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Rank-expanding satellites, Whitehead doubles, and Heegaard Floer homology

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Abstract

We show that a large class of satellite operators are rank-expanding; that is, they map some rank-one subgroup of the concordance group onto an infinite linearly independent set. Our work constitutes the first systematic study of this property in the literature and partially affirms a conjecture of the second author and Pinzón-Caicedo. More generally, we establish a Floer-theoretic condition for a family of companion knots to have infinite-rank image under satellites from this class. The methods we use are amenable to patterns that act trivially in topological concordance and are capable of handling a surprisingly wide variety of companions. For instance, we give an infinite linearly independent family of Whitehead doubles whose companion knots all have negative τ-invariant. Our results also recover and extend several theorems in this area established using instanton Floer homology.

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1 | INTRODUCTION

For any pattern knot $P \subset S^1 \times D^2$, the satellite operation $K \mapsto P(K)$ induces a map

$$P : \mathcal{C} \to \mathcal{C}$$

on the smooth (or topological) knot concordance group. These operators have been central to the study of the concordance groups in both categories; for example, see [2, 4–9, 11, 12, 15, 17, 24–26, 30, 32, 35–40, 43, 45, 51]. In this article, we investigate several questions regarding the rank of different satellite operators on the smooth concordance group. The starting point for this line of research is the following conjecture, due to the second author and Pinzón-Caicedo [32]:

Conjecture 1.1 [32, Conjecture 2]. *Every nonconstant satellite operator has infinite rank.*

Here, by the rank of P we mean the rank of the subgroup generated by the image of P, as in general P is not a homomorphism. Significant progress toward Conjecture 1.1 was made in [32, Theorem 3], where it was verified for all winding number zero patterns satisfying a certain rational linking number condition.[†] Specifically, it was shown that any such pattern maps a carefully selected sequence of torus knots to an infinite linearly independent set. Other research has focused on establishing the linear independence of explicit families of knots under patterns such as Whitehead doubling. For instance, in joint work with Kirk, the second author proved the Whitehead doubles $\{D(T_{2,2^k-1})\}_{k\geqslant 2}$ are linearly independent [24, Theorem 1]; this was extended to the entire family $\{D(T_{2,2k+1})\}_{k\in\mathbb{N}}$ by Nozaki, Sato, and Taniguchi [45, Corollary 1.13]. (See also [45, Theorem 1.12].) Linear independence of torus knots under other (Whitehead-like) satellites was studied by Pinzón–Caicedo in [51].

Given the linear independence of torus knots in $\mathcal{C}$, the above results should be thought of as examples of rank-preserving behavior for P. The existence of more exotic behavior was conjectured in [32], where the following strengthening of Conjecture 1.1 was presented:

Conjecture 1.2 [32, Conjecture 4]. *If P is a nonconstant winding number zero satellite operator, then there exists a knot K for which $\{P(nK)\}_{n\in\mathbb{Z}}$ has infinite rank.*

By the rank of $\{P(nK)\}_{n\in\mathbb{Z}}$, we again mean the rank of the subgroup generated by $\{P(nK)\}_{n\in\mathbb{Z}}$. Conjecture 1.2 states that any nontrivial satellite operator sends some rank-one subgroup of $\mathcal{C}$ surjectively onto an infinite linearly independent set. We formalize this in the following definition:

Definition 1.3. A satellite operator P is *rank-expanding* if there exists a rank-one subgroup $\{nK\}_{n\in\mathbb{Z}}$ of $\mathcal{C}$ such that $\{P(nK)\}_{n\in\mathbb{Z}}$ has infinite rank. When we wish to emphasize the knot K, we say that P is rank-expanding *along* $\{nK\}_{n\in\mathbb{Z}}$ (or sometimes just *along* K).[‡]

[†] The proof in the case of nonzero winding number is straightforward and follows from a consideration of Tristram–Levine signatures; see [32, Proposition 8].

[‡] Note that implicitly, K is required to be nontorsion in $\mathcal{C}$. One can also define rank expansion by requiring that there is some finite-rank subgroup whose image under P generates a subgroup of greater (but still possibly finite) rank; here, we have instead chosen the strongest possible notion. The authors briefly considered calling the operators of Definition 1.3 rank-*exploding*.

Note that Conjecture 1.2 is false for patterns with nonzero winding number. For example, we can take P such that $P(K)$ is the connected sum of K with the figure-eight knot; this pattern has winding number one. It it clear that $\{P(nK)\}_{n\in\mathbb{Z}}$ has rank one for any nontorsion knot K.

Prior to the current article, little was known about Conjecture 1.2, even in specific cases. Indeed, in [32] it was asked whether $\{P(nK)\}_{n\in\mathbb{N}}$ is linearly independent for P the Whitehead double and K the trefoil. We show:

Corollary 1.4. *Let $\mathcal{F}$ be any subset of $\{D(nT_{2,2k+1})\}_{n,k\in\mathbb{N}}$ whose index pairs have distinct products nk. Then $\mathcal{F}$ is linearly independent and in fact spans a $\mathbb{Z}^\infty$-summand of $\mathcal{C}$.*

Setting $k = 1$ and varying n yields the family $\mathcal{F} = \{nD(T_{2,3})\}_{n\in\mathbb{N}}$. This answers the above question in the affirmative and (in particular) shows that D is rank-expanding along $T_{2,3}$. Setting $n = 1$ and varying k yields the family $\mathcal{F} = \{D(T_{2,2k+1})\}_{k\in\mathbb{N}}$, which recovers [45, Corollary 1.13] (and thus [24, Theorem 1]). Corollary 1.4 is a consequence of a much more general result and can be extended to all multiply-clasped and twisted Whitehead doubling operators; see Theorem 1.10.

In fact, we verify Conjecture 1.2 for many other patterns and families of companions in Theorem 1.7. The prevailing belief seems to be that nontrivial satellite operators are never homomorphisms (for example, see [4, 40, 43]). Our results indicate that they are, in some quantifiable sense, maximally far from being homomorphisms. Indeed, a potentially reasonable strengthening of Conjecture 1.2 would be the following:

Conjecture 1.5. *Any nonconstant winding number zero satellite operator is rank-expanding along* every *rank-one subgroup $\{nK\}_{n\in\mathbb{Z}}$.*

It is thus natural to establish robust conditions that affirm rank expansion along different K.

The difficulty with studying the rank of satellite operators, especially for patterns such as Whitehead doubling, lies principally with a lack of effective invariants. For example, as discussed in [32, section 1], the knot Floer homology of Whitehead doubles is sufficiently constrained so that the usual suite of Floer-theoretic concordance invariants (such as τ, Υ, stable equivalence, and so on) cannot be used to establish linear independence. The most common technique to date has been to pass to the branched double covers of these knots and utilize homology cobordism invariants of the latter manifolds. In the case that these manifolds have nontrivial first homology (when the determinant of the satellite knots is not one), there are a host of Frøshov-type invariants coming out of Floer theories, or analogous Casson–Gordon signatures available in the topological category. Analyzing these invariants in conjunction with metabolizers for linking forms yields a powerful tool for studying satellite operators, and can be used to show that certain operators whose image consists of satellite knots with nonzero determinant have infinite rank, and are even rank expanding. (For example, Chuck Livingston pointed out to the authors that Casson-Gordon invariants can verify that certain *twisted* Whitehead doubles are rank-expanding. See [3, 5, 25, 26] for related results using d-invariants.)

However, when the determinant of the satellite knots is one, the branched double covers are homology spheres, and such techniques break down. To date, the only method for bypassing this has been to employ the filtration on instanton Floer homology provided by the Chern–Simons functional. As instanton Floer homology is only well-understood for a small subset of 3-manifolds, this approach has only been used to study the images of very restricted families of companion knots, such as those closely related to torus knots [12, 24, 32, 45, 51] or certain twist knots [45] (see [15] for very recent results in this direction). In particular, although the instanton approach is well-suited to understanding $\{P(K_n)\}_{n\in\mathbb{N}}$ for $\{K_n\}_{n\in\mathbb{N}}$ a family of distinct torus knots, it is not apparent how to extend this to self-connected sums of a single torus knot, with regard to Conjecture 1.2.

In this article, we use recent advances in involutive Heegaard Floer homology [14, 23, 27] to verify Conjecture 1.2 for all proper rational unknotting number one patterns satisfying a certain nonzero linking number condition; see Theorem 1.7. (This class includes all multiply-clasped and twisted Whitehead doubles.) More broadly, for such patterns we establish a general condition on a family of companion knots $\{K_n\}_{n\in\mathbb{N}}$ that guarantees that $\{P(K_n)\}_{n\in\mathbb{N}}$ has infinite rank; see Theorem 1.9. Applying this to self-connected sums of a given knot K allows us to verify Conjecture 1.2 in the cases at hand. In fact, we show that for our examples, K may be chosen to be topologically slice, so that the rank-expanding behavior of Conjecture 1.2 persists even after restricting P to $\mathcal{C}_{TS}$.

Even in the well-studied case of Whitehead doubles, our formalism can handle many new families of companion knots. In addition to the linear independence of Whitehead doubles of the form $\{D(nK)\}_{n\in\mathbb{N}}$, we give the first example of an infinite linearly independent family of (positively clasped) Whitehead doubles whose companion knots all have $\tau(K) \leqslant 0$. By work of the second author [20, Theorem 1.7] combined with that of Hom [31] and Sato [54, Theorem 1.2], if $\tau(K) \leqslant 0$ then the stable equivalence class of the knot Floer homology of $D(K)$ is trivial. This means that the (noninvolutive) knot Floer invariants of $D(K)$ contain no interesting concordance information; hence such knots are difficult to approach directly using knot Floer homology. Note that the companion knots of [24, 32, 45, 51] all have positive τ-invariant.

Corollary 1.6. *There exists a family of knots $\{K_n\}_{n\in\mathbb{N}}$ with each $\tau(K_n) \leqslant 0$ such that $\{D(K_n)\}_{n\in\mathbb{N}}$ is linearly independent. If desired, the K_n may be taken to be topologically slice.*

Finally, we provide a re-proof of a conjecture of the second author and Pinzón–Caicedo [32], who asked whether there is a knot K such that $D(K)$ and $D(-K)$ are both nonzero in concordance. This was recently answered by Lewark and Zibrowius using Khovanov homology [42, Corollary 1.13]; in Corollary 6.1 we give a general condition on K that guarantees the linear independence of $D(K)$ and $D(-K)$. In Corollary 1.6, the knots K_n can be taken so that each pair $D(K_n)$ and $D(-K_n)$ are linearly independent.

1.1 | Main theorems

We first give a rough overview of the class of patterns considered in this paper. A pattern P has *rational unknotting number one* if there exists a rational tangle T embedded in P such that replacing T with another rational tangle T' gives an unknot in the solid torus. This replacement is said to be *proper* if T' connects the same two pairs of points as T. Some examples of rational unknotting number one patterns are given in Figure 1; see Subsections 2.2 and 2.4 for further discussion and examples.

Let P be a rational unknotting number one pattern. By the Montesinos trick, a choice of unknotting tangle replacement identifies the branched double cover $\Sigma_2(P(U))$ with surgery on a strongly invertible knot J:

$$\Sigma_2(P(U)) \cong S^3_{p/q}(J).$$

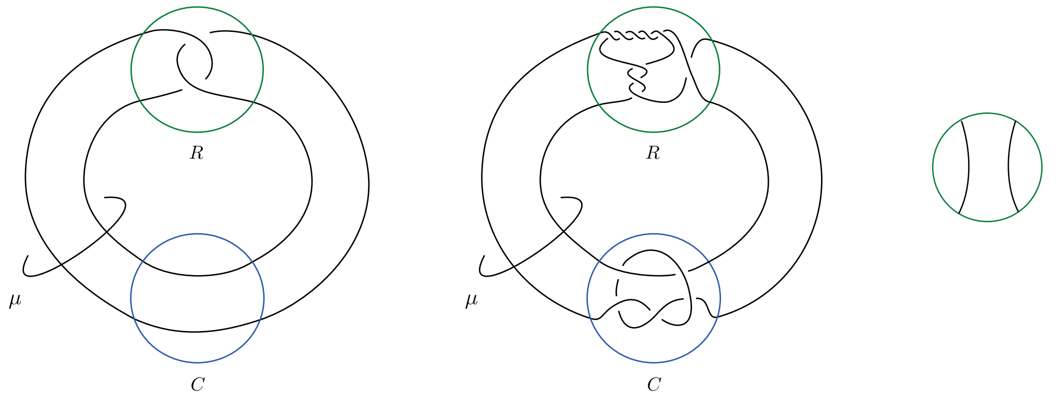

FIGURE 1 A large class of unknotting number one patterns can be formed by gluing a rational tangle R to another tangle C with unknotted (horizontal) closure; see Subsection 2.4. In such cases the unknotting tangle replacement is given by replacing R with a trivial tangle of two vertical strands.

In Subsection 2.2, we describe how to explicitly obtain J and the surgery coefficient p/q. These data depend on our choice of unknotting tangle replacement; when discussing a rational unknotting number one pattern, we will usually have a fixed tangle replacement in mind, although we suppress writing this explicitly.

Using this identification, we define an additional invariant of a rational unknotting number one pattern, which we call the *linking number* ℓ. Let μ be a meridian of the solid torus for P and $\widetilde{\mu}$ be a lift of μ to the branched cover $\Sigma_2(P(U))$. We then set $\ell = lk(J, \widetilde{\mu})$. This may be computed by using the algorithm for determining J outlined in Subsection 2.2.

We now state our main theorem:

Theorem 1.7. *Let P be a proper rational unknotting number one pattern with nonzero linking number. Then P is rank-expanding. Moreover, if K is any knot such that $V_0(nK) - V_0(-nK) \to \infty$ as $n \to \infty$, then P is rank-expanding along K.*

Again, all multiply-clasped and twisted Whitehead doubles satisfy the hypotheses of Theorem 1.7. As we may freely replace K with $-K$ for the purposes of rank expansion, note that in the latter half of the theorem it also suffices to establish $V_0(nK) - V_0(-nK) \to \infty$ as $n \to -\infty$. The condition on ℓ is equivalent to the linking number condition of [32] and is in fact a property of P, independent of the choice of tangle replacement (see Remark 3.5).

Although the Floer-theoretic condition $V_0(nK) - V_0(-nK) \to \infty$ might seem slightly opaque, there are many classes of knots for which this hypothesis is easy to verify. These include the following large families.

(1) K is any L-space knot, such as a torus knot or algebraic knot, or any linear combination of such knots of the same sign/handedness.
(2) K is any thin knot with $\tau(K) \neq 0$, such as a (quasi-)alternating knot of nonzero signature.
(3) K is any linear combination of genus one knots such that the overall connected sum satisfies $\tau(K) \neq 0$.

These examples are discussed in Section 5; note that the above list is certainly not exhaustive. The wide applicability of Theorem 1.7 may be taken as evidence that Conjecture 1.2 indeed holds along every rank-one subgroup.

Note that as any Whitehead double has genus one, in Theorem 1.7 we may take K itself to be a Whitehead double so long as $\tau(K) \neq 0$. This additional condition is quite mild, and can easily be verified using [20, Theorem 1.4] (cf. [41]). Setting (for example) $K = D(T_{2,3})$, we immediately obtain:

Corollary 1.8. *Let P be a proper rational unknotting number one pattern with nonzero linking number. Then P is rank-expanding when restricted to the subgroup C_{TS} of topologically slice knots. Setting P itself to be D (so that the image of P is contained in C_{TS}) gives an example of a rank-expanding operator $P|_{C_{TS}} : C_{TS} \to C_{TS}$.*

Theorem 1.7 is a special case of a broader statement regarding the images of general families of companions:

Theorem 1.9. *Let P be a proper rational unknotting number one pattern with nonzero linking number and $p/q > 0$. If $\{K_n\}_{n\in\mathbb{N}}$ is any family of knots such that $V_0(K_n) - V_0(-K_n) \to \infty$ as $n \to \infty$, then $\{P(K_n)\}_{n\in\mathbb{N}}$ has infinite rank.*

Theorem 1.7 follows immediately from Theorem 1.9 by setting $K_n = nK$ and (if needed) replacing P by $-P$. (If P has $p/q > 0$, then the mirrored pattern $-P$ has $p/q < 0$.) Note that the families of knots discussed after Theorem 1.7 all apply to Theorem 1.9. Previous results in the vein of Theorem 1.9 have generally focused on families of companions such as torus knots; the classes discussed in this section are significantly broader.

