

# *Algebra & Number Theory*

Volume 19  
2025  
No. 2

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An arc space of an affine cone over a projective toric variety is known to be nonreduced in general. It was demonstrated recently that the reduced scheme structure of arc spaces is very meaningful from algebro-geometric, representation-theoretic and combinatorial points of view. In this paper we develop a general machinery for the description of the reduced arc spaces of affine cones over toric varieties. We apply our techniques to a number of classical cases and explore some connections with representation theory of current algebras.

Introduction	313
1. Preliminaries and generalities	317
2. Symmetric polynomials and dual spaces of graded components	329
3. Arc rings of toric rings	333
4. Inductive construction	344
5. Examples	357
Appendix: Veronese–Segre embeddings	359
Acknowledgements	361
References	361

## Introduction

Let  $V$  be the affine cone over a normal projective toric variety. In this paper we study the arc space over  $V$ . Roughly speaking, the arc space consists of  $\mathbb{k}[[t]]$ -points of  $V$  [Nash 1995; Ein and Mustață 2009; Frenkel 2007; Mustață 2007], where  $\mathbb{k}$  is the base field. If  $V$  is embedded into an affine space with coordinates  $X_1, \dots, X_n$ , then each variable  $X_i$  produces a sequences of variables  $X_i^{(j)}$  with generating series  $X_i(s) = \sum_{j \geq 0} X_i^{(j)} s^j$ . The relations cutting out  $V$  inside the affine space are replaced with an infinite number of relations appearing as coefficients of powers of  $s$ . For example, the relation  $X_1 X_3 - X_2^2$  defining the degree two Veronese curve gives  $X_1(s)X_3(s) - X_2^2(s)$  and one is interested in the ideal, generated by the  $s$ -coefficients of this relation. We denote the arc space by  $J^\infty(V)$ . Arc spaces have attracted a lot of attention (see, e.g., [Anderson and Stapledon 2013; Arakawa and Moreau 2021; Bourqui and Sebag 2019b; de Fernex and Docampo 2020; Gorsky 2013; Ishii 2007]) due to the beauty of the theory and various connections with singularity theory, theory of motivic integration, differential algebra, representation theory, vertex algebras and other mathematical subjects. The particular case of arc spaces

MSC2020: 05E14, 14E18.

Keywords: arc spaces, toric varieties.

of toric varieties was also studied by several authors. For instance, in [Ishii and Kollár 2003] the Nash problem is positively solved for toric varieties; in [Ishii 2004] the orbits of the  $J^\infty((\mathbb{k}^\times)^n)$ -action are studied; in [Bourqui and Sebag 2019a] the finite formal model for toric singularities is constructed; for other results see also [Reguera 2023; Mourtada 2011].

One interesting feature of arc spaces is that  $J^\infty(V)$  may be nonreduced even if  $V$  is. The simplest example relevant to our studies is the cone over the degree three Veronese curve. It is defined inside  $\mathbb{A}^4$  by three quadratic equations on variables  $X_0, X_1, X_2, X_3$ , so the arc space  $J^\infty(V)$  is defined by three series of relations. However, in order to get the reduced scheme structure of  $J^\infty(V)$  one has to add another series of quadratic relations.

Studying the reduced scheme  $J_{\text{red}}^\infty(V)$  is an important problem, related to singularity theory [Mustață 2001], differential algebra [Boulier et al. 1995], representation theory [Feigin and Makedonskyi 2020; Dumanski and Feigin 2023], invariant theory [Linshaw and Song 2024], vertex algebras [Feigin and Makedonskyi 2019], and other areas. Our main motivation for the study of the reduced schemes comes from geometric representation theory due to the connection with the theory of semi-infinite flag varieties (see [Dumanski and Feigin 2023; Dumanski et al. 2021; Feigin and Makedonskyi 2020; Makedonskyi 2022]). In particular, in [Feigin and Makedonskyi 2020; Makedonskyi 2022; Dumanski and Feigin 2023] the complete list of relations is found for the arc space of certain varieties (proving, in particular, that the radical is differentially finitely generated). Let us also mention several other results. In [Boulier et al. 1995] an algorithmic approach to the problem of finding the radical of a differential ideal is developed; in [Sebag 2011; Kpognon and Sebag 2017] the question of reducedness of the arc space of a plane curve is investigated; in [Feigin and Makedonskyi 2019] a vertex algebra structure on a reduced arc-ring for flag varieties is introduced; see also [Bourqui and Haiech 2021; Morán Cañón and Sebag 2020; 2022] for other results.

Let  $V = \text{Spec}(R)$ . We denote by  $J^\infty(R)$  the ring of functions on  $J^\infty(V)$  and by  $J_{\text{red}}^\infty(R)$  the ring of functions on  $J_{\text{red}}^\infty(V)$ . The goal of this paper is two-fold. First, we develop a technique for the study of reduced arc rings for toric varieties in terms of certain spaces of symmetric polynomials. Second, we apply our construction to the description of  $J_{\text{red}}^\infty(R)$  for several classical examples. Let us describe our results in more detail.

We note that since  $V$  is an affine cone over a projective variety, there are two natural gradings on  $J_{\text{red}}^\infty(R)$ : the  $z$ -grading defined by attaching degree 1 to each variable  $X_i^{(j)}$  and the  $q$ -grading attaching degree  $j$  to  $X_i^{(j)}$ . These gradings define a graded character of the arc rings, which are series in  $z$  and  $q$ . Note that this graded character of the rings  $J^\infty(R)$  and  $J_{\text{red}}^\infty(R)$  is of great importance in singularity theory and combinatorics; see [Ait El Manssour and Pogudin 2024; Bai et al. 2020; Mourtada 2014; 2023; Bruscek et al. 2013]. The two main problems we address in this paper are as follows:

- Describe  $J_{\text{red}}^\infty(R)$  in terms of generators and relations.
- Find the graded character of  $J_{\text{red}}^\infty(R)$ .

These problems are resolved for affine cones over certain (not necessarily toric) projective varieties in [Feigin and Makedonskyi 2020; Makedonskyi 2022; Sottile and Sturmfels 2001]. For instance, for

Veronese embeddings of the projective line the  $z$ -homogeneous components of the reduced arc rings are isomorphic to the dual global Demazure modules (see [Dumanski and Feigin 2023]). More generally, in favorable situations the following nice properties hold:

- The ideal of relations is generated in degree two by explicitly given relations.
- For each  $L \geq 0$  the  $q$ -character (the Hilbert series with respect to the grading by  $q$ -degree) of the  $z$ -degree  $L$  homogeneous component of the arc ring admits the factorization  $\chi_L(q)/(q)_L$  for certain polynomial  $\chi_L(q)$ .

(Here  $(q)_L = (1-q) \cdots (1-q^L)$ .) We note that the factor  $1/(q)_L$  is the  $q$ -character of space of symmetric functions in  $L$  variables. In a favorable situation the character has the above form because

- the  $L$ -th  $z$ -homogeneous component of the arc ring admits a free action of the ring of symmetric polynomials in  $L$  variables.

For a normal lattice polytope  $P \subset \mathbb{R}^n$  we denote by  $X(P)$  the corresponding projective toric variety equipped with a projective embedding (see Section 1.1 for definitions and constructions). The variety  $X(P)$  is the projective spectrum of the toric ring  $R(P)$ . The ring  $R(P)$  is known to be generated inside a polynomial ring in  $z_1, \dots, z_n$  and  $w$  by the elements  $Y_{\bar{\alpha}^1}, \dots, Y_{\bar{\alpha}^m}$ , where  $Y_{\bar{\alpha}^i}$  are the exponents in  $z$  variables of the integer points  $\bar{\alpha}^i$  of  $P$  multiplied by an extra variable  $w$ . One also has a presentation of  $R(P)$  as the quotient of the polynomial ring in variables  $X_{\bar{\alpha}^1}, \dots, X_{\bar{\alpha}^m}$  corresponding to the elements  $Y_{\bar{\alpha}^i}$ .

Similarly to the finite picture, the reduced arc ring  $J_{\text{red}}^\infty(R(P))$  we are interested in is generated by the coefficients of the currents  $Y_{\bar{\alpha}^i}(s)$ . Thus, the reduced arc ring has two realizations: as a subring of the polynomial ring in variables  $z_i^{(j)}, w^{(j)}$  and as a quotient ring of the polynomial ring in variables  $X_{\bar{\alpha}^i}^{(j)}$ .

We introduce a family of subspaces  $A_{\bar{r}} \subset J_{\text{red}}^\infty(R(P))$  labeled by tuples  $\bar{r} = (r_1, \dots, r_m)$  of nonnegative integers. The space  $A_{\bar{r}}$  is realized inside the polynomial ring in variables  $z_i^{(j)}$ ; it is spanned as a vector space by elements of the form

$$Y_{\bar{\alpha}^1}^{(j_1^1)} \cdots Y_{\bar{\alpha}^1}^{(j_{r_1}^1)} \cdots Y_{\bar{\alpha}^m}^{(j_1^m)} \cdots Y_{\bar{\alpha}^m}^{(j_{r_m}^m)}$$

for all  $j_u^v \in \mathbb{N}$ . Since the ring  $J_{\text{red}}^\infty(R(P))$  is spanned by such elements, it can be presented as the sum of subspaces  $A_{\bar{r}}$ . This sum is clearly not direct. In order to control the ring  $J_{\text{red}}^\infty(R(P))$  we introduce an order  $\leq$  on the set of  $m$ -tuples  $\bar{r}$  and describe the subquotients

$$\sum_{\bar{r}' \leq \bar{r}} A_{\bar{r}'} / \sum_{\bar{r}' < \bar{r}} A_{\bar{r}'} \tag{0-1}$$

in terms of certain symmetric polynomials. More precisely, let  $\Lambda_{\bar{r}}(\mathbf{t}) = \mathbb{k}[t_i^{(j)}]_{i=1, \dots, m; j=1, \dots, r_i}^{\times \mathfrak{S}_{r_i}}$  be the ring of partially symmetric polynomials; see (2-2) and discussions after it for the precise definition. We construct embeddings

$$\left( \sum_{\bar{r}' \leq \bar{r}} A_{\bar{r}'} / \sum_{\bar{r}' < \bar{r}} A_{\bar{r}'} \right)^* \subset \Lambda_{\bar{r}}(\mathbf{t})$$

(here we mean the restricted dual with respect to the natural grading; see Section 2). In Section 2 (see Proposition 2.8) we prove:

**Theorem 0.1.** *There is a homogeneous element  $K \in \Lambda_{\bar{r}}(\mathbf{t})$  such that*

$$\left( \sum_{\bar{r}' \preceq \bar{r}} A_{\bar{r}'} / \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'} \right)^* \subset K \Lambda_{\bar{r}}(\mathbf{t}).$$

In Section 3 (see Corollary 3.7 and Lemma 3.17) we prove:

**Theorem 0.2.** *There is a homogeneous element  $\Gamma \in \Lambda_{\bar{r}}(\mathbf{t})$  such that*

$$\left( \sum_{\bar{r}' \preceq \bar{r}} A_{\bar{r}'} / \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'} \right)^* \supset \Gamma \Lambda_{\bar{r}}(\mathbf{t}).$$

We also prove the following corollary:

**Corollary 0.3.** *If  $K = \Gamma$ , then for all  $L \geq 0$  the degree  $L$   $z$ -homogeneous component of the reduced arc ring admits a free action of the algebra of symmetric polynomials in  $L$  variables.*

The proofs are constructive, i.e., we compute elements  $K$  and  $\Gamma$ . In the rest of the paper we provide several examples when  $K = \Gamma$ , and we call such polytopes and corresponding varieties favorable. In particular, starting from a one-dimensional case (with  $P$  being a segment and  $X(P)$  isomorphic to a projective line) we develop an inductive procedure providing a family of favorable polytopes. Namely, for a convex function  $\zeta : P \rightarrow \mathbb{R}_{\geq 0}$  with  $\zeta(\bar{\alpha}^i) = \zeta_i \in \mathbb{Z}$ , set  $\zeta_{\max} = \max\{\zeta_i\}$ . Denote by  $P^\zeta$  the polytope in  $\mathbb{R}^{n+1}$  obtained as the convex hull of  $P \times \zeta_{\max}$  and all  $\bar{\alpha}^i \times (\zeta_{\max} - \zeta_i)$ .

We prove the following theorem (see Theorem 4.20 for a precise statement).

**Theorem 0.4.** *Let  $P$  be a  $d$ -dimensional lattice polytope such that  $X(P)$  is favorable. Then  $X(P^\zeta)$  is also favorable provided certain conditions are satisfied.*

As a corollary we derive the following:

**Corollary 0.5.** *The toric varieties  $X(P)$  are favorable for  $P$  being a simplex, a parallelepiped, or a Hirzebruch trapezoid. In all these cases the radical of the ideal of the corresponding arc scheme is finitely generated as a differential ideal.*

In the Appendix we also develop a completely different technique to study the case when  $P$  is the product of simplices. This technique uses the representation theory of current algebras and the geometry of the semi-infinite flag variety.

The paper is organized as follows. In Section 1 we collect the main definitions and basic properties of the main objects of study: toric varieties, arc schemes, arc rings, supersymmetric polynomials. In Section 2 we develop a general machinery for the description of arc rings in terms of certain subspaces of symmetric polynomials. The approach is applied to the case of toric varieties in Section 3. In Section 4 we present an inductive procedure for constructing a large family of favorable toric arc spaces. The construction is applied in Section 5 where we work out explicitly several classical examples. Finally, in the Appendix we provide a representation theoretic description of the arc rings of the Veronese–Segre embeddings.

## 1. Preliminaries and generalities

We work over an algebraically closed field  $\mathbb{k}$  of characteristic zero. For variables  $z_1, \dots, z_m$  and vector  $\bar{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$  we use the multi-index notation

$$z^{\bar{a}} = z_1^{a_1} \dots z_m^{a_m}.$$

**1.1. Toric varieties.** The basic properties of toric varieties recalled here can be found in [Cox et al. 2011].

**Definition 1.1.** A toric variety  $V$  is an algebraic variety  $V$  containing the algebraic torus  $(\mathbb{k}^\times)^k$  as an open dense subset such that the natural action of the torus on itself can be extended to an action on the variety.

**Definition 1.2.** An ideal in a polynomial ring is called *toric* if it is prime and generated by binomials. A ring is toric if it is the quotient of a polynomial ring by a toric ideal.

**Proposition 1.3.** *Affine toric varieties are precisely those varieties which have the form  $\text{Spec } R$  for a toric ring  $R$ .*

For a rational cone  $\sigma \subset \mathbb{R}^m$  its set of integer points  $\sigma \cap \mathbb{Z}^m$  forms a semigroup.

**Proposition 1.4.** *Normal affine toric varieties are precisely those varieties which have the form*

$$\text{Spec}(\mathbb{k}[\sigma \cap \mathbb{Z}^m])$$

*for some  $m$  and rational cone  $\sigma \subset \mathbb{R}^m$  of dimension  $m$ .*

A convex polytope  $P \subset \mathbb{R}^m$  is said to be normal if for any integer  $k > 0$  every integer point in its dilation  $kP$  can be expressed as the sum of  $k$  (not necessarily distinct) integer points in  $P$ . In other words, this means that the set  $kP \cap \mathbb{Z}^m$  is the  $k$ -fold Minkowski sum of  $P \cap \mathbb{Z}^m$  with itself.

Consider a normal convex polytope  $P \subset \mathbb{R}^m$ . Define a polyhedral cone  $C(P) \subset \mathbb{R}^{m+1}$  to be generated by vectors  $(a, 1) \in \mathbb{R}^{m+1}$ ,  $a \in P$ . The semigroup of this cone  $C(P) \cap \mathbb{Z}^{m+1}$  consists of pairs  $(b, k)$ , where  $b$  is an integer point in  $kP$ . By normality this semigroup is generated by the points  $(a, 1)$ ,  $a \in P \cap \mathbb{Z}^m$ . The ring  $R(P) := \mathbb{k}[C(P) \cap \mathbb{Z}^{m+1}]$  is naturally graded, the  $k$ -th graded component is equal to the linear span of  $z^{\bar{a}} w^k$ ,  $\bar{a} \in kP \cap \mathbb{Z}^m$ .

**Definition 1.5.** The projective toric variety of  $P$  is  $\text{Proj}(R(P))$ .

We denote the generators  $z^{\bar{a}} w$  of the ring  $R(P)$  by  $Y_{\bar{a}}$ ,  $\bar{a} \in P \cap \mathbb{Z}^m$ .

**Example 1.6.** Let  $P \subset \mathbb{R}^2$  be the convex hull of points  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ , i.e., a unimodular simplex. Then  $R(P) \subset \mathbb{k}[z_1, z_2, w]$  is generated by monomials  $w$ ;  $z_1 w$ ;  $z_1 z_2 w$ . Clearly this algebra is freely generated by these monomials, so it is isomorphic to the polynomial ring in 3 variables. All of them have degree 1, so the corresponding variety is isomorphic to a projective plane.

**1.2. Arc spaces.** In this subsection we recall the basic properties of arc spaces; see [Chambert-Loir et al. 2018; Ein and Mustața 2009; Nash 1995].

**Definition 1.7.** Let  $V$  be a variety over  $\mathbb{k}$ . The arc (or jet- $\infty$ ) space  $J^\infty(V)$  is a scheme for which there exists a natural isomorphism of functors from  $\mathbf{CAlg}_{\mathbb{k}}$  (commutative algebras) to  $\mathbf{Set}$

$$\mathrm{Hom}(\mathrm{Spec}(-), J^\infty(V)) \simeq \mathrm{Hom}(\mathrm{Spec}(-[[s]]), V).$$

Note that if  $V = \mathrm{Spec}(R)$ , this definition is equivalent to the existence of a natural isomorphism

$$\mathrm{Hom}(\mathbb{k}[J^\infty(V)], -) \simeq \mathrm{Hom}(R, -[[s]]). \quad (1-1)$$

We immediately observe that  $J^\infty(V)$  is unique if it exists, since  $\mathbb{k}[J^\infty(V)]$  represents the functor  $\mathrm{Hom}(R, -[[s]])$ . Accordingly, the natural isomorphisms in Definition 1.7 and in (1-1) must also be unique. Throughout the paper we mostly deal with affine varieties (actually, with affine cones of projective varieties). The existence of  $J^\infty(V)$  is deduced from the following explicit construction.

