CONFORMAL LIMITS FOR PARABOLIC $\mathsf{SL}(n,\mathbb{C})$-HIGGS BUNDLES

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Abstract. In this paper we generalize the conformal limit correspondence between Higgs bundles and holomorphic connections to the parabolic setting. Under mild genericity assumptions on the parabolic weights, we prove that the conformal limit always exists and that it defines holomorphic sections of the space of parabolic λ-connections which preserve a natural stratification and foliate the moduli space. Along the way, we give a careful gauge theoretic construction of the moduli space of parabolic Higgs bundles with full flags which allows the eigenvalues of the residues of the Higgs field to vary. A number of new phenomena arise in the parabolic setting. In particular, in the generality we consider, unlike the nonparabolic case, the nonabelian Hodge correspondence does not define sections of the space of logarithmic λ-connections, and the conformal limit does not define a one-parameter family in any given moduli space.

Contents

1. Introduction

The moduli spaces of Higgs bundles and holomorphic connections on a compact Riemann surface X are holomorphic symplectic spaces. Both moduli spaces have natural stratifications with strata foliated by holomorphic affine Lagrangians [40]. The nonabelian Hodge correspondence (NAH) and the conformal limit (CL) give two very different ways of identifying these moduli spaces. For example, in rank 2, NAH identifies the Hitchin section with those holomorphic connections which arise as holonomies of hyperbolic structures on the underlying topological surface while CL identifies the Hitchin section with those holomorphic connections which arise as holonomies of $\mathbb{CP}^1$-structures with underlying Riemann surface X.

In general, NAH is a real analytic map on the entire moduli space but does not preserve the strata; it is not holomorphic and the two complex structures combine to define a hyperkähler structure. On the other hand, CL is defined on each stratum and holomorphically identifies the affine Lagrangian fibers of each stratum [12]; it does not however extend to a continuous map on the entire moduli space. Hitchin sections, which are the closed strata, define geometrically interesting strata in every rank. Under CL, Hitchin sections are identified with the space of opers [14], whereas applying NAH

Date: July 25, 2024.
2010 *Mathematics Subject Classification.* Primary: 58D27; Secondary: 14D20, 14D21, 32G13.
Key words and phrases. parabolic Higgs bundle, nonabelian Hodge correspondence, conformal limit, lambda connection.

to a Hitchin section produces connections with real holonomy known as Hitchin representations [20].

It was argued by Gaiotto in [17] that the "conformal limit" of the TBA equations from [18] should take the form of generalized Schrödinger operators. This led him to conjecture a correspondence between the Higgs bundles in the Hitchin section and the locus of holomorphic connections known as *opers*. The context of Gaiotto's conjecture was superconformal field theories of "class $\mathcal{S}$", arising from compactifications on a Riemann surface X cross a circle of radius R (see [18]). In this situation, it is natural and essential to allow for decorated punctures on X.

Without punctures, Gaiotto's conjecture was proven in [14] and generalized in [12]. In order to construct the analogous moduli spaces of Higgs bundles and holomorphic connections on noncompact Riemann surfaces, one must equip the objects with extra weighted flag data at the punctures called a parabolic structure. There is a parabolic version of the nonabelian Hodge correspondence, and the purpose of this paper is to extend the conformal limit correspondence to the parabolic setting.

A number of new phenomena arise in the conformal limit of parabolic Higgs bundles. For example, NAH changes the parabolic structure and certain relevant eigenvalues according to Simpson's table (1.2). We prove that CL changes the parabolic structure and eigenvalues according to a different table (1.5). As a result, the conformal limit takes place on a larger moduli space than does nonabelian Hodge, and the targets are different. Interestingly, this implies that the conformal limit cannot be defined as a limit of a 1-parameter family in a moduli space of parabolic logarithmic connections; rather, it must be defined as a limit in an infinite dimensional configuration space. Unlike the nonparabolic setting, nonabelian Hodge does not in general define sections of parabolic logarithmic λ-connection moduli space; we show the conformal limit does define such sections. Rank two stable parabolic Higgs bundles on the four-punctured sphere are particularly simple to describe, and in this case we describe the various strata explicitly.

1.1. **Parabolic conformal limits.** Fix a pair (X, D), where X is a compact Riemann surface with holomorphic cotangent bundle K and a divisor $D = p_1 + \cdots + p_d$. Let us briefly define notions of different parabolic objects and refer to §2 for more details. A parabolic bundle $\mathcal{E}(\alpha)$ on (X, D) is a holomorphic vector bundle $\mathcal{E} \to X$ and a choice of weighted flag $\mathcal{E}_p = \mathcal{E}_{p,1} \supset \mathcal{E}_{p,2} \supset \cdots \mathcal{E}_{p,n_p} \supset \{0\}$ for each $p \in D$, where $\mathcal{E}_{p,j}$ is given a weight $\alpha_j(p)$ satisfying $0 \leq \alpha_1(p) < \cdots < \alpha_{n_p}(p) < 1$. The case where $\dim(\mathcal{E}_{p,j}) - \dim(\mathcal{E}_{p,j+1}) = 1$ for all j and all $p \in D$ is called the *full flags case*.

A Higgs field on $\mathcal{E}(\alpha)$ is a holomorphic bundle map $\Phi : \mathcal{E} \to \mathcal{E} \otimes K(D)$, while a logarithmic connection on $\mathcal{E}(\alpha)$ is a holomorphic differential operator $\nabla : \mathcal{E} \to \mathcal{E} \otimes K(D)$ satisfying the Leibniz rule $\nabla(fs) = s \otimes \partial f + f\nabla s$. When the residues of Φ and ∇ additionally preserve the fixed flag in $\mathcal{E}_p$ for each $p \in D$, $(\mathcal{E}(\alpha), \Phi)$ and $(\mathcal{E}(\alpha), \nabla)$ are called parabolic Higgs bundles and parabolic logarithmic connections, respectively. The weights on the flags are necessary to define a notion of stability with respect to which one can form the coarse moduli spaces of semistable parabolic Higgs bundles $\mathcal{P}_0(\alpha)$, semistable strongly parabolic Higgs bundles $\mathcal{SP}_0(\alpha)$, and parabolic logarithmic connections $\mathcal{P}_1(\alpha)$ with fixed parabolic data [46].[1]

For a stable parabolic Higgs bundle $(\mathcal{E}(\alpha), \Phi)$ of parabolic degree 0, Simpson [41] proved there is a unique hermitian metric h on $\mathcal{E}|_{X\setminus D}$ which is adapted to the parabolic structure and solves the Hitchin equations (see §2.4 below). A consequence of solving the Hitchin equations is that

$$D(\mathcal{E}(\alpha), \Phi) = \bar{\partial}_E + \Phi^{*_h} + \partial_E^h + \Phi \tag{1.1}$$

is a flat connection on the restriction of the underlying smooth bundle of $\mathcal{E}$ to $X \setminus D$, where $\bar{\partial}_E + \partial_E^h$ is the Chern connection of h and Φ^{*_h} is the hermitian adjoint of Φ with respect to h.

[1]Throughout the paper we work over the complex numbers and will identify moduli schemes with their analytifications as complex analytic spaces. The universal properties will play no role except in establishing isomorphisms with the gauge theoretic constructions of Section 3.

The $(0,1)$ and $(1,0)$ parts of $D(\mathcal{E}(\alpha),\Phi))$ define a holomorphic bundle with connection on $X \setminus D$. As described in [41], the metric h determines an extension of the bundle $\mathcal{V}$ to all of X such that the connection extends to a logarithmic connection ∇ on $\mathcal{V}$. By construction, $\mathcal{V}$ has a natural parabolic structure $\mathcal{V}(\beta)$ on (X,D) and $(\mathcal{V}(\beta),\nabla)$ is a stable parabolic logarithmic connection. At a point $p \in D$, the parabolic weights α and β, and the eigenvalues of the residues of Φ and ∇ (referred to henceforth as *complex masses*) are all related by Simpson's table:

(1.2)

	$(\mathcal{E}(\alpha),\Phi)$	$(\mathcal{V}(\beta),\nabla)$
parabolic weights	α	$\beta=\alpha-(\mu+\bar{\mu})$
complex masses	μ	$\nu=\alpha+\mu-\bar{\mu}$

Unlike the nonparabolic setting, the above correspondence does not define a map on moduli spaces since the weights of $(\mathcal{V}(\beta),\nabla)$ vary with $(\mathcal{E}(\alpha),\Phi) \in \mathcal{P}_0(\alpha)$. By contrast, we show the conformal limit determines a map $\mathrm{CL} : \mathcal{P}_0(\alpha) \to \mathcal{P}_1(\alpha)$.

We now explain the conformal limit of a stable parabolic Higgs bundle $(\mathcal{E}(\alpha),\Phi)$. The metric h above actually defines a $\mathbb{C}^*$-family of flat connections on $E|_{X\setminus D}$

$$D_\lambda(\mathcal{E}(\alpha),\Phi) = \bar{\partial}_E + \lambda\Phi^{*_h} + \partial_E^h + \lambda^{-1}\Phi\ , \qquad \lambda \in \mathbb{C}^*. \tag{1.3}$$

For $R > 0$, consider $D_\lambda(\mathcal{E}(\alpha), R\cdot\Phi))$ as a two parameter family of flat connections associated to $(\mathcal{E}(\alpha),\Phi)$. If h_R is the solution to the Hitchin equations for $(\mathcal{E}(\alpha), R\Phi)$ and $\hbar = \lambda R^{-1}$, then this two parameter family can written as

$$D_{R,\hbar}(\mathcal{E}(\alpha),\Phi) = \bar{\partial}_E + \hbar R^2 \Phi^{*_{h_R}} + \partial_E^{h_R} + \hbar^{-1}\Phi. \tag{1.4}$$

The $\hbar$-conformal limit $\mathrm{CL}_\hbar(\mathcal{E}(\alpha),\Phi)$ of $(\mathcal{E}(\alpha),\Phi)$ is defined to be

$$\mathrm{CL}_\hbar(\mathcal{E}(\alpha),\Phi) = \lim_{R\to 0} D_{R,\hbar}(\mathcal{E}(\alpha),\Phi)$$

in the case such a limit exists.

The goal, then, concerns the existence of the conformal limit. Recall that in the moduli space $\mathcal{P}_0(\alpha)$ of parabolic Higgs bundles, $\lim_{\lambda\to 0}(\mathcal{E}(\alpha),\lambda\Phi)$ exists and is a special type of parabolic Higgs bundle which we call the *Hodge bundle associated to* $(\mathcal{E}(\alpha),\Phi)$. Recall also that $(\mathcal{E}(\alpha),\Phi)$ is called *strongly parabolic* if the residue of Φ maps $\mathcal{E}_{p,j}$ into $\mathcal{E}_{p,j+1}$ for all $p \in D$ and all j. In many points of the paper, we will require a certain technical hypothesis, which we state here:

Assumption A. *The objects being considered are either parabolic Higgs bundle with* **full flags or strongly parabolic** *Higgs bundles.*

The relevance of Assumption A is discussed below. First, we state the main result.

Theorem 1.1. *Let $(\mathcal{E}(\alpha),\Phi)$ be a stable parabolic Higgs bundle satisfying Assumption A whose associated Hodge bundle is stable. Then for any $\hbar \in \mathbb{C}^*$, the $\hbar$-conformal limit $\mathrm{CL}_\hbar(\mathcal{E}(\alpha),\Phi)$ exists.*

Let us note that semistability implies stability for an open and dense set of the weights α for the parabolic structure. In such a setting, $\mathcal{P}_0(\alpha)$ is smooth, and Theorem 1.1 applies to every semistable parabolic Higgs bundle satisfying Assumption A.

As in the nonparabolic case of [12], existence is proven by identifying every such $(\mathcal{E}(\alpha),\Phi)$ with a point in a particular slice through its associated Hodge bundle, and the conformal limit is computed in the slice. While the strategy here is the same, the analysis in the parabolic setting is significantly more subtle. In particular, a careful gauge theoretic construction of the moduli space is necessary. This is carried out in §3; specifically, Theorems 3.22 and 3.40. We use L^2_δ theory of Lockhart-McOwen, adapted to gauge theory on manifolds with cylindrical ends by Taubes, Mrowka, and others. In this context, many previous results exist in the literature (cf. [5, 25, 36, 13, 6, 7] and [8, 41]). To extend to arbitrary residues, we have found it necessary to make the full flags assumption. This is mostly related to the fact, discovered first by Simpson, that in the general case the harmonic metric needs to be adapted to the Jordan type of the residue, whereas the L^2_δ theory essentially requires semisimple residues. We have chosen to circumvent this issue through Assumption A. How to extend the analytic results in §3 beyond Assumption A is not immediately clear to us.

Let $\mathcal{P}_0^s(\alpha) \subset \mathcal{P}_0(\alpha)$ denote the open subset of stable points, and we assume this is a nonempty. The main result of §3 is the following.

Theorem 1.2. *Under Assumption A, for a sufficiently small parameter $\delta > 0$, there is a complex manifold $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$ constructed as an infinite dimensional quotient with the following significance:*

(1) there is a biholomorphism $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta) \xrightarrow{\sim} \mathcal{P}_0^s(\alpha)$;
(2) $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$ admits a Poisson structure;
(3) the symplectic leaves of this Poisson structure are hyperkähler manifolds, and they correspond to fixing the complex masses.

Remark 1.3. The Poisson structure on $\mathcal{P}_0(\alpha)$ was constructed in [27]. We show that this Poisson structure arises as a quotient of the holomorphic symplectic structure induced by an Atiyah-Bott-Goldman form on the moduli space of *framed* parabolic Higgs bundles (see §3.2.5). We also note that the hyperkähler metric in (3) is the natural L^2-metric. However, neither the Atiyah-Bott-Goldman form nor the L^2-metric descend to the full moduli space $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$.

1.2. **Parabolic logarithmic λ-connections.** A λ-connection on a holomorphic vector bundle $\mathcal{V}$ is a holomorphic differential operator $\nabla^\lambda : \mathcal{V} \to \mathcal{V} \otimes K$ which satisfies a λ-scaled Leibniz rule $\nabla^\lambda(fs) = \lambda\, s \otimes \partial f + f\nabla^\lambda s$. In particular, when $\lambda = 0$, we get a Higgs field, and when $\lambda = 1$ we get a holomorphic connection. In the nonparabolic setting, it is useful to think of the flat connection $D_\lambda(\mathcal{E},\Phi)$ from (1.3) as a map from $\mathbb{C}$ into the space of λ-connections defined by $\lambda \mapsto (\bar\partial_E + \lambda\Phi^{*_h}, \lambda\partial_E^h + \Phi)$. Indeed, there is a moduli space of semistable λ-connections which naturally fibers over $\mathbb{C}$, and these maps define sections which foliate the moduli space. Moreover, these sections are related to the twistor lines of the hyperkähler structure on the moduli space of Higgs bundles.

There is a straightforward parabolic generalization of λ-connections which we refer to as parabolic logarithmic λ-connections. Again, there is a moduli space $\mathcal{P}(\alpha) \to \mathbb{C}$ (resp. $\mathcal{SP}(\alpha) \to \mathbb{C}$) of semistable parabolic (resp. strongly parabolic) logarithmic λ-connections with fixed parabolic structure, and the fibers over 0 and 1 are the moduli spaces $\mathcal{P}_0(\alpha)$ and $\mathcal{P}_1(\alpha)$ of parabolic Higgs bundles and parabolic logarithmic connections, respectively. If $(\mathcal{V}^\lambda(\beta), \nabla^\lambda)$ is the associated parabolic logarithmic λ-connection associated to the flat connection $D_\lambda(\mathcal{E}(\alpha),\Phi))$ from (1.3), the λ-connection analogue of Simpson's table (1.2) has weights $\beta = \alpha - \lambda\mu - \overline{\lambda\mu}$ and complex masses $\nu = \lambda\alpha + \mu - \lambda\bar\mu$. In particular, the parabolic logarithmic λ-connections $(\mathcal{V}^\lambda(\beta),\nabla^\lambda)$ does not define a section of $\mathcal{P}(\alpha) \to \mathbb{C}$ unless the complex masses of the Higgs field all vanish, a condition that does not hold for all points of $\mathcal{P}_0(\alpha)$.

The $\hbar$-conformal limit has a natural interpretation as a limit of $\hbar$-connections. Namely, the associated $\hbar$-connection to $D_{R,\hbar}$ from (1.4) is $(\bar\partial_E + R^2\hbar\Phi^{*_{h_R}}, \hbar\partial_{h_R} + \Phi)$. We note that the both the complex masses and parabolic weights of the family $D_{R,\hbar}$ depend on R and $\hbar$, see Table 2.17. However, in the limit $R \to 0$, i.e., the $\hbar$-conformal limit, we prove the following.

Theorem 1.4. *Let $(\mathcal{E}(\alpha),\Phi)$ be a stable parabolic Higgs bundles satisfying Assumption A whose associated Hodge bundle is stable. Then the $\hbar$-conformal limit $\mathrm{CL}_\hbar(\mathcal{E}(\alpha),\Phi)$ naturally extends to a stable parabolic logarithmic $\hbar$-connection on X whose parabolic weights and complex masses are determined by the following table*

(1.5)

	$(\mathcal{E}(\alpha),\Phi)$	$\mathrm{CL}_\hbar(\mathcal{E}(\alpha),\Phi)$
parabolic weights	α	$\beta = \alpha$
complex masses	μ	$\nu = \hbar\alpha + \mu$

In particular, the map $\hbar \mapsto \mathrm{CL}_\hbar(\mathcal{E}(\alpha), \Phi)$ defines a section of the moduli space of parabolic logarithmic λ-connection $\mathcal{P}(\alpha) \to \mathbb{C}$ through $(\mathcal{E}(\alpha), \Phi) \in \mathcal{P}_0(\alpha)$.

Remark 1.5. The natural extension of $\mathrm{CL}_\hbar(\mathcal{E}(\alpha), \Phi)$ from a holomorphic $\hbar$-connection on $X \setminus D$ to a parabolic logarithmic $\hbar$-connection on X is determined by the hermitian metric h_0 at the Hodge bundle associated to $(\mathcal{E}(\alpha), \Phi)$.

Scaling the parabolic logarithmic λ-connection defines a $\mathbb{C}^*$-action on $\mathcal{P}(\alpha) \to \mathbb{C}$ which covers the standard action on $\mathbb{C}$. For $\lambda \neq 0$, this action gives a biholomorphism $\xi \cdot \mathcal{P}_\lambda(\alpha) \cong \mathcal{P}_{\xi\lambda}(\alpha)$ for all $\xi \in \mathbb{C}^*$. Generalizing Simpson's work [40], we show the limits $\xi \to 0$ always exist and hence are Hodge bundles in $\mathcal{P}_0(\alpha)$. As in [40], this is done by showing every semistable parabolic logarithmic λ-connection admits a Griffiths transverse filtration whose associated graded parabolic Higgs bundle is polystable, see Appendix A. As a result, $\mathcal{P}(\alpha)$ acquires a Białynicki-Birula stratification

$$\mathcal{P}(\alpha) = \coprod_{a \in \pi_0(\mathcal{P}(\alpha)^{\mathbb{C}^*})} \mathcal{W}^a, \tag{1.6}$$

where $\mathcal{W}^a$ is all points whose $\mathbb{C}^*$-limit is in the component labeled by $a \in \pi_0(\mathcal{P}_0(\alpha)^{\mathbb{C}^*})$, and it is a vector bundle over the smooth locus. Denote the subset of $\mathcal{W}^a$ with fixed λ by $\mathcal{W}^a_\lambda$. As in the nonparabolic setting, we prove that the conformal limit foliates $\mathcal{P}(\alpha)$ or $\mathcal{SP}(\alpha)$ with strata preserving sections and biholomorphically identifies the fibers of $\mathcal{W}^a_0$ with the fibers of $\mathcal{W}^a_\hbar$.

Theorem 1.6. *Consider the spaces $\mathcal{P}(\alpha)$ satisfying Assumption A. For each $\hbar \in \mathbb{C}$ and each connected component $\mathcal{F}^a \subset \mathcal{P}(\alpha)^{\mathbb{C}^*}$, consider the natural projection map $\mathcal{W}^a_\hbar \to \mathcal{F}^a$. For each stable $x \in \mathcal{F}^a$ denote the fiber over x by $\mathcal{W}_\hbar(x)$. Then, the $\hbar$-conformal limit*

$$\mathrm{CL}_\hbar : \mathcal{W}_0(x) \to \mathcal{P}_\hbar(\alpha)$$

is a biholomorphism onto $\mathcal{W}_\hbar(x)$. In particular, each $y \in \mathcal{W}_0(x)$ defines a section of $\mathcal{P}(\alpha) \to \mathbb{C}$ defined by $\hbar \to \mathrm{CL}_\hbar(y)$, and these sections foliate $\mathcal{W}(x)$.

Finally, the fibers $\mathcal{W}_\hbar(x)$ have a "brane" interpretation which generalizes the nonparabolic case.

Theorem 1.7. *Let $x \in \mathcal{P}^s(\alpha)^{\mathbb{C}^*}$. With respect to the Poisson structure in Theorem 1.2, the fiber $\mathcal{W}_0(x) \subset \mathcal{P}^s_0(\alpha)$ is a holomorphically embedded coisotropic submanifold. Moreover, the intersections of $\mathcal{W}_0(x)$ with the symplectic leaves of $\mathcal{P}^s_0(\alpha)$ are holomorphic Lagrangian submanifolds.*

1.3. **Organization of paper.** This paper is organized as follows. In §2 we introduce parabolic Higgs bundles and their generalization to parabolic λ-connections. We also briefly discuss the nonabelian Hodge correspondence and Simpson's table. In §3 we give a self-contained exposition of the gauge theoretic construction of moduli spaces of parabolic Higgs bundles and the de Rham moduli space of flat connections using weighted Sobolev spaces. The details are required to give a precise description of the Białynicki-Birula stratification and the proof of the existence of a conformal limit in §4. Finally, in §5 we illustrate the results of the paper in the particular case of the four-punctured sphere.

Acknowledgements. The authors thank David Alfaya, Dylan Allegretti, Sebastian Heller, Rafe Mazzeo, Motohico Mulase, Carlos Simpson and Hartmut Weiß for helpful discussions. B.C.'s research is supported by NSF grants DMS-2103685 and DMS-2337451. L.F.'s research is supported by NSF grant DMS-2005258. R.W.'s research is supported by NSF grant DMS-2204346. He is also grateful to the Max-Planck Institute for Mathematics, where a portion of this work was completed during Spring 2023. This project was initiated while all of us were visiting the Mathematical Sciences Research Institute (MSRI), now becoming the Simons Laufer Mathematical Sciences Institute (SLMath), for the semester-long program *Holomorphic Differentials in Mathematics and Physics* in 2019; the institute is supported by the National Science Foundation (Grant No. DMS-1440140).

2. Parabolic objects

For background we mostly follow [41, 27]. Fix a closed Riemann surface X of genus g and with structure sheaf $\mathcal{O}_X$ and canonical bundle K_X. Let $\{p_1, \cdots, p_d\}$ be a set of d distinct points in X such that $2g-2+d>0$, and let $D = p_1 + \cdots + p_d$ be the associated effective divisor. Let $K_X(D)$ be the line bundle whose sections are meromorphic 1-forms on X with at most simple poles at the points of D. Denote the residue map by

$$\mathrm{Res} : K_X(D) \to \bigoplus_{p\in D} \mathcal{O}_p,$$

and denote the projection onto $\mathcal{O}_p$ by Res_p.

2.1. Parabolic bundles.

Definition 2.1 (parabolic bundle)**.** Let $\mathcal{E} \to X$ be a rank n holomorphic vector bundle. For each point $p \in D$, a parabolic structure at p is a filtration $\{\mathcal{E}_\alpha\}_{\alpha\in\mathbb{R}}$ of the germs of meromorphic sections of $\mathcal{E}$ at p such that $\mathcal{E}_0$ consists of germs of holomorphic sections at p and

(1) $\alpha \leq \beta$ implies $\mathcal{E}_\alpha \supset \mathcal{E}_\beta$,
(2) for each $\alpha \in \mathbb{R}$ there is $\epsilon > 0$ so that $\mathcal{E}_{\alpha-\epsilon} = \mathcal{E}_\alpha$, and
(3) $\mathcal{E}_{\alpha+1} = \mathcal{E}_\alpha(-p)$ for all α.

There are natural parabolic structures induced on subbundles, quotients, direct sums, tensor product and exterior products. By property (3), the evaluation map $\mathcal{E}_0 \to \mathcal{E}_p$ defined by $s \mapsto s(p)$ has kernel $\mathcal{E}_1$. This defines an isomorphism $\mathcal{E}_0/\mathcal{E}_1 \cong \mathcal{E}_p$ and identifies $\mathcal{E}_{\alpha_i}$ with a linear subspace $\mathcal{E}_{p,i} \subset \mathcal{E}_p$. In particular, a parabolic structure at p is equivalent to a weighted filtration

$$\mathcal{E}|_p = \mathcal{E}_{p,1} \supset \mathcal{E}_{p,2} \supset \cdots \supset \mathcal{E}_{p,n_p} \supset \mathcal{E}_{p,n_p+1} = \{0\}, \quad 0 \leq \alpha_1(p) < \cdots < \alpha_{n(p)}(p) < 1. \tag{2.1}$$

Define the integers

$$m_j(p) = \dim(\mathcal{E}_{p,j}) - \dim(\mathcal{E}_{p,j+1}).$$

When $m_j(p) = 1$ for all j and $p \in D$, we say the parabolic structure is given by *full flags*. We will use the notation $\mathcal{E}(\alpha)$ for a vector bundle $\mathcal{E}$ equipped with a parabolic structure.

Definition 2.2 (parabolic map)**.** Given two parabolic vector bundles $\mathcal{E}(\alpha)$ and $\mathcal{F}(\beta)$, a holomorphic bundle map $f : \mathcal{E} \to \mathcal{F}$ is called *parabolic* if, $\alpha_j(p) > \beta_k(p)$ implies $f(\mathcal{E}_{p,j}) \subset \mathcal{F}_{p,k+1}$ for all $p \in D$, and *strongly parabolic* if $\alpha_j(p) \geq \beta_k(p)$ implies $f(\mathcal{E}_{p,j}) \subset \mathcal{F}_{p,k+1}$ for all $p \in D$.

If $\mathcal{F} \subset \mathcal{E}$ is a holomorphic subbundle, then the j^{th} part of the filtration $\mathcal{F}_{p,j}$ of the induced parabolic structure on $\mathcal{F}$ is given by

$$\mathcal{F}_{p,j} = \mathcal{F}_p \cap \mathcal{E}_{p,j}\ , \quad \alpha_j^{\mathcal{F}}(p) = \max_k\{\alpha_k(p) \mid \mathcal{F}_p \cap \mathcal{E}_{p,k} = \mathcal{F}_{p,j}\}.$$

The induced parabolic structure on the determinant bundle $\Lambda^n\mathcal{E}$ is just a weight

$$\|\alpha(p)\| = \sum_{j=1}^{n_p} m_j \alpha_j(p). \tag{2.2}$$

To normalize the weights, use Property (3) of Definition 2.1. Namely,

$$\det(\mathcal{E}(\alpha)) = \Big(\Lambda^n\mathcal{E} \otimes \bigotimes_{p\in D} \mathcal{O}\left(\lfloor \|\alpha(p)\| \rfloor\right)\Big)(\beta), \tag{2.3}$$

where $\beta(p) = \|\alpha(p)\| - \lfloor \|\alpha(p)\| \rfloor$. A rank n parabolic bundle $\mathcal{E}(\alpha)$ together with an isomorphism $\det(\mathcal{E}(\alpha)) \cong \mathcal{O}(0)$ between $\det(\mathcal{E}(\alpha))$ and the trivial bundle with the trivial parabolic structure is an $\mathsf{SL}(n,\mathbb{C})$-parabolic bundle. In this case, $||\alpha(p)|| \in \mathbb{N}$ for all $p \in D$.

The *parabolic degree* of $\mathcal{E}(\alpha)$ will be denoted by $\deg(\mathcal{E}(\alpha))$ and is defined by

$$\deg(\mathcal{E}(\alpha)) = \deg(\mathcal{E}) + \sum_{p\in D} \|\alpha(p)\| \ . \tag{2.4}$$

Define the parabolic slope by $\mu(\mathcal{E}(\alpha)) = \deg(\mathcal{E}(\alpha))/\operatorname{rk}(\mathcal{E})$. A parabolic bundle $\mathcal{E}(\alpha)$ is *semistable* if

$$\mu(\mathcal{F}(\alpha)) \le \mu(\mathcal{E}(\alpha)) \tag{2.5}$$

for every proper holomorphic subbundles $\mathcal{F} \subset \mathcal{E}$. A parabolic bundle $\mathcal{E}(\alpha)$ is stable if the above inequity is always strict. The parabolic degree of an $\mathsf{SL}(n,\mathbb{C})$-parabolic bundle is zero.

There is a moduli space $\mathcal{N}(\alpha) = \mathcal{N}(\alpha, \mathsf{SL}(n,\mathbb{C}))$ whose closed points correspond to $\mathcal{S}$-equivalence classes of semistable parabolic bundles $\mathsf{SL}(n,\mathbb{C})$-bundles [31]. Let $\mathcal{N}^s(\alpha) \subset \mathcal{N}(\alpha)$ denote the open subset of stable parabolic bundles. For generic values of the weights, semistability implies stability, and $\mathcal{N}^s(\alpha) = \mathcal{N}(\alpha)$. In general, $\mathcal{N}(\alpha)$, if nonempty, is a smooth projective variety of dimension

$$\dim(\mathcal{N}(\alpha)) = (g-1)(n^2-1) + \frac{1}{2}\sum_{p\in D}\left(n^2 - \sum_{j=1}^{n_p} m_j^2\right).$$

Note that the dimension depends on the parabolic structure but not the weights. There is a variant where one replaces the trivial bundle $\mathcal{O}(0)$ with any parabolic line bundle $\mathcal{L}(\beta)$ and considers the moduli space $\mathcal{N}(\alpha, \mathcal{L}(\beta))$ of semistable parabolic bundles with fixed parabolic structure together with an isomorphism $\det(\mathcal{E}(\alpha)) \cong \mathcal{L}(\beta)$.

2.2. **Parabolic Higgs bundles.**

We now define the notion of a $\mathsf{SL}(n,\mathbb{C})$-parabolic Higgs bundle.

Definition 2.3 (Parabolic Higgs bundle)**.** An $\mathsf{SL}(n,\mathbb{C})$-parabolic Higgs bundle is a pair $(\mathcal{E}(\alpha), \Phi)$ on (X, D), where

- $\mathcal{E}(\alpha)$ is a parabolic $\mathsf{SL}(n,\mathbb{C})$-bundle on X, and
- $\Phi \in H^0(\operatorname{End}(\mathcal{E}) \otimes K(D))$ such that $\operatorname{Tr}(\Phi) = 0$ and the residue $\operatorname{Res}(\Phi)$ preserves the flag in $\mathcal{E}_p$ for all $p \in D$, i.e., $\operatorname{Res}_p(\Phi)(\mathcal{E}_{p,j}) \subset \mathcal{E}_{p,j}$ for all $p \in D$ and all j.

