X}^{>1} \in \mathcal{S}_{C_p}$ as $\underline{X^{C_p}}$ interchangeably in this case.

Fact 2.1.7. Let Γ and Q be groups, $N \leq \Gamma$ a normal subgroup and $p: \Gamma \to Q$ a surjective group homomorphism. If Q acts on the topological space X, then there is a natural Γ/N equivariant homeomorphism

$$
N\backslash X \cong p(N)\backslash X.
$$

Here, X is considered as a Γ-space via p, and $p(N) \ X$ is considered as a Γ/N-space via $\Gamma/N \to Q/p(N)$. Specialising to orbit categories, we obtain the commutative diagram

$$
\mathcal{O}(Q) \xrightarrow{\text{Res}^{\mathcal{O}}} \mathcal{O}(\Gamma)
$$

$$
\downarrow_{\text{Ind}^{\mathcal{O}}} \qquad \qquad \downarrow_{\text{Ind}^{\mathcal{O}}} \qquad \downarrow_{\text{Ind}^{\mathcal{O}}} \mathcal{O}(Q/p(N)) \xrightarrow{\text{Res}^{\mathcal{O}}} \mathcal{O}(\Gamma/N).
$$

Applying $\mathrm{Psh}(-)$ with the $(-)$! functoriality, we get an equivalence of functors

$$
N\backslash \inf_{\Gamma} Q(-) \simeq \inf_{\Gamma/N} Q/p(N) \, p(N) \backslash (-) \colon \mathcal{S}_Q \to \mathcal{S}_{\Gamma/N}.
$$

Fact 2.1.8. Given any discrete group Γ and a Γ -space X , as well as a *proper* normal subgroup $N \subset \Gamma$ such that the N-action on \underline{X} is free, the inclusion $\underline{X}^{>1} \to \underline{X}$ induces equivalences

$$
(N\backslash \underline{X})^{>1}\xleftarrow{\simeq} (N\backslash \underline{X}^{>1})^{>1}\xrightarrow{\simeq} N\backslash \underline{X}^{>1}.
$$

Indeed, all functors involved commute with colimits, and the statement is clearly true for orbits, out of which every Γ–space may be built via colimits.

2.2 Genuine equivariant stable homotopy theory

Let G be a finite group. The stable category Sp_G of genuine G-spectra is a refinement of the category Sp^{BG} of spectra with G -action with better formal properties. This is a refinement in that Sp^{BG} sits fully faithfully in the category Sp_G , in fact in two different ways. One way to define Sp_G , following Barwick, is as the category $\text{Mack}_G(Sp) := \text{Fun}^\times(\text{Span}(\text{Fin}_G), Sp)$ of G–Mackey functors in spectra. A good introduction to the materials in this subsection may be found, for instance, in [\[MNN17,](#page-29-7) Part 2].

The category of genuine equivariant spectra is valuable as it is a particularly conducive environment for inductive methods enabled by many compatibility structures between these categories for different groups, expressed in terms of various adjunctions. Moreover, Sp_G should also be thought of as the "universal category of equivariant homology theories" on \mathcal{S}_G . For example, there is a symmetric monoidal colimit–preserving functor $\Sigma^\infty_+ \colon \mathcal{S}_G \to \mathrm{Sp}_G$ which is the analogue of the suspension spectrum functor nonequivariant spectra, and Sp_G is generated under colimits by $\{\Sigma_+^{\infty} G/H\}_{H\leq G}$.

Fact 2.2.1 (Restriction–(co)induction). For a subgroup $H \leq G$, we have the adjunctions

$$
\mathrm{Sp}_G \xrightarrow{\mathrm{Ind}_H^G} \mathrm{Sp}_H
$$

$$
\underset{\mathrm{Coind}_H^G}{\longrightarrow} \mathrm{Sp}_H
$$

where moreover, there is a canonical equivalence of functors ${\rm Ind}^G_H \, \simeq \, {\rm Coind}^G_H$ classically known as the Wirthmüller isomorphism.

Fact 2.2.2 (Genuine fixed points). There is a functor $(-)^G \colon \mathrm{Sp}_G \to \mathrm{Sp}$ called the *genuine* fixed points functor which, from the Mackey functors perspective, is given by evaluating at $G/G \in \text{Fin}_G$. This participates in an adjunction

$$
\text{infl}_G^e \colon \text{Sp} \xrightarrow{\longrightarrow} \text{Sp}_G : (-)^G
$$

where \inf^e_G preserves compact objects and is the unique symmetric monoidal colimit preserving functor from Sp to Sp_G. For every subgroup $H \leq G$, we may also define the genuine H –fixed points functor $(-)^H$ as the composite ${\rm Sp}_G \xrightarrow{{\rm Res}^G_H} {\rm Sp}_H \xrightarrow{(-)^H} {\rm Sp}_+$

Fact 2.2.3 (Borel fixed points). There is a standard Bousfield (co)localisation

$$
\mathrm{Sp}_G \xrightarrow{\beta_! \searrow} \mathrm{Sp}^{BG}
$$

where $\beta^* X \simeq X^e$, $\beta_!\beta^* X \simeq EG_+ \otimes X$, and $\beta_*\beta^* X \simeq F(EG_+, X)$. This well–known pair of adjunctions may for example be worked out from combining [\[MNN17,](#page-29-7) Thm. 3.9, Prop. 6.5, Prop. 6.6, Prop. 6.17]. Under the Mackey functors perspective, β^* is given by evaluating at $G/e \in \text{Fin}_G$. In particular, we see that Sp^{BG} embeds into Sp_G in two different ways, as mentioned above. Via the functor β^* as well as the homotopy orbits $(-)_{hG}$, homotopy fixed points $(-)^{hG},$ and Tate fixed points $(-)^{tG}$ functors $\mathrm{Sp}^{BG} \to \mathrm{Sp},$ we may also obtain the functors $(-)_{hG}, (-)^{hG}, (-)^{tG}\colon \mathrm{Sp}_G\to \mathrm{Sp}$ which also fit in a fibre sequence of functors $(-)_{hG} \rightarrow (-)^{hG} \rightarrow (-)^{tG}$. In particular, these functors only depend on the underlying spectrum with G–action.

Fact 2.2.4 (Geometric fixed points). There is a symmetric monoidal colimit–preserving functor $\Phi^G(-) \colon \mathrm{Sp}_G \to \mathrm{Sp}$ called the *geometric fixed points* which is uniquely characterised by sending $\sum_{+}^{\infty} \frac{G}{H}$ to 0 when $H \leq G$ and to $\mathbb S$ when $H = G$. For a subgroup $H \leq G$, we may also define Φ^H as the functor ${\rm Sp}_G \xrightarrow{{\rm Res}^G_H} {\rm Sp}_H \xrightarrow{\Phi^H} {\rm Sp.}$ The collection of functors $\Phi^H \colon \mathrm{Sp}_G \to \mathrm{Sp}$ for all $H \leq G$ is jointly conservative.

The geometric fixed points functor participates in an adjunction

$$
\Phi^G \colon \mathrm{Sp}_G \xrightarrow{\longleftarrow} \mathrm{Sp} : \Xi^G
$$

where Ξ^G is fully faithful. For $E\in \mathrm{Sp}_2\,\Xi^G E\in \mathrm{Sp}_G$ is concretely given by the $G\text{-Mackey}$ functor which assigns E to G/G and 0 to all G/H for $H \leq G$.

Furthermore, using that Sp is the initial presentably symmetric monoidal stable category, it is also not hard to see that $\Phi^G{\rm infl}_G^e\simeq{\rm id}_{\rm Sp}.$

Next, we recall the standard decomposition in the special case of genuine C_p -spectra, which is all that we will need in our work.

Fact 2.2.5 (C_p -stable recollement). Let $G = C_p$. In this case, some of the adjunctions we have seen fit into a stable recollement (also called split Verdier sequence)

$$
Sp \xleftarrow{\Phi^{C_p}} \mathbb{S}p_{C_p} \xleftarrow{\beta_!} \mathbb{S}p_{C_p} \xleftarrow{\beta_!} \mathbb{S}p^{BC_p}
$$

in that the top two layers of composites are fibre–cofibre sequences of presentable stable categories. This may be deduced, for example, from a combination of [\[MNN17,](#page-29-7) §6.4] and [\[CDH+23b,](#page-28-10) §A.2]. From this, one obtains for every $E \in \mathrm{Sp}_{C_p}$ a pullback square

$$
E^{C_p} \longrightarrow \Phi^{C_p} E
$$

$$
\downarrow \qquad \qquad \downarrow
$$

$$
E^{hC_p} \longrightarrow E^{tC_p}
$$

of spectra (c.f. for instance [\[MNN17,](#page-29-7) Thm. 6.24] or [\[CDH+23b,](#page-28-10) Prop. A.2.12]).

2.3 Equivariant Poincaré duality

Let G be a finite group. We briefly recall the theory of G -equivariant Poincaré duality spaces, which is built upon the notion of G–categories. Recall that the category Cat_G of G-categories is defined as $\text{Fun}(\mathcal{O}(G)^{\text{op}}, \text{Cat})$, akin to the category of G-spaces. This category admits an internal functor category $\underline{\text{Fun}(\mathcal{C}, \mathcal{D})}$ for each pair $\mathcal{C}, \mathcal{D} \in \text{Cat}_G$. This satisfies

$$
\underline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(G/H) \simeq \mathrm{Fun}_H(\mathrm{Res}^G_H \mathcal{L}, \mathrm{Res}^G_H \mathcal{D}),
$$

where the latter is the category of $H\text{-functors}$ from $\mathrm{Res}^G_H\underline{\mathcal{C}}$ to $\mathrm{Res}^G_H\underline{\mathcal{D}}.$ A very important G-category for us is the G-category $\underline{\mathrm{Sp}}$ of genuine G-spectra given by $\underline{\mathrm{Sp}}(G/H)=\mathrm{Sp}_H.$

Since S_G is a full subcategory of Cat_G , we may view a G –space \underline{X} as an object in Cat_G For a G -space \underline{X} we denote the unique map to the point by

$$
X\colon \underline{X}\to \underline{\ast}.
$$

Write $\underline{\mathrm{Sp}}^X = \underline{\mathrm{Fun}(X, \mathrm{Sp})}$ for the category of equivariant local systems on \underline{X} . Explicitly, that amounts to specifying a local system of H-spectra $X^H \to \text{Sp}_H$ for each subgroup $H \subset G$ plus compatibilities. Colimit, restriction and limit of local systems give two adjunctions

$$
\underbrace{\mathrm{Sp}^X} \underbrace{\underbrace{X_!}_{x*}}_{x_*} \underbrace{X_!}_{\underbrace{S}^{}_{\underbrace{S
$$

One should think of the colimit X_1E of an equivariant local system E on \underline{X} as equivariant homology of X twisted by E and similarly of the limit as equivariant twisted cohomology.