**Definition 1.8.** For a finitely generated  $\mathbb{k}$ -algebra  $R$  choose generators and relations:

$$R \simeq \mathbb{k}[X_1, \dots, X_m] / \langle f_1(X_1, \dots, X_m), \dots, f_\ell(X_1, \dots, X_m) \rangle.$$

Consider the polynomial ring  $\mathbb{k}[\{X_i^{(j)}\}]$ , where  $i = 1, \dots, m$  and  $j = 0, 1, \dots$ . The arc ring  $J^\infty(R)$  is the quotient of  $\mathbb{k}[\{X_i^{(j)}\}]$  by the ideal generated by all  $s$ -coefficients in series

$$f_k(X_1(s), \dots, X_m(s)), \quad (1-2)$$

where  $k = 1, \dots, \ell$  and  $X_i(s) := \sum_{j=0}^{\infty} X_i^{(j)} s^j$ .

**Proposition 1.9.** For an affine variety  $V = \mathrm{Spec}(R)$  the scheme

$$J^\infty(V) = \mathrm{Spec} J^\infty(R)$$

satisfies Definition 1.7. In particular,  $J^\infty(R)$  does not depend on the chosen presentation of  $R$ .

In the notation of Definition 1.8 let  $Y_i \in R$  be the image of  $X_i$  and denote by  $Y_i^{(j)} \in J^\infty(R)$  the image of  $X_i^{(j)}$ . Consider a  $\mathbb{k}$ -algebra  $A$  and a homomorphism  $\iota : R \rightarrow A[[s]]$  with

$$\iota(Y_i) = \sum_{j=0}^{\infty} a_i^{(j)} s^j. \quad (1-3)$$

Then we have a homomorphism  $\iota' : J^\infty(R) \rightarrow A$  with  $\iota'(Y_i^{(j)}) = a_i^{(j)}$ . We obtain a morphism from  $\mathrm{Hom}(J^\infty(R), A)$  to  $\mathrm{Hom}(R, A[[s]])$  given by  $\iota \mapsto \iota'$ , together these morphisms provide the functorial isomorphism from Definition 1.7. We see that  $J^\infty$  is a functor from the category of finitely generated commutative algebras to the category  $\mathbf{CAlg}_{\mathbb{k}}$ . In geometric terms we can view  $J^\infty$  as a functor from the category of varieties to the category of schemes which satisfies  $\mathrm{Spec} \circ J^\infty \simeq J^\infty \circ \mathrm{Spec}$ .

**Example 1.10.** Consider  $R = \mathbb{k}[X_{00}, X_{01}, X_{10}, X_{11}] / \langle X_{00}X_{11} - X_{10}X_{01} \rangle$ . Then the ring  $J^\infty(R)$  is defined by relations which are  $s$ -coefficients of

$$X_{00}(s)X_{11}(s) - X_{10}(s)X_{01}(s).$$



In other words, the ideal of relations is generated by the expressions

$$\sum_{i=0}^k X_{00}^{(i)} X_{11}^{(k-i)} - \sum_{i=0}^k X_{01}^{(i)} X_{10}^{(k-i)}.$$

Proposition 1.9 provides a canonical bijection from  $\text{Hom}(J^\infty(R), J^\infty(R))$  to  $\text{Hom}(R, J^\infty(R)[[s]])$ . Let  $\iota$  be the image of the identity under this bijection. For every  $r \in R$  we obtain elements  $r^{(j)} \in J^\infty(R)$  by setting

$$\iota(r) = \sum_{j=0}^{\infty} r^{(j)} s^j = r(s).$$

Explicitly, if  $R$  is generated by  $Y_1, \dots, Y_m$  and polynomial  $p$  is such that  $r = p(Y_1, \dots, Y_m)$ , then  $r(s) = p(Y_1(s), \dots, Y_m(s))$ . In particular,  $r \mapsto r(s)$  is injective.

Let  $\varphi : R \rightarrow S$  be a homomorphism of finitely generated  $\mathbb{k}$ -algebras. The homomorphism

$$J^\infty(\varphi) : J^\infty(R) \rightarrow J^\infty(S)$$

is given by

$$J^\infty(\varphi)(r^{(j)}) = \varphi(r)^{(j)}. \tag{1-4}$$

It is clear that the functor  $J^\infty$  preserves surjections. However, this functor need not preserve injections, such examples will be discussed in this paper. More precisely, we will consider situations where  $R$  is embedded into a polynomial ring  $S$  but  $J^\infty(R)$  has a nontrivial nilradical which, in particular, implies that  $J^\infty(R)$  cannot be a subring of  $J^\infty(S)$  (which is itself a polynomial ring). See, for example, Lemma 1.31.

The fact that the arc space of a reduced ring can be nonreduced is, in a sense, the only obstacle to preserving injections (at least if the rings are nice enough). We have a functor  $J_{\text{red}}^\infty$  from  $\mathbf{CAlg}_{\mathbb{k}}$  to itself which takes  $R$  to its *reduced arc ring*, i.e.,  $J^\infty(R)$  modulo its nilradical. We will now show that  $J_{\text{red}}^\infty$  preserves injections.

**Theorem 1.11.** *Suppose  $\phi : X \rightarrow Y$  is a dominant morphism of varieties. Then the corresponding morphism of arc spaces  $J^\infty(\phi) : J^\infty(X) \rightarrow J^\infty(Y)$  is dominant.*

*Proof.* We note that the restriction of  $\phi$  to the smooth locus  $X^{\text{reg}}$  gives a dominant map  $X^{\text{reg}} \rightarrow Y$ . To prove the theorem it suffices to show that  $J^\infty(X^{\text{reg}}) \rightarrow J^\infty(Y)$  is dominant. Hence, we may (and will) assume that  $X$  is smooth.

By the Chevalley constructibility theorem, the dominance is equivalent to the fact that  $\phi(X)$  contains a dense open subset of  $Y$ , denote it by  $U$ . Then the restriction of  $\phi$  to  $\phi^{-1}(U)$  can be decomposed as  $\phi^{-1}(U) \xrightarrow{\phi_1} U \xrightarrow{\phi_2} Y$  with surjective  $\phi_1$  and open embedding  $\phi_2$ . Using that  $\phi^{-1}(U) \subset X$  is smooth, we can apply the “generic smoothness on the target” theorem [Vakil 2017, Theorem 25.3.3] to  $\phi_1$ , and obtain that there is an open dense  $V \subset U$  such that  $\phi_1$  is smooth on  $\phi_1^{-1}(V)$ . Restricting to  $\phi_1^{-1}(V)$ , we have

$$\phi_1^{-1}(V) \xrightarrow{\phi_1} V \xrightarrow{\psi} U \xrightarrow{\phi_2} Y,$$

where  $\phi_1$  is surjective smooth and  $\psi, \phi_2$  are open embeddings.

Passing to arcs, we get that  $J^\infty(\phi_1)$  is surjective [Ein and Mustařă 2009, Remark 2.10] and  $J^\infty(\psi)$ ,  $J^\infty(\phi_2)$  are open embeddings [Ein and Mustařă 2009, Lemma 2.3]. Note also that by Kolchin's theorem [Ein and Mustařă 2009, Theorem 3.3]  $J^\infty(U)$ ,  $J^\infty(Y)$  are irreducible and hence the open embeddings  $J^\infty(\psi)$ ,  $J^\infty(\phi_2)$  are dominant. Therefore, the composition is dominant and the theorem is proven.  $\square$

In this paper we study the ring  $J_{\text{red}}^\infty(R)$  for a toric ring  $R$  embedded into a polynomial algebra. The following corollary implies that the reduced arc ring  $J_{\text{red}}^\infty(R)$  can be described using arc map of the embedding.

**Corollary 1.12.** *Suppose  $f : R \hookrightarrow L$  is an injective map of integral finitely generated  $\mathbb{k}$ -algebras. Then the corresponding map  $J_{\text{red}}^\infty(R) \rightarrow J_{\text{red}}^\infty(L)$  is also injective.*

*Proof.* The corresponding map  $f^* : \text{Spec } L \rightarrow \text{Spec } R$  is dominant. Applying Theorem 1.11 we obtain that  $J^\infty(f)^* : \text{Spec } J^\infty(L) \rightarrow \text{Spec } J^\infty(R)$  is dominant. But that is equivalent to the fact that  $\ker(J^\infty(f))$  is contained in the nilradical of  $J^\infty(R)$  and we are done.  $\square$

The next proposition shows how the arc space of a closure is related to the closure of the arc space.

**Proposition 1.13.** *Let  $\phi : X \rightarrow Y$  be a morphism of varieties. Then*

$$J^\infty(\overline{\text{im } \phi}) = \overline{\text{im}(J^\infty(\phi))},$$

where both sides are endowed with the reduced scheme structure.

*Proof.* Note that for any morphism of schemes  $A \rightarrow B$  there exists unique decomposition  $A \rightarrow C \rightarrow B$  such that the first arrow is dominant and the second is a closed embedding. We prove that both decompositions

$$J^\infty(X) \xrightarrow{J^\infty(\phi)} \overline{\text{im}(J^\infty(\phi))} \hookrightarrow J^\infty(Y) \quad (1-5)$$

and

$$J^\infty(X) \xrightarrow{J^\infty(\phi)} J^\infty(\overline{\text{im } \phi}) \hookrightarrow J^\infty(Y) \quad (1-6)$$

satisfy this property. This will imply the proposition.

Decomposition (1-5) satisfies the desired properties just by definition (the first map is dominant and the second is a closed embedding).

In order to obtain the same for (1-6) we first consider the decomposition  $X \rightarrow \overline{\text{im } \phi} \rightarrow Y$  of the map  $\phi$ , this decomposition has the desired properties. Applying the functor  $J^\infty$  we obtain that in decomposition (1-6) the first arrow is dominant due to Theorem 1.11 and the second is a closed embedding.  $\square$

**Corollary 1.14.** *Let  $X$  be a variety with an action of an algebraic group  $G$ . This induces the action of  $J^\infty(G) = G[[t]]$  on  $J^\infty(X)$ . Then for any point  $x \in X \hookrightarrow J^\infty(X)$  and orbits  $G.x \subset X$ ,  $G[[t]].x \subset J^\infty(X)$  one has  $J^\infty(\overline{G.x}) = \overline{G[[t]].x}$ , where both sides are endowed with the reduced scheme structure.*

*Proof.* Use Proposition 1.13 for the map  $G \rightarrow X$  defined by  $g \mapsto g.x$ .  $\square$

The corollary above implies that the arc ring  $J_{\text{red}}^\infty(R)$  of an affine toric variety, which we mainly study in this paper, is isomorphic to the coordinate ring of the cone over the closure of the orbit of the toric arc group.

To end this section we provide an example of how one may find relations in a ring  $J_{\text{red}}^\infty(R)$ . We first prove two general (and well-known) lemmas concerning an arbitrary commutative  $\mathbb{k}$ -algebra  $A$ . The lemmas can be seen to be equivalent but it is convenient for us to use them both.

**Lemma 1.15.** *Consider elements  $a_i \in A, i = 0, 1, \dots$ . Assume that the series  $a(s) = \sum_{i=0}^\infty a_i s^i$  is nilpotent. Then all the coefficients  $a_i$  are nilpotent.*

*Proof.* Clearly  $a_0$  is nilpotent. Assume that  $a_i, i < j$  are nilpotent. Then

$$a(s) - \sum_{i=0}^{j-1} a_i s^i$$

is nilpotent and thus its leading term  $a_j s^j$  is nilpotent. This completes the proof by induction.  $\square$

**Lemma 1.16.** *Let  $d$  be a  $\mathbb{k}$ -linear derivation of the algebra  $A$ . If  $a \in A$  is nilpotent, then  $d(a)$  is nilpotent.*

*Proof.* Suppose  $a^n = 0$ , then  $d^n(a^n) = 0$ . Expanding  $d^n(a^n)$  via the Leibniz rule we obtain  $n!d(a)^n + ab$  for some  $b \in A$ . Hence  $d(a)^n = -ab/n!$  is nilpotent and so is  $d(a)$ .  $\square$

**Example 1.17** [Bourqui and Sebag 2019b]. Here and further we write  $a \equiv b$  for elements  $a, b$  of a ring to denote that  $a - b$  is nilpotent. Consider the ring  $R$  generated by  $Y_1$  and  $Y_2$  satisfying the single relation  $Y_1 Y_2 = 0$ . Then the ring  $J^\infty(R)$  is generated by  $Y_1^{(j)}, Y_2^{(j)}$  satisfying the relations

$$Y_1(s)Y_2(s) = 0.$$

We show by induction that for  $n \geq 0$  the series

$$Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \tag{1-7}$$

is nilpotent in  $J^\infty(R)[[s]]$ . Indeed, assume that

$$Y_1(s) \frac{\partial^{n-1} Y_2(s)}{\partial s^{n-1}}$$

is nilpotent. Using Lemma 1.16 we have

$$Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \equiv -\frac{\partial Y_1(s)}{\partial s} \frac{\partial^{n-1} Y_2(s)}{\partial s^{n-1}}.$$

Multiplying both sides by the left-hand side part we obtain

$$\left( Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \right)^2 \equiv -Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \frac{\partial Y_1(s)}{\partial s} \frac{\partial^{n-1} Y_2(s)}{\partial s^{n-1}}.$$

By the induction hypothesis the right-hand side is nilpotent; hence so is (1-7). In fact, one may check that the coefficients of the series (1-7) span the same linear space as the elements  $Y_1^{(i)} Y_2^{(j)}$  for all  $i, j \geq 0$ . Thus the quotient of  $J^\infty(R)$  modulo its nilradical is spanned by the images of all  $Y_1^{(i_1)} \dots Y_1^{(i_M)}$  and  $Y_2^{(j_1)} \dots Y_2^{(j_N)}$ . Using this and the results of Section 2.3 one can show that the nilradical is generated by coefficients of series (1-7).

**1.3. Derivations of arc rings.** In this subsection we define two Lie algebra actions on an arbitrary arc ring, these actions will be instrumental to our construction.

Consider any commutative  $\mathbb{k}$ -algebra  $\Theta$  and an odd formal variable  $\tau$  (i.e.,  $\tau^2 = 0$ ). Let  $\varkappa : \Theta[\tau] \rightarrow \Theta$  be the natural projection, i.e.,  $\varkappa(\Theta\tau) = 0$ . In the following lemma and throughout the text we only consider  $\mathbb{k}$ -derivations.

**Lemma 1.18.** *The derivations of  $\Theta$  are in one-to-one correspondence with the homomorphisms  $\delta : \Theta \rightarrow \Theta[\tau]$  for which  $\varkappa \circ \delta = 1$ . A derivation  $d$  corresponds to the homomorphism  $1 + \tau d$ .*

*Proof.* For any derivation  $d \in \text{Der } \Theta$  one evidently has

$$\varkappa \circ (1 + \tau d) = 1.$$

Suppose that we have a homomorphism  $\delta : \Theta \rightarrow \Theta[\tau]$  such that  $\varkappa \circ \delta = 1$ . Then we have

$$\delta = 1 + \tau d$$

for a linear map  $d$ . For any  $x, y \in \Theta$ ,

$$xy + \tau d(xy) = (1 + \tau d)(xy) = (1 + \tau d)(x)(1 + \tau d)(y) = xy + \tau(xd(y) + d(x)y).$$

Hence  $d$  is a derivation. □

Fix a finitely generated  $\mathbb{k}$ -algebra  $R$  for the rest of this section. Consider a continuous derivation  $d \in \text{Der } \mathbb{k}[[s]]$ , we can extend  $d$  to a derivation of  $J^\infty(R)[[s]] = J^\infty(R) \widehat{\otimes} \mathbb{k}[[s]]$ . Here continuity is with respect to the standard topology on  $\mathbb{k}[[s]]$  and  $\widehat{\otimes}$  denotes the completed tensor product with respect to this topology; see, for instance, [Stacks]. Lemma 1.18 applied to  $\Theta = J^\infty(R)[[s]]$  and the obtained derivation provide maps

$$J^\infty(R)[[s]] \rightarrow J^\infty(R)[[s]][\tau] \rightarrow J^\infty(R)[[s]],$$

and hence the maps

$$\text{Hom}(R, J^\infty(R)[[s]]) \rightarrow \text{Hom}(R, J^\infty(R)[[s]][\tau]) \rightarrow \text{Hom}(R, J^\infty(R)[[s]]),$$

with composition equal to identity. Applying isomorphism (1-1) we have maps  $\varphi$  and  $\pi$

$$\text{Hom}(J^\infty(R), J^\infty(R)) \xrightarrow{\varphi} \text{Hom}(J^\infty(R), J^\infty(R)[\tau]) \xrightarrow{\pi} \text{Hom}(J^\infty(R), J^\infty(R)) \quad (1-8)$$

with composition equal to identity.

**Proposition 1.19.** *The image of the identity under  $\varphi$  has the form  $1 + \tau\rho(d)$  for some  $\rho(d) \in \text{Der } J^\infty(R)$ . The map  $-\rho$  defines a Lie algebra action of  $\text{Der}^c \mathbb{k}[[s]]$  (continuous derivations) on  $J^\infty(R)$ .*

*Proof.* By construction, the map  $\pi$  is obtained by applying the functor  $\text{Hom}(J^\infty(R), -)$  to the projection  $\varkappa : J^\infty(R)[\tau] \rightarrow J^\infty(R)$ , i.e., it is the composition with  $\varkappa$  on the left. In particular, if  $\delta = \varphi(\text{id}_{J^\infty(R)})$ , then  $\text{id}_{J^\infty(R)} = \varkappa\delta$ . By Lemma 1.18, we indeed have  $\delta = 1 + \tau\rho(d)$  for some  $\rho(d) \in \text{Der } J^\infty(R)$ .

