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A parametrized quasi-normal mode framework for modified Teukolsky equations

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Modifications to general relativity lead to effects in the spectrum of quasi-normal modes of black holes. In this paper, we develop a parametrized formalism to describe deviations from general relativity in the Teukolsky equation, which governs linear perturbations of spinning black holes. We do this by introducing a correction to the effective potential of the Teukolsky equation in the form of a $1/r$ expansion controlled by free parameters. The method assumes that a small deviation in the effective potential induces a small modification in the spectrum of modes and in the angular separation constants. We isolate and compute the universal linear contribution to the quasi-normal mode frequencies and separation constants in a set of coefficients, and test them against known examples in the literature (massive scalar field, Dudley-Finley equation and higher-derivative gravity). We make the coefficients publicly available for relevant overtone, angular momentum and azimuthal numbers of modes and different values of the black hole spin.

I. INTRODUCTION

Gravitational wave astronomy successfully observed more than one hundred binary black hole (BH) mergers [1]. Such an advancement in the field allows one to make precision tests of general relativity (GR). In particular, BH spectroscopy, *i.e.*, the identification of the infinite tower of quasi-normal modes (QNMs) of which the signal is composed in the linear post-merger phase known as the ringdown [2–4], has finally been applied to tens of events [5–7]. The detection of two modes simultaneously, despite being controversial for the first three observing runs [8–16], is expected to be effective for the current O4a run and future ones.

Detecting a second mode is a crucial ingredient for tests of GR. Due to no-hair theorems, the QNM spectrum of a Kerr BH depends uniquely on its mass and spin. If only one mode is detected, this can always be fitted to a QNM frequency of a Kerr BH with a certain mass and angular momentum. However, the measurement of any additional modes provides a consistency test of the Kerr QNM spectrum and hence would allow us to spot deviations from GR. This clear identification of possible beyond-GR effects makes BH spectroscopy one of the most promising ways to test GR.

Currently, ringdown tests employ blind deviations from GR in the frequencies [7], or agnostic deviations constructed assuming small-coupling and slow-spin parametrization [17–19]. On the other hand, theory-specific tests are limited to a handful of cases [20]. This is because the computation of QNMs for rotating solutions beyond-GR is incomplete. The main difficulties arise from: absence of analytic background solution, non-separability of the perturbation equations, additional fields coupled to the metric, different boundary conditions [4]. It turns out that for a vast class of theories, the first problem can be solved performing a double simultaneous expansion in the spin and in the coupling constant of the theory [21–25]. Then, one can choose whether to study perturbations giving priority to slow-spin or small coupling. The former has the advantage of being possible for metric perturbations, for which couplings between different fields are tractable, at the cost of not being able to predict precisely QNMs at high spins (which are relevant for astrophysical purposes) [26–32].

On the other hand, by assuming small-coupling for the perturbations, one can work out a modified Teukolsky equation, which is, in principle, reliable at any spin [33–35]. The disadvantage comes from the construction itself of the Teukolsky equation, which is based on curvature perturbations, and once one has some perturbations of perturbations, as in the case outlined here, metric reconstruction becomes necessary. This feature strongly hinders one from going beyond the first order in the coupling expansion.

The necessity of having reliable QNMs at high spins and the fact that observations seem to narrow down the size of deviations from GR, make the modified Teukolsky framework preferable for the study of QNMs beyond GR. The scope of this paper is to develop a general formalism for the quick computation of QNMs in the-

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ories which have a perturbative departure from GR. The framework is based on an assumption similar to one developed in spherical symmetry [36–39]. The advantage of this formalism is that it can be used also for the inverse problem, *i.e.*, if a modification to GR is detected, one wants to be able to reconstruct the potential, the metric, or even the action from which the deviation originated from [38, 40, 41]. In general, this framework opens the path to the development of a theory-informed description of QNM agnostic deviations.

The structure of the paper is as follows: we first introduce the modified Teukolsky equation and the formalism in section II; then we show how to numerically compute the coefficients with the continued fraction method and their regime of validity in section III; the formalism has already a few notable applications, which we use as a check for our computations in section IV; finally, we outline our conclusions in section V. Throughout the paper we work with mostly plus signature for the metric, and geometric units $G_{\rm N} = c = 1$. We also assume that the background, unmodified metric is a Kerr BH of mass $M = 1/2$ and spin a. It is worth noting that due to the choice of units, the spin parameter a ranges between 0 and 1/2.

II. PARAMETRIZED FORMALISM

A. The linear coefficients

Let us start from the radial Teukolsky equation for a spin s field as [42]

$$\frac{1}{\Delta^s R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[\Delta^{s+1}R'(r)\right] + V(r) = 0\,, \tag{1}$$

where the effective potential reads

$$V(r) = 2\mathrm{i}s\frac{\mathrm{d}K}{\mathrm{d}r} - \lambda_{\ell m} + \frac{1}{\Delta}\left(K^2 - \mathrm{i}sK\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)\,, \tag{2}$$

and we defined

$$\Delta = r^2 - r + a^2\,, \qquad K = (r^2+a^2)\omega - am\,, \tag{3}$$

$$\lambda_{\ell m} = B_{\ell m} + a^2\omega^2 - 2am\omega\,. \tag{4}$$

It is worth mentioning that the zeros of the function Δ determine the location of the BH inner and outer horizons, given by

$$r_\pm = \frac{1\pm\sqrt{1-4a^2}}{2}\,. \tag{5}$$

On the other hand, the angular equation reads [42]

$$\begin{aligned}\frac{1}{S(y)}\frac{\mathrm{d}}{\mathrm{d}y}\left[(1-y^2)S'(y)\right] + a^2\omega^2y^2 \\ - 2sa\omega y + B_{\ell m} + s - \frac{(m+sy)^2}{1-y^2} = 0\,,\end{aligned} \tag{6}$$

where $y = \cos\theta$.

We now assume that for a modified theory of gravity whose modifications are small with respect to GR the equation governing radial perturbations is the Teukolsky radial equation plus a correction to the potential linear in the coupling constants

$$\frac{1}{\Delta^s R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[\Delta^{s+1}R'(r)\right] + V(r) + \delta V(r) = 0\,, \tag{7}$$

and we assume that the modification is expanded in powers of r

$$\delta V(r) = \frac{1}{\Delta}\sum_{k=-K}^{4}\alpha^{(k)}\left(\frac{r}{r_+}\right)^k\,, \tag{8}$$

where K is the largest negative coefficient of the power series and $\alpha^{(k)}$ are dimensionful coefficients we assume to be small. This assumption is justified by the recent developments in obtaining modified Teukolsky equations by assuming small coupling corrections to GR [33–35].

On the other hand, we can also assume that the angular equation remains unchanged. This is due to the fact that the spheroidal harmonics are a complete basis of the 2-sphere angular variables, and it can be shown that if one forces an angular expansion of the Weyl scalars in spin-weighted spheroidal harmonics, all the mixing terms would enter at second order in the coupling constants [35, 43]. Nevertheless, a modification on the QNM frequencies will induce a modification to the separation constant $B_{\ell m}$, hence, we also need to include equation (6) in our analysis.

If we assume the couplings to be small, we are allowed to perform a Taylor expansion of QNMs and separation constants around their GR values [36, 37]. Hence, we can write

$$\begin{aligned}\omega_{n\ell m} &\simeq \omega^0_{n\ell m} + \sum_k d^{(k)}_{\omega,n\ell m}\alpha^{(k)}\,, \\ B_{\ell m}(a\omega) &\simeq B^0_{\ell m}(a\omega) + \sum_k d^{(k)}_{B,\ell m}\alpha^{(k)}\,.\end{aligned} \tag{9}$$

In the following steps, we omit the indices s, n, ℓ, m, a, ω for clarity. The linear coefficients d_ω and d_B can be identified as the derivatives of ω and B with respect to the single coupling α. To compute them, we perform the following steps. In the GR limit, one finds ω and $B_{\ell m}$ as simultaneous roots of two functions constructed from the radial and the angular equation

$$\mathcal{L}_r\left[\omega, B_{\ell m}\right] = 0\,, \qquad \mathcal{L}_\theta\left[\omega, B_{\ell m}\right] = 0\,. \tag{10}$$