In certain cases, it is possible to strengthen Theorem 1.9 by establishing linear independence of the entire image $\{P(K_n)\}_{n\in\mathbb{N}}$. For this, we restrict to the class of *rational tangle patterns*. We define the p/q-rational tangle pattern by taking the closure of a p/q-rational tangle, as discussed in Subsection 2.4. This is the simplest case of a rational unknotting number one pattern and corresponds to the case where J is an unknot.

Theorem 1.10. *Let P be a p/q-rational tangle pattern with $p/q > 0$.*

(1) *Suppose q is even. Let $\{K_n\}_{n\in\mathbb{N}}$ be any family of thin knots with $\tau(K_n)$ distinct and greater than $\lfloor(\lfloor p/q\rfloor + 1)/4\rfloor$. Then $\{P(K_n)\}_{n\in\mathbb{N}}$ is linearly independent and in fact spans a $\mathbb{Z}^\infty$-summand of C.*
(2) *Suppose q is odd. Let $\{K_n\}_{n\in\mathbb{N}}$ be any family of thin knots with $\tau(K_n)$ distinct and less than zero. Then $\{P(K_n)\}_{n\in\mathbb{N}}$ is linearly independent and in fact spans a $\mathbb{Z}^\infty$-summand of C.*

A rational tangle pattern always has rational unknotting number one. However, the tangle replacement is proper if and only if q is even. Note that every multiply-clasped and twisted Whitehead double is a rational tangle pattern; hence Theorem 1.10 recovers [51, Theorem 13]. Using the methods of this paper, it is also possible to extend Theorem 1.10 to other (nonthin) classes of companions, including certain families of torus knots or L-space knots.

1.2 | Overview

Our results employ the well-established strategy of translating the linear independence of satellites to a question about branched double covers. Recall that taking the branched double cover gives a homomorphism

$$\Sigma_2 : \mathcal{C} \to \Theta^3_{\mathbb{Z}_2}.$$

Thus, to determine whether a given family of knots is linearly independent in $\mathcal{C}$, it suffices to show that their branched double covers are linearly independent in $\Theta^3_{\mathbb{Z}_2}$. Establishing linear independence in the homology cobordism group is an old and well-explored application of Floer homology, and the results of [24, 32, 45, 51] have all relied on leveraging the Chern–Simons filtration on instanton Floer theory in this setting. In this paper, we instead use the involutive Heegaard Floer package of Hendricks and Manolescu [27]. This has already been employed by several authors to study homology cobordism; see, for example, [14, 16, 18, 22, 23, 28].

The Heegaard Floer framework is especially suited to this strategy. Indeed, let P be any rational unknotting number one pattern. We show in Section 2.3 that for any companion knot K, the branched double cover $\Sigma_2(P(K))$ is homeomorphic to p/q-surgery on a certain knot $J_{K,\widetilde{\mu}}$ constructed from J and K. To establish the linear independence of $\{P(K_n)\}_{n\in\mathbb{N}}$, it thus suffices to show that the family of $\mathbb{Z}_2$-homology spheres $\{S^3_{p/q}(J_{K_n,\widetilde{\mu}})\}_{n\in\mathbb{N}}$ is linearly independent. In joint work with Hendricks, Hom, and Zemke [23], the fourth author established a surgery formula for involutive Heegaard Floer homology. Our approach is to use this surgery formula to analyze the involutive Floer homology of $\{S^3_{p/q}(J_{K_n,\widetilde{\mu}})\}_{n\in\mathbb{N}}$ and apply existing involutive Floer techniques to show that this family has infinite rank.

1.3 | Comparison with other techniques

It may be somewhat surprising that involutive Heegaard Floer theory can be used to study the classes of satellites at hand. Indeed, prior to this article, Heegaard Floer invariants had not been successfully employed to establish that *any* winding number zero satellite operators have infinite rank. In particular, a host of Heegaard Floer theoretic invariants had failed a simple litmus test in this direction; namely, (re)proving the independence of infinite families of (untwisted) Whitehead doubles, first exhibited in [24].

It is also worth noting an interesting conceptual distinction between the instanton and Heegaard Floer homologies. An important feature in the realm of instanton Floer homology is its filtration by the Chern–Simons functional, which provides refined topological invariants that are crucial for the arguments of [12, 24, 32, 45, 51]. While instanton and Heegaard Floer homology share many formal properties, no such filtration is present on the Heegaard Floer side. Indeed, the analogous filtration on the Heegaard Floer side comes from the symplectic action functional used in the definition of Lagrangian Floer homology for the symmetric product of a Heegaard diagram. To date, however, no topological significance of this information for 3-manifolds and cobordisms between them has been discovered. Even if the action functional could be used in a similar manner, it seems unlikely that the Heegaard Floer package, being isomorphic to an abelian gauge theoretic Floer theory (Seiberg–Witten monopole Floer homology) could recover the topological information about non-abelian fundamental group representations contained in the Chern–Simons filtration. It is thus curious that the usage of involutive Heegaard Floer homology in our situation suffices to recover (and in some cases extend) previously known results established using instanton Floer theory.

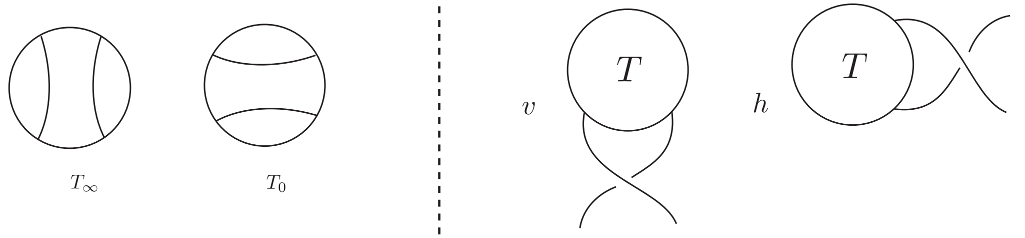

FIGURE 2 Left: The tangles $T_{1/0} = T_\infty$ and $T_{0/1} = T_0$. Right: Adding half-twists to a tangle T via the operations v and h.

Organization

In Section 2, we introduce the notion of a rational unknotting number one pattern and review the basic setup of involutive Heegaard Floer homology and local equivalence. We then prove Theorems 1.7 and 1.9 in Section 3 and Theorem 1.10 in Section 4. In Section 5, we give some examples of Theorems 1.7 and 1.9. Finally, in Section 6 we prove Corollaries 1.4 and 1.6 and discuss further applications to Whitehead doubles.

2 | BACKGROUND

In this section, we define the class of rational unknotting number one patterns and give a brief overview of the setup of involutive Heegaard Floer homology.

2.1 | Rational tangles

We first review the notion of a rational tangle. Let B^3 be a 3-ball with four marked points on its boundary. A *Conway tangle* (or sometimes just *tangle*) is a proper embedding of two disjoint arcs $T \subseteq B^3$ whose boundaries are precisely the four marked points. Two tangles are *isotopic* if there is an isotopy fixing the boundary that takes one to the other.

Definition 2.1. A tangle is *rational* if it consists of a pair of boundary-parallel arcs.

The set of rational tangles in a fixed 3-ball B^3 may be placed in (noncanonical) bijection with $\mathbb{Q} \cup \{\infty\}$, as follows. Fix a projection of B^3 and let $T_{1/0} = T_\infty$ and $T_{0/1} = T_0$ be the tangles displayed in Figure 2. Given any $p/q \in \mathbb{Q} \cup \{\infty\}$, consider the continued fraction

$$p/q = [x_1, x_2, \dots, x_n] = x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \dots + \cfrac{1}{x_n}}}$$

with each $x_i \in \mathbb{Z}$. Let h and v be the horizontal and vertical half-twist operations displayed on the right in Figure 2. We then define the p/q-rational tangle $T_{p/q}$ to be

$$T_{p/q} = \begin{cases} h^{x_1} v^{x_2} \cdots h^{x_{n-1}} v^{x_n} T_\infty & \text{for } n \text{ even} \\ h^{x_1} v^{x_2} \cdots v^{x_{n-1}} h^{x_n} T_0 & \text{for } n \text{ odd.} \end{cases}$$

$\frac{1}{2} = [0,2]$ $\frac{5}{4} = [1,4]$ $\frac{21}{16} = [1,3,5]$

FIGURE 3 Examples of rational tangles.

Conway [10] showed that up to isotopy, $T_{p/q}$ is independent of the choice of the continued fraction decomposition of p/q and that every rational tangle (on the same marked 3-ball) arises from the above construction. (Here, our sign convention is opposite to that in [19].) See Figure 3 for some examples of rational tangles.

We stress that identifying a rational tangle with an element of $\mathbb{Q} \cup \{\infty\}$ is relative to a particular projection; or, equivalently, a choice for T_∞ and T_0. (In the next subsection, we will see why being precise with this identification is so important.) Indeed, given an abstract 3-ball with four marked points, there is no canonical choice for T_∞ or T_0 without fixing a preferred projection. Instead, we declare T_∞ and T_0 to be a pair of rational tangles that connect different pairs of marked points on ∂B^3 and are simultaneously boundary-parallel. Pushing T_∞ and T_0 to the boundary of B^3 then divides ∂B^3 into two hemispheres, from which it easily follows that up to homeomorphism (not fixing ∂B^3) we may draw T_∞ and T_0 as in Figure 2. When we refer to a *p/q-rational tangle* without further elaboration, we will usually have in mind the standard projection in the sense of Figure 2.

In general, if we have fixed a projection of B^3 in which T_∞ and T_0 are not standard (in the sense of Figure 2), then in order to identify a tangle T with an element of $\mathbb{Q} \cup \{\infty\}$, we must find the homeomorphism $F : B^3 \to B^3$ that moves T_∞ and T_0 into standard position with respect to the projection. We then apply the previous discussion to the projection of $F(T)$.

2.2 | Rational unknotting number one patterns

We now define the class of patterns considered in this paper.

Definition 2.2. Let $P \subseteq S^1 \times D^2$ be a pattern. We say that P has *rational unknotting number one* if there exists a rational tangle T in P such that replacing T with another rational tangle T' gives a knot that is unknotted in the solid torus. We say that P has *proper rational unknotting number one* if T' can be taken to be a proper tangle replacement; that is, connecting the same two pairs of marked points as T.

See Figure 4 for an example of a rational unknotting one pattern. We will write P' to refer to the result of replacing T with T'; this is of course isotopic to the unknot. When we discuss a rational unknotting number one pattern, we will usually implicitly have a particular unknotting tangle replacement T' in mind, although a single pattern may admit several different unknotting replacements.

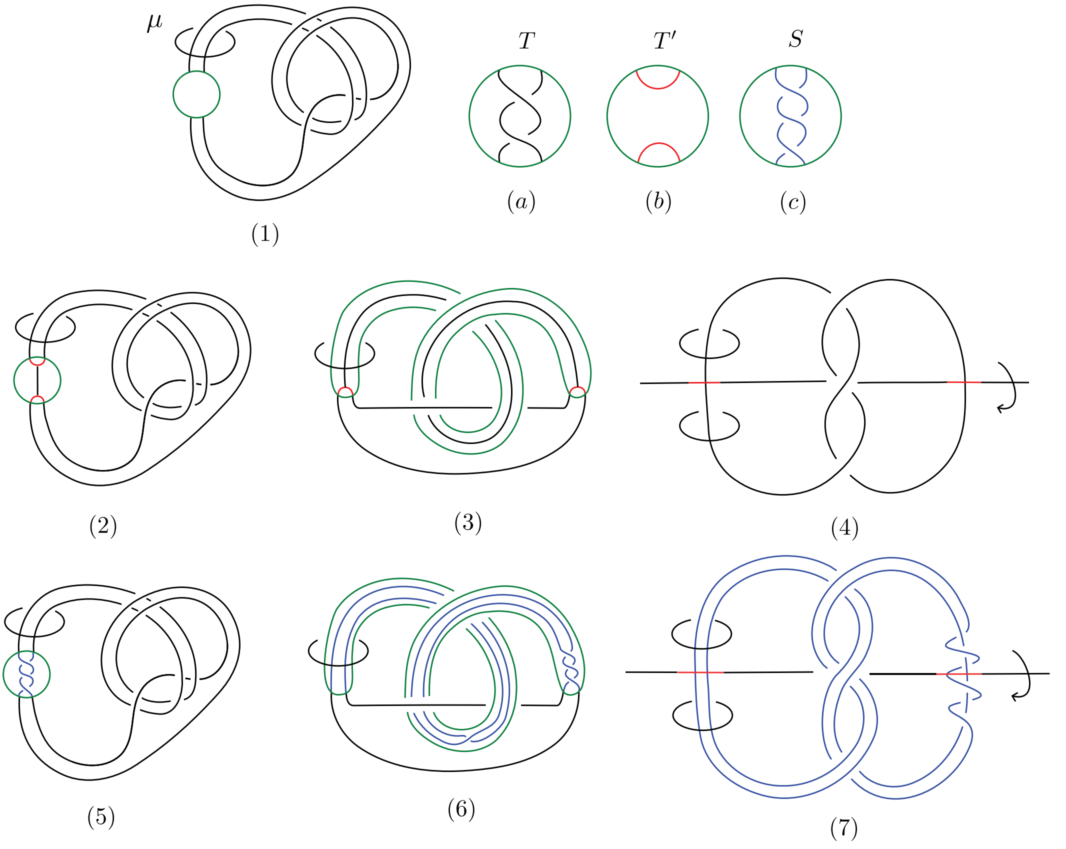

FIGURE 4 Top row: (1) the pattern P; (a), (b), and (c) are the tangles T, T', and S. Middle row: (2) the pattern P' and arc γ; (3) applying the isotopy F_1 to P' and γ; (4) the strongly invertible knot J. Note that the axis of symmetry of J passes through the crossing rather than going under it. Bottom row: (5) the tangle S; (6) applying the isotopy F_1 to S; (7) the two τ-invariant Seifert framings of J. We have isotoped one of the two framings to coincide with J as drawn in (4), so as to illustrate that the other has zero linking with J and is thus a Seifert framing.

For us, the important feature of a rational unknotting number one pattern is that its branched double cover is surgery on a strongly invertible knot. Recall that a knot J is called *strongly invertible* if there exists an orientation-preserving involution τ of S^3 that fixes J setwise and has two fixed points on J. By [57], it follows that τ is conjugate to 180° rotation about an unknotted axis. We claim that if P has rational unknotting number one, then

$$\Sigma_2(P(U)) \cong S^3_{p/q}(J)$$

for some strongly invertible knot J and surgery coefficient p/q. Moreover, this homeomorphism identifies the branched covering action on $\Sigma_2(P(U))$ with the involution on $S^3_{p/q}(J)$ induced by the strong inversion on J. Our claim is immediate from the Montesinos trick: as P' is an unknot, the branched double cover over P' is S^3. The 3-ball B^3 containing T' lifts to a solid torus in S^3, and replacing T' with T corresponds to doing surgery on the core of this solid torus.