Let us show that

$$-\rho : \text{Der}^c \mathbb{k}[[s]] \rightarrow \text{Der } J^\infty(R)$$

is a Lie algebra homomorphism. The derivation  $d$  is given by a  $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0})$ -matrix  $B$  such that

$$d(s^j) = \sum_{i \geq 0} B_{i,j} s^i$$

for any  $j \geq 0$ , i.e.,  $d$  maps a series with coefficients  $(c_0, c_1, \dots)$  to the series with coefficients  $B(c_0, c_1, \dots)^T$ . From the definitions we see that for any  $r \in R$  and  $i \geq 0$  we then have

$$\varphi(\text{id}_{J^\infty(R)})(r^{(i)}) = r^{(i)} + \tau \sum_{j \geq 0} B_{i,j} r^{(j)}.$$

In other words,  $\rho(d)$  maps a linear combination of the  $r^{(j)}$  with coefficients  $(c_0, c_1, \dots)$  to their linear combination with coefficients  $(c_0, c_1, \dots)B$ . This shows that  $\rho$  is an antihomomorphism, i.e.,  $-\rho$  is indeed a homomorphism.  $\square$

We will use the shorthand notation  $dX = -\rho(d)(X)$  for  $d \in \text{Der}^c \mathbb{k}[[s]]$  and  $X \in J^\infty(R)$ . Recall that the Lie algebra  $\text{Der}^c \mathbb{k}[[s]]$  is spanned by the derivations  $d_k = s^{k+1} \frac{\partial}{\partial s}$  with  $k \geq -1$ . These derivations act on  $J^\infty(R)$  as follows (here and further  $r^{(i)} = 0$  for any  $r \in R$  and  $i < 0$ ).

**Proposition 1.20.** *For any  $r \in R$ ,  $i \geq 0$  and  $k \geq -1$  we have*

$$d_k(r^{(i)}) = (i - k)r^{(i-k)}. \tag{1-9}$$

*Proof.* Setting  $d = d_k$  within the proof of the previous proposition we see that the matrix  $B$  is given by  $B_{i,j} = j$  when  $i - j = k$  and  $B_{i,j} = 0$  otherwise. In particular, the proof then implies that

$$d_k(r^{(i)}) = \sum_{j > 0} B_{i,j} r^{(j)} = (i - k)r^{(i-k)}. \quad \square$$

From (1-4) and Proposition 1.20 we see that  $J^\infty$  respects the action of  $\text{Der}^c \mathbb{k}[[s]]$ . In other words,  $J^\infty$  as a functor from  $\mathbf{CAlg}_{\mathbb{k}}^{\text{fin}}$  to the category of commutative  $\text{Der}^c \mathbb{k}[[s]]$ -differential algebras.

We next describe another Lie algebra action on  $J^\infty(R)$ . Fix a presentation

$$R \simeq \mathbb{k}[X_1, \dots, X_m] / \langle f_1(X_1, \dots, X_m), \dots, f_\ell(X_1, \dots, X_m) \rangle$$

and let  $Y_i \in R$  be the image of  $X_i$ .

**Definition 1.21.** For a derivation  $\mathfrak{d} \in \text{Der } R$  and  $k \geq 0$  we consider  $\mathfrak{d}^{(k)} \in \text{Der } J^\infty(R)$  defined on generators by

$$\mathfrak{d}^{(k)}(Y_i^{(j)}) := \mathfrak{d}(Y_i)^{(j-k)}.$$

**Lemma 1.22.** *The derivations  $\mathfrak{d}^{(k)}$  are well-defined and independent of the presentation.*

*Proof.* Choose polynomials  $p_1, \dots, p_m$  in  $\mathbb{k}[X_1, \dots, X_m]$  such that  $p_i(Y_1, \dots, Y_m) = \mathfrak{d}Y_i$  for all  $Y_i$ . We have a unique derivation  $\tilde{\mathfrak{d}}$  of  $\mathbb{k}[X_1, \dots, X_m]$  given by  $\tilde{\mathfrak{d}}X_i = p_i$ . We also consider the derivation  $\tilde{\mathfrak{d}}^{(k)}$  of  $\mathbb{k}[\{X_i^{(j)}\}]$  such that  $\tilde{\mathfrak{d}}^{(k)}(X_i^{(j)})$  is the coefficient of  $s^{j-k}$  in  $p_i(X_1(s), \dots, X_m(s))$  or 0 if  $j < k$ . Furthermore, we extend  $\tilde{\mathfrak{d}}^{(k)}$  to a derivation of  $\mathbb{k}[\{X_i^{(j)}\}][[s]]$  by applying it coefficientwise. Note that

$$\tilde{\mathfrak{d}}^{(k)}(X_i(s)) = s^k p_i(X_1(s), \dots, X_m(s)). \tag{1-10}$$

Now, for any  $p \in \mathbb{k}[X_1, \dots, X_m]$  its image  $\tilde{\mathfrak{d}}(p)$  projects to  $\mathfrak{d}(p(Y_1, \dots, Y_m)) \in R$ . Hence  $\tilde{\mathfrak{d}}$  preserves the ideal of relations in  $R$ , i.e., for all  $f_o$  we have polynomials  $a_q$  such that

$$\tilde{\mathfrak{d}}(f_o) = \sum_{q=1}^{\ell} a_q f_q.$$

From (1-10) we then obtain

$$\tilde{\mathfrak{d}}^{(k)}(f_o(X_1(s), \dots, X_m(s))) = s^k \sum_{q=1}^{\ell} a_q(X_1(s), \dots, X_m(s)) f_q(X_1(s), \dots, X_m(s)).$$

This means that  $\tilde{\mathfrak{d}}^{(k)} \in \text{Der } \mathbb{k}[\{X_i^{(j)}\}]$  preserves the defining ideal of  $J^\infty(R)$  generated by all coefficients of the series  $f_q(X_1(s), \dots, X_m(s))$ . By projecting onto  $J^\infty(R)$  we obtain a derivation which maps  $Y_i^{(j)}$  to the coefficient of  $s^{j-k}$  in  $p_i(Y_1(s), \dots, Y_m(s))$ . The latter coefficient is precisely  $\mathfrak{d}(Y_i)^{(j-k)}$  and the obtained derivation satisfies the definition of  $\mathfrak{d}^{(k)}$ .

To show that  $\mathfrak{d}^{(k)}$  is independent of the presentation extend  $\mathfrak{d}^{(k)}$  to a derivation of  $J^\infty(R)[[s]]$  coefficientwise. We have the embedding  $\iota$  of  $R$  into  $J^\infty(R)[[s]]$  with  $\iota(r) = r(s)$ . Now we note that

$$s^k \iota \mathfrak{d}(Y_i) = \mathfrak{d}^{(k)} \iota(Y_i),$$

i.e., the maps  $s^k \iota \mathfrak{d} \iota^{-1}$  and  $\mathfrak{d}^{(k)}$  agree on the  $\iota(Y_i)$ , and hence on all of  $\iota(R)$ . In other words,

$$\mathfrak{d}^{(k)}(r^{(j)}) = \mathfrak{d}(r)^{(j-k)} \quad (1-11)$$

for any  $r \in R$  and  $j \geq 0$  which is independent of the presentation.  $\square$

**Lemma 1.23.** *The current Lie algebra  $(\text{Der } R)[s] = \text{Der } R \otimes_{\mathbb{k}} \mathbb{k}[s]$  acts by derivations on  $J^\infty(R)$  by  $\mathfrak{d}s^k$  acting as  $\mathfrak{d}^{(k)}$ .*

*Proof.* For any two derivations  $\mathfrak{d}_1, \mathfrak{d}_2 \in \text{Der}(D)$  and  $k_1, k_2 \geq 0$  we need to prove

$$[\mathfrak{d}_1^{(k_1)}, \mathfrak{d}_2^{(k_2)}] = [\mathfrak{d}_1, \mathfrak{d}_2]^{(k_1+k_2)}.$$

However, from (1-11) we indeed have, for any  $r \in R$ ,

$$[\mathfrak{d}_1^{(k_1)}, \mathfrak{d}_2^{(k_2)}](r^{(j)}) = ([\mathfrak{d}_1, \mathfrak{d}_2](r))^{(j-k_1-k_2)} = [\mathfrak{d}_1, \mathfrak{d}_2]^{(k_1+k_2)}(r^{(j)}).$$

Since the  $r^{(j)}$  generate  $J^\infty(R)$ , the two derivations agree on all of  $J^\infty(R)$ .  $\square$

Let  $\tilde{\tau} : \mathfrak{a} \rightarrow \mathfrak{b}$  be a morphism of Lie algebras. Then there exists the morphism

$$\tilde{\tau}[s] : \mathfrak{a}[s] \rightarrow \mathfrak{b}[s], \quad as^k \mapsto \tilde{\tau}(a)s^k.$$

**Lemma 1.24.** *Let  $R, L$  be associative algebras,  $\mathfrak{g}_R, \mathfrak{g}_L$  be Lie algebras acting on them by derivations. Let  $\tau, \tilde{\tau}$  be two morphisms  $\tau : R \rightarrow L, \tilde{\tau} : \mathfrak{g}_R \rightarrow \mathfrak{g}_L$  such that  $\tau$  is equivariant with respect to  $\tilde{\tau}$ , i.e., for any  $d \in \mathfrak{g}_R, r \in R$ ,*

$$\tau(d(r)) = \tilde{\tau}(d)(\tau(r)).$$

*Then  $J^\infty(\tau)$  is equivariant with respect to  $\tilde{\tau}[s]$ .*

*Proof.* This follows from (1-11) and the definition of equivariance.  $\square$

**1.4. Supersymmetric polynomials.** In this subsection we discuss the properties of supersymmetric polynomials. Take two groups of variables  $p^{(1)}, \dots, p^{(b)}, q^{(1)}, \dots, q^{(c)}$ .

**Definition 1.25.** A polynomial  $f \in \mathbb{k}[p^{(1)}, \dots, p^{(b)}, q^{(1)}, \dots, q^{(c)}]^{\mathfrak{S}_b \times \mathfrak{S}_c}$  symmetric in the  $p^{(i)}$  and the  $q^{(j)}$  is supersymmetric if  $f|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z}$  is independent of  $Z$ . We denote the ring of such polynomials by  $\Omega_{b,c}$ .

**Proposition 1.26.** *There exists the map  $\psi_{b,c} : \Omega_{b,c} \rightarrow \Omega_{b-1,c-1}$ :*

$$\psi_{b,c} : f \mapsto f|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z}.$$

The kernel of  $\psi_{b,c}$  is the ideal

$$\prod_{1 \leq i \leq b, 1 \leq j \leq c} (p^{(i)} - q^{(j)}) \mathbb{k}[p^{(1)}, \dots, p^{(b)}, q^{(1)}, \dots, q^{(c)}]^{\mathfrak{S}_b \times \mathfrak{S}_c}.$$

*Proof.* For a polynomial  $f \in \Omega_{b,c}$  the image  $\psi_{b,c}(f)$  is symmetric in  $p^{(1)}, \dots, p^{(b-1)}$  and  $q^{(1)}, \dots, q^{(c-1)}$ . We have

$$\psi_{b,c}(f)|_{p^{(b-1)} \mapsto Z', q^{(c-1)} \mapsto Z'} = f|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z, p^{(b-1)} \mapsto Z', q^{(c-1)} \mapsto Z'}.$$

Therefore it is independent of  $Z'$ . Hence  $\psi_{b,c}(f) \in \Omega_{b-1,c-1}$ .

Assume now  $f \in \ker(\psi_{b,c})$ . Since  $f$  is divisible by  $(p^{(b)} - q^{(c)})$  and symmetric in  $p^{(1)}, \dots, p^{(b)}$  and  $q^{(1)}, \dots, q^{(c)}$ , we see that  $f$  is divisible by  $\prod_{1 \leq i \leq b, 1 \leq j \leq c} (p^{(i)} - q^{(j)})$ . Conversely

$$\prod_{1 \leq i \leq b, 1 \leq j \leq c} (p^{(i)} - q^{(j)})|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z} = 0.$$

Thus the principal ideal  $\prod_{1 \leq i \leq b, 1 \leq j \leq c} (p^{(i)} - q^{(j)}) \Omega_{b,c}$  is annihilated by  $\psi_{b,c}$ . □

The proof of the following proposition can be found in [Macdonald 1995, Section 1.3.23].

**Proposition 1.27.** *The ring  $\Omega_{b,c}$  has two sets of generators,*

$$p_k(p^{(1)}, \dots, p^{(b)}; q^{(1)}, \dots, q^{(c)}) = \sum_{i=1}^b (p^{(i)})^k - \sum_{i=1}^c (q^{(i)})^k \quad \text{and} \quad h_k(p^{(1)}, \dots, p^{(b)}; q^{(1)}, \dots, q^{(c)}),$$

where

$$\frac{\prod_{i=1}^b (1 - T p^{(i)})}{\prod_{i=1}^c (1 - T q^{(i)})} = \sum_{k=0}^{\infty} h_k(p^{(1)}, \dots, p^{(b)}; q^{(1)}, \dots, q^{(c)}) T^k$$

for a formal variable  $T$ . In particular  $\psi_{b,c}$  is surjective.

**1.5. Associated graded algebras and initial ideals.** Consider the polynomial ring  $\mathbb{k}[X_1, \dots, X_n]$ . Let  $\prec$  be a monomial order on monomials in  $X_1, \dots, X_n$ , i.e., a total order on monomials such that if  $a \prec b$  then for any monomial  $c$ :  $ac \prec bc$ . In other words,  $\prec$  is a total semigroup order on  $\mathbb{N}^n$  (see [Miller and Sturmfels 2005]).

The ring  $\mathbb{k}[X_1, \dots, X_n]$  is naturally  $\mathbb{N}^n$ -graded. All graded components are one-dimensional and are spanned by monomials:

$$\mathbb{k}[X_1, \dots, X_n][\bar{r}] = \mathbb{k} \prod X_i^{r_i}.$$

The ring  $\mathbb{J}_n = J^\infty(\mathbb{k}[X_1, \dots, X_n])$  is the polynomial ring in variables  $X_i^{(j)}$  with  $1 \leq i \leq n$  and  $j \geq 0$ . This ring has an induced  $\mathbb{N}^n$ -grading where graded component  $\mathbb{J}_n[\bar{r}]$  has the basis

$$\left\{ \prod_{1 \leq i \leq n, j=0,1,\dots} (X_i^{(j)})^{b_i^j} \mid \sum_{j=0}^{\infty} b_i^j = r_i, i = 1, \dots, n \right\}.$$

$\mathbb{J}_n$  has two filtrations by the ordered semigroup  $(\mathbb{N}^n, <)$ . The first has components

$$\mathbb{J}_n[< \bar{r}] = \bigoplus_{\bar{r}' < \bar{r}} J^\infty(\mathbb{k}[X_1, \dots, X_n][\bar{r}']),$$

while the second has components

$$\mathbb{J}_n[\leq \bar{r}] = \bigoplus_{\bar{r}' \leq \bar{r}} J^\infty(\mathbb{k}[X_1, \dots, X_n][\bar{r}']).$$

Both filtrations are multiplicative: for any  $\bar{r}_1, \bar{r}_2 \in \mathbb{N}^n$  we have

$$\mathbb{J}_n[< \bar{r}_1] \mathbb{J}_n[< \bar{r}_2] \subset \mathbb{J}_n[< \bar{r}_1 + \bar{r}_2] \quad \text{and} \quad \mathbb{J}_n[\leq \bar{r}_1] \mathbb{J}_n[\leq \bar{r}_2] \subset \mathbb{J}_n[\leq \bar{r}_1 + \bar{r}_2].$$

Let  $\mathcal{I} \subset \mathbb{J}_n$  be an ideal. Denote  $R = \mathbb{J}_n/\mathcal{I}$  and let  $\pi$  be the natural projection  $\pi: \mathbb{J}_n \rightarrow R$ . The filtrations on  $\mathbb{J}_n$  induce filtrations on  $R$  by  $R[< \bar{r}] = \pi(\mathbb{J}_n[< \bar{r}])$  and  $R[\leq \bar{r}] = \pi(\mathbb{J}_n[\leq \bar{r}])$ . These filtrations are again multiplicative and we have the associated  $\mathbb{N}^n$ -graded algebra

$$\text{gr}_{<} R = \bigoplus_{\bar{r} \in \mathbb{N}^n} R[\leq \bar{r}]/R[< \bar{r}].$$

Next, for any nonzero element  $g \in \mathbb{J}_n$  consider its decomposition

$$g = \sum_{\bar{r} \in \mathbb{N}^n} g[\bar{r}], \quad g[\bar{r}] \in \mathbb{J}_n[\bar{r}].$$

Let  $\bar{r}'$  be the largest  $\bar{r}$  with respect to  $<$  such that  $g[\bar{r}]$  is nonzero. Then  $g[\bar{r}']$  is denoted by  $\text{in}_{<} g$  and is called the *initial part* of  $g$ . The *initial ideal* of  $\mathcal{I}$  with respect to  $<$  is the linear span of  $\{\text{in}_{<} g \mid g \in \mathcal{I}\}$ . This space is denoted by  $\text{in}_{<} \mathcal{I}$ , one easily checks that it is an ideal in  $\mathbb{J}_n$ . By construction, the ideal  $\text{in}_{<} \mathcal{I}$  is homogeneous with respect to the  $\mathbb{N}^n$ -grading. Therefore, the ring  $\mathbb{J}_n/(\text{in}_{<} \mathcal{I})$  is naturally  $\mathbb{N}^n$ -graded.

Next, let us observe that  $\mathbb{J}_n$  has a further  $\mathbb{N}$ -grading  $\text{grad}$  given by  $\text{grad } X_i^{(j)} = j$ . Together with the grading considered above this turns  $\mathbb{J}_n$  into a  $\mathbb{N}^{n+1}$ -graded ring. We now assume that  $\mathcal{I}$  is homogeneous with respect to  $\text{grad}$ . This gives us an induced  $\mathbb{N}$ -grading  $\text{grad}$  on  $R$ . Note that every subspace  $\mathbb{J}_n[< \bar{r}]$  and  $\mathbb{J}_n[\leq \bar{r}]$  is also  $\text{grad}$ -homogeneous, and hence so is every  $R[< \bar{r}]$  and  $R[\leq \bar{r}]$ . As a result we obtain an induced grading  $\text{grad}$  on  $\text{gr}_{<} R$  which makes the latter  $\mathbb{N}^{n+1}$ -graded. Furthermore, since  $\mathcal{I}$  is  $\text{grad}$ -homogeneous, so is any initial ideal. Consequently,  $\text{in}_{<} \mathcal{I}$  and  $\mathbb{J}_n/\text{in}_{<} \mathcal{I}$  are  $\mathbb{N}^{n+1}$ -graded.