The following is a recollection from [\[HKK24,](#page-28-0) Sec. 4.1.]. A compact G -space X admits an equivariant dualising spectrum $D_X \in \text{Fun}_G(\underline{X}, \underline{\text{Sp}})$ which comes together with a collapse map $c: \mathbb{S}_G \to X_!D_X$. These are uniquely characterised by the property that the induced capping map

$$
c \cap_{\xi} (-): X_*(-) \to X_!(D_X \otimes -)
$$
 (3)

is an equivalence. Applying fixed points and homotopy groups, the collapse map really corresponds to a class in twisted equivariant homology such that capping with it induces an equivalence between equivariant cohomology and twisted equivariant homology. Let us just mention that there is the larger class of *twisted ambidextrous* G -spaces for which an equivalence of the form [\(3\)](#page-10-1) exists.

Definition 2.3.1. A compact (or twisted ambidextrous) G -space \underline{X} is called G -Poincaré if the dualising spectrum D_X takes values in \mathcal{P} ic(Sp).

Notation 2.3.2. Let $\xi \in \text{Fun}_G(\underline{X}, \underline{\text{Sp}})$ be a local system of G -spectra on the G -space \underline{X} . For an H-fixed point $y \in X^H$, i.e. a map $y: \underline{G/H} \to \underline{X}$, using the composition

$$
\mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}}) \xrightarrow{y^*} \mathrm{Fun}_G(\underline{G/H}, \underline{\mathrm{Sp}}) \simeq \mathrm{Sp}_H
$$

we obtain an H-spectrum that we will denote by $\xi(y)$.

Note that a compact (or twisted ambidextrous) G -space X is G -Poincaré if and only if for each $y \in X^H$ the value $D_X(y) \in \text{Sp}_H$ is an invertible H-spectrum.

Theorem 2.3.3 ([\[HKK24,](#page-28-0) Thm 4.2.9.]). Let X be a G-Poincaré space. Then for each closed subgroup $H \leq G$, the space X^H is a (nonequivariant) Poincaré space. Moreover, its dualising spectrum is given as the composite

$$
X^H \xrightarrow{D_X} \text{Sp}_H \xrightarrow{\Phi^H} \text{Sp}.
$$

Example 2.3.4 ([\[HKK24,](#page-28-0) Cor. 5.1.16.]). Let p be an odd prime and $k \ge 1$ an integer. There exists a compact C_p -space \underline{X} for which X^e is contractible while $X^{C_p} \simeq \mathbb{R}P^{2k}.$ None such C_p -space is C_p -Poincaré. In particular, there are compact G -spaces such that all fixed points are nonequivariant Poincaré spaces which are not themselves G -Poincaré.

3 Poincaré duality for the group $C_{\it n}$

In this section, we investigate equivariant Poincaré duality for the group C_p more closely. Our goal is to prove Theorem B (c.f. Theorem [3.3.3\)](#page-16-0) which gives a somewhat computable condition on a *compact* C_p *-*space \underline{X} to be C_p -Poincaré assuming that X^e and X^{C_p} are nonequivariant Poincaré spaces. This amounts to checking that $D_X : \underline{X} \to \underline{Sp}$ lands in invertible objects.

Since invertibility is a pointwise condition and since we already know that $D_{X^e}: X^e \to \text{Sp}$ lands in invertible objects, it suffices to show that $D_{\underline{X}}(x)\,\in\,\mathrm{Sp}_{C_p}$ is invertible for every $x \in X^{C_p}.$ Moreover, from our hypothesis and Theorem [2.3.3,](#page-11-1) we already know that $D_{\underline{X}}(x)^e$ and $\Phi^{C_p} D_X(x)$ are invertible spectra. This consideration leads us to record the following well–known observation.

Lemma 3.0.1. Let $E \in \mathrm{Sp}_{C_p}$ be such that E^e and $\Phi^{C_p}E$ are invertible. Then:

E is invertible
$$
\iff
$$
 E is dualisable \iff *E* is compact.

Proof. In Sp_{C_p} , dualisablity and compactness are equivalent, and invertible spectra are dualisable. So it suffices to show that if E is dualisable, then it is invertible, i.e. that the counit $E \otimes E^{\vee} \to \mathbb{S}_{C_p}$ is an equivalence. But this can be checked after applying $(-)^e$ and $\Phi^{C_p}(-)$, which are jointly conservative. As both of these functors are symmetric monoidal, the counit for E is sent to the counit for E^e and $\Phi^{C_p}E$, both of which we assumed to be equivalences. \Box

Thus, by virtue of Lemma [3.0.1,](#page-11-2) our task at hand is tantamount to ensuring that the C_p spectrum $D_X(x)$ is compact for every $x\in X^{C_p}.$ To this end, we will employ various cellular manoeuvres in [§3.1](#page-12-0) to obtain "compact approximations" to any C_p -spectrum; we then characterise compactness of a C_p –spectrum with vanishing geometric fixed points through its underlying Borel- C_p -spectrum in [§3.2.](#page-14-0) Lastly, we combine all these in [§3.3](#page-15-0) to obtain the promised recognition principle for C_p -Poincaré spaces.

Remark 3.0.2. Our work on C_p -spectra heavily drew inspiration from at least two sources. The first one being [\[Kra20\]](#page-28-11), which gives a nice computation of the Picard group of Sp_{C_p} , and whose methods we expand on. The second one is [\[KMS21\]](#page-28-12), which gives another compactness (or dualisability) criterion for C_p -spectra. Our approach is not exactly tailored to the methods in the latter source, and we will not need to refer to them, but it might be possible that they give another way of proving the main results in this section.

3.1 Cellular manoeuvres and compact approximations

Recall that a C_p -spectrum is *finite* if it lies in the stable subcategory of Sp_{C_p} generated by $\Sigma_+^\infty \underline{C}_p/e = \mathrm{Ind}_e^{C_p}$ S and $\Sigma_+^\infty \underline{C}_p/C_p = \mathrm{inf}_e^{C_p}$ S. A C_p -spectrum is compact if and only if it is a retract of a finite C_p -spectrum.

Lemma 3.1.1 (Tucking the free part). Let $X \in \mathrm{Sp}_{C_p}$ be such that X^e is bounded below and such that $\pi_k(X^e)$ is a finitely generated abelian group for $k\le N$ for some $N.$ Then there is a fiber sequence

$$
F \to X \to Y
$$

in Sp_{C_p} such that F is finite, Y^e is N-connected and $\Phi^{C_p}X\to \Phi^{C_p}Y$ is an equivalence.

Proof. Note that if $A \to B$ and $B \to C$ are maps of C_p -spectra whose fibers are compact with trivial geometric fixed points, then the composition $A \rightarrow C$ satisfies the same condition. Thus, by induction it suffices to consider the case where $N = 0$ and X^e is -1 -connected. Pick a finite set of generators $\{f_i\colon \mathbb{S}\to X^e\}$ of $\pi_0(X^e)$. Then the composition

$$
f = \left(\bigoplus_i \text{IndS} \xrightarrow{\bigoplus_i \text{Ind}f_i} \text{Ind} X^e \xrightarrow{c} X\right),
$$

where c denotes the counit, induces a surjection on π_0 upon applying $(-)^e$. Now define Y to be the cofiber of f and F its source. This finishes the proof since Ind $\mathcal S$ is compact and satisfies $\Phi^{C_p} \text{Ind} \mathbb{S} \simeq 0$. П

Remark 3.1.2. Unlike taking the appropriate connective covers, the procedure of tucking cannot be used in general to kill the homotopy groups of X^e . The reason is that the effect on the next higher homotopy group is quite brutal. However, if X^e is $(l-1)$ -connected and π_lX^e is a finitely generated free $\mathbb{Z}[C_p]$ -module, then the proof of Lemma [3.1.1](#page-12-1) shows that we can kill $\pi_l X^e$ while making sure that the next higher homology group is unchanged. On the other hand, tucking preserves the geometric fixed points whereas the aforementioned connective covers do not.

Lemma 3.1.3. Let Q be a compact spectrum, E a C_p -spectrum and $f \colon Q \to \Phi^{C_p}E$ a map. Then there exists a compact C_p -spectrum F , a map $g\colon F\to E$, and an identification $\Phi^{C_p}F\simeq Q$ under which $\Phi^{C_p} g = f$.

Proof. First we reduce to the case where E is compact. Write $E = \text{colim}_{i \in I} E_i$ as a filtered colimit of compact C_p –spectra. As Φ^{C_p} commutes with colimits and Q is compact, there is some $i\in I$ for which $f\colon Q\to \Phi^{C_p}E$ factors through the compact spectrum $\Phi^{C_p}E_i.$ If now F is a compact C_p -spectrum together with a map $F\to E_i$ which induces the map $Q\to \Phi^{C_p}E_i$ on geometric fixed points, then the composite $F \to E_i \to E$ satisfies the claim.

