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Investigation of the dynamics of an electron in a gravitational plane wave

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Abstract

This thesis examines the dynamics of electrons in a gravitational plane wave. To this end, a plane wave solution to the Dirac equation for an electron in a gravitational plane wave is first presented. A comparison of this solution with the "Volkov state" not only demonstrates that electrons behave almost identically as in a electromagnetic plane wave, but also enables the assignment of a generalized "vector potential" $-H_i^{\;\mu}$ to the gravitational wave, which couples to the initial kinetic momentum p_μ rather than to the mass. Furthermore, a solution to the Dirac equation for an electron vortex beam in a gravitational plane wave is constructed, that is, a solution with orbital angular momentum (OAM). In contrast to the plane wave solutions, this solution differs from that of a vortex beam in an electromagnetic plane wave, which is primarily due to the coupling of the generalized "vector potential" $-H_i^{\;\mu}$ to the initial kinetic momentum p_μ. Finally, a spinor is presented that represents a solution to the classical equivalent of the Dirac equation for a particle in a gravitational plane wave. The comparison with the plane wave solution to the Dirac equation reveals that the dynamics can also be accurately described by the classical spinor.

Zusammenfassung

In dieser Thesis wird die Dynamik von Elektronen in einer ebenen Gravitationswelle untersucht. Dazu wird zunächst eine ebene Wellen-Lösung der Dirac Gleichung für ein Elektron in einer ebenen Gravitationswelle präsentiert. Ein Vergleich dieser Lösung mit dem „Volkov Zustand“ zeigt dabei nicht nur, dass sich Elektronen in einer ebenen elektromagnetischen und einer ebenen Gravitationswelle nahezu identisch verhalten, sondern macht es auch möglich der Gravitationswelle ein generalisiertes „Vektorpotential“ $-H_i^{\ \mu}$ zuzuordnen, welches an den initialen kinetischen Impuls p_μ koppelt und nicht an die Masse. Darüber hinaus wird eine Lösung der Dirac Gleichung für einen Elektron Vortex Beam in einer ebenen Gravitationswelle konstruiert, d.h. eine Lösung mit Bahndrehimpuls (OAM). Im Gegensatz zu den ebene Wellen-Lösungen, unterscheidet sich diese Lösung von der für einen Vortex Beam in einer ebenen elektromagnetischen Welle, was maßgeblich darauf zurückzuführen ist, dass das generalisierte „Vektorpotential“ $-H_i^{\ \mu}$ an den initialen kinetischen Impuls p_μ koppelt. Abschließend wird ein Spinor präsentiert, welcher eine Lösung des klassischen Äquivalents der Dirac Gleichung für ein Teilchen in einer ebenen Gravitationswelle darstellt. Der Vergleich mit der ebene Wellen-Lösung der Dirac Gleichung zeigt dabei, dass die Dynamik auch vollständig und korrekt durch den klassischen Spinor beschrieben werden kann.

Contents

1 Introduction

Physical processes are described by the so-called equations of motion, normally comprising of differential equations, whose sought-after solutions are usually hard to find. Even after identifying the symmetries of the problem at hand (for instance spherical, cylindrical, etc.), there is still a considerable freedom on the exact form of the solution. Take, for instance, the Dirac equation [1], which is required for the relativistic description of spin-$\frac{1}{2}$ particles interacting with external electromagnetic fields; its solutions are given by spinors, complex four component column vectors, requiring the determination of four complex valued functions. Apart from some physical constraints, i.e. normalisation, continuity and so on, and after identifying the symmetries of the interaction (spherical, cylindrical, etc.) no other constraints are imposed on the form that each component can take. Recently a new technique for constructing solutions to the Dirac equation, the so-called "Relativistic Dynamical Inversion" (RDI), was proposed in Ref. [2], providing a different way of finding solutions to the Dirac equation in the presence of electromagnetic fields. RDI relies on the fact that relativistic interactions and the corresponding equations of motion are inextricably intertwined with geometry, i.e. group theory and the Poincaré group. This in turn fixes the form a spinor can take, thus imposing further constraints and making it easier to find solutions.
An important exact solution to the Dirac equation is the so-called "Volkov state" [3], which describes the dynamics of a free electron in an electromagnetic plane wave. In Ref. [4] RDI was used to construct a more general solution for an electron beam with orbital angular momentum along its propagation direction, a so-called vortex beam [5], in the presence of an electromagnetic plane wave. Since the Dirac equations for an electron interacting with both an electromagnetic and a gravitational plane wave are mathematically very similar, in this work RDI is applied to the latter. Given that gravity can be understood as a gauge theory [6, 7, 8], the result presented here supports the idea underlying RDI that one can construct solutions to dynamical equations by using their group symmetries. In Ref. [9] the validity of the hypothesis that all interactions of elementary particles can be described by group structures of spinor fields in spacetime was demonstrated for weak interactions. Moreover, a scheme for how such concept could be extended to the strong interaction was given. With the results of this thesis for gravitational waves, the next step would then be the application of RDI to the weak and strong interactions.
Gravitational waves were first postulated by Oliver Heaviside in 1893 as the equivalent of electromagnetic waves [10, 11]. In 1916 Albert Einstein showed

that gravitational waves come as a result of his general theory of relativity as ripples in spacetime [12, 13]. It was not until 43 years later, in 1959, that the first exact solutions to Einstein's field equations for gravitational waves were found [14]. The first experimental detection of gravitational waves was achieved another 56 years later in 2015 by the LIGO collaboration [15]. The observed gravitational wave was identified as the merger of two black holes and the measured waveform was in agreement with the predictions of the general theory of relativity.
In this thesis a solution to the Dirac equation for an electron vortex beam interacting with a gravitational plane wave is constructed. To achieve this, the known plane wave solution to the Dirac equation for an electron in a gravitational plane wave [16, 17] is first reformulated within the RDI formalism. Thus both the group symmetries and geometrical features of the solution are highlighted. Then using a procedure similar to that of Ref. [18] a wave packet with orbital angular momentum, hence an electron vortex beam, is build through superposition in momentum space of wave functions with orbital angular momentum along their propagation direction. Finally a solution of a classical equivalent of the Dirac equation [19] in a gravitational wave is presented. This generalises the result of Ref. [20], where it was shown that the classical spinor for an electromagnetic interaction, apart from an additional phase, corresponds to the Dirac spinor. The idea that RDI can also be applied to classical systems is supported by this.

This thesis is structured as follows. In section 2 the Dirac equation in Minkowski spacetime is introduced and some known solutions to it are presented. A short introduction to Spacetime Algebra then allows the Dirac equation and its solution to be converted into a Spacetime Algebra representation. How to formulate the Dirac equation in curved spacetime and how this is also useful to write the Dirac equation using curvilinear coordinates is presented in section 3. Section 4 is then devoted to the main topic of this thesis and therefore deals with solutions of the Dirac equation in a gravitational plane wave. This includes both a plane wave solution and a solution for an electron vortex beam, i.e. a solution with orbital angular momentum (OAM). The aforementioned classical equivalent of the Dirac equation and its solution for an electron in a gravitational plane wave is presented in section 5. In particular, similarities and differences between the treatment of the dynamics of an electron in a gravitational wave using classical and quantum mechanics are discussed. A summary of the results and a conclusion can be found in section 6.

2 The Dirac equation in Minkowski spacetime

In the subsequent section, the Dirac equation in Minkowski spacetime is introduced and some of its known solutions are presented, which will be used for comparison later on. In addition, the necessary mathematical foundations and notations are introduced, including in particular Spactime Algebra. The Dirac equation is also reformulated using Spacetime Algebra and its solutions are converted accordingly.

2.1 Motivating the Dirac equation

The following derivation of the Dirac equation is based on Ref. [21].
The Dirac equation represents the relativistic equivalent to the Schrödinger equation and was derived in 1928 by Paul Dirac in Ref. [1]. It describes massive spin-$\frac{1}{2}$ particles such as electrons, on which this thesis will focus.
To derive the Dirac equation one needs the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{2.1}$$

and the relativistic energy-momentum relation

$$E^2 = m^2c^4 + \boldsymbol{p}^2c^2 \tag{2.2}$$

to begin with. Here and below, $\boldsymbol{x} = (x^1, x^2, x^3)^T$ denotes the spatial components of a four-vector. By looking at these two equations one is tempted to use

$$\hat{H}_f = \sqrt{m^2c^4 + \hat{\boldsymbol{p}}^2c^2} \tag{2.3}$$

as the Hamiltonian for a free particle in the Schrödinger equation (2.1), where the hat notation is used to denote operators. But since taking the square root of a differential operator is too complicated, it would be favourable if the radicand could be written as a perfect square. Upon introducing the parameters α_i with $i = \{1, 2, 3\}$ and β one can write

$$m^2c^4 + \hat{\boldsymbol{p}}^2c^2 = \left(c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2\right)^2 \tag{2.4}$$

with the still unknown parameters α_i and β fulfilling the following relations:

$$\begin{aligned}
&\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}\mathbb{1}\,,\\
&\alpha_i\beta + \beta\alpha_i = 0\,,\\
&\alpha_i^2 = \beta^2 = \mathbb{1}\,.
\end{aligned} \tag{2.5}$$

As the relations (2.5) suggest the parameters α_i and β are not just real or complex numbers, but matrices since they do not commute.
Additionally, to ensure the hermiticity of the Hamiltonian

$$\hat{H}_f = \sqrt{m^2c^4 + \hat{\boldsymbol{p}}^2c^2} = \sqrt{\left(c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2\right)^2} = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2\ , \tag{2.6}$$

one has to impose hermiticity for the matrices α_i and β as well:

$$\begin{aligned} \alpha_i^\dagger &= \alpha_i\ , \\ \beta^\dagger &= \beta\ . \end{aligned} \tag{2.7}$$

The smallest Dimension in which all above listed requirements on the α_i and β can be met is $N = 4$ [21]. An explicit representation of these matrices is:

$$\begin{aligned} \alpha_i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\ , \\ \beta &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\ . \end{aligned} \tag{2.8}$$

Here σ_i are the 2×2 Pauli matrices and $\mathbb{1}$ is the 2×2 identity matrix.
Putting it all together one obtains the Dirac equation:

$$i\hbar\frac{\partial}{\partial t}\psi = \left(c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2\right)\psi = \left(-i\hbar c\sum_{i=1}^{3}\alpha_i\frac{\partial}{\partial x^i} + \beta mc^2\right)\psi\ . \tag{2.9}$$

Here the momentum operator $\hat{p}_i = -i\hbar\frac{\partial}{\partial x^i}$ was substituted in explicitly. Since the Dirac equation contains 4×4 matrices, it is obvious that the wave function ψ has to be a column vector of dimension $N = 4$, the so-called Dirac spinor [21]. The spinor therefore can be represented as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} r_0 - ir_3 \\ r_2 - ir_1 \\ s_3 + is_0 \\ s_1 + is_2 \end{pmatrix}\ . \tag{2.10}$$

Here the r_μ and s_μ are real valued functions of space and time [4].
Multiplying both sides of equation (2.9) with β from the left and rearranging some terms results in the today common representation of the Dirac equation for a free electron:

$$\left(\gamma^\mu\hat{p}_\mu - mc\right)\psi = \left(\gamma^\mu i\hbar\partial_\mu - mc\right)\psi = 0\ . \tag{2.11}$$

Here $p^0 = \frac{E}{c}$ as well as the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$ was used and $\mu = \{0, 1, 2, 3\}$. $\hat{p}_\mu = \left(\frac{\hat{E}}{c}, -\hat{\boldsymbol{p}}\right) = i\hbar\left(\frac{1}{c}\partial_t, \nabla\right) = i\hbar\partial_\mu$ denotes the four-momentum operator. Note that $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$. Additionally summation over repeated indices is used here and throughout the following thesis. In the Dirac representation the so-called gamma matrices are:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \\ \gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} . \end{aligned} \tag{2.12}$$

They obey the following relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4 , \tag{2.13}$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric. This relation is also called the Dirac algebra. γ^μ is treated like a normal Minkowski spacetime vector, i.e. the index is raised and lowered by means of the metric $\eta^{\mu\nu}$.
In addition, there is a fifth gamma matrix

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{2.14}$$

which can be used to write:

$$\begin{aligned} \alpha_1\alpha_2\alpha_3 &= \gamma_0\gamma_1\gamma_2\gamma_3 = \boldsymbol{i} = i\gamma_5 , \\ \boldsymbol{i}^2 &= -\mathbb{1}_4 . \end{aligned} \tag{2.15}$$

Here, the $\alpha_i = \gamma_i\gamma_0$ correspond to those in (2.8). Furthermore, the matrices $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ will be used later on.

2.1.1 Free particle solution of the Dirac equation

The next paragraph on the derivation of the free particle solution is based on Ref. [22].
The Dirac equation for a free particle can be solved by making the following plane wave ansatz:

$$\psi = N\, u(E, \boldsymbol{p}) \exp\left(\frac{-ix^\mu p_\mu}{\hbar}\right) . \tag{2.16}$$

Here N is a normalisation constant, which will not be discussed further at this point and $u(E, \boldsymbol{p})$ is a four-component bispinor. Inserting this into the Dirac equation (2.11), the Dirac equation simplifies to:

$$(\gamma^\mu p_\mu - mc)u(E, \boldsymbol{p}) = 0 \tag{2.17}$$

This is a purely algebraic equation, which can be solved by expanding the expression $(\gamma^\mu p_\mu - mc)$ using (2.12):

$$\begin{aligned}
\gamma^\mu p_\mu - mc &= \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 - mc \\
&= \frac{E}{c}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} - mc\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{E}{c} - mc\right)\mathbb{1} & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(\frac{E}{c} + mc\right)\mathbb{1} \end{pmatrix} .
\end{aligned} \tag{2.18}$$

If one then writes the four-component bispinor $u(E,\boldsymbol{p})$ as a two-component vector

$$u(E,\boldsymbol{p}) = \begin{pmatrix} u_A \\ u_B \end{pmatrix} , \tag{2.19}$$

one obtains the following coupled equations:

$$\begin{aligned}
\left(\frac{E}{c} - mc\right) u_A - \boldsymbol{\sigma}\cdot\boldsymbol{p}\, u_B &= 0 , \\
\boldsymbol{\sigma}\cdot\boldsymbol{p}\, u_A - \left(\frac{E}{c} + mc\right) u_B &= 0 .
\end{aligned} \tag{2.20}$$

These can be rearranged to:

$$\begin{aligned}
u_A &= \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\frac{E}{c} - mc} u_B , \\
u_B &= \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\frac{E}{c} + mc} u_A .
\end{aligned} \tag{2.21}$$

If one now expands the expression $\boldsymbol{\sigma}\cdot\boldsymbol{p}$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \begin{pmatrix} p^z & p^x - ip^y \\ p^x + ip^y & -p^z \end{pmatrix} , \tag{2.22}$$

the bispinor $u(E,\boldsymbol{p})$ can be constructed by making explicit choices for u_A or u_B. Choosing

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.23}$$

gives

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p^z c}{E+mc^2} \\ \frac{(p^x+ip^y)c}{E+mc^2} \end{pmatrix} \quad \text{or} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{(p^x-ip^y)c}{E+mc^2} \\ \frac{-p^z c}{E+mc^2} \end{pmatrix} . \tag{2.24}$$

These two solutions correspond to solutions with positive energy $E = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$, as can easily be checked by solving the Dirac equation again for $\boldsymbol{p} = 0$. Similarly, the two solutions

$$u_3 = \begin{pmatrix} \frac{p^z c}{E - mc^2} \\ \frac{(p^x + ip^y)c}{E - mc^2} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4 = \begin{pmatrix} \frac{(p^x - ip^y)c}{E - mc^2} \\ \frac{-p^z c}{E - mc^2} \\ 0 \\ 1 \end{pmatrix} \tag{2.25}$$

are obtained for

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{2.26}$$

These solutions correspond to negative energy $E = -\sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$, as can also be easily checked here by solving the Dirac equation again for $\boldsymbol{p} = 0$. Altogether using the "Feynman-Stückelberg interpretation" for the negative energy solutions, which means a change in the sign of E and $\boldsymbol{p}$, this results in the following four solutions:

$$\begin{aligned}
\psi_1 &= N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p^z c}{E + mc^2} \\ \frac{(p^x + ip^y)c}{E + mc^2} \end{pmatrix} \exp\left(\frac{-ix^\mu p_\mu}{\hbar}\right) \ , \\
\psi_2 &= N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{(p^x - ip^y)c}{E + mc^2} \\ \frac{-p^z c}{E + mc^2} \end{pmatrix} \exp\left(\frac{-ix^\mu p_\mu}{\hbar}\right) \ , \\
\psi_3 &= N_3 \begin{pmatrix} \frac{-p^z c}{-E - mc^2} \\ \frac{-(p^x + ip^y)c}{-E - mc^2} \\ 1 \\ 0 \end{pmatrix} \exp\left(\frac{ix^\mu p_\mu}{\hbar}\right) \ , \\
\psi_4 &= N_4 \begin{pmatrix} \frac{-(p^x - ip^y)c}{-E - mc^2} \\ \frac{p^z c}{-E - mc^2} \\ 0 \\ 1 \end{pmatrix} \exp\left(\frac{ix^\mu p_\mu}{\hbar}\right) \ .
\end{aligned} \tag{2.27}$$

Here, ψ_1 and ψ_2 represent the spin up and spin down solutions for positive energies, while ψ_3 and ψ_4 are the spin up and spin down solutions for negative energies, i.e. for antiparticles.

2.2 The Dirac equation in an external electromagnetic field A_μ

If one substitutes the kinetic momentum p_μ in (2.11) by the canonically conjugated momentum $\pi_\mu = p_\mu - eA_\mu$ [23] (this is also called "minimal coupling"), one obtains the Dirac equation for an electron moving in an electromagnetic potential A^μ:

$$(\gamma^\mu i\hbar\partial_\mu - e\gamma^\mu A_\mu - mc)\,\psi = 0\,. \tag{2.28}$$

Here e denotes the elementary charge and A^μ the four-vector potential.
For the motion of an electron in an electromagnetic plane wave, which is moving in the direction of the wave vector $\boldsymbol{k}$ with the speed of light c, the solution of the Dirac equation was found by Volkov [3]. Using the notation $a_\mu b^\mu = a \cdot b$, as well as the Feynman slash notation $\gamma^\mu a_\mu = \not{a}$, the so-called "Volkov state" is [24]:

$$\psi_p = N_p \left(1 + \frac{e}{2k \cdot p}\not{k}\not{A}\right) u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right)\,. \tag{2.29}$$

It has the phase:

$$\Phi = \int_0^{k\cdot x} \left(\frac{eA \cdot p}{k \cdot p} - \frac{e^2 A^2}{2k \cdot p}\right) \mathrm{d}\phi\,. \tag{2.30}$$

Here u_p denotes one of the previously known spinors of a free electron (cf. (2.24), (2.25)) and N_p is a normalisation constant. Use has been made of the fact that the electromagnetic potential A^μ only depends on the scalar product $\phi = k \cdot x$, i.e. $A^\mu = A^\mu(k \cdot x)$, since a plane wave is considered. Due to the fact that the electromagnetic wave moves at the speed of light, $k^2 = 0$ also applies. In addition, the Lorenz gauge $\partial_\mu A^\mu = 0$ was assumed, which can also be written as $k \cdot \partial_\phi A(\phi) = k \cdot A'(\phi) = 0$ with the previous assumption $A^\mu = A^\mu(k \cdot x)$.
A detailed derivation of this Volkov state can be found, for example, in Ref. [24]. Here, however, it will be instead checked that the Volkov state satisfies the Dirac equation by inserting it back into the equation. To do so the Dirac equation is first rearranged to:

$$(i\hbar\gamma^\mu\partial_\mu - e\not{A} - mc)\psi_p = 0\,. \tag{2.31}$$

The next step is to evaluate the partial derivatives:

$$\begin{aligned}\gamma^\mu i\hbar\partial_\mu\psi_p &= \gamma^\mu(p_\mu + \partial_\mu\Phi)\psi_p \\ &= \left(\not{p} + \not{k}\left(\frac{eA \cdot p}{k \cdot p} - \frac{e^2 A^2}{2k \cdot p}\right)\right)\psi_p\,.\end{aligned} \tag{2.32}$$

Here

$$\gamma^\mu \partial_\mu \not{k} \not{A} = \not{k} \not{k} \not{A}' \tag{2.33}$$

with $(A^\mu)' = \partial_\phi A^\mu$ and

$$\not{k}\not{k} = \frac{1}{2} k_\mu k_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = k_\mu k_\nu \eta^{\mu\nu} \mathbb{1} = k^2 \mathbb{1} = 0 \tag{2.34}$$

was used. Since $\not{p} u_p = mc u_p$ is known from the solution of the Dirac equation for a free particle , the term $\not{p}\psi_p$ is examined in more detail in the next step:

$$\begin{aligned} \not{p}\psi_p &= \not{p} N_p \left(1 + \frac{e}{2k \cdot p} \not{k}\not{A}\right) u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) , \\ &= N_p \left(\not{p} + \frac{e}{2k \cdot p} \not{p}\not{k}\not{A}\right) u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) . \end{aligned} \tag{2.35}$$

Using the identity

$$\gamma^\mu \gamma^\nu \gamma^\rho = \eta^{\mu\nu} \gamma^\rho + \eta^{\nu\rho} \gamma^\mu - \eta^{\mu\rho} \gamma^\nu - i \epsilon^{\sigma\mu\nu\rho} \gamma_\sigma \gamma^5 \tag{2.36}$$

twice, one gets

$$\not{p}\not{k}\not{A} = 2k \cdot p \not{A} - 2p \cdot A \not{k} + \not{k}\not{A}\not{p} \, . \tag{2.37}$$

This results in:

$$\begin{aligned} \not{p}\psi_p &= \left(\left(1 + \frac{e}{2k \cdot p} \not{k}\not{A}\right) \not{p} + e\not{A} - \frac{ep \cdot A}{k \cdot p} \not{k}\right) N_p u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) \\ &= mc\psi_p + \left(e\not{A} - \frac{eA \cdot p}{k \cdot p} \not{k}\right) N_p u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) . \end{aligned} \tag{2.38}$$

Furthermore, with $\not{k}^2 = 0$:

$$\begin{aligned} &\not{k} \left(\frac{eA \cdot p}{k \cdot p} - \frac{e^2 A^2}{2k \cdot p}\right) \psi_p \\ &= \left(\frac{eA \cdot p}{k \cdot p} - \frac{e^2 A^2}{2k \cdot p}\right) N_p \left(\not{k} + \frac{e}{2k \cdot p} \not{k}^2 \not{A}\right) u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) \\ &= \left(\frac{eA \cdot p}{k \cdot p} - \frac{e^2 A^2}{2k \cdot p}\right) \not{k} N_p u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) . \end{aligned} \tag{2.39}$$

Additionally, it follows from $\not{A}\not{k} = -\not{k}\not{A}$ and $\not{A}^2 = A^2 \mathbb{1}$:

$$\begin{aligned} -e\not{A}\psi_p &= -eN_p \left(\not{A} + \frac{e}{2k \cdot p} \not{A}\not{k}\not{A}\right) u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) \\ &= \left(-e\not{A} + \frac{e^2 A^2}{2k \cdot p} \not{k}\right) N_p u_p \exp\left(\frac{-i\Phi - ix \cdot p}{\hbar}\right) . \end{aligned} \tag{2.40}$$

A final comparison of equations (2.31), (2.32), (2.38), (2.39) and (2.40) shows that the Volkov state satisfies the Dirac equation (2.31) as expected.
At this point, one might ask why the solution of the Dirac equation for an electron interacting with an electromagnetic plane wave has been treated in such detail when this thesis is actually about electron dynamics in a gravitational plane wave. The reason is quite simply that the electromagnetic case and the gravitational case have some striking similarities, which are explained in what follows.