The exact form of these functions depends on the chosen numerical method. For non-zero modifications, we can perform a Taylor expansion of the two functions around $\alpha = 0$

$$\mathcal{L}_j\big|_{\rm GR} + \alpha\left.\frac{\mathrm{d}\mathcal{L}_j}{\mathrm{d}\alpha}\right|_{\rm GR} + \mathcal{O}(\alpha)^2 = 0\,, \tag{11}$$

where $j = [r,\theta]$ and we evaluate the derivative around their GR value $\omega = \omega^0$, $B = B^0$, $\alpha = 0$. By requiring that equations (11) is satisfied at each order in α, and expanding the derivative by chain rule, we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}_r}{\partial \alpha} + \frac{\partial \mathcal{L}_r}{\partial \omega} d_\omega + \frac{\partial \mathcal{L}_r}{\partial B} d_B \bigg|_{\rm GR} &= 0\,, \\ \frac{\partial \mathcal{L}_\theta}{\partial \omega} d_\omega + \frac{\partial \mathcal{L}_\theta}{\partial B} d_B \bigg|_{\rm GR} &= 0\,, \end{aligned} \tag{12}$$

where we identified d_ω and d_B from their definition in equation (9). By solving the conditions above for d_ω and d_B, we get

$$\begin{aligned} d_\omega &= -\frac{\partial \mathcal{L}_r}{\partial \alpha}\frac{\partial \mathcal{L}_\theta}{\partial B}\left(\frac{\partial \mathcal{L}_r}{\partial \omega}\frac{\partial \mathcal{L}_\theta}{\partial B} - \frac{\partial \mathcal{L}_r}{\partial B}\frac{\partial \mathcal{L}_\theta}{\partial \omega}\right)^{-1}\Bigg|_{\rm GR}\,, \\ d_B &= \frac{\partial \mathcal{L}_r}{\partial \alpha}\frac{\partial \mathcal{L}_\theta}{\partial \omega}\left(\frac{\partial \mathcal{L}_r}{\partial \omega}\frac{\partial \mathcal{L}_\theta}{\partial B} - \frac{\partial \mathcal{L}_r}{\partial B}\frac{\partial \mathcal{L}_\theta}{\partial \omega}\right)^{-1}\Bigg|_{\rm GR}\,. \end{aligned} \tag{13}$$

In section III we show how to numerically define the functions $\mathcal{L}_j$ with Leaver's continued fraction method.

B. Maximum number of independent coefficients

In reference [44], Kimura realised that in the case of spherically symmetric perturbations, there is always an ambiguity in defining the modified potential, upon a free reparametrization of the field. The same reasoning can be applied to the Teukolsky equation as well. If we perform the following transformation in equation (7)

$$R(r) \to [1 + \varepsilon X(r)]\, R(r) + \varepsilon \Delta Y(r) R'(r)\,, \tag{14}$$

assuming that $\varepsilon \ll 1$, then the equation that $R(r)$ solves is

$$\frac{1}{\Delta^s R}\frac{\mathrm{d}}{\mathrm{d}r}\left[\Delta^{s+1}R'\right] + V + \delta V + \delta\overline{V} + \delta W \frac{R'}{R} = 0\,. \tag{15}$$

By imposing $\delta W = 0$ we uniquely obtain the free function $X(r)$ as

$$X(r) = c + \frac{s}{2}\Delta' Y - \frac{1}{2}\Delta Y'\,, \tag{16}$$

which yields the "ambiguous" potential in the form

$$\begin{aligned} \delta\overline{V} = \varepsilon\Delta\Bigg[& Y'\left(\frac{(s^2-1)(\Delta')^2}{2\Delta} + 2s - 2V - 1\right) \\ & + Y\left(\frac{s(s+1)\Delta'}{\Delta} - V' - \frac{V\Delta'}{\Delta}\right) \\ & - \frac{1}{2}\Delta Y^{(3)} - \frac{3\Delta' Y''}{2}\Bigg]. \end{aligned} \tag{17}$$

Now, upon suitable choice of the function Y, we can express $\delta\overline{V}$ in the r basis. It turns out that the ansatz $Y = Y_j = y_j (r_+/r)^j$ yields

$$\delta\overline{V} = \frac{\varepsilon y_j}{\Delta}\sum_{k=-3}^{5} r_+^k \overline{A}_j^{(k)}\left(\frac{r}{r_+}\right)^{k-j}\,, \tag{18}$$

which implies that $j \geqslant 1$ since the maximum power of r in V is r^4, and the full expression of $\overline{A}_j^{(k)}$ can be found in appendix A. In general, one can take a linear combination of the free functions Y_j and still get a potential that is equivalent to the starting one. Each term of this linear combination contains the free parameter y_j, which can be used to set to 0 one of the terms $\alpha^{(k)}$ in equation (8). This reasoning allows us to fix the negative limit in the power expansion to be $K = 3$.

It is possible that by choosing a different ansatz for Y one could further reduce the number of coefficients in the equation. In fact, for the case of study of higher derivative gravity that we treat in another publication [45] the number of independent coefficients reduces to four (being $k = [-2, 0, 1, 2]$) — see also [46]. Although we could not prove this is a general feature of arbitrary modifications of the Teukolsky equation, we suspect that it was possible in that case thanks to the expansion in the spin assumed for every coefficient. Indeed, we believe that the ansatz for Y that would reduce the potential to the lowest number of terms would be, perhaps, a rational function involving powers of a and r. To date, we could not find such reduction.

III. COMPUTATION OF THE COEFFICIENTS: THE CONTINUED FRACTION METHOD

A. Continued fractions for the Teukolsky equation

We start here by recalling the Leaver method to compute the frequencies and the separation constant for a Kerr spacetime. The first step to find a continued fraction expansion is to assume an ansatz for the wavefunctions. Let us start from the radial equation, where we assume the following ansatz [47]

$$R(r) = f^{-\mathrm{i}\sigma - s}(r - r_-)^{p-1-2s}\mathrm{e}^{qr}\sum_{n=0}^{N} R_n f^n\,, \tag{19}$$

where $r_\pm$ are the zeros of Δ,

$$f = \frac{r - r_+}{r - r_-}\,, \tag{20}$$

and we defined $p = q = \mathrm{i}\omega$ and

$$\sigma = \sigma_{\rm GR} \equiv \frac{r_+(\omega - \omega_c)}{r_+ - r_-}\,, \qquad \omega_c = \frac{am}{r_+}\,, \tag{21}$$

$$r_\pm = \frac{1}{2}(1 \pm \beta)\,, \qquad \beta = \sqrt{1 - 4a^2}\,. \tag{22}$$

With these definitions, the equation (1) takes the form

$$\sum_{n=0}^{N} R_n \left(\frac{\alpha^r_{n-1}}{f} + \beta^r_n + \gamma^r_{n+1} f \right) f^n = 0\,, \tag{23}$$

where the coefficients are

$$\alpha^r_n = (n+1)\,(n+1-s-2\mathrm{i}\sigma) \tag{24}$$

$$\begin{aligned}\beta^r_n =\,& 2n(2\mathrm{i}\sigma + p + q\beta - 1) - 2n^2 - 1 - s - B_{\ell m} \\ &+ q\left(a^2 q + \beta + s\right) - 2i\sigma(p + \beta q + i\omega - 1) \\ &- p(\beta q + q + s - 1)\end{aligned} \tag{25}$$

$$\gamma^r_n = (n - p - \mathrm{i}\omega)\,(n + s - p - 2\mathrm{i}\sigma + \mathrm{i}\omega)\,. \tag{26}$$

The equation is satisfied when each term proportional to a power of f vanishes

$$\beta^r_0 R_0 + \alpha^r_0 R_1 = 0\,, \tag{27}$$

$$\gamma^r_n R_{n-1} + \beta^r_n R_n + \alpha^r_n R_{n+1} = 0 \quad \text{for } n \geqslant 1\,. \tag{28}$$

The path for the angular equation is similar. We define an ansatz to be finite at the regular singular points $y = \pm 1$ [47]

$$S(y) = (1+y)^{k_1}(1-y)^{k_2} \mathrm{e}^{a\omega y} \sum_{n=0}^{N} S_n (1+y)^n\,, \tag{29}$$

where $k_1 = |m-s|/2$ and $k_2 = |m+s|/2$. We can obtain a similar recurrence relation by inserting this ansatz into equation (6), which, with an analogous reshuffling, reads

$$\beta^\theta_0 S_0 + \alpha^\theta_0 S_1 = 0\,, \tag{30}$$

$$\gamma^\theta_n S_{n-1} + \beta^\theta_n S_n + \alpha^\theta_n S_{n+1} = 0 \quad \text{for } n \geqslant 1\,, \tag{31}$$

where the coefficients are

$$\alpha^\theta_n = -2(n+1)\,(n+1+2k_1)\,, \tag{32}$$

$$\begin{aligned}\beta^\theta_n =\,& n(n-1) + 2n(k_1 + k_2 + 1 - 2a\omega) \\ &- 2a\omega(2k_1 + s + 1) + (k_1 + k_2)(k_1 + k_2 + 1) \\ &- a^2\omega^2 - s(s+1) - B_{\ell m}\,,\end{aligned} \tag{33}$$

$$\gamma^\theta_n = 2a\omega(n + k_1 + k_2 + s)\,. \tag{34}$$

To invert the relation we can define the ladder operators which have the following property $R_{n+1} = -\Lambda^r_n R_n$ and $S_{n+1} = -\Lambda^\theta_n S_n$ as (the superscript r/θ is omitted for clarity)

$$\Lambda_n = \frac{\gamma_{n+1}}{\beta_{n+1} - \alpha_{n+1}\Lambda_{n+1}}\,. \tag{35}$$

By initializing Λ_N according to the Nollert expansion (explained in detail in appendix B), the equations one needs to solve simultaneously to obtain the eigenfrequency ω and the separation constant $B_{\ell m}$ are

$$\mathcal{L}_r = \Lambda^r_1 \alpha^r_0 - \beta^r_0 = 0\,, \tag{36}$$

$$\mathcal{L}_\theta = \Lambda^\theta_1 \alpha^\theta_0 - \beta^\theta_0 = 0\,, \tag{37}$$

which are nothing but (27) and (30).

