

Dynamical Quantum Phase Transition and Thermal Equilibrium in the Lattice Thirring Model

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Using tensor network methods, we simulate the real-time evolution of the lattice Thirring model quenched out of equilibrium in both the critical and massive phases, and study the appearance of dynamical quantum phase transitions, as non-analyticities in the Loschmidt rate. Whereas the presence of a dynamical quantum phase transition in the model does not correspond to quenches across the critical line of the equilibrium phase diagram at zero temperature, we identify a threshold in the energy density of the initial state, necessary for a dynamical quantum phase transition to be present. Moreover, in the case of the gapped quench Hamiltonian, we unveil a connection of this threshold to a transition between different regions in the finite temperature phase diagram.

I. INTRODUCTION

Out-of-equilibrium dynamics is one of the most challenging problems in the study of quantum many-body systems. This funnels the interest towards finding universal behaviours that allow a more comprehensive understanding of the dynamics. A paradigmatic scenario to investigate these questions is that of a quantum quench [1], in which a system is evolved with a Hamiltonian of interest, after having been initialized in a certain (often pure) state. In the last decade, an active direction of research in this context has been the investigation of dynamical quantum phase transitions (DQPTs) [2, 3], which appear as zeroes in the Loschmidt echo or return probability, i.e. the probability that the system is found in the initial state during the evolution.

One of the problems for studying out-of-equilibrium dynamics of such complex systems is the scarcity of appropriate tools. Only in certain cases is an analytical solution possible

that allows a complete description of the time evolution after a quench [2, 4, 5]. For the most general cases, however, the only possibility is to obtain an approximation to the dynamics by means of numerical simulations, within the regimes where these are feasible. A relevant role among numerical methods is played by tensor network state (TNS) techniques [6–11]. In particular, in one-dimensional systems, matrix product state (MPS) [12–16] algorithms provide an effective way to explore the time-dependent properties of the Loschmidt echo for systems in the thermodynamic limit [3, 17]. Although the potentially fast entanglement growth [18–20] limits the time that can be reliably simulated, TNS dynamical methods [21–23] enable an accurate picture of the quantity of interest for a moderate time after the quench, which has been exploited to obtain important results regarding DQPTs [17, 24–30].

Signatures of DQPT have been observed experimentally in trapped ion quantum simulators [31] and superconducting qubits [32]. While initial studies suggested that DQPTs occur when the parameter of the Hamiltonian is quenched across an equilibrium phase boundary [2, 3, 24], it was later realized that this is not the general case [17, 33–36], and the nature of DQPTs is more complex, and far from being completely understood [29, 37].

The study of out-of-equilibrium physics for quantum field theories (QFTs), defined in the continuum, is highly non-trivial. Equilibrium properties of such models are usually addressed

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by the path integral formalism and Monte Carlo sampling, which however encounters a sign problem in certain scenarios. This limitation has motivated an investigation of applicability of TNS methods to the realm of QFT. In particular, gauge theories in 1+1D have been systematically addressed using TNS methods (see e.g. [38–40] and references therein). Other QFTs and conformal field theories (CFTs) have also been studied in the same manner [41–44]. These investigations have focused on the equilibrium properties of the field theories. Real-time simulations are more scarce [45–48], and studying the real time dynamics of a continuum QFT with these methods remains an open problem.

The most straightforward way to study QFT problems with TNS is to use the discretized version of the model Hamiltonian, and apply standard TNS methods to find their equilibrium or dynamical properties. Such results depend on different lattice discretization schemes, but by repeating the procedure at different lattice spacings and performing the proper renormalization, it is possible to recover the continuum limit (see e.g. [45, 49]). In this work, we focus on one such spin chain, corresponding to the discretization of the 1+1 dimensional Thirring model. The resulting spin model is a specialized XXZ Hamiltonian coupled to both staggered and homogeneous magnetic fields. The ground state of the model, as determined in previous work [41] using MPS, exhibits a critical and a gapped phase, separated by a Berezinskii-Kosterlitz-Thouless (BKT) phase transition (see Fig. 1). For vanishing bare fermion mass, the model is critical for all couplings.

In this work, we use uniform MPS to study the phenomenology of DQPTs in the spin model for the discretized Thirring model. We simulate quenches in both phases from initial states with very different properties. We show that a necessary condition for a DQPT to occur within the finite time window that our numerical method can access is that the energy density of the initial state is above a certain threshold, independent of the phases of the initial and quenched Hamiltonian. However, in the case of the quench to the gapped phase we also identify a connection of DQPTs to the equilibrium phase diagram, but at finite temperature. Namely, at the temperature corresponding to the identified energy density threshold, the properties of the thermal equilibrium state undergo a substantial change, with the vanishing of a string correlator that is non-zero in the ground state. We further show that the times at which DQPTs occur as a function of the energy density of the initial state show a structure reminiscent of the complex zeros of the Loschmidt amplitude [50, 51], and several branches of DQPTs that can be explored from different initial states.

The rest of the paper is structured as follows. In Sec. II we

introduce the spin model that corresponds to the discretized Thirring Hamiltonian, and present the MPS formulation and the tools and algorithms we have used. Second, we review the equilibrium phase structure of the Thirring model from the previous research in Sec. III. In Sec. IV, we discuss the DQPT concept and the use of MPS methods to study them in the model of interest. Our results for DQPTs in different quenches, and the relation of their appearance with the finite temperature phase diagram are discussed in Sec. V. Finally, in Sec. VI we summarize our findings and discuss potential further investigations.

II. FORMALISM

A. The Thirring model as a spin model

In this work, we study the massive Thirring model in 1+1 dimensions. The action is described by

$$S_{\text{Th}}[\psi, \bar{\psi}] = \int d^2x [\bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi - \frac{g}{2} (\bar{\psi} \gamma_\mu \psi) (\bar{\psi} \gamma^\mu \psi)], \quad (1)$$

where m is the fermion mass and g is the four-fermion coupling. To facilitate the use of MPS, quantization of the classical theory described by Eq. (1) is performed through the canonical method in the Hamiltonian formalism. One subtlety in this formalism is the inclusion of effects of the anomaly in the four-fermion operator. These effects can be easily investigated through the examination of the path-integral measure in the Lagrangian formulation [52]. In the canonical procedure, they need to be accounted for via a non-local definition of the currents in the four-fermion operator [53, 54].