2.3 Fundamentals of STA

The following introduction to Spacetime Algebra (STA) is based on Refs. [19, 25] and will form the framework for some subsequent calculations and results. Here Latin letters are used to denote vectors, while Greek letters are used for scalars.
The easiest way to understand the basics of STA is to reinterpret the Dirac algebra together with the Dirac matrices. To do this, the Dirac matrices are assumed to be a orthonormal basis of Minkowski space with the signature $(+,-,-,-)$ [19]. Any vector a can then be written as $a = a^\mu \gamma_\mu$.
The inner product

$$a \cdot b = \frac{1}{2}(ab + ba) \tag{2.41}$$

and the outer product

$$a \wedge b = \frac{1}{2}(ab - ba) = -b \wedge a\,, \tag{2.42}$$

which defines so-called bivetors, can then be defined for two vectors a and b. Together, these result in the so-called "geometric product"

$$ab = a \cdot b + a \wedge b \quad . \tag{2.43}$$

A frequently required relation for vectors a, b and c is:

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b\,. \tag{2.44}$$

With this knowledge, the Dirac algebra can be expressed as follows:

$$\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu}\,. \tag{2.45}$$

Thereby $\eta_{\mu\nu}$ is the usual Minkowski metric. It also follows from this relation that the basis $\{\gamma_\mu\}$ is orthonormal. In addition, the unit pseudoscalar

$$\boldsymbol{i} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \tag{2.46}$$

must also be introduced. This can then be used to define the trivectors $\{\boldsymbol{i}\gamma_\mu\}$. A general element M of the STA, which is called a multivector, can then be written as follows:

$$M = \alpha + a + F + b\boldsymbol{i} + \beta\boldsymbol{i}\,. \tag{2.47}$$

Here, α and β are scalars, a and b are vectors, F is a bivector, $b\boldsymbol{i}$ is a trivector and $\beta\boldsymbol{i}$ is a pseudoscalar. Multivectors of higher degree are not needed. The reverse of a multivector is then defined als follows:

$$\tilde{M} = \alpha + a - F - b\boldsymbol{i} + \beta\boldsymbol{i}\,. \tag{2.48}$$

A simple multivector A_r of degree r can be constructed from a number of r pairwise distinct vectors:

$$A_r = a_1 \wedge a_2 \wedge \ \dots\ \wedge a_r\,. \tag{2.49}$$

The inner and outer product can now be generalised to any multivectors A_r and B_s with the degrees r and s:

$$A_r \cdot B_s = (-1)^{r(s+1)} B_s \cdot A_r\,, \tag{2.50}$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r\,. \tag{2.51}$$

As this will become relevant later on, the inner and outer product are now written out for a vector a and a multivector A_r of degree r:

$$a \cdot A_r = \frac{1}{2}\left(aA_r - (-1)^r A_r a\right)\,, \tag{2.52}$$

$$a \wedge A_r = \frac{1}{2}\left(aA_r + (-1)^r A_r a\right)\,. \tag{2.53}$$

The geometric product can then be written again as:

$$aA_r = a \cdot A_r + a \wedge A_r\,. \tag{2.54}$$

For any two multivectors A and B, there is also the commutator product, which should not be confused with the inner or outer product. It differs from these two other products in that its definition is independent of the degree of the multivectors under consideration:

$$A \times B = \frac{1}{2}(AB - BA)\,. \tag{2.55}$$

The so-called rotors will also become relevant. These are even multivectors and are defined for any Lorentz rotation (proper, orthochronous Lorentz transformation) $\underline{R}$ acting on the vector a as follows:

$$\underline{R}(a) = Ra\tilde{R}\,. \tag{2.56}$$

Rotors fulfil the normalisation condition

$$R\tilde{R} = 1 \tag{2.57}$$

and, as spinors, represent the spinor equivalent of the underlying Lorentz transformation. Therefore, the eigenrotor or eigenspinor of a particle corresponds to the Lorentz transformation, which connects the rest frame of the particle with the laboratory frame, thus it is describing the motion of the particle. Hence the proper velocity of a particle $v = v(\tau) = v^\mu(\tau)\gamma_\mu$ can be determined with the help of the rotor:

$$v = R\gamma_0\tilde{R} \,. \tag{2.58}$$

2.4 Formulation of the Dirac equation using STA

The below section is based on Ref. [4].
As already seen in section 2.3, the gamma matrices form an associative algebra over the real numbers according to (2.45), which is called the Spacetime Algebra (STA) by Hestenes [26, 27, 28]. The advantage of formulating the Dirac equation using STA is that the Dirac spinor ψ is independent of the representation of the gamma matrices. Thereby the α_i are understood as vectors relative to the intertial system defined by the time-like vector γ_0 [4]. The Dirac spinor, which was previously a four-dimensional column vector, is then expanded to a 4×4 matrix spinor:

$$\begin{aligned}\Psi &= r_0 + s_1\alpha_1 + s_2\alpha_2 + s_3\alpha_3 + \boldsymbol{i}(s_0 - r_1\alpha_1 - r_2\alpha_2 - r_3\alpha_3)\\ &= \begin{pmatrix}\psi_1 & -\psi_2^* & \psi_3 & \psi_4^*\\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^*\\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^*\\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^*\end{pmatrix}\,.\end{aligned} \tag{2.59}$$

As can be clearly seen, the first column of the matrix spinor Ψ corresponds to the original vector spinor ψ. It can therefore be recovered from the matrix spinor Ψ:

$$\psi = \Psi \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\,. \tag{2.60}$$

For the Hestenes-Dirac spinor Ψ, the Dirac equation can be reformulated as the Hestenes-Dirac equation [28]:

$$\hbar\gamma^\mu\partial_\mu\Psi\gamma_2\gamma_1 - e\not{A}\Psi - mc\Psi\gamma_0 = 0\,. \tag{2.61}$$

There are two things to note here. First, the Hestenes-Dirac equation is completely equivalent to the Dirac equation, i.e. solutions of the two equations can simply be reformulated into one another using (2.59) and (2.60). Second, no matrix representation needs to be assigned to the matrices γ_0, γ_1 and γ_2 *a priori*, despite the fact that these matrices appear explicitly in the equation. This means that these matrices are initially arbitrary orthonormal vectors that only receive an explicit representation by selecting a coordinate system [4]. The advantage of the Hestenes-Dirac equation is that the matrix spinor Ψ is invertible and therefore the Hestenes-Dirac equation can be solved for the vector potential $\not{A}$ for a given spinor Ψ, which is the main idea of the RDI technique. A detailed derivation of the Hestenes-Dirac equation can be found in Ref. [28].
Following Ref. [4] the matrix spinor Ψ can be brought into polar from:

$$\Psi = \sqrt{\rho}\exp(\boldsymbol{i}\beta/2)\mathcal{R}\,. \tag{2.62}$$

In this context, ρ is a non-negative scalar function that represents the probability density, β is also a scalar function, the so-called Yvon-Takabayashi angle [29, 30], and $\mathcal{R}$ is a matrix that is an element of the Lorentz group. The Lorentz transformation $\mathcal{R}$ connects the rest frame of the electron with the laboratory frame. The following applies: $\mathcal{R} = \mathcal{B}U$, where $\mathcal{B}$ is a Hermitian matrix that represents the boosts, while U is unitary and describes the rotations. The polar form thus allows for a straightforward geometric interpretation of the Hestenes-Dirac spinor Ψ [4]. Additionally, the proper velocity v^μ of the electron can be determined from the matrix spinor:

$$\begin{aligned} \Psi\gamma_0\tilde{\Psi} &= \rho\not{v}\,, \\ \text{with :}\quad \tilde{\Psi} &= \gamma_0\Psi^\dagger\gamma_0\,. \end{aligned} \tag{2.63}$$

2.4.1 Matrix spinor free particle solution

Using the general boost to a frame moving with constant velocity $\boldsymbol{v}$ in an arbitrary direction as derived in Ref. [31]

$$\begin{aligned} \mathcal{B}_g = \sqrt{\frac{E+mc^2}{2mc^2}} & \begin{pmatrix} 1 & 0 & \frac{p^z c}{E+mc^2} & \frac{p^- c}{E+mc^2} \\ 0 & 1 & \frac{p^+ c}{E+mc^2} & \frac{-p^z c}{E+mc^2} \\ \frac{p^z c}{E+mc^2} & \frac{p^- c}{E+mc^2} & 1 & 0 \\ \frac{p^+ c}{E+mc^2} & \frac{-p^z c}{E+mc^2} & 0 & 1 \end{pmatrix} \\ \text{with :}\, p^\pm = p^x \pm ip^y\,, & \end{aligned} \tag{2.64}$$

the matrix spinor solution of the Hestenes-Dirac equation (2.61) for a free electron ($A^\mu = 0,\ \forall\mu \in \{0, 1, 2, 3\}$) neglecting the normalisation can be written as:

$$\Psi_f = \mathcal{B}_g \exp\left(\frac{-\gamma_2\gamma_1 p\cdot x}{\hbar}\right) . \tag{2.65}$$

Where $\rho = 1$, $\beta = 0$ and $U = \mathbb{1}$ apply. If one compares this matrix spinor with the equations (2.24) and (2.25), it is apparent that the matrix spinor contains all four solutions, i.e. spin up and spin down for positive and negative energy respectively, and that each column of the matrix spinor corresponds to one of these solutions. Apart from this, the structure of the matrix spinor is identical to that of the vector spinors. It also becomes clear that the solution (2.65) is constructed using a boost of the at rest solution ($\boldsymbol{p} = 0$)

$$\Psi_{rest} = \exp\left(\frac{-\gamma_2\gamma_1 Et}{\hbar}\right) . \tag{2.66}$$

2.4.2 Matrix spinor Volkov state

In addition, the Volkov state (2.29) for an electromagnetic plane wave travelling in z-direction ($k^\mu = k(1,0,0,1)^T$ and $A^\mu = (0, A^1, A^2, 0)^T$) can also be expressed as a matrix spinor:

$$\Psi = \left(1 + \frac{e}{2k\cdot p}\not{k}\not{A}\right)\mathcal{B}_g \exp\left(-\gamma^2\gamma^1\frac{x\cdot p + \Phi}{\hbar}\right) . \tag{2.67}$$

If an additional Lorentz transformation $\mathcal{R}_r$ is now recognised in $\left(1 + \frac{e}{2k\cdot p}\not{k}\not{A}\right)$, the following identifications can be made $\rho = 1$, $\beta = 0$ and $\mathcal{R} = \mathcal{R}_r\mathcal{B}_g \exp\left(-\gamma^2\gamma^1\frac{x\cdot p+\Phi}{\hbar}\right)$.
In the next step, $\mathcal{R}_r$ should be divided into a rotation and a boost $\mathcal{R}_r = U_r\mathcal{B}_r$. To do this, the comparison of $\frac{e}{2k\cdot p}\not{k}\not{A}$ with equation (79) from Ref. [4] first shows that $\not{k}\wedge\not{A}$ and $\frac{e}{2k\cdot p}\not{k}\not{A}$ have the same structure in the present case and the following applies analogous to equations (80) to (89) in Ref. [4]:

$$\begin{aligned} U_r &= \exp\left(-\frac{\theta}{2}(\cos(\vartheta)\gamma^1\gamma^3 + \sin(\vartheta)\gamma^2\gamma^3)\right) , \\ \mathcal{B}_r &= \exp\left(\frac{\omega}{2}(V_1\alpha^1 + V_2\alpha^2 + V_3\alpha^3)\right) . \end{aligned} \tag{2.68}$$

The following abbreviation were used here:

$$
\begin{aligned}
\frac{\theta}{2} &= \arctan\left(k\frac{e}{2k\cdot p}\sqrt{(A^1)^2+(A^2)^2}\right) \ , \\
\cos(\vartheta) &= \frac{A^1}{\sqrt{(A^1)^2+(A^2)^2}} \ , \\
\sin(\vartheta) &= \frac{A^2}{\sqrt{(A^1)^2+(A^2)^2}} \ , \\
\frac{V_1}{2} &= \cos\left(\frac{\theta}{2}\right)\cos(\vartheta) \ , \\
\frac{V_2}{2} &= \cos\left(\frac{\theta}{2}\right)\sin(\vartheta) \ , \\
\frac{V_3}{2} &= \sin\left(\frac{\theta}{2}\right) \ , \\
\omega &= \operatorname{arctanh}\left(\frac{V_3}{2}\right) \ .
\end{aligned} \tag{2.69}
$$

Finally, it can now be checked whether U_r is unitary and $\mathcal{B}_r$ is hermitian. To do this, one must first remember that for a matrix A applies $[\exp(A)]^\dagger = \exp(A^\dagger)$. This applied to U_r results in:

$$
\begin{aligned}
U_r^\dagger &= \left[\exp\left(-\frac{\theta}{2}(\cos(\vartheta)\gamma^1\gamma^3+\sin(\vartheta)\gamma^2\gamma^3)\right)\right]^\dagger \\
&= \exp\left(-\frac{\theta}{2}(\cos(\vartheta)(\gamma^1\gamma^3)^\dagger+\sin(\vartheta)(\gamma^2\gamma^3)^\dagger)\right) \\
&= \exp\left(-\frac{\theta}{2}(\cos(\vartheta)(\gamma^3)^\dagger(\gamma^1)^\dagger+\sin(\vartheta)(\gamma^3)^\dagger(\gamma^2)^\dagger)\right) \\
&= \exp\left(-\frac{\theta}{2}(\cos(\vartheta)\gamma^3\gamma^1+\sin(\vartheta)\gamma^3\gamma^2)\right) \\
&= \exp\left(\frac{\theta}{2}(\cos(\vartheta)\gamma^1\gamma^3+\sin(\vartheta)\gamma^2\gamma^3)\right) \\
&= U_r^{-1} \ .
\end{aligned} \tag{2.70}
$$

U_r is therefore unitary as expected. For $\mathcal{B}_r$ one obtains:

$$
\begin{aligned}
\mathcal{B}_r^\dagger &= \left[\exp\left(\frac{\omega}{2}(V_1\alpha^1 + V_2\alpha^2 + V_3\alpha^3)\right)\right]^\dagger \\
&= \exp\left(\frac{\omega}{2}(V_1(\alpha^1)^\dagger + V_2(\alpha^2)^\dagger + V_3(\alpha^3)^\dagger)\right) \\
&= \exp\left(\frac{\omega}{2}(V_1(\gamma^1\gamma^0)^\dagger + V_2(\gamma^2\gamma^0)^\dagger + V_3(\gamma^3\gamma^0)^\dagger)\right) \\
&= \exp\left(\frac{\omega}{2}(V_1(\gamma^0)^\dagger(\gamma^1)^\dagger + V_2(\gamma^0)^\dagger(\gamma^2)^\dagger + V_3(\gamma^0)^\dagger(\gamma^3)^\dagger)\right) \\
&= \exp\left(-\frac{\omega}{2}(V_1\gamma^0\gamma^1 + V_2\gamma^0\gamma^2 + V_3\gamma^0\gamma^3)\right) \\
&= \exp\left(\frac{\omega}{2}(V_1\gamma^1\gamma^0 + V_2\gamma^2\gamma^0 + V_3\gamma^3\gamma^0)\right) \\
&= \exp\left(\frac{\omega}{2}(V_1\alpha^1 + V_2\alpha^2 + V_3\alpha^3)\right) \\
&= \mathcal{B}_r \; .
\end{aligned}
\tag{2.71}
$$

$\mathcal{B}_r$ is therefore hermitian and thus also fulfils the expectations.

3 The Dirac equation in curved spacetime

The subsequent section follows Refs. [32, 33].
Since this thesis primarily describes the behaviour of electrons interacting with a gravitational plane wave, which by its nature deforms spacetime, a brief introduction to how curved spacetime can be incorporated into the Dirac equation follows. From now on Greek indices are used for the spacetime manifold and Latin indices for the corresponding tangent space to the spacetime manifold at a point. This generalises (2.13) to

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \; , \tag{3.1}$$

where $g^{\mu\nu}$ is the general metric of curved spacetime. With the help of the so-called tetrads e^μ_a and e^a_μ, the spacetime manifold can be related to the tangent space. The following applies to the relationship between the metrics of the two:

$$\eta^{ab} e^\mu_a e^\nu_b = g^{\mu\nu} \; . \tag{3.2}$$

In addition, the gamma matrices of curved spacetime can also be expressed by those of flat Minkowski spacetime:

$$\gamma^\mu = \gamma^a e^\mu_a \; . \tag{3.3}$$

To be able to write the Dirac equation in curved spacetime, it must be formulated covariantly. To do this, the derivative of a spinor must be considered in more detail. If $\mathbf{\Lambda}$ is used to denote the spinor representation of a Lorentz transformation, a spinor transforms as follows:

$$\tilde{\psi} = \mathbf{\Lambda}\psi \; . \tag{3.4}$$

The derivative of a spinor, on the other hand, does not transform like a spinor:

$$\partial_\mu \tilde{\psi} = \mathbf{\Lambda}\partial_\mu\psi + \left(\partial_\mu \mathbf{\Lambda}\right)\psi \; . \tag{3.5}$$

Thus the covariant derivative

$$D_\mu \psi = \partial_\mu \psi + \mathbf{\Omega}_\mu \psi \tag{3.6}$$

is now defined so that the derivative of a spinor also transforms like a spinor. Here, $\mathbf{\Omega}_\mu$ are the spinorial connections. The covariant derivative now transforms like a spinor

$$\tilde{D}_\mu \tilde{\psi} = \partial_\mu \tilde{\psi} + \tilde{\mathbf{\Omega}}_\mu \tilde{\psi} = \mathbf{\Lambda} D_\mu \psi \tag{3.7}$$

if

$$\tilde{\mathbf{\Omega}}_\mu = \mathbf{\Lambda}\mathbf{\Omega}_\mu\mathbf{\Lambda}^{-1} - \left(\partial_\mu \mathbf{\Lambda}\right)\mathbf{\Lambda}^{-1} \tag{3.8}$$

is required as the transformation behaviour of the spinorial connections $\boldsymbol{\Omega}_\mu$ [33].
This allows the Dirac equation to be written in a covariant form, i.e. in a form appropriate to curved spacetime, by replacing the partial derivatives with the corresponding covariant derivatives:

$$(\gamma^\mu i\hbar D_\mu - mc)\,\psi = \gamma^\mu i\hbar\,(\partial_\mu + \boldsymbol{\Omega}_\mu)\,\psi - mc\psi = 0\,. \tag{3.9}$$

According to Ref. [32], the spinorial connections are calculated as follows:

$$\begin{aligned}
\boldsymbol{\Omega}_\mu &= \frac{1}{2}\Omega_{ij\mu}\sigma^{ij}\,,\\
\text{with :}\quad \Omega^i_{j\mu} &= e^\nu_j e^i_\sigma\left(\Lambda^\sigma_{\nu\mu} - e^\sigma_a \partial_\mu e^a_\nu\right)\,,\\
\Lambda^\sigma_{\alpha\nu} &= \frac{g^{\sigma\rho}}{2}\left(\partial_\alpha g_{\rho\nu} + \partial_\nu g_{\alpha\rho} - \partial_\rho g_{\alpha\nu}\right)\,,\\
\sigma^{ij} &= \frac{1}{4}[\gamma^i, \gamma^j]\,.
\end{aligned} \tag{3.10}$$

Here $\Lambda^\sigma_{\alpha\nu}$ denotes the Christoffel symbols.
Similarly, the Hestenes-Dirac equation can also be converted to covariant form by replacing the partial derivatives with covariant derivatives:

$$\hbar\gamma^\mu D_\mu \Psi\gamma_2\gamma_1 - mc\Psi\gamma_0 = \hbar\gamma^\mu\left(\partial_\mu + \boldsymbol{\Omega}_\mu\right)\Psi\gamma_2\gamma_1 - mc\Psi\gamma_0 = 0\,. \tag{3.11}$$

All information about the curvature of spacetime is then contained in the spinorial connections $\boldsymbol{\Omega}_\mu$ (3.10). Furthermore, the explicitly occurring gamma matrices γ_0, γ_1 and γ_2 are the untransformed gamma matrices of the flat Minkowski spacetime.