B. Continued fraction beyond Teukolsky

We now turn our attention to the modified Teukolsky equation. It is always possible to bring equation (8) into the following form[1]

$$\begin{aligned}\delta V(r) =\,& \frac{A^{(0)}}{\Delta} + \frac{A^{(1)}}{r_+(r-r_-)} + \frac{1}{r_+^2}\sum_{k=0}^{2} \tilde{\alpha}^{(k)} \left(\frac{r}{r_+}\right)^k \\ &+ \frac{1}{\Delta}\sum_{k=1}^{K} \alpha^{(-k)} \left(\frac{r_+}{r}\right)^k\,,\end{aligned} \tag{38}$$

where $A^{(0)}$, $A^{(1)}$ and $\tilde{\alpha}^{(k)}$ are constants that can be obtained from the constants $\alpha^{(k)}$ appearing in equation (8), as explained in appendix C. First of all, we notice that the terms multiplied by $1/\Delta$ modify the behaviour of the equation at the horizon. In order to take into account of these additional terms, we need to modify the definition of the exponent σ appearing in the ansatz (19). By requesting that the solution is regular at the horizon, we must replace the value of σ into

$$\sigma = \frac{\mathrm{i}s}{2} + \sqrt{\left(\sigma_{\mathrm{GR}} - \frac{\mathrm{i}s}{2}\right)^2 + \frac{1}{\beta^2}\sum_{k=-K}^{4} \alpha^{(k)}}\,, \tag{39}$$

where we took the positive sign of the square root in order to obtain the correct GR limit. On the other hand, the terms $\tilde{\alpha}^{(1)}$ and $\tilde{\alpha}^{(2)}$ modify the behaviour at infinity of the equation. This leads to a modification of the values of p and q into

$$q = \pm\sqrt{-\frac{\tilde{\alpha}^{(2)}}{r_+^4} - \omega^2}\,, \tag{40}$$

$$p = -\frac{r_+\tilde{\alpha}^{(1)} + \tilde{\alpha}^{(2)} - 2r_+^4\left(qs - \mathrm{i}s\omega - \omega^2\right)}{2qr_+^4}\,, \tag{41}$$

where the sign of q is chosen such that $\mathrm{Re}(q) > 0$. This asymptotic behaviour is the reason why we truncate the series in equation (8) at $k = 4$. By repeating the steps done for the GR case, we obtain a modified version of equation (23)

$$\begin{aligned}\sum_{n=0}^{N} R_n \Bigg[& \frac{\alpha^{\text{BC}}_{n-1}}{f} + \beta^{\text{BC}}_n + \gamma^{\text{BC}}_{n+1} f \\ &+ \frac{1}{\beta^2}\frac{(1-f)^2}{f}\sum_{k=1}^{K} \alpha^{(-k)} \left(\frac{1-f}{1-\eta f}\right)^k \Bigg] f^n = 0\,,\end{aligned} \tag{42}$$

[1] In section II B we showed that $K = 3$, but the following analysis works, in principle, for any value of K, hence we keep it unspecified.

where $\eta = r_-/r_+$ and

$$\alpha_n^{\text{BC}} = \alpha_n^r - \frac{1}{\beta^2}\sum_{k=1}^{K}\alpha^{(-k)}\,, \tag{43}$$

$$\beta_n^{\text{BC}} = \beta_n^r - 2\left(\sigma_{\text{GR}}-\sigma\right)\left(\sigma_{\text{GR}}+\sigma-\omega\right)\,, \\ - 2\frac{A^{(0)}}{\beta^2} + \frac{A^{(1)}}{r_+\beta} + \frac{\widetilde{\alpha}^{(0)}}{r_+^2} \tag{44}$$

$$\gamma_n^{\text{BC}} = \gamma_n^r + 2\mathrm{i}\left(\sigma_{\text{GR}}-\sigma\right)\left(s+\mathrm{i}\omega\right) \\ - \frac{1}{\beta^2}\sum_{k=1}^{K}\alpha^{(-k)} - \frac{A^{(1)}}{r_+\beta}\,, \tag{45}$$

and we used the fact that

$$\Delta = \beta^2\frac{f}{(1-f)^2}\,, \qquad r - r_- = \frac{\beta}{1-f}\,. \tag{46}$$

If we fix a single modification k, we can get rid of the rational behaviour in f by multiplying the equation by $(1-\eta f)^k$, obtaining the following expression

$$\sum_{n=0}^{N} R_n\Bigg[\left(\frac{\alpha_{n-1}^{\text{BC}}}{f} + \beta_n^{\text{BC}} + \gamma_{n+1}^{\text{BC}} f\right)(1-\eta f)^k \\ + \frac{\alpha^{(-k)}}{\beta^2}\frac{(1-f)^{k+2}}{f}\Bigg] f^n = 0\,. \tag{47}$$

We can figure out the coefficient relation (equivalent to that of equation (28)), which at a given n takes the form

$$\sum_{j=-1}^{k+1}\left(\widetilde{\gamma}_{n,j-1} + \widetilde{\beta}_{n,j} + \widetilde{\alpha}_{n,j+1}\right) R_{n-j} = 0\,. \tag{48}$$

The coefficients appearing in the relation are given by

$$\widetilde{\alpha}_{n,j} = (-\eta)^j \binom{k}{j}\alpha_{n-j}^{\text{BC}} + (-1)^j\binom{k+2}{j}\frac{\alpha^{(-k)}}{\beta^2}\,, \tag{49}$$

$$\widetilde{\beta}_{n,j} = (-\eta)^j\binom{k}{j}\beta_{n-j}^{\text{BC}}\,, \tag{50}$$

$$\widetilde{\gamma}_{n,j} = (-\eta)^j\binom{k}{j}\gamma_{n-j}^{\text{BC}}\,. \tag{51}$$

We notice that, from the definition of the binomial, $\widetilde{\alpha}_{n,j}$ is non-vanishing for $0 \leqslant j \leqslant k+2$, while $\widetilde{\beta}_{n,j}$ and $\widetilde{\gamma}_{n,j}$ are non-zero for $0 \leqslant j \leqslant k$. Now that we have a $k+3$ terms relation, we can perform a Gaussian elimination to reduce it to a three-terms relation (details can be found in the appendix of [38]). Once the three-terms relation is found, one can re-initialize the ladder operator Λ_n^r and obtain the modified frequency and separation constant from equations (36)–(37).

C. Numerical computation of the coefficients

In the previous two sections we explained how to obtain the functions $\mathcal{L}_r(\omega, B, \alpha)$ and $\mathcal{L}_\theta(\omega, B)$. To compute the coefficients d_ω and d_B as given in equation (13), we evaluate the derivatives numerically with a 4-points centered stencil. For each pair of coefficients, we initialize the ladder operators Λ_N to some arbitrary low integer N, and then increase it by one until the simultaneous relative change in d_ω and d_B is smaller than a given tolerance (which we chose to be 10^{-7}). We computed numerically all the coefficients for the following values $s=-2$, $n=[0,2]$, $\ell=[2,4]$, $m=[-\ell,\ell]$, $k=[-3,4]$ in a uniform grid in $a=[0,0.495]$ with spacing $\delta a = 0.005$. The full list of coefficients is available in a public git folder [?].

In figure 1 we show the results from this computation for the real and imaginary parts of the $d_\omega^{(k)}$ coefficients for $s=-2$, $n=0$, $\ell=2$, $m=[-2,2]$ for values of $k=[-3,4]$ and of the spin a comprised between 0 and 0.45, as well as the real and the imaginary part of $d_B^{(k)}$ for the same n, ℓ, m and $k=[-1,2]$.

To directly apply our formalism to further studies, e.g., ringdown analysis of non-linear computations or data analysis, we also provide a `python` code and a `jupyter notebook` with some examples [48]. It allows one to compute the QNMs and the separation constants as function of n, ℓ, m, a and $\alpha^{(k)}$ and can thus, in principle, be efficiently integrated in commonly used code infrastructure. The code also allows one to access some of the earlier results for the parametrized QNM framework for modifications to the Regge-Wheeler and Zerilli potentials, for which coefficients beyond the fundamental mode have been computed in reference [38]. The GR values for the QNMs have been taken from reference [3, 49]. For more details about how the code is structured and how it can be used, we refer to the provided tutorial.

In principle, one should be able to compare the coefficients for $a=0$ with those computed in [36, 38]. However, we stress that for $a=0$, equation (7) reduces to the non-spinning limit of the Bardeen-Press equation [50], whereas the formalism of [36, 38] was developed for the Regge-Wheeler and the Zerilli equation. The transformation between the Bardeen-Press potential and the Regge-Wheeler/Zerilli potentials was obtained by Chandrasekhar [51], but generalizing this to the case of the modified potential with generic $\alpha^{(k)}$ couplings is non-trivial.