The model can be discretized on a one-dimensional spatial lattice using the staggered regularization. For convenience of carrying out numerical computations in this work, the fermionic degrees of freedom in this Hamiltonian are mapped onto spin operators through the Jordan-Wigner (JW) transformation. Details of the above procedure can be found in Ref. [41]. Implementation of this strategy turns the Hamiltonian operator of the continuum Thirring model, H_{Th} , into that of the XXZ spin chain coupled to both uniform and staggered magnetic fields, up to a scaling factor,

$$H(\tilde{m}_0 a, \Delta) = -\frac{1}{2} \sum_n^{N-2} (S_n^+ S_{n+1}^- + S_{n+1}^+ S_n^-) + a \tilde{m}_0 \sum_n^{N-1} (-1)^n \left(S_n^z + \frac{1}{2} \right) + \Delta(g) \sum_n^{N-1} \left(S_n^z + \frac{1}{2} \right) \left(S_{n+1}^z + \frac{1}{2} \right), \quad (2)$$

where a is the lattice spacing, N is the total number of lattice sites, $S_n^\pm = S_n^x \pm iS_n^y$ and S_n^z are the spin matrices ($S_n^i = \sigma^i/2$ with σ^i being the Pauli matrices) at the n -th site, and $[S_n^i, S_m^j]_{n \neq m} = 0$. Here, $\tilde{m}_0 = m_0/\nu(g)$ with m_0 being the bare counterpart of the mass parameter, m , in Eq. (1). The functions $\nu(g)$ and $\Delta(g)$ are the lattice version of wavefunction renormalization and the four-fermion coupling [55],

$$\begin{aligned} \nu(g) &= \left(\frac{\pi - g}{\pi} \right) / \sin \left(\frac{\pi - g}{2} \right), \\ \Delta(g) &= \cos \left(\frac{\pi - g}{2} \right). \end{aligned} \quad (3)$$

Notice that the physical Hamiltonian is obtained by rescaling the dimensionless one in Eq. (2) by a factor $\nu(g)/a$. Throughout the rest of the paper, we will work with the lattice Hamiltonian (2) and correspondingly define dimensionless time and inverse temperature parameters, which involve rescaling the physical (dimensionful) magnitudes by $\nu(g)/a$. Additionally, we restrict our consideration to the sector with total spin (magnetization) zero [41] and find the ground state by adding the penalty term (See appendix A.)

B. Matrix product state

A generic quantum state for a system of N spins can be written in the form

$$|\Psi_N\rangle = \sum_{\sigma_1, \dots, \sigma_N} c_{\sigma_1 \dots \sigma_N} |\sigma_1 \dots \sigma_N\rangle, \quad (4)$$

which can then be approximated as a matrix product state (MPS) with bond dimension D ,

$$|\Psi_N\rangle = \sum_{\sigma_1, \dots, \sigma_N} \text{tr} [M_1^{\sigma_1} M_2^{\sigma_2} \dots M_N^{\sigma_N}] |\sigma_1 \dots \sigma_N\rangle, \quad (5)$$

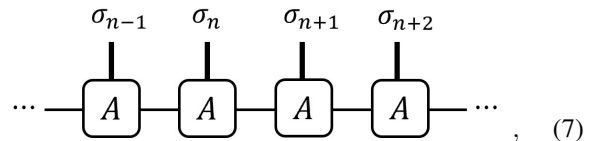
where $M_i^{\sigma_i}$ is a $D \times D$ matrix. For the open boundary condition, $M_1^{\sigma_1}$ and $M_N^{\sigma_N}$ become D -dimensional vectors. In this case, the trace in Eq. (5) becomes a simple product of matrices.

We search for DQPTs in the real-time evolution of Eq. (2). For this purpose, it is desirable to perform calculations directly in the thermodynamic limit. This can be accomplished by employing the uniform MPS (uMPS) technique, which allows for the description of the state of an infinite spin chain with translation invariance using one bulk tensor and two appropriate boundary tensors [56, 57]. The uniform MPS can be represented as

$$|\Psi(A)\rangle = \sum_{\sigma} (\dots A^{\sigma_{n-1}} A^{\sigma_n} A^{\sigma_{n+1}} A^{\sigma_{n+2}} \dots) |\sigma\rangle, \quad (6)$$

where $A^{\sigma_n} \in \mathbb{C}^{D \times D}$ for $\sigma = 1, \dots, d$. It can be represented

diagrammatically as



where A is the unit cell of the uMPS. Due to the staggered term in Eq. (2), the system is translationally invariant with a period of two. Thus, we choose the tensor A to represent two contiguous sites of the chain, and its physical dimension becomes $d = 4$.

III. EQUILIBRIUM PHASE STRUCTURE OF THE 1+1 DIMENSIONAL THIRRING MODEL

In this section, we review our study of the zero-temperature phase structure of the Thirring model using MPS [41]. An analysis of renormalization group (RG) equations of the dual sine-Gordon theory [58–60] points to the existence of two phases in the Thirring model. The massless theory is a conformal field theory at all couplings $\Delta(g)$. RG flows in the mass-coupling plane reveal that the line $m = 0$ is a fixed line under RG transformations. However, it is stable/unstable for couplings $\Delta(g)$ below/above certain $\Delta(g_*)_{m=0} = -\pi/2$. At $m > 0$, the non-zero fermion mass is an irrelevant operator for $\Delta(g) < \Delta(g_*)_m$, but becomes relevant when $\Delta(g) > \Delta(g_*)_m$, with the transition coupling $\Delta(g_*)_m$ being mass-dependent. This gives rise to two equilibrium phases:

- $\Delta(g) < \Delta(g_*)_m$ – gapless (critical) phase,
- $\Delta(g) > \Delta(g_*)_m$ – gapped (massive) phase,

separated by a BKT transition. The BKT transition was originally discussed in the context of the XY model [61, 62], which is actually dual to the sine-Gordon and Thirring theories, with the XY model temperature dual to the couplings of these theories. For a more extensive discussion of these aspects, including von Neumann entanglement entropy and chiral condensate, we refer to Ref. [41].

The quantitatively most robust analysis of the phase structure and the determination of $\Delta(g_*)_m$ is about the string correlator,

$$\begin{aligned} C_{\text{string}}(x) &= \frac{1}{N_x} \sum_n \langle S^z(n) S^z(n+1) \dots \\ &\dots S^z(n+x-1) S^z(n+x) \rangle, \end{aligned} \quad (8)$$

which is a string of S^z operators. We note that Ref. [41] erroneously states this string correlator as the fermion-antifermion one, containing $\langle S^+(n) S^z(n+1) \dots S^z(n+x-1) S^-(n+x) \rangle$ terms. At large enough distances, the correlator $C_{\text{string}}(x)$

exhibits a power-law decay in the gapless phase, given by $C_{\text{string}}(x) \propto \beta x^{-\alpha}$, and a power-exponential decay in the gapped phase, $C_{\text{string}}(x) \propto Bx^{-\eta}A^x$. The crucial distinction that makes the string correlator particularly useful for determining $\Delta(g_*)_m$ is that it decays to a constant, C , in the gapped phase. In practice, one can analyze both phases based on the power-exponential fitting ansatz:

$$C_{\text{string}}(x) = Bx^{-\eta}A^x + C, \quad (9)$$

with $A = 1$ found in fits in the gapless phase. Thus, the phase transition when going from the gapless into the gapped phase is manifested by A dropping below 1 and, in the case of the string correlator, by $C > 0$. The onset of the non-zero constant is numerically more robust than the smooth drop of A below 1, accounting for the practical usefulness of the string correlator. As we will see below, this is true also at non-zero temperatures and it will allow us to identify relations between thermal phases of the Thirring model and DQPTs.