A parabolic Higgs bundle $(\mathcal{E}(\alpha), \Phi)$ is called *strongly parabolic* if $\operatorname{Res}_p(\Phi)$ is zero on each graded piece $\mathcal{E}_{p,j}/\mathcal{E}_{p,j+1}$, i.e., $\operatorname{Res}_p(\Phi)(\mathcal{E}_{p,j}) \subset \mathcal{E}_{p,j+1}$ for all $p \in D$ and all j.

There are natural notions of stability for parabolic Higgs bundles. Namely, $(\mathcal{E}(\alpha), \Phi)$ is semistable if (2.5) holds for all proper subbundles $\mathcal{F} \subset \mathcal{E}$ such that $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes K(D)$. There is a moduli space $\mathcal{P}_0(\alpha) = \mathcal{P}_0(\alpha, \mathsf{SL}(n,\mathbb{C}))$ whose closed points correspond to $\mathcal{S}$-equivalence classes of semistable $\mathsf{SL}(n,\mathbb{C})$-parabolic Higgs bundles with fixed parabolic structure [46], the subscript will be explained in the next section. We again let $\mathcal{P}_0^s(\alpha)$ denote the locus of stable points, and it is again true that for generic choices of the weights, semistability implies stability. When nonempty, the moduli space $\mathcal{P}_0^s(\alpha)$ is a smooth quasi-projective variety of dimension

$$\dim(\mathcal{P}_0^s(\alpha)) = (n^2-1)(2g-2+d) \ . \tag{2.6}$$

In particular, the dimension is independent of the fixed parabolic structure. On the other hand, an open subset of semistable strongly parabolic Higgs bundles $\mathcal{SP}_0(\alpha) \subset \mathcal{P}_0(\alpha)$ is identified with the cotangent bundle of $\mathcal{N}^s(\alpha)$. In particular, $\dim(\mathcal{SP}_0^s(\alpha)) = 2\dim(\mathcal{N}^s(\alpha))$.

We emphasize that the only condition on the residues of the Higgs fields in $\mathcal{P}_0(\alpha)$ is that they preserve the flag structure of the parabolic bundle. For a given parabolic Higgs bundle, the residue map can be projected onto the diagonal terms of the associated graded

$$\mathfrak{l}_p = \left[\bigoplus_{j=0}^{n_p-1} \operatorname{End}(\mathcal{E}_{p,j}/\mathcal{E}_{p,j+1})\right]_0 \ ,$$

where $[\,]_0$ indicates the traceless part. Set $\mathsf{L}_p = [\prod_{j=0}^{n_p-1} \mathsf{GL}(\mathcal{E}_{p,j}/\mathcal{E}_{p,j+1})]_1$, where $[\,]_1$ indicates overall determinant $= 1$. On the moduli space $\mathcal{P}_0(\alpha)$ this induces a map

$$\mathrm{Res} : \mathcal{P}_0(\alpha) \to \bigoplus_{p\in D} \mathfrak{l}_p/\mathsf{L}_p \ . \tag{2.7}$$

By definition, $\mathrm{Res}^{-1}(0) = \mathcal{SP}_0(\alpha)$ is the strongly parabolic moduli space. There is also map to the GIT quotient $\bigoplus_{p\in D} \mathfrak{l}_p /\!\!/ \mathsf{L}_p$ which records the ordered eigenvalues of the residue. Under some assumptions, which are always satisfied in the full flags case, the fibers of Res were shown to be the symplectic leaves of a natural Poisson structure on $\mathcal{P}_0(\alpha)$ [27, §3.2.4].

Remark 2.4 (RESIDUE FOR FULL FLAGS)**.** In the full flags case, the action of L_p on $\mathfrak{l}_p$ is trivial, so Res records the eigenvalues of the residue of the Higgs field at each $p \in D$

$$\mathrm{Res} : \mathcal{P}_0(\alpha) \to \bigoplus_{p\in D} \mathfrak{l}_p \simeq \bigoplus_{p\in D} \mathbb{C}^{n-1} \ .$$

As in the nonparabolic case, choosing a basis of invariant polynomials and evaluating the Higgs field defines the Hitchin map

$$\mathcal{P}_0(\alpha) \to \mathcal{B} \cong \bigoplus_{j=2}^{n} H^0(K^j(jD)) \ . \tag{2.8}$$

Note that the strongly parabolic moduli space $\mathcal{SP}_0(\alpha)$ fibers over $\bigoplus_{j=2}^{n} H^0(K^j((j-1)D))$.

There is a natural $\mathbb{C}^*$-action on $\mathcal{P}_0(\alpha)$, given by $\xi \cdot (\mathcal{E}(\alpha), \Phi) = (\mathcal{E}(\alpha), \xi\Phi)$. This action preserves the moduli space $\mathcal{SP}_0(\alpha)$ of strongly parabolic Higgs bundles but does not preserve the other fibers of Res. In [46], the analogue of the Hitchin map for $\mathcal{P}_0(\alpha)$ is shown to be proper. This implies the $\mathbb{C}^*$-limits $\lim_{\xi\to 0}[\mathcal{E}(\alpha), \xi\Phi]$ always exist in $\mathcal{P}_0(\alpha)$. The $\mathbb{C}^*$-fixed points are systems of Hodge bundles [41]. That is, $(\mathcal{E}(\alpha), \Phi)$ is a $\mathbb{C}^*$-fixed point if and only if the parabolic bundle $\mathcal{E}(\alpha)$ decomposes as a direct sum of parabolic bundles

$$\mathcal{E}(\alpha) = \mathcal{E}_1(\beta_1) \oplus \cdots \oplus \mathcal{E}_\ell(\beta_\ell) \ ,$$

and there are $\phi_j \in H^0(\mathrm{Hom}(\mathcal{E}_j(\beta_j), \mathcal{E}_{j+1}(\beta_{j+1})) \otimes K(D))$ such that

$$\Phi = \begin{pmatrix} 0 & & & \\ \phi_1 & 0 & & \\ & \ddots & \ddots & \\ & & \phi_{\ell-1} & 0 \end{pmatrix}.$$

Remark 2.5. Even though the Higgs field at a $\mathbb{C}^*$-fixed point is nilpotent, the $\mathbb{C}^*$-fixed points in $\mathcal{P}_0(\alpha)$ are not always strongly parabolic, i.e., not necessarily in $\mathrm{Res}^{-1}(0)$. However, crucial for our later analysis, every $\mathbb{C}^*$-fixed point is strongly parabolic in the full flags case.

Example 2.6 ($\mathbb{C}^*$-FIXED POINTS IN RANK 2)**.** For $\mathrm{rk}(\mathcal{E}) = 2$, the $\mathbb{C}^*$-fixed points are easy to describe. Either $\Phi = 0$ and $\mathcal{E}(\alpha)$ is a semistable parabolic $\mathsf{SL}(2,\mathbb{C})$-bundle, or $\mathcal{E}(\alpha)$ is not stable and $\mathcal{E}(\alpha)$ is a direct sum of two parabolic line bundles $\mathcal{E}(\alpha) \cong \mathcal{L}_1(\beta_1) \oplus \mathcal{L}_2(\beta\_2)$. The filtration of $\mathcal{E}_p$ as $\mathcal{E}_p = \mathcal{E}_{p,1} \supset \mathcal{E}_{p,2} \supset \{0\}$ is compatible with these line subbundles. In particular, for each $p \in D$, $\beta_1 \neq \beta_2$ if and only if $\mathcal{E}_p = \mathcal{E}_{p,1} \supset \mathcal{E}_{p,2} \supset \{0\}$, and $\beta_i(p) > \beta_j(p)$ if and only if $\mathcal{L}_i(p) = \mathcal{E}_{p,2}$, $\beta_i(p) = \alpha_2(p)$ and $\beta_j(p) = \alpha_1(p)$. Moreover, with respect to this decomposition

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi_0 & 0 \end{pmatrix} : \mathcal{L}_1 \oplus \mathcal{L}_2 \to (\mathcal{L}_1 \otimes K(D)) \oplus (\mathcal{L}_2 \otimes K(D)) \ ,$$

where $\phi_0 \in H^0(\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2 \otimes K(D)))$ is nonzero and satisfies $\mathrm{Res}_p(\phi_0) = 0$ for each $p \in D$ such that $\beta_1(p) > \beta_2(p)$. Such a Higgs bundle is stable exactly when $\deg(\mathcal{L}_2(\beta_2)) < 0$.

Remark 2.7. It will be convenient to use invariant notation to denote the structure group, parabolics, Levi factors, and unipotent subgroups. To this end, we henceforth employ the following notation. $\mathsf{G} := \mathsf{SL}(n,\mathbb{C})$. The parabolic subgroup defined by the flag structure at $\mathcal{E}_p$ will be denoted by P_p, with unipotent subgroup U_p, and Lie algebras $\mathfrak{p}_p$ and $\mathfrak{u}_p$. For convenience, we summarize the relevant dimension formulas:

$$\begin{aligned}
\dim \mathsf{G} &= \dim \mathsf{L}_p + 2\dim \mathsf{U}_p = n^2 - 1 \\
\dim \mathsf{L}_p &= -1 + \sum_{j=0}^{n_p-1} m_j(p)^2 \\
\dim \mathsf{U}_p &= \sum_{0\le i<j\le n_p-1} m_i(p)m_j(p) \\
\dim \mathsf{P}_p &= -1 + \sum_{0\le i\le j\le n_p-1} m_i(p)m_j(p)
\end{aligned} \tag{2.9}$$

2.3. **Parabolic logarithmic λ-connections.** We now equip parabolic bundles with holomorphic differential operators known as λ-connections for $\lambda \in \mathbb{C}$.

Definition 2.8 (PARABOLIC LOGARITHMIC λ-CONNECTION)**.** An $\mathsf{SL}(n,\mathbb{C})$ parabolic logarithmic λ-connection is a triple $(\lambda, \mathcal{E}(\alpha), \nabla)$ on (X, D), where $\lambda \in \mathbb{C}$, $\mathcal{E}(\alpha)$ is a parabolic bundle on (X, D), and $\nabla : \mathcal{E} \to \mathcal{E} \otimes K(D)$ is a $\mathbb{C}$-linear sheaf map such that

1. $\nabla(fs) = \lambda\partial f \otimes s + f\nabla s$ for any locally defined holomorphic function f and section s,
2. for all $p \in D$, the residue preserves the flag in $\mathcal{E}_p$, i.e., $\mathrm{Res}_p(\nabla)(\mathcal{E}_{p,j}) \subset \mathcal{E}_{p,j} \otimes K(D)$ for all j,
3. via the isomorphism $\det(\mathcal{E}(\alpha)) \cong \mathcal{O}(0)$, the induced operator on $\det(\mathcal{E}(\alpha))$ is $\lambda\partial$.

An $\mathsf{SL}(n,\mathbb{C})$ parabolic λ-connection $(\lambda, \mathcal{E}(\alpha), \nabla)$ is called *strongly parabolic* if $\mathrm{Res}(\nabla)$ acts as multiplication by $\lambda\alpha_j(p)$ on $\mathcal{E}_{p,j+1}/\mathcal{E}_{p,j}$. That is, for all $p \in D$ and all j

$$(\mathrm{Res}_p(\nabla) - \lambda\alpha_j(p)\mathrm{Id})(\mathcal{E}_{p,j}) \subset \mathcal{E}_{p,j+1}.$$

Remark 2.9. When $\lambda = 1$, we simply refer to these objects as parabolic logarithmic connections since dropping the second and third conditions on ∇ recovers the notion of a logarithmic connection. Note that for $\lambda \neq 0$, parabolic logarithmic λ-connections can be identified with parabolic logarithmic connections via the map $(\lambda, \mathcal{E}(\alpha), \nabla) \mapsto (1, \mathcal{E}(\alpha), \lambda^{-1} \cdot \nabla)$. One the other hand, a parabolic logarithmic 0-connection is a parabolic Higgs bundle.

The stability conditions for parabolic Higgs bundles generalize immediately to parabolic λ-connections, and there is a moduli space $\mathcal{P}(\alpha)$ of $\mathsf{SL}(n,\mathbb{C})$ parabolic logarithmic λ-connections with fixed parabolic structure [1] whose points correspond to $\mathcal{S}$-equivalence classes of semistable parabolic logarithmic λ-connections.[2] Note that there is a projection map

$$\Lambda : \mathcal{P}(\alpha) \to \mathbb{C} \quad ; \quad \Lambda([\lambda, \mathcal{E}(\alpha), \nabla]) = \lambda.$$

We will denote the fibers $\Lambda^{-1}(\lambda)$ by $\mathcal{P}_\lambda(\alpha)$. In particular, $\Lambda^{-1}(0) = \mathcal{P}_0(\alpha)$ is the moduli space of $\mathsf{SL}(n,\mathbb{C})$ parabolic Higgs bundles while $\mathcal{P}_1(\alpha)$ is the moduli space of $\mathsf{SL}(n,\mathbb{C})$ parabolic logarithmic connections. There is also a moduli space $\mathcal{SP}(\alpha)$ of strongly parabolic logarithmic connections.

The moduli space $\mathcal{P}(\alpha)$ has a natural $\mathbb{C}^*$-action which is given by

$$\xi \cdot [\lambda, \mathcal{E}(\alpha), \nabla] = [\xi\lambda, \mathcal{E}(\alpha), \xi\nabla]. \tag{2.10}$$

[2]In [1, Theorem 8.4] a moduli space $\mathcal{M}_{\mathrm{Hod}}(\xi, \alpha, \bar{r})$ is constructed which is the analogue of the strongly parabolic condition on Higgs bundles. This is a closed subvariety of $\mathcal{P}(\alpha)$, the moduli space $\mathcal{P}(\alpha)$ is a special instance of the Λ-module moduli space constructed [1, Theorem 5.8].

This action preserves the Higgs bundle moduli space $\mathcal{P}_0(\alpha)$ and defines an isomorphism $\mathcal{P}_\lambda(\alpha) \cong \mathcal{P}_{\lambda \cdot \xi}(\alpha)$ for $\lambda \neq 0$. In particular, the $\mathbb{C}^*$-fixed points are exactly systems the Hodge bundles in $\mathcal{P}_0(\alpha)$ described above. The action also preserves the strongly parabolic locus $\mathcal{SP}(\alpha)$.

Generalizing the map Res from (2.7), there is a residue map Res : $\mathcal{P}(\alpha) \to \bigoplus_{p\in D} \mathfrak{l}_p/\mathsf{L}_p$, where $\mathrm{Res}([\lambda, \mathcal{E}(\alpha), \nabla])$ is the Levi projection of the residue of ∇. There is also a map $\mathcal{P}(\alpha) \to \bigoplus_{p\in D} \mathfrak{l}_p /\!\!/ \mathsf{L}_p$ which records the eigenvalues of the residue of ∇.

Definition 2.10 (COMPLEX MASSES)**.** The eigenvalues of the residue of ∇ will be referred to as the *complex masses* of $(\lambda, \mathcal{E}(\alpha), \nabla)$.

In the nonparabolic setting, Simpson showed that the $\mathbb{C}^*$-limits $\lim_{\xi\to 0}[\xi\lambda, \mathcal{E}(\alpha), \xi \cdot \nabla]$ always exist in $\mathcal{P}(\alpha)$, and thus correspond to the $\mathbb{C}^*$-fixed points [40]. In Appendix A, we show that Simpson's methods can be extended to the parabolic Higgs and logarithmic λ-connection setting. In particular, we prove the following.

Proposition 2.11 ($\mathbb{C}^*$-LIMITS EXISTS)**.** *For any* $[\lambda, \mathcal{E}(\alpha), \nabla] \in \mathcal{P}(\alpha)$*, the limit* $\lim_{\xi\to 0}[\xi\lambda, \mathcal{E}(\alpha), \xi\nabla]$ *exists.*

As an immediate consequence, the moduli space $\mathcal{P}(\alpha)$ admits a Białynicki-Birula stratification

$$\mathcal{P}(\alpha) = \coprod_{a\in\pi_0(\mathcal{P}_0(\alpha)^{\mathbb{C}^*})} \mathcal{W}^a, \tag{2.11}$$

where $\mathcal{W}^a$ consists of all points whose $\mathbb{C}^*$-limit is in the connected component of the $\mathbb{C}^*$-fixed point corresponding to $a \in \pi_0(\mathcal{P}_0(\alpha)^{\mathbb{C}^*})$.

Example 2.12 (STRATIFICATION FOR RANK 2)**.** In rank 2, the limit as $\xi \to 0$ and the stratification (2.11) are easy to describe since they are determined by the Harder–Narasimhan stratification of the underlying parabolic bundle. Namely, consider a stable λ-connection $(\lambda, \mathcal{E}(\alpha), \nabla)$.

- If $\mathcal{E}(\alpha)$ is a stable parabolic bundle, then $\lim_{\xi\to 0}[\xi\lambda, \mathcal{E}(\alpha), \nabla] = [0, \mathcal{E}(\alpha), 0]$.
- If $\mathcal{E}(\alpha)$ is a stable parabolic bundle, let $\mathcal{L}(\beta)$ be the maximal destabilize subbundle. Then
$$\lim_{\xi\to 0}[\xi\lambda, \mathcal{E}(\alpha), \nabla] = [0, \mathcal{L}(\beta) \oplus \mathcal{E}(\alpha)/\mathcal{L}(\beta), \begin{pmatrix} 0 & 0 \\ \phi_0 & 0 \end{pmatrix}],$$
where $\phi_0 : \mathcal{L}(\beta) \to \mathcal{E}(\alpha)/\mathcal{L}(\beta) \otimes K(D)$ is a nonzero holomorphic section.

2.4. **Nonabelian Hodge, Simpson's table and the Conformal Limit.** The nonabelian Hodge correspondence defines a one-to-one correspondence between polystable parabolic Higgs bundles and polystable parabolic logarithmic connections. In both directions, the correspondence is through the existence of a hermitian metric h on the underlying smooth complex vector bundle which is singular at $p \in D$. In this section we recall the main features of the correspondence, all of which were established by Simpson in [41], and define the conformal limit. More details on the analytical set-up are provided in §3.

Let $F_{(\mathcal{E},h)}$ denote the curvature of the Chern connection $A = (\mathcal{E}, h)$ of $\mathcal{E}$ with respect to h. There is a class of hermitian metrics h on $\mathcal{E}|_{X\setminus D}$ called *acceptable* which is a condition involving an upper bound on the norm of $F_{(\mathcal{E},h)}$, see [41, p. 736]. The key property of acceptable metrics is that the growth rates of local meromorphic sections induces filtration of $\mathcal{E}|_{X\setminus D}$ by coherent subsheaves, see Definition 2.1, and hence a parabolic structure on $\mathcal{E}|_p$ for each $p \in D$. Let h be an acceptable metric and z be local holomorphic coordinate centered at $p \in D$. A local meromorphic section s of $\mathcal{E}_{X\setminus D}$ is in $\mathcal{E}_{\alpha,p}$ if, for all $\epsilon > 0$

$$|s(z)|_h^2 \in O(|z|^{2\alpha-\epsilon}) .$$

An acceptable metric h is said to be compatible with a parabolic vector bundle $\mathcal{E}(\alpha)$ if the parabolic structure it induces agrees with $\mathcal{E}(\alpha)$. Suppose the parabolic structure at p is given by

$$\mathcal{E}|_p = \mathcal{E}_{p,1} \supset \mathcal{E}_{p,2} \supset \cdots \supset \mathcal{E}_{p,n_p} \supset \{0\}, \quad 0 \le \alpha_1(p) < \cdots < \alpha_{n(p)}(p) < 1,$$

and $m_j = \dim(\mathcal{E}_{p,j}) - \dim(\mathcal{E}_{p,j+1})$. Then, in a basis of the associated graded, one particularly nice adapted metric is

$$h = \operatorname{diag}\Big(\underbrace{|z|^{2\alpha_1}, \cdots, |z|^{2\alpha_1}}_{m_1}, \underbrace{|z|^{2\alpha_2}, \cdots, |z|^{2\alpha_2}}_{m_2}, \cdots, \underbrace{|z|^{2\alpha_{n_p}}, \cdots, |z|^{2\alpha_{n_p}}}_{m_{n_p}}\Big). \tag{2.12}$$

One can also have log terms on the diagonal. For example, if $\mathrm{rk}(\mathcal{E}) = 3$ and the parabolic structure is given by $\mathcal{E}_p = \mathcal{E}_{p,1} \supset \mathcal{E}_{p,2} \supset 0$ with $m_1 = 2$, then the following metric is also compatible

$$h = \operatorname{diag}(|z|^{2\alpha_1}(\log|z|)^2,\ |z|^{2\alpha_1}(\log|z|)^{-2},\ |z|^{2\alpha_2}).$$

Theorem 2.13 (Nonabelian Hodge Correspondence [41])**.** *Let $(\mathcal{E}(\alpha), \Phi)$ be a parabolic Higgs bundle with $\deg(\mathcal{E}(\alpha)) = 0$. Then $(\mathcal{E}(\alpha), \Phi)$ is polystable if and only if there exists a acceptable compatible metric h on $\mathcal{E}(\alpha)$ with Chern connection A such that*

$$F_A + [\Phi, \Phi^{*_h}] = 0. \tag{2.13}$$

Moreover, $D(\mathcal{E}(\alpha), \Phi) = d_A + \Phi + \Phi^{*_h}$ defines a flat connection on the underlying smooth bundle restricted to $X \setminus D$.

Remark 2.14. Equation (2.13) is called the Hitchin equation. The local form of the metric h solving the Hitchin equation for $(\mathcal{E}(\alpha), \Phi)$ depends on the Jordan type of the Levi projection of the residue of the Higgs field, i.e., on $\mathrm{Res}(\mathcal{E}(\alpha), \Phi)$ defined in (2.7). In particular, it is of the form (2.12) if and only if the Levi projection of the residue is diagonalizable [41]. Thus, in the cases of full flags or strongly parabolic Higgs fields, the solution metric will be of the form (2.12).

Let E be the underlying smooth bundle of $\mathcal{E}(\alpha)$, h be a compatible metric solving the Hitchin equations and let $d_A = \bar\partial_E + \partial_E^h$ be the $(0,1)$ and $(1,0)$ parts of the Chern connection. The $(0,1)$ and $(1,0)$ parts of the flat connection $D(\mathcal{E}(\alpha), \Phi)$ on $E|_{X\setminus D}$ are given by $\bar\partial_E + \Phi^{*_h}$ and $\partial_E^h + \Phi$, respectively. In [41, Theorem 2] it is proven that the metric h is acceptable for the holomorphic structure $\bar\partial_E + \Phi^{*_h}$. Hence, h determines a parabolic structure $\mathcal{V}(\beta)$ on an extension $\mathcal{V} \to X$ of the holomorphic bundle $(E_{X\setminus D}, \bar\partial_E + \Phi^{*_h})$. Moreover, it is shown that on which $\nabla = \partial_E^h + \Phi$ extends to a polystable parabolic logarithmic connection connection on $\mathcal{V}(\beta)$.

The parabolic weights of $\mathcal{E}(\alpha)$ and $\mathcal{V}(\beta)$ and complex masses (see Definition 2.10) of Φ and ∇ are related by Simpson's table (2.14)

(2.14)

	$(\mathcal{E}(\alpha), \Phi)$	$(\mathcal{V}(\beta), \nabla)$
parabolic weights	α	$\beta = \alpha - 2\mathrm{Re}(\bar\mu)$
complex masses	μ	$\nu = \alpha + \mu - \bar\mu$

Remark 2.15. This table is derived in [41, §5]. The key point is the parabolic weight changes by subtracting twice the real part of the eigenvalue of the residue of the term being added to $\bar\partial_E$ (Φ^{*_h} in this case), and the complex mass changes by adding the eigenvalue of the residue of ∂_E^h (α in this case) and the eigenvalue of the term being added to $\bar\partial_E$ (Φ^{*_h} in this case). It turns out the Jordan type of $\mathrm{Res}(\mathcal{E}(\alpha), \Phi)$ and $\mathrm{Res}(\mathcal{V}(\beta), \nabla)$ are the same. This finer structure is not be present in the strongly parabolic and full flag situations which we restrict to later.

Remark 2.16. Simpson also proved the converse of Theorem 2.13. Namely, given a polystable parabolic logarithmic connection $(\mathcal{V}(\beta), \nabla)$ on (X, D), there is a compatible acceptable metric h on $\mathcal{V}(\beta)$ such that $(\bar\partial_{\mathcal{V}}, \nabla) = (\bar\partial_E + \Phi^{*_h}, \partial_E^h + \Phi)$ for a polystable parabolic Higgs bundle $(\bar\partial_E, \Phi)$. This direction of the correspondence will not play a large role in this paper, so we do not say more about it. As mentioned in the introduction, this correspondence does not define a map from the moduli space $\mathcal{P}_0(\alpha)$ to a moduli space $\mathcal{P}_1(\beta)$ for any fixed α, β.

There is a λ-connection version of the above story. Namely, for $\lambda \in \mathbb{C}$, the metric h solving the Hitchin equations for $(\mathcal{E}(\alpha), \Phi)$ defines a holomorphic λ-connection on $E|_{X\setminus D}$ given by

$$(\lambda, \widehat{\mathcal{V}}_\lambda, \widehat{\nabla}_\lambda) = (\lambda, \bar\partial_E + \lambda\Phi^{*_h}, \lambda\partial_E^h + \Phi)$$

The metric h determines a parabolic structure $\mathcal{V}_\lambda(\beta_\lambda)$ on the extension of $\widehat{\mathcal{V}}_\lambda$, and $\widehat{\nabla}_\lambda$ extends to a polystable parabolic logarithmic λ-connection ∇_λ on $\mathcal{V}_\lambda(\beta_\lambda)$. By Remark 2.15, the parabolic weights and complex masses are related by the following λ-connection analogue of Table (2.14)

(2.15)

	$(\mathcal{E}(\alpha), \Phi)$	$(\mathcal{V}_\lambda(\beta_\lambda), \nabla^\lambda)$
parabolic weights	α	$\beta_\lambda = \alpha - 2\mathrm{Re}(\lambda\bar\mu)$
complex masses	μ	$\nu_\lambda = \lambda\alpha + \mu - \lambda\bar\mu$

We are now ready to define the conformal limit of a stable Higgs bundle $(\mathcal{E}(\alpha), \Phi)$. Since polystability is preserved by scaling the Higgs field, for $R \in \mathbb{R}^{>0}$, there is an compatible acceptable metric h_R solving the Hitchin equations for $(\mathcal{E}(\alpha), R\Phi)$. Hence, for each $\lambda \in \mathbb{C}$, we have the following family of holomorphic λ-connections defined on $E|_{X\setminus D}$

$$(\lambda, \bar\partial_E + \lambda R\Phi^{*_{h_R}}, \lambda\partial_{h_R} + R\Phi)\ .$$

Scaling this family by R^{-1} defines a family of holomorphic $R^{-1}\lambda$-connections

$$(R^{-1}\lambda, \bar\partial_E + \lambda R\Phi^{*_{h_R}}, R^{-1}\lambda\partial_{h_R} + \Phi)\ .$$

In terms of $\hbar = R^{-1}\lambda$, the above family is given by

$$D_{\hbar,R}(\mathcal{E}(\alpha), \Phi) = (\hbar, \bar\partial_E + R^2\hbar\Phi^{*_{h_R}}, \hbar\partial_{h_R} + \Phi). \tag{2.16}$$

Definition 2.17 ($\hbar$-CONFORMAL LIMIT)**.** The $\hbar$-conformal limit of the polystable parabolic Higgs bundle $(\mathcal{E}(\alpha), \Phi)$ is

$$\mathrm{CL}_\hbar(\mathcal{E}(\alpha), \Phi) = \lim_{R\to 0} D_{\hbar,R}(\mathcal{E}(\alpha), \Phi)\ .$$

It is important to note that the above limit is taken in the space of holomorhpic $\hbar$-connections on $E|_{X\setminus D}$. Thus, if the limit exists, it is a holomorphic $\hbar$-connection on $E|_{X\setminus D}$. By Remark 2.15, the analogue of Simpson's table for $D_{\hbar,R}$ is given by

(2.17)

	$(\mathcal{E}(\alpha), \Phi)$	$D_{\hbar,R}$
parabolic weights	α	$\beta_{\hbar,R} = \alpha - \mathrm{Re}(\hbar R^2\bar\mu)$
complex masses	μ	$\nu_{\hbar,R} = \hbar\alpha + \mu - \hbar R^2\bar\mu$

Remark 2.18. Note that the extension of $D_{\hbar,R}$ from a holomorphic $\hbar$-connection on $X \setminus D$ to a parabolic logarithmic $\hbar$-connection on X is determined by the metric h_R. The extension of the conformal limit is determined by the metric h_0 at the Hodge bundle associated to $(\mathcal{E}(\alpha), \Phi)$.

3. Analytic theory

The aim of this section is to prove Theorem 3.22 where we provide a gauge theoretic construction of the moduli spaces, as complex manifolds, of stable strongly parabolic bundles in general, and stable parabolic bundles *in the case of full flags*, i.e. Assumption A. The Kuranishi slice method gives local universal families, showing that these moduli spaces are analytically isomorphic to the moduli spaces discussed in Section 2. We follow the same method to produce in Theorem 3.40 the de Rham space of logarithmic connections.

Such a construction has previously been carried out for parabolic vector bundles, strongly parabolic Higgs bundles, rank 2 parabolic Higgs bundles, and wild Higgs bundles under certain assumptions (see [5, 25, 36, 13, 6, 7, 34]). The constructions generally use weighted Sobolev spaces, whose introduction into gauge theory goes back to Mrowka and Taubes [35, 43, 44]. The main result below is an extension of these constructions to the case of parabolic Higgs bundles of arbitrary rank where

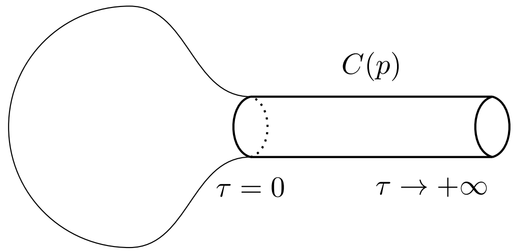

FIGURE 1. Cylindrical end

the complex masses are allowed to vary. We note that the variation of parabolic weights has been treated in [23].

An intermediate construction is that of the *framed* moduli space of parabolic bundles. We show that this admits a holomorphic symplectic form that is invariant under the change of framing. The moduli space of parabolic bundles therefore inherits a Poisson structure as a quotient, and this recovers earlier results of Logares–Martens [27] (see also [9, 29]). We prove that the symplectic leaves are exactly the moduli spaces with fixed complex masses. Moreover, we show that the holomorphic symplectic structure is actually hyperkähler, thus generalizing the result of Konno and Nakajima in the cases cited above.