3.1 The Dirac equation for a free particle in cylindrical coordinates

The previously presented covariant form of the Dirac equation can be used not only to formulate the Dirac equation in curved spacetime, but also to use curvilinear coordinates such as cylindrical coordinates (t, r, φ, z). This coordinate transformation will lead to a solution of the Dirac equation for an electron vortex beam. An electron vortex beam is an electron beam possessing well defined orbital angular momentum (OAM) along its direction of propagation. The metric tensor for a flat spacetime in cylindrical coordinates is:

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -1)\,. \tag{3.12}$$

The tetrads for this metric tensor can be calculated in the same way as the Jacobian matrix, with

$$e_a^\mu = \frac{\partial x^\mu}{\partial x^a} \tag{3.13}$$

one obtains

$$e_a^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) & 0 \\ 0 & \frac{-\sin(\varphi)}{r} & \frac{\cos(\varphi)}{r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.14}$$

This results in the gamma matrices on the manifold according to (3.3):

$$\begin{aligned} \gamma^t &= \gamma^0 \,, \quad \gamma^r = \cos(\varphi)\gamma^1 + \sin(\varphi)\gamma^2 \,, \\ \gamma^z &= \gamma^3 \,, \quad \gamma^\varphi = \frac{1}{r}\left(\cos(\varphi)\gamma^2 - \sin(\varphi)\gamma^1\right) \,. \end{aligned} \tag{3.15}$$

The Christoffel symbols can be calculated using equation (3.10). These all vanish except for the following three:

$$\Lambda^r_{\varphi\varphi} = -r \,, \quad \Lambda^\varphi_{\varphi r} = \Lambda^\varphi_{r\varphi} = \frac{1}{r} \,. \tag{3.16}$$

Applying equation (3.10) again finally results in a zero spin connection $\boldsymbol{\Omega}_\mu = 0$, $\forall \mu \in \{t, r, \varphi, z\}$.
The Dirac equation for a free electron can thus be formulated in the present case:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \,. \tag{3.17}$$

Where $\partial_\mu \in \{\partial_{ct},\ \partial_r,\ \partial_\varphi, \partial_z\}$ and $\gamma^\mu \in \{\gamma^t,\ \gamma^r,\ \gamma^\varphi,\ \gamma^z\}$ apply. Using the ansatz $\psi = (\psi_1, 0, 0, \psi_4)^{\mathrm{T}}$ results in the following four coupled differential equations:

$$\begin{aligned} \left(\frac{i\hbar}{c}\partial_t - mc\right)\psi_1 + i\hbar\, e^{-i\varphi}\left(\frac{-i\partial_\varphi}{r} + \partial_r\right)\psi_4 &= 0 \,, \\ -i\hbar\partial_z\psi_4 &= 0 \,, \\ -i\hbar\partial_z\psi_1 &= 0 \,, \\ -\left(\frac{i\hbar}{c}\partial_t + mc\right)\psi_4 + i\hbar\, e^{i\varphi}\left(\frac{-i\partial_\varphi}{r} + \partial_r\right)\psi_1 &= 0 \,. \end{aligned} \tag{3.18}$$

If $\psi_1 = \exp\left(\frac{-iEt}{\hbar}\right) F_1(\varphi)G_1(r)$ and $\psi_4 = \exp\left(\frac{-iEt}{\hbar}\right) F_4(\varphi)G_4(r)$ is also assumed with E being the inital energy of the electron, these equations are further simplified:

$$\begin{aligned} \left(\frac{E}{c} - mc\right) F_1(\varphi)G_1(r) + i\hbar\, e^{-i\varphi}\left(\frac{-i\partial_\varphi}{r} + \partial_r\right) F_4(\varphi)G_4(r) &= 0 \,, \\ -\left(\frac{E}{c} + mc\right) F_4(\varphi)G_4(r) + i\hbar\, e^{i\varphi}\left(\frac{-i\partial_\varphi}{r} + \partial_r\right) F_1(\varphi)G_1(r) &= 0 \,. \end{aligned} \tag{3.19}$$

The first of these two equations can be solved for $F_1(\varphi)G_1(r)$:

$$F_1(\varphi)G_1(r) = \frac{1}{mc - \frac{E}{c}} i\hbar\, e^{-i\varphi} \left(\frac{-i\partial_\varphi}{r} + \partial_r \right) F_4(\varphi)G_4(r)\,. \tag{3.20}$$

This result can in turn be inserted into the second of the two equations in (3.19):

$$\begin{aligned} 0 = &-\left(\frac{E}{c} + mc\right) F_4(\varphi)G_4(r) \\ &+ i\hbar\, e^{i\varphi} \left(\frac{-i\partial_\varphi}{r} + \partial_r \right) \frac{1}{mc - \frac{E}{c}} i\hbar\, e^{-i\varphi} \left(\frac{-i\partial_\varphi}{r} + \partial_r \right) F_4(\varphi)G_4(r) \\ = &-\left(\frac{E}{c} + mc\right) F_4(\varphi)G_4(r) \\ &- \frac{\hbar^2}{mc - \frac{E}{c}} \left(-\frac{G_4(r)F_4''(\varphi)}{r^2} - \frac{F_4(\varphi)G_4'(r)}{r} - F_4(\varphi)G_4''(r) \right)\,. \end{aligned} \tag{3.21}$$

With the additional ansatz $F_4(\varphi) = e^{ia\varphi}$, where the meaning of a is still to be determined, this results in the following Bessel's differential equation:

$$0 = r^2 G_4''(r) + rG_4'(r) + \left(\frac{\left(\frac{E}{c}\right)^2 - (mc)^2}{\hbar^2} r^2 - a^2 \right) G_4(r)\,. \tag{3.22}$$

For $G_4(r)$ one thus obtains

$$G_4(r) = J_a\left(\frac{r\sqrt{E^2 - m^2c^4}}{\hbar c} \right)\,, \tag{3.23}$$

where $J_a(x)$ denotes the Bessel functions of the first kind. If one inserts this result into (3.20) and uses the recurrence relation for Bessel functions $\frac{a}{x}J_a(x) = J_{a-1}(x) - J_a'(x)$, the following result is obtained:

$$\begin{aligned} F_1(\varphi)G_1(r) &= \frac{i\hbar\, e^{-i\varphi}}{mc - \frac{E}{c}} \left[\frac{a}{r} e^{ia\varphi} J_a\left(\frac{r\sqrt{E^2 - m^2c^4}}{\hbar c} \right) \right. \\ &\qquad \left. + e^{ia\varphi} \frac{\sqrt{E^2 - m^2c^4}}{\hbar c} J_a'\left(\frac{r\sqrt{E^2 - m^2c^4}}{\hbar c} \right) \right] \\ &= \frac{ie^{i(a-1)\varphi}}{mc^2 - E} \sqrt{E^2 - m^2c^4}\, J_{a-1}\left(\frac{r\sqrt{E^2 - m^2c^4}}{\hbar c} \right)\,. \end{aligned} \tag{3.24}$$

Thus the spinor

$$\psi = \exp\left(\frac{-iEt}{\hbar}\right)\begin{pmatrix} \frac{ie^{i(a-1)\varphi}}{mc^2-E}\sqrt{E^2-m^2c^4}\,J_{a-1}\left(\frac{r\sqrt{E^2-m^2c^4}}{\hbar c}\right) \\ 0 \\ 0 \\ e^{ia\varphi}J_a\left(\frac{r\sqrt{E^2-m^2c^4}}{\hbar c}\right) \end{pmatrix} . \tag{3.25}$$

solves the Dirac equation in cylindrical coordinates. Multiplying this result by $i\frac{\mathcal{N}}{B}\sqrt{E^2-m^2c^4}$ and substituting $a = l+1$, one obtains the spinor from equation (46) of Ref. [4]:

$$\psi = \exp\left(\frac{-iEt}{\hbar}\right)\frac{\mathcal{N}}{B}\begin{pmatrix} e^{il\varphi}(E+mc^2)J_l\left(\frac{r\sqrt{E^2-m^2c^4}}{\hbar c}\right) \\ 0 \\ 0 \\ ie^{i(l+1)\varphi}\sqrt{E^2-m^2c^4}\,J_{l+1}\left(\frac{r\sqrt{E^2-m^2c^4}}{\hbar c}\right) \end{pmatrix} \tag{3.26}$$

Here $M = 2l$ is the orbital angular momentum quantum number. Thus, taking into account the substitutions made, a is also somewhat equivalent to the orbital angular momentum quantum number. In addition, B is a positive real-valued number with dimensions of energy, which essentially serves to ensure that the solution is dimensionless. $\mathcal{N}$ is a normalisation constant.
As can be seen from Ref. [4], (3.26) can be written as a matrix spinor:

$$\Psi = \sqrt{\rho}\,\mathcal{B}\exp\left(-\gamma^2\gamma^1\frac{Et-\Phi}{\hbar}\right) . \tag{3.27}$$

Here the following abbreviations were used:

$$\begin{aligned} \mathcal{B} &= \exp\left(\frac{w}{2}(\gamma^0\gamma^2\cos(\varphi) - \gamma^0\gamma^1\sin(\varphi))\right) , \\ \Phi &= \hbar l\varphi , \\ \frac{w}{2} &= \operatorname{arctanh}\left(\frac{B}{2(mc^2+E)}\frac{\mathrm{d}\ \ln(f(\lambda))}{\mathrm{d}\lambda}\right) , \\ \sqrt{\rho} &= \frac{\mathcal{N}(mc^2+E)\lambda^l f(\lambda)}{B\cosh(w/2)} , \\ f(\lambda) &= \lambda^{-l}J_l\left(\frac{r\sqrt{E^2-m^2c^4}}{\hbar c}\right) , \\ \lambda &= \frac{Br}{2\hbar c} . \end{aligned} \tag{3.28}$$

This matrix spinor satisfies the Hestenes-Dirac equation in cylindrical coordinates:

$$\hbar\gamma^\mu\partial_\mu\Psi\gamma_2\gamma_1 - mc\Psi\gamma_0 = 0\,. \tag{3.29}$$

If the definitions $r = \sqrt{x^2+y^2}$ and $\varphi = \arctan\left(\frac{y}{x}\right)$ are now inserted into the spinors (3.26) or (3.27), these are transformed back to Cartesian coordinates and the spinors obtained in this way are still a solution of the free Dirac (2.11) or Hestenes-Dirac equation (2.61). This makes it clear that the spinorial connections method is not only suitable for correctly incorporating curved spacetime into the Dirac equation, but also for performing coordinate transformations. This is important because coordinate transformations, as seen here and again in section 4.1 below, often simplify the task of solving the Dirac equation.

4 Solutions of the Dirac equation for an electron in a gravitational plane wave

Now that all the necessary concepts have been introduced, the actual topic of investigation, the dynamics of electrons in a gravitational plane wave, can be addressed. An exact solution of Einstein's field equations for a gravitational plane wave was first found in Ref. [14]. As can be seen from Ref. [34], the line element for an exact gravitational plane wave propagating in the z-direction can be written as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - f^2(\phi)\mathrm{d}x^2 - g^2(\phi)\mathrm{d}y^2 - \mathrm{d}z^2 = 2\mathrm{d}v\mathrm{d}u - \tilde{f}^2(u)\mathrm{d}x^2 - \tilde{g}^2(u)\mathrm{d}y^2. \quad (4.1)$$

Here, $\phi = k_i x^i = k(ct - z)$ applies with $k^i = k(1,0,0,1)^T$ being the wave vector of the gravitational wave. In addition, the light cone coordinates $v = \frac{1}{\sqrt{2}}(ct + z)$ and $u = \frac{1}{\sqrt{2}}(ct - z)$ are introduced and, in order to simplify subsequent calculations, $f(\phi) = \tilde{f}(u)$ and $g(\phi) = \tilde{g}(u)$ is assumed.
The only remaining vacuum field equation is then:

$$\frac{f''(\phi)}{f(\phi)} + \frac{g''(\phi)}{g(\phi)} = 0 \quad \text{resp.} \quad \frac{\tilde{f}''(u)}{\tilde{f}(u)} + \frac{\tilde{g}''(u)}{\tilde{g}(u)} = 0\,. \quad (4.2)$$

Where primed functions denote derivatives with respect to ϕ resp. u.
If the light cone basis (v, x, y, u) is used, the metric tensor is therefore:

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\tilde{f}^2(u) & 0 & 0 \\ 0 & 0 & -\tilde{g}^2(u) & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,. \quad (4.3)$$

The tetrads

$$e_a^\mu = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\tilde{f}(u)} & 0 & 0 \\ 0 & 0 & \frac{1}{\tilde{g}(u)} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (4.4)$$

lead to the new gamma matrices:

$$\begin{aligned} \gamma^v &= \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3)\,, \quad \gamma^x = \frac{1}{\tilde{f}(u)}\gamma^1\,, \\ \gamma^u &= \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^3)\,, \quad \gamma^y = \frac{1}{\tilde{g}(u)}\gamma^2\,. \end{aligned} \quad (4.5)$$

The Christoffel symbols in this case are:

$$\begin{aligned}
\Lambda^x_{xu} = \Lambda^x_{ux} &= \frac{\tilde{f}'(u)}{\tilde{f}(u)}\,, \quad \Lambda^v_{xx} = \tilde{f}(u)\tilde{f}'(u)\,, \\
\Lambda^y_{yu} = \Lambda^y_{uy} &= \frac{\tilde{g}'(u)}{\tilde{g}(u)}\,, \quad \Lambda^v_{yy} = \tilde{g}(u)\tilde{g}'(u)\,.
\end{aligned} \tag{4.6}$$

The spinorial connections can be calculated from this using equation (3.10):

$$\begin{aligned}
\Omega^0_{1x} &= \frac{\tilde{f}'(u)}{\sqrt{2}}\,, \quad \Omega^1_{0x} = \frac{\tilde{f}'(u)}{\sqrt{2}}\,, \\
\Omega^1_{3x} &= -\frac{\tilde{f}'(u)}{\sqrt{2}}\,, \; \Omega^3_{1x} = \frac{\tilde{f}'(u)}{\sqrt{2}}\,, \\
\Omega^0_{2y} &= \frac{\tilde{g}'(u)}{\sqrt{2}}\,, \quad \Omega^2_{0y} = \frac{\tilde{g}'(u)}{\sqrt{2}}\,, \\
\Omega^2_{3y} &= -\frac{\tilde{g}'(u)}{\sqrt{2}}\,, \; \Omega^3_{2y} = \frac{\tilde{g}'(u)}{\sqrt{2}}\,.
\end{aligned} \tag{4.7}$$

This ultimately results in:

$$\Omega_v = 0\,, \quad \Omega_x = \frac{\tilde{f}'(u)}{\sqrt{2}}(\sigma^{01}+\sigma^{13})\,, \quad \Omega_y = \frac{\tilde{g}'(u)}{\sqrt{2}}(\sigma^{02}+\sigma^{23})\,, \quad \Omega_u = 0\,, \tag{4.8}$$

which means that the Dirac equation can be written in the form of equation (3.9).

4.1 At rest solution

If now $\psi = (\psi_1, 0, 0, 0)^T$ is assumed, the Dirac equation simplifies to the following three coupled partial differential equations:

$$\begin{aligned}
\left[i\hbar\left(\frac{\partial_u}{\sqrt{2}} + \frac{\partial_v}{\sqrt{2}} + \frac{\tilde{f}'(u)}{2\sqrt{2}\tilde{f}(u)} + \frac{\tilde{g}'(u)}{2\sqrt{2}\tilde{g}(u)}\right) - mc\right]\psi_1 &= 0\,, \\
i\hbar\left(\frac{\partial_u}{\sqrt{2}} - \frac{\partial_v}{\sqrt{2}} + \frac{\tilde{f}'(u)}{2\sqrt{2}\tilde{f}(u)} + \frac{\tilde{g}'(u)}{2\sqrt{2}\tilde{g}(u)}\right)\psi_1 &= 0\,, \\
-i\hbar\left(\frac{\partial_x}{\tilde{f}(u)} + i\frac{\partial_y}{\tilde{g}(u)}\right)\psi_1 &= 0\,.
\end{aligned} \tag{4.9}$$

If one then makes the ansatz $\psi_1 = F(v)G(u)$ and adds or subtracts the first two equations, two even more simplified and decoupled equations can be obtained:

$$\begin{aligned}\left[i\hbar\left(\sqrt{2}\partial_u+\frac{1}{\sqrt{2}}\left\{\frac{\tilde{f}'(u)}{\tilde{f}(u)}+\frac{\tilde{g}'(u)}{\tilde{g}(u)}\right\}\right)-mc\right]F(v)G(u)=0\,,\\ \left(i\hbar\sqrt{2}\partial_v-mc\right)F(v)G(u)=0\,.\end{aligned}\tag{4.10}$$

In the next step, the second of these two equations is solved first, assuming that the solutions are non-trivial, i.e. $F(v)\neq 0$ and $G(u)\neq 0$.

$$\begin{aligned}&\left(i\hbar\sqrt{2}\partial_v-mc\right)F(v)G(u)=0\\ &\Rightarrow G(u)\left(i\hbar\sqrt{2}\partial_v-mc\right)F(v)=0\\ &\Rightarrow\ \left(i\hbar\sqrt{2}\partial_v-mc\right)F(v)=0\\ &\Rightarrow F'(v)=\frac{-imc}{\hbar\sqrt{2}}F(v)\\ &\Rightarrow F(v)=A\exp\left(-\frac{imc}{\hbar\sqrt{2}}v\right)\end{aligned}\tag{4.11}$$

A similar procedure can be used for the first of the two equations from (4.10).

$$\begin{aligned}&\left[i\hbar\left(\sqrt{2}\partial_u+\frac{1}{\sqrt{2}}\left\{\frac{\tilde{f}'(u)}{\tilde{f}(u)}+\frac{\tilde{g}'(u)}{\tilde{g}(u)}\right\}\right)-mc\right]F(v)G(u)=0\\ \Rightarrow F(v)&\left[i\hbar\left(\sqrt{2}\partial_u+\frac{1}{\sqrt{2}}\left\{\frac{\tilde{f}'(u)}{\tilde{f}(u)}+\frac{\tilde{g}'(u)}{\tilde{g}(u)}\right\}\right)-mc\right]G(u)=0\\ \Rightarrow\ &\left[i\hbar\left(\sqrt{2}\partial_u+\frac{1}{\sqrt{2}}\left\{\frac{\tilde{f}'(u)}{\tilde{f}(u)}+\frac{\tilde{g}'(u)}{\tilde{g}(u)}\right\}\right)-mc\right]G(u)=0\\ \Rightarrow G'(u)&=\frac{i\left[i\hbar\frac{1}{\sqrt{2}}\left(\frac{\tilde{f}'(u)}{\tilde{f}(u)}+\frac{\tilde{g}'(u)}{\tilde{g}(u)}\right)-mc\right]}{\hbar\sqrt{2}}G(u)\\ \Rightarrow G(u)&=B\frac{\exp\left(-\frac{imc}{\hbar\sqrt{2}}u\right)}{\sqrt{\tilde{f}(u)\tilde{g}(u)}}\end{aligned}\tag{4.12}$$