However, explicitly producing J and the surgery coefficient p/q is slightly involved. An example of this procedure is given in Figure 4. Here, we have drawn the tangles T and T' in black and red, respectively, while B^3 is drawn in green. It is straightforward to check that replacing T with T' gives an unknot in the solid torus. The meridian of $S^1 \times D^2$ is labeled μ.

To draw J, let γ be a reference arc in B^3 that has one endpoint on each component of T'. In general, there are many such arcs (each looping around the components of T' multiple times); we select one by requiring γ not to intersect the disks obtained as traces of the isotopy pushing T' to ∂B^3. In the case that T' has trivial projection, γ is the obvious arc running from one component to the other, as displayed in panel (2) of Figure 4. Let F_t be an isotopy of the solid torus moving P' into a local unknot in $S^1 \times D^2$. Apply F_1 to γ and B^3, as shown in (3) of Figure 4. It is then straightforward to draw the lift of $F_1(\gamma)$ to the branched double cover over the unknot $F_1(P')$. This gives the desired strongly invertible knot J, displayed in panel (4).

Determining the surgery coefficient p/q is slightly more involved. To do this, we must find the unique rational tangle S in B^3 whose arcs lift to a pair of τ-equivariant Seifert framings of J. (The reader may easily check that, up to equivariant isotopy, there are exactly two equivariant Seifert framings of a strongly invertible knot.) This is colored blue in Figure 4. Determining S can be done by running F_t backward: in Figure 4, panel (7) shows the two τ-invariant Seifert framings for J. The quotient of these by τ is displayed in (6), while in (5) we have reversed the isotopy F_t to draw S in the original 3-ball B^3. By the Montesinos trick, the surgery coefficient p/q is then precisely the rational number identified with the original tangle T relative to the choice of reference tangles $T_\infty = T'$ and $T_0 = S$. In Figure 4, the surgery coefficient is given by $1/3$.

While the above construction illustrates the process of producing J in the context of diagrammatically taking the double branched cover, we can also describe the knot J in a more abstract manner. Indeed, the green 3-ball containing the unknotting tangle lifts to a solid torus under the double branched cover; the core of this solid torus is precisely the knot J. Thus, isotoping the black unknot (for example by F_t) only changes the knot J by an equivariant isotopy. Hence, the knot J, when produced using the above construction, is well-defined up to equivariant isotopy. Changing the tangle inside the green 3-ball (as in panel (1) of Figure 4) changes the framing of the 3-ball, which precisely corresponds to changing the framing of the lifted solid torus. The fact that there are a unique pair of equivariant Seifert framings of J determines the replacement tangle S without reference to a particular isotopy. This gives an intrinsic definition of the surgery coefficient p/q.

Definition 2.3. Let P be a rational unknotting number one tangle with a fixed choice of unknotting tangle replacement T'. As discussed above, this gives an identification

$$\Sigma_2(P(U)) \cong S^3_{p/q}(J).$$

We refer to p/q as the *coefficient* of P. Note that because P is a knot, its branched double cover is a $\mathbb{Z}_2$-homology sphere; hence p is necessarily odd. We say that P is *even* or *odd* according to the parity of q and *positive* or *negative* according to the sign of p/q.

In our setting, it turns out that even and odd rational unknotting number one patterns behave rather differently. Fortunately, even though determining the exact coefficient p/q of P is rather difficult, the parity of q can be easily read off from the tangle replacement:

Lemma 2.4 [44, Corollary 2]. *Let P be a rational unknotting number one pattern with a fixed choice of unknotting tangle replacement T'. Then q is even if and only if the tangle replacement is proper.*

Proof. As in the discussion of Subsection 2.1, let F be a homeomorphism of B^3 taking T' and S to the standard ∞- and 0-tangles T_∞ and T_0, respectively. Note that the tangle replacement T to T' is proper if and only if the tangle replacement $F(T)$ (which is by definition the p/q-tangle with respect to the standard ∞- and 0-tangles) to T_∞ is proper. It was shown in [44, Lemma 11] that the distance between $T_{p/q}$ and T_∞ is even if and only if the replacement $T_{p/q}$ to T_∞ is proper. The result follows by observing that the distance between $T_{p/q}$ and the T_∞ is precisely q. □

Note that if P is a negative pattern, then its mirror is positive. We thus lose no generality in considering the class of positive rational unknotting number one patterns.

There is one more important piece of data associated to a rational unknotting number one pattern. This is the following:

Definition 2.5. Let P be a rational unknotting number one tangle with a fixed choice of unknotting tangle replacement T'. We define the *linking number* of P by

$$\ell = lk(J, \widetilde{\mu}).$$

As $\widetilde{\mu}$ and $\tau\widetilde{\mu}$ are equally preferenced, ℓ is defined only up to sign. Note that as discussed above, the knot J is well-defined up to equivariant isotopy; it is clear that a different choice of isotopy F_t does not change the linking number of J with $\widetilde{\mu}$ (up to sign).

2.3 | Branched covers of satellites

We now extend the discussion of the previous subsection to the branched double cover of $P(K)$ for an arbitrary companion knot K. First observe that in the algorithm of Section 2.2, the meridian μ of our pattern lifts to a symmetric unlink $\widetilde{\mu} \cup \tau\widetilde{\mu}$ in S^3 disjoint from J. Here, the fact that $\widetilde{\mu} \cup \tau\widetilde{\mu}$ is a two-component unlink follows from the condition that P' is unknotted in $S^1 \times D^2$. Note that the data of P come with an orientation of μ; we give $\widetilde{\mu} \cup \tau\widetilde{\mu}$ the lifted orientation. Let K be an oriented knot in S^3. Recall that $P(K)$ can be constructed by taking the image of P inside the gluing

$$S^3 \cong (S^3 - N(\mu)) \cup_{\partial N(\mu)} (S^3 - N(K))$$

formed by a boundary identification that maps a meridian of μ to a Seifert framing of K and a longitude of μ to a meridian of K (respecting the orientations). It follows from the discussion of the previous subsection that

$$\Sigma_2(P(K)) \cong (S^3_{p/q}(J) - N(\widetilde{\mu}) - N(\tau\widetilde{\mu})) \cup_{\partial N(\widetilde{\mu})} (S^3 - N(K)) \cup_{\partial N(\tau\widetilde{\mu})} (S^3 - N(K)).$$

Note that this manifold has an obvious involution. On $S^3_{p/q}(J) - N(\widetilde{\mu}) - N(\tau\widetilde{\mu})$, this involution is induced by the strong inversion on J, while elsewhere we simply exchange the two copies of $S^3 - N(K)$.

Definition 2.6. Let J be a strongly invertible knot and $\widetilde{\mu} \cup \tau\widetilde{\mu}$ be an oriented, symmetric unlink disjoint from J. We assume that $\widetilde{\mu} \cup \tau\widetilde{\mu}$ is given a τ-invariant orientation. For any oriented knot

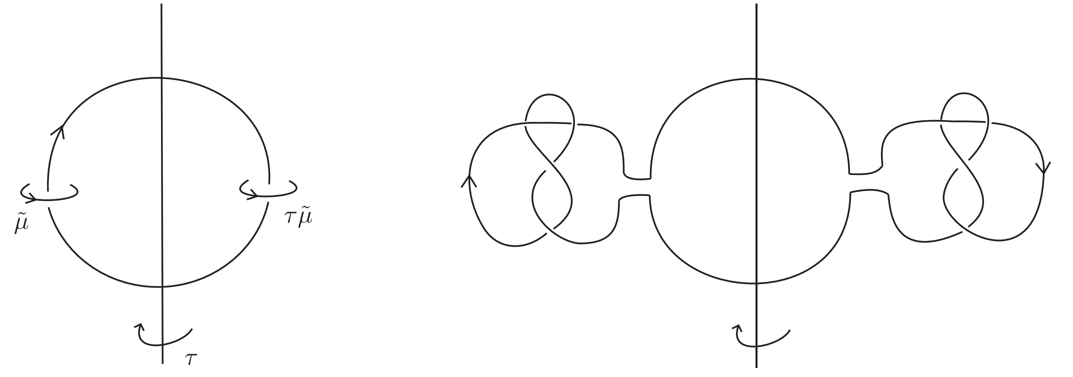

FIGURE 5 In the trivial case where J is the unknot and $\widetilde{\mu}$ is a standard meridian, we have $J_{K,\widetilde{\mu}} = K\#K^r$. Note the reversal of orientation in the second factor; this is because $\widetilde{\mu}$ and $\tau\widetilde{\mu}$ are oriented such that $lk(K,\widetilde{\mu}) = -lk(K,\tau\widetilde{\mu})$.

K, define the *double infection* $J_{K,\widetilde{\mu}}$ by infecting J twice: once using K along $\widetilde{\mu}$ and once using K along $\tau\widetilde{\mu}$. Note that $J_{K,\widetilde{\mu}}$ is a strongly invertible knot (see Figure 5).

We thus have:

$$\Sigma_2(P(K)) \cong S^3_{p/q}(J_{K,\widetilde{\mu}}).$$

Note that the surgery coefficient p/q is independent of K (and is the same as that of Definition 2.3). Moreover, once again this homeomorphism identifies the branching action on $\Sigma_2(P(K))$ with the involution on $S^3_{p/q}(J_{K,\widetilde{\mu}})$ induced by the strong inversion on $J_{K,\widetilde{\mu}}$.

To spell out the relevance of this construction, recall that our general strategy is to study the family $\{P(K_n)\}_{n\in\mathbb{N}}$ via their branched double covers $\{\Sigma_2(P(K_n))\}_{n\in\mathbb{N}}$. If P is a rational unknotting number one pattern, then this is the same as studying p/q-surgeries on the family of knots

$$J_n = J_{K_n,\widetilde{\mu}}.$$

We will use the fact that the J_n are all (double) infections of the same knot in order to derive certain structural results regarding the Floer homologies of these surgeries. This will allow us to establish the desired linear independence.

2.4 | Examples

We now give some examples of rational unknotting number one patterns. The simplest of these are *rational tangle patterns*. A rational tangle pattern is obtained by taking the horizontal closure of a p/q-rational tangle in the standard projection, as in Figure 6. Clearly, each such pattern has rational unknotting number one, with the replacement tangle T' being the standard ∞-tangle. (The resulting strongly invertible knot J is the unknot.) A rational tangle pattern has linking number ± 1 and surgery coefficient precisely p/q. Note that all multiply-clasped, multiply-twisted Whitehead doubling patterns fall into this class.

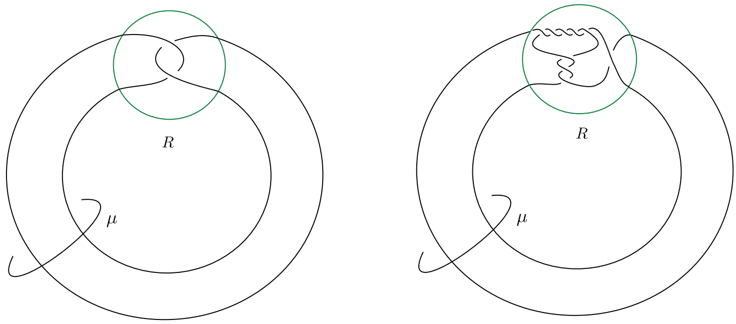

FIGURE 6 The rational tangle patterns corresponding to $1/2$ and $21/16$.

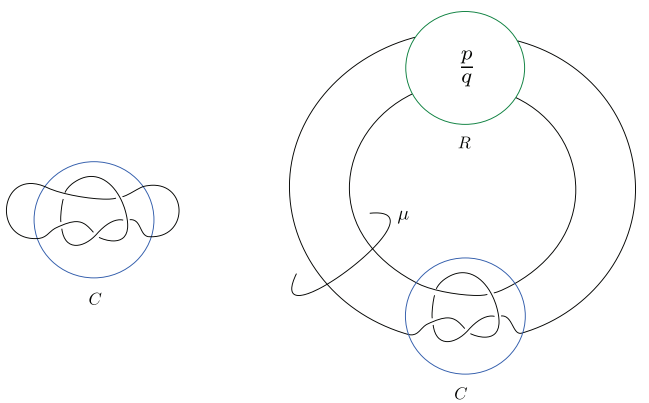

FIGURE 7 Left: A tangle C with an unknotted closure. Right: A rational unknotting number one pattern defined using C.

One particularly simple way of constructing a rational unknotting number one pattern is to start from a Conway tangle C with unknotted (horizontal) closure and glue it to any rational tangle R, as in Figure 7. We embed this in $S^1 \times D^2$ by choosing the indicated meridian μ. The resulting pattern tautologically has rational unknotting number one by replacing R with the usual ∞-tangle, and some thought shows that $\ell = \pm 1$. Note that rational tangle patterns are a special case of this construction, where C is chosen to be the trivial tangle of two horizontal strands.

In general, it is often easier to construct a rational unknotting number one pattern by working backward from the associated knot J. As in Definition 2.6, let J be a strongly invertible knot equipped with a symmetric unlink $\widetilde{\mu} \cup \tau\widetilde{\mu}$ disjoint from J. We furthermore require that $\widetilde{\mu} \cup \tau\widetilde{\mu}$ is in fact unlinked in the complement of the axis of symmetry. The algorithm of, for example, [55, section 1.1.12] allows us to express any surgery $S^3_{p/q}(J)$ as the branched double cover over a knot. We turn this knot into a pattern by taking the image of $\widetilde{\mu} \cup \tau\widetilde{\mu}$ under the quotient map. Examples of this procedure are shown along the top rows of Figures 8 and 9.

As shown along the bottom rows of Figures 8 and 9, the result may be viewed as a Conway tangle C glued to a p/q-rational tangle R in the standard projection. If $\widetilde{\mu}$ is chosen to be a meridian of J (as in Figure 8), then the meridian μ of the resulting pattern will be obviously isotopic to a curve between C and R, as in Figure 7 but in general μ may be more complicated (as in Figure 9). (Different choices for $\widetilde{\mu}$ give patterns that are the same as knots, but which may have different embeddings in the solid torus.) It is clear from the algorithm of [55, section 1.1.12] that replacing

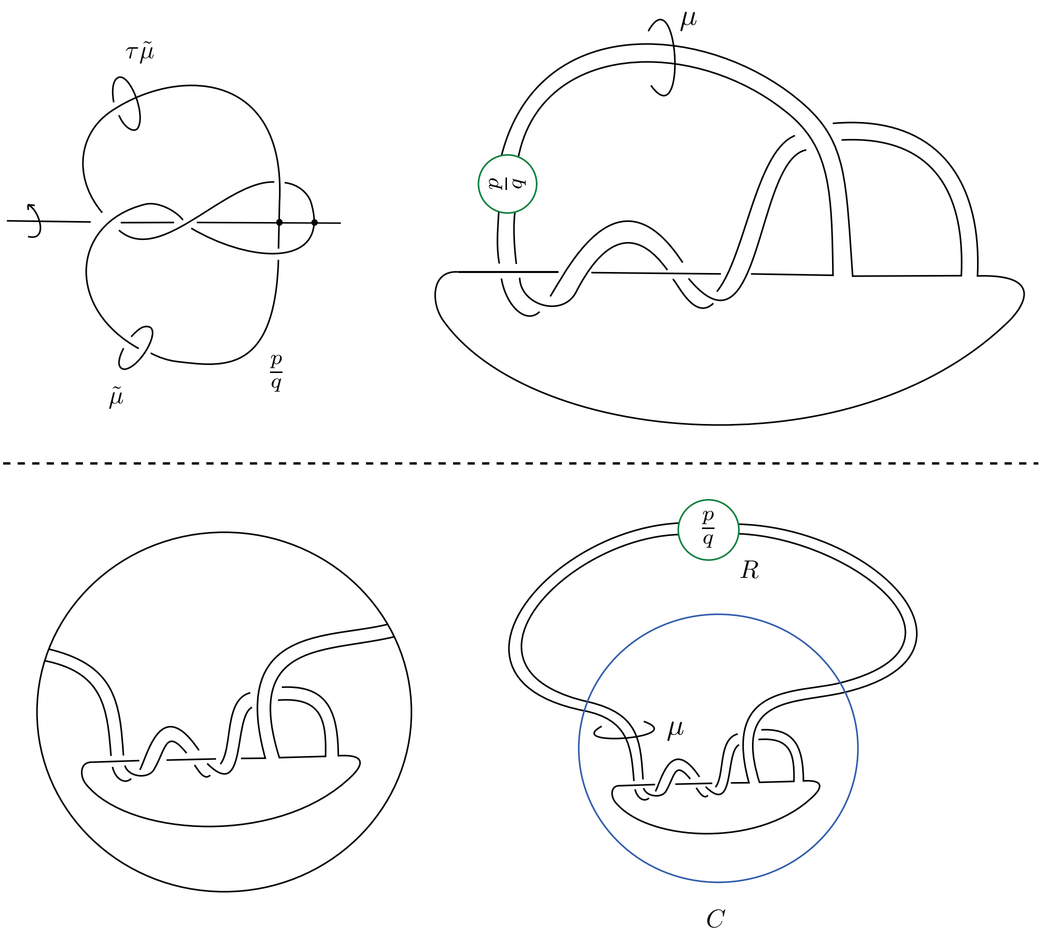

FIGURE 8 A pattern whose double branched cover is surgery on the figure-eight knot, with $\tilde{\mu}$ a standard meridian. Here, $\ell = 1$. Note that in the top-left, the axis of symmetry passes through the crossing rather than going under it.

the p/q-rational tangle with an ∞-tangle gives an unknot in the complement of μ. The same algorithm likewise shows that the tangle S coming from the Seifert framing of J is the usual 0-tangle in the standard projection. By construction, any such pattern has nonzero linking number as long as $\tilde{\mu}$ was chosen so that $lk(J, \tilde{\mu}) \neq 0$.