**Proposition 1.28.** *The rings  $\text{gr}_{\prec} R$  and  $\mathbb{J}_n / \text{in}_{\prec} \mathcal{I}$  are isomorphic as  $\mathbb{N}^{n+1}$ -graded algebras.*

*Proof.* We may consider the associated graded algebra  $\text{gr}_{\prec} \mathbb{J}_n$ , note that it is naturally identified with  $\mathbb{J}_n$ . Since the filtrations on  $R$  are defined as projections of the filtrations on  $\mathbb{J}_n$ , we obtain a surjective homomorphism  $\pi' : \text{gr}_{\prec} \mathbb{J}_n \rightarrow \text{gr}_{\prec} R$  preserving the  $\mathbb{N}^{n+1}$ -grading. In terms of this identification,  $p \in \mathbb{J}_n[\bar{r}]$  lies in  $\ker \pi'$  if and only if  $p - q \in \mathbb{J}_n[\prec \bar{r}]$  for some  $q \in \mathcal{I}$ . In other words:  $p \in \ker \pi'$  if and only if  $p = \text{in}_{\prec} q$  for some  $q \in \mathcal{I}$ . This proves the proposition.  $\square$

**Remark 1.29.** This proposition is a special case of the well-known and rather general phenomenon of isomorphisms between associated graded rings and quotients by initial ideals. See, for instance, [Kaveh and Manon 2019, Lemma 3.4] or [Makhlín 2022, Propositions 1.1 and 8.1] for statements of this form for finitely generated commutative and associative algebras.

Choose a grad-homogeneous generating set  $\{f_1, \dots, f_\ell\}$  of  $\mathcal{I}$ . We denote by  $\mathcal{J}$  the ideal generated by the polynomials  $\text{in}_{\prec} f_i$ . This ideal is  $\mathbb{N}^{n+1}$ -homogeneous and is contained in  $\text{in}_{\prec} \mathcal{I}$ . This provides a surjection  $\mathbb{J}_n / \mathcal{J} \twoheadrightarrow \mathbb{J}_n / \text{in}_{\prec} \mathcal{I}$  of  $\mathbb{N}^{n+1}$ -graded algebras. In view of Proposition 1.28, we have the following.

**Corollary 1.30.** *There exists a surjective homomorphism of  $\mathbb{N}^{n+1}$ -graded algebras from  $\mathbb{J}_n / \mathcal{J}$  to  $\text{gr}_{\prec} R$ .*

**1.6. Unit cubes.** For an integer  $\ell \geq 2$  let  $\mathbb{R}^{[\ell]}$  be the space of real-valued functions on the set of subsets of  $[\ell]$ . Consider the unit cube  $\mathcal{C}_\ell \subset \mathbb{R}^{[\ell]}$ , its vertices are the indicator functions  $\mathbf{1}_I$  with  $I \subset [\ell]$ .  $R(\mathcal{C}_\ell)$  is generated by elements  $Y_I$  which satisfy

$$Y_I Y_J - Y_{I \cap J} Y_{I \cup J}. \tag{1-12}$$

Consider the arc ring  $J^\infty(R(\mathcal{C}_\ell))$ . It is generated by variables  $Y_I^{(j)}$ ,  $j \geq 0$ , satisfying the relations

$$Y_I(s) Y_J(s) - Y_{I \cap J}(s) Y_{I \cup J}(s) = 0. \tag{1-13}$$

**Lemma 1.31.** *The coefficients of the following series are nilpotent in  $J^\infty(R(\mathcal{C}_\ell))$  for any  $0 \leq k \leq \ell - 2$ :*

$$W_{\ell,k} = \sum_{I \subset [2,\ell]} (-1)^{|I|} Y_{I \cup \{1\}}(s) \frac{\partial^k Y_{[2,\ell] \setminus I}(s)}{\partial s^k}. \tag{1-14}$$

*Proof.* We prove this lemma by induction on  $\ell$  by showing that the series  $W_{\ell,k}$  is nilpotent in  $J^\infty(R(\mathcal{C}_\ell))[[s]]$ , the lemma then follows by Lemma 1.15. In the case  $\ell = 2$  we only have  $k = 0$  and  $W_{\ell,k}$  vanishes by (1-13).

For the induction step we first consider the case  $k < \ell - 2$ . We break up  $W_{\ell,k}$  into two sums and show that each summand can be interpreted as a similar relation for  $\ell - 1$ . First consider the sum

$$\sum_{J \subset [2,\ell-1]} (-1)^{|J|+1} Y_{J \cup \{1,\ell\}}(s) \frac{\partial^k Y_{[2,\ell-1] \setminus J}(s)}{\partial s^k}. \tag{1-15}$$

Every  $I \subset [\ell]$  corresponds to a vertex  $v^I$  of  $\mathcal{C}_\ell$  such that  $v_i^I$  is 1 if  $i \in I$  and 0 otherwise. The subscripts appearing in (1-15) are those subsets which either contain both 1 and  $\ell$  or neither of them. The corresponding vertices are the vertices of an  $(\ell - 1)$ -dimensional parallelepiped  $D$ . Note that  $R(D)$  is embedded

into  $R(\mathcal{C}_\ell)$  as the subalgebra generated by the  $Y_I$  with  $v^I \in D$ . Consequently, we can view  $J^\infty(R(D))$  as the subalgebra in  $J^\infty(R(\mathcal{C}_\ell))$  generated by the corresponding  $Y_I^{(j)}$ .

Now,  $D$  can be identified with  $\mathcal{C}_{\ell-1}$  by projecting along the  $\ell$ -th coordinate. This projection is unimodular (it identifies the lattice  $\text{span}(D) \cap \mathbb{Z}^\ell$  with  $\mathbb{Z}^{\ell-1}$ ); therefore it induces isomorphisms  $R(D) \simeq R(\mathcal{C}_{\ell-1})$  and  $J^\infty(R(D)) \simeq J^\infty(R(\mathcal{C}_{\ell-1}))$ . The image of (1-15) under the latter isomorphism is  $-W_{\ell-1,k}$  which is nilpotent by the induction hypothesis.

The summands in  $W_{\ell,k}$  not found in (1-15) are those featuring subscripts which contain exactly one of 1 and  $\ell$ . The corresponding vertices again form the vertex set of an  $(\ell-1)$ -dimensional parallelepiped and the same argument shows that the remaining sum is also nilpotent.

Now we consider  $k = \ell - 2$ . First, for integers  $n \in [2, \ell]$ ,  $m \geq 0$  and  $p \in [0, m]$  consider the expression

$$\widetilde{W}_{n,m,p} = \sum_{I \subset [\ell-n+2, \ell]} (-1)^{|I|} \frac{\partial^p Y_{I \cup \{\ell-n+1\}}(s)}{\partial s^p} \frac{\partial^{m-p} Y_{[\ell-n+2, \ell] \setminus I}(s)}{\partial s^{m-p}} \in J^\infty(R(\mathcal{C}_\ell))[[s]].$$

In particular,  $\widetilde{W}_{\ell,m,0} = W_{\ell,m}$  for  $m \leq \ell - 2$ .

Let  $n$  be either  $\ell - 1$  or  $\ell$ . By, respectively, either the induction hypothesis applied to the Boolean lattice of subsets in  $[2, \ell]$  or the above case  $k < \ell - 2$ , we have  $\widetilde{W}_{n,m,0} \equiv 0$  for  $m \leq \ell - 3$  (recall that  $a \equiv b$  means that  $a - b$  is nilpotent). Then, by Lemma 1.16,

$$\frac{\partial}{\partial s} \widetilde{W}_{n,m,0} = \widetilde{W}_{n,m+1,0} + \widetilde{W}_{n,m+1,1} \equiv 0$$

for  $m \leq \ell - 3$ . We deduce that  $\widetilde{W}_{n,m,1} \equiv 0$  for  $m \leq \ell - 3$ . Iterating this process we obtain that all  $\widetilde{W}_{n,m,p} \equiv 0$  when  $m \leq \ell - 3$ . Hence, by writing the derivative of  $\widetilde{W}_{n,\ell-3,p} \equiv 0$  for any  $p \in [0, \ell - 3]$  we obtain  $\widetilde{W}_{n,\ell-2,p} = -\widetilde{W}_{n,\ell-2,p+1}$ . Together these  $\ell - 2$  equivalences provide

$$\widetilde{W}_{n,\ell-2,0} \equiv (-1)^{\ell-2} \widetilde{W}_{n,\ell-2,\ell-2} \quad \text{for } n = \ell - 1, \ell. \tag{1-16}$$

Now choose  $J \subset [2, \ell]$ . Multiplying  $W_{\ell,\ell-2}$  by  $Y_J(s)$  and using the relations (1-13) we have

$$Y_J(s) \sum_{I \subset [2, \ell]} (-1)^{|I|} Y_{I \cup \{1\}}(s) \frac{\partial^{\ell-2} Y_{[2, \ell] \setminus I}(s)}{\partial s^{\ell-2}} = Y_{J \cup \{1\}}(s) \sum_{I \subset [2, \ell]} (-1)^{|I|} Y_I(s) \frac{\partial^{\ell-2} Y_{[2, \ell] \setminus I}(s)}{\partial s^{\ell-2}}.$$

In view of (1-16) for  $n = \ell - 1$ , we have

$$\sum_{I \subset [2, \ell]} (-1)^{|I|} Y_I(s) \frac{\partial^{\ell-2} Y_{[2, \ell] \setminus I}(s)}{\partial s^{\ell-2}} = -\widetilde{W}_{\ell-1,\ell-2,0} + (-1)^{\ell-2} \widetilde{W}_{\ell-1,\ell-2,\ell-2} \equiv 0. \tag{1-17}$$

We see that  $Y_J(s)W_{\ell,\ell-2} \equiv 0$  for any  $Y_J(s)$  with  $J \subset [2, \ell]$ . By applying (1-16) for  $n = \ell$  we also have  $Y_J(s)\widetilde{W}_{\ell,\ell-2,\ell-2} \equiv 0$ . However, it is evident from the relations (1-13) that  $J^\infty(R(\mathcal{C}_\ell))[[s]]$  admits an involution exchanging  $Y_I^{(j)}$  and  $Y_{[\ell] \setminus I}^{(j)}$ . Applying this involution to  $Y_J(s)\widetilde{W}_{\ell,\ell-2,\ell-2}$  we obtain  $Y_{J'}(s)W_{\ell,\ell-2}$ , where  $1 \in J'$ , and we see that this product is also nilpotent. As a result, all  $Y_J(s)W_{\ell,\ell-2} \equiv 0$  and, since  $W_{\ell,\ell-2}^2$  is a sum of multiples of various  $Y_J(s)W_{\ell,\ell-2}$ , we have  $W_{\ell,\ell-2} \equiv 0$ .  $\square$

**Remark 1.32.** Consider  $r \leq \ell$  and an affine embedding  $\mathcal{C}_r \hookrightarrow \mathcal{C}_\ell$  mapping vertices to vertices. This induces a map  $f : J^\infty(R(\mathcal{C}_r)) \rightarrow J^\infty(R(\mathcal{C}_\ell))$  and any  $f(W_{r,k})$  will lie in the nilradical of  $J^\infty(R(\mathcal{C}_\ell))$ . It can be shown that all elements of this form generate the nilradical, more specifically, a minimal set of such relations is obtained in Section 5.1.

## 2. Symmetric polynomials and dual spaces of graded components

**2.1. Arcs over the polynomial ring.** In this subsection we use the notation from Section 1.5 and discuss the arc ring  $\mathbb{J}_n$  of a polynomial ring. Note that, similarly to  $\mathbb{J}_n$ , any arc ring  $J^\infty(R)$  has a grading grad given by  $\text{grad } r^{(d)} = d$  for any  $r \in R$ . If  $B$  is any grad-homogeneous subspace, quotient or subquotient of an arc ring, we will denote by  $B^{(d)}$  its component spanned by  $b$  with  $\text{grad } b = d$ . In particular, if all  $B^{(d)}$  are finite-dimensional (for example, when  $B = \mathbb{J}_n[\bar{a}]$ ,  $\bar{a} \in \mathbb{N}^n$ ) we will understand the dual space  $B^*$  to be the graded dual with respect to grad.

Choose  $\bar{a} \in \mathbb{N}^n$  and consider  $a_1 + \dots + a_n$  formal variables

$$s_1^{(1)}, \dots, s_1^{(a_1)}, \dots, s_n^{(1)}, \dots, s_n^{(a_n)}.$$

Let  $\epsilon_i \in \mathbb{N}^n$  be the vector with its  $i$ -th coordinate equal to 1 and all other coordinates 0. We have

$$X_i(s_i^{(j)}) \in \mathbb{J}_n[\epsilon_i] \widehat{\otimes} \mathbb{k}[s_i^{(j)}],$$

where we view any  $\mathbb{J}_n[\bar{a}]$  as a topological vector space with a neighborhood base at 0 consisting of the spaces  $\bigoplus_{d \geq N} \mathbb{J}_n[\bar{a}]^{(d)}$  for  $N \geq 0$ . Consider the expression

$$\begin{aligned} U_{\bar{a}}(X_1, \dots, X_n) &:= X_1(s_1^{(1)}) \cdots X_1(s_{a_1}^{(1)}) \cdots X_n(s_1^{(n)}) \cdots X_n(s_{a_n}^{(n)}) \\ &\in \mathbb{J}_n[\bar{a}] \widehat{\otimes} \mathbb{k}[s_1^{(1)}, \dots, s_1^{(a_1)}, \dots, s_n^{(1)}, \dots, s_n^{(a_n)}]. \end{aligned} \tag{2-1}$$

We have

$$U_{\bar{a}}(X_1, \dots, X_n) = \sum_{(k_i^j | 1 \leq i \leq n, 1 \leq j \leq a_i) \in \mathbb{N}^{a_1 + \dots + a_n}} \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} (s_i^{(j)})^{k_i^j} \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)}.$$

Collecting all terms with the same  $\prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)}$  we obtain

$$\begin{aligned} &U_{\bar{a}}(X_1, \dots, X_n) \\ &= \sum_{\substack{(k_i^j | 1 \leq i \leq n, 1 \leq j \leq a_i) \in \mathbb{N}^{a_1 + \dots + a_n} \text{ with} \\ k_i^1 \leq k_i^2 \leq \dots \leq k_i^{a_i} \text{ for every } i}} \left( \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)} \left( \sum_{(\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_n}} \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} (s_i^{(\sigma_i(j))})^{k_i^j} \right) \right). \end{aligned}$$

Therefore the coefficients at  $\prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)}$  form a linear basis of the ring

$$\Lambda_{\bar{a}}(\{s_i^{(j)}\}_{1 \leq i \leq n, 1 \leq j \leq a_i}) := \mathbb{k}[s_i^{(j)}]_{1 \leq i \leq n, 1 \leq j \leq a_i}^{\mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_n}} \tag{2-2}$$

of partially symmetric functions. Here the factor  $\mathfrak{S}_{a_i}$  permutes the variables  $s_1^{(1)}, \dots, s_1^{(a_1)}$  and so on. We will shorten the above notation to  $\Lambda_{\bar{a}}(s)$ , where we denote  $s = \{s_i^{(j)}\}_{1 \leq i \leq n, 1 \leq j \leq a_i}$ . In particular, we have

$$U_{\bar{a}}(X_1, \dots, X_n) \in \mathbb{J}_n[\bar{a}] \widehat{\otimes} \Lambda_{\bar{a}}(s). \tag{2-3}$$

Consider a linear functional  $\xi \in \mathbb{J}_n[\bar{a}]^*$  and let  $1$  be the identity isomorphism on  $\Lambda_{\bar{a}}(\mathbf{s})$ . We have the map  $\xi \widehat{\otimes} 1 : \mathbb{J}_n[\bar{a}] \widehat{\otimes} \Lambda_{\bar{a}}(\mathbf{s}) \rightarrow \Lambda_{\bar{a}}(\mathbf{s})$  and denote

$$f_\xi := (\xi \widehat{\otimes} 1)(U_{\bar{a}}(X_1, \dots, X_n)) \in \Lambda_{\bar{a}}(\mathbf{s}). \quad (2-4)$$

Set  $\mathbb{J}_n^* = \bigoplus_{\bar{a} \in \mathbb{N}^n} \mathbb{J}_n[\bar{a}]^*$ . We immediately obtain:

**Lemma 2.1.** *The map  $\gamma_{\bar{a}} : \xi \mapsto f_\xi$  is a linear isomorphism from  $\mathbb{J}_n[\bar{a}]^*$  to  $\Lambda_{\bar{a}}(\mathbf{s})$ . The map  $\gamma = \bigoplus_{\bar{a} \in \mathbb{N}^n} \gamma_{\bar{a}}$  is an isomorphism from  $\mathbb{J}_n^*$  to  $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\mathbf{s})$ .*

For vectors  $\bar{a}, \bar{b} \in \mathbb{N}^n$  we have a natural map

$$\Delta_{a,b} : \Lambda_{\bar{a}+\bar{b}}(\mathbf{s}) \hookrightarrow \Lambda_{\bar{a}}(\mathbf{s}) \otimes \Lambda_{\bar{b}}(\{s_i^{(j)}\}_{1 \leq i \leq n, 1+a_i \leq j \leq b_i+a_i}) \simeq \Lambda_{\bar{a}}(\mathbf{s}) \otimes \Lambda_{\bar{b}}(\mathbf{s}) \quad (2-5)$$

because any polynomial symmetric in  $s_i^{(1)}, \dots, s_i^{(a_i)}, s_i^{(a_i+1)}, \dots, s_i^{(a_i+b_i)}$  is symmetric separately in  $s_i^{(1)}, \dots, s_i^{(a_i)}$  and  $s_i^{(a_i+1)}, \dots, s_i^{(a_i+b_i)}$ . We choose a concrete isomorphism on the right in (2-5) by identifying the variable  $s_i^{(a_i+j)}$  in  $\Lambda_{\bar{b}}(\{s_i^{(j)}\}_{1 \leq i \leq n, 1+a_i \leq j \leq b_i+a_i})$  with the variable  $s_i^{(j)}$  in  $\Lambda_{\bar{b}}(\mathbf{s})$ . Then (2-5) provides a comultiplication map  $\Delta$  on the space  $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\mathbf{s})$ . This comultiplication is obviously coassociative.