A final solution for $\psi_1=F(v)G(u)$ was thus found:

$$\psi_1=C\frac{1}{\sqrt{\tilde{f}(u)\tilde{g}(u)}}\exp\left(-\frac{imc}{\hbar\sqrt{2}}(u+v)\right)\,.\tag{4.13}$$

Here $C = AB$ is a constant.
Similarly, solutions can also be found for cases where $\psi = (0, \psi_2, 0, 0)^T$, $\psi = (0, 0, \psi_3, 0)^T$ or $\psi = (0, 0, 0, \psi_4)^T$ are initially assumed. To summarise, the solutions found can be written as follows:

$$
\begin{aligned}
\psi_1 &= \begin{pmatrix} C\frac{1}{\sqrt{\tilde{f}(u)\tilde{g}(u)}} \exp\left(-\frac{imc}{\hbar\sqrt{2}}(u+v)\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\psi_2 &= \begin{pmatrix} 0 \\ C\frac{1}{\sqrt{\tilde{f}(u)\tilde{g}(u)}} \exp\left(-\frac{imc}{\hbar\sqrt{2}}(u+v)\right) \\ 0 \\ 0 \end{pmatrix}, \\
\psi_3 &= \begin{pmatrix} 0 \\ 0 \\ C\frac{1}{\sqrt{\tilde{f}(u)\tilde{g}(u)}} \exp\left(\frac{imc}{\hbar\sqrt{2}}(u+v)\right) \\ 0 \end{pmatrix}, \\
\psi_4 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ C\frac{1}{\sqrt{\tilde{f}(u)\tilde{g}(u)}} \exp\left(\frac{imc}{\hbar\sqrt{2}}(u+v)\right) \end{pmatrix}.
\end{aligned}
\tag{4.14}
$$

It now seems natural to convert these solutions into Cartesian coordinates by explicitly inserting the definition of v and u into the solutions. However, this must first be prepared. To do this, $\tilde{f}(u)$ and $\tilde{g}(u)$ are replaced with $f(\phi)$ and $g(\phi)$. In addition, the metric tensor

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -f^2(\phi) & 0 & 0 \\ 0 & 0 & -g^2(\phi) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.15}
$$

the tetrads

$$
e_a^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{f(\phi)} & 0 & 0 \\ 0 & 0 & \frac{1}{g(\phi)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.16}
$$

the gamma matrices

$$\begin{aligned} \gamma^t &= \gamma^0 \,, \quad \gamma^x = \frac{1}{f(\phi)}\gamma^1 \,, \\ \gamma^z &= \gamma^3 \,, \quad \gamma^y = \frac{1}{g(\phi)}\gamma^2 \,. \end{aligned} \tag{4.17}$$

and the spinorial connections

$$\Omega_t = 0, \quad \Omega_x = kf'(\phi)(\sigma^{01}+\sigma^{13}), \quad \Omega_y = kg'(\phi)(\sigma^{02}+\sigma^{23}), \quad \Omega_z = 0. \tag{4.18}$$

must also be converted to Cartesian coordinates. If the initial energy $E = mc^2$ is now also introduced, the solutions (4.14) can be written in Cartesian coordinates as follows:

$$\begin{aligned} \psi_1 &= \begin{pmatrix} C\frac{1}{\sqrt{f(\phi)g(\phi)}} \exp\left(-\frac{iE}{\hbar}t\right) \\ 0 \\ 0 \\ 0 \end{pmatrix} \,, \\ \psi_2 &= \begin{pmatrix} 0 \\ C\frac{1}{\sqrt{f(\phi)g(\phi)}} \exp\left(-\frac{iE}{\hbar}t\right) \\ 0 \\ 0 \end{pmatrix} \,, \\ \psi_3 &= \begin{pmatrix} 0 \\ 0 \\ C\frac{1}{\sqrt{f(\phi)g(\phi)}} \exp\left(\frac{iE}{\hbar}t\right) \\ 0 \end{pmatrix} \,, \\ \psi_4 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ C\frac{1}{\sqrt{f(\phi)g(\phi)}} \exp\left(\frac{iE}{\hbar}t\right) \end{pmatrix} \,. \end{aligned} \tag{4.19}$$

Using the above to Cartesian coordinates adapted gamma matrices and spinorial connections, it is easy to show by simple insertion that these solutions satisfy the Dirac equation for an electron in a gravitational plane wave in Cartesian coordinates.

By now taking a closer look at the solutions in (4.19), it also becomes clear why the supposed detour via the light cone coordinates was taken. This was done, because in Cartesian coordinates, in contrast to the light cone coordinates, the solutions are no longer separable, which means solving the

Dirac equation in light cone coordinates is considerably easier. Furthermore, it also becomes clear why the solutions found here can be called "at rest" solutions, because only the zero component of the initial kinetic momentum $p^0 = \frac{E}{c}$ is non zero. Thus the presented solutions are given in a reference frame comoving with the electron. In the next section, the goal will be to boost these solutions to an arbitrary initial kinetic momentum p^i, which has an arbitrary direction.
The vector spinor solution ψ_1 from (4.19) can also be easily converted into a matrix spinor solution of the Hestenes-Dirac equation (3.11):

$$\Psi_1 = \frac{1}{\sqrt{f(\phi)g(\phi)}} \exp\left(-\gamma^2\gamma^1 \frac{E}{\hbar} t\right) . \tag{4.20}$$

Here $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$, $\beta = 0$ and $\mathcal{R} = U = \exp\left(-\gamma^2\gamma^1 \frac{E}{\hbar} t\right)$ applies.

4.2 Solution with arbitrary initial kinetic momentum

The solution presented below was first found in Ref. [16]. In the unpublished Ref. [17], this solution was then converted into a modern notation. In the derivation of the plane wave solution, this section follows the modern notation of Ref. [17] and therefore uses Cartesian coordinates, as well as the metric, the tetrads, the gamma matrices and the spinorial connections from the equations (4.15) to (4.18).
The first step is now, analogue to Ref. [35], to boost the at rest solutions (4.19) found. For this purpose, the exponential expression is first expressed in covariant form:

$$\begin{aligned} \exp\left(-\frac{iE}{\hbar} t\right) &= \exp\left(-\frac{ix^i p_i}{\hbar}\right) = \exp\left(-\frac{ix'^j p'_j}{\hbar}\right) \\ &= \exp\left(-\frac{i(E't' - p'^{x'}x' - p'^{y'}y' - p^{z'}z')}{\hbar}\right) . \end{aligned} \tag{4.21}$$

In addition, the spinors from the equation (4.19) are multiplied by the boost from Ref. [31]:

$$\mathcal{B}_g = \sqrt{\frac{E + mc^2}{2mc^2}} \begin{pmatrix} 1 & 0 & \frac{p^z c}{E+mc^2} & \frac{p^- c}{E+mc^2} \\ 0 & 1 & \frac{p^+ c}{E+mc^2} & \frac{-p^z c}{E+mc^2} \\ \frac{p^z c}{E+mc^2} & \frac{p^- c}{E+mc^2} & 1 & 0 \\ \frac{p^+ c}{E+mc^2} & \frac{-p^z c}{E+mc^2} & 0 & 1 \end{pmatrix} , \tag{4.22}$$
$$\text{with} : p^\pm = p^x \pm ip^y .$$

Of course, it is important to note that the sign of the momentum vector p^i of the solutions with negative energies (ψ_3 and ψ_4 in equation (4.19)) must be changed in equations (4.21) and (4.22).
While it is sufficient to boost the at rest solutions of the free Dirac equation in order to get a solution with non-zero velocity, it is found here that the spinors constructed in this way do not solve the Dirac equation in the presence of a plane gravitational wave. It turns out that an extra Lorentz transformation is needed, which corresponds to an active transformation describing the motion of the electron in the gravitational plane wave. If the constant parts of the spinors from equation (4.19) and the boost from equation (4.22) are now combined to form the constant spinor u_p, for which $(p_i\gamma^i - mc)u_p = 0$ applies, the following ansatz can be made in the present case [17]:

$$\psi = \frac{1}{\sqrt{f(\phi)g(\phi)}} G(\phi) \exp\left(-\frac{ix^m p_m}{\hbar}\right) u_p \; . \tag{4.23}$$

To determine $G(\phi)$ from this, the Dirac equation is first reformulated as follows [17]:

$$\begin{gathered} \left[i\hbar\left(\gamma^\mu\partial_\mu + \chi \not{k}\right) - mc\right]\psi = 0 \; , \\ \text{with} : \chi = \frac{1}{2}\left(\frac{f'(\phi)}{f(\phi)} + \frac{g'(\phi)}{g(\phi)}\right) \quad \text{and} \quad \not{k} = k_i\gamma^i \; . \end{gathered} \tag{4.24}$$

This is possible, since explicit calculations show $\gamma^x \mathbf{\Omega}_x + \gamma^y \mathbf{\Omega}_y = \chi\not{k}$. If one also defines, following Ref. [17], $e_a^{\ \mu} = \delta_a^{\ \mu} + H_a^{\ \mu}$, so that

$$\begin{aligned} \gamma^x &= \gamma^1 + H_a^{\ x}\gamma^a \; , \\ \gamma^y &= \gamma^2 + H_a^{\ y}\gamma^a \end{aligned} \tag{4.25}$$

with

$$H_a^{\ \mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{f(\phi)} - 1 & 0 & 0 \\ 0 & 0 & \frac{1}{g(\phi)} - 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{4.26}$$

then the Dirac equation can be reformulated further

$$\left[i\hbar\left(\gamma^i\partial_i + H_a^{\ \mu}\gamma^a\partial_\mu + \chi\not{k}\right) - mc\right]\psi = 0 \; . \tag{4.27}$$

With $\gamma^i\gamma^j = \eta^{ij} + 2\sigma^{ij}$, $k^i = k(1,0,0,1)^T$, $\phi = k_i x^i$ and knowing the total contraction of a symmetric quantity with an antisymmetric quantity is zero, e.g. $p_i p_j \sigma^{ij} = 0$, one then gets:

$$\begin{aligned} &\left(\gamma^i\partial_i + H_a^{\ \mu}\gamma^a\partial_\mu + \chi\not{k}\right)^2 \\ &= \partial_i\partial^i + 2H^{i\mu}\partial_i\partial_\mu + H_l^{\ \mu}H^{l\nu}\partial_\mu\partial_\nu + 2\chi k^i\partial_i + {H'}_i^{\ \mu}\not{k}\gamma^i\partial_\mu \; . \end{aligned} \tag{4.28}$$

Here $H'^{\mu}_{i} = \partial_\phi H^{\mu}_{i}$ was used as an abbreviation.
This means that the square of the Dirac equation (4.27) is

$$
\begin{aligned}
0 &= \left[\hbar^2 \left(\gamma^i \partial_i + H^{\mu}_{a} \gamma^a \partial_\mu + \chi \not{k}\right)^2 + m^2 c^2\right] \psi \\
&= \left[\hbar^2 \left(\partial_i \partial^i + 2H^{i\mu} \partial_i \partial_\mu + H^{\mu}_{l} H^{l\nu} \partial_\mu \partial_\nu + 2\chi k^i \partial_i + H'^{\mu}_{i} \not{k} \gamma^i \partial_\mu\right) + (mc)^2\right] \psi \,,
\end{aligned}
\tag{4.29}
$$

as can be also seen in Ref. [17]. Noticing $\partial_\phi \frac{1}{\sqrt{f(\phi)g(\phi)}} = -\chi \frac{1}{\sqrt{f(\phi)g(\phi)}}$ and keeping $p^2 = p_i p^i = (mc)^2$ as well as $k^2 = k_i k^i = 0$ in mind, the individual expressions can be calculated:

$$
\begin{aligned}
\partial^i \partial_i \psi &= \partial^i \left(\left[-k_i \chi - \frac{ip_i}{\hbar}\right] \psi + k_i G'(\phi) \exp\left(-\frac{ix^m p_m}{\hbar}\right) \frac{u_p}{\sqrt{f(\phi)g(\phi)}} \right) \\
&= \left(k_i k^i \chi^2 + 2\frac{i}{\hbar} p_i k^i \chi - \frac{p^2}{\hbar^2}\right) \psi \\
&\quad + \left(-k^i k_i \chi - 2\frac{i}{\hbar} p_i k^i\right) G'(\phi) \exp\left(-\frac{ix^m p_m}{\hbar}\right) \frac{u_p}{\sqrt{f(\phi)g(\phi)}} \\
&= \left(\frac{2i}{\hbar} p_i k^i \chi - \frac{(mc)^2}{\hbar}\right) \psi - \frac{2i}{\hbar} p_i k^i G'(\phi) \exp\left(-\frac{ix^m p_m}{\hbar}\right) \frac{u_p}{\sqrt{f(\phi)g(\phi)}} \,,
\end{aligned}
\tag{4.30}
$$

$$
2H^{i\mu} \partial_i \partial_\mu \psi = -\frac{2}{\hbar^2} H^{i\mu} p_i p_\mu \psi \,, \tag{4.31}
$$

$$
H^{\mu}_{l} H^{l\nu} \partial_\mu \partial_\nu \psi = -\frac{1}{\hbar^2} H^{\mu}_{l} H^{l\nu} p_\mu p_\nu \psi \,, \tag{4.32}
$$

$$
\begin{aligned}
2\chi k^i \partial_i \psi &= 2\chi k^i \left(\left[-k_i \chi - \frac{i}{\hbar} p_i\right] \psi - k_i G'(\phi) \exp\left(-\frac{ix^m p_m}{\hbar}\right) \frac{u_p}{\sqrt{f(\phi)g(\phi)}} \right) \\
&= -\frac{2i}{\hbar} \chi k^i p_i \psi \,,
\end{aligned}
\tag{4.33}
$$

$$
H'^{\mu}_{i} \not{k} \gamma^i \partial_\mu \psi = -\frac{i}{\hbar} H'^{\mu}_{i} \not{k} \gamma^i p_\mu \psi \,. \tag{4.34}
$$

Here, $p_\mu = (\frac{E}{c}, -p^x, -p^y, -p^z) = p_i$ applies, since p_μ results from $\partial_\mu \exp\left(-\frac{x^i p_i}{\hbar}\right)$ and $\partial_\mu x^i p_i = \delta^i_\mu p_i = p_\mu$.
Inserting the ansatz (4.23) in above squared Dirac equation (4.29) therefore results in the following differential equation for $G(\phi)$:

$$
0 = \left[-2H^{i\mu} p_i p_\mu - H^{\mu}_{l} H^{l\nu} p_\mu p_\nu - i\hbar H'^{\mu}_{i} \not{k} \gamma^i p_\mu\right] G(\phi) - 2i\hbar p_i k^i G'(\phi) \,. \tag{4.35}
$$

This differential equation can be solved simply by integration:

$$
\begin{aligned}
G(\phi) &= \exp\left(\frac{i}{\hbar}\int \frac{\mathrm{d}\phi}{2p_m k^m}\left[2H^{i\mu}p_i p_\mu + H_l{}^{\mu}H^{l\nu}p_\mu p_\nu + i\hbar H'{}_i{}^{\mu}\not{k}\gamma^i p_\mu\right]\right)\\
&= \exp\left(\frac{i}{\hbar}\int \frac{\mathrm{d}\phi}{2p_m k^m}\left[2H^{i\mu}p_i p_\mu + H_l{}^{\mu}H^{l\nu}p_\mu p_\nu\right]\right)\exp\left(\frac{-H_i{}^{\mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right) .
\end{aligned}
\tag{4.36}
$$

Due to $\left(H_i{}^{\mu}\not{k}\gamma^i p_\mu\right)^2 = 0$, the matrix exponential, which is therefore also referred to as a null rotation, can be written as

$$
\exp\left(\frac{-H_i{}^{\mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right) = 1 - \frac{H_i{}^{\mu}\not{k}\gamma^i p_\mu}{2k^j p_j} . \tag{4.37}
$$

In addition, the argument of the integral can be expanded by explicitly inserting $H_i{}^{\mu}$:

$$
2H^{i\mu}p_i p_\mu + H_l{}^{\mu}H^{l\nu}p_\mu p_\nu = \left(1 - \frac{1}{f^2(\phi)}\right)(p^x)^2 + \left(1 - \frac{1}{g^2(\phi)}\right)(p^y)^2 \quad [17] . \tag{4.38}
$$

If $S = \int \frac{\mathrm{d}\phi}{2k_j p^j}\left[2H^{i\mu}p_i p_\mu + H_l{}^{\mu}H^{l\nu}p_\mu p_\nu\right]$ is defined, the solution of the Dirac equation with arbitrary initial kinetic momentum reads as follows:

$$
\boxed{\psi = \frac{1}{\sqrt{f(\phi)g(\phi)}}\left(1 - \frac{H_i{}^{\mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right)\exp\left(-\frac{ix^m p_m}{\hbar} + \frac{iS}{\hbar}\right)u_p .} \tag{4.39}
$$

This exact solution can also be found in Ref. [17].

4.2.1 Test of the solution

While a test of the solution can already be found in Ref. [17], this test will be formulated in more detail below. To do this, the above solution must first be converted into the following form:

$$
\begin{aligned}
\psi &= \left(1 - \frac{H_i{}^{\mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right)\exp\left(-\frac{ix^m p_m}{\hbar} + \frac{iF(\phi)}{\hbar}\right)u_p ,\\
&\text{with :}\\
F(\phi) &= \int \frac{\mathrm{d}\phi}{2k^m p_m}\left[2H^{i\mu}p_i p_\mu + H_l{}^{\mu}H^{l\nu}p_\mu p_\nu + 2i\hbar k^i p_i \chi\right] .
\end{aligned}
\tag{4.40}
$$

If this is now inserted into the Dirac equation, the result is:

$$
\begin{aligned}
&\left[i\hbar\left(\gamma^i\partial_i + H_a{}^{\mu}\gamma^a\partial_\mu + \chi\not{k}\right) - mc\right]\psi\\
&= \left[\not{p} + \not{k}(i\hbar\chi - F'(\phi)) + H_a^{\mu}\gamma^a p_\mu - mc\right]\psi .
\end{aligned}
\tag{4.41}
$$

Dividing this result by $\exp\left(-\frac{ix^m p_m}{\hbar} + \frac{iF(\phi)}{\hbar}\right)$ yields:

$$
\begin{aligned}
&\left[\not{p} + \not{k}(i\hbar\chi - F'(\phi)) + H_a^{\ \mu}\gamma^a p_\mu - mc\right]\left(1 - \frac{H_i^{\ \mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right) u_p \\
&= \left[\not{p} + \not{k}\left(i\hbar\chi - \frac{1}{2k^m p_m}\left[2H^{i\mu}p_i p_\mu + H_l^{\ \mu}H^{l\nu}p_\mu p_\nu + 2i\hbar k^i p_i \chi\right]\right)\right. \\
&\qquad \left. + H_a^{\ \mu}\gamma^a p_\mu - mc\right]\left(1 - \frac{H_i^{\ \mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right) u_p \\
&= \left[\not{p} - \frac{\not{k}}{2k^m p_m}\left(2H^{i\mu}p_i p_\mu + H_l^{\ \mu}H^{l\nu}p_\mu p_\nu\right)\right. \\
&\qquad \left. + H_a^{\ \mu}\gamma^a p_\mu - mc\right]\left(1 - \frac{H_i^{\ \mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right) u_p \,.
\end{aligned}
\tag{4.42}
$$