Now assume that E is compact. Then there exists $k \in \mathbb{Z}$ such that each map $Q \to T$ to a k -connected spectrum T is nullhomotopic. By Lemma [3.1.1,](#page-12-1) we can find $U\in \mathrm{Sp}_{C_p}$ together with a map $E \to U$ that has compact fiber and that induces an equivalence on geometric fixed points, such that U^e is $(k-1)$ -connected. Thus, ΣU^e is k -connected, and consequently also ΣU_{hC_p} is k -connected. Note that U is compact as well.

Consider the following diagram where the lower horizontal maps form a fiber sequence and the square is cartesian by Facts [2.2.3](#page-9-0) and [2.2.5.](#page-9-1)

The nullhomotopy of the long bent arrow induces the dashed morphism a , which in turn determines the dashed morphism b . By the adjunction from Fact [2.2.2,](#page-8-1) the map b is adjoint to a map $q\colon \, \text{infl}_{C_p}^e\,Q \to U$ which induces the map $Q \to \Phi^{C_p}U$ on geometric fixed points. This fits into a fibre sequence $F \to E \to \mathrm{cofib}(q)$ where the map $E \to \mathrm{cofib}(q)$ is induced by the map $E \to U$ from above. Note that F is compact as E is compact, U is compact as observed above, and $\mathrm{infl}_{C_p}^e$ preserves compactness. On geometric fixed points, under the identification $\Phi^{C_p} E \xrightarrow{\simeq} \Phi^{C_p} U$, this gives

$$
\Phi^{C_p} F \simeq \operatorname{fib}(\Phi^{C_p} E \to \operatorname{cofib}(Q \to \Phi^{C_p} E)) \simeq Q
$$

Corollary 3.1.4. If E is a C_p -spectrum with $\Phi^{C_p}E$ compact, then there exists a finite C_p spectrum F and a map $g: F \to E$ which induces an equivalence on geometric fixed points.

Proof. Set $f = id_{\Phi^{C_p}E}$ in Lemma [3.1.3.](#page-13-0)

The following lemma will be useful later.

Lemma 3.1.5. Consider a cospan $X \stackrel{f}{\to} Z \stackrel{g}{\leftarrow} Y$ in Sp_{C_p} where f and g induce equivalences on geometric fixed points. Additionally suppose $\Phi^{C_p} X \in \text{Sp}$ is compact. Then there exists a commutative square

with F compact such that all maps are equivalences on geometric fixed points.

Proof. Let $E = X \times Z Y$ and find a fiber sequence $F \to E \to E'$ with F compact and $\Phi^{C_p} E' \simeq 0$ as provided by Corollary [3.1.4.](#page-14-1) Then the outer quadrilateral in the diagram

has the desired properties.

3.2 Compact and induced C_p -spectra

In this section we characterise compactness for C_p -spectra with trivial geometric fixed points through their underlying Borel C_p -spectrum. Notice that C_p -spectra with vanishing geometric fixed points have the following crucial properties.

Lemma 3.2.1 (CoBorel compactness). *Let X be a* C_p *-spectrum with* $\Phi^{C_p}X \simeq 0$ *. Then*

- (1) for every $Y \in Sp_{C_p}$ the map $(-)^e \colon \text{Map}_{Sp_{C_p}}(X, Y) \to \text{Map}_{Sp^{BC_p}}(X^e, Y^e)$ is an equivalence;
- (2) the C_p -spectrum X is compact in Sp_{C_p} if and only if X^e is compact in $\mathrm{Sp}^{BC_p}.$

Proof. We use the notations from Fact [2.2.5.](#page-9-1) Observe by Fact [2.2.5](#page-9-1) that $\Phi^{C_p} X \simeq 0$ is equivalent to the condition that the adjunction counit $\beta_!\beta^*X \to X$ is an equivalence. Now for (1), just note that $\text{Map}_{\text{Sp}_{C_p}}(X, Y) \stackrel{\simeq}{\leftarrow} \text{Map}_{\text{Sp}_{C_p}}(\beta_! \beta^* X, Y) \stackrel{\simeq}{\rightarrow} \text{Map}_{\text{Sp}^{BC_p}}(\beta^* X, \beta^* Y)$ as claimed. For (2), note that $\beta_! \colon \mathrm{Sp}^{BC_p} \ \to \ \mathrm{Sp}_{C_p}$ preserves and detects compactness as it is fully faithful and admits the colimit preserving right adjoint β^* . This shows that $X^e = \beta^* X \in \text{Sp}^{BC_p}$ is compact if and only if $X \simeq \beta_!\beta^* X \in \text{Sp}_{C_p}$ is compact. \Box

 \Box

 \Box

Notation 3.2.2. Following the notation from [\[NS18,](#page-29-8) Def. I.3.7], we write ${\rm Sp}_{\rm ind}^{BC_p} \subseteq {\rm Sp}^{BC_p}$ for the smallest idempotent–complete stable subcategory generated by the image of the functor $\mathrm{Ind}_e^{C_p} \colon \mathrm{Sp} \to \mathrm{Sp}^{BC_p}.$ Similarly, we write $\mathrm{Sp}_{C_p}^{\mathrm{ind}} \subseteq \mathrm{Sp}_{C_p}$ for the smallest idempotent– complete stable subcategory containing the image of $\mathrm{Ind}_e^{C_p} \colon \mathrm{Sp} \to \mathrm{Sp}_{C_p}.$ By Lemma [3.2.1](#page-14-2) (1), the functor $(-)^e \colon \mathrm{Sp}_{C_p} \to \mathrm{Sp}^{BC_p}$ restricts to a fully faithful functor $\mathrm{Sp}_{C_p}^{\mathrm{ind}} \to \mathrm{Sp}_{\mathrm{ind}}^{BC_p},$ which is also essentially surjective (and so is an equivalence) since $(-)^e = \beta^*$ is essentially surjective and $\beta_!$ and β^* are compatible with $\mathrm{Ind}_e^{C_p}.$

Lemma 3.2.3. As full subcategories of Sp^{BC_p} , we have the equality

$$
(\text{Sp}^{\omega})^{BC_p} \cap \text{Sp}_{\text{ind}}^{BC_p} = (\text{Sp}^{BC_p})^{\omega}.
$$

Thus, if $E \in \mathrm{Sp}_{C_p}$ with $\Phi^{C_p}E = 0$, then E is compact if and only if $E^e \in (\mathrm{Sp}^\omega)^{BC_p} \cap \mathrm{Sp}^{BC_p}_{\text{ind}}$.

Proof. The inclusion $({\rm Sp}^{\omega})^{BC_p} \cap {\rm Sp}^{BC_p}_{\text{ind}} \supseteq ({\rm Sp}^{BC_p})^{\omega}_{\stackrel{}{\sim}}$ is clear since $({\rm Sp}^{BC_p})^{\omega}$ is generated under finite colimits and retracts by $\sum_{+}^{\infty} C_p/e \simeq \text{Ind}_e^{C_p}$ S. For the converse, we use that

$$
(\text{Sp}^{BC_p})^{\omega} \hookrightarrow (\text{Sp}^{\omega})^{BC_p} \twoheadrightarrow \text{stmod}_{C_p}(\mathbb{S}) := (\text{Sp}^{\omega})^{BC_p} / (\text{Sp}^{BC_p})^{\omega}
$$

is a fibre sequence of small stable categories (c.f. for instance [\[CDH+23a,](#page-28-13) Lem. A.1.8]) and that for $X, Y \in (\text{Sp}^{\omega})^{BC_p}$ we have the formula (c.f. for instance [\[Kra20,](#page-28-11) Lem. 4.2])

$$
\text{map}_{\text{stmod}_{C_p}(\mathbb{S})}(X, Y) \simeq (Y \otimes DX)^{tC_p},
$$

where DX is the pointwise Spanier–Whitehead dual in $({\rm Sp}^\omega)^{BC_p}.$

Observe that for any $X \in \text{Sp}^{BC_p}$ and $Y \in \text{Sp}_{\text{ind}}^{BC_p}$ one has $(Y \otimes \text{map}(X, \mathbb{S}))^{tC_p} \simeq 0$ owing to the fact that $(-)^{tC_p}$ vanishes on ${\rm Sp}_{\rm ind}^{BC_p}$ and that ${\rm Sp}_{\rm ind}^{BC_p}\subseteq {\rm Sp}^{BC_p}$ is a tensor–ideal. Therefore, for $Z \in (\text{Sp}^{\omega})^{BC_p} \cap \text{Sp}^{BC_p}_{\text{ind}}$, we see that

$$
\text{map}_{\text{stmod}_{C_p}(\mathbb{S})}(Z, Z) \simeq (Z \otimes DZ)^{tC_p} \simeq 0
$$

and so Z is in the kernel of the functor $(\mathrm{Sp}^\omega)^{BC_p} \to \mathrm{stmod}_{C_p}(\mathbb{S})$. Hence, by the fibre sequence above, we see that $Z\in (\mathrm{Sp}^{BC_p})^\omega$ as required.