D. Linearized regime of validity

The framework we developed is motivated by the assumption that any modification of gravity produces only slight deviations from GR in astrophysical observables. In this section, we expand on the regime of validity of the formalism, by providing a quantitative assessment of the accuracy of such approach. It is worth noting that we can only asses the error made by restricting to linear corrections to the frequencies, as defined in equation (9) and not taking into account higher-order corrections to the potential, which are beyond the scope of this paper.

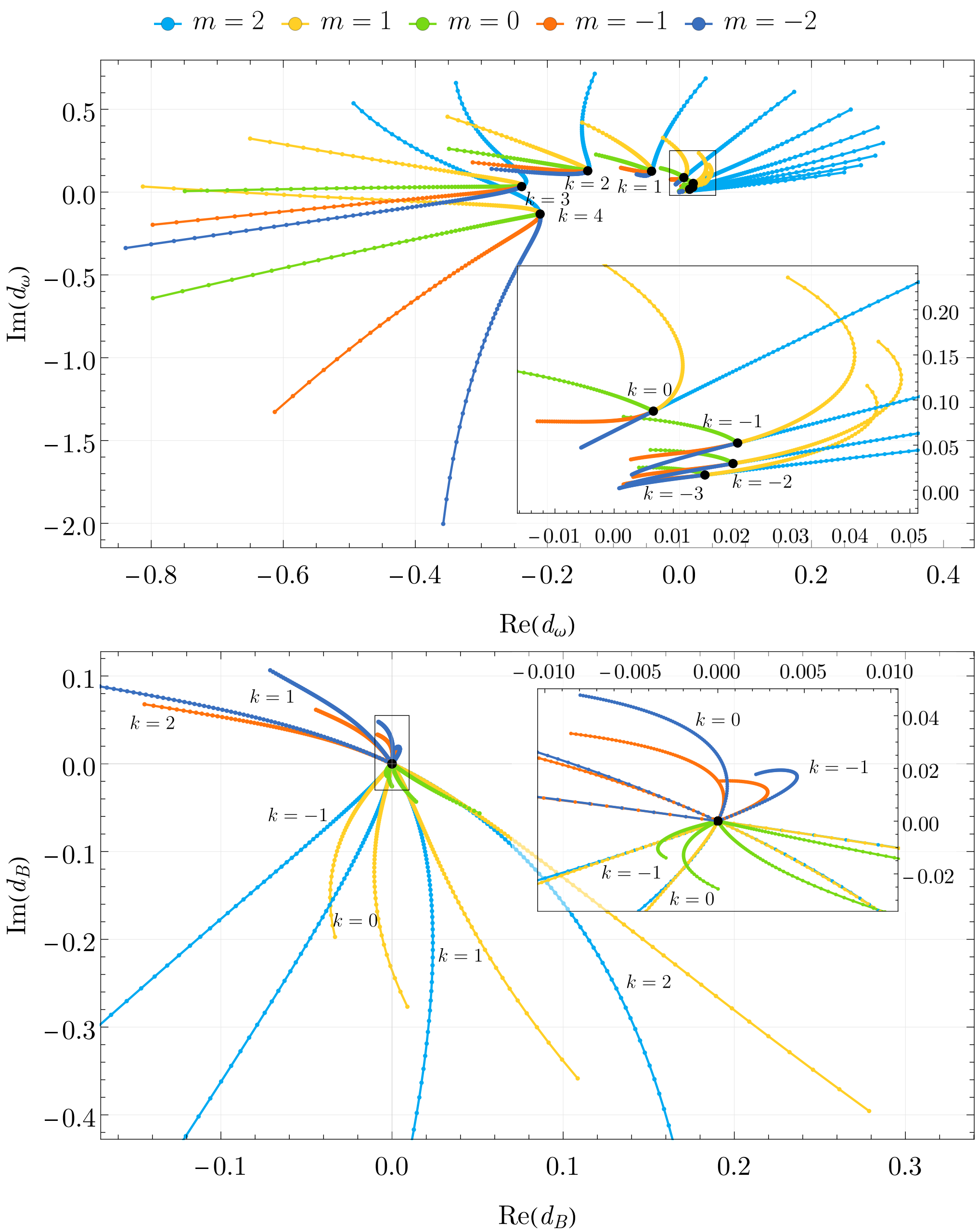

FIG. 1. In the top panel we show the real and the imaginary part of d_ω for $n=0$, $\ell=2$, $m=[-2,2]$, $k=[-3,4]$ and values of the spin from $a=0$ to $a=0.45$, and each point is on a step of $\delta a=0.005$. The inset focuses around the coefficients with $k=[-3,0]$. In the bottom panel we show the real and the imaginary part of d_B for $n=0$, $\ell=2$, $m=[-2,2]$, $k=[-1,2]$ and same values of the spin. The inset focuses around the coefficients with $m\leqslant 0$ and $k=[-1,0]$. In both plots the black dot signals the coefficient value for $a=0$.

First of all, we give a heuristic motivation on the maximum size of the coefficients, by requesting that the perturbation equation is not strongly modified at the boundaries of our dominion and that $\omega\simeq\mathcal{O}(1)$. At $r\to\infty$, we have seen from equations (40) and (41) that the only coefficients modifying the asymptotic structure of the potential are $\alpha^{(3)}$ and $\alpha^{(4)}$. With some simple algebra, we can infer

$$\alpha^{(3)}\lesssim 1\,,\qquad \alpha^{(4)}\lesssim 1\,. \tag{52}$$

On the other hand, the modifications in the potential affect the near-horizon expansion as in equation (39). The condition is such that the sum of coefficients must behave

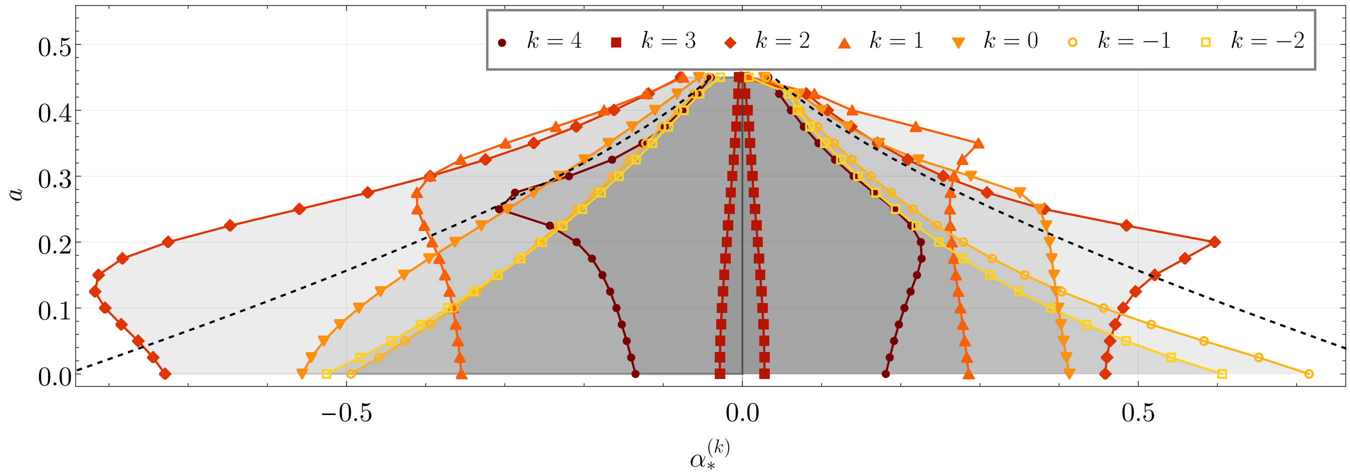

FIG. 2. We plot the threshold values $\alpha_*^{(k)}$ against the spin for different values of k. With $\alpha_*^{(k)}$, we identify the limit value of a coupling at which an error of 1% the a linear and a non-linear approximation is obtained. The plot uses as non-linear estimates the full continued fraction results. The error is evaluated for the $n = 0$, $\ell = m = 2$ mode and different values of k and of the spin a. The black dashed lines correspond to the estimate (53)

as

$$\sum_k \alpha^{(k)} \lesssim \left| 2\beta^2 \sigma_{\rm GR} \left(\sigma_{\rm GR} - \frac{\mathrm{i}s}{2} \right) \right| . \quad (53)$$

It is worth noting that when the superradiant condition $\omega = \omega_c$ is activated, one has $\sigma_{\rm GR} = 0$, and we expect that the formalism is valid only if the sum of the $\alpha^{(k)}$ is approximately 0.

In general, however, each power of k affects in a different way the effective potential. In order to have a more quantitative estimate of the allowed regime of validity, we perform two separate analysis. First, we compare the QNM frequencies computed with the linear approximation against those obtained with a full continued-fraction method discussed above. We estimated the error on the frequencies as

$$\Delta_\omega \equiv \sqrt{\left(\frac{\Delta\omega_R}{\omega_R}\right)^2 + \left(\frac{\Delta\omega_I}{\omega_I}\right)^2} , \quad (54)$$

where $\omega_{R,I}$ are respectively the real and imaginary parts of the QNM. We computed the error for several real positive and negative values of the couplings $\alpha^{(k)}$ and extracted the threshold values $\alpha_*^{(k)}$ at which the error reaches 1%. The results are represented in figure 2 for the mode $n = 0$, $\ell = m = 2$, for selected values of the spins between $a = 0$ and $a = 0.45$ and for $k = [-2, 4]$.