As a summary of our findings for the zero-temperature phase structure, we show Fig. 1. The color-coded regions correspond to the equilibrium phase on the coupling-mass plane through the correlator analyses from our previous work [41]. In particular, we fitted Eq. (9) and extracted values of the constant C . Since the results depend, in general, on the fitting interval, we adopted a systematic procedure to assess the systematic uncertainty of C (dominating with respect to other uncertainties, coming from the finite bond dimension and the finite volume). For details of this procedure, we refer again to Ref. [41]. Depending on the value of C , we identified three situations. With $C < 0.001$ or $C > 0.01$, one unambiguously concludes either the critical phase (purple region) or the gapped phase (orange region), respectively. If $C \in [0.001, 0.01]$ (grey region), the systematic uncertainty does not allow us to draw conclusions about a (non)vanishing value of C . This region of ambiguous values of C is indicated by grey shading and the BKT transition must occur there. We note the onset a nonzero value of C is very smooth at small fermion masses, while it becomes rather sharp with an increasing mass. At very small masses, the implied transition point is consistent with the RG flow analysis ($\Delta(g) \approx -\pi/2$) and at large masses, it moves towards more negative values.

IV. THE LOSCHMIDT ECHO AND THE MIXED TRANSFER MATRIX

A. Quantum quench and DQPT

In this work, we investigate DQPT in a quantum quench. A quantum quench is the paradigmatic scenario where to study non-equilibrium dynamics of a quantum system. The system is assumed to be prepared in a certain initial state $|\Psi_1\rangle$, typically the ground state of a Hamiltonian H_1 . From time $t = 0$

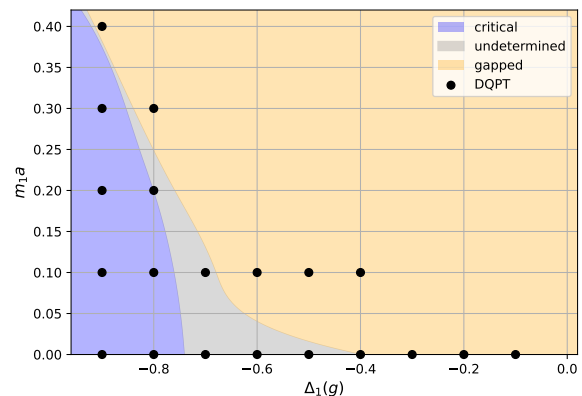


FIG. 1. Zero-temperature phase diagram of the Thirring model in the coupling-mass plane. The color-coded areas on the plot depict the equilibrium phase through correlator analyses [41]. The blue region is the critical region, where the constant C (defined in Eq. (9)) is less than 0.001; the orange region is the gapped phase region, where $C > 0.01$; the grey region is undetermined, where the corresponding $C \in [0.001, 0.01]$. The black point marks the location where we observe the DQPT, starting from the ground state of the Hamiltonian $H_1(m_1 a, \Delta_1)$ and quenching to $H_2(m_2 a = 0.1, \Delta_2 = 0.9)$.

on, it is evolved with a different Hamiltonian H_2 . The evolved state can be written as $|\Psi(t)\rangle = e^{-itH_2}|\Psi_1\rangle$. The key ingredient in the study of DQPT is the Loschmidt amplitude,

$$G(t) = \langle \Psi_1 | \Psi(t) \rangle = \langle \Psi_1 | e^{-iH_2 t} | \Psi_1 \rangle, \quad (10)$$

which is analogous to the boundary partition function [63] $Z_B(z) = \langle \Psi_1 | e^{-zH_2} | \Psi_1 \rangle$ in the complex plane $z \in \mathbb{C}$. The non-analytical points of the free energy density $f(z) = -\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_B(z)$ are believed to indicate phase transitions [64]. The squared modulus of Loschmidt amplitude in Eq. (10), the Loschmidt echo, quantifies the return probability, i.e. the probability of the system being found in the initial state after a certain time t . In the thermodynamic limit ($N \rightarrow \infty$), the Loschmidt echo will approach $e^{-Nr(t)}$, with the rate function

$$r(t) = -\lim_{N \rightarrow \infty} \frac{1}{N} \log |G(t)|^2. \quad (11)$$

This function can be interpreted as a dynamical free energy, and can exhibit non-analyticities. Dynamical quantum phase transitions (DQPT) are defined as the times when $r(t)$ is non-analytical [2]. In some cases, DQPTs are associated with quenches across an equilibrium phase transition [2, 65], but such connection to equilibrium is not always present [33], which motivates the identification of DQPTs as a genuinely non-equilibrium effect. The potential observable signature of DQPTs in local and string quantities has also been a subject of intensive studies. [66, 67] The complex phenomenology of DQPTs has motivated studies in multiple models, but the phenomenon is yet far from a full theoretical understanding [3, 17, 27, 68, 69].