The statement about compact C_p -spectra follows by combining the first part with Lemma [3.2.1](#page-14-2) (2). \Box

3.3 Recognising C_p -Poincaré spaces

Construction 3.3.1 (Contravariant functoriality of dualising spectra). Consider a map $f\colon \underline{Y}\to \underline{X}$ in $\mathcal{S}_{C_p}^{\omega}.$ We explain how to construct a canonical "wrong–way" map

$$
\text{BC}^f: D_{\underline{X}} \longrightarrow f_! D_{\underline{Y}}.\tag{4}
$$

Combining the contravariant functoriality of cohomology from [\[HKK24,](#page-28-0) Construction 3.4.1] with the defining property of the dualising spectrum, we obtain the natural transformation

$$
X_!(D_{\underline{X}} \otimes -) \simeq X_*(-) \xrightarrow{\mathrm{BC}^f_*} Y_* f^*(-) \simeq Y_!(D_{\underline{Y}} \otimes f^*(-)) \simeq X_!(f_!D_{\underline{Y}} \otimes -)
$$

By the classification of colimit preserving functors, see [\[Cno23,](#page-28-14) Corollary 2.30] or [\[HKK24,](#page-28-0) Theorem 2.1.37], this is induced by a map $\mathrm{BC}^f\colon D_{\underline{X}}\to f_!D_{\underline{Y}}.$

Now consider $\underline{X}\in \mathcal{S}_{C_p}^\omega.$ For the inclusion $\varepsilon\colon \underline{X^{C_p}}\to \underline{X}$ of the singular part from Con-struction [2.1.5](#page-7-0) we thus obtain a map $\mathrm{BC}^\varepsilon\colon D_{\underline{X}}\to \varepsilon_!D_{\underline{X}^{C_p}}.$ Applying ε^* yields the map

$$
\varepsilon^* \mathcal{B} \mathcal{C}^\varepsilon \colon \varepsilon^* D_{\underline{X}} \longrightarrow \varepsilon^* \varepsilon_! D_{X^{C_p}},\tag{5}
$$

which may be viewed as a morphism in the nonparametrised functor category $\mathrm{Fun}(X^{C_p}, \mathrm{Sp}_{C_p}) \ \simeq \ \mathrm{Fun}_{C_p}(\underline{X^{C_p}}, \underline{\mathrm{Sp}})$ - this equivalence may be obtained by applying [\[HKK24,](#page-28-0) Lem. 2.1.16] to the adjunction $\inf_G^e : \mathcal{S} \rightleftharpoons \mathcal{S}_G : (-)^G$.

The wrong–way map [\(5\)](#page-16-1) satisfies the following key vanishing result permitting our characterisation of C_p –Poincaré spaces. By virtue of the lemma, the cofibre of ([5\)](#page-16-1) may be viewed as measuring the "geometric free part" of the dualising sheaf D_X .

Lemma 3.3.2. Let $\underline{X} \in \mathcal{S}^{\omega}_{C_p}$ and let $\nu \colon \mathrm{Sp}_{C_p} \to \mathcal{D}$ be an exact functor which vanishes on $\mathrm{Sp}_{C_p}^{\mathrm{ind}}$. Then the map

$$
\nu(\varepsilon^* D_{\underline{X}}) \longrightarrow \nu(\varepsilon^* \varepsilon_! D_{\underline{X}^{C_p}})
$$

in Fun (X^{C_p}, \mathcal{D}) induced by [\(5\)](#page-16-1) is an equivalence.

Proof. We have to show, for any $x \in X^{C_p}$, that the map $\nu(D_{\underline{X}}(x)) \stackrel{\simeq}{\to} \nu(f_!D_{X^{C_p}}(x))$ is an equivalence. First, let us show that for any compact C_p -space \underline{X} the map $\varepsilon \colon \overline{\overline{X^{C_p}}} \to \underline{X}$ induces an equivalence

$$
\nu(X_*(-)) \xrightarrow{\simeq} \nu(X_*^{C_p} \varepsilon^*(-)).
$$

If $\varepsilon: \underline{X^{C_p}} \to \underline{X}$ is an equivalence, this is a tautology. The class of spaces for which the assertion is true is moreover stable under pushouts, retracts and contains $\underline{Y} = C_p/e$, as

$$
0 \simeq \nu(\emptyset_*\epsilon^*(-)) \simeq \nu(\underline{C_p/e}_*(-)) \simeq \nu(\underline{C_p/e}_!(-)) \simeq \nu(\mathrm{Ind}_e^{C_p}(-)) \simeq 0.
$$

Using that $X_* \simeq X_! (D_X \otimes -)$ and $X^{C_p}_* \epsilon^* \simeq X_!^{C_p}$ $\frac{C_p}{\Gamma} (D_{\underline{X}^{C_p}} \otimes \epsilon^* -) \simeq X_! (\epsilon_! D_{\underline{X}^{C_p}} \otimes -)$ we obtain the equivalence

$$
\overline{\nu}(X_!(D_{\underline{X}}\otimes -))\stackrel{\simeq}{\longrightarrow} \overline{\nu}(X_!(\varepsilon_!D_{\underline{X}^{C_p}}\otimes -)).
$$

Now consider a fixed point $x\colon *\to X^{C_p}$ (which we also view as $x\colon *\to \underline{X}$). Note that the projection formula provides an equivalence, natural in $E \in \text{Sp}^{\underline{X}}$

$$
X_!(E \otimes x_!(\mathbb{S}_{C_p})) \simeq X_!x_!(x^*E) \simeq x^*E = E(x).
$$

Thus, for any $x \in X^{C_p}$, the map $\nu(D_{\underline{X}}(x)) \stackrel{\simeq}{\longrightarrow} \nu(\epsilon_! D_{\underline{X}^{C_p}}(x))$ is an equivalence, whence the result. \Box

We are now ready to prove our main result characterising C_p –Poincaré spaces. Note that, unlike Lemma [3.3.2,](#page-16-2) the key characterising property is given solely in terms of $D_{X^{C_p}}$ and does not involve D_{X} .

Theorem 3.3.3. Let \underline{X} be a compact C_p -space for which X^e and X^{C_p} are (nonequivariant) Poincaré spaces. Then \underline{X} is C_p -Poincaré if and only if the cofiber

$$
\operatorname{cofib}(D_{\underline{X}^{C_p}} \to \varepsilon^* \varepsilon_! D_{\underline{X}^{C_p}})^e \in \operatorname{Fun}(X^{C_p}, \operatorname{Sp}^{BC_p})
$$

pointwise lies in the stable subcategory $\mathrm{Sp}_{\mathrm{ind}}^{BC_p} \subseteq \mathrm{Sp}^{BC_p}.$

Proof. As X^e is assumed to be Poincaré the specta $D_X(y) = D_{X^e}(y) \in \text{Sp}$ are invertible for all $y \in X^e$. Furthermore, as X^{C_p} is Poincaré we know that $\Phi^{C_p} D_{\underline{X}}(x) = D_{X^{C_p}}(x)$ is invertible for all $x \in X^{C_p}.$ It now follows from Lemma [3.0.1](#page-11-2) that \underline{X} is C_p -Poincaré if and only if the C_p -spectrum $D_X(x)$ is compact for all points $x \in X^{C_p}$.

Note that all maps in the bottom right cospan in the diagram

$$
F \xrightarrow{\qquad f' \qquad \qquad} D_{X^{C_p}}(x)
$$

\n
$$
\downarrow g'
$$

\n
$$
D_{\underline{X}}(x) \xrightarrow{\qquad f} \varepsilon_! D_{X^{C_p}}(x)
$$

\n(6)

induce equivalences on geometric fixed points: the map f by Lemma [3.3.2,](#page-16-2) and the map q by [\[HKK24,](#page-28-0) Lem. 4.2.3]. We can use Lemma [3.1.5](#page-14-3) to complete [\(6\)](#page-17-1) to a commutative square of C_p -spectra where F is compact and all maps are equivalences on geometric fixed points. Consider the exact functor

$$
\nu = \left(\mathrm{Sp}_{C_p} \xrightarrow{(-)^e} \mathrm{Sp}^{BC_p} \to \mathrm{Sp}^{BC_p} / \mathrm{Sp}_{\mathrm{ind}}^{BC_p} \right).
$$

Note that as g' and f' are maps between compact C_p -spectra that induce an equivalence on geometric fixed points, Lemma [3.2.3](#page-15-1) shows that $\nu(\text{cofib}(f')) \simeq \nu(\text{cofib}(g')) \simeq 0$, so $\nu(f')$ and $\nu(g')$ are equivalences.

Let us first assume that X is C_p -Poincaré. It follows from Lemma [3.3.2](#page-16-2) that $\nu(\text{cofib}(f)) \simeq$ 0, so also $\nu(f)$ is an equivalence. Thus, $\nu(g)$ is an equivalence from which we obtain $\nu(\text{cofib}(q)) \simeq 0$, proving one direction of the claim.

For the other direction, assume $\nu(\text{cofib}(q)) \simeq 0$, i.e. that $\nu(q)$ is an equivalence. As before, as $\nu(f)$ and $\nu(f')$ are equivalences we obtain that $\nu(\text{cofib}(g')) \simeq 0$. By definition of ν , this means $\mathrm{cofib}(g')^e \in \mathrm{Sp}_{\mathrm{ind}}^{BC_p}.$ Now, $F \in \mathrm{Sp}_{C_p}$ and $D_{\underline{X}}(x) \in \mathrm{Sp}_{C_p}$ both have compact underlying spectra. Hence, it follows from Lemma [3.2.3](#page-15-1) that $\mathrm{cofib}(g') \in \mathrm{Sp}_{C_p}$ is compact. But then $D_X(x)$ is compact too, as was to be shown. \Box

4 Genuine virtual Poincaré duality groups

In this section, we define a refinement of the classical notion of virtual Poincaré duality groups. Recall that a Poincaré duality group is a discrete group π such that $B\pi$ is a Poincaré space, and a group is a virtual Poincaré duality group if it contains a Poincaré duality group of finite index. In this case, every finite index torsionfree subgroup will be a Poincaré duality group.

Now, if Γ is a discrete group and π a finite index torsionfree normal subgroup, then the space $B\pi$ can be enhanced to a Γ/π -space in a canonical way by viewing it as the quotient $\pi\backslash E_\Gamma$ Fin of the universal space for the family of finite subgroups of Γ. One might naturally wonder if that Γ/π -space is Γ/π -Poincaré, in which case we call Γ a genuine virtual Poincaré duality group. In fact, we will give a slightly more general definition that also includes the case where Γ is a Lie group. Heuristically, genuine virtual Poincaré duality groups are those which capture the homotopical properties of groups for which the universal space of proper actions admits a smooth manifold model.

4.1 Universal spaces for proper actions

Let us first collect some examples and constructions for universal spaces for proper actions from the literature.