It can be seen that there is a complicated dependence of the thresholds on the type of modification we introduce in the potential. However, a main qualitative feature can be read off, i.e. that, for any modification, the threshold tends to get smaller for higher spins. The physical interpretation of that, is that, for a given beyond-GR effect in the modified Teukolsky equation, rotation tends to exacerbate the deviation of the linear approximation with respect to the true values of the QNMs. Bearing this caveat in mind, we will still show in the next section that the linear approximation provides very good results in a couple of known models of perturbation of rotating BHs with deviations from Kerr, also for high spin.

Since the computation of QNMs with the continued fraction method is not immediate nor straightforward to implement, we want to provide a quick estimate for the errors of the single-k contributions. In this respect, we compute the diagonal quadratic corrections, as explained in appendix D. We checked the estimate $\alpha_*^{(k)}$ by computing the error Δ_ω assuming that the non-linear frequencies are obtained including quadratic coefficients. By a qualitative comparison, the quadratic estimate works well to capture the error except for $k = 3$, and partially for $k = 4$. Even though it is not as precise as the full non-linear comparison, the quadratic coefficients can be used as a quick way to understand what threshold value to take for the couplings.

Lastly, we want to stress that the thresholds that we provided in this section, are referred to the contribution of a single modification. Hence, it could be that, depending on the values of the coefficients, the combination of multiple k would need larger or smaller threshold values. This means that for a theory-specific case, the bounds on $\alpha^{(k)}$ might differ from what we inferred in this section, and need to be addressed case-by-case.

IV. APPLICATIONS

A. Massive scalar perturbations

The first example that we provide to test our formalism is for the computation of the QNMs of a massive scalar field, a case extensively studied in the literature [52–54].

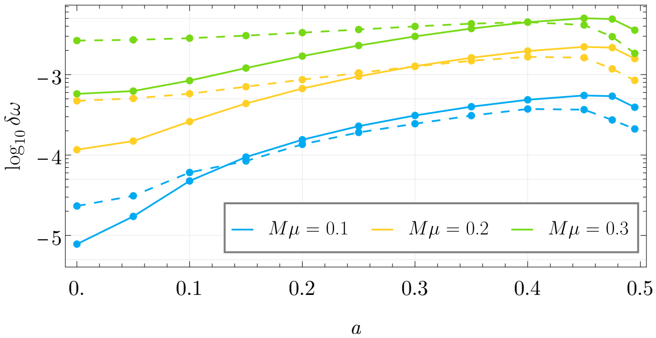

FIG. 3. Relative difference for the real part (solid line) and imaginary part (dashed line) of the fundamental $\ell = m = 2$ mode for a massive scalar perturbation computed with the linear approximation against the non-linear results of [54].

The radial and angular perturbation equations for a massive scalar field ($s = 0$) with mass μ are

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\Delta R'(r)\right] + \left(\frac{K^2}{\Delta} - \lambda_{\ell m} - \mu^2 r^2\right) R(r) = 0\,, \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\left(1-y^2\right) S'(y)\right] + \left[a^2\left(\omega^2-\mu^2\right)y^2 + B_{\ell m} - \frac{m^2}{1-y^2}\right] S(y) = 0\,. \tag{56}$$

First of all, we bring the angular equation into the form of equation (6) by transforming $\omega \to \omega + \frac{\mu^2}{2\omega}$. Then, by assuming $\mu \ll 1$, the radial equation is automatically brought in the form of (7), with the only non-zero $\alpha^{(k)}$ being

$$\alpha^{(1)} = \mu^2 a r_+ \left(a - \frac{m}{\omega_0}\right)\,, \qquad \alpha^{(3)} = \mu^2 r_+^3\,, \tag{57}$$

where ω_0 is the unperturbed Kerr frequency. The effect of the mass on the frequency at linear order in μ^2 is given by

$$\omega_{\text{L}} = \omega_0 + \frac{\mu^2}{2\omega_0} + d_{(1)}\alpha^{(1)} + d_{(3)}\alpha^{(3)}\,. \tag{58}$$

In figure 3 we show the difference $\delta\omega = |\omega_{\text{L}} - \omega_{\text{NL}}|$ between the linear results in (58) and the nonlinear QNMs ω_{NL} computed in [54] for $\ell = m = 2$ modes.

B. The Dudley-Finley equation

As a second example, we would like to test our formalism against gravitational perturbation of a Kerr-Newman (KN) BH in the limit of small charge, since the QNMs for a generic electric charge Q have been computed numerically in [55] and fits are available in [20]. Unfortunately, the KN perturbation equation is not explicitly separable, not even in the limit of small charge [56] in which at least electromagnetic and gravitational perturbations decouple. One could apply the algorithm of [33–35] to obtain a modified Teukolsky operator for the KN solution, but it goes beyond the scopes of this paper. For the sake of testing the method, we can restrict ourselves to the Dudley-Finley (DF) equation, a proxy equation for the perturbations of the Kerr-Newman metric [57–59]. The DF equation is obtained by taking equation (1) and performing the following substitution

$$\Delta \to \Delta + Q^2\,. \tag{59}$$

We can then rescale the equation into the form (7) by assuming that $Q \ll 1/2$ and by defining a new spin parameter

$$\bar{a} = a + \frac{Q^2}{2a}\,, \tag{60}$$

such that $\bar{a}^2 \simeq a^2 + Q^2$. Retaining only the terms quadratic in Q, we can see that the only non-zero $\alpha^{(k)}$ terms that contribute to the equation (8) are

$$\alpha^{(0)} = Q^2\left[\frac{is}{2\bar{a}}(m - 2\bar{a}\bar{\omega}_0) - (m - \bar{a}\bar{\omega}_0)^2\right]\,, \tag{61}$$

$$\alpha^{(1)} = Q^2 r_+ \left[\bar{\omega}_0 \frac{m - \bar{a}\omega_0}{\bar{a}} - \frac{is}{\bar{a}}(m - 2\bar{a}\bar{\omega}_0)\right]\,, \tag{62}$$

$$\alpha^{(2)} = -Q^2 r_+^2 \bar{\omega}_0^2\,, \tag{63}$$

where $\bar{\omega}_0$ is the Kerr frequency evaluated at spin $\bar{a}$. The DF linear frequencies at spin a are obtained by

$$\omega_{\text{L}} = \bar{\omega}_0 + \sum_{k=0}^{2} \alpha^{(k)} d_{(k)}\,. \tag{64}$$

In Figure 4 we show the real and imaginary part of the absolute difference $\delta\omega = |\omega_{\text{L}} - \omega_{\text{NL}}|$ between the linear results in (64) and the nonlinear QNMs ω_{NL} computed via the Leaver method in [59], for various spins and different values of the electric charge. We make the comparison with $\ell = 2$ modes, with all values of $m = [-2, 2]$, for the fundamental and first two overtones. The plot clearly shows that the discrepancy between the linearized QNMs and the full non-linear results scales with the charge, and the approximation remains valid for all the different values of (n, ℓ, m) surveyed.

Finally, we comment on the fact that the errors grow for small values of the spin. This is due to the fact that in order to bring the equation in the form of (7), we performed the transformation (60), which brings a term $1/a$ to the denominator when $m \neq 0$. In other words, this transformation is valid as long as $|Q| \ll |a|$. Nevertheless, the smallness of the universal coefficients $d_{(k)}$ is such that the combination in frequency (64) is finite and faithful to the non-linear value.

C. Higher derivative gravity

Now we want to check the prediction of QNMs in higher derivative gravity using the parametrized method