3.1. **Gauge theory with weighted Sobolev spaces.**

3.1.1. *Connections and gauge groups.* In this section we define the space of connections and the group of gauge transformations. The material here is largely based on [30], [4, 5], and [13, Sec. 3], but specific details are required for the application to conformal limits. For the sake of completeness, we therefore give a precise and self-contained presentation.

We continue with the notation of §2. Set $X^\times := X \setminus D$, and let $E \to X^\times$ be a trivial rank n bundle with hermitian metric h_0. Denote the endomorphism bundle of E by $\mathfrak{gl}_E$, and the bundle of skew-hermitian endomorphisms by $\mathfrak{u}_E$. Their traceless versions are denoted by $\mathfrak{sl}_E$ and $\mathfrak{su}_E$, respectively. For each $p \in D$, choose local conformal coordinates z on open disks $\Delta(p)$ centered at p. We assume $\overline{\Delta}(p) \cap \overline{\Delta}(p') = \emptyset$ for $p \neq p'$, so that $X_0 := X \setminus \cup_{p\in D}\Delta(p)$ is a Riemann surface with boundary. Let $\Delta^\times(p) := \Delta(p)\setminus\{p\}$. Write $z = re^{i\theta}$, and identify $\Delta^\times(p)$ with a semi-infinite cylinder $C(p)$ via: $(r,\theta) \mapsto (\tau,\theta)$, where $\tau = -\log r$ (see Figure 1). We suppose that $X^\times$ is endowed with a fixed conformal metric that is euclidean $d\tau^2 + d\theta^2$ on every $C(p)$. Let $d\mu$ denote the associated area form. The Lefschetz operator $\Lambda : \Omega^2(X^\times) \to \Omega^0(X^\times)$ is the complex linear extension obtained from setting $\Lambda(d\mu) = 1$, and by convention Λ vanishes on functions and 1-forms. We also extend the collection of coordinates τ on the union of cylinders $C(p)$ to a smooth function, also denoted τ, on all of $X^\times$. For a function f on $X^\times$, by $\lim\limits_{\tau\to+\infty} f(\tau,\theta)$ we mean the collective limits on each $C(p)$, $p \in D$, when they exist.

We also choose a fixed framing of E on each $C(p)$; that is, a unitary frame $\{e_i(p)\}_{i=1}^n$ for each $p \in D$ (we will typically denote this simply $\{e_i\}_{i=1}^n$ when the point is implicit). Fix the data $\alpha(p)$ for each $p \in D$ as in §2.1. Let A_0 denote a fixed unitary connection which, in each local frame $e_1,\ldots,e_n$ on $C(p)$, has the form $d_{A_0} = d + \sqrt{-1}\hat{\alpha}(p)d\theta$, where

$$\hat{\alpha}(p) = \begin{pmatrix} \alpha_1(p) & & \\ & \ddots & \\ & & \alpha_{n_p}(p) \end{pmatrix}$$

As before, $0 \le \alpha_1(p) < \cdots < \alpha_{n_p}(p) < 1$, and in the matrix above each $\alpha_j(p)$ is repeated $m_j(p)$ times. The frame $\{e_i(p)\}$ gives an identification of the restriction of the bundle $\mathfrak{gl}_E$ to $C(p)$ with

$\mathfrak{gl}_n$. The collections $\{\alpha(p), m(p)\}$ define parabolic, Levi, and unipotent subalgebras $\mathfrak{p}_p, \mathfrak{l}_p, \mathfrak{u}_p \subset \mathfrak{gl}_n$, respectively.

We denote by ∇_0 the covariant derivative on E-valued tensors obtained from the Levi-Civita connection on $X^\times$ and A_0. For $\delta \in \mathbb{R}$, define the weighted Sobolev spaces $L^p_{k,\delta}$ of sections of E (and associated bundles) by completing the space $C_0^\infty(E)$ of smooth compactly supported sections on $X^\times$ in the norm

$$\|\sigma\|_{L^p_{k,\delta}} = \left\{ \int_{X^\times} d\mu \, e^{\tau\delta} \left(|\nabla_0^{(k)}\sigma|^p + \cdots + |\nabla_0\sigma|^p + |\sigma|^p \right) \right\}^{1/p}$$

where norms $|\cdot|$ will be understood to be taken with respect to the background hermitian structure on E and the conformal metric on X. Weighted Sobolev spaces have the usual embedding and multiplication properties (cf. [26, Lemma 7.2]).

Define the space of unitary connections:

$$\mathcal{A}_\delta = d_{A_0} + L^2_{1,\delta}\left(\mathfrak{su}_E \otimes T^*X^\times\right) \tag{3.1}$$

The complexification of this bundle splits

$$(\mathfrak{su}_E \otimes T^*X) \otimes \mathbb{C} \simeq (\mathfrak{sl}_E \otimes K_X) \oplus (\mathfrak{sl}_E \otimes \overline{K}_X)$$

and for $A \in \mathcal{A}_\delta$, the decomposition into $(1,0)$ and $(0,1)$ type will be denoted: $d_A = \partial_A + \bar{\partial}_A$. Formal L^2-adjoints are denoted d_A^*, ∂_A^*, and $\bar{\partial}_A^*$, and we have the Kähler identities:

$$\bar{\partial}_A^* = -i[\Lambda, \partial_A] \quad , \quad \partial_A^* = +i[\Lambda, \bar{\partial}_A] \ . \tag{3.2}$$

On $\Delta^\times(p)$, a holomorphic frame for $\bar{\partial}_{A_0}$ is given by $s_i = |z|^{\alpha_k(p)} e_i$, $k = 1, \ldots, n_p$, for

$$\sum_{j \le k-1} m_j(p) < i \le \sum_{j \le k} m_j(p)$$

(we have set $m_0(p) = 0$).

Consider the following spaces.

$$\mathcal{R}_\delta = \left\{ \eta \in L^2_{2,loc.}(\mathfrak{gl}_E) \mid \|d_{A_0}\eta\|_{L^2_{1,\delta}} < +\infty \right\}$$
$$\mathcal{H}_\delta = \left\{ \eta \in \mathcal{R}_\delta \mid d_{A_0}^*(e^{\tau\delta} d_{A_0}\eta) = 0 \right\}$$

We refer to $\mathcal{H}_\delta$ as infinitesimal *harmonic* gauge transformations. Let $\nabla_{\partial_p} = \dfrac{d}{d\theta} + i\hat{\alpha}(p)$ denote the boundary operator (i.e. the restriction of d_{A_0}) acting on sections of E (and the associated bundle $\mathfrak{gl}_E$) over the component of ∂X_0 intersecting $C(p)$.

Lemma 3.1. *The kernel* $\ker \nabla_{\partial_p}$ *consists of the constant sections in* $\mathfrak{l}_p \subset \mathfrak{gl}_E$.

Proof. The Lie algebra $\mathfrak{l}_p$ may be identified with the centralizer in $\mathfrak{gl}_n$ of $\hat{\alpha}(p)$. If $\psi \in \ker \nabla_{\partial_p}$, then write $\psi(\theta) = \sum_{m \in \mathbb{Z}} \psi_m e^{im\theta}$, where $\psi_m \in \mathfrak{gl}_n$. It follows that $m\psi_m + [\widehat{\alpha}(p), \psi_m] = 0$, or in terms of the unitary frame $\{e_i\}$, $(m + \alpha_j(p) - \alpha_k(p))(\psi_m)_{jk} = 0$. This implies $\psi_m = 0$ if $m \neq 0$, i.e. $\psi(\theta)$ is constant, and it commutes with $\widehat{\alpha}(p)$. □

Remark 3.2. More generally, the spectrum of ∇_{∂_p} acting on sections of the bundle $\mathfrak{gl}_E$ restricted to the boundary consists of

$$\{\sqrt{-1}\lambda_{ij}^m \mid \lambda_{ij}^m = m + \alpha_i(p) - \alpha_j(p) \ , \ m \in \mathbb{Z}\} \ .$$

In particular, the spectrum is symmetric about the origin.

Let $\lambda(p)$ denote the smallest (positive) nonzero eigenvalue of ∇_{∂_p}. Explicitly,

$$\begin{aligned}\lambda(p) &= \min\left\{|\lambda^m_{ij}| \mid m,i,j,\ \lambda^m_{ij}\neq 0\right\}\\ &= \min\left\{\{|\alpha_i(p)-\alpha_j(p)| \mid \alpha_i(p)\neq\alpha_j(p)\},\{1-|\alpha_i(p)-\alpha_j(p)| \mid i,j=1,\dots,n\}\right\}\end{aligned}$$

We shall choose δ to satisfy

$$0<\delta<\min_{p\in D}\lambda(p) \tag{3.3}$$

This is the key assumption made throughout this section. The definition of gauge groups relies on the following result.

Proposition 3.3. *There is a direct sum decomposition $\mathcal{R}_\delta = L^2_{2,\delta}(\mathfrak{gl}_E)\oplus\mathcal{H}_\delta$. Moreover, there is a continuous map*

$$\mathbf{b}:\mathcal{R}_\delta\longrightarrow\ker\nabla_\partial : \eta\mapsto\lim_{\tau\to+\infty}\eta(\tau,\theta)$$

The map $\mathbf{b}$ satisfies $\mathbf{b}^{-1}(0)=L^2_{2,\delta}(\mathfrak{gl}_E)$, and the restriction $\mathbf{b}:\mathcal{H}_\delta\to\ker\nabla_\partial$ is an isomorphism.

Proof. We first define the isomorphism

$$\mathbf{b}:\mathcal{H}_\delta\xrightarrow{\sim}\bigoplus_{p\in D}\mathfrak{l}_p \tag{3.4}$$

Let $\eta\in\mathcal{H}_\delta$ and consider its restriction to $C(p)$. Write η locally as $\sum_\lambda f_\lambda(\tau)\psi_\lambda(\theta)$, where ψ_λ is an eigenfunction of the boundary operator with eigenvalue $\sqrt{-1}\lambda$, $\lambda\in\mathbb{R}$ (we suppress the notation for multiplicities in the eigenvalues). Then harmonicity of η implies

$$(e^{\tau\delta}f'_\lambda)'=\lambda^2 e^{\tau\delta}f_\lambda\ .$$

The general solution is:

$$f_\lambda = c^\pm_\lambda\exp\left(\frac{\tau}{2}(-\delta\pm\sqrt{4\lambda^2+\delta^2})\right)\ ,\ c^\pm_\lambda \text{ constant.}$$

Now since $d_{A_0}\eta\in L^2_\delta$, we have $f'_\lambda\in L^2_\delta$. This means that $c^+_\lambda=0$, unless $\lambda=0$. Hence, we may define

$$\mathbf{b}(\eta):=c^+_0=\lim_{\tau\to+\infty}\eta(\tau,\theta)$$

which exists on each $C(p)$ and lies in $\mathfrak{l}_p$. Notice that $e^{\tau\delta}d_{A_0}\eta$ is also bounded. If $\eta\in\ker\mathbf{b}$ we can integrate by parts

$$0=\langle\eta,d^*_{A_0}(e^{\tau\delta}d_{A_0}\eta)\rangle_{L^2}=\|d_{A_0}\eta\|^2_{L^2_\delta}$$

where $d^*_{A_0}$ denotes the formal L^2 adjoint of d_{A_0}, so η is covariantly constant. But since it also vanishes on the boundary it must vanish identically. Therefore, $\mathbf{b}$ is injective.

Now suppose $\eta\in\mathcal{R}_\delta$. The operator $\bar\partial_{A_0}$ has closed range (cf. [26, Thm. 1.3], and note the restriction (3.3)), and so we may apply the Hodge theorem to write:

$$\bar\partial_{A_0}\eta=\beta+\bar\partial_{A_0}u$$

where β is harmonic with respect to the weighted adjoint: $\bar\partial^*_{A_0}(e^{\tau\delta}\beta)=0$, and $u\in L^2_{1,\delta}$. Then $\bar\partial^*_{A_0}(e^{\tau\delta}\bar\partial_{A_0}(\eta-u))=0$. Since the curvature $F_{A_0}=0$ on the cylinders, a comparison of the Laplacians for $\bar\partial_{A_0}$ and d_{A_0} on the cylinder involves only the boundary operator. An argument like the one above then says that $\eta-u$ has a well-defined limit as $\tau\to+\infty$. Hence, $\mathbf{b}(\eta)$ is well defined on all of $\mathcal{R}_\delta$ by setting

$$\mathbf{b}(\eta):=\lim_{\tau\to+\infty}(\eta-u)(\tau,\theta)$$

This agrees with the previous definition in the case $\eta\in\mathcal{H}_\delta$, since then $\bar\partial_{A_0}u=0$, and it is straightforward to show that $\lim_{\tau\to+\infty}u(\tau,\theta)=0$. If $c\in\mathfrak{l}_p$, then let η be a smooth section of $\mathfrak{gl}_E$ that is constant $=c$ on $C(p)$, and vanishes on $C(p')$ for all $p'\neq p$. Since $d_{A_0}\eta$ is then compactly supported, it follows that $\eta \in \mathcal{R}_\delta$. Since $u \in L^2_{1,\delta}$, it is easy to show that $\mathbf{b}(\eta) = c$, and this proves surjectivity of $\mathbf{b}$. Now if $\eta \in \mathcal{R}_\delta \cap \ker \mathbf{b}$, then $\eta \in L^2_{2,\delta}$. It is also easy to show that elements $\eta \in L^2_{2,\delta}$ are continuous and $\lim_{\tau\to+\infty} e^{\tau\delta}\eta(\tau,\theta) = 0$ (cf. [5, p. 222, *Remarque*]). Hence, the kernel of $\mathbf{b}$ is exactly $L^2_{2,\delta}$.

Finally, we claim that for any $\eta \in \mathcal{R}_\delta$ there is $u \in L^2_{2,\delta}$ such that $\eta + u \in \mathcal{H}_\delta$. To show this, let us first note the Poincaré inequality: there is a constant $C > 0$ so that

$$\|u\|_{L^2_\delta} \leq C\|d_{A_0}u\|_{L^2_\delta} \tag{3.5}$$

for all $u \in L^2_{1,\delta}$. For if such an inequality did not hold, then we could find a sequence $u_j \in L^2_{1,\delta}$ with $\|u_j\|_{L^2_\delta} = 1$, and $\|d_{A_0}u_j\|_{L^2_\delta} \to 0$. Notice that this implies that $\|u_j\|_{L^2_{1,\delta}}$ is uniformly bounded. We may therefore assume that $u_j \to u$ weakly in $L^2_{1,\delta}$.

We shall use the following several times: Fix a smooth function ϕ on $\mathbb{R}$ such that

$$\phi(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}$$

For $R \geq 0$, define the cut-off function $\phi_R \in C^\infty(X^\times)$ by

$$\phi_R(z) := \phi(\tau(z) - R) \tag{3.6}$$

For $\varepsilon > 0$, we may choose R sufficiently large so that $\|(d\phi_R)u\|_{L^2_\delta} \leq \varepsilon$. By the compact embedding $L^2_1 \hookrightarrow L^4$ on the complement of the cylinders, we may assume $(1-\phi_R)u_j \to (1-\phi_R)u$ strongly in L^2_δ. We also have

$$\|d_{A_0}(\phi_R u_j)\|_{L^2_\delta} \longrightarrow \|(d\phi_R)u\|_{L^2_\delta}\ .$$

Now the inequality (3.5) holds on each $C(p)$ by an integration by parts argument (cf. [5, Théorème 1.2]). Applied to $\phi_R u_j$, we see that if ε is chosen sufficiently small, and R accordingly, then $\|\phi_R u_j\|_{L^2_\delta} \leq 1/2$ for large j, and therefore $\|(1-\phi_R)u\| \geq 1/2$. This now is a contradiction, because we must have $d_{A_0}u = 0$, so $|u|$ is constant in L^2_δ, and therefore zero.

With this understood, choose $u_j \in L^2_{2,\delta}$ so that

$$\|d_{A_0}(\eta + u_j)\|_{L^2_{1,\delta}} \longrightarrow L := \inf_{v\in L^2_{2,\delta}} \|d_{A_0}(\eta+v)\|_{L^2_{1,\delta}}$$

Using (3.5), we have a uniform bound on $\|u_j\|_{L^2_{2,\delta}}$. We can therefore extract a subsequence (denoted the same) so that $u_j \to u$ weakly in $L^2_{2,\delta}$. Let $\widetilde{\eta} = \eta + u$. Then $\|d_{A_0}(\widetilde{\eta})\|_{L^2_{1,\delta}} = L$. In particular, $\langle d_{A_0}\widetilde{\eta}, d_{A_0}v\rangle_{L^2_{1,\delta}} = 0$ for all smooth, compactly supported v. It follows that $\widetilde{\eta} \in \mathcal{H}_\delta$. This completes the proof of Proposition 3.3. □

Later on we will need the following remark. Clearly, the definition of the boundary map $\mathbf{b}$ is valid for any $\eta \in L^2_{2,loc.}$, $d_{A_0}\eta \in L^2_{1,\delta}$, that is harmonic on each portion of $C(p)$ where $\tau > R$, for some $R \geq 0$. Then elements $\ell \in \mathfrak{l}_p$ may be extended to smooth sections of $\mathfrak{gl}_E$ as $\phi_R \cdot \ell$. Note that $\bigoplus_{p\in D} \mathfrak{l}_p$ has an invariant metric. The next result is straightforward from the eigensection expansion used above. We omit the proof.

Lemma 3.4. *Suppose $\eta \in L^2_{2,loc.}(\mathfrak{gl}_E)$, $d_{A_0}\eta \in L^2_{1,\delta}(\mathfrak{gl}_E)$, is harmonic on each portion of $C(p)$ where $\tau > R$, for some $R \geq 0$. Then there is a constant $C(R)$ depending upon R but not η, so that:*

(1) $|\mathbf{b}(\eta)| \leq C(R)\|\eta\|_{L^2_{-\delta}}$;
(2) $\|\eta - \phi_R \cdot \mathbf{b}(\eta)\|_{L^2_\delta} \leq C(R)\|\eta\|_{L^2_{-\delta}}$.

The following is also clear.

Lemma 3.5. *The norm*

$$\|\eta\|^2_{\mathcal{R}_\delta} = \|d_{A_0}\eta\|^2_{L^2_{1,\delta}} + |\mathbf{b}(\eta)|^2$$

gives $\mathcal{R}_\delta$ a Banach space structure for which the projections onto $L^2_{2,\delta}(\mathfrak{gl}_E)$ and $\mathcal{H}_\delta$ are continuous. Moreover, pointwise multiplication $\mathcal{R}_\delta \times \mathcal{R}_\delta \to \mathcal{R}_\delta$ is well-defined and continuous, and $\mathbf{b} : \mathcal{R}_\delta \to \ker \nabla_\partial$ is a continuous linear map.

We are now in a position to define the gauge groups. Let

$$\mathcal{G}_\delta := \{\eta \in \mathcal{R}_\delta \mid \det \eta = 1\} \quad , \quad \mathcal{G}_{\delta,*} := \{\eta \in \mathcal{G}_\delta \mid \mathbf{b}(\eta) = I\}$$

Then $\mathcal{G}_\delta$ and $\mathcal{G}_{\delta,*}$ are complex Banach Lie groups with Lie algebras

$$\mathrm{Lie}\,\mathcal{G}_\delta = \mathcal{R}^0_\delta := \{\eta \in \mathcal{R}_\delta \mid \mathrm{tr}\,\eta = 0\} \quad , \quad \mathrm{Lie}\,\mathcal{G}_{\delta,*} = L^2_{2,\delta}(\mathfrak{sl}_E) \tag{3.7}$$

It is clear that with these definitions there is a smooth action of $\mathcal{G}_\delta$ on $\mathcal{A}_\delta$ defined by pullback: $\bar\partial_{g(A)} := g \circ \bar\partial_A \circ g^{-1}$. Note also that $\mathcal{G}_{\delta,*} \subset \mathcal{G}_\delta$ is a normal subgroup, and the center $Z \subset \mathsf{SL}(2,\mathbb{C})$ embeds into $\mathcal{G}_\delta$. We denote the quotient

$$\overline{\mathbf{L}} := \mathcal{G}_\delta / Z \times \mathcal{G}_{\delta,*} \tag{3.8}$$

Notice that the defining conditions for $\mathcal{R}_\delta$ are closed with respect to hermitian conjugation. For future reference, we therefore define the closed Banach subgroups of $\mathcal{G}_\delta$ (resp. $\mathcal{G}_{\delta,*}$):

$$\mathcal{K}_\delta = \{g \in \mathcal{G}_\delta \mid gg^* = I\} \quad , \quad \mathcal{K}_{\delta,*} = \{g \in \mathcal{G}_{\delta,*} \mid gg^* = I\}$$

These have Lie algebras

$$\mathrm{Lie}\,\mathcal{K}_\delta = \mathcal{U}_\delta := \{\eta \in \mathcal{R}^0_\delta \mid \eta = -\eta^*\} \quad , \quad \mathrm{Lie}\,\mathcal{K}_{\delta,*} = L^2_{2,\delta}(\mathfrak{su}_E)$$

We end this section by discussing holomorphic structures. In the following, K_X (resp. $\overline{K}_X$) denotes the complex line bundle of $(1,0)$-forms (resp. $(0,1)$-forms) on X. The conformal metric on $X^\times$ gives a hermitian structure on K_X, $\overline{K}_X$, on $X^\times$. The Levi-Civita connection and A_0 give a connection, still denoted A_0, on tensors on associated bundles to E tensored by K_X or $\overline{K}_X$. With this understood, we have

Lemma 3.6. *Given $A \in \mathcal{A}_\delta$, there is $g \in \mathcal{G}_{\delta,*}$ such that $g(A)$ is smooth, and $\bar\partial_{g(A)} = \bar\partial_{A_0}$ on each $C(p)$.*

Proof. This is essentially [4, Prop. II.7 and Lemme III.5], which itself is modeled on [3, Lemma 14.8]. The key point is to show that

$$\bar\partial_A : L^2_{2,\delta}(\mathfrak{sl}_E) \longrightarrow L^2_{1,\delta}(\mathfrak{sl}_E \otimes \overline{K}_X)$$

is Fredholm. By [26, Thm. 1.1], the corresponding operator (which is translation invariant on each $C(p)$)

$$\bar\partial_{A_0} : L^2_{2,\delta}(\mathfrak{sl}_E) \longrightarrow L^2_{1,\delta}(\mathfrak{sl}_E \otimes \overline{K}_X)$$

is Fredholm. There is a constant $\varepsilon > 0$ so that $\bar\partial_{A_0} + B$ is Fredholm (of the same index), for any bounded map B with $\|B\| < \varepsilon$ ([38, Thm. 2.9]). Write $\beta = \bar\partial_A - \bar\partial_{A_0} \in L^2_{1,\delta}(\mathfrak{sl}_E \otimes \overline{K}_X)$. Then we have the continuous inclusion $L^2_{1,\delta} \hookrightarrow L^4_\delta$ and multiplication maps $L^4_\delta \times L^4_\delta \to L^2_\delta$. Recall the cut-off function ϕ_R. Since there is a constant C (independent of η) satisfying

$$\sup |\eta| \le C \|\eta\|_{L^2_{2,\delta}} \ , \tag{3.9}$$

we can choose R sufficiently large (depending upon β) so that the map

$$L^2_{2,\delta}(\mathfrak{sl}_E) \longrightarrow L^2_{1,\delta}(\mathfrak{sl}_E \otimes \overline{K}_X) : \eta \mapsto \phi_R[\beta, \eta]$$

is bounded of norm less than ε. Now the map $\eta \mapsto (1 - \phi_R)[\beta, \eta]$ defines a compact operator between the same spaces. This follows from the compact inclusion $L^2_1 \hookrightarrow L^4$ on the support of

$1-\phi_R$. Since adding a compact operator to a Fredholm operator is still Fredholm with the same index ([38, Thm. 2.10]), the result follows by writing $\bar\partial_A = \bar\partial_{A_0} + (1-\phi_R)\cdot\beta + \phi_R\cdot\beta$. □

3.1.2. *Higgs fields.* We begin with the key definition.

Definition 3.7 (ADMISSIBLE HIGGS FIELDS)**.** The set of admissible Higgs fields is defined to be

$$\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X) = \left\{\Phi\in L^2_{-\delta}(\mathfrak{sl}_E\otimes K_X)\mid \bar\partial_{A_0}\Phi\in L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)\right\}$$

Since $\bar\partial_{A_0}$ is a closed operator, the domain $\mathscr{D}_\delta\subset L^2_{-\delta}$ becomes a Banach space with respect to the graph norm, which we denote by

$$\|\Phi\|^2_{\mathscr{D}_\delta} := \|\Phi\|^2_{L^2_{-\delta}} + \|\bar\partial_{A_0}\Phi\|^2_{L^2_\delta}\ . \tag{3.10}$$

From the inclusion $L^2_\delta\hookrightarrow L^2_{-\delta}$, there is a continuous embedding

$$L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)\hookrightarrow \mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\ . \tag{3.11}$$

We will need the following useful decomposition.

Proposition 3.8 (DECOMPOSITION OF HIGGS FIELDS)**.** *Given $\Phi\in\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)$, there is an expression $\Phi=\Phi_0+\Phi_1$, where $\Phi_1\in L^2_{1,\delta}$, and Φ_0 is continuous and bounded. There a well-defined limit*

$$\lim_{\tau\to+\infty}\Phi_0 = \ell\in\bigoplus_{p\in D}\mathfrak{l}_p$$

where by the limit we mean with respect to the trivialization dz/z of K_X and the fixed unitary frame $\{e_i\}$ on $C(p)$. Moreover, there is a constant C, depending on A_0 and δ but independent of Φ, such that

$$|\ell|\le\sup|\Phi_0|\le C\|\Phi\|_{\mathscr{D}_\delta}\ ; \tag{3.12}$$

$$\|\Phi_1\|_{L^2_{1,\delta}}\le C\|\Phi\|_{\mathscr{D}_\delta}\ . \tag{3.13}$$

In particular, this defines a continuous map

$$\mathbf{r}:\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\longrightarrow\bigoplus_{p\in D}\mathfrak{l}_p:\Phi\mapsto\mathbf{r}(\Phi):=\ell$$

with $\ker\mathbf{r}=L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)$.

Remark 3.9. We emphasize that while the decomposition $\Phi=\Phi_0+\Phi_1$ in Proposition 3.8 is not unique, the limit ℓ is.

Proof of Proposition 3.8. By the definition of $\mathscr{D}_\delta$ we have $\bar\partial_{A_0}\Phi\in L^2_\delta$. We may therefore use the Hodge decomposition as in the previous section to write:

$$\bar\partial_{A_0}\Phi=\Omega+\bar\partial_{A_0}\psi$$

with Ω harmonic and $\psi\in L^2_{1,\delta}$. Then

$$\bar\partial^*_{A_0}(e^{\tau\delta}\bar\partial_{A_0}(\Phi-\psi))=0$$

Trivializing by dz/z and using the same analysis as in the previous section, we see that $\Phi-\psi$ has a well defined limit ℓ as $\tau\to+\infty$, where ℓ lies in the Levi factors. Moreover, $\ell=0$ implies $\Phi\in L^2_{1,\delta}$. The first assertion of the Proposition then follows by setting $\Phi_0=\Phi-\psi$ and $\Phi_1=\psi$. By Lemma 3.4, we have

$$|\ell|\le C\|\Phi_0\|_{L^2_{-\delta}} \tag{3.14}$$

Next, since Ω is the harmonic projection,

$$\|\Omega\|_{L^2_\delta}\le\|\bar\partial_{A_0}\Phi\|_{L^2_\delta} \tag{3.15}$$

Consider the Laplacian

$$\bar\partial_{A_0}\bar\partial^{*_\delta}_{A_0} : L^2_{2,\delta}(\mathfrak{sl}_E\otimes K_X\otimes \overline{K}_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes \overline{K}_X)\ ,$$

where $\bar\partial^{*_\delta}_{A_0} := e^{-\tau\delta}\bar\partial^*_{A_0}e^{\tau\delta}$. By [26, Thm. 1.1], this is a Fredholm operator, and in fact it has index 0. Let $G^{(1)}_\delta$ denote its Green's operator. Then by definition, $\Phi_1 = \bar\partial^{*_\delta}_{A_0}G^{(1)}_\delta(\bar\partial_{A_0}\Phi)$. It follows from the elliptic estimate that

$$\|\Phi_1\|_{L^2_{1,\delta}} \le C\|\bar\partial_{A_0}\Phi\|_{L^2_\delta} \tag{3.16}$$

which gives (3.13). This also implies trivially that

$$\|\Phi_0\|_{L^2_{-\delta}} \le \|\Phi\|_{L^2_{-\delta}} + \|\Phi_1\|_{L^2_{-\delta}} \le \|\Phi\|_{L^2_{-\delta}} + C\|\bar\partial_{A_0}\Phi\|_{L^2_\delta} \le C\|\Phi\|_{\mathscr{D}_\delta}\ , \tag{3.17}$$

and the estimate on ℓ in (3.12) follows from this and (3.14).

To prove the estimate on Φ_0 in (3.12), let $\widetilde{\ell} = \phi_R\cdot\ell$ be the smooth extension of the constant sections $\ell_p\otimes(dz/z)$ on each $C(p)$, obtained by multiplying by a cut-off function as above. Set $\widetilde{\Phi}_0 = \Phi_0 - \widetilde{\ell}$. By (3.14) and (3.17), it suffices to prove a bound on $\widetilde{\Phi}_0$. We first claim that

$$\|\widetilde{\Phi}_0\|_{L^2_{1,\delta}} \le C\|\Phi\|_{\mathscr{D}_\delta} \tag{3.18}$$

for a constant C which depends on the choice of cut-off function but is otherwise independent of Φ. Indeed, since $\widetilde{\Phi}_0$ vanishes as $\tau\to\infty$, we can integrate by parts to obtain

$$\|\partial_{A_0}\widetilde{\Phi}_0\|^2_{L^2_\delta} = \|\bar\partial_{A_0}\widetilde{\Phi}_0\|^2_{L^2_\delta} + \langle\widetilde{\Phi}_0, i\Lambda F_{A_0}\widetilde{\Phi}_0\rangle_{L^2_\delta} + \langle\widetilde{\Phi}_0, i\delta\Lambda(d\tau\wedge d_{A_0}\widetilde{\Phi}_0)\rangle_{L^2_\delta}$$

It follows that

$$\|\nabla_0\widetilde{\Phi}_0\|^2_{L^2_\delta} \le C(\|\bar\partial_{A_0}\widetilde{\Phi}_0\|^2_{L^2_\delta} + \|\widetilde{\Phi}_0\|^2_{L^2_\delta}) \le C(\|\Omega\|^2_{L^2_\delta} + \|\Phi_0\|^2_{L^2_{-\delta}}) \le C\|\Phi\|^2_{\mathscr{D}_\delta} \tag{3.19}$$

where we have used (3.15), (3.17), and Lemma 3.4 (2). The claim (3.18) now follows from (3.19) and (3.5). Next, a bound on $\sup_{X_0}|\widetilde{\Phi}_0|$ follows from elliptic regularity and the fact that $\bar\partial^{*_\delta}_{A_0}\bar\partial_{A_0}\Phi_0 = 0$. Finally, on each $C(p)$, $\bar\partial^{*_\delta}_{A_0}\bar\partial_{A_0}\widetilde{\Phi}_0 = 0$, and so we have

$$\|\bar\partial^*_{A_0}\bar\partial_{A_0}\widetilde{\Phi}_0\|_{L^2_\delta} \le C\|\widetilde{\Phi}_0\|_{L^2_{1,\delta}}$$

Since the operator $\bar\partial^*_{A_0}\bar\partial_{A_0}$ is translation invariant on $C(p)$, the usual a priori estimate (cf. [26, eq. (2.4)]) along with (3.18) imply an estimate

$$\|\widetilde{\Phi}_0\|_{L^2_{2,\delta}} \le C\|\Phi\|^2_{\mathscr{D}_\delta}\ ,$$

and the L^∞ bound follows as in (3.9). This completes the proof of the existence of a decomposition.