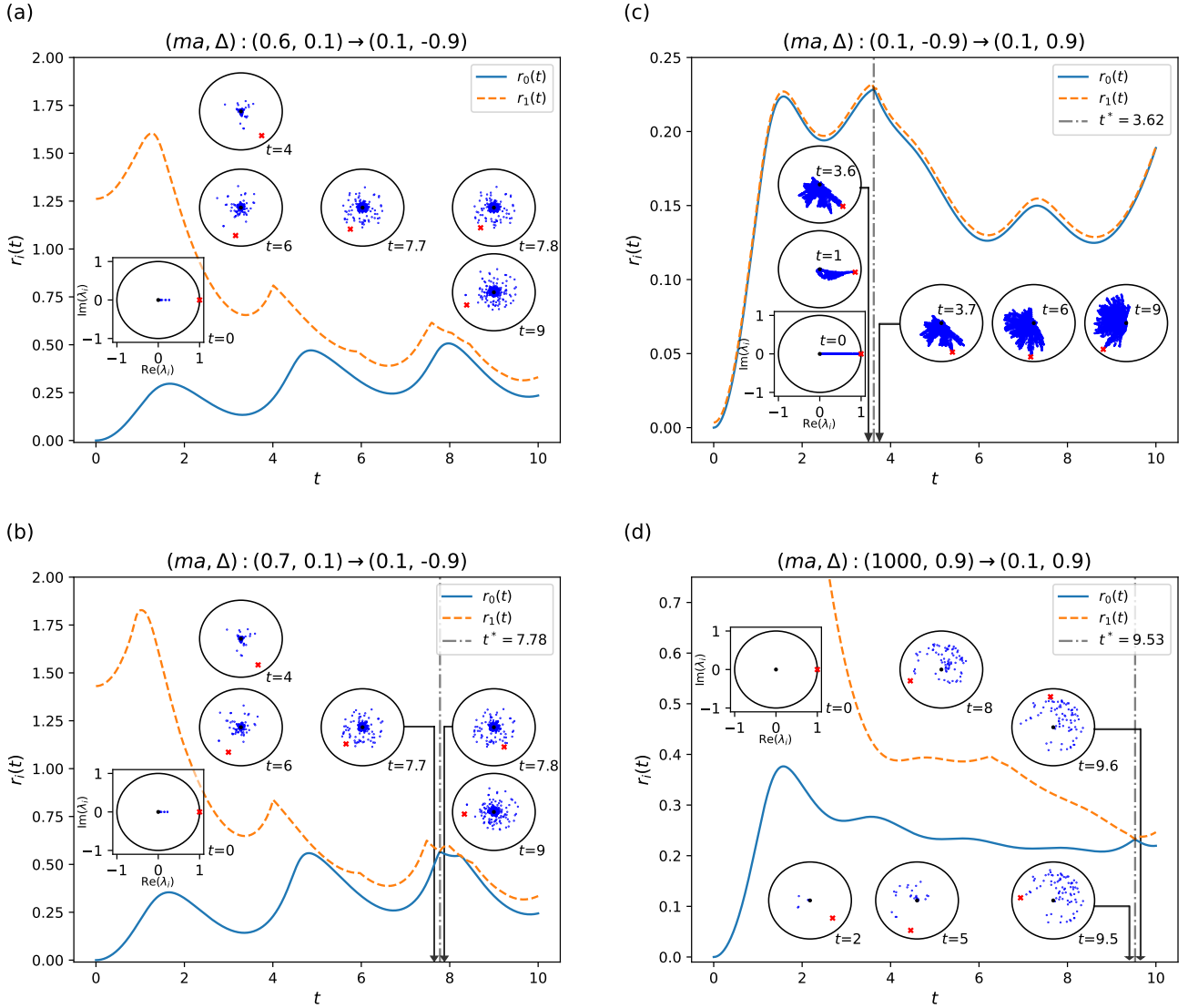


FIG. 2. The return rate function and the full spectrum of the mixed transfer matrix $T(t)$. The quantity r_i is defined in Eq. (15), while r_0 corresponds to the return rate function defined in Eq. (11). The insets display the full spectrum of the mixed transfer matrix $T(t)$ defined in Eq. (14), with the red cross representing the corresponding dominant eigenvalue. The time t^* (indicated by the grey dash-dotted line) denotes when the first DQPT occurs. This plot shows how the non-analyticities in the return rate $r_0(t)$ correspond to discontinuous jumps in the phase of the dominant eigenvalue of $T(t)$ (red point). We can notice that in (a), with $(m_1 a, \Delta_1) = (0.6, 0.1)$ and $(m_2 a, \Delta_2) = (0.1, -0.9)$, there are no DQPTs. However, in (b), a small increase in $m_1 a$ to 0.7, with the same quench Hamiltonian, induces a DQPT at $t^* = 7.78$.

The transfer matrix formalism allows for a systematic exploration of the character of DQPTs [17, 25–27, 29]. We focus here on the case where the initial state is a MPS, and consider directly the translationally invariant case, i.e., the initial state is parametrized by a single rank-3 tensor A . The Loschmidt amplitude, for a translationally invariant system of N sites, can then be expressed as

$$G(t) = \langle l | T^N(t) | r \rangle = \lambda_0^N, \quad (12)$$

where $T(t)$ is the mixed transfer matrix defined by

$$T(t) := \sum_i \bar{A}(0)^\sigma \otimes A(t)^\sigma, \quad (13)$$

and $A(t)$ represents the tensor describing the evolved state. The vectors $|l\rangle$ and $|r\rangle$ are the left and right eigenvectors with

the leading eigenvalue λ_0 , satisfying $(l|r) = 1$. Graphically,

$$T(t) = \begin{array}{c} \text{---} \boxed{\bar{A}(0)} \text{---} \\ | \\ \text{---} \boxed{A(t)} \text{---} \end{array} \quad (14)$$

The Loschmidt rate is thus determined by the spectrum of the (in general non-Hermitian) matrix $T(t)$ and, in the thermodynamic limit, by its dominant eigenvalue $\lambda_0(t)$. DQPTs appear when there is an eigenvalue crossing and the eigenvector associated to the dominant eigenvalue changes [17, 70].

B. Investigation of DQPT in the lattice version of the Thirring model

Here we focus on the Hamiltonian Eq. (2) and consider quenches from the ground state $|\Psi_1\rangle$ of the model with a given set of parameters, $H_1 = H(m_1a, \Delta_1)$. The quench Hamiltonian, with which the state is evolved, will be denoted as $H_2 = H(m_2a, \Delta_2)$. For a fixed H_2 , we study the real-time evolution of different initial states, $|\Psi_1\rangle$. The latter are represented by their MPS approximation, obtained using the variational uMPS (VUMPS) algorithm [57], and the real time evolution according to H_2 can be numerically simulated using a standard tensor network algorithm [7, 11, 71]. In particular, we use time-dependent variational principle (TDVP) algorithm [21, 72] to obtain the time-dependent tensors of the MPS representation, $A(t)$. We obtain the spectrum of the mixed transfer matrix, $T(t)$, using Eq. (13) and explore the appearance of DQPTs for various quench parameters. To investigate how DQPTs occur, we introduce the following quantity,

$$r_i(t) = -\log |\lambda_i(t)|^2, \quad (15)$$

where $\{\lambda_i\}$ is the spectrum of the mixed transfer matrix $T(t)$, satisfying $|\lambda_0| \geq |\lambda_1| \geq \dots$. Notice that r_0 is exactly return rate function defined in Eq. (11). Fig. 2 shows the results for a selection of simulation parameters. The first DQPT occurs when $t = t^*$. The insets show snapshots of the spectrum of $T(t)$, which demonstrate that the non-analytical points in the return rate function in Eq. (11) are associated with a discontinuous jump in the phase of the dominant eigenvalue of this mixed transfer matrix [70].

By choosing parameters, (m_1a, Δ_1) and (m_2a, Δ_2) , to be in a different or in the same equilibrium phase, we can simulate quenches across the phase diagram, which we determined in our previous study [41]. We find that, as observed in other models [17, 33], the equilibrium phases of H_1 and H_2 does not determine the presence of DQPTs, which can appear in both types of scenario. In Fig. 1, we illustrate that the existence of DQPTs is not equivalent to the equilibrium phase.

DQPTs have been demonstrated to occur in quenches that happen entirely within either of the equilibrium phases.

V. DQPTS AND FINITE TEMPERATURE PHASE DIAGRAM

A. Energy threshold and the inverse temperature

As discussed in the previous section, the appearance of a DQPT after a certain quench is not determined by the fact that the quench crossed an equilibrium phase boundary. Hence the ground state of H_2 is not the most adequate equilibrium reference for all initial states. Since the mean energy of the initial state $E_1 := \langle \Psi_1 | H_2 | \Psi_1 \rangle$ is typically much higher than the ground state energy of the quench Hamiltonian H_2 , one does not necessarily expect the time evolved state to be close to the ground state of H_2 , even at the level of local observables. Instead, it is more meaningful to compare the local properties to those of the thermal equilibrium state at the same energy density. If the system equilibrates (in the sense of locally resembling an equilibrium ensemble), the thermal state should correspond to the maximal entropy state compatible with the conserved quantities. Our particular model conserves energy and total magnetization, and we are considering initial states in which the latter vanishes. Thus, the relevant equilibrium state is the Gibbs state with the same energy E_1 , restricted to the subspace of vanishing total magnetization; namely (and up to normalization) $\rho(\beta) \propto P_{s_z=0} e^{-\beta H_2} P_{s_z=0}$, where $P_s = \sum_{n, S_z |n\rangle = s |n\rangle} |n\rangle \langle n|$ is the projector onto the sector of total magnetization s . The inverse temperature $\beta_1 := \beta(E_1)$ corresponding to the state is fixed by the condition

$$E_1 = E(\beta_1) := \frac{\text{tr}(\rho_{\beta_1} H_2)}{\text{tr}(\rho_{\beta_1})}. \quad (16)$$