Definition 4.1.1. Let Γ be a Lie group. By Cpt we denote the family of compact subgroups of Γ. The universal space for proper actions is the universal space for the family Cpt, and is denoted by E_{Γ} Cpt.

If Γ is discrete, the family of compact subgroups of Γ agrees with the family of finite subgroups, and we denote it by Fin.

Example 4.1.2 ($[Abe78]$, Thm 4.15.). Assume the Lie group Γ acts properly, smoothly and isometrically on a simply connected complete Riemannian manifold M with nonpositive sectional curvature. Then M with its Γ-action provides a model for E_{Γ} Cpt.

Example 4.1.3 ([\[MS02,](#page-29-9) Theorem 1]). Suppose, Γ is a hyperbolic (discrete) group. Then a barycentric subdivision of the *Rips complex* of Γ for sufficiently high $\delta > 0$ for a word metric on Γ provides a finite Γ-CW model for E_Γ Fin. In particular, the Γ-space E_Γ Fin is compact.

More examples for geometrically interesting models for the universal spaces for proper actions can be found in the survey $[Lüc05]$ and the references therein. For the next statement, recall that for a subgroup $H \leq \Gamma$, its normaliser is defined as $N_{\Gamma} H \coloneqq \{g \in \Gamma \mid gHg^{-1} = H\}$ and its Weyl group as $W_{\Gamma}H := N_{\Gamma}H/H$.

Theorem 4.1.4 (Lück-Weiermann's decomposition, [[LW12,](#page-29-4) Thm. 2.3, Cor. 2.10]). Let Γ be a discrete group such that each nontrivial finite subgroup is contained in a unique maximal finite subgroup. Let M be a set of representatives of conjugacy classes of maximal finite subgroups of Γ. Then the square

$$
\coprod_{F \in \mathcal{M}} \operatorname{Ind}_{N_{\Gamma}F}^{\Gamma} \underline{EN_{\Gamma}F} \longrightarrow \underline{E\Gamma}
$$
\n
$$
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad (\Box)
$$
\n
$$
\coprod_{F \in \mathcal{M}} \operatorname{Ind}_{N_{\Gamma}F}^{\Gamma} \underline{E}_{N_{\Gamma}F} \underline{\text{Fin}} \longrightarrow \underline{E_{\Gamma}\text{Fin}}
$$

is a pushout in S_{Γ} .

Remark 4.1.5. To prove Theorem [4.1.4](#page-18-1) one checks that (\square) is a pushout on all fixed points by distinguishing the three cases $H = 1$, $H \neq 1$ finite and H infinite. Examples of groups for which it applies are discrete groups Γ for which there exists π torsionfree and an extension

$$
1 \to \pi \to \Gamma \to C_p \to 1.
$$

If Γ is assumed pseudofree, then we see that $\underline{E_{\Gamma}} \mathrm{Fin}^{>1}$ is discrete.

Example 4.1.6. Let us give an illustrative geometric example. Consider the group $\Gamma = p3$, the symmetry group of the wallpaper depicted in Figure [1,](#page-0-1) with its action on the euclidean plane. This action is isometric and has finite stabilisers, so by Example [4.1.2](#page-18-3) it is a model for the universal space of proper action of $p3$.

Note that the translations form a normal torsion free subgroup of $p3$ of index 3. We can apply Theorem [4.1.4](#page-18-1) to obtain that the subspace of the plane with nontrivial Γ -isotropy is Γ-homotopy equivalent to $\coprod_F \mathrm{Ind}_{N_\Gamma F}^\Gamma \underline{E}_{N_\Gamma F}$ Fin . Now from the picture it is easy to read off that the singular part is indeed discrete and consists of three Γ -orbits. We conclude that Γ has precisely three conjugacy classes of nontrivial finite subgroups, and each such nontrivial finite $F \subset \Gamma$ satisfies $N_{\Gamma} F = F$.

Figure 3: Three points with C_3 -symmetry, corresponding to three conjugacy classes of finite subgroups of $p3$.

4.2 Equivariant Poincaré duality for groups

For the following, a closed subgroup $\pi \subset \Gamma$ of a Lie group is *cocompact* if the topological space $\pi\backslash\Gamma$ is compact. If $\pi\subset\Gamma$ is normal, this is equivalent to Γ/π being a compact Lie group.

Definition 4.2.1. Let Γ be a Lie group and let $\pi \subset \Gamma$ be a cocompact torsionfree discrete normal subgroup. We write

$$
\underline{B_{\Gamma}\pi} \coloneqq \pi \backslash \underline{E_{\Gamma} \text{Cpt}} \in \mathcal{S}_{\Gamma/\pi}
$$

for the quotient of E_{Γ} Cpt by the action of π .

The following definition is supposed to capture the homological properties of Lie groups Γ that admit a cocompact smooth manifold model for E_{Γ} Cpt (and a discrete torsionfree normal cocompact subgroup). If Γ is torsionfree, it reduces to Γ being a Poincaré duality group. Here, and only here we refer to Poincaré duality for compact Lie groups as also developed in [\[HKK24\]](#page-28-0), but the reader mainly interested in discrete group actions can assume Γ to be discrete throughout.

Definition 4.2.2. Let Γ be a Lie group. Then Γ is called a genuine virtual Poincaré duality group if it has a cocompact torsionfree normal subgroup, and if for any such cocompact torsionfree normal subgroup $\pi \subset \Gamma$, the Γ/π -space $B_{\Gamma}\pi$ is a Γ/π -Poincaré space.

Proposition 4.2.3. Suppose Γ is a Lie group with torsionfree cocompact normal subgroups $\pi, \pi' \subset \Gamma$ whose intersection is again cocompact. Then

$$
\underline{B_{\Gamma}\pi}
$$
 is Γ/π -Poincaré $\iff \underline{B_{\Gamma}\pi'}$ is Γ/π' -Poincaré.

Proof. If suffices to consider the case where $\pi \subseteq \pi'$. Note that the normal subgroup $\pi'/\pi \subseteq \Gamma/\pi$ acts freely on $\underline{B_{\Gamma}\pi}$. Applying [\[HKK24,](#page-28-0) Corollary 4.3.13], we see that $\underline{B_{\Gamma}\pi}$ is Γ/π-Poincaré if and only if $(\pi' / \pi) \setminus \underline{B_{\Gamma} \pi} \simeq \underline{B_{\Gamma} \pi'}$ is Γ/π $'$ -Poincaré. \Box

Note that, in general, the intersection of two cocompact subgroups is not again cocompact, e.g. for **Z**, 2**Z** ⊂ **R**. In the case where Γ is discrete, the intersection of two finite index subgroups is again finite, from which we obtain the following result.

Corollary 4.2.4. Suppose, Γ is a discrete group. Then the following are equivalent.

- (1) The group Γ is a genuine virtual Poincaré duality group.
- (2) There exists some torsionfree finite index normal subgroup $\pi \subset \Gamma$ such that the Γ/π -space B_{Γ} π is Γ/π -Poincaré.

5 Extensions by C_n

In this section, we study genuine virtual Poincaré duality groups sitting in an extension

$$
1 \to \pi \to \Gamma \to C_p \to 1 \tag{7}
$$

more closely. In [§5.1](#page-20-2) we will prove the characterisation from Theorem [C;](#page-5-0) in [§5.2](#page-24-0) we will use this to prove property (H) for pseudofree extensions.

5.1 Characterisation of genuine virtual Poincaré duality groups

Lemma 5.1.1. Consider an extension of groups of the form [\(7\)](#page-20-3) where π is torsionfree. Write M for a complete set of representatives of the conjugacy classes of nontrivial finite subgroups of Γ.

- (1) If $F \leq \Gamma$ is a nontrivial subgroup with $\pi \cap F = e$ (e.g. F is finite), then the composition of $F \rightarrow \Gamma \rightarrow C_p$ is an isomorphism. In particular, F is a maximal finite subgroup of Γ .
- (2) There is an equivalence of spaces

$$
\underline{B_{\Gamma}\pi}^{C_p} \simeq \coprod_{F \in \mathcal{M}} BW_{\Gamma}F.
$$

Proof. Point (1) follows as the kernel of $\Gamma \to C_p$ is torsionfree, so every finite subgroup of Γ will inject into C_p . For (2), observe that if π acts freely on a Γ-space \underline{Y} , then the map

$$
(\pi \backslash \underline{Y}^{>1})^{C_p} \to (\pi \backslash \underline{Y})^{C_p}
$$

is an equivalence by Fact [2.1.8.](#page-8-2) Thus, since applying $(\pi \backslash -)^{C_p}$ to the top row in the pushout square ([□](#page-18-2)) yields the map of empty spaces, we get an identification

$$
\underline{B_{\Gamma}\pi}^{C_p} \simeq \left(\pi\backslash \coprod_{F\in \mathcal{M}}\mathrm{Ind}_{N_{\Gamma}F}^{\Gamma}\underline{E}_{N_{\Gamma}F}\underline{\mathrm{Fin}}\right)^{C_p} \simeq \coprod_{F\in \mathcal{M}}(\pi\backslash \mathrm{Ind}_{N_{\Gamma}F}^{\Gamma}\underline{E}_{N_{\Gamma}F}\underline{\mathrm{Fin}})^{C_p}.
$$

Consider the surjective composition of group homomorphisms $N_{\Gamma}F \subset \Gamma \to C_p$. By Observation [2.1.4,](#page-7-1) we get

$$
\pi\backslash \mathrm{Ind}_{N_{\Gamma}F}^{\Gamma}\underline{E}_{N_{\Gamma}F}\underline{\mathrm{Fin}}\simeq(\pi\cap N_{\Gamma}F)\backslash \underline{E}_{N_{\Gamma}F}\underline{\mathrm{Fin}}.
$$