**Proposition 2.2.** *The comultiplication map  $\Delta$  is dual to the multiplication in the ring  $\mathbb{J}_n$  with respect to  $\gamma$ .*

*Proof.* Consider the comultiplication  $\widetilde{\Delta} = (\gamma^{-1} \otimes \gamma^{-1})\Delta\gamma$  on  $\mathbb{J}_n^*$ . We need to show that for any  $X, Y \in \mathbb{J}_n$  and  $\xi \in \mathbb{J}_n^*$  we have  $\xi(XY) = \widetilde{\Delta}(\xi)(X \otimes Y)$ . By linearity it suffices to consider the case when  $X$  and  $Y$  are monomials and  $\xi(Z) = 1$  for some monomial  $Z$  while  $\xi$  vanishes on all other monomials. We are to check that  $\widetilde{\Delta}(\xi)(X \otimes Y)$  equals 1 if  $XY = Z$  and 0 otherwise. This follows directly from the definitions of  $\gamma$  and  $\Delta$ .  $\square$

In the following corollary and below we often identify  $\mathbb{J}_n[a]^*$  with  $\Lambda_{\bar{a}}(\mathbf{s})$  via  $\gamma_{\bar{a}}$ .

**Corollary 2.3.** (a) *Consider an ideal  $I \subset \mathbb{J}_n$ . Then the subspace in  $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\mathbf{s})$  which annihilates  $I$  is a subcoalgebra.*

(b) *Consider a subalgebra  $S \subset \mathbb{J}_n$ . Then the subspace in  $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\mathbf{s})$  which annihilates  $S$  is a coideal.*

**2.2. Action of the current Lie algebra on dual components.** Since the polynomial algebra  $R = \mathbb{k}[X_1, \dots, X_n]$  is  $\mathbb{N}^n$ -graded, it is acted upon by an  $n$ -dimensional abelian Lie algebra  $\mathfrak{h} = \langle h_1, \dots, h_n \rangle$ . For  $r \in R[\bar{a}]$  we have  $h_i(p) = a_i p$ . The Lie algebra  $\mathfrak{h}$  acts by derivations and, applying Definition 1.21, we obtain derivations  $h_i^{(k)}$  on  $\mathbb{J}_n$  given by

$$h_{i_1}^{(k)}(X_{i_2}^{(j)}) = \delta_{i_1, i_2} X_{i_2}^{(j-k)}.$$

By Lemma 1.23 we have an action of  $\mathfrak{h}[s]$  on  $\mathbb{J}_n$  with  $h_i s^k$  acting as  $h_i^{(k)}$ . This  $\mathfrak{h}[s]$ -action is faithful; hence  $h_i s^k \mapsto h_i^{(k)}$  provides an embedding  $\mathfrak{h}[s] \hookrightarrow \text{Der } \mathbb{J}_n$  and allows us to identify  $h_i s^k$  with  $h_i^{(k)}$ . This action also preserves the  $\mathbb{N}^n$ -grading, i.e.,

$$\mathfrak{h}[s](\mathbb{J}_n[\bar{a}]) \subset \mathbb{J}_n[\bar{a}].$$

We may extend the action of  $\mathfrak{h}[s]$  to  $\mathbb{J}_n[[s]]$  (coefficientwise) and  $\mathbb{J}_n \widehat{\otimes} \Lambda_{\bar{a}}(s)$  (by acting on the first factor). In particular, we have

$$h_{i_1}^{(k)} X_{i_2}(s) = \delta_{i_1, i_2} s^k X_{i_2}(s). \tag{2-6}$$

**Lemma 2.4.** 
$$h_i^{(k)}(U_{\bar{a}}(X_1, \dots, X_n)) = U_{\bar{a}}(X_1, \dots, X_n) \sum_{j=1}^{a_i} (s_i^{(j)})^k.$$

*Proof.* This follows from (2-6) by the Leibniz rule. □

**Proposition 2.5.** *The action of  $\mathfrak{h}[s]$  on  $\Lambda_{\bar{a}}(s)$  dual (with respect to  $\gamma_{\bar{a}}$ ) to the  $\mathfrak{h}[s]$ -action on  $\mathbb{J}_n[\bar{a}]$  is given by*

$$h_i^{(k)}(f) = - \sum_{j=1}^{a_i} (s_i^{(j)})^k f$$

for any  $f \in \Lambda_{\bar{a}}(s)$ .

*Proof.* Choose  $f \in \Lambda_{\bar{a}}(s)$  and let  $\xi = \gamma_{\bar{a}}^{-1}(f)$ . Then for the dual  $\mathfrak{h}[s]$ -action we have

$$\begin{aligned} h_i^{(k)}(f) &= (h_i^{(k)}(\xi) \widehat{\otimes} 1)(U_{\bar{a}}(X_1, \dots, X_n)) = -(\xi \widehat{\otimes} 1)(h_i^{(k)}(U_{\bar{a}}(X_1, \dots, X_n))) \\ &= -(\xi \widehat{\otimes} 1)(U_{\bar{a}}(X_1, \dots, X_n)) \sum_{j=1}^{a_i} (s_i^{(j)})^k = - \sum_{j=1}^{a_i} (s_i^{(j)})^k f. \end{aligned} \tag{2-7}$$

This completes the proof. □

The action (1-9) of the Lie algebra  $\text{Der}^c \mathbb{k}[[s]]$  also preserves graded components of the arc ring  $\mathbb{J}_n$ .

**Proposition 2.6.** *The action of  $\text{Der}^c \mathbb{k}[[s]]$  on  $\Lambda_{\bar{a}}(s)$  dual to the action on  $\mathbb{J}_n[\bar{a}]$  is given by*

$$d_k(f) = - \left( \sum_{1 \leq i \leq n, 1 \leq j \leq a_i} (s_i^{(j)})^{k+1} \frac{\partial}{\partial s_i^{(j)}} \right) f.$$

*Proof.* We have

$$d_k(X_i(s_i^{(j)})) = (s_i^{(j)})^{k+1} \frac{\partial}{\partial s_i^{(j)}} X_i(s_i^{(j)}).$$

Therefore, by the Leibniz rule,

$$d_k(U_{\bar{a}}(X_1, \dots, X_n)) = \left( \sum_{1 \leq i \leq n, 1 \leq j \leq a_i} (s_i^{(j)})^{k+1} \frac{\partial}{\partial s_i^{(j)}} \right) U_{\bar{a}}(X_1, \dots, X_n).$$

The proof is now completed similarly to the previous proposition. □

**2.3. Arcs and quadratic monomial ideals.** Consider a set of pairs

$$\pi = \{(\alpha_1, \beta_1), \dots, (\alpha_p, \beta_p)\}, \quad \alpha_i, \beta_i \in [n], \alpha_i < \beta_i$$

and the monomial ideal  $\mathcal{M}_\pi = \langle X_{\alpha_i} X_{\beta_i} \rangle_{i=1, \dots, p}$  in  $\mathbb{k}[X_1, \dots, X_n]$ . The goal of this subsection is to describe the space dual to the ring  $J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)$  in terms of symmetric functions.

We have the surjection  $\rho : \mathbb{J}_n \rightarrow J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)$  with kernel generated by coefficients of the series

$$X_{\alpha_i}(s)X_{\beta_i}(s), \quad (\alpha_i, \beta_i) \in \pi. \quad (2-8)$$

This kernel is  $\mathbb{N}^n$ -homogeneous; hence, the ring  $J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)$  is  $\mathbb{N}^n$ -graded and  $\rho$  restricts to  $\mathbb{N}^n$ -homogeneous surjections  $\rho[\bar{a}]$  from  $\mathbb{J}_n[\bar{a}]$  to  $J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]$ . Therefore, the (graded) dual of the homogeneous component  $J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]$  can be naturally viewed as a subspace in  $\mathbb{J}_n[\bar{a}]^*$ .

The kernel of  $\rho$  is  $\mathfrak{h}[s]$  invariant. Therefore, the map  $\rho$  is a homomorphism of  $\mathfrak{h}[s]$ -modules and thus the dual of the homogeneous component  $J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]$  has a structure of a  $\mathfrak{h}[s]$ -submodule.

**Proposition 2.7.** *The subspace*

$$\gamma_{\bar{a}}(J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]^*) \subset \Lambda_{\bar{a}}(\mathbf{s})$$

is the principal ideal generated by

$$\prod_{\substack{(\alpha, \beta) \in \pi \\ 1 \leq i \leq a_\alpha, 1 \leq j \leq a_\beta}} (s_\alpha^{(i)} - s_\beta^{(j)}). \quad (2-9)$$

This principal ideal is a cyclic  $\mathfrak{h}[s]$ -module generated by (2-9) that is isomorphic to  $\Lambda_{\bar{a}}(\mathbf{s})$ .

*Proof.* In view of (2-8), the kernel of  $\rho[\bar{a}]$  is the linear span of coefficients of the series

$$U_{\bar{a}-e_\alpha-e_\beta}(X_1, \dots, X_n)X_\alpha(t)X_\beta(t) = U_{\bar{a}}(X_1, \dots, X_n)|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t} \quad (2-10)$$

for  $(\alpha, \beta) \in \pi$ , where  $e_\alpha$  is the  $\alpha$ -th basis vector in  $\mathbb{N}^n$ . For  $\xi \in \mathbb{J}_n[\bar{a}]^*$  we have

$$\begin{aligned} (\xi \widehat{\otimes} 1)(U_{\bar{a}}(X_1, \dots, X_n)|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t}) &= (\xi \widehat{\otimes} 1)(U(X_1, \dots, X_n)\bar{a})|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t} \\ &= \gamma_{\bar{a}}(\xi)|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t}. \end{aligned}$$

Thus,  $\xi(\ker \rho[\bar{a}]) = 0$  if and only if  $\gamma_{\bar{a}}(\xi)|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t} = 0$ . By symmetry,  $\gamma_{\bar{a}}(\xi)$  must then be divisible by (2-9). The last claim follows from (2-7). This completes the proof.  $\square$

We also need the following generalization of this proposition. For each pair of indices  $(\alpha, \beta)$ ,  $1 \leq \alpha < \beta \leq n$  choose a number  $k(\alpha, \beta) \in \mathbb{N}$ . Denote the collection of these numbers by  $\vec{k}$ . Consider the ideal  $\mathcal{I}_{\vec{k}}$  generated by the coefficients of

$$\left\langle X_{\alpha_i}(s) \frac{\partial^{k'} X_{\beta_i}(s)}{\partial s^{k'}} \right\rangle, \quad 1 \leq \alpha < \beta \leq n, \quad 0 \leq k' < k(\alpha, \beta). \quad (2-11)$$

We denote by  $\rho$  the natural surjection  $\mathbb{J}_n \twoheadrightarrow \mathbb{J}_n/\mathcal{I}_{\vec{k}}$  and let  $\rho[\bar{a}] : \mathbb{J}_n[\bar{a}] \twoheadrightarrow (\mathbb{J}_n/\mathcal{I}_{\vec{k}})[\bar{a}]$  be its  $\mathbb{N}^n$ -homogeneous components. As before  $\mathcal{I}_{\vec{k}}$  is  $\mathfrak{h}[s]$ -invariant. Therefore,  $\rho[\bar{a}]$  is a map of  $\mathfrak{h}[s]$ -modules. As in the previous case we identify  $(\mathbb{J}_n/\mathcal{I}_{\vec{k}})[\bar{a}]^*$  with a principal ideal in  $\Lambda_{\bar{a}}(\mathbf{s})$ .

**Proposition 2.8.** *The subspace  $\gamma_{\bar{a}}((\mathbb{J}_n/\mathcal{I}_{\bar{k}})[\bar{a}]^*) \subset \Lambda_{\bar{a}}(\mathfrak{s})$  is the principal ideal generated by*

$$\prod_{\substack{1 \leq \alpha < \beta \leq n \\ 1 \leq i \leq a_\alpha, 1 \leq j \leq a_\beta}} (s_\alpha^{(i)} - s_\beta^{(j)})^{k(\alpha, \beta)}. \tag{2-12}$$

*This principal ideal is a cyclic  $\mathfrak{h}[s]$ -module generated by (2-12) that is isomorphic to  $\Lambda_{\bar{a}}(\mathfrak{s})$ .*

*Proof.* By definition,  $\ker \rho[\bar{a}]$  is the linear span of coefficients of the series

$$U_{\bar{a}-e_\alpha-e_\beta}(X_1, \dots, X_n) X_\alpha(t) \frac{\partial^{k'} X_\beta(t)}{\partial t^{k'}} = \frac{\partial^{k'} U_{\bar{a}}(X_1, \dots, X_n)}{(\partial s_\beta^{(a_\beta)})^{k'}} \Big|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t}, \tag{2-13}$$

where  $0 \leq k' < k$ . We have, for any  $\xi \in \mathbb{J}_n[\bar{a}]^*$ ,

$$\begin{aligned} (\xi \widehat{\otimes} 1) \left( \frac{\partial^{k'} U_{\bar{a}}(X_1, \dots, X_n)}{(\partial s_\beta^{(a_\beta)})^{k'}} \Big|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t} \right) &= \frac{\partial^{k'} \xi(U_{\bar{a}}(X_1, \dots, X_n))}{(\partial s_\beta^{(a_\beta)})^{k'}} \Big|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t} \\ &= \frac{\partial^{k'} \gamma_{\bar{a}}(\xi)}{(\partial s_\beta^{(a_\beta)})^{k'}} \Big|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t}. \end{aligned}$$

Thus,  $\xi(\ker \rho[\bar{a}]) = 0$  if and only if for any  $\alpha < \beta$  and  $0 \leq k' < k(\alpha, \beta)$  we have

$$\frac{\partial^{k'} \gamma_{\bar{a}}(\xi)}{(\partial s_\beta^{(a_\beta)})^{k'}} \Big|_{s_\alpha^{(a_\alpha)} \mapsto t, s_\beta^{(a_\beta)} \mapsto t} = 0.$$

By symmetry, such  $\gamma_{\bar{a}}(\xi)$  are precisely those divisible by  $(s_\alpha^{(i)} - s_\beta^{(j)})^{k(\alpha, \beta)}$  for any  $\alpha < \beta$ ,  $1 \leq i \leq a_\alpha$  and  $1 \leq j \leq a_\beta$ . The last claim is straightforward.  $\square$

**Remark 2.9.** Consider a finite set  $S \subset \mathbb{N}^n$  and the ideal  $I_S \subset \mathbb{k}[X_1, \dots, X_n]$  generated by monomials  $X^{\bar{b}}$ ,  $\bar{b} \in S$ . Then the subspace  $\gamma_{\bar{a}}(J^\infty(\mathbb{k}[X_1, \dots, X_n]/I_S)[\bar{a}]^*) \subset \Lambda_{\bar{a}}(\mathfrak{s})$  is the ideal

$$\{f \text{ such that } f|_{s_1^{(1)}, \dots, s_1^{(b_1)}, \dots, s_n^{(1)}, \dots, s_n^{(b_n)} \mapsto t} = 0 \text{ for all } \bar{b} \in S\}.$$

This generalizes Propositions 2.7 and 2.8, the proof being similar. We note that in this general case the dual space is still an ideal in  $\Lambda_{\bar{a}}(\mathfrak{s})$  which, however, need not be principal.

### 3. Arc rings of toric rings

**3.1. Graded components of toric rings.** Consider a normal lattice polytope  $P \subset \mathbb{R}_{\geq 0}^n$  and the corresponding toric ring  $R(P)$ . Note that  $R(P)$  is invariant under shifts of  $P$  so we do not lose generality by assuming that  $P$  lies in the positive orthant.

We have an embedding

$$\eta : R(P) \hookrightarrow \mathbb{k}[z_1, \dots, z_n, w]$$

with  $\eta(Y_{\bar{\alpha}}) = z^{\bar{\alpha}} w$  for  $\bar{\alpha} \in P \cap \mathbb{Z}^n$ . We study the corresponding map of arc rings

$$J^\infty(\eta) : J^\infty R(P) \rightarrow J^\infty(\mathbb{k}[z_1, \dots, z_n, w]).$$

By definition, applying  $J^\infty(\eta)$  coefficientwise we have  $J^\infty(\eta)(Y_{\bar{\alpha}}(s)) = z(s)^{\bar{\alpha}} w(s)$ . Let  $\{\bar{\alpha}^1, \dots, \bar{\alpha}^m\}$  be the set of integer points in  $P$ . We have a surjection

$$\tau : \mathbb{k}[X_{\bar{\alpha}^1}, \dots, X_{\bar{\alpha}^m}] \twoheadrightarrow R(P),$$

where  $\tau(X_{\bar{\alpha}^i}) = Y_{\bar{\alpha}^i}$ . We also have the map

$$\eta \circ \tau : \mathbb{k}[X_{\bar{\alpha}^1}, \dots, X_{\bar{\alpha}^m}] \rightarrow \mathbb{k}[z_1, \dots, z_n, w]$$

and, for a vector  $\bar{r} = (r_i) \in \mathbb{N}^m$ ,

$$\eta \circ \tau(X^{\bar{r}}) = z^{\sum_{i=1}^m r_i \bar{\alpha}^i} w^{\sum_{i=1}^m r_i}.$$

Thus,  $\eta \circ \tau$  maps monomials in the  $X_{\bar{\alpha}^i}$  to monomials in the  $z_i$  and  $w$ . For  $\bar{a} \in \mathbb{N}^n$ ,  $L \in \mathbb{N}$  we define

$$\mathcal{R}(\bar{a}, L) = \left\{ \bar{r} \in \mathbb{N}^m \mid \sum_{i=1}^m r_i \bar{\alpha}^i = \bar{a} \text{ and } \sum_{i=1}^m r_i = L \right\}.$$

Then  $\eta \circ \tau(X^{\bar{r}}) = z^{\bar{a}} w^L$  if and only if  $\bar{r} \in \mathcal{R}(\bar{a}, L)$ .

The map  $J^\infty(\tau) : \mathbb{J}_m \rightarrow J^\infty R(P)$  is surjective, since  $J^\infty(\tau)(X_{\bar{\alpha}^i}^{(j)}) = Y_{\bar{\alpha}^i}^{(j)}$  (more generally,  $J^\infty$  is easily seen to preserve epimorphisms). Therefore, we have

$$J^\infty(\eta)(J^\infty R(P)) = J^\infty(\eta \circ \tau)(\mathbb{J}_m) \subset J^\infty(\mathbb{k}[z_1, \dots, z_n, w])$$

and we have a surjective map

$$\nu : J_{\text{red}}^\infty(R(P)) \twoheadrightarrow J^\infty(\eta)(J^\infty R(P)) = J^\infty(\eta \circ \tau)(\mathbb{J}_m). \quad (3-1)$$

By Corollary 1.12,  $\nu$  is an isomorphism.