If we have two such expressions:

$$\Phi = \Phi_0 + \Phi_1 = \Phi_0' + \Phi_1'$$

then $\Phi_0 - \Phi_0'$ is bounded, continuous, and in L^2_δ, and therefore

$$\lim_{\tau\to+\infty}\Phi_0 - \Phi_0' = 0\ .$$

Hence, the limit ℓ is independent of the decomposition. ☐

Remark 3.10. Notice that for any $A\in\mathcal{A}_\delta$, $\Phi\in\mathscr{D}_\delta$, we have $\bar\partial_A\Phi\in L^2_\delta$. Indeed, $\bar\partial_A - \bar\partial_{A_0} = \beta\in L^2_{1,\delta}$, and with respect to the decomposition in Proposition 3.8,

$$[\beta,\Phi] = [\beta,\Phi_0] + [\beta,\Phi_1]$$

Since Φ_0 is bounded, the first term on the right hand side above is in L^2_δ. Since $\Phi_1\in L^2_{1,\delta}$, the second term is in L^2_δ via the inclusion $L^2_{1,\delta}\hookrightarrow L^4_\delta$. Hence, $\bar\partial_A\Phi\in L^2_\delta \iff \bar\partial_{A_0}\Phi\in L^2_\delta$.

Definition 3.11 (Higgs pairs)**.** A parabolic Higgs pair is a couple $(A,\Phi)\in\mathcal{A}_\delta\times\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)$ satisfying: $\bar\partial_A\Phi=0$. We say that (A,Φ) is strongly parabolic if $\Phi\in L^2_{1,\delta}$. We let

$$\mathcal{B}^{par}_\delta\subset\mathcal{A}_\delta\times\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\quad,\quad\mathcal{B}^{spar}_\delta\subset\mathcal{A}_\delta\times L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)$$

denote the spaces of parabolic and strongly parabolic Higgs pairs, respectively.

The following key example gives the link Higgs pairs and parabolic Higgs bundles.

Example 3.12. Let $(A,\Phi)\in\mathcal{B}^{par}_\delta$, and assume that $A=A_0$ on $C(p)$, $p\in D$. In the local holomorphic frame $\{s_i\}$ from §3.1.1, write

$$\Phi s_i=\sum_{j=1}^n\Phi_{ij}s_j=\sum_{j=1}^n P_{ij}(z)s_j\otimes\frac{dz}{z}$$

Then the $P_{ij}(z)$ are holomorphic functions on $\Delta^\times(p)$. In terms of the unitary frame,

$$\Phi e_i=\sum_{j=1}^n\widehat\Phi_{ij}e_j=\sum_{j=1}^n\Phi_{ij}|z|^{\alpha_j(p)-\alpha_i(p)}e_j=\sum_{j=1}^n P_{ij}(z)|z|^{\alpha_j(p)-\alpha_i(p)}e_j\otimes\frac{dz}{z}$$

Now $\widehat\Phi_{ij}\in L^2_{-\delta}$. Recalling that $e^{-\tau\delta}=|z|^\delta$, and that dz/z has norm 1, we have

$$\int_{\Delta^\times(p)}\frac{|dz|^2}{|z|^2}\,|P_{ij}(z)|^2|z|^{2(\alpha_j(p)-\alpha_i(p))+\delta}<+\infty$$

By the condition (3.3), the above bound means that the $P_{ij}(z)$ are regular on $\Delta(p)$. Moreover, $P_{ij}(0)=0$ unless $2(\alpha_j(p)-\alpha_i(p))+\delta>0$, which occurs only if $\alpha_j(p)\ge\alpha_i(p)$, again by (3.3). In other words, Φ can have only simple poles, and $\mathrm{Res}_p\Phi$ is upper triangular with respect to the canonical frame.

Proposition 3.13. *Assume full flags. Then the following maps are continuous:*

$$\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\times\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X):(\varphi,\psi)\mapsto[\varphi^*,\psi]\tag{3.20}$$

$$L^2_{2,\delta}(\mathfrak{sl}_E)\times\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_{1,\delta}(\mathfrak{gl}_E\otimes K_X):(\eta,\varphi)\mapsto\eta\varphi\tag{3.21}$$

$$L^2_{2,\delta}(\mathfrak{sl}_E)\times\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X):(\eta,\varphi)\mapsto[\varphi,[\varphi^*,\eta]]\tag{3.22}$$

Proof. We shall prove that under the assumptions there is a constant C, independent of η,φ,ψ, such that

$$\|[\varphi^*,\psi]\|_{L^2_\delta}\le C\|\varphi\|_{\mathscr{D}_\delta}\|\psi\|_{\mathscr{D}_\delta}\ ;\tag{3.23}$$

$$\|\eta\varphi\|_{L^2_{1,\delta}}\le C\|\eta\|_{L^2_{2,\delta}}\|\varphi\|_{\mathscr{D}_\delta}\ .\tag{3.24}$$

$$\|[\varphi,[\varphi^*,\eta]]\|_{L^2_\delta}\le C\|\eta\|_{L^2_{2,\delta}}\|\varphi\|^2_{\mathscr{D}_\delta}\ .\tag{3.25}$$

For (3.23), use the decomposition from Proposition 3.8 to write: $\varphi=\varphi_0+\varphi_1$, $\psi=\psi_0+\psi_1$. Consider the terms on the right hand side in the expansion

$$[\varphi^*,\psi]-[\varphi_0^*,\psi_0]=[\varphi_0^*,\psi_1]+[\varphi_1^*,\psi_0]+[\varphi_1^*,\psi_1]\ .$$

Because φ_0 and ψ_0 are bounded, and φ_1 and ψ_1 are in $L^2_{1,\delta}$, the first two terms are clearly in L^2_δ, and the last term is in L^2_δ because of the embedding $L^2_{1,\delta}\hookrightarrow L^4_\delta$. The term $[\varphi_0^*,\psi_0]$ is not, in general, in L^2_δ. Let m_0 (resp. ℓ_0) denote the limits of φ_0 (resp. ψ_0) from Proposition 3.8, and let $\widetilde m_0:=\phi_1\cdot m_0$ (resp. $\widetilde\ell:=\phi_1\cdot\ell_0$) be smooth extensions to $X^\times$ (see (3.6)). Then in particular, $\widetilde m_0$, $\widetilde\ell$ and their derivatives have pointwise norms bounded by the norms of m_0 and ℓ_0. We further expand:

$$[\varphi_0^*,\psi_0]=[\varphi_0^*-\widetilde m_0^*,\psi_0-\widetilde\ell_0]+[\varphi_0^*-\widetilde m_0^*,\widetilde\ell_0]+[\widetilde m_0^*,\psi_0-\widetilde\ell_0]+[\widetilde m_0^*,\widetilde\ell_0]\tag{3.26}$$

Since we assume full flags, $\mathfrak{l}_p$ is abelian for every $p \in D$, so the last term is compactly supported. Moreover, by our choice of extension $\widetilde{\ell}_0$ and the second part of Proposition 3.8, it is bounded by the graph norm of ψ. The other terms on the right hand side of (3.26) are bounded by the same graph norm by using Lemma 3.4: the first term by (3.16) and the multiplication theorem; and the second and third terms by our choice of extensions and (3.17). This proves the estimate (3.23).

For (3.24), again using the embedding $L^2_{1,\delta} \hookrightarrow L^4_\delta$, along with Proposition 3.8, we have

$$\|\eta\varphi\|_{L^2_\delta} \leq \|\eta\|_{L^2_\delta} \sup|\varphi_0| + C\|\eta\|_{L^2_{1,\delta}}\|\varphi_1\|_{L^2_{1,\delta}} \leq C\|\eta\|_{L^2_{2,\delta}}\|\varphi\|_{\mathscr{D}_\delta} .$$

On the other hand,

$$\begin{aligned}\|\bar\partial_{A_0}(\eta\varphi)\|_{L^2_\delta} &\leq \|\bar\partial_{A_0}(\eta)\varphi_0\|_{L^2_\delta} + \|\bar\partial_{A_0}(\eta)\varphi_1\|_{L^2_\delta} + \|\eta\bar\partial_{A_0}\varphi\|_{L^2_\delta} \\ &\leq \|\eta\|_{L^2_{1,\delta}} \sup|\varphi_0| + C\|\eta\|_{L^2_{2,\delta}}\|\varphi_1\|_{L^2_{1,\delta}} + \sup|\eta|\|\varphi\|_{\mathscr{D}_\delta} \\ &\leq C\|\eta\|_{L^2_{2,\delta}}\|\varphi\|_{\mathscr{D}_\delta}\end{aligned}$$

The argument for (3.25) is similar. This completes the proof of the Proposition. ☐

Corollary 3.14. *Defining*

$$\mathcal{G}_\delta \times \mathcal{A}_\delta \times \mathscr{D}_\delta \longrightarrow \mathcal{A}_\delta \times \mathscr{D}_\delta : (g,(A,\Phi)) \mapsto (g(A), g\Phi g^{-1})$$

gives a smooth (left) action of the gauge group $\mathcal{G}_\delta$ on $\mathcal{A}_\delta \times \mathscr{D}_\delta$ which preserves the space $\mathcal{B}^{par}_\delta$.

By Lemma 3.6, after a gauge transformation in $\mathcal{G}_{\delta,*}$, we may always assume a Higgs pair (A,Φ) is smooth and of the form (A_0,Φ) on each $C(p)$. Therefore, Example 3.12 shows there is a well defined map:

$$\mathrm{Res} : \mathcal{B}^{par}_\delta \longrightarrow \bigoplus_{p\in D} \mathfrak{l}_p \tag{3.27}$$

taking Φ to the collection $\{P_{ij}(0)\}_{\alpha_i(p)=\alpha_j(p)}$. Note that this does not depend on the representative in the $\mathcal{G}_{\delta,*}$-orbit of (A,Φ). It is also clear that $\mathcal{B}^{spar}_\delta = \mathrm{Res}^{-1}(0)$.

Let us also note the following. Recall the definition of $\mathbf{r}$ from Proposition 3.8.

Proposition 3.15. *For $(A,\Phi) \in \mathcal{B}^{par}_\delta$, $\mathrm{Res}(A,\Phi) = \mathbf{r}(\Phi)$.*

Proof. First, note that $\mathbf{r}(g\Phi g^{-1}) = \mathbf{r}(\Phi)$ for $g \in \mathcal{G}_{\delta,*}$. Now if $A = A_0$ on each $C(p)$, then we may take

$$\Phi_0 = \sum_{j=1}^n P_{ij}(0)|z|^{\alpha_j(p)-\alpha_i(p)} e_j \otimes \frac{dz}{z}$$

near each $p \in D$. Then Φ_0 has the same limit as the one defined in 3.12, and by using the expression for Φ in the example one checks that $\Phi - \Phi_0 \in L^2_{1,\delta}$. Conversely, since both $\mathbf{r}$ and $\mathrm{Res}_{\mathfrak{l}}$ are gauge invariant, we may always reduce to this case. ☐

Remark 3.16. If $A_0^{(1)}$ and $A_0^{(2)}$ are model connections on bundles $E^{(1)}$ and $E^{(2)}$ as in §3.1.1, then clearly $E^{(1)} \oplus E^{(2)}$ inherits a model connection. In particular, the results of this section apply, *mutatis mutandi*, to Higgs fields $\Phi \in L^2_{1,\delta}(\mathrm{Hom}(E^{(1)},E^{(2)}) \otimes K_X)$.

3.1.3. *Relation with parabolic Higgs bundles and nonabelian Hodge.* We now make the connection between Higgs pairs and Higgs bundles. Recall the configuration spaces $\mathcal{B}^{par}_\delta$ and $\mathcal{B}^{spar}_\delta$ defined in Definition 3.11.

Proposition 3.17 (Higgs pairs and Higgs bundles)**.** *Associated to each $\mathcal{G}_{\delta,*}$ orbit in $\mathcal{B}^{par}_\delta$ (resp. $\mathcal{B}^{spar}_\delta$) there is a unique isomorphism class of framed parabolic (resp. strongly parabolic) Higgs bundles. Moreover, the residue maps defined in eqs. (2.7) and (3.27) agree.*

Proof. By Lemma 3.6, a $\mathcal{G}_{\delta,*}$-orbit in $\mathcal{B}_\delta^{par}$ contains a pair (A,Φ), where $\bar\partial_A$ is a smooth $\bar\partial$-operator $\bar\partial_A$ that is equal to $\bar\partial_{A_0}$ on each $C(p)$. This defines a holomorphic bundle $\mathcal{E}^0$ on $X^\times$ with a preferred holomorphic frame $\{s_i\}$ on each $C(p)$. Gluing to the trivial bundle on $\Delta(p)$ uniquely determines an extension of $\mathcal{E}^0$ to a holomorphic bundle $\mathcal{E}$ on X. For each $0\leq\alpha<1$, define $\mathcal{E}_\alpha$ to be the sheaf of germs generated by $\{s_i\}$ for $\alpha\leq\alpha_i(p)$. This defines a parabolic structure on $\mathcal{E}$ with jumps $\alpha(p)$ at p. According to the discussion in Example 3.12, $\Phi\in H^0(\mathrm{End}(\mathcal{E}(\alpha))\otimes K(D))$, with given residues. Hence, $(\mathcal{E},\Phi)$ is a parabolic Higgs bundle in the sense of Definition 2.3. Given two such pairs (A_1,Φ_1) and (A_2,Φ_2) in the same $\mathcal{G}_{\delta,*}$ orbit, then the element g such that $g(A_1,\Phi_1)=(A_2,\Phi_2)$ extends as a holomorphic isomorphism of parabolic Higgs bundles $(\mathcal{E}_1(\alpha),\Phi_1)\simeq(\mathcal{E}_2(\alpha),\Phi_2)$. □

In view of Proposition 3.17, we have the following.

Definition 3.18. We call a Higgs pair $(A,\Phi)\in\mathcal{B}_\delta^{par}$ stable (resp. semistable) according to the stability (resp. semistability) of any of the isomorphic parabolic Higgs bundles in its $\mathcal{G}_{\delta,*}$-orbit. Let $\mathcal{B}_\delta^{par,s}$ (resp. $\mathcal{B}_\delta^{spar,s}$) denote the open subset of stable parabolic (resp. stable strongly parabolic) Higgs bundles.

At this point, we have made no assumption on the weights α, other than the full flags assumption for the general parabolic case. Consider now the parabolic bundle $\mathcal{E}(\alpha)$ determined by extending $\bar\partial_{A_0}$ to a bundle $\mathcal{E}$ with parabolic structure, as in Proposition 3.17 (the Higgs field is not relevant here). We now suppose the weights α and the background connection A_0 have been chosen so that $\mathcal{E}(\alpha)$ has parabolic degree zero, and that in fact $\Lambda^n(\mathcal{E}(\alpha))$ is holomorphically trivial. Hence, in particular, $\|\alpha(p)\|\in\mathbb{N}$ for all $p\in D$.

Let $\mathbf{1}_{D,\|\alpha\|}$ denote a section of

$$\mathcal{L}_{D,\|\alpha\|}:=\bigotimes_{p\in D}\mathcal{O}(\|\alpha(p)\|p)$$

defined by the divisor. This gives a trivialization, unique up to a nonzero multiplicative constant, of $\mathcal{L}_{D,\|\alpha\|}$ on $X^\times$. Use this to define a singular hermitian metric on $\mathcal{L}_{D,\|\alpha\|}$ by declaring $\|\mathbf{1}_{D,\|\alpha\|}\|=1$. Then the product of this with the background metric on $E\to X^\times$ combine to give a smooth metric on $\Lambda^n\mathcal{E}(\alpha)$. Since we assume this is trivial, there is a modification of the product metric, unique up to a nonzero multiplicative constant, which makes $\Lambda^n\mathcal{E}(\alpha)$ flat. We can then choose a trivialization of $\Lambda^n\mathcal{E}(\alpha)$ by taking a global nonzero flat section.

Since all the bundles in $\mathcal{B}_\delta^{par}$ (resp. $\mathcal{B}_\delta^{spar}$) induce the same structure on $\Lambda^n\mathcal{E}(\alpha)$, the trivialization may be fixed once and for all. We record this discussion in the following.

Proposition 3.19. *Assume the weights and background connection A_0 have been chosen so that $\Lambda^n(\mathcal{E}(\alpha))$ is trivial. Then associated to each $\mathcal{G}_{\delta,*}$ orbit in $\mathcal{B}_\delta^{par}$ (resp. $\mathcal{B}_\delta^{spar}$) there is a unique isomorphism class of framed parabolic (resp. strongly parabolic) $\mathsf{SL}(n,\mathbb{C})$-Higgs bundles.*

Finally, we end this subsection by quoting the important result on the existence of Hermitian-Einstein metrics on parabolic Higgs pairs, proved by Simpson, Biquard, and Mochizuki. For $A\in\mathcal{A}_\delta$, write $A=A_0+a$, with $a\in L^2_{1,\delta}(\mathfrak{su}_E\otimes T^*X^\times)$. Let

$$F_A=F_{A_0}+d_{A_0}a+a\wedge a$$

denote the curvature. Then since F_{A_0} vanishes in a neighborhood of D, it follows that $F_A\in L^2_\delta$. We denote by

$$F_A^\perp:=F_A-\frac{1}{n}\,\mathrm{tr}(F_A)\cdot\mathbf{I}$$

the traceless part of F_A. Given $\Phi\in\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)$, and assuming full flags, we also have from (3.23) that $[\Phi,\Phi^*]\in L^2_\delta$. The *Hitchin equations* for a parabolic Higgs pair $(A,\Phi)\in\mathcal{B}_\delta^{par}$ are:

$$F_A^\perp+[\Phi,\Phi^*]=0\ . \tag{3.28}$$

We have the following parabolic version of the Hitchin-Simpson theorem.

Theorem 3.20 (Nonabelian Hodge Theorem)**.** *Let $(A,\Phi)\in\mathcal{B}^{par,s}_{\delta}$. Then there is $g\in\mathcal{G}_\delta$ such that the pair $g(A,\Phi)$ satisfies (3.28). Moreover, g is unique up to the action of $\mathcal{K}_\delta$.*

Sketch of proof. In the context of weighted Sobolev spaces, this version of the theorem is due to Biquard [6, Thm. 8.1]. Let us note the following. First, by Proposition 3.3 we have that $\mathcal{R}_\delta\simeq\widehat{L}^2_{2,\delta}$, where the latter space is defined in [6, p. 53]. Choose any $\widetilde{\delta}>\delta$ still satisfying the assumption (3.3). On [6, p. 77] it is shown that there is a solution to (3.28) after acting by a gauge transformation in $\widehat{L}^p_{2,\widetilde{\delta}}$ for any $p>2$. Now by the embedding $L^p_{2,\widetilde{\delta}}\subset L^2_{2,\delta}$ for p sufficiently close to 2 (see [6, Lemme 4.5]) we conclude that there is in fact a solution in the orbit of $\mathcal{G}_\delta$. The uniqueness statement is standard. ☐

As in the case of gauge theory on closed manifolds, we may alternatively regard (3.28) as an equation for a metric. Recall that h_0 denotes the hermitian metric on $E\to X^\times$. Then for each $g\in\mathcal{G}_\delta$, we define a new metric $g(h_0)$ by the rule: $\langle u,v\rangle_{g(h_0)}:=\langle gu,gv\rangle_{h_0}$. We define the space of admissible metrics:

$$\mathcal{M}_\delta:=\{g(h_0)\mid g\in\mathcal{G}_\delta\}\simeq\mathcal{G}_\delta/\mathcal{K}_\delta \tag{3.29}$$

Then we have the following reformulation of Theorem 3.20.

Theorem 3.21 (Harmonic metric)**.** *Let $(A,\Phi)\in\mathcal{B}^{par,s}_{\delta}$ and $(\mathcal{E}(\alpha),\Phi)$ a stable parabolic Higgs bundle associated to (A,Φ) from Proposition 3.17. Then there exists a unique metric $h\in\mathcal{M}_\delta$ satisfying*

$$F^{\perp}_{(\mathcal{E},h)}+[\Phi,\Phi^{*_h}]=0\ ,$$

where $(\mathcal{E},h)$ denotes the Chern connection for $\mathcal{E}$ with the metric h.

The metric in the theorem above is called the *harmonic metric* for $(\mathcal{E}(\alpha),\Phi)$.

3.2. Analytic Dolbeault moduli spaces.

Fix α, and suppose δ satisfies (3.3). Set

$$\mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta)=\mathcal{G}_\delta\backslash\mathcal{B}^{par,s}_{\delta}\ ,\ \mathbf{M}^{spar,s}_{\mathsf{Dol}}(\alpha,\delta)=\mathcal{G}_\delta\backslash\mathcal{B}^{spar,s}_{\delta}$$

The main goal of this section is to prove the following:

Theorem 3.22 (Moduli spaces of Higgs bundles)**.**

(1) If nonempty, the moduli space $\mathbf{M}^{spar,s}_{\mathsf{Dol}}(\alpha,\delta)$ is a smooth complex manifold of complex dimension

$$\dim\mathbf{M}^{spar,s}_{\mathsf{Dol}}(\alpha,\delta)=(2g-2)\dim\mathsf{G}+2\sum_{p\in D}\dim(\mathsf{G}/\mathsf{P}_p)\ .$$

(2) Assume full flags. If nonempty, the moduli space $\mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta)$ is a smooth complex manifold of complex dimension

$$\dim\mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta)=(2g-2+d)\dim\mathsf{G}=\dim\mathbf{M}^{spar,s}_{\mathsf{Dol}}(\alpha,\delta)+\sum_{p\in D}\dim(\mathsf{L}_p)\ .$$

(3) The assignment of a parabolic Higgs bundle to (A,Φ) (Proposition 3.17) induces biholomorphisms

$$\mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta)\xrightarrow{\sim}\mathcal{P}^s_0(\alpha)\quad,\quad\mathbf{M}^{spar,s}_{\mathsf{Dol}}(\alpha,\delta)\xrightarrow{\sim}\mathcal{SP}^s_0(\alpha)\ .$$

We note that previous constructions exist in the strongly parabolic case (e.g. [25]) and for rank 2 parabolic bundles [36]. Below we consider the general case of parabolic bundles under Assumption A of the introduction. We will actually first construct the *framed* moduli spaces

$$\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)=\mathcal{G}_{\delta,*}\backslash\mathcal{B}^{par,s}_{delta}\quad,\quad\mathbf{M}^{spar,s}_{\mathsf{Dol},*}(\alpha,\delta)=\mathcal{G}_{\delta,*}\backslash\mathcal{B}^{spar,s}_{delta}$$

as finite dimensional complex manifolds. It can be shown (cf. [24, Prop. 7.1.14]) that the residual gauge group $\overline{\mathbf{L}}$ acts freely and properly on $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$ and $\mathbf{M}^{spar,s}_{\mathsf{Dol},*}(\alpha,\delta)$. The moduli spaces are then the quotients

$$\mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta)=\overline{\mathbf{L}}\backslash\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta) \quad , \quad \mathbf{M}^{spar,s}_{\mathsf{Dol}}(\alpha,\delta)=\overline{\mathbf{L}}\backslash\mathbf{M}^{spar,s}_{\mathsf{Dol},*}(\alpha,\delta)$$

and inherit a complex manifold structure [11].

3.2.1. *Index computations.* The natural operators in the deformation complexes of Higgs bundles are $D''=\bar\partial_A+\Phi$ and $D'=\partial_A+\Phi^*$. By the Kähler identities, D' is the formal L^2-adjoint of D'' (see [39, §1]). In the context of strongly parabolic Higgs bundles in the weighted Sobolev spaces we are using, it is more natural to consider the L^2_δ-adjoint $D'_\delta:=e^{-\tau\delta}D'e^{\tau\delta}$. The adjoint for parabolic Higgs bundles (not necessarily strongly parabolic) would be more complicated. However, in light of Proposition 3.13, we may instead use the same operators as in the strongly parabolic case. The goal of this section is to prove index formulas for the Dirac type operators $D''+D'_\delta$.

Proposition 3.23. *Let $(A,\Phi)\in\mathcal{B}^{spar}_\delta$ be a smooth parabolic Higgs bundle.*
(1) Let

$$T^{spar}_{(A,\Phi)}:L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E)\oplus L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)$$

be the operator defined by

$$T^{spar}_{(A,\Phi)}(\beta,\varphi)=(e^{-\tau\delta}\bar\partial_A^*(e^{\tau\delta}\beta)-i\Lambda[\Phi^*,\varphi],\bar\partial_A\varphi+[\Phi,\beta])$$

Then $T^{spar}_{(A,\Phi)}$ is Fredholm of index

$$\operatorname{index}(T^{spar}_{(A,\Phi)})=(2g-2+d)\dim\mathsf{G}$$

(2) Consider the same operator as above, but with different domain:

$$T^{par}_{(A,\Phi)}:L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E)\oplus L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)$$

Then $T^{par}_{(A,\Phi)}$ is Fredholm of index

$$\operatorname{index}(T^{par}_{(A,\Phi)})=(2g-2+d)\dim\mathsf{G}+\sum_{p\in D}\dim\mathsf{L}_p$$

Proof. We first note that the operators $T^{spar}_{(A,\Phi)}$ and $T^{spar}_{(A,\Phi)}$ are well-defined by Proposition 3.13. Also, as in the proof of Lemma 3.6, to prove Fredholmness and compute the index, we may drop the terms involving Φ and Φ^* and replace A with A_0. So it suffices to consider the decoupled operators

$$\bar\partial_{A_0}:L^2_{1,\delta}(\mathfrak{sl}_E)\longrightarrow L^2_\delta(\mathfrak{sl}_E\overline{K}_X):\eta\mapsto\bar\partial_{A_0}\eta$$
$$T_\beta:L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E):\beta\mapsto e^{-\tau\delta}\bar\partial^*_{A_0}(e^{\tau\delta}\beta)$$
$$T_\varphi:L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X):\varphi\mapsto\bar\partial_{A_0}\varphi$$

We have:

$$\operatorname{index}(T^{spar}_{(A,\Phi)})=\operatorname{index}(T_\beta)+\operatorname{index}(T_\varphi) \tag{3.30}$$

The operators $\bar\partial_{A_0}$ and T_φ are Fredholm ([26, Thm. 1.1]). The systems are equivalent to the L^2-extended operator of Atiyah-Patodi-Singer (see [44, p. 56] and [13, Prop. 3.7]). It then follows from

[2, Thm. 3.10] that the indices are

$$\operatorname{index}(\bar\partial_{A_0}) = -(g-1+d/2)\dim\mathsf{G} - \frac{1}{2}\sum_{p\in D}\dim\mathsf{L}_p$$

$$\operatorname{index}(T_\varphi) = (g-1+d/2)\dim\mathsf{G} - \frac{1}{2}\sum_{p\in D}\dim\mathsf{L}_p \tag{3.31}$$

Here, we have used the fact from Remark 3.2 that the boundary operator has a symmetric spectrum, and therefore the η-function vanishes identically (see [2, eq. (1.7)]). The adjoint has index

$$\operatorname{index}(T_\beta) = -\operatorname{index}(\bar\partial_{A_0}) = (g-1+d/2)\dim\mathsf{G} + \frac{1}{2}\sum_{p\in D}\dim\mathsf{L}_p \tag{3.32}$$

The proof of part (1) then follows from eqs. (3.30), (3.31), and (3.32). For part (2), let $\widehat{T}_\varphi$ denote the same operator T_φ but with domain $\mathscr{D}_\delta$. The proof of part (2) then follows immediately from the following claim:

Claim. $\operatorname{index}(\widehat{T}_\varphi) = \operatorname{index}(T_\varphi) + \sum_{p\in D}\dim\mathsf{L}_p$.

The Claim follows from [2, eq. (3.25)]. Alternatively, consider the same operator with different domains and ranges:

$$\widetilde{T}_\varphi : L^2_{1,-\delta}(\mathfrak{sl}_E\otimes K_X)\longrightarrow L^2_{-\delta}(\mathfrak{sl}_E\otimes K_X\otimes \overline{K}_X):$$

Clearly, $\ker\widetilde{T}_\varphi = \ker\widehat{T}_\varphi$. For the cokernel, let $\beta\in L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)$ satisfy $\bar\partial^*_{A_0}(e^{\tau\delta}\beta)=0$. Then there is a well-defined limit

$$\ell = \lim_{\tau\to\infty} e^{\tau\delta}\beta \in \bigoplus_{p\in D}\mathfrak{l}_p$$

Now $\widetilde{\ell} := \phi_R\cdot\ell\in\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)$, and $\langle\bar\partial_{A_0}\ell,\beta\rangle_{L^2_\delta} = 2\pi|\ell|^2$. Therefore, for $\beta\in\operatorname{coker}\widehat{T}_\varphi$, $\ell=0$, and so $e^{\tau\delta}\beta\in L^2_\delta$. Let $\Omega = e^{2\tau\delta}\beta$. Then $\bar\partial^*_{A_0}(e^{-\tau\delta}\Omega)=0$, $\Omega\in L^2_{-\delta}$, and indeed $\Omega\in\operatorname{coker}\widetilde{T}_\varphi$. Conversely, given $\Omega\in\operatorname{coker}\widetilde{T}_\varphi$, we have

$$\lim_{\tau\to\infty} e^{-\tau\delta}\Omega = 0$$

and so we may define $\beta = e^{-2\tau\delta}\Omega\in\operatorname{coker}\widehat{T}_\varphi$. Hence, $\operatorname{coker}\widetilde{T}_\varphi\simeq\operatorname{coker}\widehat{T}_\varphi$, and $\operatorname{index}(\widetilde{T}_\varphi) = \operatorname{index}(\widehat{T}_\varphi)$. Finally, by [26, Thm. 1.2],

$$\operatorname{index}(\widetilde{T}_\varphi) = \operatorname{index}(T_\varphi) + \sum_{p\in D}\dim\mathsf{L}_p$$

This proves the Claim and completes the proof of the Proposition. □

We shall also need the following.