Now note that, by (1), the only finite subgroups of $N_{\Gamma}F$ are e and F. This implies $\underline{E}_{N_{\Gamma}F}\text{Fin}\simeq$ $\inf_{W_\Gamma F}^{N_\Gamma F} \underline{EW}_\Gamma F$. Also, the composition $\pi \cap N_\Gamma F \, \subset \, N_\Gamma F \, \to \, W_\Gamma F$ is an isomorphism. Indeed, it is injective, as it has at most finite kernel and π is torsionfree, and it is surjective as $N_{\Gamma}F$ is generated by F and $\pi \cap N_{\Gamma}F$, and F maps to zero in $W_{\Gamma}F$. We thus obtain

$$
(\pi \cap N_{\Gamma} F)\backslash \underline{E_{N_{\Gamma} F} \underline{\mathrm{Fin}}}\simeq (\pi \cap N_{\Gamma} F)\backslash \mathrm{infl}_{N_{\Gamma} F}^{W_{\Gamma} F} \underline{E W_{\Gamma} F}\simeq \mathrm{infl}_{C_p}^e BW_{\Gamma} F,
$$

the second equivalence being an instance of Fact [2.1.7](#page-7-2) for $G = N_{\Gamma}F$, $Q = W_{\Gamma}F$ and $N =$ $N_{\Gamma}F \cap \pi$. This finishes the proof of the second assertion. \Box

Lemma 5.1.2. In the situation of Lemma [5.1.1,](#page-20-0) for each $x\colon\underline{\ast}\to\underline{B_\Gamma\pi}^{C_p}$ there is a pullback

$$
\underline{T} \xrightarrow{a} \underline{B_{\Gamma} \pi^{C_p}} \qquad (8)
$$
\n
$$
\underline{\psi} \xrightarrow{b} \underline{B_{\Gamma} \pi}
$$

of C_p -spaces where \underline{T} is a disjoint union of C_p -orbits and has exactly one fixed point. Furthermore, $x \simeq as$ where $s: \underline{*} \to \underline{T}$ denotes the section coming from the fixed point of \underline{T} .

Proof. Note that the π -actions on \underline{E}_{Γ} Fin as well as on \underline{E}_{Γ} Fin^{>1} are free, so that by [\[HKK24,](#page-28-0) Lem. 2.2.38] we have a cartesian square

$$
\underbrace{E_{\Gamma} \text{Fin}}_{\text{F} \text{in}} > 1 \longrightarrow \text{inf}_{\Gamma}^{C_p} \underbrace{B_{\Gamma} \pi^{C_p}}_{\text{F} \text{in}} \tag{9}
$$
\n
$$
\underbrace{E_{\Gamma} \text{Fin}} \longrightarrow \text{inf}_{\Gamma}^{C_p} \underbrace{B_{\Gamma} \pi}.
$$

Now, recallin Fact [2.1.8,](#page-8-2) we have $\underline{B_\Gamma \pi}^{C_p} \simeq \pi \backslash \underline{E_\Gamma \mathrm{Fin}}^{>1}.$ The point x gives rise to a map of Γ-spaces $h\colon\underline{\ast}\to\inf_{\Gamma}^{C_p}\underline{B_\Gamma\pi}^{C_p},$ and so by [\[HKK24,](#page-28-0) Lem. 2.2.39.], we get a commuting square

$$
\frac{\Gamma/F \xrightarrow{f} E_{\Gamma}\text{Fin}^{>1}}{\downarrow}
$$
\n
$$
\xrightarrow{\ast} \frac{h = \pi \setminus f}{\inf_{\Gamma} C_p} \frac{C_p}{B_{\Gamma} \pi^{C_p}}
$$
\n(10)

for F a subgroup with $\pi\backslash(\Gamma/F) \simeq *$ (so that F is nontrivial) and $\pi \cap F = e$. From Lemma [5.1.1](#page-20-0) we now learn that the map $F \subset \Gamma \to C_p$ is an isomorphism. Hence, F can be used to define a section $s: C_p \to \Gamma$. We will now construct the following diagram.

Restricting the pullback [\(9\)](#page-21-0) along s, we obtain the outer cartesian square of C_p -spaces in

$$
\operatorname{Res}_{C_p}^{\Gamma} \coprod_{F' \in \mathcal{M}} \Gamma/N_{\Gamma} F' \xrightarrow{\simeq} \operatorname{Res}_{C_p}^{\Gamma} \underline{E_{\Gamma}} \underline{\operatorname{Fin}}^{>1} \longrightarrow \operatorname{Res}_{C_p}^{\Gamma} \operatorname{inf}_{\Gamma}^{C_p} \underline{B_{\Gamma} \pi}^{C_p} \xrightarrow{\simeq} \underline{B_{\Gamma} \pi}^{C_p}
$$
\n
$$
\downarrow \qquad (A) \qquad \qquad \downarrow \qquad (B) \qquad \qquad \downarrow \qquad (C) \qquad \qquad \downarrow
$$
\n
$$
\star \xrightarrow{\simeq} \operatorname{Res}_{C_p}^{\Gamma} \underline{E_{\Gamma}} \underline{\operatorname{Fin}} \longrightarrow \operatorname{Res}_{C_p}^{\Gamma} \operatorname{inf}_{\Gamma}^{C_p} \underline{B_{\Gamma} \pi} \xrightarrow{\simeq} \underline{B_{\Gamma} \pi}.
$$

Here, the lower equivalence in square (A) comes from the observation that for any group G, the space E_G Fin becomes equivariantly contractible when restricted to a finite subgroup. The upper equivalence combines Theorem [4.1.4](#page-18-1) with the observation, that $\text{Res}_{C_p}^{\Gamma} \text{Ind}_{N_{\Gamma}F}^{\Gamma} \underline{E}_{N_{\Gamma}F} \text{Fin}\to \text{Res}_{C_p}^{\Gamma} \text{Ind}_{N_{\Gamma}F}^{\Gamma} \underline{\ast} \,=\, \text{Res}_{C_p}^{\Gamma} \Gamma/N_{\Gamma}F$ is an equivalence, which is checked easiest by looking at the map on underlying spaces and C_p -fixed points. The square labeled (B) is obtained by restricting the pullback [\(9\)](#page-21-0) along the section $s: C_p \to \Gamma$. By virtue of s being a section of $\Gamma\to C_p,$ and as inflation is nothing but restriction along a surjective group homomorphism, we have ${\rm Res}^{C_p}_\Gamma\,{\rm inf}_{\Gamma}^{C_p}\simeq{\rm id}_{\mathcal{S}_{C_p}}$, explaining the identifications in square (C) . It now follows from Lemma [5.1.3](#page-22-0) below that the upper left corner has exactly one fixed point, as required.

To see the last statement that $x \simeq as$, observe that the section s was chosen to identify C_p with a specific finite subgroup $F \subset \Gamma$ having the property that the composition

$$
\underline{\ast} \xrightarrow{eF} \text{Res}_{C_p}^{\Gamma} \underline{\Gamma/F} \xrightarrow{\text{Res}_{C_p}^{\Gamma} f} \text{Res}_{C_p}^{\Gamma} \underline{E_{\Gamma}\text{Fin}}^{>1} \to \underline{B_{\Gamma}\pi}^{C_p}
$$

 \Box

is the point x by [\(10\)](#page-22-1).

Lemma 5.1.3. Let Γ be a group for which each nontrivial finite subgroup is contained in a unique maximal finite subgroup. Then for maximal finite subgroups $F, F' \subseteq \Gamma$ we have that F acts freely on $\Gamma/N_{\Gamma}F'$ if F and F' are not conjugate and $(\Gamma/N_{\Gamma}F')^F=*$ if F and F' are conjugate.

Proof. Suppose there is $f \in F \backslash e$ and $g \in \Gamma$ such that $fgN_{\Gamma}F' = gN_{\Gamma}F'$. Then $g^{-1}fg \in F'$ $N_{\Gamma}F'$ so the subgroup generated by F' and $g^{-1}fg$ is finite. By maximality of $F',$ we obtain $g^{-1}fg \in F'.$ The nontrivial element f lies in both maximal finite subgroups F and $gF'g^{-1}$

of Γ which agree by uniqueness. This shows that the F-action on $\Gamma/N_{\Gamma}F'$ is free if F and F' are not conjugate.

In the other case it suffices to show that $(\Gamma/N_{\Gamma}F)^F=*$. One fixed point is clearly given by $eN_\Gamma F$. Suppose that there are $f \in F \backslash e$ and $g \in \Gamma$ such that $fgN_\Gamma F = gN_\Gamma F$. The argument from above shows that $F=gFg^{-1}$, i.e. $g\in N_\Gamma F.$ \Box

We now come to the proof of our main characterisation result for genuine virtual Poincaré duality groups coming from C_p -extensions.

Theorem 5.1.4. Consider an extension of groups

$$
1 \to \pi \to \Gamma \to C_p \to 1
$$

where π is a Poincaré duality group, and assume that the Γ -space E_{Γ} Fin is compact. Then the following are equivalent.

- (1) The group Γ is a genuine virtual Poincaré duality group.
- (2) For each nontrivial finite subgroup $F \subset \Gamma$, the Weyl group $W_{\Gamma}F$ is a Poincaré duality group.

Proof. First of all, note that since $\pi \setminus (-)$ preserves compact objects – it admits a right adjoint with a further right adjoint – the C_p -space $\underline{B_\Gamma \pi} = \pi \backslash \underline{E\mathrm{Fin}_\Gamma}$ is compact. As such, $\underline{B_\Gamma \pi}^{C_p}$ is also compact.

To prove that (1) implies (2), recall that by Lemma [5.1.1](#page-20-0) we get an equivalence

$$
\underline{B}_{\Gamma}\underline{\pi}^{C_p} \simeq \coprod_{F \in \mathcal{M}} BW_{\Gamma}F \tag{11}
$$

where F runs through a set M of representatives of the conjugacy classes of nontrivial finite subgroups. If a space is Poincaré, then each individual component is Poincaré. So we learn that $W_{\Gamma}F$ is a Poincaré duality group for each $F \in \mathcal{M}$. As conjugate subgroups have isomorphic Weyl groups, this implies that the conclusion holds for each nontrivial finite subgroup F .