For the rest of this section we fix a monomial order  $\prec$  on  $\mathbb{k}[X_{\bar{\alpha}^1}, \dots, X_{\bar{\alpha}^m}]$  as well as the corresponding order on  $\mathbb{N}^m$ . As discussed in Section 1.5, we have two  $(\mathbb{N}^m, \prec)$ -filtrations with components  $\mathbb{J}_m[\prec \bar{r}]$  and  $\mathbb{J}_m[\leq \bar{r}]$ . Since both  $J_{\text{red}}^\infty(R(P))$  and  $J^\infty(\eta \circ \tau)(\mathbb{J}_m)$  are surjective images of  $\mathbb{J}_m$  they admit induced  $(\mathbb{N}^m, \prec)$ -filtrations which are seen to be identified by the isomorphism  $\nu$ . Since  $\nu$  respects the grading  $\text{grad}$ , we obtain:

**Lemma 3.1.** *For any  $d \in \mathbb{N}$  and  $\bar{r} \in \mathbb{N}^m$  the map  $\nu$  induces an isomorphism of vector spaces*

$$\text{gr}_{\prec} J_{\text{red}}^\infty R(P)[\bar{r}]^{(d)} \simeq \text{gr}_{\prec} J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{r}]^{(d)}.$$

**3.2. Dual of the inclusion map.** Retaining notation from the previous subsection consider  $\bar{r} \in \mathcal{R}(\bar{a}, L) \subset \mathbb{N}^m$  for some  $\bar{a} \in \mathbb{N}^n$  and  $L \in \mathbb{N}$ . Consider the subspace  $A_{\bar{r}} \subset J^\infty(\mathbb{k}[z_1, \dots, z_n, w])$  spanned by elements of the form

$$\eta(Y_{\bar{\alpha}^1})^{(j_1^1)} \cdots \eta(Y_{\bar{\alpha}^1})^{(j_{r_1}^1)} \cdots \eta(Y_{\bar{\alpha}^m})^{(j_1^m)} \cdots \eta(Y_{\bar{\alpha}^m})^{(j_{r_m}^m)} = J^\infty(\eta) \left( \prod_{i=1}^m \prod_{k=1}^{r_m} Y_{\bar{\alpha}^i}^{(j_k^i)} \right)$$

for all  $j_u^v \in \mathbb{N}$ . This subspace coincides with  $J^\infty(\eta \circ \tau)(\mathbb{J}_m[\bar{r}])$ .



In this subsection we study the dual map to the inclusion

$$\varphi_{\bar{r}} : A_{\bar{r}} \hookrightarrow J^\infty(\mathbb{k}[z_1, \dots, z_n, w][\bar{a}, L]), \tag{3-2}$$

where on the right we have a component of the natural  $\mathbb{N}^{n+1}$ -grading.

We consider the following series in variables  $t_i^{(j)}$  with  $1 \leq i \leq m$  and  $1 \leq j \leq r_i$ :

$$\begin{aligned} V_{\bar{r}} &:= \eta(Y_{\bar{\alpha}^1})(t_1^{(1)}) \cdots \eta(Y_{\bar{\alpha}^1})(t_1^{(r_1)}) \cdots \eta(Y_{\bar{\alpha}^m})(t_m^{(1)}) \cdots \eta(Y_{\bar{\alpha}^m})(t_m^{(r_m)}) \\ &= (z^{\bar{\alpha}^1} w)(t_1^{(1)}) \cdots (z^{\bar{\alpha}^1} w)(t_1^{(r_1)}) \cdots (z^{\bar{\alpha}^m} w)(t_m^{(1)}) \cdots (z^{\bar{\alpha}^m} w)(t_m^{(r_m)}). \end{aligned} \tag{3-3}$$

From the definitions we have:

**Proposition 3.2.** *The space  $A_{\bar{r}}$  is spanned by the coefficients of  $V_{\bar{r}}$  as of a series in the  $t_i^{(j)}$ .*

By Lemma 2.1 we obtain a linear isomorphism

$$\gamma_{\bar{a},L} : J^\infty(\mathbb{k}[z_1, \dots, z_n, w][\bar{a}, L])^* \rightarrow \Lambda_{\bar{a}}(\mathbf{s}) \otimes \Lambda_L(\{s_w^{(j)}\}_{j=1,\dots,L}).$$

We will denote the right-hand side above by  $\Lambda_{\bar{a},L}(\mathbf{s})$ .

Recall that  $\bar{a} = \sum_{j=1}^m r_j \bar{\alpha}^j$  and  $L = \sum_{i=1}^m r_i$ . Consider the homomorphism of polynomial algebras

$$\varphi_{\bar{r}}^\vee : \mathbb{k}[s_i^{(j)}, s_w^{(j')}]_{\substack{1 \leq i \leq n \\ 1 \leq j \leq a_i \\ 1 \leq j' \leq L}} \rightarrow \mathbb{k}[t_i^{(j)}]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq r_i}}$$

given by

$$\varphi_{\bar{r}}^\vee(s_i^{(j)}) = \varphi_{\bar{r}}^\vee(s_w^{(j')}) = \begin{cases} t_1^{(1)} & \text{if } j \in [1, \alpha_i^1] \text{ and } j' = 1, \\ \vdots & \\ t_1^{(r_1)} & \text{if } j \in [(r_1 - 1)\alpha_i^1 + 1, r_1\alpha_i^1] \text{ and } j' = r_1, \\ t_2^{(1)} & \text{if } j \in [r_1\alpha_i^1 + 1, r_1\alpha_i^1 + \alpha_i^2] \text{ and } j' = r_1 + 1, \\ \vdots & \\ t_2^{(r_2)} & \text{if } j \in [r_1\alpha_i^1 + (r_2 - 1)\alpha_i^2 + 1, r_1\alpha_i^1 + r_2\alpha_i^2] \text{ and } j' = r_1 + r_2, \\ \vdots & \end{cases} \tag{3-4}$$

In other words, for a given  $1 \leq i \leq n$  we divide the tuple  $s_i^{(1)}, \dots, s_i^{(a_i)}$  into  $L$  groups of consecutive elements so that the first  $r_1$  groups contain  $\alpha_i^1$  elements, the next  $r_2$  groups contain  $\alpha_i^2$  elements, etc. Then  $\varphi_{\bar{r}}^\vee$  maps the elements in the first  $r_1$  groups to, respectively,  $t_1^{(1)}, \dots, t_1^{(r_1)}$ , the elements in the next  $r_2$  groups to, respectively,  $t_2^{(1)}, \dots, t_2^{(r_2)}$  and so on. As for the variables  $s_w^{(1)}, \dots, s_w^{(L)}$ , the first  $r_1$  of them are mapped to, respectively,  $t_1^{(1)}, \dots, t_1^{(r_1)}$ , the next  $r_2$  are mapped to, respectively,  $t_2^{(1)}, \dots, t_2^{(r_2)}$ , etc.

**Remark 3.3.** The map  $\varphi_{\bar{r}}^\vee$  can be described in the following way. The ring  $\Lambda_{\bar{a},L}(\mathbf{s})$  is generated by power sums

$$p_k(s_i^{(1)}, \dots, s_i^{(a_i)}) = \sum_{j=1}^{a_i} (s_i^{(j)})^k \quad \text{and} \quad p_k(s_w^{(1)}, \dots, s_w^{(L)}) = \sum_{j=1}^L (s_w^{(j)})^k.$$

We have

$$\varphi_{\bar{r}}^{\vee}(p_k(s_i^{(1)}, \dots, s_i^{(a_i)})) = \sum_{j=1}^m \alpha_i^j p_k(t_j^{(1)}, \dots, t_j^{(r_j)}) \quad \text{and} \quad \varphi_{\bar{r}}^{\vee}(p_k(s_w^{(1)}, \dots, s_w^{(L)})) = \sum_{j=1}^m p_k(t_j^{(1)}, \dots, t_j^{(r_j)}).$$

In particular,  $\varphi_{\bar{r}}^{\vee}(\Lambda_{\bar{a},L}(\mathbf{s}))$  is generated as a ring by the elements  $\sum_{j=1}^m \alpha_i^j p_k(t_j^{(1)}, \dots, t_j^{(r_j)})$ ,  $i = 1, \dots, n$  and  $\sum_{j=1}^m p_k(t_j^{(1)}, \dots, t_j^{(r_j)})$ .

**Example 3.4.** Let  $P$  be the convex hull of  $\{(0, 0), (1, 2), (2, 1)\}$  and let  $\alpha_1 = (0, 0)$ ,  $\alpha_2 = (1, 1)$ ,  $\alpha_3 = (2, 1)$ ,  $\alpha_4 = (1, 2)$ . Consider  $\bar{r} = (1, 1, 1, 1) \in \mathcal{R}((4, 4), 4)$ . Then  $\varphi_{\bar{r}}^{\vee}$  is given by

$$s_w^{(1)} \mapsto t_1^{(1)}, \quad s_w^{(2)}, s_1^{(1)}, s_2^{(1)} \mapsto t_2^{(1)}, \quad s_w^{(3)}, s_1^{(2)}, s_1^{(3)}, s_2^{(2)} \mapsto t_3^{(1)}, \quad s_w^{(4)}, s_1^{(4)}, s_2^{(3)}, s_2^{(4)} \mapsto t_4^{(1)}$$

and we have

$$\begin{aligned} \varphi_{\bar{r}}^{\vee}(p_k(s_w^{(1)}, s_w^{(2)}, s_w^{(3)}, s_w^{(4)})) &= p_k(t_1^{(1)}) + p_k(t_2^{(1)}) + p_k(t_3^{(1)}) + p_k(t_4^{(1)}), \\ \varphi_{\bar{r}}^{\vee}(p_k(s_1^{(1)}, s_1^{(2)}, s_1^{(3)}, s_1^{(4)})) &= p_k(t_2^{(1)}) + 2p_k(t_3^{(1)}) + p_k(t_4^{(1)}), \\ \varphi_{\bar{r}}^{\vee}(p_k(s_2^{(1)}, s_2^{(2)}, s_2^{(3)}, s_2^{(4)})) &= p_k(t_2^{(1)}) + p_k(t_3^{(1)}) + 2p_k(t_4^{(1)}). \end{aligned}$$

**Lemma 3.5.** *The image  $\varphi_{\bar{r}}^{\vee}(\Lambda_{\bar{a},L}(\mathbf{s}))$  is contained in the subring  $\Lambda_{\bar{r}}(\mathbf{t})$ .*

*Proof.* The ring  $\Lambda_{\bar{a},L}(\mathbf{s})$  decomposes into a product of its subspaces as

$$\Lambda_{\bar{a},L}(\mathbf{s}) = \Lambda_{a_1}(\{s_1^{(j)}\}_{j=1, \dots, a_1}) \cdots \Lambda_{a_n}(\{s_n^{(j)}\}_{j=1, \dots, a_n}) \Lambda_L(\{s_w^{(j)}\}_{j=1, \dots, L}).$$

The image under  $\varphi_{\bar{r}}^{\vee}$  of a polynomial in  $\Lambda_{a_i}(\{s_i^{(j)}\}_{j=1, \dots, a_i})$  is evidently symmetric in  $t_{\ell}^{(1)}, \dots, t_{\ell}^{(r_{\ell})}$  for each  $\ell$ . The same holds for any polynomial in the image of  $\Lambda_L(\{s_w^{(j)}\}_{j=1, \dots, L})$  and the lemma follows.  $\square$

Recall the inclusion map  $\varphi_{\bar{r}}$  defined in (3-2). We denote by

$$\varphi_{\bar{r}}^* : J^{\infty}(\mathbb{k}[z_1, \dots, z_n, w])[\bar{a}, L]^* \rightarrow A_{\bar{r}}^*$$

the (graded) dual map for this inclusion.

**Theorem 3.6.** *The annihilator of  $A_{\bar{r}}$  in  $J^{\infty}(\mathbb{k}[z_1, \dots, z_n, w])[\bar{a}, L]^*$  is the subspace  $\ker(\varphi_{\bar{r}}^{\vee} \gamma_{\bar{a},L})$ . In other words, there exists a linear isomorphism*

$$\varepsilon_{\bar{r}} : A_{\bar{r}}^* \xrightarrow{\sim} \varphi_{\bar{r}}^{\vee}(\Lambda_{\bar{a},L}(\mathbf{s}))$$

such that  $\varepsilon_{\bar{r}} \varphi_{\bar{r}}^* = \varphi_{\bar{r}}^{\vee} \gamma_{\bar{a},L}$ .

*Proof.* The two claims are equivalent because the annihilator of  $A_{\bar{r}}$  is  $\ker \varphi_{\bar{r}}^*$ , i.e., both state that  $\varphi_{\bar{r}}^*$  and  $\varphi_{\bar{r}}^{\vee} \gamma_{\bar{a},L}$  have the same kernel.

Recall the expression  $U_{\bar{a}}(z_1, \dots, z_n)$  defined in (2-1) and consider

$$U = U_{\bar{a}}(z_1, \dots, z_n) w(s_w^{(1)}) \cdots w(s_w^{(L)}) \in J^{\infty}(\mathbb{k}[z_1, \dots, z_n, w])[\bar{a}, L] \widehat{\otimes} \Lambda_{\bar{a},L}(\mathbf{s}).$$

By definition we have  $(1 \widehat{\otimes} \varphi_{\bar{r}}^{\vee})(U) = V_{\bar{r}}$ .

Consider a functional  $\xi \in J^\infty(\mathbb{k}[z_1, \dots, z_n, w])[\bar{a}, L]^*$ . We are to show that  $\xi(A_{\bar{r}}) = 0$  if and only if  $\varphi_{\bar{r}}^\vee \gamma_{\bar{a}, L}(\xi) = 0$ . Note that we may view  $V_{\bar{r}}$  as an element of

$$J^\infty(\mathbb{k}[z_1, \dots, z_n, w])[\bar{a}, L] \widehat{\otimes} \Lambda_{\bar{r}}(\mathfrak{t})$$

and that, in view of Proposition 3.2,  $\xi(A_{\bar{r}}) = 0$  if and only if  $(\xi \widehat{\otimes} 1)(V_{\bar{r}}) = 0$ . However,

$$(\xi \widehat{\otimes} 1)(V_{\bar{r}}) = (\xi \widehat{\otimes} 1)(1 \widehat{\otimes} \varphi_{\bar{r}}^\vee)(U) = (1 \widehat{\otimes} \varphi_{\bar{r}}^\vee)(\xi \widehat{\otimes} 1)(U) = \varphi_{\bar{r}}^\vee(\gamma_{\bar{a}, L}(\xi)). \quad \square$$

Recall that the monomial order  $\prec$  induces a filtration on and an associated graded for  $J^\infty(\eta \circ \tau)(\mathbb{J}_m)$ . We can write the filtration components as

$$J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\leq \bar{r}] = \sum_{\bar{r}' \leq \bar{r}} A_{\bar{r}'}$$

and similarly for  $J^\infty(\eta \circ \tau)(\mathbb{J}_m)[< \bar{r}]$ . This lets us write the components of the associated graded as

$$\text{gr}_{\prec} J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{r}] = A_{\bar{r}} / \left( A_{\bar{r}} \cap \sum_{\bar{r}' \prec \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} A_{\bar{r}'} \right).$$

We can now apply Theorem 3.6 to identify the (graded) dual space  $\text{gr}_{\prec} J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{r}]^*$  with an image of  $\varphi_{\bar{r}}^\vee$ .

**Corollary 3.7.** *We have an isomorphism of grad-graded spaces*

$$\text{gr}_{\prec} J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{r}]^* \simeq \varphi_{\bar{r}}^\vee \left( \bigcap_{\bar{r}' \prec \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} \ker(\varphi_{\bar{r}'}^\vee) \right),$$

where  $\ker$  denotes the kernel in  $\Lambda_{\bar{a}, L}(\mathfrak{s})$ .

**3.3. Current algebra action and  $\mathcal{A}$ -freeness.** In this subsection we denote  $R = R(P)$ . Recall that each ring  $S \in \{R, J^\infty(R), J_{\text{red}}^\infty(R)\}$  possesses an  $\mathbb{N}^{n+1}$ -grading with components  $S[\bar{a}, L]$ . The ring  $R$  admits a natural action of a one-dimensional Lie algebra  $\mathfrak{h} = \mathbb{k}h$  given by  $hr = Lr$  for  $r \in R[\bar{a}, L]$ . In view of Lemma 1.23 we obtain an  $\mathfrak{h}[s]$ -action on  $J^\infty(R)$  given by  $hs^k(r^{(j)}) = (hr)^{(j-k)}$ . By Lemma 1.16 we have an induced  $\mathfrak{h}[s]$ -action on  $J_{\text{red}}^\infty(R)$ . Clearly, the components  $J_{\text{red}}^\infty(R)[\bar{a}, L]$  are preserved by the action of  $\mathfrak{h}[s]$ .

Next, consider the algebra  $\mathbb{k}[z_1, \dots, z_n, w]$ , it also admits an  $\mathfrak{h}$ -action given by  $h(M) = LM$  for  $M = z_{i_1} \dots z_{i_N} w^L$ . By Lemma 1.23 we obtain an action of  $\mathfrak{h}[s]$  on  $J^\infty \mathbb{k}[z_1, \dots, z_n, w]$ . Now recall the isomorphism (3-1),

$$\nu : J_{\text{red}}^\infty(R) \rightarrow J^\infty(\eta \circ \tau)(\mathbb{J}_m).$$

Since the right-hand side above is a subspace in  $J^\infty \mathbb{k}[z_1, \dots, z_n, w]$ , we will view  $\nu$  as an embedding into the latter. It is then seen from the definitions that  $\nu$  is a homomorphism of  $\mathfrak{h}[s]$ -algebras with respect to the  $\mathfrak{h}[s]$ -action on  $J_{\text{red}}^\infty(R)$  defined in the previous paragraph.

**Lemma 3.8.** *Every subspace  $A_{\bar{r}} \subset \nu(J_{\text{red}}^\infty(R))$  is preserved by the  $\mathfrak{h}[s]$ -action.*

*Proof.* For any  $\bar{\alpha}^i$  we have  $hs^k((z^{\bar{\alpha}^i} w)^{(j)}) = (z^{\bar{\alpha}^i} w)^{(j-k)}$ . Hence, applying  $hs^k$  coefficientwise we have  $hs^k((z^{\bar{\alpha}^i} w)(t)) = t^k(z^{\bar{\alpha}^i} w)(t)$ . Therefore,

$$hs^k(V_{\bar{r}}) = \left( \sum_{i=1}^m \sum_{\ell=1}^{r_i} (t_i^{(\ell)})^k \right) V_{\bar{r}}. \tag{3-5}$$

Proposition 3.2 now provides the claim. □

Let us fix  $L \in \mathbb{N}$  and  $\bar{a} \in \mathbb{N}^n$ .