2.5 | Local equivalence

In this section, we give a brief overview of the involutive Heegaard Floer package and the machinery of local equivalence. Let Y be a 3-manifold equipped with a self-conjugate spin^c-structures $\mathfrak{s} = \bar{\mathfrak{s}}$. In [27], Hendricks and Manolescu defined a homotopy involution on the Heegaard Floer chain complex coming from the conjugation symmetry present in a Heegaard diagram:

$$\iota : CF^-(Y, \mathfrak{s}) \to CF^-(Y, \mathfrak{s}).$$

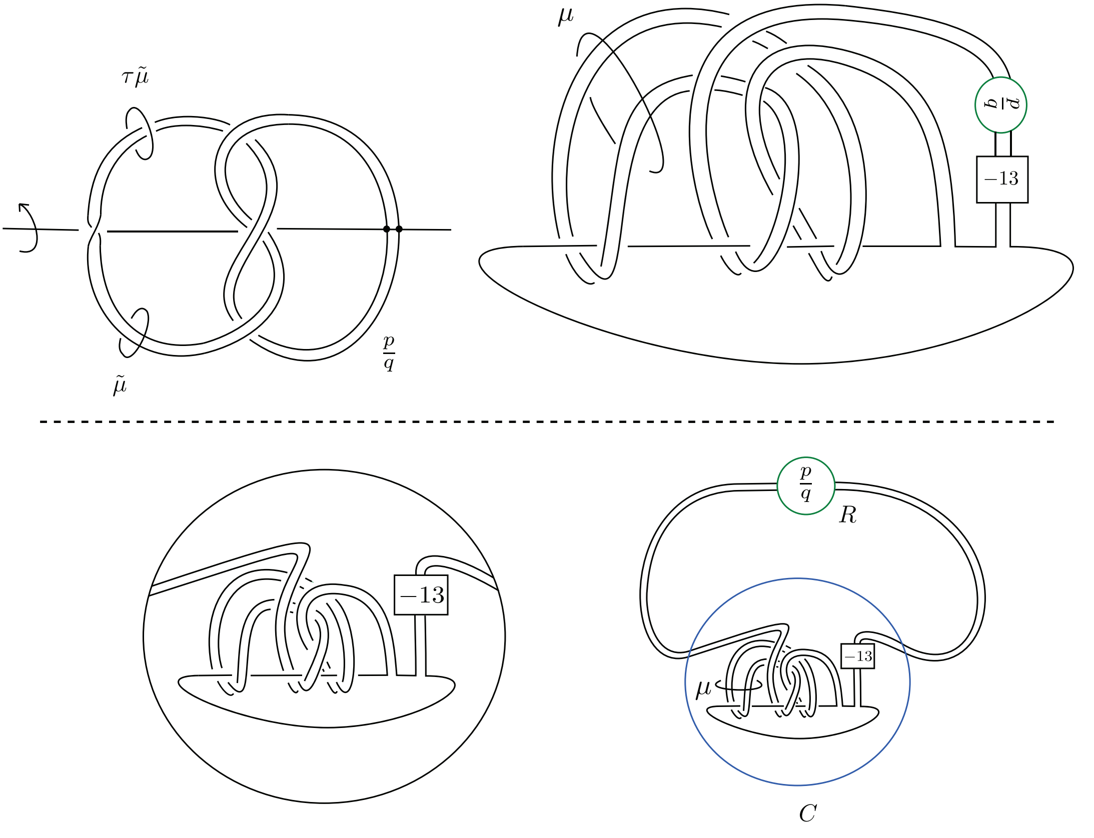

FIGURE 9 A pattern whose double branched cover is surgery on the (2,1)-cable of the trefoil, with $\tilde{\mu}$ as indicated. Here, $\ell = 2$. Note that in the top-left, the axis of symmetry passes through the crossing rather than going under it. The (−13)-twist box is due to the writhe of the knot and comes from the need to find an equivariant Seifert framing; see the algorithm in [55, section 1.1.12].

These additional data provide an enhancement of the usual Heegaard Floer invariant of Ozsváth and Szabó [48, 49]. More precisely, associated to $(Y, \mathfrak{s})$, we may consider the pair $(CF^-(Y, \mathfrak{s}), \iota)$; up to an appropriate notion of homotopy equivalence, this is a diffeomorphism invariant of Y.

Although involutive Heegaard Floer homology is defined for all 3-manifolds, we will mainly be concerned with rational homology spheres. In this case, we formalize the resulting algebraic structure in the following definition:

Definition 2.7 [28, Definition 8.1]. An *ι-complex* is a pair (C, ι), where:

(1) C is a (free, finitely generated) chain complex over $\mathbb{F}[U]$ with

$$U^{-1}H_*(C) \cong \mathbb{F}[U, U^{-1}],$$

here, $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We require C to be graded by a coset of $\mathbb{Z}$ in $\mathbb{Q}$ and U to be of degree −2;

(2) $\iota : C \to C$ is a $\mathbb{F}[U]$-equivariant, grading-preserving homotopy involution; that is,

$$\iota^2 \simeq \mathrm{id}$$

via a U-equivariant chain homotopy.

In [28], it is shown that if Y is a rational homology sphere equipped with a self-conjugate spin^c-structures $\mathfrak{s}$, then the pair $(CF^-(Y, \mathfrak{s}), \iota)$ is an ι-complex.

To study homology cobordism, we introduce the following equivalence relation:

Definition 2.8 [28, Definition 8.3]. Two ι-complexes (C, ι) and (C', ι') are called *locally equivalent* if there exist $\mathbb{F}[U]$-equivariant, grading-preserving chain maps

$$f : C \to C' \quad \text{and} \quad g : C' \to C$$

such that

$$f \circ \iota \simeq \iota' \circ f \quad \text{and} \quad g \circ \iota' \simeq \iota \circ g$$

and f and g induce isomorphisms on homology after localizing with respect to U. We call a map f as above a *local map* from (C, ι) to (C', ι'), and similarly we refer to g as a local map in the other direction.

To see the relevance of Definition 2.8 to homology cobordism, let $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$ be two rational homology spheres equipped with self-conjugate spin^c-structures and let W be a rational homology cobordism from Y_1 and Y_2. Suppose W admits a self-conjugate spin^c-structure $\mathfrak{s}$ restricting to $\mathfrak{s}_i$ on each Y_i. Then the Heegaard Floer cobordism map $F_{W,\mathfrak{s}}$ (together with its reverse) constitutes a local equivalence between the ι-complexes associated to $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$.

In the setting of $\mathbb{Z}$- or $\mathbb{Z}_2$-homology cobordism, we always have a unique self-conjugate spin^c-structure on Y or W. In this case, we may unambiguously associate to Y its *local equivalence class*; this is a homology cobordism invariant. Denote

$$\mathfrak{I} = \{\text{all } \iota\text{-complexes}\} \,/\, \text{local equivalence}.$$

We then obtain a map $h : \Theta^3_{\mathbb{Z}_2} \to \mathfrak{I}$ given by sending

$$[Y] \mapsto h([Y]) = [(CF^-(Y, \mathfrak{s})[-2], \iota)].$$

Here, $\mathfrak{s}$ is the unique self-conjugate spin^c-structure on Y and the $[-2]$ is a formal (unimportant) grading shift. In [28, section 8.3], it is shown that $\mathfrak{I}$ is an abelian group with the operation of tensor product. The identity element is given by the trivial complex $\mathbb{F}[U]$ and inverses are given by dualizing; see [28, section 8.3] for details.[†] Moreover, it is shown that h is a well-defined homomorphism.

It is thus possible to show that a given family of $\mathbb{Z}_2$-homology spheres is linearly independent by computing their local equivalence classes and establishing their linear independence in $\mathfrak{I}$. However, this requires an analysis of the algebraic structure of $\mathfrak{I}$. Although in general $\mathfrak{I}$ is very complicated, techniques for carrying out this strategy have been developed in (for example)

[†] Strictly speaking, our notation $\mathfrak{I}$ is not quite the group $\mathfrak{I}$ of [28, section 8.3]. The difference is that here, we allow our ι-complex to have gradings valued in $\mathbb{Q}$, rather than $\mathbb{Z}$.

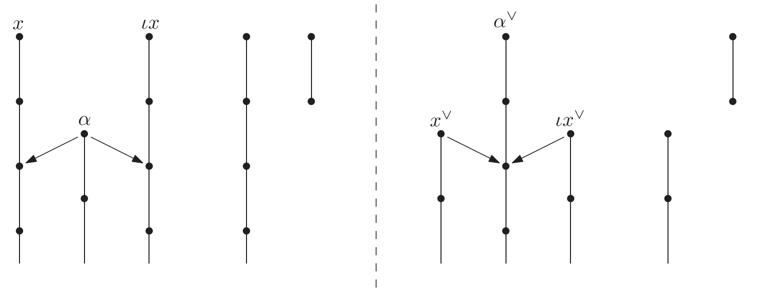

FIGURE 10 Left: The complex X_2 and its homology. Right: The complex $X_2^\vee$ and its homology. In both cases, generators over $\mathbb{F}$ are represented by dots; the action of U is given by following the vertical line segments downward. For the two chain complexes, the action of ∂ is given by extending the indicated arrows U-equivariantly.

[14, 16, 18, 22]. The results of the current article will depend on several such calculations, which we outline in the next subsection.

2.6 | Linear independence in $\mathfrak{I}$

We will need to be familiar with the following especially simple class of ι-complexes:

Definition 2.9. For $i \in \mathbb{N}$, define X_i to be the ι-complex generated over $\mathbb{F}[U]$ by three elements: x, ιx, and $\alpha = \iota\alpha$. These have gradings given by $\mathrm{gr}(x) = \mathrm{gr}(\iota x) = 0$ and $\mathrm{gr}(\alpha) = -2i + 1$; the differential is defined by $\partial\alpha = U^i(x + \iota x)$. We likewise have the dual complex $X_i^\vee$, which is generated by $x^\vee, \iota x^\vee$, and $\alpha^\vee = \iota\alpha^\vee$. These have gradings given by $\mathrm{gr}(x^\vee) = \mathrm{gr}(\iota x^\vee) = 0$ and $\mathrm{gr}(\alpha^\vee) = 2i - 1$; the differential is defined by $\partial x^\vee = \partial \iota x^\vee = U^i\alpha^\vee$. The complexes X_i and $X_i^\vee$ are displayed in Figure 10, along with their homologies. We will often also write X_i for the local equivalence class of X_i in $\mathfrak{I}$, and similarly for $X_i^\vee$.

The X_i turn out to be fundamental for understanding the structure of $\mathfrak{I}$ and occur as the local equivalence classes of several families of homology spheres; see, for example, [16]. We have the following basic fact regarding the X_i:

Theorem 2.10 [16, Proof of Theorem 1.7]. *For $i \in \mathbb{N}$, the classes X_i are linearly independent in $\mathfrak{I}$.*

We will also need a generalization of Theorem 2.10 that follows from the proof of [13, Proof of Theorem 1.3]. To state this, we recall some notation. If (C, ι) is an ι-complex, then it follows from the first condition of Definition 2.7 that $H_*(C)$ is isomorphic to an $\mathbb{F}[U]$-module of the form $\mathbb{F}[U] \oplus (U\text{-torsion})$. The grading of the uppermost generator of the copy of $\mathbb{F}[U]$ is well-defined and gives the d-invariant $d(C)$.

Definition 2.11. If (C_1, ι_1) and (C_2, ι_2) are two ι-complexes with $d(C_1) = d(C_2)$, then we write $(C_1, \iota_1) \leqslant (C_2, \iota_2)$ if there is a local map from (C_1, ι_1) to (C_2, ι_2).

We will often suppress writing ι_1 and ι_2 in the inequality (and when discussing local equivalence). Note that if C_1 is locally equivalent to C_2, then automatically $d(C_1) = d(C_2)$. However, in general, a local map from C_1 to C_2 only guarantees $d(C_1) \leqslant d(C_2)$; hence Definition 2.11 is stronger than the presence of a local map.† It turns out that Definition 2.11 defines a partial order on $\mathfrak{I}$; for a discussion of the importance of Definition 2.11, see [14].

Theorem 2.12 [13, Proof of Theorem 1.3]. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of ι-complexes. Suppose there exists a sequence $(n_i)_{i\in\mathbb{N}}$ such that $n_i \to \infty$ and $C_i \leqslant X_{n_i}$ for each i. Then $\{C_i\}_{i\in\mathbb{N}}$ has infinite rank in $\mathfrak{I}$.*

Proof. We sketch the proof for the convenience of the reader. It was shown in [13, Lemma 7.11] that for any local equivalence class C_i with $C_i \leqslant X_{n_i}$, the connected homology $HF_{\text{conn}}(mC_i)$ (see [22] for a definition) has a U-torsion tower of length at least n_i whenever $m \neq 0$. We then build a infinite linearly independent subsequence of the C_i as follows. At the pth stage, let i_p be any integer for which n_{i_p} is larger than the maximal U-torsion tower length appearing among $HF_{\text{conn}}(C_{i_1})$, $HF_{\text{conn}}(C_{i_2}), \dots, HF_{\text{conn}}(C_{i_{p-1}})$. It follows $HF_{\text{conn}}(mC_{i_p})$ has a U-torsion tower of length at least n_{i_p}. According to [13, Lemma 7.10], this implies that mC_{i_p} does not lie in the the span of C_{i_1}, $C_{i_2}, \dots, C_{i_{p-1}}$ for any $m \neq 0$. Hence, $\{C_i\}_{i\in\mathbb{N}}$ has infinite rank in $\mathfrak{I}$. □

For concreteness, in Definition 2.9 we have normalized the X_i such that each $d(X_i) = 0$. However, Theorems 2.10 and 2.12 hold more generally upon applying a grading shift to each X_i or X_{n_i}. More precisely, for any sequence of integers d_i, the grading-shifted classes $X_i[d_i]$ are linearly independent. Similarly, if $\{C_i\}_{i\in\mathbb{N}}$ is a sequence of ι-complexes with $C_i \leqslant X_{n_i}[d_i]$, then $\{C_i\}_{i\in\mathbb{N}}$ again has infinite rank in $\mathfrak{I}$. Note that in the latter case, Definition 2.11 requires $d_i = -d(C_i)$; otherwise Theorem 2.12 is false.‡ These minor extensions easily follow from considering the splitting

$$\mathfrak{I} = \mathfrak{I}_0 \oplus \mathbb{Z},$$

where $\mathfrak{I}_0$ is the subgroup of $\mathfrak{I}$ consisting of all ι-complexes with d-invariant zero.