Since $(\not{p} - mc)u_p = 0$ and $\not{k}\not{k} = k^i k_i = 0$, this can be further simplified as follows:

$$
\begin{aligned}
&\left[\not{p} - \frac{\not{k}}{2k^m p_m}\left(2H^{i\mu}p_i p_\mu + H_l^{\ \mu}H^{l\nu}p_\mu p_\nu\right)\right. \\
&\qquad \left. + H_a^{\ \mu}\gamma^a p_\mu - mc\right]\left(1 - \frac{H_i^{\ \mu}\not{k}\gamma^i p_\mu}{2k^j p_j}\right) u_p \\
&= \left[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l^{\ \mu}H^{l\nu}p_\mu p_\nu\right) + H_a^{\ \mu}\gamma^a p_\mu\right. \\
&\qquad \left. - (\not{p} + H_a^{\ \mu}\gamma^a p_\mu - mc)\frac{H_i^{\ \nu}\not{k}\gamma^i p_\nu}{2k^j p_j}\right] u_p \\
&= \left[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l^{\ \mu}H^{l\nu}p_\mu p_\nu\right) + H_a^{\ \mu}\gamma^a p_\mu\right. \\
&\qquad \left. + (mc - H_a^{\ \mu}\gamma^a p_\mu)\frac{H_i^{\ \nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - \frac{H_i^{\ \nu}\not{p}\not{k}\gamma^i p_\nu}{2k^j p_j}\right] u_p \\
&= \left[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l^{\ \mu}H^{l\nu}p_\mu p_\nu\right) + H_a^{\ \mu}\gamma^a p_\mu\right. \\
&\qquad \left. + (mc - H_a^{\ \mu}\gamma^a p_\mu)\frac{H_i^{\ \nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - \frac{H_i^{\ \nu}p_a k_b \gamma^a\gamma^b\gamma^i p_\nu}{2k^j p_j}\right] u_p \,.
\end{aligned}
\tag{4.43}
$$

Using the identity (2.36) twice, gives:

$$\begin{aligned}
&\Big[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l{}^{\mu} H^{l\nu} p_\mu p_\nu\right) + H_a{}^{\mu}\gamma^a p_\mu \\
&\quad + (mc - H_a{}^{\mu}\gamma^a p_\mu)\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - \frac{H_i{}^{\nu} p_a k_b \gamma^a\gamma^b\gamma^i p_\nu}{2k^j p_j}\Big] u_p \\
&= \Big[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l{}^{\mu} H^{l\nu} p_\mu p_\nu\right) \\
&\quad + H_a{}^{\mu}\gamma^a p_\mu + (mc - H_a{}^{\mu}\gamma^a p_\mu)\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j} \\
&\quad - \frac{H_i{}^{\nu} p_a k_b p_\nu}{2k^j p_j}(\eta^{ab}\gamma^i + \eta^{bi}\gamma^a - \eta^{ai}\gamma^b + \gamma^b\gamma^i\gamma^a - \eta^{bi}\gamma^a - \eta^{ia}\gamma^b + \eta^{ab}\gamma^i)\Big] u_p \\
&= \Big[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l{}^{\mu} H^{l\nu} p_\mu p_\nu\right) + H_a{}^{\mu}\gamma^a p_\mu \\
&\quad + (mc - H_a{}^{\mu}\gamma^a p_\mu)\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - \frac{H_i{}^{\nu} p_a k_b p_\nu}{2k^j p_j}(2\eta^{ab}\gamma^i + \gamma^b\gamma^i\gamma^a - 2\eta^{ai}\gamma^b)\Big] u_p \\
&= \Big[-\frac{\not{k}}{2k^j p_j}\left(2H^{i\mu}p_i p_\mu + H_l{}^{\mu} H^{l\nu} p_\mu p_\nu\right) + H_a{}^{\mu}\gamma^a p_\mu \\
&\quad + (mc - H_a{}^{\mu}\gamma^a p_\mu)\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - H_i{}^{\nu}\gamma^i p_\nu - \frac{H_i{}^{\nu}\not{k}\gamma^i\not{p}p_\nu}{2k^j p_j} + \frac{\not{k}H^{a\nu}p_a p_\nu}{k^j p_j}\Big] u_p \\
&= \Big[-\frac{\not{k}}{2k^j p_j}H_l{}^{\mu} H^{l\nu} p_\mu p_\nu + (mc - H_a{}^{\mu}\gamma^a p_\mu)\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - \frac{H_i{}^{\nu}\not{k}\gamma^i\not{p}p_\nu}{2k^j p_j}\Big] u_p \,.
\end{aligned} \tag{4.44}$$

If one uses $(\not{p} - mc)u_p = 0$ again, this can be simplified further:

$$\begin{aligned}
&\Big[-\frac{\not{k}}{2k^j p_j}H_l{}^{\mu} H^{l\nu} p_\mu p_\nu + (mc - H_a{}^{\mu}\gamma^a p_\mu)\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j} - \frac{H_i{}^{\nu}\not{k}\gamma^i\not{p}p_\nu}{2k^j p_j}\Big] u_p \\
&= \Big[-\frac{\not{k}}{2k^j p_j}H_l{}^{\mu} H^{l\nu} p_\mu p_\nu - H_a{}^{\mu}\gamma^a p_\mu\frac{H_i{}^{\nu}\not{k}\gamma^i p_\nu}{2k^j p_j}\Big] u_p \,.
\end{aligned} \tag{4.45}$$

If now the fact that $a \in \{1,2\}$ and $\not{k} = k(\gamma^0 - \gamma^3)$ is used, then $\gamma^a \not{k} = -\not{k}\gamma^a$ and it follows:

$$\begin{aligned}
&\left[-\frac{\not{k}}{2k^j p_j} H_l^{\ \mu} H^{l\nu} p_\mu p_\nu - H_a^{\ \mu} \gamma^a p_\mu \frac{H_i^{\ \nu} \not{k} \gamma^i p_\nu}{2k^j p_j}\right] u_p \\
&= \left[-\frac{\not{k}}{2k^j p_j} H_l^{\ \mu} H^{l\nu} p_\mu p_\nu + \frac{\not{k}}{2k^j p_j} H_a^{\ \mu} H_i^{\ \nu} \gamma^a \gamma^i p_\mu p_\nu\right] u_p \\
&= \left[-\frac{\not{k}}{2k^j p_j} H_l^{\ \mu} H^{l\nu} p_\mu p_\nu + \frac{\not{k}}{2k^j p_j} H_a^{\ \mu} H_i^{\ \nu} (\eta^{ai} + 2\sigma^{ai}) p_\mu p_\nu\right] u_p \\
&= \left[-\frac{\not{k}}{2k^j p_j} H_l^{\ \mu} H^{l\nu} p_\mu p_\nu + \frac{\not{k}}{2k^j p_j} H_a^{\ \mu} H^{a\nu} p_\mu p_\nu\right] u_p \\
&= 0\,.
\end{aligned} \tag{4.46}$$

It was used that due to the asymmetry of σ^{ai}, the contraction of σ^{ai} with a symmetrical quantity is zero.
The solution (4.39) presented above therefore satisfies the Driac equation, as expected.

4.2.2 Matrix spinor form

This result can now be easily written as a matrix spinor that satisfies the Hestenes-Dirac equation (3.11). The construction chosen here using the boost is particularly advantageous, as u_p can simply be replaced by $\mathcal{B}_g$ from equation (4.22). In addition, i must be replaced with $\gamma^2\gamma^1$, as is usual for the construction of the matrix spinors:

$$\Psi = \frac{1}{\sqrt{f(\phi)g(\phi)}} \left(1 - \frac{H_i^{\ \mu} \not{k} \gamma^i p_\mu}{2k^j p_j}\right) \mathcal{B}_g \exp\left(-\gamma^2\gamma^1 \frac{x^m p_m - S}{\hbar}\right) . \tag{4.47}$$

The comparison with equation (4.20) makes it clear that this solution emerges from the at rest solution by applying the boost $\mathcal{B}_g$ and a further Lorentz transformation $\mathcal{R}_r = 1 - \frac{H_i^{\ \mu} \not{k} \gamma^i p_\mu}{2k^j p_j}$, which also results in an additional phase $S = \int \frac{\mathrm{d}\phi}{2k_j p^j} \left[2H^{i\mu} p_i p_\mu + H_l^{\ \mu} H^{l\nu} p_\mu p_\nu\right]$. Now one can make the following identifications $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$, $\beta = 0$ and $\mathcal{R} = \mathcal{R}_r \mathcal{B}_g \exp\left(-\gamma^2\gamma^1 \frac{x^m p_m - S}{\hbar}\right)$.
The aim in the next step is to separate $\mathcal{R}_r$ into a rotation and a boost $\mathcal{R}_r = U_r \mathcal{B}_r$. To do this, $\frac{H_i^{\ \mu} \not{k} \gamma^i p_\mu}{2k^j p_j}$ must first be considered a little more closely

by explicitly using $H_i^{\ \mu}$:

$$\begin{aligned}
&\frac{H_i^{\ \mu}\not{k}\gamma^i p_\mu}{2k^j p_j} \\
&= -\frac{1}{2k^j p_j}\left[\left(\frac{1}{f(\phi)}-1\right)p^x k(\gamma^0-\gamma^3)\gamma^1+\left(\frac{1}{g(\phi)}-1\right)p^y k(\gamma^0-\gamma^3)\gamma^2\right] \\
&= -\frac{1}{2k^j p_j}\left[\left(\frac{1}{f(\phi)}-1\right)p^x k(\gamma_1\gamma_0-\gamma_3\gamma_1)+\left(\frac{1}{g(\phi)}-1\right)p^y k(\gamma_2\gamma_0-\gamma_3\gamma_2)\right] \\
&= -\frac{1}{2k^j p_j}\left[\left(\frac{1}{f(\phi)}-1\right)p^x k(\alpha_1-\gamma_3\gamma_1)+\left(\frac{1}{g(\phi)}-1\right)p^y k(\alpha_2+\gamma_2\gamma_3)\right] \ .
\end{aligned} \tag{4.48}$$

If this result is now compared with equation (79) from Ref. [4], a comparable structure of the two matrices can be recognised. Therefore, equations (80) to (89) from Ref. [4] are adapted below for the case analysed here:

$$\begin{aligned}
U_r &= \exp\left(-\frac{\theta}{2}(\cos(\vartheta)\gamma^1\gamma^3+\sin(\vartheta)\gamma^2\gamma^3)\right) \ , \\
\mathcal{B}_r &= \exp\left(\frac{\omega}{2}(V_1\alpha^1+V_2\alpha^2+V_3\alpha^3)\right) \ .
\end{aligned} \tag{4.49}$$

The abbreviations used are as follows:

$$\begin{aligned}
\frac{\theta}{2} &= \arctan\left(k\frac{1}{2k^j p_j}\sqrt{(A^1)^2+(A^2)^2}\right) \ , \\
\cos(\vartheta) &= \frac{A^1}{\sqrt{(A^1)^2+(A^2)^2}} \ , \\
\sin(\vartheta) &= \frac{A^2}{\sqrt{(A^1)^2+(A^2)^2}} \ , \\
\frac{V_1}{2} &= \cos\left(\frac{\theta}{2}\right)\cos(\vartheta) \ , \\
\frac{V_2}{2} &= \cos\left(\frac{\theta}{2}\right)\sin(\vartheta) \ , \\
\frac{V_3}{2} &= \sin\left(\frac{\theta}{2}\right) \ , \\
\omega &= \operatorname{arctanh}\left(\frac{V_3}{2}\right) \ , \\
A^1 &= -\left(\frac{1}{f(\phi)}-1\right)p^x \ , \\
A^2 &= -\left(\frac{1}{g(\phi)}-1\right)p^y \ .
\end{aligned} \tag{4.50}$$

A rotation U_r and a boost $\mathcal{B}_r$ were thus found, so that $\mathcal{R}_r = U_r\mathcal{B}_r$ holds. This means that the matrix spinor can be written as follows:

$$\Psi = \frac{1}{\sqrt{f(\phi)g(\phi)}} U_r\mathcal{B}_r\mathcal{B}_g \exp\left(-\gamma^2\gamma^1\frac{x^m p_m - S}{\hbar}\right) . \tag{4.51}$$

4.2.3 Comparison with the electromagnetic case and interpretation

In order to compare the case of the gravitational wave with that of the electromagnetic wave, the corresponding Dirac and Hestenes Dirac equations must first be compared. For the case of an electromagnetic plane wave propagating in the z-direction ($k^\mu = k(1,0,0,1)^T$, $A^\mu = (0, A^1, A^2, 0)^T$) this reads:

$$\hbar\gamma^i\partial_i\Psi\gamma_2\gamma_1 - e\gamma^i A_i\Psi - mc\Psi\gamma_0 = 0\,. \tag{4.52}$$

In the case of the gravitational wave, however, the following applies in adapted notation:

$$\hbar(\gamma^i\partial_i + H_i^{\ \mu}\gamma^i\partial_\mu + \chi\gamma^i k_i)\Psi\gamma_2\gamma_1 - mc\Psi\gamma_0 = 0\,. \tag{4.53}$$

At first glance, when comparing the two equations, it becomes obvious that the Hestenes-Dirac equation has an additional term $\chi\gamma^i k_i\Psi\gamma_2\gamma_1$ in the case of the gravitational wave. This follows directly from the spinorial connections and means that $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$ applies to all solutions, since $\gamma^i\partial_i\frac{1}{\sqrt{f(\phi)g(\phi)}} = -\chi\gamma^i k_i\frac{1}{\sqrt{f(\phi)g(\phi)}}$, while $\sqrt{\rho} = 1$ applies in the case of the electromagnetic wave, as the comparison of the equations (4.51) and (2.67) also shows. Here it should also be noted that $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}} = \frac{1}{\sqrt[4]{|\det(g_{\mu\nu})|}}$ applies. This at least suggest that this expression reflects the deformation of spacetime as a result of the gravitational wave.

Apart from this, however, the two Hestnes-Dirac equations have a similar structure, although there are still three minor differences. In the gravitational wave case, the term $\hbar H_i^{\ \mu}\gamma^i\partial_\mu\Psi\gamma_2\gamma_1$ contains partial derivatives, while in the electromagnetic case the term of the vector potential $-e\gamma^i A_i\Psi$ does not contain any derivatives. Due to the chosen ansatz $\Psi = \frac{1}{\sqrt{f(\phi)g(\phi)}}G(\phi)\mathcal{B}_g$ $\exp\left(-\frac{-\gamma^2\gamma^1 x^m p_m}{\hbar}\right)$, the derivative in the case of the gravitational wave can be replaced by p_μ, as will be explained in more detail below. Furthermore the expression $\hbar H_i^{\ \mu}\gamma^i\partial_\mu\Psi\gamma_2\gamma_1$, in contrast to $-e\gamma^i A_i\Psi$, contains the gamma matrices γ_2 and γ_1, which in combination and applied from the right to Ψ correspond to the imaginary unit i in the "normal" Dirac equation.

This is due to the fact that the term $\hbar H_i^{\ \mu}\gamma^i\partial_\mu\Psi\gamma_2\gamma_1$ is part of the covariant derivative in the case of the gravitational wave. This product $\gamma_2\gamma_1$ is cancelled by the expression $\exp\left(-\frac{-\gamma^2\gamma^1 x^m p_m}{\hbar}\right)$, since the derivative of this exponential term $\partial_\mu \exp\left(-\frac{-\gamma^2\gamma^1 x^m p_m}{\hbar}\right) = -\frac{p_\mu}{\hbar}\exp\left(-\frac{-\gamma^2\gamma^1 x^m p_m}{\hbar}\right)\gamma^2\gamma^1$ and $-\gamma^2\gamma^1\gamma_2\gamma_1 = -\gamma_2\gamma_1\gamma_2\gamma_1 = \mathbb{1}$ applies. Thus, based on the chosen ansatz, the expression $\hbar H_i^{\ \mu}\gamma^i\partial_\mu\Psi\gamma_2\gamma_1$ simplifies to $p_\mu H_i^{\ \mu}\gamma^i\Psi$, which is analogous to the term $-e\gamma^i A_i\Psi$ in the electromagnetic case. In addition, the signs of the terms $\hbar H_i^{\ \mu}\gamma^i\partial_\mu\Psi\gamma_2\gamma_1$ and $-e\gamma^i A_i\Psi$ differ, which leads to differences in the signs of the solutions, which will become clearer later on when the solutions are considered in more detail.
However, since the two Hestenes-Dirac equations are identical apart from this, it is clear from a purely mathematical point of view that the solutions to these two equations must be very similar and have a comparable structure. Now that the comparison of the two Hestenes-Dirac equations has been completed, the next step is to compare the matrix spinors (4.51) and (2.67). To do this, the two Lorentz transformations $\mathcal{R}_r$ are first compared in the form before they were split into a boost and a rotation. For the electromagnetic wave, this Lorentz transformation is as follows:

$$\mathcal{R}_r = \left(1 + \frac{e}{2k\cdot p}\not{k}\not{A}\right) . \tag{4.54}$$

While for the gravitational wave

$$\mathcal{R}_r = \left(1 - \frac{H_i^{\ \mu}\not{k}\gamma^i p_\mu}{2k\cdot p}\right) \tag{4.55}$$

applies. At first glance, it can be seen that these two matrices have the same structure, apart from the sign between the unit matrix and the remainder of the Lorentz transformation. This difference in sign is due to the fact that the sign of the $\hbar H_i^{\ \mu}\gamma^i\partial_\mu\Psi\gamma_2\gamma_1 = p_\mu H_i^{\ \mu}\gamma^i\Psi$ and $-e\gamma^i A_i\Psi$ terms in the two Hestenes-Dirac equations is different. If one now identifies $-eA_i$ with $p_\mu H_i^{\ \mu}$, not only do the two expressions in the Hestenes Dirac equations coincide, but the two Lorentz transformations (4.54) and (4.55) are now also completely identical. As a result, the rotation U_r and the boost $\mathcal{B}_r$ are also the same for both the electromagnetic wave and the gravitational wave, see equations (2.68) and (2.69) compared to (4.49) and (4.50).
However, not only are the two Lorentz transformations to be regarded as identical due to this identification, but the additional phases as a result of the gauge freedom are now also equivalent. This is because, taking into account the fact that $eA\cdot p \mathrel{\hat{=}} -H^{i\mu}p_i p_\mu$ and $e^2A^2 \mathrel{\hat{=}} H_l^{\ \mu}H^{l\nu}p_\mu p_\nu$ apply with

the previously made identification,

$$-\Phi = -\int \left(\frac{eA \cdot p}{k \cdot p} - \frac{e^2 A^2}{2k \cdot p} \right) \mathrm{d}\phi \tag{4.56}$$

becomes

$$S = \int \frac{\mathrm{d}\phi}{2k \cdot p} \left[2H^{i\mu} p_i p_\mu + H_l{}^\mu H^{l\nu} p_\mu p_\nu \right]. \tag{4.57}$$

Thus, considering the identification $-eA_i \mathrel{\hat{=}} p_\mu H_i{}^\mu$, apart from the additional scaling $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$ as a result of the spinorial connections, not only the Hestenes-Dirac equations for the case of the electromagnetic wave and the gravitational wave coincide, but also the corresponding spinors that solve them.
In conclusion, it can be stated that $-H_i{}^\mu$ is the generalised "vector potential" of the gravitational wave, so to speak, and the initial kinetic momentum p_μ is the quantity to which this "vector potential" couples, analogous to the the charge e in the electromagnetic case. Note that this is a physical observation, which, at first, is non-trivial. One would naively expect that the coupling is with the mass. Rather, it is directly with the momentum and indirectly with the mass (on-shell constraint). But, with the same thought one concludes that this should be the case since light also is affected by gravity. Thus momentum is more fundamental here.