The determination of β_1 can be achieved using standard TNS methods, exploiting the fact that the thermal equilibrium state for a local Hamiltonian can be approximated efficiently by a matrix product operator (MPO) [73, 74]. More concretely, given a value of the inverse temperature β , the canonical purification of the thermal equilibrium ensemble restricted to the vanishing magnetization sector can be written as $|\Psi(\beta)\rangle := e^{-\beta H_2/2} \sum_{n, S_z |n\rangle = 0} |n\rangle_S |n\rangle_A$, where the sum runs over a basis restricted to the sector $S_z = 0$, and the subscripts, S and A, refer respectively to the system and ancillary degrees of freedom. This state, also known as thermofield double state, can be approximated by a MPS using standard tensor network algorithms [6, 7]. This is achieved by preparing a maximally-mixed state of the system and the ancilla, expressed as a simple MPS, where each system site is entangled only with one, next neighbor, ancillary site, and evolving it in imaginary time with an approximation of the operator $e^{-\beta H_2/2}$. In order to restrict the ensemble to a sector of fixed magnetization, we

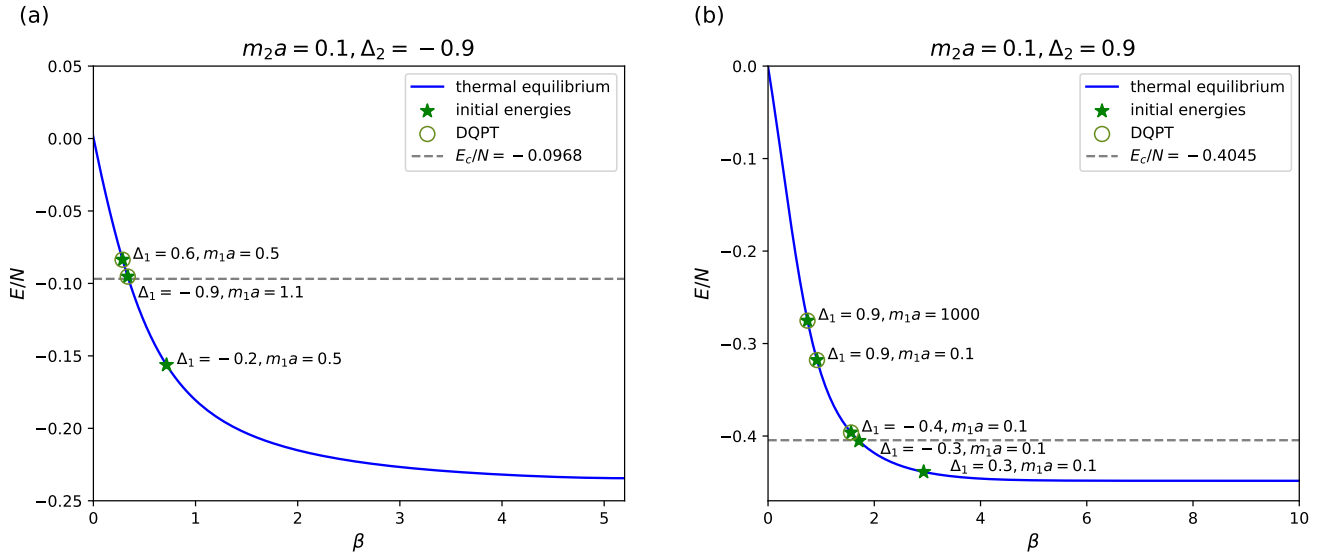


FIG. 3. Relation $E(\beta)$ between the mean energy and the inverse temperature in the Gibbs state restricted to the zero magnetization sector for $(m_2a, \Delta_2)=(0.1, -0.9)$ (left) and $(0.1, 0.9)$ (right). We display the energy densities of several explored quenches using star symbols, and circles indicate the initial states where our simulations reveal the occurrence of at least one DQPT. Additionally, the dashed line denotes the lowest value of energy density (E_c/N) at which a DQPT was observed (see also Fig. 4). For $(m_2a, \Delta_2)=(0.1, -0.9)$ (left), $E_c/N = -0.0968$ and the corresponding inverse temperature $\beta_c \approx 0.34$; for $(m_2a, \Delta_2)=(0.1, 0.9)$ (right), $E_c/N = -0.4045$ and $\beta_c \approx 1.70$.

write the projector onto the sector $S_z = 0$ as a MPO, and use as initial mixed state, instead of the identity. Additionally, this MPO can also be applied to the intermediate state during the evolution to ensure that truncation errors due to the approximations do not induce undesired magnetization components. For a finite system of size N , the projector onto vanishing magnetization has an exact MPO representation, with bond dimension $N/2$. Applying the aforementioned TNS evolution algorithm in imaginary time to the vectorized projector provides a MPS approximation to $|\Psi(\beta)\rangle$ for varying β , from which the thermal expectation value of a physical observable O can be computed directly as

$$\langle O \rangle_\beta := \frac{\text{tr}(O\rho_\beta)}{\text{tr}\rho_\beta} = \frac{\langle \Psi(\beta) | O \otimes \mathbb{I} | \Psi(\beta) \rangle}{\langle \Psi(\beta) | \Psi(\beta) \rangle}, \quad (17)$$

where the second factor in the tensor product is the identity acting on the ancillary system. In particular, computing the energy along the imaginary time evolution provides an estimate of the function $E(\beta)$, from which we can extract the value β_1 corresponding to a given initial state by inverting it numerically.

For concreteness, in the rest of the section we focus on two quenches, corresponding to Hamiltonians in each of the phases (see Fig. 1). Here, we choose $(m_2a, \Delta_2) = (0.1, -0.9)$ in the gapless phase, and $(m_2a, \Delta_2) = (0.1, 0.9)$ in the gapped one. For each of them we approximate the thermal equilibrium state in a system of $N = 300$ sites, sufficiently large to neglect finite size effects in this analysis. The corresponding $E(\beta)$ curves are shown in Fig. 3.

For each of the simulated initial states, we can now extract the effective temperature from these curves. The ground states of the different initial Hamiltonians H_1 correspond to initial states with different effective temperatures. Since the latter only depends on the energy E_1 , states in very distant regions of the equilibrium phase diagram may have the same temperature. As an illustration, we show the energy densities of some of the explored quenches with star symbols in Fig. 3. We furthermore indicate with circles the initial states for which our simulations show the appearance of (at least one) DQPT. From these results we observe that for DQPTs to be present, the effective temperature of the initial state needs to surpass a certain threshold. This seems to suggest that a decisive factor for the occurrence or not of a DQPT is the amount of excitation energy in the initial state.

To verify this observation, we estimated the value of E_1 for a grid of (m_1a, Δ_1) values, $0 < m_1a \leq 1.5$ and $-1 \leq \Delta_1 \leq 1$, and identified the regions of parameters for which the threshold (denoted as E_c) is surpassed (see Fig. 4). We conducted this study for both quench Hamiltonians mentioned above. For the gapless Hamiltonian, $(m_2a, \Delta_2) = (0.1, -0.9)$, the value of E_1 increases smoothly with both m_1a and Δ_1 , and the plane of constant $E_c/N \approx -0.0968$, corresponding to the lowest value for which a DQPT was observed. As indicated in Fig. 4a, this divides the parameter space in two connected regions. Consistent with this, we find DQPTs for all initial states sampled from the region with higher energy.