2.7 | Involutive surgery formula

Let K be an oriented knot in S^3. In [27], Hendricks and Manolescu defined a grading-preserving, skew-filtered homotopy involution

$$\iota_K : CFK^\infty(K) \to CFK^\infty(K)$$

on the knot Floer complex of K. As in the 3-manifold case, the (filtered) homotopy equivalence class of the pair $(CFK^\infty(K), \iota_K)$ is a diffeomorphism invariant of K. Although in general the action of ι_K is difficult to compute, there are a wide variety of cases in which ι_K is determined for formal reasons; these include thin knots and L-space knots (see [27, section 7] and [27, section 8]).

In this paper, we will mainly utilize the involutive knot Floer package in the context of the involutive surgery formula. To state this, we first review some notation from the usual knot Floer

† This is essentially matter of notational convention. Elsewhere in the literature, it is often assumed that $d(C) = 0$ for convenience, in which case an inequality is indeed equivalent to the existence of a local map.

‡ Our convention for the grading-shift notation is that an element of C in grading zero has grading $-\Delta$ in $C[\Delta]$.

surgery formula. For $s \in \mathbb{Z}$, let $A_s^-(K)$ be the subcomplex of $CFK^\infty(K)$ given by:

$$A_s^-(K) = \text{span}_{\mathbb{F}[U]}\{[\mathbf{x}, i, j], \text{ such that } i \leqslant 0, j \leqslant s\}.$$

Let $B_s^-(K)$ be the subcomplex of $CFK^\infty(K)$ given by:

$$B_s^-(K) = \text{span}_{\mathbb{F}[U]}\{[\mathbf{x}, i, j], \text{ such that } i \leqslant 0\}.$$

Note that $H_*(B_s^-(K)) \cong \mathbb{F}[U]$ for any s. We also have the inclusion map

$$v : A_s^-(K) \to B_s^-(K).$$

See [47, 52] for further discussion.

In [27, section 6], Hendricks and Manolescu established a large surgery formula by showing that for any integer $p \geqslant g(K)$, there is a relatively graded homotopy equivalence

$$(CF^-(S_p^3(K), [0]), \iota) \simeq (A_0^-(K), \iota_K).$$

Note that ι_K preserves $A_0^-(K)$. In [23], Hendricks, Hom, Zemke, and the fourth author extended this to a general surgery formula for computing $(CF^-(S^3_{p/q}(K), [0]), \iota)$. The local equivalence class of the resulting ι-complex is easily described. The following will be the main technical tool used in this paper:

Theorem 2.13 [23, Proposition 22.9]. *Let p and q be positive, relatively prime integers. Suppose p is odd, so that [0] is the unique self-conjugate* spin^c*-structure on* $S^3_{p/q}(K)$*. Then:*

(1) *if q is odd,* $(CF^-(S^3_{p/q}, [0]), \iota)$ *is locally equivalent to* $(A_0^-(K), \iota_K)$*.*
(2) *if q is even,* $(CF^-(S^3_{p/q}, [0]), \iota)$ *is locally equivalent to truncated mapping cone complex below:*

$$A^-_{[p/2q]} \xrightarrow{\ v\ } B^-_{[p/2q]} \xleftarrow{\ v\ } A^-_{[p/2q]}$$

Here, $[n]$ *represents the integer closest to* n.[†] *The* ι*-action on this complex is given by interchanging the two copies of* $A^-_{[p/2q]}$ *and fixing* $B^-_{[p/2q]}$*.*

Note that if q is even, up to local equivalence the action of ι does not in fact depend on ι_K.

3 | PROOF OF THEOREMS 1.7 AND 1.9

We now turn to the proof of Theorem 1.9, which will quickly imply Theorem 1.7.

[†] No half-integers for n appear in this article.

3.1 Families of surgeries

We begin with a general theorem regarding the linear independence of families of homology spheres obtained by odd-over-even surgeries on knots.

Theorem 3.1. *Let p and q be positive integers with p odd and q even. Let $\{J_n\}_{n\in\mathbb{N}}$ be any family of knots in S^3. If $V_0(J_n) \to \infty$ as $n \to \infty$, then the family of $\mathbb{Z}_2$-homology spheres $\{S^3_{p/q}(J_n)\}_{n\in\mathbb{N}}$ has infinite rank in $\Theta^3_{\mathbb{Z}_2}$.*

Proof. For convenience, write $s = \lfloor p/(2q) \rfloor$. Denote $A_s^- = A_s^-(J_n)$ and $B_s^- = B_s^-(J_n)$, so that the ι-complex C_n of $S^3_{p/q}(J_n)$ is locally equivalent to the complex $A_s^- \oplus A_s^- \to B_s^-$ of Theorem 2.13. The structure of knot Floer homology implies we have a relatively graded isomorphism

$$H_*(A_s^-) \cong \mathbb{F}[U] \oplus (U\text{-torsion}) \quad \text{and} \quad H_*(B_s^-) \cong \mathbb{F}[U].$$

The map v induces an injection from the $\mathbb{F}[U]$-tower of $H_*(A_s^-)$ to $H_*(B_s^-) \cong \mathbb{F}[U]$ which is modeled on multiplication by some power of U.[†] This power of U is precisely the knot Floer concordance invariant $V_s(J_n)$ defined in [46].

Although in general it is difficult to completely understand C_n, we will show that the above discussion suffices to produce an inequality

$$X^\vee_{V_s(J_n)}[-d(C_n)] \leqslant C_n.$$

We then dualize and apply Theorem 2.12. This will give the linear independence of the $C_n^\vee$, and hence the C_n.

Let $a \in A_s^-$ be a cycle generating the $\mathbb{F}[U]$-tower in $H_*(A_s^-)$ and let $b \in B_s^-$ be a cycle generating $H_*(B_s^-) \cong \mathbb{F}[U]$. Write a_1 for the copy of a in the first summand of $A_s^- \oplus A_s^-$ and a_2 for the copy of a in the second. Note that $v(a)$ is homologous to $U^{V_s(J_n)}b$; let $c \in B_s^-$ be such that $\partial c = v(a) + U^{V_s(J_n)}b$.

Define a (grading-homogenous) map from $X^\vee_{V_s(J_n)}$ to C_n by sending

$$x^\vee \mapsto a_1 + c, \quad \iota x^\vee \mapsto a_2 + c, \quad \text{and} \quad \alpha^\vee \mapsto b$$

This is an ι-equivariant chain map; the situation is schematically depicted in Figure 11. The d-invariant of C_n is given by the grading of $a_1 + a_2$. Applying a grading shift to $X^\vee_{V_s(J_n)}$ so that $x^\vee$ and $\iota x^\vee$ have this grading then gives the claimed inequality.

Dualizing, we obtain an inequality from $C_n^\vee$ to some grading shift of $X_{V_s(J_n)}$. By [46, 53], we have that

$$V_0(J_n) - s \leqslant V_s(J_n) \leqslant V_0(J_n).$$

As s is independent of n, the condition $V_0(J_n) \to \infty$ implies that $V_s(J_n) \to \infty$. Theorem 2.12 then shows that the span of the $C_n^\vee$, and hence the span of the C_n, has infinite rank in $\mathfrak{I}$. ☐

† The decomposition $H_*(A_s^-) \cong \mathbb{F}[U] \oplus (U\text{-torsion})$ is of course not canonical, but this statement holds for any choice of decomposition.

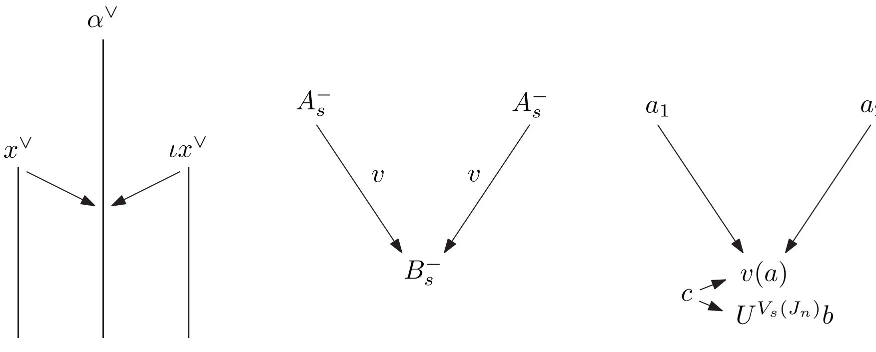

FIGURE 11 Left: The complex $X^\vee_{V_s(J_n)}$. Middle: the complex $A_s^- \oplus A_s^- \to B_s^-$ afforded by Theorem 2.13. Right: A subcomplex of $A_s^- \oplus A_s^- \to B_s^-$ isomorphic to $X^\vee_{V_s(J_n)}$.

In fact, an examination of the proof of Theorem 3.1 establishes a slightly stronger claim: we may even allow the surgery coefficient p/q to vary, so long as $s = [p/(2q)]$ remains bounded. (This extension will not be used in the present paper.) Theorem 3.1 immediately gives a general result regarding rational unknotting number one patterns. For completeness, we record this here:

Theorem 3.2. *Let P be a positive proper rational unknotting number one pattern with associated knot J. For any family of knots $\{K_n\}_{n\in\mathbb{N}}$ in S^3, let $J_n = J_{K_n,\tilde{\mu}}$ be the corresponding doubly infected family. If $V_0(J_n) \to \infty$ as $n \to \infty$, then $\{P(K_n)\}_{n\in\mathbb{N}}$ has infinite rank.*

Proof. As explained in Section 2, by passing to branched double covers and applying the discussion of Subsection 2.3, it suffices to prove that $\{S^3_{p/q}(J_n)\}_{n\in\mathbb{N}}$ has infinite rank in $\Theta^3_{\mathbb{Z}_2}$. As explained in Subsection 2.2, the fact that P is proper means that q is even, while p is always odd. The claim then follows verbatim from Theorem 3.1. □

Theorem 3.2 is trivially a specialization of Theorem 3.1: so far, we have not imposed any condition on $\ell = lk(J, \widetilde{\mu})$, nor have we used any details of the definition of a rational tangle pattern other than the fact that their branched double covers are surgeries on knots. However, while in principle Theorem 3.2 is entirely general, in practice it may be difficult to check the condition $V_0(J_n) \to \infty$, as the knots J_n are extremely complicated.

Our approach will thus be to estimate $V_0(J_n)$ in terms of the invariants of the companion knots K_n. We show that if $\ell \neq 0$, then we can bound $V_0(J_n)$ below in terms of $V_0(K_n) - V_0(-K_n)$, which will establish Theorem 1.9. The desired inequality will follow from the construction of a certain negative-definite cobordism whose incoming end is positive surgery on J_n. The outgoing end of our cobordism will be the connected sum of three pieces: positive surgery on K_n, negative surgery on K_n, and a third fixed manifold that is independent of K_n. A similar cobordism was considered in [32]. We provide an elementary discussion of this technique in the next subsection.

3.2 | Construction of the cobordism

Fix any nonzero integer M. In Figure 12, we have displayed an alternative surgery diagram for M-surgery on $J_{K,\widetilde{\mu}}$. This consists of a copy of J with surgery coefficient M, together with K, τK,

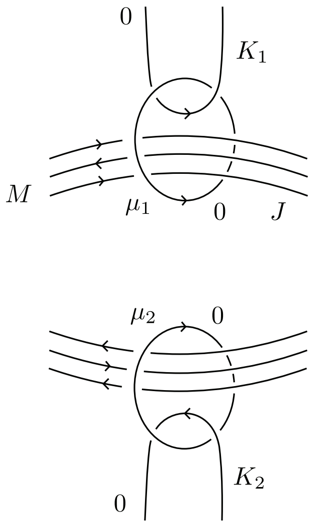

FIGURE 12 A surgery diagram for M-surgery on $J_{K,\widetilde{\mu}}$. In this example, $\ell = 1$.

μ, and $\tau\mu$, all with surgery coefficient zero. For convenience, we denote these by K_1, K_2, μ_1, and μ_2, respectively. For ease of bookkeeping, we give K_2 and μ_2 the reversed orientation as compared to their pushforward orientations under τ. (Note that in the discussion of Subsection 2.2, both of these are given the pushforward orientation. Hence, we may simultaneously reverse orientation on both at no cost.) Then

$$lk(\mu_i, K_i) = 1 \quad \text{and} \quad lk(\mu_i, J) = \ell.$$

for $i = 1, 2$.

To see that the diagram of Figure 12 is correct, slide the strands of J that pass through μ_i over K_i, via $J \mapsto J - \ell K_1 - \ell K_2$. This changes J into $J_{K,\widetilde{\mu}}$ (with surgery coefficient M) and unlinks J from each μ_i. We then use μ_i to separate K_i from the rest of the diagram and delete both pairs μ_i and K_i.

Now fix any pair of integers N_1 and N_2. Construct a cobordism W from M-surgery on $J_{K,\widetilde{\mu}}$ by attaching a pair of 2-handles along the curves γ_1 and γ_2 indicated on the left in Figure 13. These have framings $-N_1$ and $-N_2$, respectively. For concreteness, we orient γ_1 and γ_2 such that the nonzero linking numbers are given by

$$lk(\gamma_i, \mu_i) = 1 \quad \text{and} \quad lk(\gamma_i, K_i) = -N_i.$$

The outgoing end of this cobordism is homeomorphic to the connected sum

$$S^3_{N_1}(K) \# Y(J, \mu, M) \# S^3_{N_2}(K),$$

where $Y(J, \mu, M)$ is a 3-manifold that depends only on J, μ, M, N_1 and N_2, although we suppress the dependence on N_1 and N_2. (In particular, it does not depend on K.) To see this, slide K_i over γ_i, as in Figure 13.

We now investigate under what conditions this cobordism is negative definite:

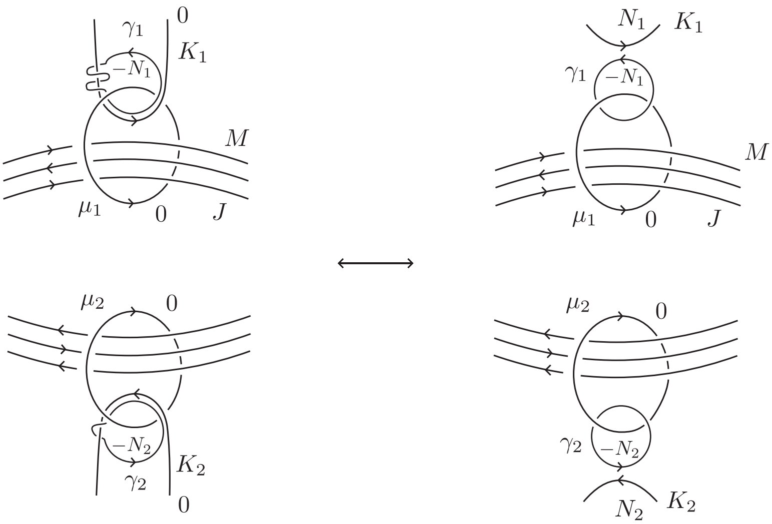

FIGURE 13 Left: Constructing a cobordism by attaching a pair of 2-handles along the curves γ_1 and γ_2. Right: The outgoing end of this cobordism. The equivalence between the left- and right-hand pictures is most easily envisioned by going from right-to-left, in which case the map is given by sliding K_i over γ_i via $K_i \mapsto K_i + \gamma_i$. The map from left-to-right corresponds to the slide $K_i \mapsto K_i - \gamma_i$. The 3-manifold $Y(J, \mu, M)$ is surgery on the sublink on the right consisting of J, μ_i, and γ_i. In this example, $\ell = 1$, $N_1 = 2$, and $N_2 = -1$.