4.3 Solution for an electron vortex beam with OAM

The next step in the discussion of electron dynamics in a gravitational plane wave is to find a solution to the Dirac equation for an electron vortex beam with orbital angular momentum (OAM) along its axis of propagation in the presence of a gravitational plane wave. To achieve this, one might be tempted to generalise the solution with OAM for an electron in an electromagnetic plane wave from Eq. (96) of Ref. [4]. However, due to the differences between the Dirac respectively Hestenes-Dirac equations described in section 4.2.3 for the case of an electromagnetic and a gravitational wave, this turns out to be more difficult than expected. This is due in particular to the fact that the term $i\hbar H_i{}^\mu \gamma^i \partial_\mu \psi$ contains an additional partial derivative in the case of the gravitational wave, which does not occur in the corresponding term $-e\gamma^i A_i \psi$ for the case of the electromagnetic wave.
So instead a superposition in momentum space of the plane wave solutions (4.39) is considered to find the desired solution for an electron vortex beam, which is a very similar procedure compared to Ref. [18]. In momentum

space, cylindrical coordinates are used and $p^z = 0$ is assumed, i.e. $\boldsymbol{p} = (p^x, p^y, p^z)^T = p\ (\cos(\theta), \sin(\theta), 0)^T$. Here p is the constant absolute value of the momentum. It should also be pointed out that $E^2 = p^2c^2 + m^2c^4$ applies here for the magnitude p of the momentum. The spinor u_p takes then the form

$$u_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{cp\, e^{i\theta}}{E+mc^2} \end{pmatrix} \tag{4.58}$$

and the solution (4.39) can be written as follows:

$$\begin{aligned} \psi_p &= \frac{\exp(i\Phi)}{\sqrt{f(\phi)g(\phi)}} \mathcal{R}_r u_p \\ &= \frac{\exp(i\Phi)}{\sqrt{f(\phi)g(\phi)}} \left[1 + \frac{\not{k}pc}{2kE} \left\{ \left(\frac{1}{f(\phi)} - 1 \right) \cos(\theta)\gamma^1 \right.\right. \\ &\quad \left.\left. + \left(\frac{1}{g(\phi)} - 1 \right) \sin(\theta)\gamma^2 \right\} \right] \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{cp\, e^{i\theta}}{E+mc^2} \end{pmatrix} . \end{aligned} \tag{4.59}$$

Here the phase Φ is:

$$\begin{aligned} \Phi &= -\frac{1}{\hbar} \left[Et - p\{x\cos(\theta) + y\sin(\theta)\} \right] + \left[a\cos^2(\theta) + b\sin^2(\theta) \right] , \\ a &= \frac{p^2c}{\hbar} \int \frac{\mathrm{d}\phi}{2kE} \left(1 - \frac{1}{f^2(\phi)} \right) , \\ b &= \frac{p^2c}{\hbar} \int \frac{\mathrm{d}\phi}{2kE} \left(1 - \frac{1}{g^2(\phi)} \right) . \end{aligned} \tag{4.60}$$

The desired wave packet, which is, as already mentioned, obtained through a superposition, then looks as follows:

$$\psi = \frac{1}{2\pi} \int_0^{2\pi} \psi_p \, \mathrm{d}\theta . \tag{4.61}$$

In order to solve this integral and thus determine the superposition, the product of the Lorentz transformation $\mathcal{R}_r$ and the constant spinor u_p in equation (4.61) is first calculated explicitly:

$$
\begin{aligned}
\mathcal{R}_r u_p &= \begin{pmatrix} 1 + \frac{p^2c^2e^{i\theta}\left[\cos(\theta)\left(\frac{1}{f(\phi)}-1\right)-i\sin(\theta)\left(\frac{1}{g(\phi)}-1\right)\right]}{2E(E+mc^2)} \\ -\frac{pc\left[\cos(\theta)\left(\frac{1}{f(\phi)}-1\right)+i\sin(\theta)\left(\frac{1}{g(\phi)}-1\right)\right]}{2E} \\ \frac{p^2c^2e^{i\theta}\left[\cos(\theta)\left(\frac{1}{f(\phi)}-1\right)-i\sin(\theta)\left(\frac{1}{g(\phi)}-1\right)\right]}{2E(E+mc^2)} \\ \frac{pc\,e^{i\theta}}{E+mc^2} + \frac{pc\left[\cos(\theta)\left(\frac{1}{f(\phi)}-1\right)+i\sin(\theta)\left(\frac{1}{g(\phi)}-1\right)\right]}{2E} \end{pmatrix} \\
&= \begin{pmatrix} 1 + \frac{p^2c^2e^{i\theta}\left[\left(e^{i\theta}+e^{-i\theta}\right)A-\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E(E+mc^2)} \\ -\frac{pc\left[\left(e^{i\theta}+e^{-i\theta}\right)A+\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E} \\ \frac{p^2c^2e^{i\theta}\left[\left(e^{i\theta}+e^{-i\theta}\right))A-\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E(E+mc^2)} \\ \frac{pc\,e^{i\theta}}{E+mc^2} + \frac{pc\left[\left(e^{i\theta}+e^{-i\theta}\right)A+\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E} \end{pmatrix} .
\end{aligned}
\tag{4.62}
$$

The abbreviations

$$
\begin{aligned}
A &= \frac{1}{f(\phi)} - 1 \, , \\
B &= \frac{1}{g(\phi)} - 1
\end{aligned}
\tag{4.63}
$$

were introduced here. The spinor ψ_p can then be written as follows:

$$
\psi_p = \frac{\exp(i\Phi)}{\sqrt{f(\phi)g(\phi)}} \begin{pmatrix} 1 + \frac{p^2c^2e^{i\theta}\left[\left(e^{i\theta}+e^{-i\theta}\right)A-\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E(E+mc^2)} \\ -\frac{pc\left[\left(e^{i\theta}+e^{-i\theta}\right)A+\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E} \\ \frac{p^2c^2e^{i\theta}\left[\left(e^{i\theta}+e^{-i\theta}\right))A-\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E(E+mc^2)} \\ \frac{pc\,e^{i\theta}}{E+mc^2} + \frac{pc\left[\left(e^{i\theta}+e^{-i\theta}\right)A+\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E} \end{pmatrix} . \tag{4.64}
$$

To further transform ψ_p so that the integral (4.61) can be solved, the Jacobi-Anger expansion (see e.g. Ref. [36]) is now required:

$$
e^{iz\cos(\theta)} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta} \, . \tag{4.65}
$$

Here z is any complex-valued function that does not depend on θ and $J_n(z)$ is the n-th Bessel function of the first kind.

Considering the term $\exp\left(i\left[a\cos^2(\theta)+b\sin^2(\theta)\right]\right)$ in more detail, gives

$$
\begin{aligned}
&\exp\left(i\left[a\cos^2(\theta)+b\sin^2(\theta)\right]\right) \\
&= \exp\left(i\left[\frac{a}{2}(1+\cos(2\theta))+\frac{b}{2}(1-\cos(2\theta))\right]\right) \\
&= \exp\left(i\,\frac{a+b}{2}\right)\exp\left(i\frac{a-b}{2}\cos(2\theta)\right) \\
&= \exp\left(i\,\frac{a+b}{2}\right)\sum_{n=-\infty}^{\infty} i^n J_n\left(\frac{a-b}{2}\right)e^{i2n\theta}\ ,
\end{aligned}
\tag{4.66}
$$

where the Jacobi-Anger expansion (4.65) was used in the last step.
If this is applied to ψ_p, the result is:

$$
\begin{aligned}
\psi_p = \frac{\exp\left(i\tilde{\Phi}\right)}{\sqrt{f(\phi)g(\phi)}}\exp\left(i\,\frac{a+b}{2}\right)\sum_{n=-\infty}^{\infty} i^n J_n\left(\frac{a-b}{2}\right)e^{i2n\theta} \\
\begin{pmatrix}
1+\frac{p^2c^2e^{i\theta}\left[\left(e^{i\theta}+e^{-i\theta}\right)A-\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E(E+mc^2)} \\
-\frac{pc\left[\left(e^{i\theta}+e^{-i\theta}\right)A+\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E} \\
\frac{p^2c^2e^{i\theta}\left[\left(e^{i\theta}+e^{-i\theta}\right))A-\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E(E+mc^2)} \\
\frac{pc\,e^{i\theta}}{E+mc^2}+\frac{pc\left[\left(e^{i\theta}+e^{-i\theta}\right)A+\left(e^{i\theta}-e^{-i\theta}\right)B\right]}{4E}
\end{pmatrix}\ .
\end{aligned}
\tag{4.67}
$$

The above spinor has the new phase

$$
\begin{aligned}
\tilde{\Phi} &= -\frac{1}{\hbar}\left[Et-p\{x\cos(\theta)+y\sin(\theta)\}\right] \\
&= -\frac{1}{\hbar}\left[Et-p\,r\cos(\theta-\varphi)\right]\ .
\end{aligned}
\tag{4.68}
$$

Here the spatial coordinates were also converted into cylindrical coordinates, i.e. $r=\sqrt{x^2+y^2}$ and $\varphi=\arctan\left(\frac{y}{x}\right)$.
In the last step of the transformation of ψ (4.61), the integral over θ is now to be interchanged with the sum of the Jacobi-Anger expansion (4.65) of ψ_p, this is possible because the Jacobi-Anger expansion converges uniformly [37].

This results in integrals of the following form:

$$
\begin{aligned}
&\int_0^{2\pi} \exp\left(i\frac{p}{\hbar}r\cos(\theta-\varphi)+im\theta\right)\mathrm{d}\theta \\
&= \int_{-\varphi}^{2\pi-\varphi} \exp\left(i\frac{p}{\hbar}r\cos(\alpha)+im(\alpha+\varphi)\right)\mathrm{d}\alpha \\
&= \exp(im\varphi)\int_{-\varphi}^{2\pi-\varphi} \exp\left(i\frac{p}{\hbar}r\cos(\alpha)+im\alpha\right)\mathrm{d}\alpha \\
&= 2\pi\, i^m \exp(im\varphi) J_m\left(\frac{p}{\hbar}r\right)\,.
\end{aligned}
\tag{4.69}
$$

In the last step of the calculation, the integral representation of the Bessel functions of the first kind was used (see for instance Ref. [38]). It should also be noted that m is an integer in this case.
Integration can now be performed in equation (4.61) and the result obtained is:

$$
\begin{aligned}
\psi &= \frac{\exp\left(-i\frac{Et}{\hbar}\right)}{\sqrt{f(\phi)g(\phi)}}\exp\left(i\,\frac{a+b}{2}\right)\sum_{n=-\infty}^{\infty} i^n J_n\left(\frac{a-b}{2}\right) i^{2n}e^{i2n\varphi} \\
&\quad\begin{pmatrix}
J_{2n}(\lambda)+\frac{p^2c^2\left[i^2e^{i2\varphi}J_{2n+2}(\lambda)(A-B)+J_{2n}(\lambda)(A+B)\right]}{4E(E+mc^2)} \\
-\frac{pc\left[ie^{i\varphi}J_{2n+1}(\lambda)(A+B)+i^{-1}e^{-i\varphi}J_{2n-1}(\lambda)(A-B)\right]}{4E} \\
\frac{p^2c^2\left[i^2e^{i2\varphi}J_{2n+2}(\lambda)(A-B)+J_{2n}(\lambda)(A+B)\right]}{4E(E+mc^2)} \\
\frac{pc\,ie^{i\varphi}J_{2n+1}(\lambda)}{E+mc^2}+\frac{pc\left[ie^{i\varphi}J_{2n+1}(\lambda)(A+B)+i^{-1}e^{-i\varphi}J_{2n-1}(\lambda)(A-B)\right]}{4E}
\end{pmatrix} \\
&= \frac{\exp\left(-i\frac{Et}{\hbar}\right)}{\sqrt{f(\phi)g(\phi)}}\exp\left(i\,\frac{a+b}{2}\right)\sum_{n=-\infty}^{\infty} (-i)^n J_n\left(\frac{a-b}{2}\right) e^{i2n\varphi} \\
&\quad\begin{pmatrix}
J_{2n}(\lambda)+\frac{p^2c^2\left[-e^{i2\varphi}J_{2n+2}(\lambda)(A-B)+J_{2n}(\lambda)(A+B)\right]}{4E(E+mc^2)} \\
-\frac{pc\left[ie^{i\varphi}J_{2n+1}(\lambda)(A+B)-ie^{-i\varphi}J_{2n-1}(\lambda)(A-B)\right]}{4E} \\
\frac{p^2c^2\left[-e^{i2\varphi}J_{2n+2}(\lambda)(A-B)+J_{2n}(\lambda)(A+B)\right]}{4E(E+mc^2)} \\
\frac{pc\,ie^{i\varphi}J_{2n+1}(\lambda)}{E+mc^2}+\frac{pc\left[ie^{i\varphi}J_{2n+1}(\lambda)(A+B)-ie^{-i\varphi}J_{2n-1}(\lambda)(A-B)\right]}{4E}
\end{pmatrix},
\end{aligned}
\tag{4.70}
$$

where $\lambda = \frac{p}{\hbar}r$.
By slightly adapting the calculation, however, the case can also be treated in which the electron vortex beam initially has an OAM with the quantum number l. The procedure is again similar to that in Ref. [18]. In this case, a superposition of the form

$$
\psi_l = \frac{1}{2\pi i^l}\int_0^{2\pi} e^{il\theta}\psi_p\,\mathrm{d}\theta \tag{4.71}
$$

is considered. If the above procedure is repeated, the additional term $e^{il\theta}$ only influences the integer m in the integrals of the form (4.69) and the following result is finally obtained:

$$\boxed{\psi_l = \frac{\exp\left(-i\frac{Et}{\hbar}\right)}{\sqrt{f(\phi)g(\phi)}} \exp\left(i\,\frac{a+b}{2}\right) \sum_{n=-\infty}^{\infty} (-i)^n J_n\left(\frac{a-b}{2}\right) e^{i(2n+l)\varphi} \begin{pmatrix} J_{2n+l}(\lambda) + \frac{p^2c^2\left[-e^{i2\varphi}J_{2n+2+l}(\lambda)(A-B)+J_{2n+l}(\lambda)(A+B)\right]}{4E(E+mc^2)} \\ -\frac{pc\left[ie^{i\varphi}J_{2n+1+l}(\lambda)(A+B)-ie^{-i\varphi}J_{2n-1+l}(\lambda)(A-B)\right]}{4E} \\ \frac{p^2c^2\left[-e^{i2\varphi}J_{2n+2+l}(\lambda)(A-B)+J_{2n+l}(\lambda)(A+B)\right]}{4E(E+mc^2)} \\ \frac{pc\,ie^{i\varphi}J_{2n+1+l}(\lambda)}{E+mc^2} + \frac{pc\left[ie^{i\varphi}J_{2n+1+l}(\lambda)(A+B)-ie^{-i\varphi}J_{2n-1+l}(\lambda)(A-B)\right]}{4E} \end{pmatrix}}\,. \tag{4.72}$$

There are two things to note. First, the result (4.72) can also be generalised relatively easily to the case $p^z = \text{const.} \neq 0$. To do this, the term Et in the phase is replaced by $Et - p^z z$ and $k \cdot p = k\left(\frac{E}{c} - p^z\right)$ is used. In addition, the spinor u_p then changes to

$$u_p = \begin{pmatrix} 1 \\ 0 \\ \frac{p^z}{E+mc^2} \\ \frac{cp\,e^{i\theta}}{E+mc^2} \end{pmatrix}, \tag{4.73}$$

which results in additional terms proportional to p^z in the final result. The general procedure remains the same.

Second, only one of the four constant spinors u_p, which solve the equation $(\not{p} - mc)u_p = 0$ (see equations (2.24) and (2.25)), was used here. Using one of the other three spinors u_p would result in similar solutions for an electron vortex beam with OAM.

4.3.1 Comparison with the free and the electromagnetic case

When comparing the result (4.72) with the solution of the free Dirac equation for an electron vortex beam (3.26), it becomes immediately apparent that both solutions are rotationally symmetric. In the case of the free electron vortex beam, the symmetry axis corresponds to the propagation direction of the electron beam, whereas, in the case of the electron vortex beam interacting with the gravitational wave, the symmetry axis is the propagation direction of the gravitational wave. According to Noether's theorem [39], every symmetry leads to a conserved quantity, thus the cylindrical coordinates, which reflect the rotational symmetry, are directly associated with the

conservation of angular momentum and thus OAM.
On closer comparison of the two solutions, equation (3.26) can also be recognized in the first summands of the 1st and 4th components of the vector part of equation (4.72), apart from the additional terms that are added as a result of the gravitational wave, since these two summands have the OAM quantum number l and $l+1$ respectively. The reason for this is that the two solutions (4.72) and (3.26) are based on the same constant spinor u_p (4.58). In fact, although the solution (3.26) was found by transforming the free Dirac equation to cylindrical coordinates, this solution could also have been found using a procedure analogous to that in section 4.3. This has the direct consequence that the free electron vortex beam (3.26) can be recovered from the result (4.72) by "switching off" the gravitational wave, i.e. setting $f(\phi) = g(\phi) = 1$ or $A = B = a = b = 0$.
Comparing the solution (4.72) with the solution for an electron vortex beam interacting with an electromagnetic plane wave in equations (6) and (7) of Ref. [18], more differences can be recognised in the general structure of the two solutions, apart from the series representation, which is similar in both cases, than one might have expected due to the fact that the procedure for obtaining these solutions is almost identical. First, it is noticeable that in the case of the gravitational wave, the Lorentz transformation $\mathcal{R}_r$ cannot be extracted from the integral of the superposition and thus also not from the series, as is the case in Ref. [18]. The reason for this is found when comparing the two Lorentz transformations $\mathcal{R}_r$ in equations (4.54) and (4.55). This is because while the Lorentz transformation in the case of the electromagnetic wave is independent of the momenta p^x and p^y, the Lorentz transformation in the case of the gravitational wave depends directly on these two momenta, which means that the Lorentz transformation must be taken into account in the integral (4.61). This dependence of the Lorentz transformation on the two momenta is, as already described in section 4.2.3, a direct consequence of the fact that the term $\hbar H_i^{\ \mu} \gamma^i \partial_\mu \Psi \gamma_2 \gamma_1$ depends on the partial derivatives ∂_x and ∂_y. Alternatively, this could also have been realised with the identification $-eA_i \mathrel{\hat{=}} p_\mu H_i^{\ \mu}$ made in section 4.2.3.
It is noticeable that in the solution (4.72), compared to Ref. [18], all n of the series representation, apart from those in $(-i)^n J_n\left(\frac{a-b}{2}\right)$, which is the term corresponding to $i^n J_n(f_0)$ in Ref. [18], have been replaced by $2n$. The reason for this can also be found in the identification $-eA_i \mathrel{\hat{=}} p_\mu H_i^{\ \mu}$, which is equivalent to generalised "vector potential" $-H_i^{\ \mu}$ coupling to initial kinetic momentum p_μ. As a result, the phase S of the plane wave solution of the Dirac equation in the presence of a gravitational wave depends on $(p^x)^2$ and $(p^y)^2$ and not only on p^x and p^y as in the case of the Volkov state (2.29). This results in double angle functions when applying the Jacobi Anger ex-

pansion to the phase S (see equation (4.66)), while the application of the Jacobi Anger expansion to the phase of the Volkov state does not result in double angle functions (see ref. [18]).
In conclusion, it can be stated that within the framework of the Dirac equation, the dynamics of electron vortex beams in an electromagnetic plane wave differ from that in a gravitational plane wave, while in the case of plane wave solutions of the two Dirac equations, the dynamics are under the identification $-eA_i \hat{=} p_\mu H_i^{\ \mu}$ identical. The reason for this is that the gravitational wave does not simply exert a force on the electron vortex beam like the electromagnetic wave, but deforms spacetime. The deformation of spacetime mainly results in the coupling of the generalised "vector potential" $-H_i^{\ \mu}$ to the initial kinetic momentum p_μ, which is expressed through the additional partial derivatives in the term $i\hbar H_i^{\ \mu}\gamma^i\partial_\mu\psi$ and has an additional influence on the electron vortex beam. Furthermore, it must also be noted at this point that the deformation of spacetime is again additionally expressed in the solution (4.72) by the density $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$.

5 Classical Spinor solution for an electron in a gravitational plane wave

The final step to complete this discussion of the electron dynamics in a gravitational plane wave is to present a classical equivalent to the Dirac spinor (4.51) described above, i.e. one that is not based on quantum mechanics. Doing so will indicate that the RDI technique can also be applied to classical systems, further emphasizing the versatility of the method for solving dynamical problems. Spacetime Algebra (STA) as formulated by David Hestenes is used for this purpose, which is explained in section 2.3.

5.1 STA in curved spacetime

The subsequent additions to STA taking curved spacetime into account are based on Refs. [19, 40, 41].
The equations of motion in curved spacetime are the so-called geodesic equations [42]:

$$0 = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Lambda^\mu_{\ \alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = \frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} + \Lambda^\mu_{\ \alpha\beta}v^\alpha v^\beta \,. \tag{5.1}$$

Here τ is the proper time, $\Lambda^\mu_{\ \alpha\beta}$ are the Christoffel symbols and v^μ is the proper velocity.
In equation (185) of Ref. [19], these are translated into a rotor equation, which reads as follows:

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = -\frac{1}{2}\omega(v)R\,. \tag{5.2}$$

$\omega(v)$ is a bivector that describes the surrounding geometry of spacetime. In comparison to equation (185) of Ref. [19], $\Omega = 0$ was set, which is possible, because Ω is a bivector describing the electromagnetic fields in which the electron moves, but here only the interaction with a gravitational wave is to be considered. The complete derivation of this equation can be found in sections 5.4 and 5.5 of Ref. [19].
The aim is now to solve this equation for an electron in a gravitational plane wave. To do this, however, one must first describe how the bivector $\omega(v)$ can be determined, which requires reformulating STA in curved spacetime.
Using Ref. [40], it becomes clear that the metric $g_{\mu\nu}$ of curved spacetime can be written as follows:

$$g_{\mu\nu} = g_\mu \cdot g_\nu\,. \tag{5.3}$$

A comparison with equation (3.1) shows that the so-called coordinate frame $\{g_\mu\}$ corresponds to the transformed gamma matrices $\{\gamma_\mu\}$ in the case of the

Dirac equation. This also makes it clear that a distinction between Latin and Greek indices to distinguish the manifold from the tangent space is not necessary. In the case of STA, the transformation between $\{\gamma_\mu\}$ and $\{g_\mu\}$ is carried out using the so-called fiducial tensor h^μ_ν according to:

$$g_\mu = h^\nu{}_\mu \gamma_\nu \, . \tag{5.4}$$

In this case, a comparison with equation (3.3) shows that h^μ_ν corresponds to the tetrads e^a_μ from before. If one uses orthogonal coordinates, which will be the case in the following in the form of Cartesian coordinates, one can also define a tensor h_μ from h^μ_ν as

$$h^\mu_\nu = h_\nu \delta^\mu_\nu \, , \tag{5.5}$$

which in the end simply corresponds to the diagonal elements of h^μ_ν. Here δ^μ_ν is the usual Kronecker delta. This defines the most important quantities that are subsequently required to determine $\omega(v)$; for a more detailed explanation of STA in curved spacetime, please refer to Refs. [19, 40].
According to equation (35) of Ref. [41],

$$\omega(v) = v^\mu \omega_\mu \tag{5.6}$$

now applies. Here, v^μ is the proper velocity, which is calculated using the geodesic equations. The ω_μ, on the other hand, is a bivector that is defined as follows according to equation (30) of Ref. [40]:

$$\omega_\mu = \gamma_\mu \wedge \gamma^a h_a^{-1} \partial_a h_\mu \, . \tag{5.7}$$

It should be noted that ω_μ corresponds to the spinorial connections $\mathbf{\Omega}_\mu$ from equation (3.10). This is because, as can be seen from equations (119) and (120) of Ref. [40], the covariant derivative of a spinor ψ or a multivector M is:

$$\begin{aligned} D_\mu \psi &= \left(\partial_\mu + \frac{1}{2} \omega_\mu \right) \psi \, , \\ D_\mu M &= \partial_\mu M + \omega_\mu \times M \, . \end{aligned} \tag{5.8}$$

Therefore, one can make the identification $\mathbf{\Omega}_\mu = \frac{1}{2}\omega_\mu$.