Proposition 3.24. *Recall the definition of* $\mathcal{R}^0_\delta$ *from* (3.7). *Then the operator*

$$(D'')^*D'' : \mathcal{R}^0_\delta\longrightarrow L^2_\delta(\mathfrak{sl}_E)$$

has index zero. Moreover, $\ker((D'')^*D'') = \ker D''$.

Proof. As in the proof of Proposition 3.23, we have

$$\operatorname{index}((D'')^*D'') = \operatorname{index}(T_{A_0})$$

where $T_{A_0} = \bar\partial^*_{A_0}\bar\partial_{A_0}$. Now consider the same operator with different domain:

$$\widetilde{T}_{A_0} : L^2_{2,\delta}(\mathfrak{sl}_E)\longrightarrow L^2_\delta(\mathfrak{sl}_E)$$

As in the proof of [26, Lemma 7.3],

$$\operatorname{index}(\widetilde{T}_{A_0}) = \dim\ker_{L^2_{2,\delta}}(\bar\partial^*_{A_0}\bar\partial_{A_0}) - \dim\ker_{L^2_{2,-\delta}}(\bar\partial^*_{A_0}\bar\partial_{A_0})$$

Expanding in terms of Fourier modes as in the proof of Proposition 3.3, we see that

$$\operatorname{index}(\widetilde{T}_{A_0}) = -\dim\bigoplus_{p\in D}\mathfrak{l}_p \tag{3.33}$$

Moreover, if $\eta\in\ker_{L^2_{2,-\delta}}(\bar\partial^*_{A_0}\bar\partial_{A_0})$, then $\eta\in\mathcal{R}^0_\delta$, and conversely by Proposition 3.3, $\mathcal{R}^0_\delta\subset L^2_{2,-\delta}$. The first statement then follows.

For the second statement, suppose $\eta\in\mathcal{R}^0_\delta$ satisfies $(D'')^*D''\eta=0$. Then it suffices to show $D''\eta\in L^2_\delta$. For if this is the case, then for the cut-off function ϕ_R as in (3.6), and using the fact that η is bounded (cf. Proposition 3.3), we have

$$\lim_{R\to+\infty}\left|\langle D''\eta,(d\phi_R)\eta\rangle_{L^2}\right| \le \lim_{R\to+\infty}\|D''\eta\|_{L^2_\delta}\|(d\phi_R)\eta\|_{L^2_{-\delta}}=0$$

It follows that

$$\begin{aligned}0&=\lim_{R\to+\infty}\langle (D'')^*D''\eta,(1-\phi_R)\eta\rangle_{L^2}\\&=\lim_{R\to+\infty}\left\{\langle D''\eta,(1-\phi_R)D''\eta\rangle_{L^2}-\langle D''\eta,(d\phi_R)\eta\rangle_{L^2}\right\}\\&=\|D''\eta\|^2_{L^2}\end{aligned}$$

and so $D''\eta=0$.

Now to show $D''\eta\in L^2_\delta$, first note that by the definition of $\mathcal{R}_\delta$, $\bar\partial_{A_0}\eta\in L^2_{1,\delta}$. We have $\bar\partial_A=\bar\partial_{A_0}+\beta$, with $\beta\in L^2_{1,\delta}$, and so $\bar\partial_A\eta\in L^2_\delta$ because η is also bounded. Write $\Phi=\Phi_0+\Phi_1$ according to Proposition 3.8. Then since b is bounded, $[\Phi_1,b]\in L^2_\delta$. Because of Assumption A, as in the proof of (3.23), we also have $[\Phi_0,b]\in L^2_\delta$. Hence, $D''\eta=(\bar\partial_A+\Phi)\eta\in L^2_\delta$, the proof is complete. ☐

3.2.2. *The Hodge slice.* Let (A,Φ) be a *smooth* parabolic Higgs pair. Consider the deformation complex

$$\begin{aligned}&\mathcal{C}^{par}_\delta(A,\Phi):\\&L^2_{2,\delta}(\mathfrak{sl}_E)\xrightarrow{d_1}L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\xrightarrow{d_2}L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)\end{aligned}\tag{3.34}$$

where

$$\begin{aligned}d_1(\eta)&=(\bar\partial_A\eta,[\Phi,\eta])\\d_2(\beta,\varphi)&=\bar\partial_A\varphi+[\Phi,\beta]\end{aligned}\tag{3.35}$$

Lemma 3.25 (Stable implies simple)**.** *Consider the extension of d_1 to the larger domain $\mathcal{R}^0_\delta$. If (A,Φ) is stable then* $\ker d_1=\{0\}$.

Proof. Let $\eta\in\ker d_1$. Let $(\mathcal{E},\Phi)$ be the parabolic bundle corresponding to (A,Φ) as in Proposition 3.17. Without loss of generality, we may assume $A=A_0$ on each cylinder $C(p)$. We now proceed as in Example 3.12. Namely, in the local unitary (resp. holomorphic) frame $\{e_i\}$ (resp. $\{s_i\}$) we write

$$\eta e_i=\sum_{j=1}^n\widehat{\eta}_{ij}e_j=\sum_{j=1}^n\eta_{ij}(z)|z|^{\alpha_j(p)-\alpha_i(p)}e_j$$

where the $\eta_{ij}(z)$ are holomorphic functions on $\Delta^\times(p)$. Since $\widehat{\eta}_{ij}$ is bounded (cf. Proposition 3.3), $\eta_{ij}(z)$ is regular on $\Delta(p)$. Moreover, $\eta_{ij}(0)=0$ if $\alpha_i(p)>\alpha_j(p)$. Hence, η defines a parabolic endomorphism. Since stability implies simplicity, this forces $\eta\equiv0$. ☐

Define the cohomology groups of the complex:

$$H^i(\mathcal{C}_\delta^{par}(A,\Phi)) = \begin{cases} \ker d_1 & i=0 \\ \ker d_2/\operatorname{im} d_1 & i=1 \\ \operatorname{coker} d_2 & i=2 \end{cases}$$

We have the formal L^2 adjoints:

$$d_1^*(\beta,\varphi) = \bar\partial_A^*\beta - i\Lambda[\Phi^*,\varphi]$$
$$d_2^* B = ([\Phi^*, i\Lambda B], \bar\partial_A^* B)$$

For the weighted Sobolev spaces, the natural adjoints are $d_1^{*\delta} := e^{-\tau\delta} d_1^* e^{\tau\delta}$ and $d_2^{*\delta} := e^{-\tau\delta} d_2^* e^{\tau\delta}$. We have the following immediate but important consequence of Lemma 3.25.

Proposition 3.26. *The space $\mathcal{B}_\delta^{par,s}$ (resp. $\mathcal{B}_\delta^{spar,s}$) is a smooth Banach submanifold of $L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)$ (resp. $L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)$).*

Proof. By the implicit function theorem, it suffices to prove that d_2 is surjective if (A,Φ) is stable. After a complex gauge transformation we may assume that $A=A_0$ on each C_i. Then as in the proof of Proposition 3.23, we see that if $B\in(\operatorname{im} d_2)^\perp$, then $e^{\tau\delta}B\in\mathcal{R}^0_\delta$ and indeed, $e^{\tau\delta}i\Lambda B^*\in\ker d_1$. Hence, $\operatorname{coker} d_2=\{0\}$ by Lemma 3.25. ☐

Definition 3.27. Given $(A,\Phi)\in\mathcal{B}^{par}_\delta$ smooth, let

$$\mathbf{H}^{par}_\delta(A,\Phi) = \{(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\mid d_2(\beta,\varphi)=0\ ,\ d_1^{*\delta}(\beta,\varphi)=0\}$$

Lemma 3.28. *Given $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$ smooth, we have:*

$$H^1(\mathcal{C}^{par}_\delta(A,\Phi))\simeq\mathbf{H}^{par}_\delta(A,\Phi)$$

Proof. Let $(\beta,\varphi)\in\ker d_2$. Because of Proposition 3.13, $d_1^*(\beta,\varphi)\in L^2_\delta$. By Lemma 3.25,

$$d_1^{*\delta}d_1 : L^2_{2,\delta}(\mathfrak{sl}_E)\longrightarrow L^2_\delta(\mathfrak{sl}_E)$$

is surjective, so we can find η so that $d_1^{*\delta}d_1\eta=-d_1^{*\delta}(\beta,\varphi)$. Then $(\beta,\varphi)+d_1\eta\in\mathbf{H}^{par}_\delta(A,\Phi)$. ☐

By the same argument as in the proof of Proposition 3.26 (in this case $e^{\tau\delta}i\Lambda B^*\in L^2_{2,\delta}$), we have

Lemma 3.29 (Serre duality)**.** *For $(A,\Phi)\in\mathcal{B}^{par}_\delta$, we have:*

$$H^2(\mathcal{C}^{par}_\delta(A,\Phi))\simeq H^0(\mathcal{C}^{par}_\delta(A,\Phi))^*$$

By Lemma 3.29 and Proposition 3.23, it follows that

$$\dim\mathbf{H}^{par}_\delta(A,\Phi) = (2g-2+d)\dim\mathsf{G}+\sum_{p\in D}\dim\mathsf{L}_p \tag{3.36}$$

for all $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$.

Let us also recall the strongly parabolic deformation complex. Let $(A,\Phi)\in\mathcal{B}^{par}_\delta$ be smooth and define:

$$\begin{aligned}&\mathcal{C}^{spar}_\delta(A,\Phi):\\ &L^2_{2,\delta}(\mathfrak{sl}_E)\xrightarrow{d_1}L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)\xrightarrow{d_2}L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)\end{aligned}\tag{3.37}$$

Note that by (3.21) the complex is well-defined, i.e. we *do not* need to assume that (A,Φ) is strongly parabolic. Define the harmonics

$$\mathbf{H}^{spar}_\delta(A,\Phi)\simeq H^1(\mathcal{C}^{spar}_\delta(A,\Phi)) := \ker d_2/\operatorname{im} d_1$$

in the same way as (3.27). Then as in the proof of Proposition 3.26, for stable Higgs bundles we have vanishing of H^0 and H^2 of the complex (3.37), and so

$$\begin{aligned}\dim \mathbf{H}^{spar}_\delta(A,\Phi) &= (2g-2)\dim \mathsf{G} + 2\sum_{p\in D}\dim(\mathsf{G}/\mathsf{P}_p) + \sum_{p\in D}\dim \mathsf{L}_p \\ &= \dim \mathbf{H}^{par}_\delta(A,\Phi) - \sum_{p\in D}\dim \mathsf{L}_p\end{aligned} \tag{3.38}$$

for all $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$ (cf. (3.36)).

We have the following useful consequence. From Proposition 3.8, there is a well-defined residue map

$$\mathbf{H}^{par}_\delta(A,\Phi)\longrightarrow\bigoplus_{p\in D}\mathfrak{l}_p : [(\beta,\varphi)]\mapsto \lim_{\tau\to+\infty}\varphi_0 \tag{3.39}$$

The kernel is exactly $\mathbf{H}^{spar}_\delta(A,\Phi)$. The dimension count above (3.38) then implies the next result.

Corollary 3.30. *The map (3.39) is surjective.*

The space $\mathbf{H}^{par}_\delta(A,\Phi)$ (resp. $\mathbf{H}^{spar}_\delta(A,\Phi)$) is identified with the tangent space to $\mathbf{M}^{par,s}_{\mathsf{Dol},*}$ (resp. $\mathbf{M}^{spar,s}_{\mathsf{Dol},*}$) for $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$ (resp. $(A,\Phi)\in\mathcal{B}^{spar,s}_\delta$) at the point $[(A,\Phi)]$. We now proceed to define coordinate neighborhoods via the Kuranishi method. For $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$, following [12, Def. 3.1], we define the *Hodge slice* by

$$\begin{aligned}&\mathcal{S}^{par}_\delta(A,\Phi) = \\ &\quad\left\{(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus\mathscr{D}_\delta \mid d_2(\beta,\varphi)+[\beta,\varphi]=0\ ,\ d_1^{*_\delta}(\beta,\varphi)=0\right\}\end{aligned} \tag{3.40}$$

Consider the operator

$$d_2d_2^{*_\delta} : L^2_{2,\delta}(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)\longrightarrow L^2_\delta(\mathfrak{sl}_E\otimes K_X\otimes\overline{K}_X)$$

Again using [26, Thm. 1.1], along with (3.22) and the argument in the proof of Lemma 3.6, this is a Fredholm operator of index 0, and in fact by Lemma 3.29 it is invertible. Let $G^{(2)}_\delta$ denote its Green's operator. We then define the *Kuranishi map*

$$\mathbf{k} : \ker d_1^{*_\delta}\longrightarrow \ker d_1^{*_\delta} : (\beta,\varphi)\mapsto(\beta,\varphi)+d_2^{*_\delta}G^{(2)}_\delta([\beta,\varphi]) \tag{3.41}$$

An application of the inverse function theorem shows that $\mathbf{k}$ defines a homeomorphism of neighborhoods U,V of the origin from

$$\mathbf{k} : \mathcal{S}^{par}_\delta(A,\Phi)\supset U \xrightarrow{\ \sim\ } V\subset \mathbf{H}^{par}_\delta(A,\Phi)$$

Similarly, if $(A,\Phi)\in\mathcal{B}^{spar,s}_\delta$ we define

$$\begin{aligned}&\mathcal{S}^{spar}_\delta(A,\Phi) = \\ &\left\{(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\oplus L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X) \mid d_2(\beta,\varphi)+[\beta,\varphi]=0\ ,\ d_1^{*_\delta}(\beta,\varphi)=0\right\}\end{aligned} \tag{3.42}$$

In this case, the Kuranishi map $\mathbf{k}$ defines a homeomorphism of neighborhoods of the origin from $\mathcal{S}^{spar}_\delta(A,\Phi)$ to $\mathbf{H}^{spar}_\delta(A,\Phi)$.

3.2.3. *Proof of Theorem 3.22.* The existence of a complex manifold structure on the moduli space is now very standard (cf. [22] and [24, Ch. VII]).

Proposition 3.31 (Local slice)**.** *Let $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$ be a smooth, stable, parabolic Higgs pair. Then the map*

$$p_{\mathrm{H}} : \mathcal{S}^{par}_\delta(A,\Phi)\longrightarrow\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta) : (\beta,\varphi)\mapsto[(\bar\partial_A+\beta,\Phi+\varphi)]$$

is a local homeomorphism from an open neighborhood of the origin in $\mathcal{S}^{par}_\delta(A,\Phi)$ to an open neighborhood of $[(A,\Phi)]$. This gives a local coordinate chart on $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$. The transition functions for the local coordinate charts constructed in this way are holomorphic.

We first need a generalization of the Poincaré inequality.

Lemma 3.32. *Fix (A,Φ) be as in Proposition 3.31, and let d_1 denote the operator defined in (3.35). Then there is a constant $C>1$, depending on (A,Φ), such that for any $\eta\in L^2_{2,\delta}(\mathfrak{sl}_E)$,*

$$\|\eta\|_{L^2_{1,\delta}}\leq C\|d_1\eta\|_{\mathscr{D}_\delta}\ .$$

Proof. To the contrary, suppose the existence of a sequence η_j with $\|\eta_j\|_{L^2_{1,\delta}}=1$ and $\|d_1\eta_j\|_{\mathscr{D}_\delta}\to 0$. Notice that from the elliptic estimate this implies that $\|\eta_j\|_{L^2_{2,\delta}}$ is uniformly bounded. We may therefore assume $\eta_j\to\eta$ weakly in $L^2_{2,\delta}$. Choose a cut-off function ψ vanishing on X_0 and identically $=1$ for τ large. Since A_0 is flat on the cylinders $C(p)$, (3.5) implies

$$\|\psi\eta_j\|_{L^2_{1,\delta}}\leq C\|d_{A_0}(\psi\eta_j)\|_{L^2_\delta}\leq 2C\|\bar\partial_{A_0}(\psi\eta_j)\|_{L^2_\delta} \tag{3.43}$$

Now $A=A_0+\beta$ with $\beta\in L^2_{1,\delta}$. By the embedding $L^2_{1,\delta}\hookrightarrow L^4_\delta$, we have

$$\|[\beta,\psi\eta_j]\|_{L^2_\delta}\leq C\|\beta\|_{L^2_{1,\delta}(\mathrm{supp}(\psi))}\|\psi\eta_j\|_{L^2_{1,\delta}}$$

Hence, we may choose ψ so that the inequality (3.43) holds with A_0 replaced by A (and a different constant). By readjusting ψ, we conclude as in the proof of (3.5), that $\|\eta\|_{L^2_{1,\delta}}\geq 1/2$ and $\bar\partial_A\eta=0$. Since $\|[\Phi,\eta_j]\|_{L^2_{-\delta}}\to 0$ and $\eta_j\to\eta$ strongly on compact sets, we have $[\Phi,\eta]=0$. This then contradicts the assumption that $\ker d_1=\{0\}$. □

Proof of Proposition 3.31. The first assertion now follows exactly as in [22, Lemma 1.7]. For the sake of completeness, we repeat the argument in our case. Fix a smooth (A,Φ). If $(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\times\mathscr{D}_\delta$, and $g\in\mathcal{G}_{\delta,*}$, denote by $g(\beta,\varphi):=(\widetilde\beta,\widetilde\varphi)$, where

$$g(\bar\partial_A+\beta,\Phi+\varphi)=(\bar\partial_A+\widetilde\beta,\Phi+\widetilde\varphi)\ .$$

With this understood, define the map

$$\Psi: L^2_{2,\delta}(\mathfrak{sl}_E)\times L^2_{1,\delta}(\mathfrak{sl}_E\otimes\overline{K}_X)\times\mathscr{D}_\delta\longrightarrow L^2_\delta(\mathfrak{sl}_E)\ :\ (\eta,\beta,\varphi)\mapsto\ d_1^{*\delta}[e^\eta(\beta,\varphi)]$$

Then the derivative $D_1\Psi(0,0,0)$ with respect to η, evaluated at the origin $(\eta,\beta,\varphi)=(0,0,0)$, is $d_1^{*\delta}d_1$. Since (A,Φ) is stable, by Lemma 3.25, $\ker d_1=\{0\}$, and $D_1\Psi(0,0,0)$ is therefore an isomorphism. By the implicit function theorem, for (β,φ) in a neighborhood U of the origin there is a unique $g\in\mathcal{G}_{\delta,*}$ in a neighborhood V of the identity so that $g(\beta,\varphi)\in\ker d_1^{*\delta}$.

Suppose now that $g(\beta,\varphi)=(\widetilde\beta,\widetilde\varphi)$ for some $g\in\mathcal{G}_{\delta,*}$, where $(\beta,\varphi),(\widetilde\beta,\widetilde\varphi)\in\ker d_1^{*\delta}$. If we define η by $g=1+\eta$, then by Proposition 3.3, $\eta\in L^2_{2,\delta}(\mathfrak{gl}_E)$. We compute:

$$d_1\eta=(\beta-\widetilde\beta+\eta\beta-\widetilde\beta\eta,\varphi-\widetilde\varphi+\eta\varphi-\widetilde\varphi\eta)\ .$$

Using Proposition 3.13 (which holds equally well for $\mathfrak{gl}_E$-valued sections), we obtain the estimate

$$\|d_1\eta\|_{\mathscr{D}_\delta}\leq C'(1+\|\eta\|_{L^2_{2,\delta}})(\|(\beta,\varphi)\|_{\mathscr{D}_\delta}+\|(\widetilde\beta,\widetilde\varphi)\|_{\mathscr{D}_\delta}) \tag{3.44}$$

for a constant C'. The usual elliptic estimate gives

$$\|\eta\|_{L^2_{2,\delta}}\leq C(\|\eta\|_{L^2_{1,\delta}}+\|\bar\partial_{A_0}\eta\|_{L^2_{1,\delta}})\ .$$

By patching the elliptic estimate on a compact set to the argument using multiplication properties as above, we can replace A_0 with A in the above equation to obtain

$$\|\eta\|_{L^2_{2,\delta}}\leq C(\|\eta\|_{L^2_{1,\delta}}+\|d_1\eta\|_{\mathscr{D}_\delta})\ .$$

Finally, using this, along with (3.44) and Lemma 3.32, we have

$$\|\eta\|_{L^2_{2,\delta}}\leq\frac{C'(\|(\beta,\varphi)\|_{\mathscr{D}_\delta}+\|(\widetilde\beta,\widetilde\varphi)\|_{\mathscr{D}_\delta})}{1-CC'(\|(\beta,\varphi)\|_{\mathscr{D}_\delta}+\|(\widetilde\beta,\widetilde\varphi)\|_{\mathscr{D}_\delta})}$$

If (β,φ) and $(\widetilde{\beta},\widetilde{\varphi})$ are sufficiently small, $g\in V$ and is therefore the identity. This proves that the slice is homeomorphic onto its image. The second assertion on the holomorphicity follows similarly, as in [22, Lemma 2.3]. ☐

Completion of the Proof of Theorem 3.22. By the discussion at the end of the previous section, $\mathcal{S}^{par}_\delta(A_0,\Phi_0)$ is locally homeomorphic to a domain in complex euclidean space of the correct dimension. This gives a local coordinate chart on $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$. As we also have that the transition functions are holomorphic, this therefore gives $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$ the structure of a complex manifold. The same arguments apply to the strongly parabolic case. This proves statements (1) and (2) of the theorem.

For (3), we claim that the based moduli spaces $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$ and $\mathbf{M}^{spar,s}_{\mathsf{Dol},*}(\alpha,\delta)$ represent the moduli functors for families of *framed* parabolic and strongly parabolic Higgs bundles introduced in [27]. Below we shall only give a few details in the parabolic case, for example.

An important first step is the existence of a universal family on $X\times\mathcal{S}^{par,s}_\delta(A_0,\Phi_0)$. Let $\widehat{E}\to X^\times\times\mathcal{S}^{par,s}_\delta(A_0,\Phi_0)$ be the pullback of the bundle $E\to X^\times$ above. Consider the $\bar\partial$-operator given at the point (β,φ)

$$\bar\partial_{\widehat{E}}=\bar\partial_{\mathcal{S}}+\bar\partial_{A_0}+\beta$$

Here, $\bar\partial_{\mathcal{S}}$ is short-hand for the Cauchy-Riemann operator on $\mathcal{S}^{par,s}_\delta(A_0,\Phi_0)$, which acts on sections of $\widehat{E}$ because it is a pullback from $X^\times$. Notice that β, regarded as a form on $\mathcal{S}^{par,s}_\delta(A_0,\Phi_0)$, varies holomorphically. Hence, $\bar\partial^2_{\widehat{E}}=0$, and this therefore gives a holomorphic structure on $\widehat{\mathcal{E}}$ on $\widehat{E}$. As in Proposition 3.17, there is a natural extension, also denoted $\widehat{\mathcal{E}}$, of $\widehat{E}$ as a holomorphic bundle on $X\times\mathcal{S}^{par,s}_\delta(A_0,\Phi_0)$. This extension depends on a choice of trivial holomorphic bundle on $D\times\mathcal{S}^{par,s}_\delta(A_0,\Phi_0)$. The parabolic structure is given by the growth of holomorphic sections with respect to the background metric on $\widehat{E}$. The associated graded bundle on $D\times\mathcal{S}^{par,s}_\delta$ is naturally trivialized by the germs of the s_i. Finally, the relative logarithmic Higgs field is defined by $\widehat{\Phi}_0+\varphi$, where $\widehat{\Phi}_0$ is the pullback of Φ_0 and φ is the tautological form defined as with β.

Using the existence of the universal family one can show as in [37, Sec. 6], or the detailed exposition in [16, Sec. 4.2], that $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$ locally represents the analytic functor of framed parabolic bundles. The construction of an algebraic space, $\mathcal{P}_0(\alpha)$, representing the algebraic functor is outlined in [27]. The isomorphism of $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$ with the analytification of $\mathcal{P}_0(\alpha)$ then follows as in [33]. Finally, it is clear that the identification is equivariant with respect to the action of $\overline{\mathbf{L}}$, and this in turn proves assertion (3) of Theorem 3.22. ☐

3.2.4. *Variation of the harmonic metric.* Having defined the manifold structure on the framed moduli space $\mathbf{M}^{par,s}_{\mathsf{Dol},*}(\alpha,\delta)$ and its quotient $\mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta)$, we can now prove a statement about the variation of the harmonic metric. Notice that the space (3.29) of admissible metrics has the structure of a Banach manifold with tangent space

$$T_{h_0}\mathcal{M}_\delta\simeq\sqrt{-1}\,\mathcal{U}_\delta\ .$$

Theorem 3.33 (Variation of harmonic metric). *Let $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$ be a solution to (3.28) for the metric h_0, and let $\mathscr{H}:\mathcal{S}^{par}_\delta(A,\Phi)\to\mathcal{M}_\delta$ denote the family of harmonic metrics from Theorem 3.21. Then $\mathscr{H}$ is a C^∞ map. Moreover, the derivative $\mathscr{H}$ at the origin vanishes.*

Proof. Define the map

$$\begin{aligned}\mathscr{N}:\mathcal{S}^{par}_\delta(A,\Phi)\times\mathcal{M}_\delta&\longrightarrow L^2_\delta(i\mathfrak{su}_E)\\ (\beta,\varphi;h)&\mapsto i\Lambda\left(F_{(\bar\partial_E+\beta,h)}+[\Phi+\varphi,(\Phi+\varphi)^{*_h}]\right)\end{aligned}$$

Then the derivative of $\mathcal{N}$ with respect to $\mathcal{M}_\delta$ at the origin is

$$d_2\mathcal{N}_{(0,0;h_0)} : \sqrt{-1}\,\mathcal{U}_\delta \longrightarrow L^2_\delta(i\mathfrak{su}_E) : \dot{h} \mapsto i\Lambda\left(\bar{\partial}_E\partial_E^{h_0}(\dot{h}) + [\Phi,[\Phi^*_{h_0},\dot{h}]]\right)$$

A calculation shows that $d_2\mathcal{N}_{(0,0;h_0)}(\dot{h}) = (D'')^*D''(\dot{h})$. By Proposition 3.24 (note that $(D'')^*D''$ preserves the hermitian subspace) and Lemma 3.25, we conclude that $d_2\mathcal{N}_{(0,0;h_0)}$ is an isomorphism. The smoothness of $\mathscr{H}$ then follows from the implicit function theorem. The second statement follows exactly as in [12, Prop. 3.12]. □

3.2.5. *The hyperkähler metric.* Recall the residue map (see (3.27)

$$\mathrm{Res} : \mathbf{M}^{par,s}_{\mathsf{Dol}}(\alpha,\delta) \longrightarrow \bigoplus_{p\in D}\mathfrak{l}_p$$

(as well as Assumption A). The following is an immediate consequence of Corollary 3.30.

Proposition 3.34. *The map* $\mathrm{Res}_{\mathfrak{l}}$ *is a holomorphic submersion.*

The goal of this section is to prove the following.

Theorem 3.35 (Hyperkähler structure)**.** *For any* $\ell \in \bigoplus_{p\in D}\mathfrak{l}_p$*, the fiber* $\mathrm{Res}_{\mathfrak{l}}^{-1}(\ell)$ *is a hyperkähler manifold.*

This generalizes the results of Konno [25] and Nakajima [36]. The first step is the definition of a holomorphic 2-form. On $\mathcal{B}^{par,s}_\delta$, we set:

$$\Omega((\beta_1,\varphi_1),(\beta_2,\varphi_2)) := i\int_{X^\times}\mathrm{tr}(\varphi_2\wedge\beta_1 - \varphi_1\wedge\beta_2) \tag{3.45}$$

This is well-defined, since we have a natural duality $L^2_{-\delta}\times L^2_\delta \hookrightarrow L^1$. Clearly, Ω is holomorphic and closed. We have the following:

Proposition 3.36. *The form* Ω *descends to a holomorphic symplectic form* Ω_* *on* $\mathbf{M}^{par,s}_{\mathsf{Dol},*}$.

We first prove the following.

Lemma 3.37. *Let* $(A,\Phi)\in\mathcal{B}^{par,s}_\delta$ *be a smooth Higgs pair, and let* $(\beta,\varphi)\in L^2_{1,\delta}$ *be in the kernel of* d_2*. Then there is a unique* $\eta\in\mathcal{R}^0_\delta$ *such that* $(\beta,\varphi)+d_1\eta\in\ker d_1^*$.

Proof. It suffices to solve $d_1^*d_1\eta = -d_1^*(\beta,\varphi)$. By Proposition 3.24, this is possible if $\ker(d_1^*d_1) = \{0\}$ on $\mathcal{R}^0_\delta$. As in the proof of Proposition 3.3, it follows easily that $\ker(d_1^*d_1) = \ker d_1$, and hence the desired result is a consequence of Lemma 3.25. □

Proof of Proposition 3.36. We have

$$\Omega((\beta,\varphi),d_1\eta) := -i\int_{X^\times}\mathrm{tr}(d_2(\beta,\varphi)\eta) + i\int_{X^\times}d\,\mathrm{tr}(\varphi\eta) \tag{3.46}$$

The second term on the right hand side vanishes for $\eta\in L^2_{2,\delta}$ (cf. (3.21)). Hence, Ω descends, and we denote by Ω_* the resulting holomorphic form on $\mathbf{M}^{par,s}_{\mathsf{Dol},*}$. Now suppose $(\beta,\varphi)\in\ker d_2$ is such that $\Omega_*((\beta,\varphi),\cdot)\equiv 0$. Recall from Proposition 3.8 that there is a decomposition $\varphi = \varphi_0+\varphi_1$, where $\varphi_1\in L^2_{1,\delta}$, and a well-defined limit $\varphi_\infty := \lim_{\tau\to+\infty}\varphi_0$. If $\ell\in\bigoplus_{p\in D}\mathfrak{l}_p$ is arbitrary, and $\eta\in\mathcal{R}^0_\delta$ such that $\lim_{\tau\to+\infty}\eta = \ell$, then the calculation in (3.46) shows that

$$0 = \Omega((\beta,\varphi),d_1\eta) := \mathrm{tr}(\varphi_\infty\ell)$$

Since ℓ is arbitrary, we see that $\varphi_\infty = 0$, and hence $\varphi\in L^2_{1,\delta}$. By Lemma 3.37, we can then find a different $\eta\in\mathcal{R}^0_\delta$ such that

$$(\widetilde{\beta},\widetilde{\varphi}) = (\beta,\varphi)+d_1\eta\in\ker d_1^*$$

Notice that $(-\widetilde{\varphi}^*, \widetilde{\beta}^*) \in \ker d_2$. The condition

$$\Omega((\widetilde{\beta}, \widetilde{\varphi}), (\widetilde{\varphi}^*, -\widetilde{\beta}^*)) = 0$$

implies that $(\widetilde{\beta}, \widetilde{\varphi}) = 0$, or that $(\beta, \varphi) = -d_1\eta$. Recall that η has a well-defined limit $\eta_\infty := \lim_{\tau\to+\infty} \eta$. By Corollary 3.30, for arbitrary $\ell \in \bigoplus_{p\in D} \mathfrak{l}_p$ we can find $(\beta_1, \varphi_1) \in \ker d_2$ such that $(\varphi_1)_\infty = \ell$. Again using (3.46), we have

$$0 = \Omega_*((\beta, \varphi), (\beta_1, \varphi_1)) = \mathrm{tr}(\eta_\infty \ell)$$

Since ℓ is arbitrary, we conclude that $\eta_\infty = 0$, and so by Proposition 3.3, $\eta \in L^2_{2,\delta}$. But then the class of (β, η) is zero. Hence, Ω_* is nondegenerate. □

Proposition 3.38. *The symplectic form Ω induces a Poisson structure on $\mathbf{M}^{par,s}_{\mathsf{Dol}}$ via the fibration*

$$p_* : \mathbf{M}^{par,s}_{\mathsf{Dol},*} \longrightarrow \mathbf{M}^{par,s}_{\mathsf{Dol}}$$

The symplectic leaves of $\mathbf{M}^{par,s}_{\mathsf{Dol}}$ are the fibers $\mathrm{Res}_{\mathfrak{l}}^{-1}(\ell)$, for $\ell \in \bigoplus_{p\in D} \mathfrak{l}_p$.