Lemma 3.3. *The cobordism of Figure 13 is negative definite if and only if*

$$\ell^2(N_1^2 + N_2^2) - (N_1 + N_2)M > 0 \quad \text{and} \quad MN_1N_2(-\ell^2(N_1 + N_2) + M) > 0.$$

Proof. As $M \neq 0$, the incoming end of the cobordism is a rational homology sphere with first homology $\mathbb{Z}/M\mathbb{Z}$. Over $\mathbb{Q}$, the second homology of the cobordism is thus clearly of rank two and is generated by the cores of the 2-handles attached along γ_1 and γ_2. However, in order to calculate the self-intersections of these generators, we must perform handleslides on the γ_i to algebraically unlink them from the rest of the diagram. In addition, the γ_i are not null-homologous in general; instead, we are only guaranteed that they are M-torsion in first homology.

We thus instead consider the curves

$$C_1 = M\gamma_1 + (N_1\ell^2 - M)K_1 + MN_1\mu_1 - N_1\ell J + N_1\ell^2 K_2$$

$$C_2 = M\gamma_2 + (N_2\ell^2 - M)K_2 + MN_2\mu_2 - N_2\ell J + N_2\ell^2 K_1.$$

These are obtained by sliding $M\gamma_1$ and $M\gamma_2$ over the other curves in the diagram to make them algebraically unlinked from the left-hand side of Figure 13. Indeed, the reader should verify that the linking numbers between C_1 and the five curves K_1, μ_1, J, μ_2, and K_2 are zero, and similarly for C_2. For convenience, we recall that the nonzero linking numbers are given by

$$lk(\mu_i, K_i) = lk(\mu_i, \gamma_i) = 1, \quad lk(\mu_i, J) = \ell, \quad \text{and} \quad lk(\gamma_i, K_i) = -N_i,$$

and

$$lk(\gamma_i, \gamma_i) = -N_i \quad \text{and} \quad lk(J, J) = M.$$

The self-linking of C_1 is given by

$$\begin{aligned} lk(C_1, C_1) &= M lk(C_1, \gamma_1) \\ &= M\big(M(-N_1) + (N_1\ell^2 - M)(-N_1) + MN_1\big) \\ &= MN_1(M - N_1\ell^2). \end{aligned}$$

Similarly, $lk(C_2, C_2) = MN_2(M - N_2\ell^2)$. Meanwhile,

$$lk(C_1, C_2) = M lk(C_1, \gamma_2) = -MN_1N_2\ell^2.$$

Thus, the intersection form of the cobordism is proportional to

$$\begin{pmatrix} N_1(M - N_1\ell^2) & -N_1N_2\ell^2 \\ -N_1N_2\ell^2 & N_2(M - N_2\ell^2) \end{pmatrix}.$$

This has characteristic polynomial

$$t^2 + \big(\ell^2(N_1^2 + N_2^2) - (N_1 + N_2)M\big)t + \big(MN_1N_2(-\ell^2(N_1 + N_2) + M)\big).$$

The roots of the characteristic polynomial are simultaneously negative if and only if their sum is negative and their product is positive; that is,

$$\ell^2(N_1^2 + N_2^2) - (N_1 + N_2)M > 0 \quad \text{and} \quad MN_1N_2(-\ell^2(N_1 + N_2) + M) > 0,$$

as desired. □

In our situation, we will be interested in large positive surgery on $J_{K,\widetilde{\mu}}$, as $V_0(J_{K,\widetilde{\mu}})$ is (up to an overall shift) given by the d-invariant of such a manifold. We thus assume that M is positive. Suppose in addition that $\ell \neq 0$. Then we have:

Lemma 3.4. *Let $M > 0$ and $\ell \neq 0$. Then for any $N_1 \gg 0$ and $N_2 < 0$ with N_2 small in magnitude compared to N_1, the cobordism W is negative definite.*

Proof. Under the hypotheses of the lemma, the conditions of Lemma 3.3 are equivalent to

$$N_1^2 + N_2^2 - (N_1 + N_2)\frac{M}{\ell^2} > 0 \quad \text{and} \quad N_1N_2\left(-(N_1 + N_2) + \frac{M}{\ell^2}\right) > 0.$$

The first condition is clearly satisfied as long as at least one of N_1 and N_2 is sufficiently large in magnitude. In addition, if N_1 is sufficiently positive compared to the magnitude of N_2, then the factor $-(N_1 + N_2) + M/\ell^2$ in the second condition is negative. As $N_1 > 0$ and $N_2 < 0$, this gives the claim. □

Remark 3.5. Let P be any pattern with winding number zero. The meridian μ of P lies on the boundary of the solid torus $S^1 \times D^2$ for P and thus inherits a normal framing as a curve on $\partial(S^1 \times D^2)$. Lifting this normal framing to the branched double cover defines a pushoff of $\widetilde{\mu}$, which we denote by $\widetilde{\mu}'$. In [32], the authors consider the rational linking number $lk(\widetilde{\mu}, \widetilde{\mu}')$ and impose the condition $lk(\widetilde{\mu}, \widetilde{\mu}') \neq 0$ as a hypothesis of [32, Theorem 3]. In our context, $\widetilde{\mu}'$ may be obtained by taking the Seifert framing of $\widetilde{\mu}$ before surgering along J in Figure 13; the quantity $lk(\widetilde{\mu}, \widetilde{\mu}')$ is the rational linking number of $\widetilde{\mu}$ and $\widetilde{\mu}'$ in the surgered manifold. It is easily checked that this is nonzero if and only if $\ell = lk(\widetilde{\mu}, J)$ is nonzero. Thus, the linking number requirement we impose in this paper is the same as that of [32]; moreover, this characterization shows that the condition $\ell \neq 0$ depends only on P (and not the choice of unknotting tangle replacement).

3.3 | Completion of proof

We now finally conclude the proof of Theorem 1.9. We recall the statement for the convenience of the reader:

Theorem 1.9. *Let P be a proper rational unknotting number one pattern with nonzero linking number and $p/q > 0$. If $\{K_n\}_{n\in\mathbb{N}}$ is any family of knots such that $V_0(K_n) - V_0(-K_n) \to \infty$ as $n \to \infty$, then $\{P(K_n)\}_{n\in\mathbb{N}}$ has infinite rank.*

Proof. Fix any positive integer M and integers N_1 and N_2 satisfying the conditions of Lemma 3.4. It will be useful to assume that M, N_1, and N_2 are odd. We obtain a negative-definite cobordism W from

$$S^3_M(J_n) = S^3_M(J_{K_n,\widetilde{\mu}})$$

to

$$S^3_{N_1}(K_n)\#Y\#S^3_{N_2}(K_n),$$

where $Y = Y(J, \mu, M)$ is independent of K_n. We claim that there exists a spin^c-structure $\mathfrak{s}$ on W that restricts to $[0]$ on $S^3_M(J_n)$ and $[0]$ on both factors $S^3_{N_1}(K_n)$ and $S^3_{N_2}(K_n)$. This can be shown in many ways. For example, note that as M, N_1, and N_2 are odd, the spin^c-structures on $S^3_M(J_n)$, $S^3_{N_1}(K_n)$, and $S^3_{N_2}(K_n)$ are parameterized by the Chern classes of their determinant line bundles. Let $\mathfrak{t}$ be any spin^c-structure on W with determinant line bundle L. Let

$$2E + 1 = MN_1N_2.$$

Then the tensor product $\mathfrak{s} = \mathfrak{t} \otimes L^E$ is a spin^c-structure on W with first Chern class $(2E + 1)c_1(L)$; this trivially vanishes when restricted to $S^3_M(J_n)$, $S^3_{N_1}(K)$, and $S^3_{N_2}(K)$. We thus obtain an equality of d-invariants:

$$d\big(S^3_M(J_n), [0]\big) + \Delta(W, \mathfrak{s}) \leqslant d\Big(S^3_{N_1}(K_n), [0]\Big) + d(Y, \mathfrak{s}|_Y) + d\Big(S^3_{N_2}(K_n), [0]\Big). \tag{1}$$

Here, $\Delta(W, \mathfrak{s})$ is the Heegaard Floer grading shift associated to W and $\mathfrak{s}$. Crucially, note that $\Delta(W, \mathfrak{s})$ and $d(Y, \mathfrak{s}|_Y)$ do not depend on the index n.

Now, as M and N_1 are positive, we have the standard equality

$$d(S_M(J_n),[0]) = \frac{M-1}{4} - 2V_0(J_n) \quad \text{and} \quad d(S_{N_1}(K),[0]) = \frac{N_1-1}{4} - 2V_0(K_n).$$

As N_2 is negative, we have

$$d(S_{N_2}(K_n),[0]) = -d(S_{-N_2}(-K_n),[0]) = \frac{N_2+1}{4} + 2V_0(-K_n).$$

Substituting these into our inequality (1) for d-invariants and collecting terms, we obtain

$$V_0(J_n) \geqslant V_0(K_n) - V_0(-K_n) + C,$$

where C is a constant not depending on n. Hence, we see that the condition $V_0(K_n) - V_0(-K_n) \to \infty$ in fact guarantees $V_0(J_n) \to \infty$. Applying Theorem 3.2 then gives the result. □

Remark 3.6. The reader may wonder whether the condition $\ell \neq 0$ is necessary. This is crucial for the argument: note that if $\ell = 0$, then the conditions of Lemma 3.3 become

$$-(N_1+N_2)M > 0 \quad \text{and} \quad M^2N_1N_2 > 0.$$

If $M > 0$, these conditions are only satisfied when N_1 and N_2 are both less than zero. In this case, the resulting inequality bounds $V_0(J_n)$ below by a constant plus $-2V_0(-K_n)$, which is not generally useful (as V_0 is positive). Similarly, the reader may wonder whether more judicious choices for N_1 and N_2 might produce different inequalities. For example, if we could choose N_1 and N_2 to both be positive, we would bound $V_0(J_n)$ below by a constant plus $2V_0(K_n)$. Unfortunately, this is also impossible: if N_1 and N_2 are positive, the conditions of Lemma 3.3 become

$$N_1^2 + N_2^2 - (N_1+N_2)\frac{M}{\ell^2} > 0 \quad \text{and} \quad -(N_1+N_2) + \frac{M}{\ell^2} > 0.$$

It is straightforward to verify that this is impossible.

This immediately completes the proof of Theorem 1.7:

Theorem 1.7. *Let P be a proper rational unknotting number one pattern with nonzero linking number. Then P is rank-expanding. More specifically, if K is any knot such that $V_0(nK) - V_0(-nK) \to \infty$ as $n \to \infty$, then P is rank-expanding along K.*

Proof. Let P be a proper rational unknotting number one pattern with $\ell \neq 0$. If P is positive, then setting $K_n = nK$ and applying Theorem 1.9 immediately gives the claim. Otherwise, consider the (mirrored, orientation-reversed) pattern $-P$. This is also a proper rational unknotting number one pattern with $\ell \neq 0$; considering the branched double cover shows that $-P$ is positive. Again setting $K_n = nK$, Theorem 1.9 implies $(-P)(nK)$ (for $n > 0$) has infinite rank. But this means $P(-nK)$ (for $n > 0$) has infinite rank. Hence, P is certainly rank-expanding along $\{nK\}_{n\in\mathbb{Z}}$. □

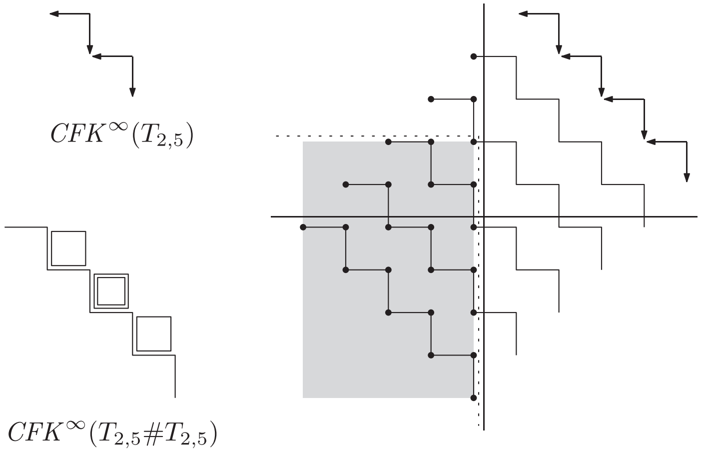

FIGURE 14 Top left: The knot Floer complex of the thin knot $T_{2,5}$; $\tau(T_{2,5}) = 2$. Bottom left: the knot Floer complex of $T_{2,5}\#T_{2,5}$ (after a change of basis). Right: The subcomplex of $CFK^\infty(T_{2,5}\#T_{2,5})$ spanned by the staircase generators. We have schematically depicted $(A_s^-)'$ and $(B_s^-)'$ for $s = 2$. The generators in $(B_s^-)'$ are drawn as dots (i.e., to the left of the vertical axis); the generators in $(A_s^-)'$ are the subset of these dots lying in the shaded region.

4 | PROOF OF THEOREM 1.10

We now turn to the proof of Theorem 1.10.

Theorem 1.10. *Let P be a p/q-rational tangle pattern with $p/q > 0$.*

(1) *Suppose q is even. Let $\{K_n\}_{n\in\mathbb{N}}$ be any family of thin knots with $\tau(K_n)$ distinct and greater than $\lfloor(\lfloor p/q\rfloor + 1)/4\rfloor$. Then $\{P(K_n)\}_{n\in\mathbb{N}}$ is linearly independent and in fact spans a $\mathbb{Z}^\infty$-summand of $\mathcal{C}$.*
(2) *Suppose q is odd. Let $\{K_n\}_{n\in\mathbb{N}}$ be any family of thin knots with $\tau(K_n)$ distinct and less than zero. Then $\{P(K_n)\}_{n\in\mathbb{N}}$ is linearly independent and in fact spans a $\mathbb{Z}^\infty$-summand of $\mathcal{C}$.*

Proof. Let P be a p/q-rational tangle pattern with $p/q > 0$. We start by showing that the surgered family $\{S^3_{p/q}(K_n\#K_n)\}_{n\in\mathbb{N}}$ is linearly independent in $\Theta^3_{\mathbb{Z}_2}$.

We first consider the case when q is even. Following the proof of Theorem 1.7, we again analyze the surgery complex afforded by Theorem 2.13 and compare it to some $X_i^\vee$. However, because the knot Floer homology of a thin knot is very simple, in this case we will be able to establish an explicit local equivalence, rather than just an inequality. This will allow us to utilize Theorem 2.10 rather than Theorem 2.12, and thus conclude linear independence.

As before, write $s = [p/2q]$ and denote $A_s^- = A_s^-(K_n\#K_n)$ and $B_s^- = B_s^-(K_n\#K_n)$, so that the ι-complex of $S^3_{p/q}(K_n\#K_n)$ is locally equivalent to the complex $A_s^- \oplus A_s^- \to B_s^-$ defined in Theorem 2.13.