5.1.1 Equivalence between the Rotor equation and the geodesic equations

In the following, it will be shown that the rotor equation (5.2) is actually equivalent to the geodesic equations (5.1). Although the notation introduced

in the previous section 5.1 is very compact and, as will be shown later, helpful, only in this section the original notation will be used once again, as the following calculation is based on a great deal of index calculation, which is clearer in the original notation. This means that in this section, Latin indices again refer to the tangent space and Greek indices to the spacetime manifold. Thus the $\{g_\mu\}$ from the previous section are replaced with the $\{\gamma_\mu\}$ and the basis of the tangent space is now $\{\gamma_a\}$. The transformation between the tangent space and the manifold is therefore carried out using the tetrads e^a_μ according to equation (3.3).
Starting with equation (2.58) and differentiating it with respect to the proper time, the result is:

$$\dot{v} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(R\gamma_0\tilde{R}\right) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}\,. \tag{5.9}$$

If the rotor equation (5.2) is now inserted into this result and the fact that $\omega(v)$ is a bivector is taken into account, from which $\tilde{\omega}(v) = -\omega(v)$ follows according to equation (2.48), this simplifies to:

$$\begin{aligned}
\dot{v} &= -\frac{1}{2}\omega(v)R\gamma_0\tilde{R} - \frac{1}{2}R\gamma_0\tilde{R}\tilde{\omega}(v) \\
&= -\frac{1}{2}\left[\omega(v)\,v + v\,\tilde{\omega}(v)\right] \\
&= -\frac{1}{2}\left[\omega(v)\,v - v\,\omega(v)\right] \\
&= -\omega(v)\cdot v\,,
\end{aligned} \tag{5.10}$$

$$\Rightarrow \dot{v} + \omega(v)\cdot v = 0\,. \tag{5.11}$$

Both the definition of the proper velocity through the rotor (2.58) and the definition of the inner product for a vector and a bivector (2.52) were used. This already shows that the rotor equation is equivalent to equation (5.11). The second step is to show that equation (5.11) is equivalent to the geodesic equations. For this purpose, the definition of $\omega(v)$ (5.6), the relationship $\mathbf{\Omega}_\mu = \frac{1}{2}\omega_\mu$, the definition of $\mathbf{\Omega}_\mu$ according to equation (3.10), the relation (2.44) and the definition of the Dirac algebra (2.45) are used below. In addition, η_{ij} denotes the Minkowski metric as usual.

$$
\begin{aligned}
0 &= \dot{v} + \omega(v) \cdot v \\
&= \dot{v} + v^\mu v^\nu \omega_\mu \cdot \gamma_\nu \\
&= \dot{v} + v^\mu v^\nu \frac{1}{2} \left(\omega_\mu \gamma_\nu - \gamma_\nu \omega_\mu\right) \\
&= \dot{v} + v^\mu v^\nu \left(\boldsymbol{\Omega}_\mu \gamma_\nu - \gamma_\nu \boldsymbol{\Omega}_\mu\right) \\
&= \dot{v} + v^\mu v^\nu \frac{1}{2} \Omega_{ij\mu} \left(\sigma^{ij} \gamma_\nu - \gamma_\nu \sigma^{ij}\right) \\
&= \dot{v} + v^\mu v^\nu \frac{1}{4} \Omega_{ij\mu} \left((\gamma^i \wedge \gamma^j)\gamma_\nu - \gamma_\nu(\gamma^i \wedge \gamma^j)\right) \\
&= \dot{v} + v^\mu v^\nu \frac{1}{4} \Omega^a_{j\mu} \eta^{jb} \left((\gamma_a \wedge \gamma_b)\gamma_\nu - \gamma_\nu(\gamma_a \wedge \gamma_b)\right) \\
&= \dot{v} + v^\mu v^\nu \frac{1}{4} \Omega^a_{j\mu} \eta^{jb} e^c_\nu \left((\gamma_a \wedge \gamma_b)\gamma_c - \gamma_c(\gamma_a \wedge \gamma_b)\right) \\
&= \dot{v} - v^\mu v^\nu \frac{1}{2} \Omega^a_{j\mu} \eta^{jb} e^c_\nu \gamma_c \cdot (\gamma_a \wedge \gamma_b) \\
&= \dot{v} - v^\mu v^\nu \frac{1}{2} \Omega^a_{j\mu} \eta^{jb} e^c_\nu \left[(\gamma_c \cdot \gamma_a)\gamma_b - (\gamma_c \cdot \gamma_b)\gamma_a\right] \\
&= \dot{v} - v^\mu v^\nu \frac{1}{2} \Omega^a_{j\mu} \eta^{jb} e^c_\nu \left[\eta_{ca}\gamma_b - \eta_{cb}\gamma_a\right]
\end{aligned}
\tag{5.12}
$$

The next step is to take a closer look at the term $\frac{1}{2}\Omega^a_{j\mu}\eta^{jb}e^c_\nu\left[\eta_{ca}\gamma_b - \eta_{cb}\gamma_a\right]$. It is required to know that according to equation (3.10)

$$
\Omega^i_{j\mu} e^j_\nu = e^i_\sigma \Lambda^\sigma_{\nu\mu} - \partial_\mu e^i_\nu \tag{5.13}
$$

holds. In addition, the following calculation uses the fact that $\Omega_{ij\mu} = -\Omega_{ji\mu}$ applies. This then leads to:

$$
\begin{aligned}
&\frac{1}{2} \Omega^a_{j\mu} \eta^{jb} e^c_\nu \left[\eta_{ca}\gamma_b - \eta_{cb}\gamma_a\right] \\
&= \frac{1}{2} \left(\Omega^a_{j\mu} \eta^{jb} e^c_\nu \eta_{ca}\gamma_b - \Omega^a_{j\mu} \eta^{jb} e^c_\nu \eta_{cb}\gamma_a\right) \\
&= \frac{1}{2} \left(\Omega_{cj\mu} e^c_\nu \gamma^j - \Omega^a_{j\mu} e^c_\nu \delta^j_c \gamma_a\right) \\
&= \frac{1}{2} \left(-\Omega^j_{c\mu} e^c_\nu \gamma_j - \Omega^a_{c\mu} e^c_\nu \gamma_a\right) \\
&= -\Omega^j_{c\mu} e^c_\nu \gamma_j \\
&= -\left(\Lambda^\beta_{\mu\nu} e^j_\beta - \partial_\mu e^j_\nu\right)\gamma_j \\
&= -\left(\Lambda^\beta_{\mu\nu} \gamma_\beta - \partial_\mu \gamma_\nu\right) \ .
\end{aligned}
\tag{5.14}
$$

Inserting this back into equation (5.12), writing out the product rule for $\dot{v}$ and noticing $v^\mu \partial_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau}$, results in:

$$\begin{aligned}
0 &= \dot{v} - v^\mu v^\nu \frac{1}{2}\Omega^a_{j\mu}\eta^{jb}e^c_\nu \left[\eta_{ca}\gamma_b - \eta_{cb}\gamma_a\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(v^\mu\gamma_\mu\right) + v^\mu v^\nu \left(\Lambda^\beta_{\mu\nu}\gamma_\beta - \partial_\mu\gamma_\nu\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(v^\mu\right)\gamma_\mu + v^\mu\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\gamma_\mu\right) + v^\mu v^\nu \Lambda^\beta_{\mu\nu}\gamma_\beta - v^\nu\frac{\mathrm{d}}{\mathrm{d}\tau}(\gamma_\nu) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(v^\mu\right)\gamma_\mu + v^\alpha v^\beta \Lambda^\mu_{\alpha\beta}\gamma_\mu \,.
\end{aligned} \tag{5.15}$$

However, since the $\{\gamma_\mu\}$ are generally not equal to zero, one gets:

$$0 = \frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} + \Lambda^\mu_{\alpha\beta}v^\alpha v^\beta \,. \tag{5.16}$$

These, though, are exactly the geodesic equations (5.1), which finally proves the equivalence between the rotor equation (5.2) and the geodesic equations (5.1).

5.2 Solution of the Rotor equation for an electron in a gravitational plane wave

The notation from section 5.1 is now used in this paragraph, which deals with solving the rotor equation (5.2) in presence of a gravitational plane wave. Using (4.15), the line element of the gravitational wave is:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - f^2(\phi)\mathrm{d}x^2 - g^2(\phi)\mathrm{d}y^2 - \mathrm{d}z^2 \,. \tag{5.17}$$

If this is now compared with equation (33) of Ref. [40], the result is:

$$h_t = 1\,, \quad h_x = f(\phi)\,, \quad h_y = g(\phi)\,, \quad h_z = 1\,. \tag{5.18}$$

Alternatively, this could have been determined using the tetrads (4.16) as well. Equation (5.18) can now be used to determine the bivectors from equation (5.7):

$$\begin{aligned}
&\omega_t = 0\,, \\
&\omega_x = kf'(\phi)\gamma_1 \wedge (\gamma^0 - \gamma^3) = kf'(\phi)\gamma_1 \wedge (\gamma_0 + \gamma_3) = kf'(\phi)\gamma_1(\gamma_0 + \gamma_3)\,, \\
&\omega_y = kg'(\phi)\gamma_2 \wedge (\gamma^0 - \gamma^3) = kg'(\phi)\gamma_2 \wedge (\gamma_0 + \gamma_3) = kg'(\phi)\gamma_2(\gamma_0 + \gamma_3)\,, \\
&\omega_z = 0\,.
\end{aligned} \tag{5.19}$$

Once again primed functions denote derivatives with respect to ϕ. It should also be noted that the outer product can be omitted in the last transformation step, as the $\{\gamma_\mu\}$ are orthonormal. The comparison with equation (4.18), in fact, shows that $\mathbf{\Omega}_\mu = \frac{1}{2}\omega_\mu$.
This means that $\omega(v) = v^x\omega_x + v^y\omega_y$ is already half determined. However, before v^x and v^y can be determined using the geodesic equations, a relationship between ϕ and τ must first be established. To do so, a similar argument as in section 3 of Ref. [43] or in Ref. [44] is applied. $\omega(v)$ is inserted into the rotor equation in the above general form:

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}\tau} &= -\frac{1}{2}(v^x\omega_x + v^y\omega_y)R \\ &= -\frac{k}{2}\left(v^x f'(\phi)\gamma_1(\gamma_0+\gamma_3) + v^y g'(\phi)\gamma_2(\gamma_0+\gamma_3)\right)R\,.\end{aligned} \tag{5.20}$$

This is now multiplied by the wave vector $\bar{k} = k^\mu\gamma_\mu = k(\gamma_0+\gamma_3) = k(\gamma^0 - \gamma^3)$, the bar notation is used here to distinguish the wave vector from its magnitude, which results in:

$$\frac{\mathrm{d}\bar{k}R}{\mathrm{d}\tau} = \frac{\mathrm{d}R\bar{k}}{\mathrm{d}\tau} = 0\,. \tag{5.21}$$

This result is obtained because:

$$(\gamma_0+\gamma_3)(\gamma_0+\gamma_3) = (\gamma_0+\gamma_3)\cdot(\gamma_0+\gamma_3) + (\gamma_0+\gamma_3)\wedge(\gamma_0+\gamma_3) = 0\,. \tag{5.22}$$

Here the inner product evaluates to zero because the $\{\gamma_\mu\}$ are orthonormal and outer product evaluates to zero because of the asymmetry of the outer product. Thus, the wave vector $\bar{k}_{rest} = R\bar{k}\tilde{R}$ is constant in the rest frame, since:

$$\frac{\mathrm{d}\bar{k}_{rest}}{\mathrm{d}\tau} = \dot{R}\bar{k}\tilde{R} + R\bar{k}\dot{\tilde{R}} = 0\,. \tag{5.23}$$

However, it follows from this that the rest frame frequency ω_{rest} is also constant, which, taking into account the fact that Lorentz transformations preserve the scalar product (here inner product), establishes the desired relationship between ϕ and τ:

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= \dot{\phi} = \frac{\mathrm{d}\bar{k}_{rest}\cdot x_{rest}}{\mathrm{d}\tau} = \frac{\mathrm{d}k_{rest}c\tau}{\mathrm{d}\tau} = k_{rest}c = \omega_{rest} = \text{const.}\,, \\ &\Rightarrow \phi = \omega_{rest}\tau\,.\end{aligned} \tag{5.24}$$

The geodesic equations (5.1) can now be solved in the present case. First the required Christoffel symbols are calculated:

$$\begin{aligned}
\Lambda^x_{\ xt} &= \Lambda^x_{\ tx} = k\frac{f'(\phi)}{f(\phi)}\,, \quad \Lambda^t_{\ xx} = kf'(\phi)f(\phi)\,,\\
\Lambda^x_{\ xz} &= \Lambda^x_{\ zx} = -k\frac{f'(\phi)}{f(\phi)}\,, \ \Lambda^z_{\ xx} = kf'(\phi)f(\phi)\,,\\
\Lambda^y_{\ yt} &= \Lambda^x_{\ ty} = k\frac{g'(\phi)}{g(\phi)}\,, \quad \Lambda^t_{\ yy} = kg'(\phi)g(\phi)\,,\\
\Lambda^y_{\ yz} &= \Lambda^y_{\ zy} = -k\frac{g'(\phi)}{g(\phi)}\,, \ \Lambda^z_{\ yy} = kg'(\phi)g(\phi)\,.
\end{aligned} \tag{5.25}$$

The gesodesic equations (5.1) are therefore as follows, where dotted functions are derivatives with respect to the proper time τ and primed functions are derivatives with respect to ϕ:

$$\begin{aligned}
\text{(I)}\ 0 &= \dot{v}^t + kf'(\phi)f(\phi)(v^x)^2 + kg'(\phi)g(\phi)(v^y)^2\,,\\
\text{(II)}\ 0 &= \dot{v}^x + v^x 2k\frac{f'(\phi)}{f(\phi)}(v^t - v^z)\,,\\
\text{(III)}\ 0 &= \dot{v}^y + v^y 2k\frac{g'(\phi)}{g(\phi)}(v^t - v^z)\,,\\
\text{(IV)}\ 0 &= \dot{v}^z + kf'(\phi)f(\phi)(v^x)^2 + kg'(\phi)g(\phi)(v^y)^2\,.
\end{aligned} \tag{5.26}$$

At first glance, it becomes clear that $\dot{v}^t = \dot{v}^z$, from which follows $v^t = v^z + \text{const.} = v^z + b$. If one now inserts this into equation (II), the result is:

$$\text{(II}'\text{)}\ 0 = \dot{v}^x + v^x 2k\frac{f'(\phi)}{f(\phi)}b = \omega_{rest}v'^x + v^x 2k\frac{f'(\phi)}{f(\phi)}b\,. \tag{5.27}$$

If the initial conditions are now chosen such that $b = \frac{\omega_{rest}}{k}$, the result for the proper velocity in x-direction is:

$$v^x = \frac{v^x(\phi_0)f^2(\phi_0)}{f^2(\phi)} = \frac{v_0^x}{f^2(\phi)}\,. \tag{5.28}$$

And analogously for v^y:

$$v^y = \frac{v^y(\phi_0)g^2(\phi_0)}{g^2(\phi)} = \frac{v_0^y}{g^2(\phi)}\,. \tag{5.29}$$

Although v^t and v^z are no longer needed for determining $\omega(v)$, the differential equations for these should also be solved for the sake of completeness. If the expressions for v^x and v^y are inserted into (IV), the following results:

$$\text{(IV}'\text{)}\ 0 = \omega_{rest}v'^z + k\frac{f'(\phi)}{f^3(\phi)}(v_0^x)^2 + k\frac{g'(\phi)}{g^3(\phi)}(v_0^y)^2\,. \tag{5.30}$$

Again this can be integrated immediately:

$$v^z = \frac{k}{2\omega_{rest}} \left(\frac{(v_0^x)^2}{f^2(\phi)} + \frac{(v_0^y)^2}{g^2(\phi)} \right) + v_0^z \,. \tag{5.31}$$

Here v_0^z is the constant of integration.
From the relationship between v^t and v^z then follows:

$$v^t = \frac{k}{2\omega_{rest}} \left(\frac{(v_0^x)^2}{f^2(\phi)} + \frac{(v_0^y)^2}{g^2(\phi)} \right) + v_0^z + \frac{\omega_{rest}}{k} \,. \tag{5.32}$$

Finally, it can now be checked whether the choice $b = \frac{\omega_{rest}}{k}$ was correct by checking whether $v^2 = v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu = c^2$ applies. Here, $g_{\mu\nu}$ is the metric of the gravitational plane wave (4.15). It holds:

$$\begin{aligned} v^2 &= v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu \\ &= (v^t)^2 - f^2(\phi)(v^x)^2 - g^2(\phi)(v^y)^2 - (v^z)^2 \\ &= (v^z)^2 + 2\frac{\omega_{rest}}{k} v^z + \frac{\omega_{rest}^2}{k^2} - f^2(\phi)(v^x)^2 - g^2(\phi)(v^y)^2 - (v^z)^2 \\ &= \frac{(v_0^x)^2}{f^2(\phi)} + \frac{(v_0^y)^2}{g^2(\phi)} + 2v_0^z \frac{\omega_{rest}}{k} + \frac{\omega_{rest}^2}{k^2} - \frac{(v_0^x)^2}{f^2(\phi)} - \frac{(v_0^y)^2}{g^2(\phi)} \\ &= 2v_0^z \frac{\omega_{rest}}{k} + \frac{\omega_{rest}^2}{k^2} \,. \end{aligned} \tag{5.33}$$

With $v_0^z = \frac{c^2 k}{2\omega_{rest}} - \frac{\omega_{rest}}{2k}$, $v^2 = c^2$ then applies. This not only shows that the choice $b = \frac{\omega_{rest}}{k}$ was correct, but also that this choice was mandatory for v^2 to be generally constant in the first place. If b had been chosen differently, v^x and v^y would depend on different powers of the functions $f(\phi)$ and $g(\phi)$ and v^2 would therefore not be constant in general.
Overall, this results in the following expression for $\omega(v)$:

$$\begin{aligned} \omega(v) &= k \left(v_0^x \frac{f'(\phi)}{f^2(\phi)} \gamma_1 \wedge (\gamma_0 + \gamma_3) + v_0^y \frac{g'(\phi)}{g^2(\phi)} \gamma_2 \wedge (\gamma_0 + \gamma_3) \right) \\ &= k \left(v_0^x \frac{f'(\phi)}{f^2(\phi)} \gamma_1 (\gamma_0 + \gamma_3) + v_0^y \frac{g'(\phi)}{g^2(\phi)} \gamma_2 (\gamma_0 + \gamma_3) \right) \\ &= k \left(v_0^x \frac{f'(\phi)}{f^2(\phi)} (\gamma^0 - \gamma^3)\gamma^1 + v_0^y \frac{g'(\phi)}{g^2(\phi)} (\gamma^0 - \gamma^3)\gamma^2 \right) \,. \end{aligned} \tag{5.34}$$

It is possible to transform the outer product into the geometric product, as the $\{\gamma_\mu\}$ are orthonormal. If this expression for $\omega(v)$ is now inserted into the rotor equation (5.2), the following differential equation for the rotor R is obtained, taking into account the chain rule:

$$\frac{\mathrm{d}R}{\mathrm{d}\phi} = -\frac{k}{2\omega_{rest}} \left(v_0^x \frac{f'(\phi)}{f^2(\phi)} (\gamma^0 - \gamma^3)\gamma^1 + v_0^y \frac{g'(\phi)}{g^2(\phi)} (\gamma^0 - \gamma^3)\gamma^2 \right) R \,. \tag{5.35}$$