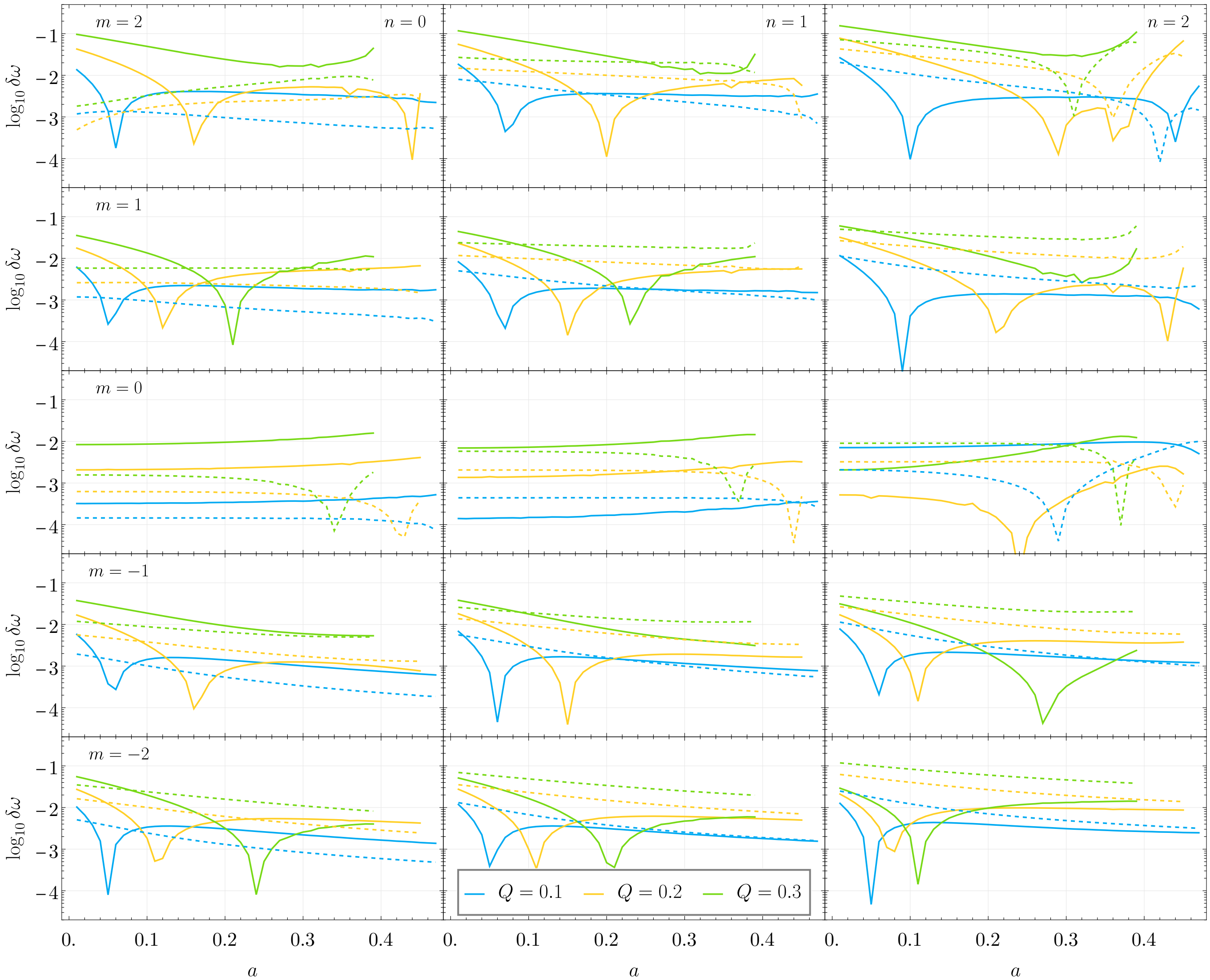

FIG. 4. Difference between the real (solid lines) and imaginary (dashed lines) part of the Dudley-Finley QNMs computed either with the linear perturbative approach or the full continued fraction method. We show results for $n=0$ (left panels), $n=1$ (central panels) and $n=2$ (right panels), $\ell=2$, $m=[-2,2]$ (bottom to top panels) for different values of the spin and of the electric charge. Note that the three curves have different endpoints, as for a given Q the maximum value of a is $a_{\text{MAX}}=\frac{1}{2}\sqrt{1-4Q^2}$

against the results presented in [46]. In a companion paper [45], focused on the analysis of QNMs in higher-derivative gravity, we show how to reduce the radial perturbation equation to the form of equation (7), with the only non-vanishing values of $\alpha^{(k)}$ being $k=k^{\text{HD}}=[-2,0,1,2]$

$$\delta V^{\pm}=\lambda\sum_{k\in k^{\text{HD}}}\alpha_{\pm}^{(k)}\left(\frac{r}{r_+}\right)^{k}\,, \tag{65}$$

where the $\pm$ refers to the polarization of the perturbation and we collected out λ, the normalized coupling constant of the theory.[2] From this, we can compute the frequencies deviations, normalized by the coupling constant λ

$$\delta\omega^{\pm}=\frac{\omega^{\pm}-\omega^{\text{Kerr}}}{\lambda}=\sum_{k\in k^{\text{HD}}}\alpha_{\pm}^{(k)}d_{(k)}\,, \tag{66}$$

for each parity, and each realization of the theory. In figure 5, we compare our results against the fits $\delta\omega^{\text{fit}}$ given

[2] *cfr.* equation (30) of [45], for which $\alpha^{(k)}=A^{(k)}r_+^k$, and the coupling constant has been previously factorized out. Here we use λ to refer to the coupling constant, to avoid misunderstanding with the α_{q} used in [45].

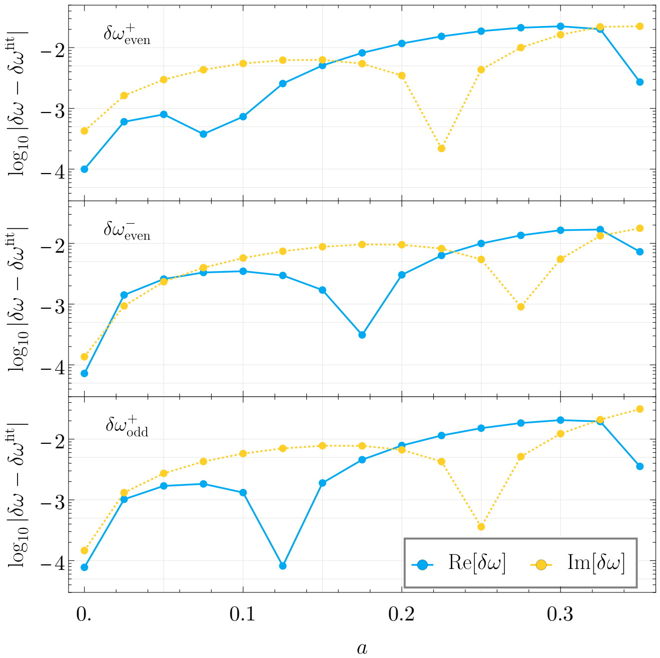

FIG. 5. Real and imaginary part of the absolute difference between the fits of [46] and the frequencies computed with the parametrized formalism for the $\pm$ polarizations of even cubic modes and the + polarization of the odd cubic mode

in [46]. We truncate the analysis at spin $a = 0.35$, since the fits are valid only up to this value. The plot shows remarkable agreement between the corrections computed with two different methods, strengthening the validity of the parametrized formalism. In figure 5 we limited to show polar and axial, $\ell = m = 2$ values for the even-parity cubic theory, labelled as $\delta\omega^{\pm}_{\rm even}$, as well as polar, $\ell = m = 2$ values for the odd-parity cubic theory, labelled as $\delta\omega^{+}_{\rm odd}$. Details on the definition of these modes can be found in [46] and in the companion paper where we perform an extensive study of QNMs of rotating BHs in higher derivative gravity [45].

V. CONCLUSIONS

In this paper we have shown how to connect small deviations parametrized by powers of the radial coordinate r in the Teukolsky equation to small deviations in the eigenfrequencies and in the separation constants of modified Kerr BHs. We proved that for each value of n, ℓ, m there are up to nine independent coefficients in the radial parametrization, but for specific cases they could be less. We presented a robust method to compute the coefficients that control the linear corrections to the QNM frequencies and separation constants through a generalization of Leaver's continued fractions, and used it to compute them for $n = [0, 2]$, $\ell = [2, 4]$, $m = [-\ell, \ell]$ and $k = [-3, 4]$ in a range of spins between 0 and 0.495.[3] These results are available online in a public git repository [48], together with a `python` code and a `jupyter notebook` to compute the QNMs and separation constants. There we also provide a tutorial demonstrating how to use the code, which can in principle be applied to compute QNMs with arbitrary n, l, m, k and angular momentum besides those we computed explicitly here.

We checked the quality of the predictions against three cases known in the literature: perturbations of a massive scalar field around Kerr, the Dudley-Finley equation and the QNMs of BHs in higher-derivative gravity. For all the three cases the frequencies predicted by the formalism show great agreement with respect to the results in the literature.

These coefficients will be particularly useful for the computation of QNMs of rotating BHs in alternative theories of gravity for which a modified Teukolsky equation is obtained, in the spirit of the method developed in [33–35]. So far, this method has been successfully applied to higher-derivative gravity [46], but other theories like scalar-Gauss-Bonnet gravity and dynamical-Chern-Simons gravity [60] are good candidates for this computation. In order to study those cases, it would be interesting to generalize our parametrized formalism by including couplings between the Teukolsky equation and a scalar field, analogous to the analysis of [37] in the case of static BHs.

The modified Teukolsky approach of [33–35] is currently limited by the fact that metric reconstruction is only available for GR, meaning that it is not yet possible to extend the method beyond first order in the coupling. For this reason, we did not explore further the quadratic coefficients as it was done in the non-rotating case [37, 38], and we limited the computation of the diagonal ones just to have a quick estimate of the error of the method itself.

Let us also remark that a different approach to study beyond-GR QNMs based on spectral methods has recently been introduced and successfully demonstrated for a wide range of spins in Refs. [61–65]. These methods have the advantage of being more flexible, but on the other hand, they have a much higher computational complexity and cost than the standard perturbative approaches. In this regard, the results of our method applied to specific theories may be useful to validate the spectral approaches.