It is a standard fact that if K_n is thin, then the connected sum $K_n\#K_n$ is also thin. Hence, the knot Floer complex of $K_n\#K_n$ consists of a step-length-one staircase, together with a number of side-length-one boxes, as schematically displayed on the left in Figure 14. The staircase has total height $2\tau(K_n)$. The fact that $\tau(K_n) > 0$ (together with the fact that $\tau(K_n\#K_n) = 2\tau(K_n)$ is even) shows that the staircase opens toward the south-west, as in Figure 14.

We argue that up to local equivalence, we can successively simplify $CFK^\infty(K_n\#K_n)$ until it is the same as some $X_i^\vee$. The first simplification is as follows. Consider the subcomplex of $CFK^\infty(K_n\#K_n)$ spanned by the staircase generators. Let the intersection of this subcomplex with A_s^- be denoted $(A_s^-)'$, and define $(B_s^-)'$ similarly. (See the right of Figure 14.) This gives an obvious subcomplex $(A_s^-)' \oplus (A_s^-)' \to (B_s^-)'$ of $A_s^- \oplus A_s^- \to B_s^-$. The inclusion and projection maps for this subcomplex are easily checked to be local equivalences, so without loss of generality we may replace the original complex $A_s^- \oplus A_s^- \to B_s^-$ with $(A_s^-)' \oplus (A_s^-)' \to (B_s^-)'$.

An examination of Figure 14 shows that $H_*((A_s^-)')$ and $H_*((B_s^-)')$ are both copies of $\mathbb{F}[U]$. Thus, the associated graded complex

$$H_*((A_s^-)') \oplus H_*((A_s^-)') \to H^*((B_s^-)')$$

with the induced map $v_* \oplus v_*$ is certainly isomorphic to a grading-shifted copy of $X^\vee_{V_s(K_n\#K_n)}$. It is moreover easy to show that in this case, the associated graded complex is ι-equivariantly homotopy equivalent to the original. Hence, we obtain the desired local equivalence

$$X^\vee_{V_s(K_n\#K_n)}[-d(C_n)] \simeq C_n.$$

An examination of Figure 14 shows

$$V_s(K_n\#K_n) = \lceil(\tau(K_n\#K_n) - s)/2\rceil = \tau(K_n) - \lfloor s/2\rfloor$$

so long as the right-hand side is positive, and $V_s(K_n\#K_n) = 0$ otherwise. (Consider the copy of the staircase on the right of Figure 14 that intersects the left-half plane in a single dot. Then $V_s(K_n\#K_n)$ is the number of diagonal translations needed for this staircase to intersect the shaded region.) Some numerological casework shows that

$$\lfloor s/2\rfloor = \lfloor[p/(2q)]/2\rfloor = \lfloor(\lfloor p/q\rfloor + 1)/4\rfloor.$$

The hypotheses of the theorem thus imply that the ι-complexes of the $S^3_{p/q}(K_n\#K_n)$ are locally equivalent to grading-shifted copies of $X_i^\vee$, with i positive and distinct. As these are linearly independent in $\mathfrak{I}$, this completes the proof.

We now turn to the case when q is odd. By Theorem 2.13, the ι-complex of $S^3_{p/q}(K_n\#K_n)$ is locally equivalent to the large surgery complex $(A_0^-(K_n\#K_n), \iota_K)$. We attempt to understand $A_0^-(K_n\#K_n)$ explicitly. Much of this computation follows from [27, section 8], so we will be brief.

In [27, section 8], Hendricks and Manolescu calculate the ι_K-complex of all thin knots. Their result shows that up to local equivalence, the ι_K-complex of a thin knot is locally equivalent either to a staircase or a staircase plus a single side-length-one box. These possibilities are displayed in Figure 15; note that we now assume $\tau(K_n\#K_n)$ is negative. In the former case, the action of ι_K is the obvious reflection map on the staircase generators. In the latter, we have the slight modification (in the notation of Figure 15):

$$\iota_K d = d + b, \quad \iota_K b = b + e, \quad \iota_K c = c' + a', \quad \text{and} \quad \iota_K c' = c + a,$$

with ι_K acting by reflection on all other generators. We abuse notation and write A_0^- for this simplified representative of the local equivalence class of $A_0^-(K_n\#K_n)$.

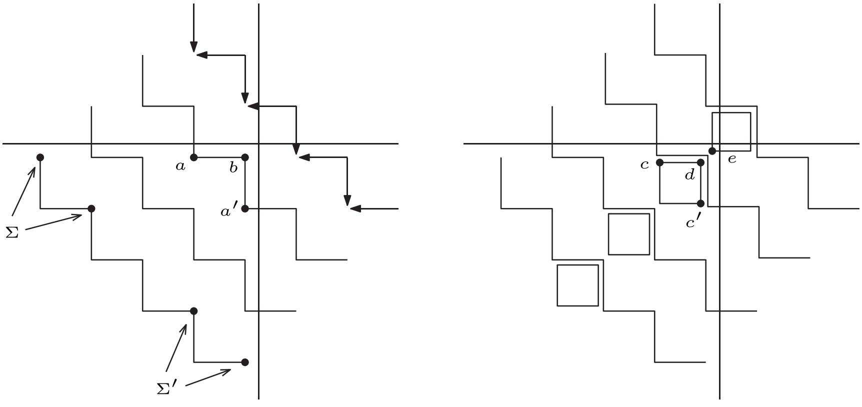

FIGURE 15 Left: Staircase with no box, with generators a, a' and b labeled. We have also labeled sums-of-generators Σ and Σ'. To define these, consider the first copy of the staircase contained in the lower-left quadrant. Note that there are an odd number of noncycle generators in this staircase. Let Σ be the sum of such generators in the (strict) upper-half of this staircase and Σ' be the reflection of Σ. Right: Staircase with a single box; several further generators labeled. (The generators labels on the left are meant to carry over in the obvious way.)

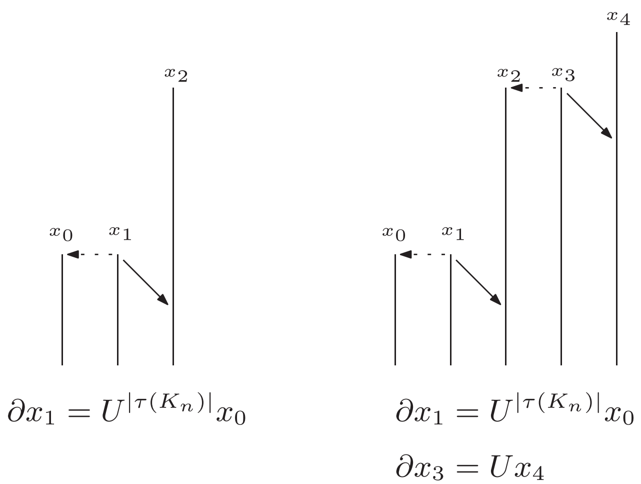

FIGURE 16 Left: An ι-complex with three generators. The dashed arrow represents the action of $\omega = 1 + \iota$; here $\omega x_1 = x_0$ and otherwise vanishes. The solid arrow represents ∂; here $\partial x_1 = U^{|\tau(K_n)|} x_2$ and otherwise vanishes. Right: An ι-complex with five generators. Here $\omega x_1 = x_0$ and $\omega x_3 = x_2$. The differential is given by $\partial x_1 = U^{|\tau(K_n)|} x_2$ and $\partial x_3 = U x_4$.

We show that for a staircase with no box, (A_0^-, ι_K) is homotopy equivalent to (a grading-shifted copy of) $X^\vee_{|\tau(K_n)|}$. For this, consider the subcomplex S of A_0^- spanned over $\mathbb{F}[U]$ by a, a', and b, together with the sums-of-generators Σ and Σ'. The reader may check that this is a ι_K-equivariant subcomplex of A_0^- which is homotopy equivalent to the original. Moreover, we claim that S is homotopy equivalent to the complex on the left in Figure 16. One direction of this homotopy equivalence is given by the map

$$f(x_0) = \Sigma + U^{|\tau(K_n)|} b + \Sigma', \quad f(x_1) = \Sigma, \quad \text{and} \quad f(x_2) = a.$$

This does not intertwine ι_K with the ι-action in Figure 16 on the nose, but if we set

$$H(x_0) = 0, \quad H(x_1) = 0, \quad \text{and} \quad H(x_2) = b$$

then $f\iota + \iota_K f = \partial H + H\partial$. We leave it to the reader to produce the homotopy equivalence in the other direction. A quick change-of-basis shows that up to grading shift, the left-hand complex in Figure 16 is precisely $X^\vee_{|\tau(K_n)|}$, giving the claim. For further discussion, see [14, Example 2.6].

We now turn to understanding the case of a staircase with box. In this case, it turns out that the (A_0^-, ι_K) is not locally equivalent to a copy of $X_i^\vee$. However, it is still possible to understand its local equivalence class. To see, this we modify our subcomplex S from before by additionally including the generators c, c', d, and e, as displayed on the right in Figure 16. Once again, the reader can check that S is a subcomplex of A_0^- which is homotopy equivalent to the original. We further claim that S is homotopy equivalent to the complex on the right in Figure 16. To see this, map

$$f(x_0) = \Sigma + U^{|\tau(K_n)|} b + \Sigma', \quad f(x_1) = \Sigma, \quad f(x_2) = a, \quad f(x_3) = c', \quad \text{and} \quad f(x_4) = e.$$

Setting

$$H(x_0) = U^{|\tau(K_n)|-1} c, \quad H(x_1) = 0, \quad H(x_2) = b, \quad H(x_3) = d, \quad \text{and} \quad H(x_4) = 0,$$

exhaustive evaluation on each generator gives $f\iota + \iota_K f = \partial H + H\partial$. We leave it to the reader to produce the map in the other direction.

After a change-of-basis, the complex on the right of Figure 16 is locally equivalent (up to grading shift) to

$$X^\vee_{|\tau(K_n)|} \otimes X_1^\vee.$$

The relevant computation follows from [18, Lemma 5.2]; see also [14, Theorem 8.1]. Thus, for each n, we have that the ι-complex of $S^3_{p/q}(K_n \# K_n)$ is locally equivalent to a grading-shifted copy of either $X^\vee_{|\tau(K_n)|}$ or $X^\vee_{|\tau(K_n)|} \otimes X_1^\vee$. As the $\tau(K_n)$ are distinct, this gives the claim.

Finally, we may upgrade the statement of linear independence to the spanning of a $\mathbb{Z}^\infty$-summand using the work of [14]. In [14], the authors construct an infinite family of linearly independent homomorphisms from $\Theta^3_{\mathbb{Z}_2}$ to $\mathbb{Z}$, factoring through the homomorphism $h : \Theta^3_{\mathbb{Z}_2} \to \mathfrak{I}$. More precisely, they construct an algebraically defined auxiliary group $\widehat{\mathfrak{I}}$ with a homomorphism $\widehat{h} : \mathfrak{I} \to \widehat{\mathfrak{I}}$ and define a family of linearly independent homomorphisms

$$\phi_n : \widehat{\mathfrak{I}} \to \mathbb{Z}.$$

Composing everything with the double branched cover homomorphism, we obtain a linearly independent family of homomorphisms

$$\mathcal{C} \xrightarrow{\Sigma_2} \Theta^3_{\mathbb{Z}_2} \xrightarrow{h} \mathfrak{I} \xrightarrow{\widehat{h}} \widehat{\mathfrak{I}} \xrightarrow{\phi_n} \mathbb{Z}.$$

Moreover, in [14] it is shown that $\phi_i(\widehat{h}(X_j)) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. This suffices to establish the claim. □

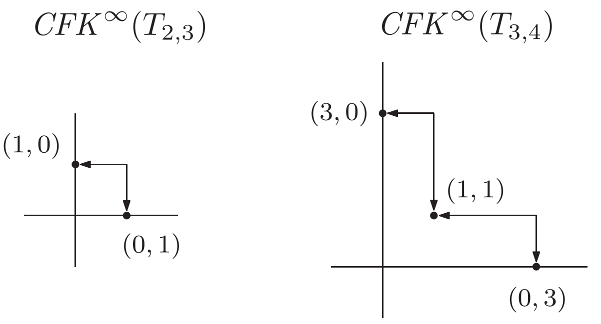

FIGURE 17 The knot Floer complexes of $T_{2,3}$ and $T_{3,4}$.

5 | THIN KNOTS AND L-SPACE KNOTS

We now turn to some applications and examples of Theorems 1.7 and 1.9. The first order of business is to understand the general condition $V_0(K_n) - V_0(-K_n) \to \infty$ of Theorem 1.9. In practice, $V_0(-K_n)$ may often be known to be bounded, or even zero; for example, if K is a positive L-space knot, or K is a thin knot with $\tau(K) > 0$. Moreover, if K_1 and K_2 have $V_0(-K_1) = V_0(-K_2) = 0$, then by the sub-additivity of V_0 their connected sum has this property also. Thus, if we assume that the family $\{K_n\}_{n\in\mathbb{N}}$ is drawn from the monoid generated by positive L-space knots, or the monoid of thin knots with $\tau(K) > 0$, then the hypothesis of Theorem 1.9 simplifies to $V_0(K_n) \to \infty$.

It is also natural to search for related conditions that do not explicitly reference any Floer-theoretic invariants. For the class of thin knots, this is straightforward: if K is thin, then $V_0(K) = \max\{0, \lceil \tau(K)/2 \rceil\}$. Moreover, for all known examples of thin knots (including all alternating and quasi-alternating knots), we have $\tau(K) = -\sigma(K)/2$. In this case we may thus re-write the hypothesis of Theorem 1.9 as $\sigma(K_n) \to -\infty$.

In the setting of L-space knots, finding a topological condition is slightly more involved. For this, it will be helpful for us to recall the work of Borodzik and Livingston [1] concerning the calculation of the V_0-invariant of a connected sum of positive L-space knots. We describe their algorithm below. Let $C_1, \dots, C_n$ be a sequence of positive staircase complexes corresponding to positive L-space knots $K_1, \dots, K_n$. For each i, place C_i in the first quadrant and let S represent the set of all generators of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in C_1 \otimes C_2 \otimes \cdots \otimes C_n$, where each a_i varies among the generators of C_i with Maslov grading zero. Let $\underline{S}$ be the set of pairs of integers obtained from the bigradings of the generators in S. From [1, Proposition 5.1], we have:

$$V_0(K_1 \# K_2 \# \cdots \# K_n) = \min_{(\alpha,\beta)\in\underline{S}} \{\max(\alpha, \beta)\}.$$

For the convenience of the reader, we include a sample computation:

Example 5.1. Let $K = T_{2,3} \# T_{2,3}$. Placing the knot Floer complex of $T_{2,3}$ in the first quadrant, as in Figure 17, we see that there are two generators with Maslov grading 0. Let us denote the generator in bigrading (1,0) by a and the generator in bigrading (0,1) by b. The elements of S are then given by $\{a \otimes a, a \otimes b, b \otimes a, b \otimes b\}$. Hence, we get

$$\min_{(\alpha,\beta)\in\underline{S}} \{\max(\alpha, \beta)\} = \min_{(\alpha,\beta)\in\underline{S}} \{2, 1, 1, 2\} = 1.$$

Thus, $V_0(K) = 1$.