The differential equation (5.35) can be solved using an exponential ansatz and the result is:

$$R = \exp\left[\frac{\not{k}}{2\omega_{rest}}\left(v_0^x\left\{\frac{1}{f(\phi)} - \frac{1}{f(\phi_0)}\right\}\gamma^1 + v_0^y\left\{\frac{1}{g(\phi)} - \frac{1}{g(\phi_0)}\right\}\gamma^2\right)\right] R(\phi_0)\,. \tag{5.36}$$

Here, as before, $\not{k} = k^\mu\gamma_\mu$. Furthermore, $R(\phi_0)$ is an arbitrary constant rotor. If one again uses the fact that $(\gamma^0 - \gamma^3)^2 = 0$, the exponential can be written out as follows:

$$\boxed{\begin{aligned} R = \Bigg[1 + \frac{\not{k}}{2\omega_{rest}}\Bigg(& v_0^x\left\{\frac{1}{f(\phi)} - \frac{1}{f(\phi_0)}\right\}\gamma^1 \\ & + v_0^y\left\{\frac{1}{g(\phi)} - \frac{1}{g(\phi_0)}\right\}\gamma^2\Bigg)\Bigg] R(\phi_0)\,. \end{aligned}} \tag{5.37}$$

This is the final result for the rotor of a point particle in a gravitational plane wave. If the first part is now identified as the Lorentz transformation $R_r = \left[1 + \frac{\not{k}}{2\omega_{rest}}\left(v_0^x\left\{\frac{1}{f(\phi)} - \frac{1}{f(\phi_0)}\right\}\gamma^1 + v_0^y\left\{\frac{1}{g(\phi)} - \frac{1}{g(\phi_0)}\right\}\gamma^2\right)\right]$ acting on the constant spinor $R(\phi_0)$, this result can also be written as follows:

$$R = R_r R(\phi_0) \tag{5.38}$$

To get back the solutions of the geodesic equations ((5.28), (5.29), (5.32), (5.31)) via equation (2.58), the constant spinor $R(\phi_0)$ takes the form

$$R(\phi_0) = \sqrt{c}\,\exp\left[\frac{\not{k}}{2\omega_{rest}}\left(v_0^x\frac{1}{f(\phi_0)}\gamma^1 + v_0^y\frac{1}{g(\phi_0)}\gamma^2\right)\right] \tag{5.39}$$

and in this case $\frac{\omega_{rest}}{k} = c$ and $v_0^z = 0$ holds. Equation (5.37) then takes the form:

$$R = \sqrt{c}\left[1 + \frac{\not{k}}{2\omega_{rest}}\left(v_0^x\frac{1}{f(\phi)}\gamma^1 + v_0^y\frac{1}{g(\phi)}\gamma^2\right)\right]\,. \tag{5.40}$$

It is important to note that a renormalization of the rotors has occurred, such that the relation $R\tilde{R} = c$ now applies instead of $R\tilde{R} = 1$. This adjustment is due to the fact that the calculations are not being performed in natural units, where $c = 1$.

5.2.1 Comparison with the Quantum case

In the next step, the rotor from equation (5.37) shall now be compared with the Hestenes-Dirac spinor from equation (4.47). Equation (4.48) is particularly helpful here.

First of all, it is noticeable that the rotor (5.37) has the same structure as the matrix spinor (4.47), consisting of a free solution that undergoes an additional Lorentz transformation due to the gravitational wave. Furthermore, the comparison of R_r with $\mathcal{R}_r$ also shows strong similarities. However, two decisive differences can also be recognised. First, the Lorentz transformation of the Rotor R_r contains the proper velocity and the rest frame frequency ω_{rest} instead of the momentum p^μ and the scalar product $k \cdot p$ in the case of $\mathcal{R}_r$ and second, for the Lorentz transformation of the Hestenes-Dirac spinor $\mathcal{R}_r$ applies $f(\phi_0) = g(\phi_0) = 1$.
If one now takes a closer look at the first of these two differences, one realises that there is actually no real difference. This is because $p^\mu = mv^\mu(\phi_0)$ and $\omega_{rest} = k_\mu v^\mu(\phi_0)$ follows from equation (5.24). Here p^μ denotes the initial kinetic momentum of the electron, as it also occurs in the Lorentz transformation $\mathcal{R}_r$ (4.55). From this one can obtain directly:

$$\frac{v^\mu(\phi_0)}{\omega_{rest}} = \frac{m\, v^\mu(\phi_0)}{m\, \omega_{rest}} = \frac{p^\mu}{k^\mu p_\mu} = \frac{p^\mu}{k \cdot p} \,. \tag{5.41}$$

The Lorentz transformation of the rotor R_r can therefore also be written like this:

$$\begin{aligned} R_r &= \left[1 + \frac{\not{k}}{2k \cdot p}\left(p^x \left\{\frac{f^2(\phi_0)}{f(\phi)} - \frac{f^2(\phi_0)}{f(\phi_0)}\right\}\gamma^1 + p^y \left\{\frac{g^2(\phi_0)}{g(\phi)} - \frac{g^2(\phi_0)}{g(\phi_0)}\right\}\gamma^2\right)\right] \\ &= \left[1 + \frac{\not{k}}{2k \cdot p}\left(p^x \left\{\frac{f^2(\phi_0)}{f(\phi)} - f(\phi_0)\right\}\gamma^1 + p^y \left\{\frac{g^2(\phi_0)}{g(\phi)} - g(\phi_0)\right\}\gamma^2\right)\right] . \end{aligned} \tag{5.42}$$

The second difference, on the other hand, cannot be eliminated so easily and it is generally true that the Lorentz transformations of the Hestenes-Dirac spinor and the rotor differ by the spinor

$$\frac{\not{k}}{2k \cdot p}\left(p^x \left\{\frac{1 - f^2(\phi_0)}{f(\phi)} + f(\phi_0) - 1\right\}\gamma^1 + p^y \left\{\frac{1 - g^2(\phi_0)}{g(\phi)} + g(\phi_0) - 1\right\}\gamma^2\right) . \tag{5.43}$$

However, if one generally sets $f(\phi_0) = g(\phi_0) = 1$, this difference also disappears, and the equality $R_r = \mathcal{R}_r$ holds. The case $R_r = \mathcal{R}_r$ means that the particle dynamics of the electron are almost identical in the classical and quantum mechanical cases, since R_r respectively $\mathcal{R}_r$ are the Lorentz transformations that connect the rest frame of the electron with the laboratory frame.
Overall, it can therefore be stated that the classical solution in the form of the rotor (5.37) corresponds, under the constraint $f(\phi_0) = g(\phi_0) = 1$,

to the quantum mechanical solution in the form of the matrix Dirac spinor (4.47), apart from the additional phase $S = \int \frac{\mathrm{d}\phi}{2p_j k^j} \left[2H^{i\mu}p_i p_\mu + H_l{}^{\mu} H^{l\nu} p_\mu p_\nu\right]$ and the scaling $\frac{1}{\sqrt{f(\phi)g(\phi)}}$ of the matrix Dirac spinor. This correspondence is striking, since it means that an electron in a gravitational wave behaves almost identically in classical and quantum mechanical terms, where it is important to note that the same correspondence can be observed between the rotor and the Hestens-Dirac spinor in the case of an electromagnetic plane wave [44]. Furthermore, it is also possible to understand why the classical solution does not have a phase compared to the Dirac spinor. To do this, it is necessary to take a closer look at the two underlying equations of motion. The rotor equation (5.2) is invariant under gauge transformations, since in this equation only one derivative occurs with respect to the invariant proper time, whereby the entire equation is gauge invariant. This in turn means that the solution to the rotor equation, the classical eigenspinor or rotor, is also gauge invariant.

The Dirac equation (4.27) or Hestenes-Dirac equation (4.53), on the other hand, is covariant, as previously noted, but not invariant under gauge transformations. The gauge freedom lies in the choice of tetrads (4.16), as these are not unique, which makes it immediately clear why these equations are not gauge invariant, when considering the transformation of the gamma matrices (3.3). In order to satisfy this gauge freedom, the Dirac or Hestenes-Dirac spinors contain the additional phase $S = \int \frac{\mathrm{d}\phi}{2p_j k^j} \left[2H^{i\mu}p_i p_\mu + H_l{}^{\mu} H^{l\nu} p_\mu p_\nu\right]$.

In addition to this rather mathematical justification for the additional phase in the quantum mechanical case, this can also be justified with a rather simple physical argument. The rotor equation is the spinor equivalent of the geodesic equations, which was shown in Section 5.1.1. Therefore the rotor equation describes classical rigid point particles, which have, at least in Lagrangian and Hamiltonian mechanics, no wave properties and therefore no phase.

It is noteworthy that classical mechanics can also be described by a wave function on a Hilbert space in the so-called Koopman-von Neumann (KvN) formalism. The KvN mechanics is therefore a form of classical statistical mechanics and makes probability statements that are equivalent to Liouville's theorem. A short and very understandable introduction to the KvN formalism can be found, for example, in Ref. [45]. Section 5.1 of Ref. [45] also makes it clear why the absence of the phase in the case of the classical spinor (respectively rotor) is also plausible in the context of the KvN formalism. This is because while in quantum mechanics the phase and amplitude of the wave function are coupled (see equations (77) and (78) of Ref. [45]), phase and amplitude in KvN mechanics are separate from each other (see equations (79) and (80) of Ref. [45]). Thus, the phase in KvN formalism does not influence the particle dynamics.

The Dirac equation or Hestenes-Dirac equation, on the other hand, is a quantum mechanical equation that describes quantum systems, in this case electrons, which have both particle and wave properties and thus also a phase.

Overall, it can be stated that the result found here is in agreement with Ref. [20], because there it was shown in a general way that in the case of electromagnetic interaction the rotor, apart from an additional phase, coincides with the Dirac or Hestenes-Dirac spinor. This statement can therefore be extended to the interaction with gravitational plane waves. Moreover, this suggests that the RDI technique may also be applicable to classical systems.

6 Conclusion

In this thesis the dynamics of electrons in a gravitational plane wave were investigated. First, in analogy to Ref. [17], a plane wave solution of the Dirac equation for an electron moving in a gravitational plane wave was constructed (see equation (4.39)). A comparison of this solution with the so-called "Volkov state" (2.29) revealed striking similarities between the behaviour of an electron in a electromagnetic and a gravitational plane wave. With the identification $-eA_i \hat{=} p_\mu H_i^{\ \mu}$, the two solutions are even exactly identical, apart from the probability density $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$ in the gravitational case. As a result, the dynamics of the electron, i.e. the Dirac current density or respectively the classical proper velocity v^μ, is identical in both cases, apart from the aforementioned density $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$, which influences the Dirac current density but not the classical proper velocity. Thus, the gravitational wave can also be assigned a generalised "vector potential" in the form of $-H_i^{\ \mu}$. However, this now does not couple to the electric charge e, but to the initial kinetic momentum p_μ. In fact, this physical observation contradicts the naive expectation one might have at first glance that the generalised "vector potential" couples to the mass. However, upon closer examination, it becomes evident that the coupling should indeed apply to the initial kinetic momentum p_μ, as gravity, for instance, also affects massless photons.
A reformulation of the plane wave solution into a matrix spinor (4.51) then allowed a more precise analysis of the dynamics and a geometric interpretation. It was found that the solution essentially consists of a general Lorentz boost in arbitrary direction and with arbitrary initial kinetic momentum (4.22) and an additional active Lorentz transformation as a result of the gravitational wave (4.55). A closer look at this additional Lorentz transformation then revealed that it can again be divided into a boost and a rotation (4.49). This means that the solution of the Dirac equation in the presence of a gravitational plane wave is boosted and rotated compared to the solution of the free Dirac equation. Furthermore, as a result of the gauge freedom of the tetrads, an additional phase occurs and the density $\sqrt{\rho} = \frac{1}{\sqrt{f(\phi)g(\phi)}}$ describes the deformation of spacetime. Moreover, this supports the idea that the RDI technique is a general framework to construct solutions for dynamical problems based on their underlying symmetries.
Additionally, a solution to the Dirac equation was constructed for an electron vortex beam interacting with a gravitational plane wave (4.72), i.e. a solution with OAM. It was found that in the case of the electron vortex beam, the dynamics for interaction with the electromagnetic plane wave and the

gravitational plane wave are not identical, as is the case for the plane wave solutions. The reason for this is that the previously identified generalised "vector potential" $-H_i^{\mu}$ of the gravitational wave does not couple to the electric charge e, but to the initial kinetic momentum p_μ. As a result, not only does the Lorentz transformation $\mathcal{R}_r$ become dependent on the momenta p^x and p^y, but these momenta now also enter the phase of the plane wave solution quadratically instead of linearly. This significantly changes the result of the superposition (4.71) compared to the electromagnetic case. At this point, it would be interesting for future work to convert the solution (4.72) into a form that is analogous to that in equation (96) of Ref. [4], i.e. to replace the series representation of the solution (4.72) with an explicit representation by means of a suitable change of coordinates $(x, y) \to (x', y')$. This would allow for a better understanding of how the deformation of spacetime induced by the gravitational plane wave influences the dynamics of the electron vortex beam, compared to the force exerted by the electromagnetic plane wave.

Finally solving the rotor equation revealed that the spinor (5.37), which describes an electron in a gravitational plane wave within the framework of classical mechanics, i.e. based on the geodesic equations, is almost identical to the solution of the Dirac equation. Only an additional phase is missing in the classical case, when the Lorentz transformations R_r and $\mathcal{R}_r$ are compared and $f(\phi_0) = g(\phi_0) = 1$ is imposed. The missing phase in the classical spinor results from the fact that the rotor equation is gauge invariant, whereas the Dirac equation is only covariant. This in turn means that the dynamics of an electron can also be described correctly within the framework of classical mechanics. The rotor and the Dirac spinors being almost identical is therefore a result that agrees with Ref. [20], since it has already been described there that, for electromagnetic interactions, the Hestenes-Dirac spinor, apart from an additional phase, corresponds to the spinor that solves the classical rotor equation. The similarities between the rotor and the Hestenes-Dirac spinor further indicate that the RDI technique is also applicable to classical systems.

References

[1] Dirac P. A. M., *The quantum theory of the electron*, Proc. Roy. Soc. Lon. A, 117, 610 (1928)

[2] Campos A. G., Cabrera R., Rabitz H. A., Bondar D. I., *Analytic Solutions to Coherent Control of the Dirac Equation*, Phys. Rev. Lett. 119, 173203 (2017)

[3] Wolkow D.M., *Über eine Klasse von Lösungen der Diracschen Gleichung*, Z. Physik 94, 250–260 (1935)

[4] Campos A. G., Hatsagortsyan K. Z., Keitel C. H., *Construction of Dirac spinors for electron vortex beams in background electromagnetic fields*, Phys. Rev. Res. 3, 013245 (2021)

[5] Verbeeck J., Tian H., Schattschneider P., *Production and application of electron vortex beams*, Nature (London) 467, 301 (2010)

[6] Utiyama R., *Invariant Theoretical Interpretation of Interaction*, Phys. Rev. 101, 1597 (1956)

[7] Kibble T. W. B., *Lorentz Invariance and the Gravitational Field*, J. Math. Phys. 2, 212–221 (1961)

[8] Hehl F. W., *Gauge Theory of Gravity and Spacetime*, In: Lehmkuhl D., Schiemann G., Scholz E. (eds) Towards a Theory of Spacetime Theories, Einstein Studies, vol 13. Birkhäuser, New York, NY (2017)

[9] Hestenes D., *Space-Time Structure of Weak and Electromagnetic Interactions*, Found Phys 12, 153–168 (1982)

[10] Heaviside O., *A Gravitational and Electromagnetic Analogy. Part I*, The Electrician 31, 281-282 (1893)

[11] Heaviside O., *A Gravitational and Electromagnetic Analogy. Part II*, The Electrician 31, 359 (1893)

[12] Einstein A., *Näherungsweise Integration der Feldgleichungen der Gravitation*, Sitzungsberichte der Königlichen Preußischen Akademie der Wissenschaft (Berlin), 688-696 (1916)

[13] Einstein A., *Über Gravitationswellen*, Sitzungsberichte der Königlichen Preußischen Akademie der Wissenschaft (Berlin), 154-176 (1918)

[14] Bondi H., Pirani F. A. E., Robinson I., *Gravitational waves in general relativity III. Exact plane waves*, Proc. R. Soc. Lond. (1959)

[15] LIGO Scientific Collaboration and Virgo Collaboration, *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. 116, 061102 (2016)

[16] Villalba V. M., *Exact solution to the Dirac equation in the presence of an exact gravitational plane wave*, Phys. Lett. A, 136:197–199 (1989)

[17] Audagnotto G., *Notes on the solution of the Dirac equation in an exact gravitational plane wave*, unpublished

[18] Hayrapetyan A. G., Matula O., Aiello A., Surzhykov A., Fritzsche S., *Interaction of Relativistic Electron-Vortex Beams with Few-Cycle Laser Pulses*, Phys. Rev. Lett. 112, 134801 (2014)

[19] Hestenes D., *Gauge Theory Gravity with Geometric Calculus*, Found. Phys. 35, 903–970 (2005)

[20] Krüger H., *Classical limit of real Dirac theory: Quantization of relativistic central field orbits*, Found. Phys. 23, 1265–1288 (1993)

[21] Greiner W., *Relativistic Quantum Mechanics-Wave Equations*, Springer, Berlin, Heidelberg (1990), p. 75-80

[22] Greiner W., *Relativistic Quantum Mechanics-Wave Equations*, Springer, Berlin, Heidelberg (1990), p. 82-83

[23] Jackson J. D., *Classical Electrodynamics (Third edition)*, Wiley, New York (1999), Sec. 12.1

[24] Greiner W., *Quantum Electrodynamics*, Springer, Berlin, Heidelberg (2009), p. 234-239

[25] Hestenes D., *Space-Time Algebra*, Birkhäuser Cham, 2015

[26] Hestenes D., *Real spinor fields*, J. Math. Phys. 8, 798 (1967)

[27] Hestenes D., *Local observables in the Dirac theory*, J. Math. Phys. 14, 893 (1973)

[28] Hestenes D., *Observables, operators, and complex numbers in the Dirac theory*, J. Math. Phys. 16, 556 (1975)

[29] Takabayasi T., *Relativistic Hydrodynamics of the Dirac Matter. Part I. General Theory*, Prog. Theor. Phys. Suppl. 4, 1 (1957)

[30] Yvon J., *Equations de Dirac-Madelung*, J. Phys. Radium 1, 18 (1940)

[31] Greiner W., *Relativistic Quantum Mechanics-Wave Equations*, Springer, Berlin, Heidelberg (1990), p. 131-132

[32] Campos A. G., Fabbri L., *Relativistic dynamical inversion in manifestly covariant form*, Phys. Rev. Research 4, 023140 (2022)

[33] Collas P., Klein D., *The Dirac Equation in Curved Spacetime: A Guide for Calculations*, Springer, Cham (2019), p. 15-20

[34] Schutz B., *A First Course in General Relativity (Second Edition)*, Cambridge University Press (2009), p. 210-212

[35] Greiner W., *Relativistic Quantum Mechanics-Wave Equations*, Springer, Berlin, Heidelberg (1990), p. 127-130

[36] Watson G. N., *A Treatise on the Theory of Bessel Functions (Second Edition)*, Cambridge University Press, Cambridge (1966), Sec. 2.22

[37] Lorenzutta S., Maino G., Dattoli G., Torre A., Chiccoli C., *Infinite-variable Bessel functions of the Anger type and the Fourier expansions*, Rep. Math. Phys., vol. 39, no. 2, p. 163-176 (1997)

[38] Watson G. N., *A Treatise on the Theory of Bessel Functions (Second Edition)*, Cambridge University Press, Cambridge (1966), Sec. 2.2

[39] Noether E., *Invariante Variationsprobleme*, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse, 235–257 (1918)

[40] Hestenes D., *Curvature calculations with spacetime algebra*, Int J Theor Phys 25, 581–588 (1986)

[41] Hestenes D., *Spinor approach to gravitational motion and precession*, Int J Theor Phys 25, 589–598 (1986)

[42] Schutz B., *A First Course in General Relativity (Second Edition)*, Cambridge University Press (2009), p. 156-157

[43] Hestenes D., *Proper dynamics of a rigid point particle*, J. Math. Phys. 15, 1778–1786 (1974)

[44] Campos A. G., *Geometric derivation of the Dirac equation*, unpublished

[45] Piasecki D., *Introduction to Koopman-von Neumann Mechanics*, BSc. thesis, Tulane University (2021)

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