In contrast, for the quench into the gapped phase, $(m_2a, \Delta_2) = (0.1, 0.9)$, the dependence of E_1 on (m_1a, Δ_1)

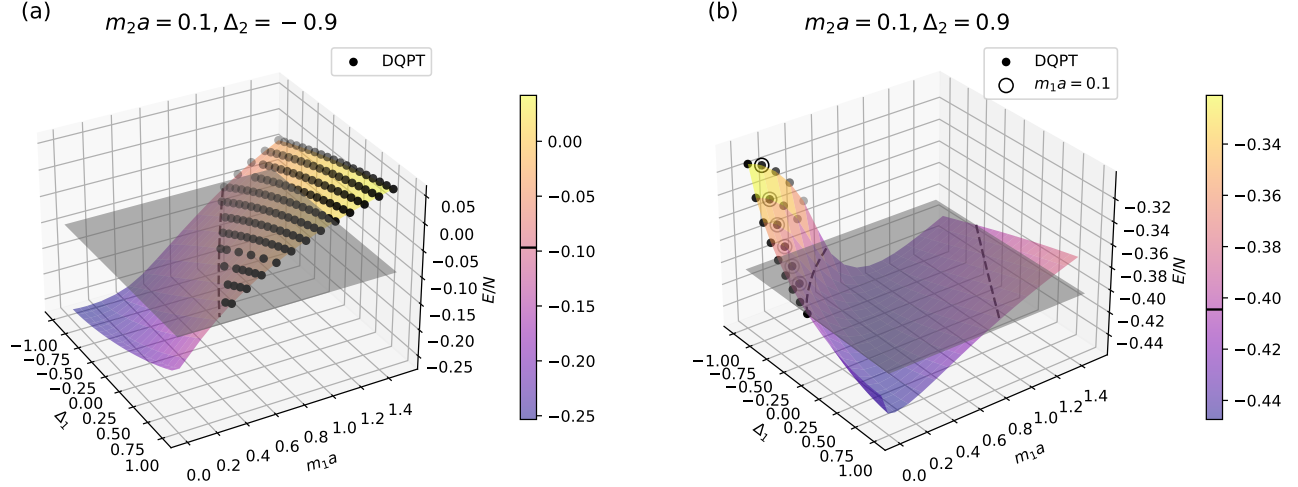


FIG. 4. Energy densities corresponding to ground states of varying m_1a and Δ_1 , as a function of these parameters, for the quench to ($m_2a = 0.1$, $\Delta_2 = -0.9$) (left) and ($m_2a = 0.1$, $\Delta_2 = 0.9$) (right). In each case, the dots indicate where DQPT has been observed. The horizontal plane shows the energy threshold, E_c/N , which is also marked as a black line on the color bars. On the right panel, the circles indicate the family of quenches from constant $m_1a = 0.1$.

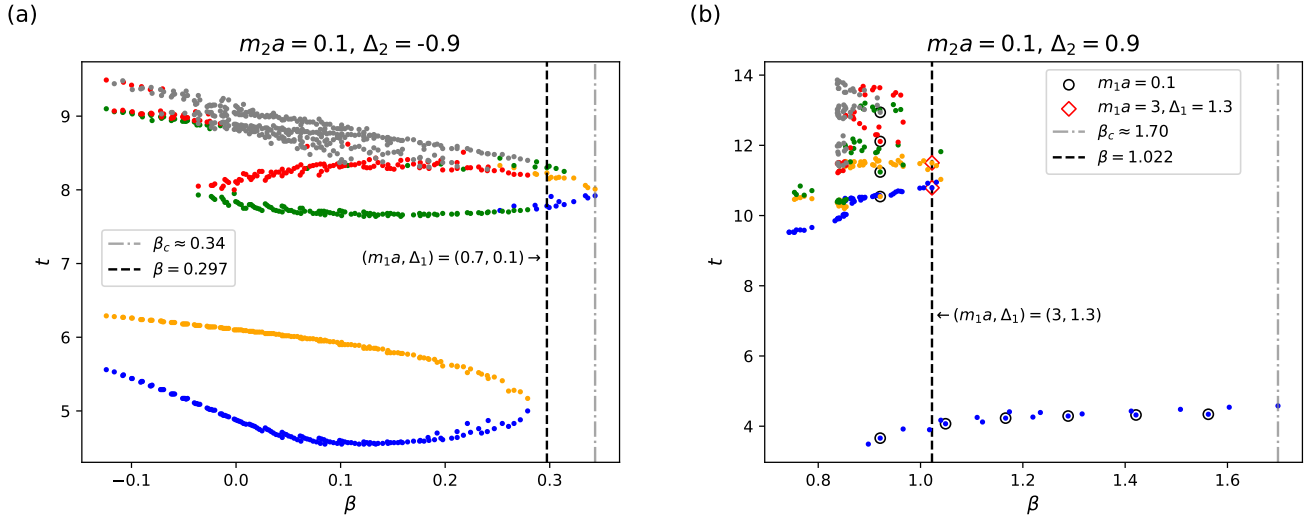


FIG. 5. The DQPTs on the inverse temperature-time plane. The first, second, third, and fourth occurrences of DQPTs are colored blue, orange, green, and red, respectively. Grey indicates occurrences beyond the fourth. The grey dash-dotted line denotes the critical inverse temperature β_c . (a) We simulate the evolution parameters $m_2a = 0.1$ and $\Delta_2 = -0.9$ and perform a scan for the initial parameters ($m_1 = 0.1, 0.2, \dots, 1.5$; $\Delta_1 = -0.9, -0.8, \dots, 0.9$). For example, the dashed line is the case for $m_1a = 0.7$ and $\Delta_1 = 0.1$, which is also specifically shown in Fig. 2b. (b) The simulation results for the evolution parameters $m_2a = 0.1$ and $\Delta_2 = 0.9$, scanning for the initial parameters as follows: ($m_1 = 0$; $\Delta_1 = -0.9, -0.8, \dots, -0.1, 0.9$), ($m_1 = 0.1, 0.2, \dots, 1.5$; $\Delta_1 = -0.9, -0.8, \dots, 0.9$), ($m_1 = 1, 2, \dots, 9$; $\Delta_1 = 1, 1.1, 1.3, 1.5, \dots, 1.9$) and ($m_1 = 10, 20, 30, 40, 60, 80, 100, 1000$; $\Delta_1 = 0.9$). The black dashed line and the red diamond symbols represent the case for $m_1a = 1.3$ and $\Delta_1 = 3$, where DQPT does not occur at the lowest branch near $t \approx 4$.

is not monotonic. We find two disjoint regions of parameters with energies above the identified threshold $E_c/N \approx -0.4045$, lying on opposite corners of the parameter space as can be seen in Fig. 4b. One region with negative Δ_1 and small m_1a (overlapping but not coinciding with the critical

phase, as shown in Fig. 1), and the other region with positive large Δ_1 and m_1a , deep within the gapped regime. We simulate quenches from both regions, as well as intermediate parameters with energy E_1 below the threshold. For the first high-energy region, i.e. negative Δ_1 and small m_1a , we

consistently observe DQPTs for energies above the threshold. The second region, with large positive Δ_1 and m_1a , however, seems to exhibit DQPTs only at much later times than the first. An extreme case is shown in Fig. 2d, for the quench from $m_1a = 1000$, $\Delta_1 = 0.9$. We observe that as m_1a values decrease, the time at which the first DQPT is observed increases. Due to limitations imposed by truncation errors, we can reliably explore times up to $t \lesssim 14$. Within this window, we only observe DQPTs for points in the parameter regime with masses outside our original grid. We presume that longer times would identify DQPTs also in the rest of the region above the threshold.