To prove that (2) implies (1), since $\underline{B_{\Gamma}\pi}^{C_p}$ is compact by the first paragraph, there must only be finitely many components in the decomposition [\(11\)](#page-23-1). By the hypothesis of (2), each component is Poincaré, implying that $\underline{B_\Gamma\pi}^{C_p}$ is (nonequivariantly) Poincaré. We are thus in the situation of Theorem [3.3.3.](#page-16-0) To apply it, we have to show that for all $x\in \underline{B_\Gamma\pi}^{C_p},$ the cofiber of the map $D_{B_{\Gamma}\pi^{C_p}}(x)^e\to j_!D_{B_{\Gamma}\pi^{C_p}}(x)^e$ lies in the perfect subcategory ${\rm Sp}^{BC_p}_{\rm ind}\subset{\rm Sp}^{BC_p}$ generated by induced spectra, where $j\colon \underline{B_{\Gamma}\pi}^{C_p} \to \underline{B_{\Gamma}\pi}$ denotes the inclusion.

For ease of notation, let us write $\underline{X} := B_{\Gamma} \pi$ in the following. From Lemma [5.1.2,](#page-21-1) we get a cartesian square of C_p -spaces

$$
\underbrace{T}_{s} \xrightarrow{\begin{array}{c} a \\ \uparrow \\ \downarrow \end{array}} \underbrace{X^{C_p}}_{\downarrow j}
$$
\n
$$
\underbrace{\begin{array}{c} \uparrow \\ \downarrow \end{array}}_{\underline{\ast} \longrightarrow \underline{X}} \qquad (12)
$$

where $\underline{T} = \underline{*} \coprod \underline{S}$ and \underline{S} is a disjoint union of free C_p -orbits where, furthermore, the point x corresponds to the image of the composite as, where $s: \underline{*} \to \underline{T}$ denotes the section coming from the fixed point of \underline{T} .

Now observe that the map $D_{X^{C_p}}(x) \rightarrow j^* j_! D_{X^{C_p}}(x)$ identifies with the map $s^*a^*D_{X^{C_p}} \to s^*p^*p_!a^*D_{X^{C_p}} \simeq \text{colim}_{\mathcal{I}} a^*D_{X^{C_p}}$ induced by the unit id $\to p^*p_!$. The decomposition $\underline{T} \simeq \underline{S} \coprod \underline{*}$ now provides a splitting

$$
\operatornamewithlimits{colim}_{\underline{T}} a^*D_{\underline{X^{C_p}}}\simeq s^*a^*D_{\underline{X^{C_p}}}\oplus \operatornamewithlimits{colim}_{\underline{S}} a^*D_{\underline{X^{C_p}}}|_{\underline{S}}
$$

and the induced map

$$
s^*a^*D_{\underline{X}^{C_p}} \to s^*a^*D_{\underline{X}^{C_p}} \oplus \operatorname{colim}_{{\underline{S}}} a^*D_{\underline{X}^{C_p}}|_{{\underline{S}}} \tag{13}
$$

is an equivalence on the first component by functoriality of colimits. The C_p -spectrum $\text{colim}_{\underline{S}} a^* D_{\underline{X}^{C_p}}|_{\underline{S}}$ is induced as \underline{S} is free. Together this shows that the cofiber of the map $s^*a^*D_{X^{C_p}}\to s^*a^*j^*j_!D_{X^{C_p}}$ lies in the subcategory ${\rm Sp}_{\rm ind}^{BC_p}\subset{\rm Sp}^{BC_p}$ as was to be shown. \Box

Remark 5.1.5. Instead of explicitly identifying the map $\epsilon\colon D_{X^{C_p}}(x)\to j^*j_!D_{X^{C_p}}(x)$ in the last step of the argument above, one can also finish the proof using the following trick. Using the splitting [\(13\)](#page-24-1), one can reduce to showing that the cofiber of a selfmap f of the invertible spectrum $D_{X^{C_p}}(x)$ is induced. As j is an equivalence on C_p -fixed points, one sees that $\Phi^{C_p}(\epsilon)$ is an equivalence. This implies that the selfmap f in question also is an equivalence on geometric fixed points. The Burnside congruences show that $\text{cofib}(f)^e$ is *n*-torsion for n congruent to 1 mod p, in particular n coprime to p. But every compact spectrum with C_p -action which is *n*-torsion for *n* coprime to *p* vanishes in stmod(C_p), so it is induced.

5.2 Condition (H)

In this section we will prove an abstract version of Condition (H) for general C_p -Poincaré spaces with discrete fixed points and see how this implies Condition (H) from Condition [1.3.](#page-3-0) Essential for this is the theory of singular parts and equivariant fundamental classes, especially the gluing class, introduced in [\[HKK24,](#page-28-0) §4.5]. Let us recall the relevant notions and constructions here.

Construction 5.2.1 ([\[HKK24,](#page-28-0) Cons. 4.5.4]). For $\xi \in \text{Fun}_G(\underline{X}, \text{Sp})$, there is a preferred map

 $(X_{!}\xi)^{hG} \to \Sigma (X_{!}^{>1}\varepsilon^*\xi)_{hG}$. It is defined as the blue composite in the commuting diagram

$$
(X_1^{>1}\varepsilon^*\xi)_{hG} \longrightarrow (X_1^{>1}\varepsilon^*\xi)^{hG} \longrightarrow (X_1^{>1}\varepsilon^*\xi)^{tG} \longrightarrow \Sigma(X_1^{>1}\varepsilon^*\xi)_{hG}
$$

\n
$$
(X_1\xi)_{hG} \longrightarrow (X_1\xi)^{hG} \longrightarrow (X_1\xi)^{tG}
$$

\n
$$
\downarrow \qquad \qquad \downarrow
$$

\n
$$
Q_{hG} \longrightarrow Q^{hG}
$$

\n
$$
\downarrow \qquad \qquad (14)
$$

\n
$$
\Sigma(X_1^{>1}\varepsilon^*\xi)_{hG}
$$

where the horizontal and vertical sequences are cofiber sequences and where we used the shorthand $Q \coloneqq \mathrm{cofib}\left(X_{!}^{>1}\varepsilon^*\xi \to X_{!}\xi\right)$. By [\[HKK24,](#page-28-0) Lemma B.0.1], the red and blue composite in [\(14\)](#page-25-1) are equivalent up to a sign.

Construction 5.2.2 (Gluing classes, [\[HKK24,](#page-28-0) Cons. 4.5.5]). Let $\underline{X} \in S_G$ be a G-Poincaré space with dualising spectrum $D_X \in \text{Fun}_G(\underline{X}, \underline{\text{Sp}})$ and collapse map $c: \mathbb{S}_G \to X_! D_X$. The gluing class of \underline{X} is defined to be the composite

$$
\mathbb{S} \xrightarrow{\text{can}} \mathbb{S}_G^{hG} \xrightarrow{c^{hG}} (X_! D_{\underline{X}})^{hG} \longrightarrow \Sigma (X_!^{>1} \varepsilon^* D_{\underline{X}})_{hG},
$$

where the last map is the blue composite from (14) . The *linearised gluing class* is obtained by postcomposing with the map induced by $\mathbb{S} \to \mathbb{Z}$

$$
\Sigma(X_!^{>1} \varepsilon^* D_{\underline{X}})_{hG} \to \Sigma(X_!^{>1} \varepsilon^* D_{\underline{X}} \otimes \mathbb{Z})_{hG}.
$$

We now specialise to the case $G = C_p.$ Recall from Construction [2.1.5](#page-7-0) that for $\underline{X} \in \mathcal{S}_{C_p},$ we have $\underline{X}^{>1} \simeq \underline{X}^{C_p}$ coming from the fact that $X^{>1} \simeq X^{C_p}.$

Proposition 5.2.3 (Abstract condition (H)). Let \underline{X} be a C_p -Poincaré space with discrete fixed points such that each component of X^e has positive dimension. Then the linearised gluing class

$$
\mathbb{S} \to \Sigma(X_1^{>1} \varepsilon^* D_{\underline{X}} \otimes \mathbb{Z})_{hC_p} \xleftarrow{\simeq} \bigoplus_{y \in X^{C_p}} \Sigma(D_{\underline{X}}(y) \otimes \mathbb{Z})_{hC_p}
$$

maps a generator of $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$ to a generator of $\pi_0 \Sigma(D_X(y) \otimes \mathbb{Z})_{hC_p} \simeq \mathbb{Z}/p$ in each summand.

Recollections 5.2.4 (Invertible C_p -spectra and group (co)homology). For the proof of Propo-sition [5.2.3,](#page-25-0) recall the following facts about the homology of invertible C_p -spectra. Recall that for an abelian group with G -action M , writing $M[d]$ for the corresponding object in $\text{Mod}^{\mathit{BG}}_{\mathbb{Z}}$ concentrated in degree d, we have

$$
\pi_*M[d]_{hG} \simeq H_{*-d}(G;M), \ \ \pi_*M[d]^{hG} \simeq H^{d-*}(G;M), \ \ \pi_*M[d]^{tG} \simeq \widehat{H}^{d-*}(G;M).
$$

For $E\in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_{C_p})$ there are integers d^e and d^f such that $E^e\otimes\mathbb{Z}\simeq \mathbb{Z}[d^e]$ after forgetting the C_p -action and $\Phi^{C_p}E\otimes\mathbb Z\,\simeq\,\mathbb Z[d^f].$ We write $\mathbb Z$ for the trivial C_p -representation and \mathbb{Z}^{σ} for the sign C_2 -representation. From [\[Kra20,](#page-28-11) Sec. 8.1.] and elementary group homology computations we obtain Table [1.](#page-26-0) In each case, if $d^e + 1 \leq d^f$, another group homological computation together with Table [1](#page-26-0) shows that the map

$$
\pi_{d^f}(E^e\otimes\mathbb{Z})^{tC_p}\to \pi_{d^f-1}(E^e\otimes\mathbb{Z})_{hC_p}
$$

is an isomorphism between cyclic groups of order p.