**Corollary 3.9.** *For any  $\bar{r} \in \mathcal{R}(\bar{a}, L)$  we have an  $\mathfrak{h}[s]$ -action on the subquotient*

$$\sum_{\bar{r}' \preceq \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} A_{\bar{r}'} / \sum_{\bar{r}' \prec \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} A_{\bar{r}'}$$

Recall that for every  $L$  the space  $J_{\text{red}}^\infty(R)[\bar{a}, L]$  is graded by grad and the homogeneous components are finite-dimensional. We denote by  $J_{\text{red}}^\infty(R)[\bar{a}, L]^*$  the graded dual space. We denote by  $\mathcal{A}_L \subset U(\mathfrak{h}[s])$  the subalgebra generated by  $hs, \dots, hs^L$ . We will be interested in situations when the action of  $\mathcal{A}_L$  on  $J_{\text{red}}^\infty(R)[\bar{a}, L]^*$  is free for all  $\bar{a}$  and  $L$ .

**Remark 3.10.** An analogous freeness property holds for the homogeneous coordinate ring of the Plücker embedding and it has important representation theoretic meaning (see [Fourier and Littelmann 2007; Feigin and Makedonskyi 2020; Kato 2018]). This property also holds for the Veronese curve (see [Dumanski and Feigin 2023]). It is interesting to investigate the graded rank of these free modules. For example, in the case of the Plücker embedding they are equal to Macdonald polynomials for  $t = 0$ ; see [Ion 2003; Naoi 2012].

For  $\bar{r} \in \mathcal{R}(\bar{a}, L)$  we define the following  $\mathfrak{h}[s]$ -action on the algebra  $\Lambda_{\bar{r}}(t)$ :

$$hs^\ell(f) = \left( \sum_{i=1}^m \sum_{k=1}^{r_i} (t_i^{(k)})^\ell \right) f. \tag{3-6}$$

**Proposition 3.11.** *The space  $\varphi_{\bar{r}}^\vee(\Lambda_{\bar{a}, L}(s))$  is invariant under the action (3-6). The action on this subspace is dual to the action of  $\mathfrak{h}[s]$  on  $A_{\bar{r}}$  with respect to the isomorphism  $\varepsilon_{\bar{r}}$  (Theorem 3.6).*

*Proof.* To prove the first claim note that the first factor in the right-hand side of (3-6) lies in  $\varphi_{\bar{r}}^\vee(\Lambda_{\bar{a}, L}(s))$  by Remark 3.3. The second claim follows from (3-5). □

**Lemma 3.12.** *Assume that the natural action of  $\mathcal{A}_L$  is free on  $\varphi_{\bar{r}}^\vee(\bigcap_{\bar{r}' \prec \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} \ker(\varphi_{\bar{r}'}^\vee))$  for every  $\bar{r} \in \mathcal{R}(\bar{a}, L)$ . Then it is free on the space  $J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{a}, L]^*$ .*

*Proof.* One sees that the isomorphism in Corollary 3.7 is one of  $\mathfrak{h}[s]$ -modules; thus  $J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{a}, L]^*$  has a filtration with free subquotients. However, free modules are projective; hence  $J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar{a}, L]^*$  is isomorphic to the direct sum of its subquotients. □

We will need the following general fact about symmetric polynomials.

**Lemma 3.13.** *Let  $n = n_1 + \dots + n_a$ . The ring  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_a}}$  is free of  $q$ -rank  $\binom{n}{n_1 \dots n_a}_q$  over its subring  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ .*

*Proof.* We are going to prove that it is a direct summand of a free module. Consider the tower of ring extensions,

$$\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_n} \subset \mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_a}} \subset \mathbb{k}[x_1, \dots, x_n].$$

Note that the rightmost ring in this tower is isomorphic to

$$\bigotimes_{i=1}^a \mathbb{k}[x_{n_1+\dots+n_{i-1}+1}, \dots, x_{n_1+\dots+n_i}]$$

and hence is free of  $q$ -rank  $[n_1]_q! \dots [n_a]_q!$  over the second ring in this tower, which is isomorphic to

$$\bigotimes_{i=1}^a \mathbb{k}[x_{n_1+\dots+n_{i-1}+1}, \dots, x_{n_1+\dots+n_i}]^{\mathfrak{S}_{n_i}}$$

(here we use the classical fact that a polynomial ring is free over its subring of symmetric polynomials). Therefore,  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_a}}$  is a direct summand in  $\mathbb{k}[x_1, \dots, x_n]$  as a  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_a}}$ -module, and, in particular, as a  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ -module. Hence,  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_a}}$  is a projective module over  $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_n}$  which by the Quillen–Suslin theorem implies freeness.

Since the ranks of free modules in a tower of extensions multiply, the  $q$ -rank of our extension can be expressed as a quotient of ranks

$$\frac{[n]_q!}{[n_1]_q! \dots [n_a]_q!} = \binom{n}{n_1 \dots n_a}_q. \quad \square$$

Denote by  $I$  the kernel of the projection  $\mathbb{J}_m \rightarrow J_{\text{red}}^\infty(R(P))$ . By Proposition 1.28 the quotient  $\mathbb{J}_m / \text{in}_< I$  is isomorphic to  $\text{gr}_< J_{\text{red}}^\infty(R(P))$ .

**Theorem 3.14.** *Assume that  $\text{in}_< I$  is generated by relations of the form (2-11). Then*

$$\text{gr}_<(J_{\text{red}}^\infty(R(P)))[\bar{a}, L] \simeq J_{\text{red}}^\infty(R(P))[\bar{a}, L]$$

as a  $\mathcal{A}_L$ -module and the dual of this module is free.

*Proof.* Consider  $\bar{r} \in \mathcal{R}(\bar{a}, L)$ . By Proposition 2.8 the space  $\text{gr}_<(J_{\text{red}}^\infty(R(P)))[\bar{r}]^*$  is a free module over  $\Lambda_{\bar{r}}(\mathfrak{t})$ . By (3-6) the algebra  $\mathcal{A}_L$  can be embedded into  $\Lambda_{\bar{r}}(\mathfrak{t})$  as the subalgebra of polynomials symmetric in all  $t_i^{(k)}$ . Therefore, applying Lemma 3.13, we deduce that  $\text{gr}_<(J_{\text{red}}^\infty(R(P)))[\bar{r}]^*$  is free over  $\mathcal{A}_L$ . Lemma 3.12 now shows that  $\text{gr}_<(J_{\text{red}}^\infty(R(P)))[\bar{a}, L]^*$  is free.  $\square$

**3.4. Technical lemma.** Consider integers  $m \geq 1$  and  $\zeta_1, \dots, \zeta_m \geq 0$  together with a vector  $\bar{a} = (a_1, \dots, a_m, a) \in \mathbb{N}^{m+1}$ . We also consider variables  $u_i^{(j)}$  with  $1 \leq i \leq m, 1 \leq j \leq a_i$  and  $s^{(j)}$  with  $1 \leq j \leq a$ . We work with the ring

$$\Lambda_{\bar{a}}(\mathbf{u}, \mathbf{s}) := \mathbb{k}[u_1^{(1)}, \dots, u_1^{(a_1)}, \dots, u_m^{(1)}, \dots, u_m^{(a_m)}, s^{(1)}, \dots, s^{(a)}]^{\mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_m} \times \mathfrak{S}_a}.$$

Within the set of collections of positive integers  $\bar{r} = (r_{i,j})_{1 \leq i \leq m, 0 \leq j \leq \zeta_i}$  we distinguish the subset

$$\mathcal{R}(\bar{a}) := \left\{ \bar{r} \mid \sum_{j=0}^{\zeta_i} r_{i,j} = a_i, \quad \sum_{1 \leq i \leq m, 0 \leq j \leq \zeta_i} r_{i,j} = a \right\}.$$

For each  $\bar{r} \in \mathcal{R}(\bar{a})$  we have the ring

$$\Lambda_{\bar{r}}(\mathbf{t}) := \mathbb{k}[t_{i,j}^{(1)}, \dots, t_{i,j}^{(r_{i,j})}]_{1 \leq i \leq m, 0 \leq j \leq \zeta_i}^{\times_{i,j} \mathfrak{S}_{r_{i,j}}}.$$

We now define a map  $\psi_{\bar{r}} : \Lambda_{\bar{a}}(\mathbf{u}, \mathbf{s}) \rightarrow \Lambda_{\bar{r}}(\mathbf{t})$ . As for  $\varphi_{\bar{r}}^{\vee}$  above we define  $\psi_{\bar{r}}$  as a map between rings of all polynomials and then restrict to subrings of symmetric polynomials, the fact that the image lies in  $\Lambda_{\bar{r}}(\mathbf{t})$  is checked similarly to Lemma 3.5. Consider a variable  $u_i^{(k)}$  and note that we have a unique  $j_0 \in [0, \zeta_i]$  for which  $\sum_{j=1}^{j_0-1} r_{i,j} < k \leq \sum_{j=1}^{j_0} r_{i,j}$ . Denote  $\ell = k - \sum_{j=1}^{j_0} r_{i,j}$ , then we set  $\psi_{\bar{r}}(u_i^{(k)}) = t_{i,j_0}^{(\ell)}$ . Now consider a variable  $s^{(k)}$ . We have a unique  $i_0 \in [1, m]$  for which

$$\sum_{1 \leq i \leq i_0-1, 0 \leq j \leq \zeta_i} jr_{i,j} < k \leq \sum_{1 \leq i \leq i_0, 0 \leq j \leq \zeta_i} jr_{i,j}.$$

Denote the first sum above by  $S$ . We also have a unique  $j_0 \in [0, \zeta_{i_0}]$  for which

$$S + \sum_{j=1}^{j_0-1} jr_{i_0,j} < k \leq S + \sum_{j=1}^{j_0} jr_{i_0,j}.$$

Finally, denote

$$\ell = \left\lceil \frac{k - S - \sum_{j=1}^{j_0-1} jr_{i_0,j}}{j_0} \right\rceil,$$

note that  $\ell \in [1, r_{i_0,j_0}]$ . We set  $\psi_{\bar{r}}(s^{(k)}) = t_{i_0,j_0}^{(\ell)}$ .

Note that  $\psi_{\bar{r}}$  establishes a bijection between all variables of the form  $u_i^{(k)}$  and all variables of the form  $t_{i,j}^{(\ell)}$  for every  $i$ . Meanwhile, the number of variables of the form  $s^{(k)}$  which are mapped into a given  $t_{i,j}^{(\ell)}$  is equal to  $j$ .

**Remark 3.15.** The map  $\psi_{\bar{r}}$  appears naturally in the following context. Consider lattice polytopes  $P \subset \mathbb{R}^n$  and  $Q \subset \mathbb{R}^{n+1}$  such that the following holds. Let  $\bar{\alpha}^1, \dots, \bar{\alpha}^m$  be the integer points of  $P$ . Then the integer points of  $Q$  are precisely the points  $\bar{\beta}_{i,j} = \bar{\alpha}^i \times j$  with  $1 \leq i \leq m, 0 \leq j \leq \zeta_i$ . In particular, the projection of  $Q$  along the  $(n+1)$ -st coordinate is  $P$ . The resulting embedding  $Q \hookrightarrow P \times \mathbb{R}$  gives rise to the map

$$\mathbb{k}[X_{i,j}]_{1 \leq i \leq m, 0 \leq j \leq \zeta_i} \rightarrow \mathbb{k}[X_i]_{1 \leq i \leq m} \otimes \mathbb{k}[z] \quad (3-7)$$

taking  $X_{i,j}$  to  $X_i z^j$ . We may now successively apply  $J^\infty$  to this map, restrict to the component of grading  $\bar{r}$  on the left and the component of grading  $\bar{a}$  on the right and then dualize both sides. We obtain a map  $\psi_{\bar{r}} : \Lambda_{\bar{a}}(\mathbf{u}, \mathbf{s}) \rightarrow \Lambda_{\bar{r}}(\mathbf{t})$ , this is precisely the map defined above.

Consider a total order  $<$  on pairs  $(i, j)$ ,  $1 \leq i \leq m, 0 \leq j \leq \zeta_i$  with the property  $(i, j) < (i', j')$  if  $j < j'$ . One has an induced lexicographic order on vectors  $\bar{r}_{i,j}$  such that  $\bar{r} < \bar{r}'$  if and only if the  $<$ -minimal  $(i, j)$  for which  $r_{i,j} \neq r'_{i,j}$  satisfies  $r_{i,j} < r'_{i,j}$ .

**Definition 3.16.** For  $(i, j) \prec (i', j')$  let  $\kappa((i, j), (i', j'))$  be the number of  $\ell$  such that  $(i, j) \prec (i', \ell)$  and  $\ell < j'$ .

In other words, let  $e_{\min}$  be the minimal number  $e$  such that  $(i, j) \prec (i', e)$ . Then  $\kappa((i, j), (i', j')) = j' - e_{\min}$ . The key claim of this subsection is as follows.

**Lemma 3.17.** 
$$\psi_{\bar{r}} \left( \bigcap_{\bar{r} \succ \bar{r}'} \ker(\psi_{\bar{r}'}) \right) \supset \prod_{\substack{(i,j) \prec (i',j'), \\ 1 \leq \ell \leq r_{i,j}, \\ 1 \leq \ell' \leq r_{i',j'}}} (t_{i,j}^{(\ell)} - t_{i',j'}^{(\ell')})^{\kappa((i,j),(i',j'))} \Lambda_{\bar{r}}(\mathbf{t}).$$

This lemma will be proved by induction reducing the case of vectors  $\bar{a}$  and  $\bar{r}$  to smaller vectors defined as follows. Choose  $1 \leq \tilde{i} \leq m$  with  $\zeta_{\tilde{i}} > 0$ . For  $\max(a_{\tilde{i}} - a, 0) \leq \ell \leq a_{\tilde{i}}$  we define the truncation operator

$$\text{tc}_{\ell}^{\tilde{i}}(\bar{a}) = (a_1, \dots, a_{\tilde{i}-1}, a_{\tilde{i}} - \ell, a_{\tilde{i}+1}, \dots, a_m, a - a_{\tilde{i}} + \ell) \in \mathbb{N}^{m+1}.$$

The vector  $\bar{r}$  is transformed by the map  $\text{tc}^{\tilde{i}}$  with

$$\text{tc}^{\tilde{i}}(\bar{r})_{i,j} = \begin{cases} r_{i,j} & \text{if } i \neq \tilde{i}, \\ r_{i,j+1} & \text{if } i = \tilde{i}, \end{cases}$$

where  $1 \leq i \leq m$  and  $1 \leq j \leq \zeta_i$  for  $i \neq \tilde{i}$  while  $1 \leq j \leq \zeta_{\tilde{i}} - 1$  otherwise. Note that we have

$$\text{tc}^{\tilde{i}}(\bar{r}) \in \mathcal{R}(\text{tc}_{r_{\tilde{i},0}}^{\tilde{i}}(\bar{a})).$$

We now show that  $\psi_{\bar{r}}$  factors through  $\psi_{\text{tc}^{\tilde{i}}(\bar{r})}$ .

We define the map

$$\chi_{\ell}^{\tilde{i}} : \Lambda_{\bar{a}}(\mathbf{u}, \mathbf{s}) \rightarrow \Lambda_{\text{tc}_{r_{\tilde{i},0}}^{\tilde{i}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell)}]^{\mathfrak{S}_{\ell}}$$

by

$$u_i^{(j)} \mapsto u_i^{(j)} \quad \text{if } i \neq \tilde{i}, \quad u_{\tilde{i}}^{(j)} \mapsto \begin{cases} u_{\tilde{i}}^{(j-\ell)} & \text{if } j > \ell, \\ t_{\tilde{i},0}^{(j)} & \text{if } j \leq \ell, \end{cases} \quad s^{(j)} \mapsto \begin{cases} u_{\tilde{i}}^{(j)} & \text{if } j \leq a_{\tilde{i}} - \ell, \\ s^{(j-a_{\tilde{i}}+\ell)} & \text{if } j > a_{\tilde{i}} - \ell. \end{cases}$$

Let  $\text{sh}_{\tilde{i}}$  be the index shift operator,

$$\text{sh}_{\tilde{i}}(t_{i,j}) = t_{i,j} \quad \text{if } i \neq \tilde{i}, \quad \text{sh}_{\tilde{i}}(t_{\tilde{i},j}) = t_{\tilde{i},j+1}.$$

Then we have the identity

$$\psi_{\bar{r}} = ((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}) \circ \chi_{r_{\tilde{i},0}}^{\tilde{i}}. \tag{3-8}$$

For clarity let us point out the following. As mentioned, the maps  $\psi_{\bullet}$ ,  $\chi_{\bullet}^{\bullet}$  and  $\text{sh}_{\bullet}$  are defined as maps between polynomial rings and then restricted to symmetric polynomials. It should be noted that identity (3-8) does not hold for these maps between polynomial rings (only for the restrictions) because the left-hand side and the right-hand side may not coincide on some  $s^{(i)}$  (but always coincide up to the action of the corresponding symmetric group).

**Remark 3.18.** Identity (3-8) can be interpreted in terms of Remark 3.15 as follows. The map (3-7) decomposes as

$$\mathbb{k}[X_{i,j}]_{1 \leq i \leq m, 0 \leq j \leq \zeta_i} \rightarrow \mathbb{k}[X_i]_{1 \leq i \leq m} \otimes \mathbb{k}[z] \otimes \mathbb{k}[X_{\tilde{i},0}] \rightarrow \mathbb{k}[X_i]_{1 \leq i \leq m} \otimes \mathbb{k}[z].$$

Here the left arrow maps  $X_{i,j}$  with  $i \neq \tilde{i}$  to  $X_i z^j$ , maps  $X_{\tilde{i},j}$  with  $j \geq 1$  to  $X_{\tilde{i}} z^{j-1}$  and preserves  $X_{\tilde{i},0}$ . The right arrow preserves  $z$  and  $X_i$  with  $i \neq \tilde{i}$  while mapping  $X_{\tilde{i}}$  to  $X_{\tilde{i}} z$  and  $X_{\tilde{i},0}$  to  $X_{\tilde{i}}$ . We again apply  $J^\infty$ , restrict to components of gradings  $\bar{r}$ ,  $\text{tc}_{r_{\tilde{i},0}^{\tilde{i}}}^{\tilde{i}}(\bar{a}) \oplus r_{\tilde{i},0}^{\tilde{i}}$  and  $\bar{a}$  and then dualize. One may check that on the right we will obtain the map  $\chi_{r_{\tilde{i},0}^{\tilde{i}}}^{\tilde{i}}$  and on the left we will obtain  $(\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}$  (the latter can be seen by further decomposing the left arrow).