Proof. Since the residual gauge group $\overline{\mathbf{L}}$ preserves the symplectic form Ω_*, it is standard that the Poisson structure on $\mathbf{M}^{par,s}_{\mathsf{Dol},*}$ induces one on the quotient $\mathbf{M}^{par,s}_{\mathsf{Dol}}$. It is easy to see using Proposition 3.8 and (3.46) that the fibers of p_* are isotropic. It follows that the rank of the Poisson structure on $\mathbf{M}^{par,s}_{\mathsf{Dol}}$ coincides with the dimension of the fibers $\mathrm{Res}_{\mathfrak{l}}^{-1}(\ell)$. Hence, to show these are the symplectic leaves it suffices to show that Ω_* descends to a symplectic form Ω_ℓ on each $\mathrm{Res}_{\mathfrak{l}}^{-1}(\ell)$. The tangent space to a fiber of

$$\mathbf{M}^{par,s}_{\mathsf{Dol},*} \longrightarrow \bigoplus_{p\in D} \mathfrak{l}_p$$

consists of $(\beta, \varphi) \in \ker d_2$ with $\varphi \in L^2_{2,\delta}$. The fact that the restriction of Ω_* to this fiber descends follows again from (3.46). To prove nondegeneracy, note that by Lemma 3.37 we may assume $(\beta, \varphi) \in \ker d_1^*$. Then $(-\varphi^*, \beta^*) \in \ker d_2$, and

$$\Omega_*((\beta, \varphi), (-\varphi^*, \beta^*)) = \|\beta\|^2_{L^2} + \|\varphi\|^2_{L^2}$$

Nondegeneracy of Ω_ℓ follows, and this completes the proof. □

Proof of Theorem 3.35. This now follows as in the case of Higgs bundles on closed Riemann surfaces. Namely, on $\mathrm{Res}^{-1}(\ell)$ we choose representatives for tangent vectors $(\beta, \varphi) \in \ker d_2 \cap \ker d_1^*$. That we may do this follows from Lemma 3.37. We then define almost complex structures

$$I(\beta, \varphi) = (i\beta, i\varphi)$$
$$J(\beta, \varphi) = (i\varphi^*, -i\beta)$$
$$K(\beta, \varphi) = (-\varphi^*, \beta^*)$$

Then with respect the L^2 metric the associated fundamental forms $\Omega_I = \Omega_\ell$, ω_J, and ω_K are closed. The integrabilty of J and K then follows from [19, Lemma 6.8]. The complex structure we have defined on $\mathrm{Res}_{\mathfrak{l}}^{-1}(x)$ coincides with I. □

3.3. deRham moduli spaces.

3.3.1. *Construction of moduli of flat bundles.* The method used in §3.2 to construct the Dolbeault moduli spaces of parabolic Higgs bundles may be used to form the deRham moduli spaces. As in the case of Higgs bundles, we only describe the set-up for the strongly parabolic case, or the case of full flags. Since the details should be clear from the previous discussion, we omit proofs.

In parallel with Definition 3.11, consider the following

Definition 3.39. Let

$$\mathcal{F}_\delta^{par} = \{(A,\Phi)\in\mathcal{A}_\delta\times\mathscr{D}_\delta(\mathfrak{sl}_E\otimes K_X)\mid \nabla=d_A+\Phi \text{ is projectively flat}\}$$
$$\mathcal{F}_\delta^{spar} = \left\{(A,\Phi)\in\mathcal{A}_\delta\times L^2_{1,\delta}(\mathfrak{sl}_E\otimes K_X)\mid \nabla=d_A+\Phi \text{ is projectively flat}\right\}$$

The flatness condition is

$$F_\nabla^\perp = F_A^\perp + \bar\partial_A\Phi = 0 \tag{3.47}$$

which makes sense as an equation in L^2_δ because of the defining property of $\mathscr{D}_\delta$ (see Remark 3.10). The group $\mathcal{G}_\delta$ acts on $\mathcal{F}_\delta^{par}$ by pulling back the complex connection ∇ and rewriting in terms of a unitary and $(1,0)$ part. Explicitly, for $g\in\mathcal{G}_\delta$,

$$g(\nabla) := g\circ\nabla\circ g^{-1} = d_{g(A)} + g^{-1}\Phi g - \partial_A(g)g^{-1} - (g^*)^{-1}\partial_A(g^*)\ .$$

By Lemma 3.6, we can find a gauge transformation such that $\bar\partial_\nabla=\bar\partial_A$ is a smooth $\bar\partial$ operator. Hence, F_A is smooth, and it then follows from elliptic regularity and (3.47) that Φ is smooth. So ∇ defines a holomorphic connection on $X^\times$. Using Lemma 3.6 once again, we may assume $A=A_0$ on the cylinders. Since A_0 is flat on the cylinders, (3.47) implies $\bar\partial_{A_0}\Phi=0$ on each $C(p)$. From Example 3.12 one sees that ∇ defines a logarithmic connection on X on a holomorphic bundle with a parabolic structure, and $\mathrm{Res}_p(\nabla)=\alpha(p)+\mathrm{Res}_p(\Phi)$ for each $p\in D$.

Set

$$\mathbf{M}_{\mathsf{dR}}^{spar,s}(\alpha,\delta) := \mathcal{G}_\delta\backslash\mathcal{F}_\delta^{spar,s}\ ,\quad \mathbf{M}_{\mathsf{dR}}^{par,s}(\alpha,\delta) := \mathcal{G}_\delta\backslash\mathcal{F}_\delta^{par,s}\ .$$

Then we have the following:

Theorem 3.40 (Moduli spaces of flat connections)**.** *Fix α, and suppose δ satisfies (3.3). Then if nonempty the moduli spaces $\mathbf{M}_{\mathsf{dR}}^{spar,s}(\alpha,\delta)$ and $\mathbf{M}_{\mathsf{dR}}^{par,s}(\alpha,\delta)$ are smooth complex manifolds of complex dimensions*

$$\dim\mathbf{M}_{\mathsf{dR}}^{spar,s}(\alpha,\delta) = (2g-2)\dim\mathsf{G} + 2\sum_{p\in D}\dim(\mathsf{G}/\mathsf{P}_p)$$
$$\dim\mathbf{M}_{\mathsf{dR}}^{par,s}(\alpha,\delta) = (2g-2+d)\dim\mathsf{G}\ .$$

Moreover, the assignment of a logarithmic connection to (A,Φ) described above induces biholomorphism

$$\mathbf{M}_{\mathsf{dR}}^{spar,s}(\alpha,\delta)\xrightarrow{\sim}\mathcal{SP}_1(\alpha)\quad,\quad \mathbf{M}_{\mathsf{dR}}^{par,s}(\alpha,\delta)\xrightarrow{\sim}\mathcal{P}_1(\alpha)\ .$$

3.3.2. *Joint parametrization by the Hodge slice.* Let $(A,\Phi)\in\mathcal{B}_\delta^{par,s}$ satisfy the Hitchin equations, so that $D=D''+D'$, where $D''=\bar\partial_A+\Phi$, $D'=\partial_A+\Phi^*$ is a flat connection: $D\in\mathcal{F}_\delta^{par,s}$. Set:

$$D'_\delta := e^{-\tau\delta}D'e^{\tau\delta}\quad,\quad D_\delta = D''+D'_\delta$$

Then: $F_{D_\delta}=\delta\bar\partial\partial\tau\cdot id$. Fix a U(1) connection ∇_δ^0 on the trivial line bundle $\underline{\mathbb{C}}$ with curvature $-\delta\bar\partial\partial\tau$, i.e. the Chern connection for the metric $e^{-\tau\delta}$. Then clearly, under the identification $\underline{\mathbb{C}}\otimes E\simeq E$ given by the trivialization of $\underline{\mathbb{C}}$, $D=\nabla_\delta^0\otimes D_\delta$. We now define the map:

$$p_{\mathsf{dR}}:\mathcal{S}_\delta^{par}(A,\Phi)\to\mathbf{M}_{\mathsf{dR},*}^{par,s}(\alpha,\delta):(\beta,\varphi)\mapsto[\nabla_\delta^0\otimes(D_\delta+\beta+\varphi)] \tag{3.48}$$

Then p_{dR} is well-defined on a neighborhood of the origin in the slice. We have the following

Proposition 3.41. *The map p_{dR} gives a biholomorphism of a neighborhood of the origin in $\mathcal{S}_\delta^{par}(A,\Phi)$ to a neighborhood of $[D]\in\mathbf{M}_{\mathsf{dR},*}^{par,s}(\alpha,\delta)$.*

By the discussion in §3.3.1, there is a well-defined residue map

$$\mathrm{Res}:\mathbf{M}_{\mathsf{dR}}^{par,s}(\alpha,\delta)\longrightarrow\bigoplus_{p\in D}\mathfrak{l}_p:\nabla\mapsto\bigoplus_{p\in D}\mathrm{Res}_p(\nabla)$$

We state the following without proof (cf. proof of Theorem 3.35).

Theorem 3.42. *For any $\ell \in \bigoplus_{p\in D} \mathfrak{l}_p$, the spaces $\mathrm{Res}^{-1}(\ell) \subset \mathbf{M}_{\mathsf{dR}}^{par,s}(\alpha,\delta)$ are holomorphic symplectic manifolds.*

4. Stratifications and conformal limits

4.1. **Hodge pairs.** The goal of this section is to define a generalization of the Białynicki-Birula decomposition of the Dolbeault moduli space in the parabolic setting. This is based on Simpson's notion of a Hodge bundle [42], which in the moduli space is a fixed point for the $\mathbb{C}^*$-action. In our analytic setting, there is a subtlety regarding the relationship between the Hodge and parabolic filtrations. We address this in §4.1.1. In §4.1.2 we define the Białynicki-Birula (or BB) slices, that were first introduced in [12] in the nonparabolic setting.

4.1.1. *The parabolic and Hodge filtrations.* There is a natural action of $\mathbb{C}^*$ on $\mathcal{B}_\delta^{par}$ given by $\lambda \cdot (A,\Phi) = (A, \lambda\Phi)$. Since this commutes with the action of $\mathcal{G}_\delta$, we have a $\mathbb{C}^*$-action on $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$. Suppose $[(A,\Phi)] \in \mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$ is a fixed point of $\mathbb{C}^*$. Then as in the closed surface case (cf. [19, eq. (7.2)]), there is a representative (A,Φ) of $[(A,\Phi)]$ so that for every ϑ there is $g_\vartheta \in \mathcal{K}_\delta$, with the property that $d_A = g_\vartheta(d_A)$ and $e^{i\vartheta}\Phi = g_\vartheta \Phi g_\vartheta^{-1}$. Note that this implies that g_ϑ has constant eigenvalues. It follows that $\mathcal{E} = (E, \bar\partial_A)$ splits holomorphically and isometrically as

$$\begin{aligned} &\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell \\ &\Phi : \mathcal{E}_a \to \mathcal{E}_{a+1} \otimes K_X \ , \ a = 1, \ldots, \ell-1 \end{aligned} \tag{4.1}$$

and Φ annihilates $\mathcal{E}_\ell$. The important point is that for generic ϑ the E_i are precisely the eigenbundles of g_ϑ. Following Simpson, we call such a Higgs pair a *Hodge pair*.

The following compares the quasi-parabolic structure with the Hodge splitting.

Proposition 4.1. *Let (A,Φ) be a Hodge pair. Then after possibly acting by an element of $\mathcal{K}_{\delta,*}$, the C^∞ splitting (4.1) is preserved by d_{A_0} and is translation invariant on each $C(p)$, $p \in D$.*

Proof. Let

$$\mathcal{K}_\delta^c = \{ g \in \mathcal{K}_\delta \mid \mathbf{b}(g) = \mathbf{b}(g_\vartheta) \ , \ \mathsf{char}(g) \text{ is constant } \}$$

where char denotes the (pointwise) characteristic polynomial. Then $\mathcal{K}_\delta^c \subset \mathcal{K}_\delta$ is a Banach submanifold with tangent space

$$T_g \mathcal{K}_\delta^c = \left\{ [g,H] \mid H \in L^2_{2,\delta}(\mathfrak{su}_E) \right\}$$

Choose $g_0 \in \mathcal{K}_\delta^c$ to be an element that is constant on each $C(p)$ (with respect to the fixed unitary frame $\{e_i\}$). Define:

$$F : \mathcal{K}_{\delta,*} \to \mathcal{K}_\delta^c : h \mapsto h g_0 h^{-1}$$

Then

$$DF(1) : T_{id}\mathcal{K}_{\delta,*} \simeq L^2_{2,\delta}(\mathfrak{su}_E) \longrightarrow T_{g_0}\mathcal{K}_\delta^c : H \mapsto [H, g_0]$$

is surjective, and so by the implicit function theorem F is locally an open mapping. It follows that if g is sufficiently close to g_0, then $g = h g_0 h^{-1}$ for some h.

Now returning to the case at hand, we can find an h_0 which is the identity outside a large compact set, such that $h_0 g_\vartheta h_0^{-1}$ is sufficiently close to g_0. By the previous paragraph, we can then find $h \in \mathcal{K}_\delta$ so that $h g_\vartheta h^{-1} = g_0$. Hence, after a gauge transformation, we may assume $g_\vartheta = \mathbf{b}(g_\vartheta)$ is constant on each $C(p)$. Write $d_A = d_{A_0} + b$. Since g_ϑ is constant on each $C(p)$, we have

$$0 = d_A(g_\vartheta) = d_{A_0}(g_\vartheta) + [b, g_\vartheta] = [\sqrt{-1}\hat\alpha \, d\theta + b, g_\vartheta]$$

But $b \in L^2_\delta$, so taking limits we conclude that $[\hat\alpha, g_\vartheta] = 0$. Hence, A_0 preserves the splitting given by g_ϑ, and since g_ϑ is constant, the splitting is translation invariant. The result follows. ☐

Remark 4.2. (1) We note that the only $\mathbb{C}^*$ fixed points in the *framed* moduli space $\mathbf{M}_{\mathsf{Dol},*}^{par,s}(\alpha,\delta)$ lie in the locus where $\Phi \equiv 0$. Indeed, by the discussion above, g_ϑ has constant eigenvalues, and so if g_ϑ were in the based gauge group $\mathcal{G}_{\delta,*}$ it would necessarily be the identity.

(2) It also follows that in the case of full flags, a Hodge pair is strongly parabolic. Indeed, by Proposition 4.1 and the form (4.1), it follows from Example 3.12 that $\mathrm{Res}(\Phi)$ is strictly upper triangular.

As a consequence of Proposition 4.1, we may find a connection $\widetilde{A}_0$ that preserves the splitting and agrees with A_0 on each $C(p)$, $p \in D$. From now on we just assume $\widetilde{A}_0 = A_0$ everywhere. Then A_0 induces connections $A_0^{(a)}$ on the bundles E_a that are translation invariant. We may therefore define the weighted Sobolev space of connections $\mathcal{A}_\delta^{(a)}$ for each bundle E_a. We then have the inclusion:

$$\mathcal{A}_\delta^{(1)} \times \cdots \times \mathcal{A}_\delta^{(\ell)} \longrightarrow \mathcal{A}_\delta$$

The Hodge pair structure induces connections $A^{(a)}$ which are not a priori translation invariant, but up to gauge they can be written as $d_{A^{(a)}} = d_{A_0^{(a)}} + b^{(a)}$, with $b^{(a)} \in L^2_{1,\delta}(\mathfrak{u}_{E_a} \otimes T^*X^\times)$. That is, $A^{(a)} \in \mathcal{A}_\delta^{(a)}$. Applying Lemma 3.6 for each a, we can assume that the connection on the Hodge bundle agrees with A_0 on each $C(p)$. We shall say that the Hodge pair is *in good gauge*.

Now suppose (A,Φ) is a Hodge pair in good gauge. Following [12] we consider bundles

$$\mathfrak{n}_E^+ = \bigoplus_{b>a} \mathrm{Hom}(E_b, E_a)\ ,\ \mathfrak{h}_E = \bigoplus_{a=1}^{\ell} \mathrm{End}E_a \cap \mathfrak{sl}_E$$

with the induced connections from the $A^{(a)}$. Because the Hodge splitting is translation invariant on each $C(p)$, it makes sense to consider the subbundles $\mathfrak{n}_E^+ \cap \mathfrak{l}_p$ and $\mathfrak{h}_E \cap \mathfrak{l}_p$. Note that:

$$\dim \mathfrak{l}_p = 2\,\mathrm{rank}(\mathfrak{n}_E^+ \cap \mathfrak{l}_p) + \mathrm{rank}(\mathfrak{h}_E \cap \mathfrak{l}_p)\ . \tag{4.2}$$

We then define the subcomplex of (3.34):

$$\begin{gathered}\mathcal{C}_\delta^{par,+}(A,\Phi): \\ L^2_\delta(\mathfrak{n}_E^+) \xrightarrow{d_1} L^2_{1,\delta}(\mathfrak{n}_E^+ \otimes \overline{K}_X) \oplus \mathscr{D}_\delta((\mathfrak{h}_E \oplus \mathfrak{n}_E^+) \otimes K_X) \xrightarrow{d_2} L^2_\delta(\mathfrak{h}_E \oplus \mathfrak{n}_E^+)\end{gathered} \tag{4.3}$$

Similarly, we have the subcomplex of (3.37):

$$\begin{gathered}\mathcal{C}_\delta^{spar,+}(A,\Phi): \\ L^2_\delta(\mathfrak{n}_E^+) \xrightarrow{d_1} L^2_{1,\delta}(\mathfrak{n}_E^+ \otimes \overline{K}_X) \oplus L^2_{1,\delta}((\mathfrak{h}_E \oplus \mathfrak{n}_E^+) \otimes K_X) \xrightarrow{d_2} L^2_\delta(\mathfrak{h}_E \oplus \mathfrak{n}_E^+)\end{gathered} \tag{4.4}$$

Lemma 4.3. *Let $(A,\Phi) \in \mathcal{B}_\delta^{spar,s}$ be a Hodge pair in good gauge. Then*

$$\dim H^1(\mathcal{C}_\delta^{par,+}(A,\Phi)) = (g-1)\dim \mathsf{G} + \sum_{p\in D} \dim(\mathsf{G}/\mathsf{P}_p) + \sum_{p\in D} \dim \mathsf{L}_p \tag{4.5}$$

$$\dim H^1(\mathcal{C}_\delta^{spar,+}(A,\Phi)) = (g-1)\dim \mathsf{G} + \sum_{p\in D} \dim(\mathsf{G}/\mathsf{P}_p) + \sum_{p\in D} \mathrm{rank}(\mathfrak{n}_E^+ \cap \mathfrak{l}_p) \tag{4.6}$$

Under the assumption of full flags,

$$\dim H^1(\mathcal{C}_\delta^{spar,+}(A,\Phi)) = (g-1)\dim \mathsf{G} + \sum_{p\in D} \dim(\mathsf{G}/\mathsf{P}_p) \tag{4.6'}$$

Proof. As in the proof of Proposition 3.23, elements of cohomology have representatives in harmonic spaces $\mathbf{H}_\delta^{par,+}(A,\Phi)$ and $\mathbf{H}_\delta^{spar,+}(A,\Phi)$. The dimension count then reduces to computing the sum of the indices of the decoupled operators

$$\begin{aligned} T_\beta^+ &: L^2_{1,\delta}(\mathfrak{n}_E^+ \otimes \overline{K}_X) \longrightarrow L^2_\delta(\mathfrak{n}_E) \\ T_\varphi^+ &: L^2_{1,\delta}((\mathfrak{h}_E \oplus \mathfrak{n}_E) \otimes K_X) \longrightarrow L^2_\delta(\mathfrak{h}_E \oplus \mathfrak{n}_E) \end{aligned}$$

The operator T_β^+ is the adjoint of

$$\bar\partial_{A_0} : L^2_{1,\delta}(\mathfrak{n}_E^+) \longrightarrow L^2_\delta(\mathfrak{n}_E \otimes \overline{K}_X)$$

As above, using [2] we have

$$i(\bar\partial_{A_0}) = \deg \mathfrak{n}_E^+ - \operatorname{rank}(\mathfrak{n}_E^+)(g-1+d/2) - \frac{1}{2}\sum_{p\in D}\operatorname{rank}(\mathfrak{n}_E^+ \cap \mathfrak{l}_p) - \frac{\eta^+(0)}{2}$$

Here, $\eta^+(s)$ is the η-function for the boundary operator on $\mathfrak{n}_E^+$, and by $\deg \mathfrak{n}_E^+$ we mean the integral of the α_0 term in [2, Thm. 3.10 (i)]. Similarly,

$$i(T_\varphi^+) = \deg \mathfrak{n}_E^+ + \deg \mathfrak{h}_E + (\operatorname{rank}\mathfrak{n}_E^+ + \operatorname{rank}\mathfrak{h}_E)(g-1+d/2) - \frac{1}{2}\sum_{p\in D}\operatorname{rank}((\mathfrak{n}_E^+ \oplus \mathfrak{h}_E)\cap \mathfrak{l}_p) - \frac{\eta^+(0)}{2}$$

Here, we have used the fact that the spectrum of the boundary operator on $\mathfrak{h}$ is symmetric about the origin (cf. Remark 3.2), so its contribution to the η-function vanishes. Now $\deg \mathfrak{h}_E = 0$, since the curvature form is traceless. Noting that $2\operatorname{rank}\mathfrak{n}_E^+ + \operatorname{rank}\mathfrak{h}_E = \dim \mathsf{G} = n^2-1$, and using (4.2), we have

$$\begin{aligned}
i(T_\beta^+) + i(T_\varphi^+) &= (g-1+d/2)\dim\mathsf{G} - \frac{1}{2}\sum_{p\in D}\operatorname{rank}(\mathfrak{h}_E\cap\mathfrak{l}_p) \\
&= (g-1)\dim\mathsf{G} + \frac{1}{2}\sum_{p\in D}(\dim\mathsf{G} - \operatorname{rank}(\mathfrak{h}_E\cap\mathfrak{l}_p)) \\
&= (g-1)\dim\mathsf{G} + \frac{1}{2}\sum_{p\in D}(\dim(\mathsf{G}/\mathsf{L}_p) + 2\operatorname{rank}(\mathfrak{n}_E^+\cap\mathfrak{l}_p)) \\
&= (g-1)\dim\mathsf{G} + \sum_{p\in D}(\dim(\mathsf{G}/\mathsf{P}_p) + \operatorname{rank}(\mathfrak{n}_E^+\cap\mathfrak{l}_p))
\end{aligned} \tag{4.7}$$

This proves (4.6). For the parabolic case, we again let $\widetilde{T}_\varphi^+$ be the operator T_φ^+, but with domain $\mathscr{D}_\delta((\mathfrak{l}_E\oplus\mathfrak{n}_E)\otimes K_X)$. As in the proof of Proposition 3.23,

$$i(\widetilde{T}_\varphi^+) = i(T_\varphi^+) + \sum_{p\in D}\operatorname{rank}((\mathfrak{n}_E^+\oplus\mathfrak{h}_E)\cap\mathfrak{l}_p)$$

Now (4.5) follows from (4.6) and (4.2). However, note that since we assume full flags in the parabolic case, in fact $\mathfrak{n}_E^+\cap\mathfrak{l}_p = \{0\}$ and $\mathfrak{l}_p\subset\mathfrak{h}_E$, so the equality is automatic. ☐

4.1.2. *The Białynicki-Birula slice.* Following [12, Def. 3.7], we define

Definition 4.4 (BB slice)**.** Let $(A,\Phi)\in\mathcal{B}^{spar,s}_\delta$ be a smooth Hodge pair in good gauge. The BB-slice at (A,Φ) in the strongly parabolic case is defined by

$$\mathcal{S}^{spar,+}_\delta(A,\Phi) = \big\{(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{n}_E^+\otimes\overline{K}_X)\oplus L^2_{1,\delta}((\mathfrak{l}_E\oplus\mathfrak{n}_E^+)\otimes K_X) \mid d_2(\beta,\varphi)+[\beta,\varphi]=0\ ,\ d_1^{*\delta}(\beta,\varphi)=0\big\}$$

In case the system of weights $\{\alpha(p)\}_{p\in D}$ corresponds to full flags, we define the BB-slice in the parabolic case to be

$$\mathcal{S}^{par,+}_\delta(A,\Phi) = \big\{(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{n}_E^+\otimes\overline{K}_X)\oplus \mathscr{D}_\delta((\mathfrak{l}_E\oplus\mathfrak{n}_E^+)\otimes K_X) \mid d_2(\beta,\varphi)+[\beta,\varphi]=0\ ,\ d_1^{*\delta}(\beta,\varphi)=0\big\}$$

We have the analog of [12, Prop. 4.2].

Proposition 4.5. *Let $(A,\Phi)\in\mathcal{B}_\delta^{spar,s}$ be a smooth Hodge pair For each*

$$(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{n}_E^+\otimes\overline{K}_X)$$

satisfing $d_2(\beta,\varphi)+[\beta,\varphi]=0$, there is a unique $f\in L^2_{2,\delta}(\mathfrak{n}_E^+)$ such that the complex gauge transformation $g=\mathbf{I}+f\in\mathcal{G}_{\delta,*}$ takes $(\bar\partial_E+\beta,\Phi+\varphi)$ into the slice $\mathcal{S}_\delta^{par,+}(A,\Phi)$.

The proof follows exactly as in the reference above. The key step is to use the invertibility of $d_1^{*\delta}d_1$ on $L^2_{2,\delta}$. This follows from Lemma 3.25. We omit the details.

Example 4.6. Consider a rank 2 Hodge bundle (A_0,Φ_0) and an element (β,φ) in the BB-slice. On $C(p)$ we write:

$$\bar\partial_A=\begin{pmatrix}\bar\partial_{L_1} & b\\ 0 & \bar\partial_{L_2}\end{pmatrix}\ ,\ \Phi=\begin{pmatrix}\varphi_1 & \varphi_2\\ \varphi_0 & -\varphi_1\end{pmatrix}\otimes\frac{dz}{z}\tag{4.8}$$

Note that while φ_0 is holomorphic and possibly nonzero at $z=0$ (depending upon the relation between the Hodge splitting and the parabolic structure), φ_1 is not holomorphic unless $b=0$ or $\varphi_0=0$. Similarly, φ_2 is not holomorphic unless $b=0$ or $\varphi_1=0$. Nevertheless, we claim that the limits $\lim_{z\to 0}\varphi_i$ exist for $i=1,2$. Moreover, identifying $\mathfrak{l}/W$ with $\mathbb{C}/\pm 1$:

$$\mathrm{Res}_p([\bar\partial_A,\Phi])=\lim_{z\to 0}\varphi_1(z)$$

To prove the claim, we first bring $\bar\partial_A$ into the standard form $\bar\partial_{A_0}$ (modulo permuting the factors). See Example 3.12. This is done by a based gauge transformation

$$g=\begin{pmatrix}1 & -u\\ 0 & 1\end{pmatrix}$$

where $g^{-1}\circ\bar\partial_A\circ g=\bar\partial_{A_0}$, and $u\in L^2_{2,\delta}$ is such that $\bar\partial u=b$ on $C(p)$. Then

$$g^{-1}\Phi g=\begin{pmatrix}\varphi_1+u\varphi_0 & \varphi_2-u(2\varphi_1+u\varphi_0)\\ \varphi_0 & -(\varphi_1+u\varphi_0)\end{pmatrix}\otimes\frac{dz}{z}$$

Notice that $\varphi_1+u\varphi_0$ is holomorphic. But we have a bound $|u(z)|\leq C|z|^\delta$, and φ_0 is holomorphic at $z=0$, hence

$$\mathrm{res}_{\mathfrak{l}}([\bar\partial_A,\Phi])=\lim_{z\to 0}(\varphi_1+u\varphi_0)(z)=\lim_{z\to 0}\varphi_1(z)$$

Similarly, $\varphi_2-u(2\varphi_1+u\varphi_0)$ is holomorphic, and so

$$\lim_{z\to 0}\left(\varphi_2-u(2\varphi_1+u\varphi_0)\right)=\lim_{z\to 0}\varphi_2$$

exists.

Let $(A,\Phi)\in\mathcal{B}_\delta^{spar,s}$ be a smooth Hodge pair as above. We define the stable manifold of (A,Φ) for the action of $\mathbb{C}^*$ on $\mathbf{M}_{\mathsf{Dol}}^{par,s}$.

$$\mathcal{W}_0(A,\Phi)=\left\{[(\widetilde{A},\widetilde{\Phi})]\in\mathbf{M}_{\mathsf{Dol}}^{par,s}\mid\lim_{t\to 0}[(\widetilde{A},t\widetilde{\Phi})]=[(A,\Phi)]\right\}\tag{4.9}$$

Like Proposition 4.5 above, the proof of [12, Cor. 4.3] can be adapted to give the following result and proof of Theorem 1.7.

Proposition 4.7. *Let $(A,\Phi)\in\mathcal{B}_\delta^{par,s}$ be a smooth, stable, Hodge pair. Then the map*

$$p_{\mathrm{H}}:\mathcal{S}_\delta^{par,+}(A,\Phi)\longrightarrow\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta):(\beta,\varphi)\mapsto[(\bar\partial_A+\beta,\Phi+\varphi)]$$

is a biholomorphism onto $\mathcal{W}_0(A,\Phi)$. Moreover, $\mathcal{W}_0(A,\Phi)$ is coisotropic with respect to the Poisson structure on $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$, and for any $\ell\in\bigoplus_{p\in D}\mathfrak{l}_p$, $\mathcal{W}_0(A,\Phi)\cap\mathrm{Res}^{-1}(\ell)$ is a holomorphic Lagrangian with respect to Ω_ℓ.