	$d^e - d^f$ even	$d^e - d^f$ odd
p odd	$\pi_* E^e \otimes \mathbb{Z} \simeq \mathbb{Z}[d^e]$ $\pi_{d} f(E^e \otimes \mathbb{Z})^{tC_p} \simeq \mathbb{Z}/p$	impossible
$p=2$	$\pi_* E^e \otimes \mathbb{Z} \simeq \mathbb{Z}[d^e]$ $\pi_{d} f(E^e \otimes \mathbb{Z})^{tC_p} \simeq \mathbb{Z}/p$	$\pi_* E^e \otimes \mathbb{Z} \simeq \mathbb{Z}^\sigma [d^e]$ $\pi_{d^f}(E^e\otimes \mathbb{Z})^{tC_p}\simeq \mathbb{Z}/p$

Table 1: Homological information on invertible C_p -spectra

Observation 5.2.5. Consider a map $f: \underline{Y} \to \underline{X}$ in \mathcal{S}_G . We claim that there is a commuting diagram of functors $Sp^{\underline{X}} \to Sp^{\Phi \widetilde{\mathcal{P}}}$

$$
X_! \Phi \stackrel{c}{\longleftarrow} X_! f_! f^* \Phi
$$

\n
$$
\downarrow \simeq \qquad \qquad \downarrow \simeq
$$

\n
$$
\Phi X_! \stackrel{c}{\longleftarrow} \Phi X_! f_! f^*,
$$
\n(15)

where the vertical maps are the Beck-Chevalley transformations, which are equivalences as the geometric fixed points functor $\Phi\colon \underline{\text{Sp}} \to \underline{\text{Sp}}^{\Phi\mathcal{P}}$ from [\[HKK24,](#page-28-0) Construction 2.2.31] preserves parametrised colimits, and the horizontal maps are induced by the adjunction counit c: $f_! f^* \to id$. This follows immediately from [\[Hil24,](#page-28-16) Lemma 2.2.6], again using that Φ preserves parametrised colimits. Importantly, the top map (and hence also the bottom map) is an equivalence by [\[HKK24,](#page-28-0) Lem. 4.2.3].

Notice also that, by naturality of Beck-Chevalley transformations, if we have a decomposition $\underline{Y} = \underline{Y_1} \coprod \underline{Y_2}$, the right vertical map in [\(15\)](#page-26-1) is compatible with this splitting.

Proof of Proposition [5.2.3.](#page-25-0) By construction, the gluing class factors through the map

$$
(X_!^{>1} \varepsilon^* D_{\underline{X}} \otimes \mathbb{Z})^{tC_p} \to \Sigma (X_!^{>1} \varepsilon^* D_{\underline{X}} \otimes \mathbb{Z})_{hC_p}
$$

which happens to be an isomorphism on π_0 using Recollection [5.2.4](#page-25-2) and that $\underline{X}^{>1}$ is a discrete C_p -space (so that $d^f = 0$) with trivial C_p -action. It thus suffices to show that the gluing class maps to a generator in each summand of

$$
\pi_0(X_!^{>1} \varepsilon^* D_{\underline{X}} \otimes \mathbb{Z})^{tC_p} \simeq \bigoplus_{y \in X^{C_p}} \pi_0(D_{\underline{X}}(y) \otimes \mathbb{Z})^{tC_p} \simeq \bigoplus_{X^{C_p}} \mathbb{Z}/p.
$$

We have the following commutative diagram.

$$
\begin{array}{ccc}\nS & \xrightarrow{c_{X}^{C_{p}}} & X_{1}^{>1}D_{X^{C_{p}}} \longleftarrow \text{Id} & X_{1}^{>1}D_{X^{C_{p}}} \\
\downarrow \xrightarrow{\downarrow} & \downarrow \xrightarrow{\downarrow} & \downarrow \xrightarrow{\downarrow} \\
\Phi^{C_{p}}S & \xrightarrow{\Phi^{C_{p}}(X_{1}D_{X})} & \xrightarrow{\sim} & \Phi^{C_{p}}(X_{1}^{>1}\varepsilon^{*}D_{X}) \longrightarrow \Phi^{C_{p}}(X_{1}^{>1}\varepsilon^{*}D_{X} \otimes \text{inf}_{C_{p}}^{e} \mathbb{Z}) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{Set}^{C_{p}} & \xrightarrow{c_{X}^{C_{p}}} & (X_{1}D_{X})^{tC_{p}} \longleftarrow \xrightarrow{\sim} & (X_{1}^{>1}\varepsilon^{*}D_{X})^{tC_{p}} \longrightarrow (X_{1}^{>1}\varepsilon^{*}D_{X} \otimes \mathbb{Z})^{tC_{p}} \\
\uparrow & \uparrow & \downarrow & \downarrow & \downarrow \\
\text{Set}^{C_{p}} & \xrightarrow{\wedge^{C_{p}}} & \uparrow & \downarrow & \downarrow \\
\text{Set}^{C_{p}} & \xrightarrow{\wedge^{C_{p}}} & (X_{1}D_{X})^{tC_{p}} \longrightarrow (X_{1}^{>1}\varepsilon^{*}D_{X})^{tC_{p}} \longrightarrow \text{Set}^{C_{p}} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{Set}^{C_{p}} & \xrightarrow{\wedge^{C_{p}}} & (X_{1}D_{X})^{tC_{p}} \longrightarrow \text{Set}^{C_{p}} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow &
$$

The rightmost part is induced by the ring map $\mathbb{S} \to \mathbb{Z}$. The violet square is obtained from Observation [5.2.5](#page-26-2) applied to $f = \varepsilon \colon \underline{X^{C_p}} \simeq \underline{X}^{>1} \to \underline{X}$ and $D_{\underline{X}}$, using the equivalence $\Phi^{C_p} D_{\underline{X}} \simeq D_{X^{C_p}}$ from Theorem [2.3.3](#page-11-1) and $\varepsilon^{C_p} = \mathrm{id}_{X^{C_p}}.$

By definition, the blue route recovers the gluing class. Following the upper route to the same object gives a class having the desired properties. Indeed, on π_0 the upper route reads

$$
\mathbb{Z} \xrightarrow{\Delta} \bigoplus_{X^{C_p}} \mathbb{Z} \longrightarrow \bigoplus_{X^{C_p}} \mathbb{Z}
$$
\n
$$
\downarrow \simeq
$$
\n
$$
\bigoplus_{X^{C_p}} \mathbb{Z} \xrightarrow{\simeq} \bigoplus_{X^{C_p}} \mathbb{Z}
$$
\n
$$
\downarrow \qquad \qquad \downarrow
$$
\n
$$
\bigoplus_{X^{C_p}} \mathbb{Z}/p
$$

where proj refers to the sum of the projection maps $\mathbb{Z} \to \mathbb{Z}/p$. Here, all maps in sight preserve the individual summands of $\bigoplus_{X^{C_p}} \mathbb{Z}$: the only potentially nonobvious case is the vertical red map, which is dealt with in Observation [5.2.5.](#page-26-2) \Box

Proof of Theorem [D.](#page-5-2) Recall that Condition (H) from Condition [1.3](#page-3-0) asks about surjectivity of the upper composite in the diagram

$$
H_d^{\Gamma}(\underline{E}_{\Gamma}\underline{\mathrm{Fin}}, \underline{E}_{\Gamma}\underline{\mathrm{Fin}}^{>1}) \xrightarrow{\partial} H_{d-1}^{\Gamma}(\underline{E}_{\Gamma}\underline{\mathrm{Fin}}^{>1}) \longrightarrow H_{d-1}^{\Gamma}(\underline{\ast})
$$

$$
\downarrow \simeq \qquad \qquad \downarrow \simeq \qquad \qquad \downarrow \simeq
$$

$$
H_d^{\Gamma/\pi}(\pi \backslash \underline{E}_{\Gamma}\underline{\mathrm{Fin}}, \pi \backslash \underline{E}_{\Gamma}\underline{\mathrm{Fin}}^{>1}) \xrightarrow{\partial} H_{d-1}^{\Gamma/\pi}(\pi \backslash \underline{E}_{\Gamma}\underline{\mathrm{Fin}}^{>1}) \longrightarrow H_{d-1}^{\Gamma/\pi}(\underline{\ast}),
$$

where the right horizontal maps are induced from the the projection onto the F -component in the splitting $\underline{E_{\Gamma}}\text{Fin}^{>1}=\coprod_{F'\in \mathcal{M}}\Gamma/F'.$ We may thus equivalently show surjectivity of the lower horizontal composite. Denote $\underline{X} = \pi \backslash \underline{E_{\Gamma} \text{Fin}}$, which is C_p -Poincaré by Theorem [5.1.4](#page-23-0) since in this case, $W_\Gamma F \cong \{e\}$ and $E_\Gamma \text{Fin}$ is compact by Example [4.1.3.](#page-18-4)

Now by definition, the bottom composite in the diagram above is obtained by postcom- $\mathrm{posing~cofib}(X_\cdot^{>1}\epsilon^*D_{\underline{X}}\otimes\mathbb{Z}\to X_\cdot D_{\underline{X}}\otimes\mathbb{Z})_{hC_p}\to \Sigma(X_\cdot^{>1}\epsilon^*D_{\underline{X}}\otimes\mathbb{Z})_{hC_p}$ with projection to a component of $X^{C_p}.$ Thus, by Proposition [5.2.3](#page-25-0) and with the alternative description of the gluing class via the red route in Construction [5.2.1,](#page-24-2) we obtain the required surjectivity. \Box

References