The most intriguing open problem from our analysis is whether one can find a better way to exploit the potential ambiguity, as done in section II B. With the choice we made, we could reduce the number of independent coefficients $\alpha^{(k)}$ to 8. For the case of higher derivative gravity we have been able to numerically reduce the number of free coefficients to just 4 [45]. One may wonder whether

[3] We recall that in our conventions extremality corresponds to $a = 1/2$

higher derivative gravity has a special structure of the equations, or if there is a fundamental transformation of the potential that could diminish the number of free parameters.

Such discussion is relevant especially if one wants to use this formalism in a theory-agnostic setup, *e.g.*, to perform ringdown tests of GR or the inverse problem. Since the coefficients $d^{(k)}$ have a spin dependence, it would be interesting to map them to ParSpec [17]. Another useful mapping would be with the WKB deviation coefficients, as done in [40]. Finally, in the upcoming analysis, it would be interesting to compare the detectability of beyond-Teukolsky effects against that of second order QNMs [66–68].

ACKNOWLEDGMENTS

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Appendix A: Ambiguity of the potential modifications

Here we list the explicit form of the coefficients of equation (18) when one transforms the radial Teukolsky function as (14). These values hold for $s=-2$:

$$\overline{A}_j^{(-3)} = \frac{a^6}{2} j(j+1)(j+2)\,, \qquad \overline{A}_j^{(-2)} = -\frac{3a^4}{2} j(j+1)^2\,, \tag{A1}$$

$$\overline{A}_j^{(-1)} = \frac{a^4}{2} j\left[3j(j+1) - 4B + 4m^2 - 8i\omega + 10\right] + 4ia^3 jm + \frac{3}{2}a^2 j\left(j^2+j-1\right)\,, \tag{A2}$$

$$\begin{aligned}\overline{A}_j^{(0)} &= a^3(2j-1)\left[a\omega(\omega-4i) - 2m(\omega+2i)\right] \\ &\quad - a^2\left[3j^3 - j\left(4B - 2m^2 + 4i\omega - 1\right) + B + 2\right] - 4iajm - \frac{1}{2}j\left(j^2-4\right)\,,\end{aligned} \tag{A3}$$

$$\begin{aligned}\overline{A}_j^{(1)} &= 2a^4(j-1)\omega^2 + \frac{1}{2}a^2\Big[3j^2(j-1) + 4B + 2\omega(\omega-16i) + 8 + j\left(-8B + 4m^2 - 4\omega(\omega-8i) + 2\right)\Big] \\ &\quad + 2am\left[2j(\omega+3i) - \omega - 2i\right] + \frac{1}{2}j\left[3(j-1)j - 4B - 11\right] + B + 2\,,\end{aligned} \tag{A4}$$

$$\overline{A}_j^{(2)} = a^2\omega\left[\omega - 16i(j-1)\right] - 2a(2j-1)m(\omega+2i) + \frac{1}{2}j(-3(j-2)j + 8B - 24i\omega + 13) - 3(B - 4i\omega + 2)\,, \tag{A5}$$

$$\overline{A}_j^{(3)} = 2a^2(2j-3)\omega^2 + \frac{1}{2}(j-1)\left[(j-2)j - 4(B+2)\right] + 4i(5j-6)\omega\,, \tag{A6}$$

$$\overline{A}_j^{(4)} = 2\omega\left[2(\omega+3i) - j(\omega+4i)\right]\,, \qquad \overline{A}_j^{(5)} = 2(j-2)\omega^2\,. \tag{A7}$$

Appendix B: Nollert's improvements of continued fraction

The procedure to numerically solve Teukolsky equation through Leaver's method requires in practice an initialization for the radial ladder operator Λ_n^r. Such quantity can be expanded for large initialization number N as

$$\Lambda_N^r = \sum_{j=0}^{J} C_j N^{-j/2} + \mathcal{O}\left(N\right)^{-(J+1)/2}\,. \tag{B1}$$

One can initialize the ladder operator just retaining the first term $C_0=-1$. However, this approximation requires in general a very high initial value for N (which means long computational time) and appears to be insufficient for frequencies with large imaginary part (namely higher overtones). In [69], it was shown that adding further corrections to the initial Λ_N^r improves the accuracy of the method, also allowing to capture higher overtones. The $(k+3)$-terms recurrence relation of equation (48) for

$n = N$ can be expressed as

$$\sum_{j=-1}^{k+1} M_{N,j}\, R_{N-j} = 0\,, \tag{B2}$$

where we defined

$$M_{N,j} \equiv \widetilde{\gamma}_{N,j-1} + \widetilde{\beta}_{N,j} + \widetilde{\alpha}_{N,j+1}\,. \tag{B3}$$

Dividing it by R_{N-k-1} and using the definition of ladder operator $\Lambda^r_N = -a_{N+1}/a_N$ one obtains the equation

$$\sum_{j=-1}^{k+1} (-1)^j\, M_{N,j} \prod_{i=j+1}^{k+2} \Lambda^r_{N-i} = 0\,. \tag{B4}$$

Plugging the definition (B1) into the above formula, one can solve for the coefficients order by order. By the definition of $\widetilde{\gamma}_{N,j}$, $\widetilde{\beta}_{N,j}$ and $\widetilde{\alpha}_{N,j}$, they scale with N as

$$\widetilde{\alpha}_{N,j} = N^2 + \widetilde{\alpha}_1 N + \widetilde{\alpha}_0 \tag{B5}$$
$$\widetilde{\beta}_{N,j} = -2N^2 + \widetilde{\beta}_1 N + \widetilde{\beta}_0 \tag{B6}$$
$$\widetilde{\gamma}_{N,j} = N^2 + \widetilde{\gamma}_1 N + \widetilde{\gamma}_0 \tag{B7}$$

If we fix $C_0 = -1$, then the other coefficients up to $J = 5$ can be written as

$$C_1 = \pm\sqrt{-\widetilde{\alpha}_1 - \widetilde{\beta}_1 - \widetilde{\gamma}_1}\,, \tag{B8}$$
$$C_2 = \widetilde{\alpha}_1 + \frac{\widetilde{\beta}_1}{2} - \frac{1}{4}\,, \tag{B9}$$
$$C_3 = \frac{C_2^2}{2C_1} - \frac{C_2}{4C_1} - \frac{\widetilde{\alpha}_0 + \widetilde{\beta}_0 + \widetilde{\gamma}_0}{2C_1} - \frac{1+2\widetilde{\alpha}_1}{4}C_1\,, \tag{B10}$$
$$C_4 = \widetilde{\alpha}_0 - \frac{4\widetilde{\alpha}_1 - 8\widetilde{\beta}_0 + 1}{16} - \frac{1+4\widetilde{\alpha}_1}{4}C_2 - \frac{C_3}{2C_1}\,, \tag{B11}$$
$$\begin{aligned} C_5 = &\frac{2\widetilde{\alpha}_1 + 3}{4C_1}C_2^2 - \frac{8\widetilde{\alpha}_0 + 4\widetilde{\alpha}_1 + 3}{16}C_1 - \frac{C_3^2}{2C_1} \\ &- \frac{4\widetilde{\alpha}_1 + 3}{16C_1}C_2 + C_3\left(\frac{C_2}{2C_1^2} - \widetilde{\alpha}_1 - 1\right) \\ &+ \left(\frac{C_2}{C_1} - \frac{3}{4C_1}\right)C_4\,, \end{aligned} \tag{B12}$$

and the sign of C_1 is chosen such that $\mathrm{Re}\,(C_1) > 0$.

We can do the same expansion for the angular part, by expanding

$$\Lambda^\theta_N = \sum_{j=0}^{J} D_j N^{-j/2} + \mathcal{O}\,(N)^{-(J+1)/2}\,. \tag{B13}$$

From equations (32)–(34) we can schematically say that

$$\alpha^\theta_N = -2N^2 + \alpha_1 N + \alpha_0 \tag{B14}$$
$$\beta^\theta_N = N^2 + \beta_1 N + \beta_0 \tag{B15}$$
$$\gamma^\theta_N = \gamma_1 N + \gamma_0 \tag{B16}$$

By solving perturbatively in $1/N$ the relation (35), we obtain the following expression for the coefficients D_j up to $J = 4$

$$D_0 = 0 \tag{B17}$$
$$D_1 = \gamma_1 \tag{B18}$$
$$D_2 = \gamma_0 - \gamma_1\,(1 + \beta_1 + 2\gamma_1) \tag{B19}$$
$$\begin{aligned} D_3 = &\gamma_1^2(\alpha_1 - 2) - \gamma_1(1 + \beta_0 + \beta_1) \\ &- D_2(\beta_1 + 4\gamma_1 + 2) \end{aligned} \tag{B20}$$
$$\begin{aligned} D_4 = &\alpha_0\gamma_1^2 + \frac{2D_2^2\,(\beta_1 + 3\gamma_1 + 2)}{\gamma_1} \\ &+ D_2\,(\beta_0 + \beta_1 + 2\gamma_1 + 1) \\ &+ D_3\left(\frac{2D_2}{\gamma_1} - \beta_1 - 4\gamma_1 - 2\right) \end{aligned} \tag{B21}$$