Remark 4.8. Note that the image of $\mathcal{S}_\delta^{par,+}(A,\Phi)$ in the framed moduli space $\mathbf{M}_{\mathsf{Dol},*}^{par,s}(\alpha,\delta)$ is a holomorphic Lagrangian; hence the quotient in $\mathbf{M}_{\mathsf{Dol}}^{par,s}(\alpha,\delta)$ is coisotropic.

4.1.3. *Relation with Simpson's partial stratification.* We define $\mathcal{W}_1(A,\Phi)$ to be the set of all $[D]\in \mathbf{M}_{\mathsf{dR},*}^{par,s}(\alpha,\delta)$ satisfying the following condition. Let $(\mathcal{E}(\alpha),\nabla)$ be a holomorphic bundle with logarithmic connection ∇ associated to $[D]$ under the identification from Theorem 3.40. By Proposition A.1 we know there is an associated Hodge bundle $(Gr_{\mathcal{A}}(\mathcal{E})(\alpha),\Phi_{\mathcal{A}})$. Then we say $[D]\in\mathcal{W}_1(A,\Phi)$ if the isomorphism class of Higgs pairs associated to $(Gr_{\mathcal{A}}(\mathcal{E})(\alpha),\Phi_{\mathcal{A}})$ via Theorem 3.22 coincides with $[(A,\Phi)]$.

Recall the map p_{dR} from (3.48).

Proposition 4.9. *Let $(A,\Phi)\in\mathcal{B}_\delta^{par,s}$ be a smooth, stable, Hodge pair. Then the map*

$$p_{\mathsf{dR}}:\mathcal{S}_\delta^{par,+}(A,\Phi)\longrightarrow\mathbf{M}_{\mathsf{dR},*}^{par,s}(\alpha,\delta)$$

is a biholomorphism onto $\mathcal{W}_1(A,\Phi)$.

This is [12, Cor. 4.9]. The proof there relies on the following:

Lemma 4.10. *Let (A,Φ) be as above, and let D be the associated flat connection. Suppose that*

$$(\beta,\varphi)\in L^2_{1,\delta}(\mathfrak{n}_E^+\otimes\overline{K})\oplus\mathscr{D}_\delta((\mathfrak{l}_E\oplus\mathfrak{n}_E^+)\otimes K_X)$$

is such that $D+\beta+\varphi$ is flat. Then there exists a unique smooth section $f\in L^2_{2,\delta}(\mathfrak{n}_E^+)$ such that if $g=1+f\in\mathcal{G}_{\delta,*}$, and

$$g(D+\beta+\varphi)=D+\widetilde{\beta}+\widetilde{\varphi}$$

then $(\widetilde{\beta},\widetilde{\varphi})\in\mathcal{S}_\delta^{par,+}(A,\Phi)$.

Proof. The proof follows the recursive argument in [12, Prop. 3.11]. The key step is to invert the Laplacian $D'_\delta D''$ (see [12, eq. (3.9)]). By the assumption of stability of the Hodge bundle, the kernel, and hence also cokernel, of this operator vanishes (see 3.25). Hence, the same proof as in that reference applies here as well. □

4.2. **Existence of conformal limits.** In this section we prove the main result on the existence of a conformal limit. Fix a smooth stable Hodge pair $(A,\Phi)\in\mathcal{B}_\delta^{par}$ in good gauge. We assume, without loss of generality, that the hermitian metric h on E satisfies the Hitchin equations. Let $D=\nabla_\delta^0\otimes D_\delta$ denote the corresponding flat connection. To keep with the notation of the references, for $\mathbf{u}=(\beta,\varphi)\in\mathcal{S}_\delta^{par,+}(A,\Phi)$, let $\bar\partial_{\mathbf{u}}=\bar\partial_A+\beta$, $\Phi_{\mathbf{u}}=\Phi+\varphi$. Fix $R>0$. Let $h(\mathbf{u},R)$ denote the harmonic metric for the Higgs pair $(\bar\partial_{\mathbf{u}},R\Phi_{\mathbf{u}})$. Consider the flat connection

$$D_{(\mathbf{u},R)}=\Phi_{\mathbf{u}}+\bar\partial_{\mathbf{u}}+\partial_{\mathbf{u}}^{h(\mathbf{u},R)}+R^2\Phi_{\mathbf{u}}^{*h(\mathbf{u},R)}\ . \tag{4.10}$$

Notice that if $\mathbf{u}\in\mathcal{S}_\delta^{spar,+}(A,\Phi)$, then $D_{(\mathbf{u},R)}$ is a family in $\mathcal{F}_\delta^{spar}$. However, if $\mathbf{u}\in\mathcal{S}_\delta^{par,+}(A,\Phi)$, it is not always true that $D_{(\mathbf{u},R)}$ lies in $\mathcal{F}_\delta^{par}$. Nevertheless, we prove the following.

Proposition 4.11 (Conformal limit). $\lim_{R\to 0}D_{(\mathbf{u},R)}=\nabla_\delta^0\otimes(D_\delta+\beta+\varphi)$. *Here, the convergence is C^∞ on compact subsets of $X^\times$.*

Proof. Given the set-up of the previous sections, the proof is formally the same that of [14] (see especially, [12, pp. 1223-4]). As in these references, the key is to find an expansion in R for the metric $h(\mathbf{u},R)$. We first modify h to get a metric h'_R by an action of

$$g=\begin{pmatrix}R^{m_1/2} & & 0\\ & \ddots & \\ 0 & & R^{m_\ell/2}\end{pmatrix}$$

where $m_j - m_{j+1} = 2$, $j = 1, \dots, \ell-1$, and $\sum_{j=1}^{\ell}(\operatorname{rank} E_j)m_j = 0$. Notice that since $d_{A_0}g$ is compactly supported, in particular we have $g \in \mathcal{R}^0_\delta$, and hence g is an element of the gauge group $\mathcal{G}_\delta$. Then as in [12, eq. (5.4)],

$$\lim_{R\to 0}\left\{\Phi_{\mathbf{u}} + \bar\partial_{\mathbf{u}} + e^{-\tau\delta}\partial_{\mathbf{u}}^{h'_R}e^{\tau\delta} + R^2\Phi_{\mathbf{u}}^{*h'_R}\right\} = D_\delta + \beta + \varphi\ .$$

The goal now is express $h(\mathbf{u},R) = g_R(h'_R)$, $g_R = \exp(f_R) \in \mathcal{G}_\delta$, where f_R is a traceless h'_R-hermitian endomorphism. The Hitchin equations

$$N_{(\mathbf{u},R)}(f_R) := i\Lambda(F_{(\bar\partial_u, h(\mathbf{u},R)} + [\Phi_{\mathbf{u}}, \Phi_{\mathbf{u}}^{*h(\mathbf{u},R)}]) = 0$$

are now a function of f_R. As in [12, p. 1224], we may view $N_{(\mathbf{u},R)}$ as a map

$$N_{(\mathbf{u},R)} : \mathcal{R}_\delta(\mathfrak{h}_E \oplus \mathfrak{n}_E^+) \longrightarrow L^2_\delta(\mathfrak{h}_E \oplus \mathfrak{n}_E^+)$$

The linearization at $R = 0$, $f_R = 0$, may be computed:

$$\frac{1}{2}dN_{(\mathbf{u},0)}(0)\dot f = \bar\partial_{\mathbf{u}}\partial_E \dot f + [\Phi_{\mathbf{u}}, [\Phi^*, \dot f]]\ .$$

As an operator on the $\dot f$ variable, it follows as in the proof of Lemma 3.6 and Proposition 3.24 that $dN_{(\mathbf{u},0)}(0)$ is Fredholm of index 0; hence, surjectivity follows from injectivity. With respect to the splitting of the bundle, decompose $\dot f = \dot f_{\mathfrak{h}} + \dot f_+$. Following the proof of [12, Lemma 5.2], an element of of $\ker dN_{(\mathbf{u},0)}(0)$ would satisfy $(D'')^*D''\dot f_{\mathfrak{h}} = 0$. Since we assume the Hodge pair is stable, by Lemma 3.25 and Proposition 3.24, the operator $(D'')^*D''$ has no kernel, and so $\dot f_{\mathfrak{h}} = 0$. Repeating this argument for the upper trianguler components $\dot f_+$ then shows that $\ker dN_{(\mathbf{u},0)}(0) = \{0\}$. The implicit function theorem can then be applied to $N_{(\mathbf{u},R)}$ to find the solution f_R for small R. The rest of the proof follows as in the references cited above. □

4.3. **Conformal limit in Rank 2.** Recall from Example 2.6 that there are two types of fixed points in rank two. Namely, $\mathcal{E}(\alpha)$ is a stable parabolic bundle with $\Phi = 0$ or

$$(\mathcal{E}(\alpha), \Phi) \cong (\mathcal{L}_1(\beta_1) \oplus \mathcal{L}_2(\beta_2), \begin{pmatrix} 0 & 0 \\ \phi_0 & 0\end{pmatrix})\ , \tag{4.11}$$

where $\mathcal{L}_i(\beta_i)$ are parabolic line bundles and $\phi_0 : \mathcal{L}_1 \to \mathcal{L}_2 \otimes K(D)$ is not zero and $\operatorname{Res}_p(\phi_0) = 0$ whenever $\beta_1(p) > \beta_2(p)$.

In the case when $\mathcal{E}(\alpha)$ is stable and $\Phi = 0$, the the BB-slice consists of all parabolic Higgs bundles $(\mathcal{E}(\alpha), \varphi)$ with underlying bundle $\mathcal{E}(\alpha)$. The $\hbar$-conformal limit in this case is

$$\mathrm{CL}_\hbar(\bar\partial_E, 0 + \varphi) = (\hbar, \bar\partial_E, \hbar\partial_h + \varphi), \tag{4.12}$$

where $\bar\partial_E + \partial_h$ is the flat unitary logarithmic connection associated to the stable bundle $\mathcal{E}(\alpha)$.

For the fixed points with $\Phi \neq 0$, let h be the harmonic metric and $D = \bar\partial_E + \Phi^{*_h} + \partial_E^h + \Phi$ be the associated parabolic logarithmic connection. The splitting $\mathcal{L}_1 \oplus \mathcal{L}_2$ is orthogonal with respect to h, so $h = h_1 \oplus h_2$. As in Example 4.6, points in the BB-slice through $(\bar\partial_E, \Phi)$ have the form $(\bar\partial_E + \beta, \Phi + \varphi)$, where

$$\beta = \begin{pmatrix} 0 & b \\ 0 & 0\end{pmatrix}, \quad \varphi = \begin{pmatrix}\varphi_1 & \varphi_2 \\ 0 & -\varphi_1\end{pmatrix}. \tag{4.13}$$

Being in slice means $\bar\partial_E(\varphi) + [\beta, \Phi] + [\beta, \varphi] = 0$ and $e^{-\tau\delta}\partial_E^h(e^{\tau\delta}\beta) + [\Phi^{*_h}, \varphi] = 0$. Specifically,
(4.14)

$$\bar\partial_E\varphi_1 + b\wedge\phi_0 = 0, \qquad \bar\partial_E\varphi_2 - 2b\wedge\varphi_1 = 0 \qquad \text{and} \qquad e^{-\tau\delta}\partial_E^h(e^{\tau\delta}b) - 2\phi_0^{*_h}\wedge\varphi_1 = 0\ .$$

For such Higgs bundles, the $\hbar$-conformal limit is $(\hbar, \bar\partial_E + \hbar\Phi^{*_h} + \beta, \hbar\partial_E^h + \Phi + \varphi)$. More explicitly,

$$\mathrm{CL}_\hbar(\bar\partial_E + \beta, \Phi + \varphi) = \left(\hbar, \begin{pmatrix} \bar\partial_1 & \phi_0^{*_h} + b \\ 0 & \bar\partial_2 \end{pmatrix}, \begin{pmatrix} \hbar\partial_1^h + \varphi_1 & \varphi_2 \\ \phi_0 & \hbar\partial_2^h - \varphi_1 \end{pmatrix}\right). \tag{4.15}$$

Remark 4.12. As noted in Example 4.6, the residue of the associated Higgs field at $p \in D$ is given by $\lim_{z\to p}\varphi_1$. In particular, if $\mathrm{Res}(\Phi) \neq 0$, then $\varphi_1 \neq 0$, and the slice equations (4.14) imply $b \neq 0$. Hence, the loci of the slice with $b = 0$ is in the strongly parabolic moduli space. Note also that $b = 0$ if and only if φ_2 is holomorphic.

5. A detailed study of the four-punctured sphere

In this section we discuss our main results in the special case of rank 2 on the four-punctured sphere where things are relatively explicit. Many aspects of the Higgs bundle moduli space in this special case were studied by Fredrickson–Mazzeo–Swoboda–Weiß in [15]. We focus on the fixed points which are analogous to the uniformizing points in the unpunctured case. In particular, the harmonic metric of these fixed points is related to a metric on $\mathbb{CP}^1$ with constant negative curvature and conical singularities. Furthermore, in these cases the intersection of the BB-slice with the hyperkahler moduli spaces with fixed complex masses parameterize sections of the Hitchin map, and the conformal limit of such objects share many features with the set of opers.

5.1. **4-punctured sphere moduli space.** Consider the Riemann surface $X = \mathbb{CP}^1$ and fix an effective divisor $D = p_1 + p_2 + p_3 + p_4$. Without loss of generality we assume $p_i \neq \infty$ for all i. Throughout this section, all local computations are done in the affine chart $\mathbb{CP}^1 \setminus \{\infty\}$.

Fix a rank 2 vector bundle $E \to \mathbb{CP}^1$ with degree -4. For each $p_i \in D$, fix a weighted filtration

$$\begin{matrix} E_{p_i} \supset & F_{p_i} & \supset 0 \\ 0 < \alpha(p_i) < 1 - \alpha(p_i) < 1, \end{matrix}$$

where $\alpha(p_i) \in (0, \frac{1}{2})$. For notational convenience, we write $\alpha(p_i) = \alpha_i$ and $F_{p_i} = F_i$. The parabolic weights are determined by the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (0, \frac{1}{2})^4$. This puts us in the case of full flags and trivial determinant since, given a parabolic bundle $\mathcal{E}(\alpha)$ with this data, equation (2.3) gives an isomorphism $\det(\mathcal{E}(\alpha)) \cong \mathcal{O}(-4)(D) \cong \mathcal{O}(0)$. We assume that the parabolic weights are generic, so that semistability implies stability. With this fixed data, we define the moduli space $\mathcal{P}_0(\alpha)$. It is a six dimensional complex manifold with Hitchin map $-\det : \mathcal{P}_0(\alpha) \to \mathcal{B} = H^0(K^2(2D))$.

The residue map $\mathrm{Res} : \mathcal{P}_0(\alpha) \to \oplus_{p\in D}\mathfrak{l}_p$ (see Remark 2.4) is determined by the vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, where $\mu_i \in \mathbb{C}$ is the complex mass at $p_i \in D$. Here, the subspace F_i is the eigenspace of the residue $\mathrm{res}_{p_i}\Phi$ with eigenvalue μ_i. Denote the fiber by

$$\mathcal{P}_0(\alpha, \mu) = \mathrm{Res}^{-1}(\mu) \subset \mathcal{P}_0(\alpha).$$

For each μ, $\mathcal{P}_0(\alpha, \mu)$ is a smooth hyperkähler manifold of complex dimension two. Let $\mathcal{B}(\mu)$ denote the image of $\mathcal{P}_0(\alpha, \mu)$ under the Hitchin map

$$\begin{aligned} -\det :\ & \mathcal{P}_0(\alpha, \mu) \longrightarrow \mathcal{B}(\mu) \subset H^0(K^2(2D)) . \\ & [\mathcal{E}(\alpha), \Phi] \longmapsto -\det(\Phi) \end{aligned}$$

The space $\mathcal{B}(\mu)$ consists of all elements of $H^0(K^2(2D))$ whose residue[3] at $p_i \in D$ is μ_i^2. In particular, $\mathcal{B}(\mu)$ is affine over $H^0(K^2(D)) \cong \mathbb{C}$.

Since we are in the case of the full flags, the strongly parabolic moduli space $\mathcal{SP}_0(\alpha)$ corresponds to setting all complex masses $\mu_i = 0$. Moreover, all $\mathbb{C}^*$-fixed points in $\mathcal{P}_0(\alpha)$ lie in $\mathcal{SP}_0(\alpha)$. In this case, the base $\mathcal{B}(\alpha, 0)$ is given by

$$\mathcal{B}(\alpha, 0) \cong H^0(K^2(D)) \cong H^0(\mathcal{O}).$$

[3] in the sense of [45, eq. 1.2.19]

The nilpotent cone, i.e., the zero fiber of $\mathcal{SP}_0(\alpha) \to \mathcal{B}(\alpha, 0)$, is the unique singular fiber.

For generic parabolic weights α, the nilpotent cone consists of five spheres in an affine D_4 arrangement. There are five connected components of $\mathbb{C}^*$-fixed points, these are shown in red in Figure 2. In particular, there are four isolated fixed points which determine the 'exterior spheres' and one component of fixed points isomorphic to $\mathbb{CP}^1$ which is the 'interior sphere.'

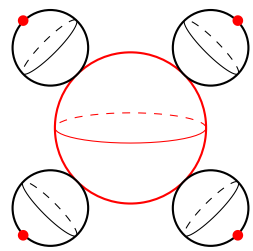

FIGURE 2. The nilpotent cone of the strongly parabolic Hitchin moduli space on $(\mathbb{CP}^1, D)$. The $\mathbb{C}^*$-fixed points are shown in red.

Consequently, the BB-stratification of the moduli space $\mathcal{P}(\alpha)$ from (2.11) is

$$\mathcal{P}(\alpha) = \coprod_{a \in \pi_0(\mathcal{P}_0(\alpha)^{\mathbb{C}^*})} \mathcal{W}^a = \mathcal{W}^{cent} \cup \mathcal{W}^{ext}, \tag{5.1}$$

where $\mathcal{W}^{cent}$ is the stratum labeled by the central sphere in Figure 2, and $\mathcal{W}^{ext}$ consists of the four strata labeled by the exterior fixed points. The volumes of the spheres with respect to the hyperkähler metric on $\mathcal{SP}_0(\alpha)$ vary with the parabolic weights $\alpha \in (0, \frac{1}{2})^4$, and degenerate to zero on certain 3-dimensional 'walls' of $(0, \frac{1}{2})^4$ [15]. These walls divide the hypercube $(0, \frac{1}{2})^4$ into twenty-four chambers. As shown in [32, 15] and discussed below, the Higgs bundles corresponding to the $\mathbb{C}^*$-fixed points is chamber-dependent.

5.2. $\mathbb{C}^*$**-fixed points.** Consider a stable $\mathbb{C}^*$-fixed point of the form (4.11). Recall, for each $p \in D$, the subspace F_p is either $\mathcal{L}_1|_p$ or $\mathcal{L}_2|_p$. Let D_I and D_{I^c} be the effective subdivisors of D for which the subspace F_p is $\mathcal{L}_2|_p$ and $\mathcal{L}_1|_p$, respectively. Thus,

$$D = D_I + D_{I_c}.$$

Here I denotes the subset of $\{1, 2, 3, 4\}$ determined by the support of D_I, and I^c is its complement. The parabolic weights $\beta_1(p_i)$ and $\beta_2(p_i)$ are given by

$$\beta_1(p_i) = \begin{cases} \alpha_i & i \in I \\ 1 - \alpha_i & i \in I^c \end{cases} \quad \text{and} \quad \beta_2(p_i) = \begin{cases} 1 - \alpha_i & i \in I \\ \alpha_i & i \in I^c \end{cases}. \tag{5.2}$$

The map ϕ_0 in the Higgs field is a parabolic map $\mathcal{L}_1(\beta_1) \to \mathcal{L}_2(\beta_2) \otimes K_X(D)$. By Definition 2.2, this means that ϕ is a meromorphic section of $\mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes K_X$ with a worst simple poles at $p \in D$ and whose residue is zero whenever $\beta_1(p_i) > \beta_2(p_i)$. Since $\beta_1(p_i) > \beta_2(p_i)$ implies $p_i \in D_{I^c}$, we have

$$\phi_0 \in H^0(\mathcal{L}_1^* \mathcal{L}_2 K(D_I)).$$

The $\mathbb{C}^*$-fixed point under consideration is stable whenever $\phi_0 \in H^0(\mathcal{L}_1^* \mathcal{L}_2 K(D_I))$ is nonzero and $\deg(\mathcal{L}_2(\beta_2)) < 0$. Equivalently, we have

$$\begin{gathered} \deg(\mathcal{L}_2(\beta_2)) = \deg(\mathcal{L}_2) + \deg(D_I) - \textstyle\sum_{i \in I} \alpha_i + \sum_{j \in I^c} \alpha_j < 0 \\ \deg(\mathcal{L}_1^* \mathcal{L}_2 K(D_I)) = 2 + 2\deg(\mathcal{L}_2) + \deg(D_I) \geq 0 \,. \end{gathered} \tag{5.3}$$

A straight forward computation shows that stability forces the degrees of $\mathcal{L}_1$ and $\mathcal{L}_2$ to be $-3,-2,-1$. Hence $\mathcal{L}_2$ is isomorphic to $\mathcal{O}(-1),\mathcal{O}(-2)$ or $\mathcal{O}(-3)$ and $\mathcal{L}_1\cong\mathcal{O}(-4)\mathcal{L}_2^*$. There are five cases determined by the degree of D_I. The following table gives the conditions which are direct consequences of (5.3).

(5.4)

$\deg(D_I)$	$\mathcal{L}_2$	condition on $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$	$\mathcal{L}_1^*\mathcal{L}_2K(D_I)$
0	$\mathcal{O}(-1)$	$\sum_{i=1}^4\alpha_i<1$	$\mathcal{O}$
1	$\mathcal{O}(-1)$	$-\alpha_i+\sum_{j\in I^c}\alpha_j<0$, where $D_I=p_i$	$\mathcal{O}(1)$
2	$\mathcal{O}(-2)$	$-\sum_{i\in I}\alpha_i+\sum_{j\in I^c}\alpha_j<0$	$\mathcal{O}$
3	$\mathcal{O}(-2)$	$\alpha_j-\sum_{i\in I}\alpha_i<-1$, where $D_{I^c}=p_j$	$\mathcal{O}(1)$
4	$\mathcal{O}(-3)$	$\sum_{i=1}^4\alpha_i>1$	$\mathcal{O}$

The five cases are described succinctly in following proposition.

Proposition 5.1. *Consider a stable $\mathbb{C}^*$-fixed point* $(\mathcal{E}(\alpha),\Phi)=\left(\mathcal{L}_1(\beta_1)\oplus\mathcal{L}_2(\beta_2),\begin{pmatrix}0&0\\\phi_0&0\end{pmatrix}\right)$. *Let $D_I\subset D$ be the effective subdivisor determined by the points $p\in D$ for which $F_p=\mathcal{L}_2|_p$. Then,*

$$\mathcal{L}_1\oplus\mathcal{L}_2\cong\mathcal{O}(d-3)\oplus\mathcal{O}(-1-d),\tag{5.5}$$

where $d=\left\lfloor\frac{\deg(D_I)}{2}\right\rfloor$, *and ϕ_0 is a nonzero section of* $\mathrm{Hom}(\mathcal{L}_1,\mathcal{L}_2)\otimes K(D_I)\cong\mathcal{O}(\deg(D_I)-2d)$.

5.2.1. *The conformal limit for the central sphere.* The central sphere in Figure 2 either consists of stable parabolic bundles with zero Higgs field or corresponds to the case where $\deg(D_I)=1$ or $\deg(D_I)=3$ in Proposition 5.1. Of the 24 chambers in the space of weights, 16 have stable parabolic bundles as their central sphere. From Table (5.4), these occur when $0<-\alpha_k+\sum_{i\neq k}\alpha_i<1$ for all k.

In the remaining eight chambers, there are no stable parabolic bundles $\mathcal{E}(\alpha)$. Instead, the central sphere consists of Higgs bundles described in Table (5.4); take $\deg(D_I)=1$ if there is a weight α_k with $-\alpha_k+\sum_{i\neq k}\alpha_i<0$ or $\deg(D_I)=3$ if there is a weight α_k with $\alpha_k-\sum_{i\neq k}\alpha_i>-1$. In both cases, the central sphere is parameterized by $\mathbb{P}(H^0(\mathcal{O}(1))\setminus\{0\})$. For $u\in\mathbb{CP}^1$, the bundle is $\mathcal{O}(d-3)\oplus\mathcal{O}(-1-d)$, where $d=\lfloor\frac{\deg(D_I)}{2}\rfloor$ and the Higgs field is determined by a nonzero section ϕ_u of $\mathcal{O}(2-2d)K(D_I)\cong\mathcal{O}(1)$ which vanishes at u and is defined up to scale.

In these cases, the BB-slice is given by

$$(\bar\partial_E,\Phi)=\left(\begin{pmatrix}\bar\partial_{d-3}&b\\0&\bar\partial_{-d-1}\end{pmatrix},\begin{pmatrix}\varphi_1&\varphi_2\\\phi_0&-\varphi_1\end{pmatrix}\right),$$

where b and φ_i satisfy (4.14). The $\hbar$-conformal limit of such a Higgs bundle is given by (4.15).

Remark 5.2. Note that for these fixed points, the term b is always nonzero. Indeed, by Remark 4.12, $b=0$ implies $\varphi_1=0$ and φ_2 is holomorphic. Hence, φ_2 is a holomorphic map

$$\varphi_2:\mathcal{O}(-d-1)\to\mathcal{O}(d-3)\otimes K(D_{I^c}).$$

But $d+1+d-3+-2+\deg(D_{I^c})=-1$, so such a map must be zero.

5.3. **Exterior fixed points: Hitchin sections and opers.** Each of the four exterior fixed points in Figure 2 is described by a case in Table (5.4) where $\deg(D_I)$ is even. The condition on the parabolic weights in (5.3) implies that there is exactly one fixed point with $\deg(D_I)=0$ or $\deg(D_I)=4$, determined by the sign of $-1+\sum\alpha_i$, and three fixed points with $\deg(D_I)=2$, determined by the three effective subdivisors D_I with $-\sum_{i\in I}\alpha_i+\sum_{j\in I^c}\alpha_j<0$. In particular, each stratum in $\mathcal{W}^{ext}$ from (5.1) is labeled by the subdivisor D_I. Denote the corresponding stratum by $\mathcal{W}^{ext_I}$. The

$\mathbb{C}^*$-fixed point which labels the stratum $\mathcal{W}^{ext_I}$ is

$$(\mathcal{E}(\alpha),\Phi) = \left(\mathcal{O}(d-3)\oplus\mathcal{O}(-1-d), \begin{pmatrix} 0 & 0 \\ \phi_0 & 0\end{pmatrix}\right), \tag{5.6}$$

where $d = \frac{\deg(D_I)}{2}$ and ϕ_0 is a nowhere zero section of $\mathcal{O}(2-2d)K(D_I)\cong\mathcal{O}$, defined up to scale.

In these cases, the BB-slice parameterizes $\mathcal{W}_0^{ext_I}$ and is given by

$$(\bar\partial_E,\Phi) = \left(\begin{pmatrix}\bar\partial_{d-3} & b \\ 0 & \bar\partial_{-d-1}\end{pmatrix}, \begin{pmatrix}\varphi_1 & \varphi_2 \\ \phi_0 & -\varphi_1\end{pmatrix}\right),$$

where b and φ_i satisfy (4.14). Unlike the case when $\deg(D_I)$ is odd in Remark 5.2, the $b=0$ locus of the slice is nontrivial when $\deg(D_I)$ is even. In fact, the $b=0$ locus of each slice defines a section

$$s_I : H^0(K^2(D)) \to \mathcal{SP}_0(\alpha)$$

of the Hitchin map for the strongly parabolic moduli space. Namely, for $q\in H^0(K^2(D))$ and $\phi_0 : \mathcal{O}(d-3)\to\mathcal{O}(-d-1)\otimes K(D_I)$ nonzero, we have q/ϕ_0 defines a holomorphic section of $\mathcal{O}(2d-2)\otimes K(D_{I^c})$. For the exterior fixed point labeled by D_I, the Hitchin section s_I is

$$s_I(b) = (\mathcal{O}(d-3)\oplus\mathcal{O}(-d-1), \begin{pmatrix} 0 & \frac{q}{\phi_0} \\ \phi_0 & 0\end{pmatrix}).$$

Remark 5.3. The difference between the $\deg(D_I)=0$ and $\deg(D_I)=4$ cases is which component of the Higgs field can vanish. The bundle, flag structure and form of the Higgs field are the same.

The $\hbar$-conformal limit of such a Higgs bundle is given by

$$\mathrm{CL}_\hbar\left(\begin{pmatrix}\bar\partial_{d-3} & b \\ 0 & \bar\partial_{-d-1}\end{pmatrix}, \begin{pmatrix}\varphi_1 & \varphi_2 \\ \phi_0 & -\varphi_1\end{pmatrix}\right) = (\hbar, \begin{pmatrix}\bar\partial_1 & \phi_0^{*_h}+b \\ 0 & \bar\partial_2\end{pmatrix}, \begin{pmatrix}\hbar\partial_1^h+\varphi_1 & \varphi_2 \\ \phi_0 & \hbar\partial_2^h-\varphi_1\end{pmatrix}).$$

Let $(\mathcal{V}(\alpha),\nabla_\hbar)$ denote the corresponding parabolic bundle and parabolic logarithmic $\hbar$-connection of the conformal limit. From the form of the conformal limit, $\mathcal{V}(\alpha)$ is an extension

$$\mathcal{O}(d-3)\to\mathcal{V}(\alpha)\to\mathcal{O}(-1-d).$$

Moreover, the parabolic map

$$\mathcal{O}(d-3)\xrightarrow{\nabla_\hbar}\mathcal{V}\otimes K(D)\longrightarrow\mathcal{O}(-d-1)\otimes K(D)$$

is given by ϕ_0, and hence can only have simple poles at D_I. In particular, the parabolic logarithmic $\hbar$-connection induces an isomorphism $\mathcal{O}(d-3)\cong\mathcal{O}(-d-1)\otimes K(D_I)$. In the nonparabolic case, this is the property that defines the set of opers, and opers are exactly the the image of the Hitchin section under the conformal limit [14]. As a result, such objects can be viewed as parabolic opers.

Remark 5.4. Note that if we forget the parabolic structure and just consider the associated logarithmic $\hbar$-connection, then only the case $D_I = D$ satisfies the oper condition. Indeed, ϕ_0 is an isomorphism between the subbundle and the quotient twisted by $K(D)$ only when $D = D_I$.