We introduce the following two (rather trivial) lemmas:

Lemma 5.2. *Let K be a positive L-space knot and n be the number of nonzero terms in the Alexander polynomial Δ_K. Then*

$$V_0(K) \geqslant \lfloor (n-1)/4 \rfloor.$$

Proof. As is well-known to experts in Floer theory, it is straightforward to determine the knot Floer complex of K (and thus the value of V_0) from Δ_K in the case that K is an L-space knot [50]. Let

$$\Delta_K = (-1)^m + \sum_{i=1}^{m} (-1)^{m-i}(t^{n_i} + t^{-n_i})$$

for $0 < n_1 < n_2 < \cdots < n_m$. Then (as is recorded in [27, section 7]),

$$V_0(K) = n_m - n_{m-1} + \cdots + (-1)^{m-2} n_2 + (-1)^{m-1} n_1.$$

As each pair $n_k - n_{k-1}$ is at least one, we of course have $V_0(K) \geqslant \lfloor m/2 \rfloor = \lfloor (n-1)/4 \rfloor$. □

Lemma 5.3. *Let $K_1, \ldots, K_n$ be any collection of positive L-space knots. Then*

$$V_0(K_1 \# \cdots \# K_n) \geqslant \max\{[n/2], V_0(K_1), \ldots, V_0(K_n)\}.$$

Proof. It is easy to check that

$$V_0(K_1 \# \cdots \# K_n) \geqslant V_0(K_i)$$

for each i. Indeed, by sub-additivity,

$$V_0(K_i) - V_0(-K_1 \# \cdots \# -K_{i-1} \# -K_{i+1} \# \cdots \# -K_n) \leqslant V_0(K_1 \# \cdots \# K_n),$$

but the second term is zero. The claim that $V_0(K_1 \# \cdots \# K_n)$ is at least $\lfloor n/2 \rfloor$ follows from work of Borodzik and Livingston [1]. Indeed, take an arbitrary element $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in S$ and assume that the (i,j)-coordinates of a_l is (x_l, y_l). Let k be the number of times the i-coordinate of (x_l, y_l) is 0. Now observe that

$$\sum_{l=1}^{n} x_l \geqslant n - k \quad \text{and} \quad \sum_{l=1}^{n} y_l \geqslant k.$$

Hence, we obtain

$$\max\left\{ \sum_{l=1}^{n} x_l, \sum_{l=1}^{n} y_l \right\} \geqslant \left[\frac{n}{2} \right].$$

As the choice of the element $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in S$ was arbitrary, we obtain,

$$V_0(K_1 \# K_2 \# \cdots \# K_n) \geqslant [n/2].$$

completing the proof. □

Now suppose the family $\{K_n\}_{n\in\mathbb{N}}$ in Theorem 1.9 is drawn from the monoid generated by positive L-space knots. By Lemmas 5.2 and 5.3, it follows that $\{P(K_n)\}_{n\in\mathbb{N}}$ has infinite rank so long as either:

(1) the number of summands in K_n is unbounded as $n \to \infty$; or
(2) the set of Alexander polynomial lengths (that is, number of nonzero terms in each Alexander polynomial) occurring among summands of the K_n is unbounded as $n \to \infty$.

This leads to the following:

Corollary 5.4. *Let P be a proper rational unknotting number one pattern with $\ell \neq 0$. Let $\mathcal{M}$ be the monoid of positive linear combinations of (right-handed) torus knots. If $\{K_n\}_{n\in\mathbb{N}}$ is any infinite subset of $\mathcal{M}$, then $\{P(K_n)\}_{n\in\mathbb{N}}$ has infinite rank.*

Proof. As $\mathcal{M}$ is a submonoid of the monoid generated by positive L-space knots, it suffices to show that any infinite family of elements of $\mathcal{M}$ must satisfy either (1) or (2) above. Suppose $\{K_n\}_{n\in\mathbb{N}}$ does not satisfy (1). Then the set of distinct individual torus knots $T_{p,q}$ that appear as summands in the K_n must be infinite, and in particular have unbounded indices. (That is, either p is unbounded or q is unbounded.)

In [56], it is shown that the number of nonzero terms in $\Delta_{T_{p,q}}$ is given by $vx + uy$, where x, y, u, and v are positive integers such that $vx - uy = 1$, $p = x + y$, and $q = u + v$. As u and v are at least one, we certainly have $vx + uy \geqslant x + y = p$. Similarly, $vx + uy \geqslant v + u = q$. Hence, any infinite family of torus knots has unbounded Alexander polynomial length (as measured by number of nonzero terms). □

For the patterns considered in [32], it follows from the proof [32, Theorem 3] that any infinite family of distinct torus knots has infinite-rank image. In Corollary 5.4, we extend this by allowing the family of companion knots to be drawn from sums of torus knots, rather than individual torus knots.

We now prove the applicability of Theorem 1.7 to the following three families of knots discussed in the introduction.

(1) K is any L-space knot, such as a torus knot or algebraic knot, or any linear combination of such knots of the same sign/handedness.
(2) K is any thin knot with $\tau(K) \neq 0$, such as a (quasi-)alternating knot of nonzero signature.
(3) K is any linear combination of genus one knots such that the overall connected sum satisfies $\tau(K) \neq 0$.

The first and second families are immediate from the discussion of this section; note that if K is a positive L-space knot, then $V_0(nK) \geqslant [n/2]$ by Lemma 5.3. Thus, the only nontrivial case is the third claim.

Using the concordance invariant ν^+ introduced by Hom and Wu in [33], Hom [31] and Kim and Park [34] defined an equivalence relation on the set of knot Floer complexes called ν^+*-equivalence*.

The ν^+-equivalence class of a knot is a concordance invariant that is well-defined with respect to connected sums/tensor products. Moreover, the numerical invariants $V_0(K)$ and $\tau(K)$ may both be computed from the ν^+-equivalence class of $CFK(K)$. (Note that ν^+-equivalence is the same as the stable equivalence of [31].) In [54], Sato determined the ν^+-equivalence class of all genus one knots. According to [54, Theorem 1.2], if K is a genus one knot, then

$$CFK^\infty(K) \sim_{\nu^+} \begin{cases} CFK^\infty(T_{2,3}) & \text{if } \tau(K) = 1 \\ CFK^\infty(U) & \text{if } \tau(K) = 0 \\ CFK^\infty(-T_{2,3}) & \text{if } \tau(K) = -1. \end{cases}$$

This implies that if K is a linear combination of genus one knots, then $CFK^\infty(K) \sim_{\nu^+} cT_{2,3}$, where $c = \tau(K)$. Hence, for the purposes of calculation $V_0(nK)$, we may assume that K is $cT_{2,3}$. But $cT_{2,3}$ is a thin knot, for which we have already established the desired claim.

6 | WHITEHEAD DOUBLES

We close by proving the applications to Whitehead doubles mentioned in the introduction. First, note that Corollary 1.4 is an immediate consequence of Theorem 1.10:

Proof of Corollary 1.4. We have $\tau(nT_{2,2k+1}) = nk$; applying Theorem 1.10 gives the claim. □

We thus turn to Corollary 1.6. This is a particularly simple case of the setup of Section 3, in the sense that

$$\Sigma_2(D(K)) = S^3_{1/2}(K\#K^r)$$

for any K. The same application of Theorem 2.13 as in the proof of Theorem 3.1 shows that up to an overall grading shift, the ι-complex C of $\Sigma_2(D(K))$ thus satisfies

$$C^\vee \leqslant X_{V_0(K\#K^r)}.$$

Note that here, $[p/(2q)] = [1/4] = 0$. In contrast to the proof of Theorem 1.9, we forego the negative-definite cobordism of Subsection 3.2 and instead utilize $V_0(K\#K^r)$ directly.

Proof of Corollary 1.6. This is a special case of Theorem 3.1. Let $\{K_n\}_{n\in\mathbb{N}}$ be any family of companion knots and let $C_n^\vee$ be the ι-complex of $\Sigma_2(D(K_n))$. Up to overall grading shift, we have

$$C_n^\vee \leqslant X_{V_0(K_n\#K_n^r)}$$

for each n. If $V_0(K_n\#K_n^r) \to \infty$, then by Theorem 2.12 there exists an infinite linearly independent subset of $\{D(K_n)\}_{n\in\mathbb{N}}$. It thus suffices to find a family of companion knots with $V_0(K_n\#K_n^r) \to \infty$ but each $\tau(K_n) \leqslant 0$. This is provided in Example 6.3. □

The above discussion can also be used to answer a conjecture of the second author and Pinzón-Caicedo [32], who asked whether there is a knot K such that the Whitehead doubles $D(K)$ and $D(-K)$ are nonzero in concordance. We prove the following general condition:

Corollary 6.1. *Let K be any knot with $V_0(K\#K^r) > 0$ and $\tau(K) < 0$. Then $D(K)$ and $D(-K)$ are linearly independent.*

Proof of Corollary 6.1. If $V_0(K\#K^r) > 0$, then a trivial application of the proof of Theorem 2.12 shows that the ι-complex C of $\Sigma_2(D(K))$ is nontorsion. Hence, $D(K)$ is nontorsion in $\mathfrak{C}$. As $\tau(K) < 0$, [20, Theorem 1.4] implies that $\tau(D(K)) = 0$ and $\tau(D(-K)) = 1$. As τ is a homomorphism, this shows that $D(-K)$ is also nontorsion in $\mathfrak{C}$ and that it is linearly independent with $D(K)$. ☐

We now give several infinite families of knots for which $V_0(K\#K^r) > 0$ and $\tau(K) \leqslant 0$. This condition turns out to be fairly common; we give a flexible recipe for constructing a wide class of examples below. Let A and B be a pair of knots such that

$$V_0(2A) > V_0(2B) \quad \text{and} \quad \tau(A) \leqslant \tau(B).$$

Then we claim that $K = A\# - B$ is a knot with the desired properties. To see this, first note that as (noninvolutive) knot Floer homology is insensitive to orientation reversal, we may replace $K\#K^r$ with $2K$. Subadditivity of V_0 then gives the lower bound

$$0 < V_0(2A) - V_0(2B) \leqslant V_0(2A\# - 2B) = V_0(2K),$$

while $\tau(K) = \tau(A) - \tau(B) \leqslant 0$. The advantage of the ansatz $K = A\# - B$ is that if A and B are sums of positive L-space knots (or are locally equivalent to such sums), then the quantities $V_0(2A)$ and $V_0(2B)$ are easily computed via the algorithm of [1, Proposition 5.1] (described in the proof of Lemma 5.3).

Example 6.2. As a first example, we illustrate the above procedure for the example $K = 5T_{2,3}\# - 2T_{3,4}$ of Lewark and Zibrowius [42, Corollary 1.13]. In this case, $A = 5T_{2,3}$ and $B = 2T_{3,4}$. The knot Floer complexes of $T_{2,3}$ and $T_{3,4}$ are displayed in Figure 17. Applying the algorithm of [1, Proposition 5.1] easily shows that $V_0(2A) = V_0(10T_{2,3}) = 5$. (Alternatively, one can use the fact that $10T_{2,3}$ is thin.) Similarly, the algorithm of [1] shows that $V_0(2B) = V_0(4T_{3,4}) = 4$. On the other hand, $\tau(A) = \tau(5T_{2,3}) = 5$ while $\tau(B) = \tau(2T_{3,4}) = 6$.

Many similar examples can be constructed by forming the difference of sums of torus knots in the style of Example 6.2; for instance, $\{n(n-1)T_{2,3}\# - 2T_{n,n+1}\}_{n\geqslant 2}$ or $\{T_{2,2(n^2-n-1)+1}\# - T_{n,n+1}\}_{n\geqslant 2}$. We also provide an example of an infinite family where the companion knots are topologically slice.

Example 6.3. Let $D = D(T_{2,3})$ be the Whitehead double of $T_{2,3}$. For any $n \geqslant 2$ and nonnegative integers p and q, let

$$K_{n,p,q} = p(T_{2,2n+1} - D_{2,2n+1}) + qD.$$

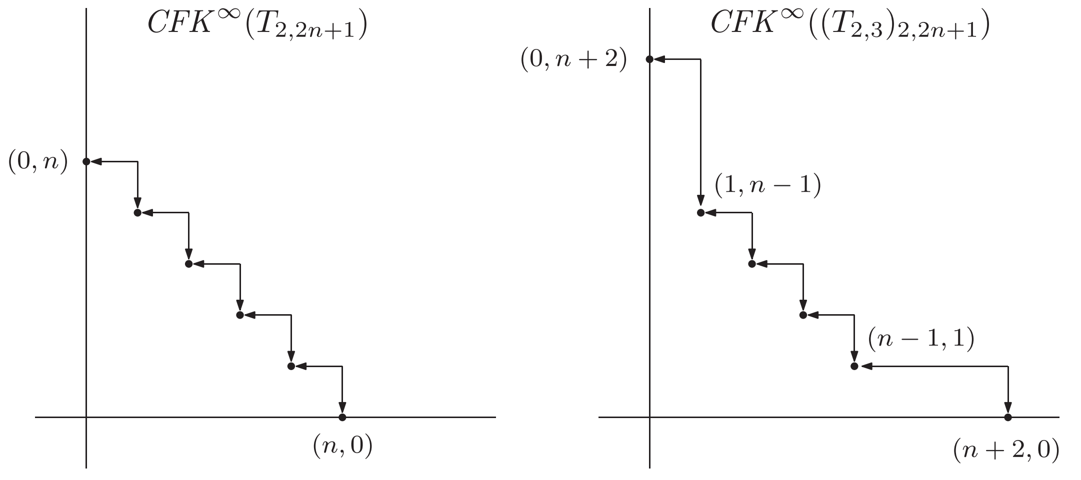

FIGURE 18 The knot Floer complexes of $T_{2,2n+1}$ and $(T_{2,3})_{2,2n+1}$. The longer vertical arrows on the right have length three, while all other arrows have length one; note that $n \geqslant 2$. Here we have drawn the case of $n = 5$.

We claim that $K_{n,p,q}$ is topologically slice. To see this, note that D is topologically concordant to the unknot; hence its $(2, 2n + 1)$-cable $D_{2,2n+1}$ is topologically concordant to $T_{2,2n+1}$. Thus, both of the summands $p(T_{2,2n+1} - D_{2,2n+1})$ and qD above are topologically slice.

We fit the family $K_{n,p,q}$ into our ansatz by setting

$$A = pT_{2,2n+1} + qD \quad \text{and} \quad B = pD_{2,2n+1}.$$

Although the knots D and $D_{2,2n+1}$ are somewhat complicated, their knot Floer complexes are locally equivalent to L-space complexes. Indeed, [54, Theorem 1.2] implies that for the purposes of computing V_0, we may replace D with $T_{2,3}$, both in A and in the cable $D_{2,2n+1}$. (For an explanation of the latter, see [29, Proposition 4].) The complexes of $T_{2,2n+1}$ and $(T_{2,3})_{2,2n+1}$ are displayed in Figure 18. To compute the complex of $(T_{2,3})_{2,2n+1}$, we use [21, Theorem 1.10], which implies that $(T_{2,3})_{2,2n+1}$ is an L-space knot. The computation then follows from the behavior of the Alexander polynomial under cabling.

Applying the algorithm of [1, Proposition 5.1] shows that

$$V_0(2A) = V_0(2pT_{2,2n+1} + 2qT_{2,3}) = pn + q \quad \text{and} \quad V_0(2B) = V_0(2p(T_{2,3})_{2,2n+1}) = pn.$$

Hence, $V_0(2A) - V_0(2B) = q$. On the other hand,

$$\tau(A) = \tau(pT_{2,2n+1} + qT_{2,3}) = pn + q \quad \text{and} \quad \tau(B) = \tau(p(T_{2,3})_{2,2n+1}) = p(n + 2),$$

showing that $\tau(K_1) - \tau(K_2) = q - 2p$. We may thus choose any infinite family of (p, q) with $0 < q \leqslant 2p$ and q unbounded to guarantee an infinite linearly independent subset and complete the proof of Corollary 1.6. If strict inequality $q < 2p$ holds, then $D(K_{n,p,q})$ and $D(-K_{n,p,q})$ are moreover linearly independent in each case. Note that if $K_{n,p,q}$ satisfies the above properties, then any positive multiple of $K_{n,p,q}$ does as well; hence we obtain rank-expansion along such $K_{n,p,q}$.

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JOURNAL INFORMATION

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