Appendix C: Splitting of the potential

In this section of the appendix we show how to transform the potential (8) into the potential (38). We start by splitting equation (8) into

$$\begin{aligned} \delta V(r) &= \frac{1}{\Delta}\sum_{k=-K}^{4} \alpha^{(k)}\left(\frac{r}{r_+}\right)^k \\ &= \frac{1}{\Delta}\sum_{k=0}^{4} \alpha^{(k)}\left(\frac{r}{r_+}\right)^k + \frac{1}{\Delta}\sum_{k=1}^{K} \alpha^{(-k)}\left(\frac{r_+}{r}\right)^k \end{aligned} \tag{C1}$$

For $k \geqslant 1$, the first generic term in k of the sum can be rewritten as

$$\begin{aligned} \frac{\alpha^{(k)}}{\Delta}\left(\frac{r}{r_+}\right)^k &= \frac{\alpha^{(k)}}{\Delta}\left(\frac{r^k - r_+^k}{r_+^k} + 1\right) \\ &= \alpha^{(k)}\left[\frac{1}{\Delta} + \frac{1}{r_+(r - r_-)}\sum_{j=0}^{k-1}\left(\frac{r}{r_+}\right)^j\right] \end{aligned} \tag{C2}$$

For $k \geqslant 2$ we can further simplify this term as

$$\begin{aligned}
\frac{\alpha^{(k)}}{\Delta}\left(\frac{r}{r_+}\right)^k &= \alpha^{(k)}\left[\frac{1}{\Delta}+\frac{1}{r_+(r-r_-)}\sum_{j=0}^{k-1}\frac{r^j-r_-^j+r_-^j}{r_+^j}\right] \\
&= \alpha^{(k)}\left[\frac{1}{\Delta}+\frac{1}{r_+(r-r_-)}\sum_{j=0}^{k-1}\left(\frac{r_-}{r_+}\right)^j+\frac{1}{r_-r_+}\sum_{j=1}^{k-1}\left(\frac{r_-}{r_+}\right)^j\sum_{n=0}^{j-1}\left(\frac{r}{r_-}\right)^n\right] \\
&= \alpha^{(k)}\left[\frac{1}{\Delta}+\frac{1}{r_+(r-r_-)}\sum_{j=0}^{k-1}\left(\frac{r_-}{r_+}\right)^j+\frac{1}{r_+^2}\sum_{j=0}^{k-2}\left(\frac{r}{r_-}\right)^j\sum_{n=j}^{k-2}\left(\frac{r_-}{r_+}\right)^n\right]
\end{aligned} \tag{C3}$$

where we obtained the last line by expanding the series and collecting the terms in r to the same power. Summing over all the non-negative values of k yields

$$\frac{1}{\Delta}\sum_{k=0}^{4}\alpha^{(k)}\left(\frac{r}{r_+}\right)^k=\frac{1}{\Delta}\sum_{k=0}^{4}\alpha^{(k)}+\frac{1}{r_+(r-r_-)}\sum_{k=1}^{4}\alpha^{(k)}\sum_{j=0}^{k-1}\left(\frac{r_-}{r_+}\right)^j+\frac{1}{r_+^2}\sum_{k=0}^{2}\left(\frac{r}{r_+}\right)^k\sum_{j=k}^{2}\alpha^{(j+2)}\sum_{n=0}^{j}\left(\frac{r_-}{r_+}\right)^n \tag{C4}$$

Now, we can perform a mapping between the coefficients $\alpha^{(k)}$ of equation (8) and the coefficients $A^{(0)}$, $A^{(1)}$ and $\tilde{\alpha}^{(k)}$ introduced in equations (38). From a direct comparison we have

$$A^{(0)}=\sum_{k=0}^{4}\alpha^{(k)} \tag{C5}$$

$$A^{(1)}=\sum_{k=1}^{4}\alpha^{(k)}\sum_{j=0}^{k-1}\left(\frac{r_-}{r_+}\right)^j \tag{C6}$$

$$\tilde{\alpha}^{(k)}=\sum_{j=k}^{2}\alpha^{(j+2)}\sum_{n=0}^{j}\left(\frac{r_-}{r_+}\right)^n \tag{C7}$$

Appendix D: Diagonal quadratic coefficients

We show here how to compute the quadratic diagonal coefficients, defined from the next-to-leading-order expansion

$$\begin{aligned}
\omega &\simeq \omega^0+\sum_k d_\omega^{(k)}\alpha^{(k)}+\frac{1}{2}e_\omega^{(k)}{\alpha^{(k)}}^2\,, \\
B &\simeq B^0+\sum_k d_B^{(k)}\alpha^{(k)}+\frac{1}{2}e_B^{(k)}{\alpha^{(k)}}^2\,.
\end{aligned} \tag{D1}$$

By extending the Taylor expansion (11) to the second order in α, we obtain

$$\mathcal{L}_j|_{\rm GR}+\alpha\left.\frac{\mathrm{d}\mathcal{L}_j}{\mathrm{d}\alpha}\right|_{\rm GR}+\frac{\alpha^2}{2}\left.\frac{\mathrm{d}^2\mathcal{L}_j}{\mathrm{d}\alpha^2}\right|_{\rm GR}+\mathcal{O}(\alpha)^3=0\,. \tag{D2}$$

By expanding with the chain rule the total derivative $\mathrm{d}^2\mathcal{L}_j/\mathrm{d}\alpha^2|_{\rm GR}$, one can read off the quadratic coefficients as

$$\begin{aligned}
e_\omega =& \left(\frac{\partial\mathcal{L}_r}{\partial B}\frac{\partial\mathcal{L}_\theta}{\partial\omega}-\frac{\partial\mathcal{L}_r}{\partial\omega}\frac{\partial\mathcal{L}_\theta}{\partial B}\right)^{-1}\left[\frac{\partial^2\mathcal{L}_r}{\partial\alpha^2}\frac{\partial\mathcal{L}_\theta}{\partial B}+2d_\omega\frac{\partial^2\mathcal{L}_r}{\partial\alpha\partial\omega}\frac{\partial\mathcal{L}_\theta}{\partial B}+2d_B\frac{\partial^2\mathcal{L}_r}{\partial\alpha\partial B}\frac{\partial\mathcal{L}_\theta}{\partial B}\right. \\
&\left.-d_\omega^2\left(\frac{\partial^2\mathcal{L}_\theta}{\partial\omega^2}\frac{\partial\mathcal{L}_r}{\partial B}-\frac{\partial^2\mathcal{L}_r}{\partial\omega^2}\frac{\partial\mathcal{L}_\theta}{\partial B}\right)-d_B^2\left(\frac{\partial^2\mathcal{L}_\theta}{\partial B^2}\frac{\partial\mathcal{L}_r}{\partial B}-\frac{\partial^2\mathcal{L}_r}{\partial B^2}\frac{\partial\mathcal{L}_\theta}{\partial B}\right)+2d_\omega\,d_B\left(\frac{\partial^2\mathcal{L}_r}{\partial\omega\partial B}\frac{\partial\mathcal{L}_\theta}{\partial B}-\frac{\partial^2\mathcal{L}_\theta}{\partial\omega\partial B}\frac{\partial\mathcal{L}_r}{\partial B}\right)\right], \\
e_B =& -\left(\frac{\partial\mathcal{L}_r}{\partial B}\frac{\partial\mathcal{L}_\theta}{\partial\omega}-\frac{\partial\mathcal{L}_r}{\partial\omega}\frac{\partial\mathcal{L}_\theta}{\partial B}\right)^{-1}\left[\frac{\partial^2\mathcal{L}_r}{\partial\alpha^2}\frac{\partial\mathcal{L}_\theta}{\partial\omega}+2d_\omega\frac{\partial^2\mathcal{L}_r}{\partial\alpha\partial\omega}\frac{\partial\mathcal{L}_\theta}{\partial\omega}+2d_B\frac{\partial^2\mathcal{L}_r}{\partial\alpha\partial B}\frac{\partial\mathcal{L}_\theta}{\partial\omega}\right. \\
&\left.-d_\omega^2\left(\frac{\partial^2\mathcal{L}_r}{\partial\omega^2}\frac{\partial\mathcal{L}_\theta}{\partial\omega}-\frac{\partial^2\mathcal{L}_\theta}{\partial\omega^2}\frac{\partial\mathcal{L}_r}{\partial\omega}\right)-d_B^2\left(\frac{\partial^2\mathcal{L}_r}{\partial B^2}\frac{\partial\mathcal{L}_\theta}{\partial\omega}-\frac{\partial^2\mathcal{L}_\theta}{\partial B^2}\frac{\partial\mathcal{L}_r}{\partial\omega}\right)+2d_\omega\,d_B\left(\frac{\partial^2\mathcal{L}_r}{\partial\omega\partial B}\frac{\partial\mathcal{L}_\theta}{\partial\omega}-\frac{\partial^2\mathcal{L}_\theta}{\partial\omega\partial B}\frac{\partial\mathcal{L}_r}{\partial\omega}\right)\right],
\end{aligned} \tag{D3}$$

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