This exploration of the parameter space indicates that for states with similar effective temperature, but corresponding to very different values of (m_1a, Δ_1) , the DQPTs may occur in very different ways. In order to obtain a more complete picture, we plot in Fig. 5, the times at which we observe DQPTs as a function of the effective temperature of the initial state for all the simulated cases. For each of the initial states, we determine β from the energy expectation value $\langle \Psi_1 | H_2 | \Psi_1 \rangle$ through Eq. (16). The figure shows not only the first observed non-analyticity (blue dots), but the times of all subsequently observed DQPTs (with colors indicating the number of the DQPT). These figures are reminiscent of the non-analyticities shown by the complex Loschmidt rate, when defined on the complex t plane for a fixed initial state [3]. In such case, the non-analyticities of the rate in the complex-time plane are analogous to Fisher zeros of a boundary partition function, and determine the presence and time of occurrence of DQPTs [50, 51]. In our case, the initial state varies and we do not evolve it in imaginary time, but the results suggest that our effective temperature does play an equivalent role, allowing us to explore such complex plane from purely real-time data. However, we also observe that some initial states are not sensitive to some of the DQPT *branches*. For instance, if we select $(m_1a, \Delta_1) = (3, 1.3)$ and quenches to $(m_2a, \Delta_2) = (0.1, 0.9)$ (indicated by the black dashed line and red diamond marks in Fig. 5b), the DQPT does not occur at the lowest branch near $t \approx 4$. The generality of this connection for other models thus needs further investigation.

B. String correlator and DQPT

Some DQPTs have been connected to equilibrium phase diagrams [2, 3, 24, 75]. In Ref. [41], we identified the ground state phase structure by analyzing the behavior of the string correlator (8) (see Sec. III). Namely, the zero-temperature gapless phase is characterized by the string correlator decaying to zero at large distances ($C = 0$), while the gapped phase evinces a decay of this correlator to a non-zero value ($C > 0$). In the present work, we use similar methodology to connect the occurrence of DQPT to the thermal string correlator, pertaining to the effective temperature of the initial state. As in

Ref. [41], we fit the distance-dependent string correlators with the power-exponential ansatz given by Eq. (9). To establish the connection to DQPT, we note that, keeping the mass fixed, $m_1a = m_2a = 0.1$, and varying Δ_1 , each value of the coupling corresponds to a different effective β , as illustrated in Fig. 3a (for $\Delta_2 = -0.9$) and Fig. 3b (for $\Delta_2 = 0.9$).

We start the discussion with the latter, $\Delta_2 = 0.9$. We observe the following correspondence (see also Fig. 6):

- for $\beta \leq 1.563$ ($\Delta_1 \leq -0.4$), the constant C is consistent with zero (at $\beta = 1.563$, $C = 0.000004(4)$) and DQPT is observed,
- for $\beta \geq 1.712$ ($\Delta_1 \geq -0.3$), a non-zero constant emerges (at $\beta = 1.712$, $C = 0.000046(10)$) and we find no DQPT.

The observed correspondence pertains to DQPTs pictured in the lowest branch of Fig. 5b, explicitly marked with circles when $m_1a = m_2a$. Thus, this kind of DQPT is related to a substantial change in the properties of the thermal state, characterized by a different value of C . The former/latter may be the thermal counterpart of the zero-temperature critical/gapped phase, but a detailed characterization of finite temperature phases of the model is beyond the scope of our work.

The picture is somewhat different for $\Delta_2 = -0.9$. In this case, $C = 0$ at all temperatures, regardless of the occurrence of DQPT, found at $\beta \lesssim 0.34$ (see Fig. 5a). Note, however, that none of the DQPTs represented in Fig. 5a occurs at $m_1a = m_2a$, i.e. a prerequisite for DQPT at $\Delta_2 = -0.9$ is to have a quench in fermion mass. Thus, the analyzed thermal string correlator is not related in any way to the presence

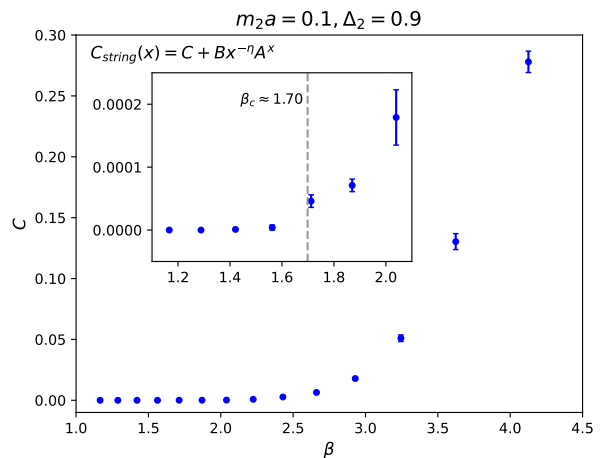


FIG. 6. Value of the constant C , defined in Eq. (9), that characterizes the long distance behavior of the string correlator (8), for the thermal equilibrium state of the Hamiltonian $(m_2a, \Delta_2) = (0.1, 0.9)$ as a function of the inverse temperature β . The grey dashed line in the inset denotes the critical inverse temperature β_c .

of a DQPT for the final Hamiltonian $(m_2a, \Delta_2)=(0.1,-0.9)$. However, it cannot be excluded that this DQPT is related to a phase transition between other types of phases, characterized by a different observable. We leave a more detailed investigation of finite-temperature phases and their potential relation to DQPTs for future work.

Ultimately, we conclude that there is no direct correspondence between the presence of a DQPT and the phase structure probed by the decay of the string correlator. It is plausible that the DQPTs observed for $m_2a = 0.1$ and $\Delta_2 = -0.9$ or $\Delta_2 = 0.9$ are of a different type, with only the latter related to the large-distance properties described by the string correlator.

VI. CONCLUSION

Using standard TNS methods, we have investigated the occurrence of DQPTs in the lattice discretization of the Thirring model, which corresponds to a spin-1/2 XXZ chain coupled to a uniform and a staggered magnetic field, with two free dimensionless parameters, respectively related to the coupling and the mass. Using the uniform MPS ansatz to work directly in the thermodynamic limit, we have simulated quenches where the evolution is given by a constant Hamiltonian and the initial state is chosen as the ground state for different parameters, in the same or a different phase. We then studied the time evolution of the spectrum of the mixed transfer operator, whose dominant eigenvector directly determines the Loschmidt rate. DQPTs are found when another eigenvalue crosses with the largest one.