Similarly, forgetting the parabolic structure on the Higgs bundle defines a $K(D)$-twisted Higgs bundle on $\mathbb{CP}^1$ with determinant $\mathcal{O}(-4)$, i.e., $\Phi:\mathcal{E}\to\mathcal{E}\otimes K(D)$ is holomorphic. There is a natural notion of stability for such objects and a corresponding moduli space $\mathcal{M}_{K(D)}$. Taking $-\det(\Phi)$ defines a Hitchin map $\mathcal{M}_{K(D)}\to H^0(K^2(2D))$, which has a natural section

$$s_{K(D)} : \ H^0(K^2(2D))\longrightarrow\mathcal{M}_{K(D)} \tag{5.7}$$

defined by $s_{K(D)}(q) = \left[\mathcal{O}(-1)\oplus\mathcal{O}(-3), \begin{pmatrix}0 & q\\ 1 & 0\end{pmatrix}\right]$.

We prove that the intersection of each stratum $\mathcal{W}_0^{ext_I}$ with the fixed complex mass moduli spaces $\mathcal{P}_0(\alpha,\mu)$ parameterizes a section of the Hitchin map $-\det : \mathcal{P}_0(\alpha,\mu)\to\mathcal{B}(\mu)$.

Theorem 5.5. *Consider the stratum $\mathcal{W}_0^{ext_I}$ of any exterior fixed point. Then, for each fixed complex mass vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, there is a section*

$$s_I^\mu : \mathcal{B}(\mu) \to \mathcal{P}_0(\alpha, \mu)$$

of the Hitchin map whose image is $\mathcal{W}_0^{ext_I} \cap \mathcal{P}_0(\alpha, \mu)$. Furthermore, the underlying $K(D)$-twisted Higgs bundle of $s_I^\mu(q)$ is given by $s_{K(D)}(q)$ from (5.7) when $\deg(D_I) = 4$.

Remark 5.6. The maps s_I^μ do not assemble to define section of the Hitchin map for the full moduli space $\mathcal{P}_0(\alpha)$. This is because the base $\mathcal{B}(\mu)$ only determines μ_i^2 at each $p_i \in D$. Hence, fixing $\mathcal{B}(\mu)$ and D_I defines 2^k sections $s_I^\mu : \mathcal{B}(\mu) \to \mathcal{P}_0(\alpha)$, where k is the number of $p_i \in D$ with $\mu_i \neq 0$. Alternatively, let $\hat{\mathcal{B}}$ be the image of the product map

$$(\mathrm{Res}, -\det) : \mathcal{P}_0(\alpha) \to \hat{\mathcal{B}} \subset \mathbb{C}^4 \times H^0(K^2(2D)),$$

and $\pi : \hat{\mathcal{B}} \to H^0(K^2(2D))$ be the natural projection map. Then, the sections $\pi^* s_I^\mu$ do assemble to define a section $S_I : \hat{\mathcal{B}} \to \mathcal{P}_0(\alpha)$. Namely, for $(\mu, q) \in \hat{\mathcal{B}} \subset \mathbb{C}^4 \times \mathcal{B}$ we have

$$S_I(\mu, q) = s_I^\mu(q).$$

The space $\hat{\mathcal{B}}$ is biholomorphic to $\mathbb{C}^5$ and $\pi : \hat{\mathcal{B}} \to H^0(K^2(2D))$ is a 16 to 1 branched cover ramified at the points where $\mu_i = 0$ for some i. This perspective is discussed further in [15]. A similar subtlety motivates Bridgeland–Smith's moduli space of framed quadratic differentials [10].

The parameterization of s_I^μ given in the proof is not by the BB-slice. However, since both objects parameterize $\mathcal{W}_0^{ext_I}$ we have the following corollary.

Corollary 5.7. *For each exterior stratum $\mathcal{W}_0^{ext_I}$ and each complex mass vector μ, the intersection of the BB-slice with $\mathcal{P}_0(\alpha, \mu)$ parameterizes the section s_I^μ. Hence, the conformal limit of the points in the image of the sections s_I^μ can be viewed as parabolic opers.*

Remark 5.8. In [17], Gaiotto conjectures that in the conformal limit, the image of "a canonical complex Lagrangian submanifold which is a section of the torus fibration" is a complex Lagrangian submanifold of a certain complex manifold. The above Corollary is in the spirit of Gaiotto's conjecture, as indeed the image of the BB-slice in $\mathcal{P}_0(\alpha, \mu)$ is a holomorphic Lagrangian submanifold (Theorem 1.7), and the image of the section s_I^μ in the Corollary is a holomorphic Lagrangian inside of $\mathcal{P}_\hbar(\alpha, \hbar\alpha + \mu)$. Note that [17] concerns $\mathsf{PU}(2)$-Hitchin moduli spaces rather than $\mathsf{SU}(2)$-Hitchin moduli spaces; $\mathsf{PU}(2)$-Hitchin moduli spaces on the four-punctured sphere have a single Hitchin section.

Proof of Theorem 5.5. There are three cases determined by $\deg(D_I) = 0, 2, 4$. First assume $\deg(D_I) = 4$. Denote the restriction of the section s from (5.7) to $\mathcal{B}(\mu)$ by $s_\mu^I(q)$. This $K(D)$-twisted Higgs bundle admits a parabolic structure such that the parabolic Higgs bundle is stable and defines a point in $\mathcal{P}_0(\alpha, \mu)$. Indeed, define the flag $F_i \subset (\mathcal{O}(-1) \oplus \mathcal{O}(-3))|_{p_i}$ with weight $1 - \alpha_i$ by

$$F_i = \text{the } \mu_i\text{-eigenspace of } \mathrm{Res}_{p_i}\left(\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right).$$

Now assume $\deg(D_I) = 2$, without loss of generality assume $D_I = \{p_3, p_4\}$. To define the section s_μ^I, we first show that each $q \in \mathcal{B}_\mu$ and $\phi \in H^0(K(D_I))$ determines unique sections $s_1 \in H^0(K^2(D))$ and $s_2 \in H^0(\mathcal{O}(D_{I^C}) \otimes \mathcal{O}(p_4))$ such that

$$q = s_1^2 + \phi s_2, \tag{5.8}$$

Since s_1 is uniquely determined by specifying its residue at three points, define s_1 by

$$\mathrm{Res}_{p_1}(s_1) = \mu_1, \qquad \mathrm{Res}_{p_2}(s_1) = \mu_2 \qquad \text{and} \qquad \mathrm{Res}_{p_3}(s_1) = -\mu_3\ .$$

Hence, $\mathrm{Res}_{p_4}(s_1) = \mu_3 - \mu_1 - \mu_2$. For s_1 and s_2 to solve (5.8), we must have

$$\mathrm{Res}_{p_4}(s_2) = -\frac{(\mu_3 - \mu_2 - \mu_1)^2 + \mu_4^2}{\mathrm{Res}_{p_4}(\phi)} \ .$$

For such an s_2 and any $q \in \mathcal{B}(\mu)$, we have $q - (s_1^2 + \phi s_2) \in H^0(K^2(D))$. Since $B(\mu)$ is affine over $H^0(K^2(D))$, for each q there is a unique s_2 with the above residue at p_4 which solves (5.8).

For $q \in \mathcal{B}(\mu)$ and s_1, s_2 as above, define the following $K(D)$-twisted Higgs bundle

$$s_I^\mu(q) = \left(\mathcal{O}(-2) \oplus \mathcal{O}(-2), \begin{pmatrix} s_1 & s_2 \\ \phi & -s_1 \end{pmatrix}\right).$$

Denote the Higgs field by Φ, and note that $-\det(\Phi) = q$ by (5.8). The parabolic structure so that $s_I^\mu(q) \in \mathcal{P}_0(\alpha, \mu)$ is defined as follows. By construction, we have

$$\mathrm{Res}_{p_1}(\Phi) = \begin{pmatrix} \mu_1 & \mathrm{Res}_{p_1}(s_2) \\ 0 & -\mu_1 \end{pmatrix}, \quad \mathrm{Res}_{p_2}(\Phi) = \begin{pmatrix} \mu_2 & \mathrm{Res}_{p_2}(s_2) \\ 0 & -\mu_2 \end{pmatrix}, \quad \mathrm{Res}_{p_3}(\Phi) = \begin{pmatrix} -\mu_3 & 0 \\ \mathrm{Res}_{p_3}(\phi) & \mu_3 \end{pmatrix}.$$

Thus, the flag $F_i \subset \mathcal{O}(-2) \oplus \mathcal{O}(-2)$ must be the first summand if $i = 1, 2$ and the second summand for $i = 3$. For p_4, we have

$$\mathrm{Res}_{p_4}(\Phi) = \begin{pmatrix} \mu_3 - \mu_1 - \mu_2 & \mathrm{Res}_{p_4}(s_2) \\ \mathrm{Res}_{p_4}(\phi) & \mu_1 + \mu_2 - \mu_3 \end{pmatrix},$$

with eigenvalues $\pm\mu_4$. Hence, the subspace F_4 must be the μ_4-eigenspace of $\mathrm{Res}_{p_4}(\Phi)$. The parabolic weights on the subspaces F_i are $1 - \alpha_i$.

Finally, assume $\deg(D_I) = 0$, a nonzero $\phi : \mathcal{O}(-3) \to \mathcal{O}(-1)K$ determines the associated fixed point. If $\sum \mu_i = 0$, then there is a unique section $t_1 \in H^0(K(D))$ with $\mathrm{Res}_{p_i}(t_1) = \mu_i$ for all i. Since ϕ has no poles, there is a unique $t_2 \in H^0(K^2(D))$ with $q - t_1^2 = \phi t_2$. Define s_I^μ by

$$s_I^\mu(q) = \left(\mathcal{O}(-3) \oplus \mathcal{O}(-1), \begin{pmatrix} t_1 & t_2 \\ \phi & -t_1 \end{pmatrix}\right).$$

Since the residue at each p_i is upper triangular, for all i, take $F_i = \mathcal{O}(-3)|_{p_i}$ with weights $1 - \alpha_i$.

If $\sum \mu_i \neq 0$, then the section will be given by

$$s_I^\mu(q) = (\mathcal{O}(-2) \oplus \mathcal{O}(-2), \begin{pmatrix} t_1 & t_2 \\ t_3 & -t_1 \end{pmatrix}),$$

where $t_j \in H^0(K(D))$ for $j = 1, 2, 3$ and $q = t_1^2 + t_2 t_3$. Each t_j is determined by its residue at 3 points, and these residues are uniquely defined by the condition that all F_i are contained in a fixed subbundle isomorphic to $\mathcal{O}(-3)$ and $q \in \mathcal{B}(\mu)$. Consider the holomorphic subbundle

$$\mathcal{O}(-3) \xrightarrow{(z-p_1, z-p_2)} \mathcal{O}(-2) \oplus \mathcal{O}(-2).$$

For $i = 1, 2$, the condition that $\mathrm{Res}_{p_i}(\Phi)$ has $\mathcal{O}(-3)$ as its μ_i-eigenspace implies

$$\mathrm{Res}_{p_1}(t_1) = -\mu_1 \ , \qquad \mathrm{Res}_{p_1}(t_3) = 0 \qquad \mathrm{Res}_{p_2}(t_1) = \mu_2 \qquad \text{and} \qquad \mathrm{Res}_{p_2}(t_2) = 0 \ .$$

Moreover, the condition $q = t_1^2 + t_2 t_3$ fixes $\mathrm{Res}_{p_1}(t_2)$ and $\mathrm{Res}_{p_2}(t_3)$. For $i = 3$, $\mathrm{Res}_{p_i}(\Phi)$ has the fixed $\mathcal{O}(-3)$ as μ_3-eigenspace implies

$$\mathrm{Res}_{p_3}(t_2) = (\mu_3 - \mathrm{Res}_{p_3}(t_1)) \cdot \frac{p_3 - p_1}{p_3 - p_2} \qquad \text{and} \qquad \mathrm{Res}_{p_3}(t_3) = (\mu_3 + \mathrm{Res}_{p_3}(t_1)) \cdot \frac{p_3 - p_2}{p_3 - p_1} \ .$$

Hence, specifying $\mathrm{Res}_{p_3}(t_1)$ and $q = t_1^2 + t_2 t_3$ determines all t_j. But the residue of q and $t_1^2 + t_2 t_3$ are the same for any choice of $\mathrm{Res}_{p_3}(t_1)$. Thus, $q = t_1^2 + t_2 t_3$ uniquely determines all t_j.

It remains to show that the Higgs bundles described above are stable and in the correct stratum $\mathcal{W}_0^{ext_I}$. First suppose $\deg(D_I) = 4$. For $\lambda \in \mathbb{C}^*$, consider the scaled Higgs field $\lambda\Phi$ of $s_I^\mu(q)$. The gauge transformation $g_\lambda = \operatorname{diag}(\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}})$ is holomorphic and acts on $\lambda\Phi$ as

$$g_\lambda(\lambda\Phi)g_\lambda^{-1} = \begin{pmatrix} 0 & \lambda^2 q \\ 1 & 0 \end{pmatrix} ,$$

Since $F_i \neq \mathcal{O}(-1)|_{p_i}$ for all $p_i \in D$, we have $\lim_{\lambda\to 0} g_\lambda \cdot F_i = \mathcal{O}(-3)|_{p_i}$. For λ sufficiently small, this Higgs bundle is in an open neighborhood of the $\mathbb{C}^*$ fixed point in $\mathcal{W}_0^{ext_I}$. Since stability is open, preserved by the $\mathbb{C}^*$ action and gauge transformation, $s_I(q)$ is stable and in $\mathcal{W}_0^{ext_I}$.

The cases $\deg(D_I) = 2$ and $\deg(D_I) = 0$ with $\sum \mu_i = 0$ are similar to the $\deg(D_I) = 4$ case. Finally, suppose $\deg(D_I) = 0$ with $\sum \mu_i \neq 0$. In this case, the holomorphic bundle is a nonsplit extension of $\mathcal{O}(-1)$ by $\mathcal{O}(-3)$. In a smooth splitting $\mathcal{O}(-3) \oplus \mathcal{O}(-1)$ the Higgs bundle is given by

$$(\bar\partial_E, \Phi) = \left(\begin{pmatrix} \bar\partial_{-3} & b \\ 0 & \bar\partial_{-1} \end{pmatrix}, \begin{pmatrix} \varphi_1 & \varphi_2 \\ \phi & -\varphi_1 \end{pmatrix} \right),$$

where $\phi : \mathcal{O}(-3) \to \mathcal{O}(-1) \otimes K(D)$ is nonzero and holomorphic. The gauge transformation $g_\lambda = \operatorname{diag}(\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}})$ acts on $(\bar\partial_E, \lambda\Phi)$ as

$$g_\lambda \cdot (\bar\partial_E, \lambda\Phi) = \left(\begin{pmatrix} \bar\partial_{-3} & \lambda b \\ 0 & \bar\partial_{-1} \end{pmatrix}, \begin{pmatrix} \lambda\varphi_1 & \lambda^2\varphi_2 \\ \phi & -\lambda\varphi_1 \end{pmatrix} \right) .$$

Moreover, $g_\lambda \cdot F_i = F_i$ since $F_i = \mathcal{O}(-3)|_{p_i}$ for all i. By the same arguement as the $\deg(D_I) = 4$ case, it follows that $s_I^\mu(q)$ is stable and in $\mathcal{W}_0^\mu$. $\square$

Remark 5.9. Switching the roles of p_3, p_4 in the $\deg(D_I) = 2$ case, or choosing a different $\mathcal{O}(-3)$ subbundle in the $\deg(D_I) = 0$ case with $\sum \mu_i \neq 0$ defines isomorphic parabolic Higgs bundles.

We end the paper by showing that, analogous to the Hitchin section in the nonparabolic case, the harmonic metric at an exterior fixed point comes from a constant negative curvature metric on $\mathbb{CP}^1$ with conical singularities at each $p_i \in D$. As above, write the parabolic bundle associated to an exterior $\mathbb{C}^*$-fixed point (5.6) as $\mathcal{L}_1(\beta_1) \oplus \mathcal{L}_2(\beta_2)$, where β_1 and β_2 are defined in (5.2). Then

$$\mathcal{L}_1(\beta_1)^2 \cong \mathcal{O}(-2)(\gamma) = K(\gamma), \qquad \text{where} \qquad \gamma(p_i) = \begin{cases} 2\alpha_i & p_i \in D_I \\ 1 - 2\alpha_i & p_i \in D_{I^c} \end{cases} .$$

Since $\deg(K(\gamma)) = 2\deg(\mathcal{L}_1(\beta_1)) > 0$, the harmonic metric on $K^{-1}(-\gamma)$ is a constant negative curvature singular metric g on the tangent bundle K^{-1} which is smooth on $\mathbb{CP}^1 \setminus D$ and $|z|^{2\gamma}h$ extends as a smooth metric across D. The hermitian metric g gives a Riemannian of constant negative curvature on $\mathbb{CP}^1$ which has conical singularities at D with

$$\text{cone angle of } g = \begin{cases} 2\pi(1 - 2\alpha_i) & \text{if } p_i \in D_I \\ 4\pi\alpha_i & \text{if } p_i \in D_{I^c} \end{cases} .$$

Consider the square root $K(\gamma)^{\frac{1}{2}} = \mathcal{O}(-1)(\frac{\gamma}{2})$, its dual is given by

$$K(\gamma)^{-\frac{1}{2}} = \mathcal{O}(1)(-\frac{\gamma}{2}) \cong (\mathcal{O}(1) \otimes \mathcal{O}_D^{-1})(1 - \frac{\gamma}{2}) \cong \mathcal{O}(-3)(1 - \frac{\gamma}{2}).$$

With this setup, the following lemma is immediate.

Lemma 5.10. *Let $d = \frac{\deg(D_I)}{2}$ and $K(\gamma)^{\frac{1}{2}}$ be as above. Consider the parabolic line bundle $\mathcal{L}_3(\beta_3)$, where $\mathcal{L}_3 = \mathcal{O}(-2 + d)$ and $\beta_3(p_i) = 0$ if $p_i \in D_I$ and $\beta_3(p_i) = \frac{1}{2}$ if $p_i \in D_{I^c}$. Then the parabolic bundle for the exterior fixed point from Proposition 5.1 is given by*

$$\mathcal{O}(d-3)(\beta_1) \oplus \mathcal{O}(-1-d)(\beta_2) \cong (\mathcal{L}_3(\beta_3) \otimes K(\gamma)^{\frac{1}{2}}) \oplus (\mathcal{L}_3(\beta_3)^{-1} \otimes K(\gamma)^{-\frac{1}{2}}).$$

Denote the harmonic metric on $\mathcal{L}_3(\beta_3)$ by h_3. Since $\deg(\mathcal{L}_3(\beta_3)) = 0$, h_3 is flat. Note that $g^{\frac{1}{2}}$ defines a metric on $K(\gamma)^{-\frac{1}{2}}(-\frac{\gamma}{2})$ and $g^{-\frac{1}{2}}$ defines a metric on $K(\gamma)^{\frac{1}{2}}$. The trivial metric on trivial parabolic bundle $\mathcal{O}(0)$ defines a metric $h_{\det}$ on the weighted line bundle $\mathcal{O}_D^{-1}(1)$, and $g^{\frac{1}{2}}h_{\det}$ is a compatible metric on the parabolic line bundle $\mathcal{O}(-3)(1-\frac{\gamma}{2})$.

Proposition 5.11. *With the notation above, the harmonic metric on the exterior $\mathbb{C}^*$-fixed point*

$$\left(\mathcal{O}(d-3)(\beta_1) \oplus \mathcal{O}(-1-d)(\beta_2), \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}\right)$$

is given by $h = (h_3 \cdot g^{-\frac{1}{2}}) \oplus (h_3^{-1} \cdot g^{\frac{1}{2}} \cdot h_{\det})$.

Proof. Since the metrics $h_{\det}$ and h_3 are flat, and the harmonic metric is diagonal at $\mathbb{C}^*$-fixed points, the metric h solves the Hitchin equations $F_{A_h} + [\Phi, \Phi^{*_h}] = 0$ if and only if

$$0 = F_{A_g} + 2\phi \wedge \phi^* = F_{A_g} + 2gh_3^{-2}h_{\det}\phi \wedge \bar{\phi}.$$

The curvature of the Levi-Civita connection on the holomorphic tangent bundle is $-4i(\frac{1}{4}K_g\omega_g)$, where ω_g is the Kähler form and K_g is the Gauss curvature of g which is -4. To complete the proof, we show $2\phi \wedge \phi^* = -4i\omega_g$. Note that if $g = \lambda^2 \, \mathrm{d}z \otimes \mathrm{d}\overline{z}$, then $\omega_g = \lambda^2 \, \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. In the trivialization of $\mathcal{O}(d-3) \oplus \mathcal{O}(-1-d)$ given by $(\mathrm{d}z)^{\frac{3-d}{2}} \oplus (\mathrm{d}z)^{\frac{1+d}{2}}$, the Higgs field is given by $\phi = \mathrm{d}z^{d-1} \otimes \prod_{p_i \in D_I}(z-p_i)^{-1}\mathrm{d}z$ and its adjoint is

$$\phi^* = gh_3^{-2}h_{\det}\bar{\phi} = \underbrace{\lambda^2 \, \mathrm{d}z \otimes \mathrm{d}\bar{z}}_{g} \underbrace{\prod_{p_i \in D} |z-p_i|^{-2}|\mathrm{d}z|^{-2(d-2)}}_{h_3^{-2}(\mathrm{d}z^{d-2}, \mathrm{d}z^{d-2})} \underbrace{\prod_{p_i \in D_{I^c}} |z-p_i|^2|\mathrm{d}z|^{-4}}_{h_{\det}(\mathrm{d}z^2, \mathrm{d}z^2)} \left(\mathrm{d}\bar{z}^{d-1} \otimes \prod_{p_i \in D_I} (\overline{z-p_i})^{-1}\mathrm{d}\bar{z}\right).$$

Hence $\phi \wedge \phi^* = \lambda^2 \, \mathrm{d}z \wedge \mathrm{d}\overline{z} = -2i\omega_g$. ∎

Appendix A. Simpson's stratification

In this appendix we show that Simpson's iterative process of [40] generalizes to parabolic logarithmic λ-connections. The main difference with [40] is that stable parabolic connections are not always irreducible. When the parabolic logarithmic connections are irreducible, the associated stratifications were studied in [40] and[28]. We note that the discussion below also applies to Higgs bundles in both the parabolic and nonparabolic setting. For additional details on Simpson's construction we follow [21, §2.2.1].

Let $(\mathcal{E}(\alpha), \nabla)$ be a parabolic logarithmic λ-connection. A filtration

$$\mathcal{E}(\alpha) = \mathcal{A}^0(\alpha) \supset \mathcal{A}^1(\alpha) \supset \cdots \supset \mathcal{A}^\ell(\alpha) \supset 0$$

is called Griffiths transverse (with respect to ∇) if $\nabla(\mathcal{A}^j(\alpha)) \subset \mathcal{A}^{j-1}(\alpha) \otimes K(D)$ for all j. For example, if $\mathcal{A}^1(\alpha)$ is the maximal destabilizing subbundle of $\mathcal{E}(\alpha)$, then $\mathcal{A}^1(\alpha) \subset \mathcal{A}^0(\alpha)$ is a Griffiths transverse filtration. Given such a filtration, the associated graded is

$$Gr_{\mathcal{A}}(\mathcal{E}) = \bigoplus_{j=1}^{\ell} \mathcal{A}_j(\alpha) \quad \text{where} \quad \mathcal{A}_j(\alpha) = \mathcal{A}^j(\alpha)/\mathcal{A}^{j+1}(\alpha).$$

The parabolic connection ∇ induces a parabolic morphism $\phi_j : \mathcal{A}_j(\alpha) \to \mathcal{A}_{j-1}(\alpha)) \otimes K(D)$, and hence defines a system of Hodge bundles

$$(Gr_{\mathcal{A}}(\mathcal{E})(\alpha), \Phi_{\mathcal{A}}) = \left(\mathcal{A}_\ell(\alpha) \oplus \cdots \oplus \mathcal{A}_0(\alpha) \; , \; \begin{pmatrix} 0 & & & \\ \phi_\ell & 0 & & \\ & \ddots & \ddots & \\ & & \phi_1 & 0 \end{pmatrix}\right). \tag{A.1}$$

As in [40, Lemma 4.1], we have the following proposition.

Proposition A.1. *If the system of Hodge bundles $(Gr_{\mathcal{A}}(\mathcal{E})(\alpha), \Phi_{\mathcal{A}})$ from (A.1) is semistable, then*

$$\lim_{\xi\to 0}[\xi\lambda, \mathcal{E}(\alpha), \xi\nabla)] = [Gr_{\mathcal{A}}(\mathcal{E})(\alpha), \Phi_{\mathcal{A}}] \ .$$

If the parabolic system of Hodge bundles (A.1) is not semistable, let $(\bigoplus \hat{\mathcal{A}}_j(\alpha), \Phi_{\hat{\mathcal{A}}})$ be the maximal destabilizing system of Hodge bundles. Define the following two invariants of a Griffiths transverse filtration $\mathcal{A}^\bullet$ whose associated system of Hodge bundles is not semistable:

$$\zeta(\mathcal{A}^\bullet) = \mu(\textstyle\bigoplus_j \hat{\mathcal{A}}_j(\alpha)) \qquad \text{and} \qquad \eta(\mathcal{A}^\bullet) = \mathrm{rk}(\textstyle\bigoplus_j \hat{\mathcal{A}}_j(\alpha)) \ .$$

Following [40], define a new filtration $\mathcal{B}^\bullet$ of $\mathcal{E}$ by

$$\mathcal{B}^j = \ker(\mathcal{E} \to (\mathcal{E}/\mathcal{A}^j)/\hat{\mathcal{A}}_{j-1})$$

This new filtration is again Griffiths transverse, and the summands $\mathcal{B}_j(\alpha) = \mathcal{B}^j(\alpha)/\mathcal{B}^{j+1}(\alpha)$ of the associated graded fit in an exact sequence

$$0 \to \mathcal{A}_j(\alpha)/\hat{\mathcal{A}}_j(\alpha) \to \mathcal{B}_j(\alpha) \to \hat{\mathcal{A}}_{j-1}(\alpha) \to 0. \tag{A.2}$$

If the parabolic system of Hodge bundles $(Gr_{\mathcal{B}}(\mathcal{E})(\alpha), \Phi_{\mathcal{B}})$ is semistable, then we have identified the limit $\xi \to 0$, if it is not, then the process can be repeated to obtain a new system of Hodge bundles. Simpson's key observation is that (ζ, η) decreases at each step of the above iterative process.

Proposition A.2. *Suppose $(Gr_{\mathcal{B}}(\mathcal{E})(\alpha), \Phi_{\mathcal{B}})$ is not semistable and let $\bigoplus_j \hat{\mathcal{B}}_j$ be the maximal destabilizing system of Hodge bundles. Then*

(1) $\zeta(\mathcal{B}^\bullet) \leq \zeta(\mathcal{A}^\bullet)$,
(2) if $\zeta(\mathcal{B}^\bullet) = \zeta(\mathcal{A}^\bullet)$, then $\eta(\mathcal{B}^\bullet) \leq \eta(\mathcal{A}^\bullet)$, and
(3) if $\zeta(\mathcal{B}^\bullet) \leq \zeta(\mathcal{A}^\bullet)$ and $\eta(\mathcal{B}^\bullet) \leq \eta(\mathcal{A}^\bullet)$, then $\hat{\mathcal{B}}_j(\alpha) \cong \hat{\mathcal{A}}_{j-1}(\alpha)$ for all j.

Proof. For the nonparabolic setting, see [21, §2.2.1]. The only changes that need to be made are to consider all objects and destabilizing subobjects in the parabolic category. □

Proposition A.3. *Let $(\mathcal{E}(\alpha), \nabla)$ be a semistable parabolic λ-connection, then there exists a Griffiths transverse filtration such that the associated parabolic system of Hodge bundles is semistable. In particular, the limit $\lim_{\xi\to 0}[\xi\lambda, \mathcal{E}(\alpha), \xi\nabla)]$ exists.*

Proof. We will show that the semistability assumption implies that the iterative process described above terminates. By Proposition A.2, the invariants (η, ζ) decrease in lexicographically at each step. We will show (η, ζ) can only take finitely many values and that semistability implies (η, ζ) can remain unchanged by only finitely many consecutive step of the process.

The invariant η can clearly only take finitely many values. To see that ζ can also only take finitely many values, note that ζ is bounded above by the slope of the maximal destabilizing subbundle of $\mathcal{E}(\alpha)$ and bounded below by the slope of $\mathcal{E}(\alpha)$. Since the parabolic weights are fixed, the parabolic degree of the maximal destabilizing subbundle can only take finitely many values, and hence ζ can take only finitely many values.

Let $\mathcal{A}^\bullet$ be a Griffiths transverse filtration of $\mathcal{E}$, suppose the associated system of Hodge bundles $(\bigoplus_j \mathcal{A}_j, \Phi_{\mathcal{A}})$ given by (A.1) is unstable. Consider the new system of Hodge bundles $(\bigoplus_j \mathcal{B}_j, \Phi_{\mathcal{B}})$ given by (A.2). Let $\bigoplus_j \hat{\mathcal{A}}_j(\alpha)$ and $\bigoplus_j \hat{\mathcal{B}}_j(\alpha)$ be the maximal destabilizing subobjects. Suppose $\eta(\mathcal{B}^\bullet) = \eta(\mathcal{A}^\bullet)$ and $\zeta(\mathcal{B}^\bullet) = \zeta(\mathcal{A}^\bullet)$. Then $\hat{\mathcal{B}}_j(\alpha) \cong \hat{\mathcal{A}}_{j-1}(\alpha)$ by part (3) of Proposition A.2. Hence, the exact sequence (A.2) splits and

$$\mathcal{B}_j(\alpha) \cong \mathcal{A}_j(\alpha)/\hat{\mathcal{A}}_j(\alpha) \oplus \hat{\mathcal{A}}_{j-1}(\alpha)$$

with maximal destabilizing subobject $\hat{\mathcal{A}}_{j-1}$. In particular, the maximal destabilizing subobject is shifted to the left 1-step in the grading (A.1). If the invariants (η,ζ) remain unchanged for infinitely many consecutive steps, the associated grade will eventually be of the form

$$\mathcal{E}_k \oplus \mathcal{E}_{k-1} \oplus \cdots \oplus \mathcal{E}_{k-\ell} \oplus \mathcal{E}_{k-\ell-2} \oplus \cdots \oplus \mathcal{E}_0,$$

where $\bigoplus \hat{\mathcal{A}}_j \cong \mathcal{E}_k \oplus \mathcal{E}_{k-1} \oplus \cdots \oplus \mathcal{E}_{k-\ell}$. Since there is a gap in the grading, the $\phi_{k-\ell-1}$ term in the Higgs field vanishes. By Griffiths transversality, this means the maximal destabilizing subbundle is ∇-invariant, contradicting $(\mathcal{E}(\alpha),\nabla)$ being semistable. $\square$

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