We have identified a threshold in the energy density of the initial state below which DQPTs do not occur, at least during the finite times reachable by our simulations. The threshold is observed both for quenches in the gapless and the gapped phases. The energy density of each initial state corresponds to an effective temperature: that of the thermal equilibrium state with the same energy per spin, restricted to the sector of zero magnetization. This thermal state is the one expected to describe the equilibrium values of local observables after the quench in the long time limit, since energy and magnetization are the only local conserved quantities in the problem.

We have analyzed the position of each observed DQPT (or non-analyticity of the Loschmidt rate) in the time-effective-temperature plane. This reveals structures of non-analyticities reminiscent of the zeros in the complex-time plane of the Loschmidt rate defined for a fixed initial state. Our results show that DQPTs for different initial states lie in multiple well-defined branches. Initial states in different phases can appear on the same branch, which thus is not related to the zero-temperature phase diagram. All DQPT branches start at some minimal effective temperature. In most cases, the lowest temperature edge of a branch corresponds to a smooth merge

with another one. However, in the case of the quench with the gapped Hamiltonian, we identify a particular branch, with DQPTs happening at the shortest times, with a sharp end at $\beta_c \sim 1.7$. We have found that this energy threshold corresponds very accurately with a substantial change in the properties of the thermal equilibrium states at the corresponding effective temperature. More concretely, below this temperature, a certain string correlator attains a non-zero value at long distances, a property that also characterizes the ground state in the gapped phase. For this branch, we have thus established a relation between the presence of DQPTs and the equilibrium phase diagram at finite T. We have not found a similar correspondence for the smooth thresholds of branches that merge into each other.

Our study thus suggests that observing the dynamics and analyzing the DQPTs from different initial states can probe properties of the finite temperature phase diagram, in particular when a sharp threshold is found for the occurrence of these non-analyticities. Since our simulations are limited to finite time windows, due to the accumulation of truncation error, and we have not mapped out the complete thermal equilibrium properties at all temperatures, further investigation is needed to determine whether the other observed branches correspond also to some quantifiable equilibrium property. It is also interesting to investigate whether similar effects appear in other spin models, and the precise correspondence between Fisher zeros and the DQPT lines in our time-effective temperature representation.

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Appendix A: Penalty term and Matrix Product Operator

Since the z -component of the total spin corresponds to the total fermion number in the Thirring model, and we are only interested in the zero-charge sector. To enforce this, we intro-

duce a penalty term in the Hamiltonian,

$$\bar{H}^{\text{penalty}} = H + \lambda \left(\sum_{n=0}^{N-1} S_n^z \right)^2. \quad (\text{A1})$$

Here, λ should be sufficiently large (100 in this work) to ensure that the ground state obtained *via* a variational search is in the sector of vanishing total S^z [76], that is $\langle S_{tot}^z \rangle = \langle \sum_n S_n^z \rangle = 0$. This enables us to interpret our results in terms of the dual SG theory and the XY model. However, the penalty term will not be turned on during the real time evolution simulations since $\langle S_{tot}^z \rangle$ is a conserved quantity. In our simulations, this quantity is well-preserved, i.e., $\langle S_{tot}^z(t) \rangle = 0$.

In TN algorithms, it is useful to express the Hamiltonian as the matrix product operator (MPO). The Hamiltonian of Thirring model in Eq. (A1) can be expressed as an MPO:

$$W^{[n]} = \begin{pmatrix} \mathbb{I} & -\frac{1}{2}S^+ & -\frac{1}{2}S^- & 2\lambda S^z & \Delta S^z & \beta_n S^z + \alpha \mathbb{I} \\ 0 & 0 & 0 & 0 & 0 & S^- \\ 0 & 0 & 0 & 0 & 0 & S^+ \\ 0 & 0 & 0 & \mathbb{I} & 0 & S^z \\ 0 & 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I} \end{pmatrix}. \quad (\text{A2})$$

where $\beta_n = \Delta + (-1)^n \tilde{m}_0 a$, $\alpha = \frac{1}{4}(\lambda + \Delta)$. The β_n contains the staggered term. To simplify the simulation, we also combine two-site MPOs into one-site, as we did with the uMPS, represented as $W_1 W_2 \rightarrow W$.

Appendix B: Brief introduction of VUMPS and TDVP

To prepare the ground state in uMPS form, we use the VUMPS algorithm [57]. In VUMPS, we need to construct the effective Hamiltonian through the MPO and the boundary tensors L, R , which satisfy the fixed-point equations (See Fig. 7).

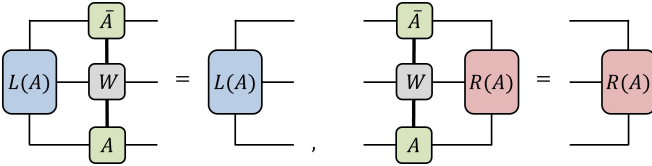
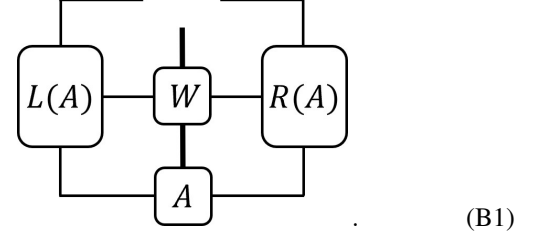


FIG. 7. The fixed-point equations of the boundary tensors L and R .

The tensor diagram of the effective Hamiltonian operating on

the unit cell can be expressed as:



The boundary tensors L and R are functions of A . We could use the Lanczos algorithm to obtain the optimal A and then update L and R until it reach the ground state. In our simulation, we turned on the penalty term λ in Eq. (A2) to guarantee that the ground state we obtained is in the sector of vanishing total $\langle S_z \rangle$.

For real-time evolution, we choose the TDVP algorithm, which preserves conserved quantities well [72]. TDVP corresponds to a projection of the evolution state of the Schrödinger equation onto the tangent space of the uMPS manifold \mathcal{M} at the current state (see Fig. 8,). We can solve the time derivative of the uMPS $|\Psi(A)\rangle$ through the TDVP equation:

$$\frac{\partial}{\partial t} |\Psi(A)\rangle = -i \hat{P}_{T_{|\Psi(A)\rangle} \mathcal{M}} \hat{H} |\Psi(A)\rangle. \quad (\text{B2})$$

The left-hand side of Eq. (B2) gives the time derivative of the tensor A , while the right-hand side gives the projection result: $-iB$, where B is also a tensor that has the same shape as tensor A . Thus, we will get $\dot{A} = -iB$, which can be solved using the Runge-Kutta (RK4) method. Note that for real-time evolution, we turn off the penalty term λ in Eq. (A2) (setting $\lambda = 0$).

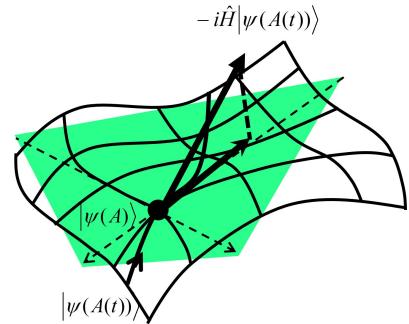


FIG. 8. An illustration of the TDVP. The curve is the manifold of the uMPS and the colored plane is the tangent space of the current state.

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