For  $\ell < a_{\tilde{i}}$  we have

$$\ker \chi_\ell^{\tilde{i}} \supset \ker \chi_{\ell+1}^{\tilde{i}}.$$

Moreover, we have  $\chi_\ell^{\tilde{i}} = \tau \circ \chi_{\ell+1}^{\tilde{i}}$ , where

$$\begin{aligned} \tau : \Lambda_{\text{tc}_{\ell+1}^{\tilde{i}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell+1)}]^{\mathfrak{S}_{\ell+1}} &\rightarrow \Lambda_{\text{tc}_{\ell+1}^{\tilde{i}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[u_{\tilde{i}}^{(a_{\tilde{i}}-\ell)}] \otimes \mathbb{k}[t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell)}]^{\mathfrak{S}_\ell}, \\ \tau(t_{\tilde{i},0}^{(\ell+1)}) &= \tau(s^{(a_{\tilde{i}}-\ell+1)}) = u_{\tilde{i}}^{(a_{\tilde{i}}-\ell)} \end{aligned}$$

and  $\tau$  sends any variable (on the left) different from  $t_{\tilde{i},0}^{(\ell+1)}$  and  $s^{(a_{\tilde{i}}-\ell+1)}$  to the same variable on the right.

**Lemma 3.19.**  $\chi_{\ell+1}^{\tilde{i}}(\ker \chi_\ell^{\tilde{i}}) \supset \prod_{\substack{j=1, \dots, a_{\tilde{i}}-\ell+1, \\ j'=1, \dots, \ell+1}} (s^{(j)} - t_{\tilde{i},0}^{(j')}) (\Lambda_{\text{tc}_{\ell+1}^{\tilde{i}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell+1)}]^{\mathfrak{S}_{\ell+1}}).$

*Proof.* One has

$$\begin{aligned} \chi_{\ell+1}^{\tilde{i}}(p_k(u_{\tilde{i},0}^{(1)}, \dots, u_{\tilde{i},0}^{(a_{\tilde{i}})})) &= p_k(u_{\tilde{i},0}^{(1)}, \dots, u_{\tilde{i},0}^{(a_{\tilde{i}}-\ell-1)}) + p_k(t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell+1)}), \\ \chi_{\ell+1}^{\tilde{i}}(p_k(s^{(1)}, \dots, s^{(a)})) &= p_k(u_{\tilde{i},0}^{(1)}, \dots, u_{\tilde{i},0}^{(a_{\tilde{i}}-\ell-1)}) + p_k(s^{(1)}, \dots, s^{(a_{\tilde{i}}-\ell+1)}). \end{aligned}$$

Therefore, the image of  $\chi_{\ell+1}^{\tilde{i}}$  contains the differences

$$p_k(t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell+1)}) - p_k(s^{(1)}, \dots, s^{(a_{\tilde{i}}-\ell+1)})$$

which generate the ring  $\Omega_{\ell+1, a_{\tilde{i}}-\ell+1}$  of supersymmetric polynomials in variables  $t_{\tilde{i},0}^{(1)}, \dots, t_{\tilde{i},0}^{(\ell+1)}$  and  $s^{(1)}, \dots, s^{(a_{\tilde{i}}-\ell+1)}$  (Proposition 1.27). We deduce that the image of  $\chi_{\ell+1}^{\tilde{i}}$  contains the ring

$$S = \Omega_{\ell+1, a_{\tilde{i}}-\ell+1} \otimes \Lambda_{a_{\tilde{i}}-\ell-1}(\mathbf{u}_{\tilde{i}}) \otimes \bigotimes_{i \neq \tilde{i}} \Lambda_{a_i}(s_i).$$

The right-hand side in the statement of our lemma is equal to  $\ker \tau \cap S$  due to Proposition 1.26. However,  $\chi_{\ell+1}^{\tilde{i}}(\ker \chi_\ell^{\tilde{i}}) = \text{im } \chi_{\ell+1}^{\tilde{i}} \cap \ker \tau \supset S \cap \ker \tau$ . □

*Proof of Lemma 3.17.* Let  $(\tilde{i}, 0)$  be the smallest pair  $(i, j)$  in the order  $\prec$ . By definition we have

$$\bar{r} \succ \bar{r}' \iff r_{\tilde{i},0} > r'_{\tilde{i},0} \text{ or } (r_{\tilde{i},0} = r'_{\tilde{i},0} \text{ and } \text{tc}^{\tilde{i}}(\bar{r}) \succ \text{tc}^{\tilde{i}}(\bar{r}')),$$

where  $\text{tc}^{\tilde{i}}(\bar{r})$  and  $\text{tc}^{\tilde{i}}(\bar{r}')$  are also compared lexicographically.



Using Lemma 3.19 and identity (3-8) we have

$$\chi_{r_{i,0}^{\tilde{i}}} \left( \bigcap_{\bar{r}': r_{i,0}^{\tilde{i}} > r_{i,0}^{\bar{r}'}} \ker(\psi_{\bar{r}'}) \right) \supset \prod_{\substack{\ell=1, \dots, a-a_i+r_{i,0}, \\ \ell'=1, \dots, r_{i,0}}} (s^{(\ell)} - t_{i,0}^{(\ell')}) \Lambda_{\text{tc}_{r_{i,0}^{\tilde{i}}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[t_{i,0}^{(1)}, \dots, t_{i,0}^{(r_{i,0}^{\tilde{i}})}]_{\mathfrak{S}_{r_{i,0}^{\tilde{i}}}}. \quad (3-9)$$

We proceed by induction on  $\sum_{i=1}^m \zeta_i$ . By the induction hypothesis we have

$$\psi_{\text{tc}^{\tilde{i}}(\bar{r})} \left( \bigcap_{\text{tc}^{\tilde{i}}(\bar{r}) > \text{tc}^{\tilde{i}}(\bar{r}')} \ker(\psi_{\text{tc}^{\tilde{i}}(\bar{r}')}) \right) \supset \prod_{\substack{(i,j) < (i',j'), \\ 1 \leq \ell \leq \text{tc}^{\tilde{i}}(r)_{i,j}, \\ 1 \leq \ell' \leq \text{tc}^{\tilde{i}}(r)_{i',j'}}} (t_{i,j}^{(\ell)} - t_{i',j'}^{(\ell')})^{\times((i,j),(i',j'))} \Lambda_{\text{tc}^{\tilde{i}}(\bar{r})}(\mathbf{t}).$$

Let  $\bar{r}' < \bar{r}$  be such that  $r_{i,0}^{\bar{r}'} = r_{i,0}^{\tilde{i}}$ . Consider the set  $F_{\bar{r}'}$  consisting of all

$$f \in \Lambda_{\text{tc}_{r_{i,0}^{\tilde{i}}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[t_{i,0}^{(1)}, \dots, t_{i,0}^{(r_{i,0}^{\tilde{i}})}]_{\mathfrak{S}_{r_{i,0}^{\tilde{i}}}}$$

such that  $(\psi_{\text{tc}^{\tilde{i}}(\bar{r}')}) \otimes \text{id} f = 0$ . Then, by (3-8),

$$\psi_{\bar{r}}(\ker(\psi_{\bar{r}'})) = ((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}) \circ \chi_{r_{i,0}^{\tilde{i}}}(\ker(\psi_{\bar{r}'})),$$

and hence we get (note that  $\text{sh}_{\tilde{i}}$  is bijective)

$$\begin{aligned} \psi_{\bar{r}}(\ker(\psi_{\bar{r}'})) &= ((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id})(\ker(\psi_{\text{tc}^{\tilde{i}}(\bar{r}')}) \otimes \text{id}) \cap (\text{im } \chi_{r_{i,0}^{\tilde{i}}}) \\ &\supset ((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}) \prod_{\substack{\ell=1, \dots, a-a_i+r_{i,0}, \\ \ell'=1, \dots, r_{i,0}}} (s^{(\ell)} - t_{i,0}^{(\ell')}) F_{\bar{r}'} \end{aligned}$$

due to Lemma 3.19. Taking the intersection one gets

$$\psi_{\bar{r}} \left( \bigcap_{\bar{r}' < \bar{r}, r_{i,0}^{\bar{r}'} = r_{i,0}^{\tilde{i}}} \ker(\psi_{\bar{r}'}) \right) \supset ((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}) \prod_{\substack{\ell=1, \dots, a-a_i+r_{i,0}, \\ \ell'=1, \dots, r_{i,0}}} (s^{(\ell)} - t_{i,0}^{(\ell')}) \bigcap_{\bar{r}' < \bar{r}, r_{i,0}^{\bar{r}'} = r_{i,0}^{\tilde{i}}} F_{\bar{r}'}$$

Therefore, using (3-9) we obtain

$$\psi_{\bar{r}} \left( \bigcap_{\bar{r}' < \bar{r}} \ker(\psi_{\bar{r}'}) \right) \supset ((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}) \prod_{\substack{\ell=1, \dots, a-a_i+r_{i,0}, \\ \ell'=1, \dots, r_{i,0}}} (s^{(\ell)} - t_{i,0}^{(\ell')}) \bigcap_{\bar{r}' < \bar{r}, r_{i,0}^{\bar{r}'} = r_{i,0}^{\tilde{i}}} F_{\bar{r}'}$$

Definition 3.16 implies

$$(\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id} \left( \prod_{\substack{\ell=1, \dots, a-a_i+r_{i,0}, \\ \ell'=1, \dots, r_{i,0}}} (s^{(\ell)} - t_{i,0}^{(\ell')}) \right) = \prod_{(i,j) > (\tilde{i},0)} \prod_{\substack{\ell=1, \dots, r_{i,j}, \\ \ell'=1, \dots, r_{i,0}}} (t_{i,j}^{(\ell)} - t_{i,0}^{(\ell')})^{\times((i,j),(\tilde{i},0))}.$$

By induction hypothesis we have

$$\begin{aligned} &((\text{sh}_{\tilde{i}} \circ \psi_{\text{tc}^{\tilde{i}}(\bar{r})}) \otimes \text{id}) \left( \bigcap_{\bar{r}' < \bar{r}, r_{i,0}^{\bar{r}'} = r_{i,0}^{\tilde{i}}} F_{\bar{r}'} \right) \\ &\supset \prod_{(i,j) > (i',j') > (\tilde{i},0)} \prod_{\substack{\ell=1, \dots, r_{i,j}, \\ \ell'=1, \dots, r_{i',j'}}} (t_{i,j}^{(\ell)} - t_{i',j'}^{(\ell')})^{\times((i,j),(i',j'))} \Lambda_{\text{tc}_{r_{i,0}^{\tilde{i}}}(\bar{a})}(\mathbf{u}, \mathbf{s}) \otimes \mathbb{k}[t_{i,0}^{(1)}, \dots, t_{i,0}^{(r_{i,0}^{\tilde{i}})}]_{\mathfrak{S}_{r_{i,0}^{\tilde{i}}}}. \quad \square \end{aligned}$$

**4. Inductive construction**

**4.1. Veronese curve.** In this subsection we study the arc schemes corresponding to the segments  $[0, \zeta]$  (recall that a one-dimensional polytope is affine equivalent to a segment). By definition, the Veronese ring  $R([0, \zeta])$  is generated by  $Y_\alpha = z^\alpha w$ ,  $\alpha = 0, \dots, \zeta$ . The ring  $J_{\text{red}}^\infty(R([0, \zeta]))$  of reduced arcs over  $R([0, \zeta])$  was studied in [Dumanski and Feigin 2023]. Here we study this ring by methods developed above. Namely, we use the results from Section 1.6 on the structure of the reduced arc schemes of the cubes and Lemma 3.17.

Recall the notation  $Y_I$  for the generators of the arc rings of cubes.

For any  $0 \leq \alpha < \beta \leq \zeta$  we define the linear map of polytopes

$$[0, 1]^{\beta-\alpha} \rightarrow [0, \zeta], \quad (d_1, \dots, d_{\beta-\alpha}) \mapsto \alpha + \sum_{i=1}^{\beta-\alpha} d_i.$$

This map defines a homomorphism of corresponding toric rings  $R([0, 1]^{\beta-\alpha}) \rightarrow R([0, \zeta])$ :

$$\iota_{\alpha, \beta} : Y_I \mapsto Y_{\alpha+|I|}.$$

**Lemma 4.1.** *For integers  $0 \leq \alpha < \beta \leq \zeta$  and  $0 \leq k' \leq \beta - \alpha - 2$  the coefficients of the following series are zero in the ring  $J_{\text{red}}^\infty(R([0, \zeta]))$ :*

$$W'_{\alpha, \beta, k'} = \sum_{i=0}^{\beta-\alpha-1} (-1)^i \binom{\beta-\alpha-1}{i} \frac{\partial^{k'} Y_{\alpha+i}(s)}{\partial s^{k'}} Y_{\beta-i}(s).$$

*Proof.* Consider the map  $J^\infty(\iota_{\alpha, \beta}) : J^\infty(R([0, 1]^{\beta-\alpha})) \rightarrow J^\infty(R([0, \zeta]))$ . Recall the elements  $W_{\beta-\alpha, k'} \in J^\infty(R([0, 1]^{\beta-\alpha}))$ . By Lemma 1.31 the coefficients of  $W_{\beta-\alpha, k'}$  are nilpotent. However,

$$J^\infty(\iota_{\alpha, \beta})(W_{\beta-\alpha, k'}) = W'_{\alpha, \beta, k'}.$$

Thus the coefficients of  $W'_{\beta-\alpha, k'}$  are nilpotent. Hence they are equal to 0 in  $J_{\text{red}}^\infty(R([0, \zeta]))$ . □

**Definition 4.2.** The ring  $\overline{J_{\text{red}}^\infty(R([0, \zeta]))}$  is the quotient of  $\mathbb{k}[X_i^{(j)}]_{0 \leq i \leq \zeta, j \geq 0}$  modulo the ideal generated by coefficients of the series

$$\sum_{i=0}^{\beta-\alpha-1} (-1)^i \binom{\beta-\alpha-1}{i} \frac{\partial^{k'} X_{\alpha+i}(s)}{\partial s^{k'}} X_{\beta-i}(s) \tag{4-1}$$

for all  $0 \leq \alpha < \beta \leq \zeta, 0 \leq k' \leq \beta - \alpha - 2$ .

Clearly, there is a surjective homomorphism  $\overline{J_{\text{red}}^\infty(R([0, \zeta]))} \rightarrow J_{\text{red}}^\infty(R([0, \zeta]))$ .

Consider the following (weight-restricted lexicographic) order on the semigroup  $\mathbb{N}^{\zeta+1}$ : we say that  $(u_0, \dots, u_\zeta)$  is less than  $(v_0, \dots, v_\zeta)$  if  $u_0 + \dots + u_\zeta < v_0 + \dots + v_\zeta$  or  $u_0 + \dots + u_\zeta = v_0 + \dots + v_\zeta$  and there exists an  $i$  such that  $u_i < v_i$  and  $u_j = v_j$  for  $j < i$ . We have an induced order  $\prec$  on the graded components of  $J^\infty(\mathbb{k}[X_0, \dots, X_\zeta])$ . The next lemma is obvious.

**Lemma 4.3.** *The initial part of (4-1) is equal to*

$$\frac{\partial^{k'} X_\alpha(s)}{\partial s^{k'}} X_\beta(s). \tag{4-2}$$

We define the ring  $\widehat{J_{\text{red}}^\infty(R([0, \zeta]))}$  to be a quotient of  $\mathbb{k}[X_i^{(j)}]_{i=0, \dots, \zeta, j \geq 0}$  by the ideal generated by coefficients of (4-2). These relations are homogeneous with respect to the grading  $\prec$ . Thus the ring  $\widehat{J_{\text{red}}^\infty(R([0, \zeta]))}$  has a natural  $\mathbb{N}^{\zeta+1}$ -grading and we denote the  $\bar{r}$ -th graded component by  $\widehat{J_{\text{red}}^\infty(R([0, \zeta]))}[\bar{r}]$ . Proposition 2.8 applied to this graded component gives us the following.

**Proposition 4.4.** *We have the following isomorphism of grad-graded spaces:*

$$\widehat{J_{\text{red}}^\infty(R([0, \zeta]))}[\bar{r}]^* \simeq \prod_{0 \leq \alpha < \beta \leq \zeta, 1 \leq i \leq r_\alpha, 1 \leq j \leq r_\beta} (t_\alpha^{(i)} - t_\beta^{(j)})^{\beta-\alpha-1} \mathbb{k}[t_\alpha^{(j)}]_{0 \leq \alpha \leq \zeta, 1 \leq j \leq r_\alpha}^{\mathfrak{S}_{r_0} \times \dots \times \mathfrak{S}_{r_\zeta}}$$

Taking into account the grading  $\text{grad}$  we extend the  $\mathbb{N}^{\zeta+1}$ -grading above to the  $\mathbb{N}^{\zeta+2}$ -grading. Corollary 1.30 implies:

**Lemma 4.5.** *There is the following surjective map of  $\mathbb{N}^{\zeta+2}$ -graded rings:*

$$\widehat{J_{\text{red}}^\infty(R([0, \zeta]))} \twoheadrightarrow \text{gr}_\prec J_{\text{red}}^\infty(R([0, \zeta])).$$

We take a vector  $\bar{r} = (r_0, \dots, r_\zeta)$  and denote

$$\sum_{i=0}^{\zeta} i r_i = a, \quad \sum_{i=0}^{\zeta} r_i = L.$$

Recall the notation  $\bar{r} \in \mathcal{R}(a, L)$ . We consider the rings of symmetric functions  $\Lambda_{a,L}(s)$  and  $\Lambda_{\bar{r}}(t)$ . We have the map  $\varphi_{\bar{r}} : A_{\bar{r}} \rightarrow J^\infty(\mathbb{k}[z_1, w])[a, L]$  for  $\bar{r} \in \mathcal{R}(a, L)$  and the map  $\varphi_{\bar{r}}^\vee : \Lambda_{a,L}(s) \rightarrow \Lambda_{\bar{r}}(t)$  defined by the following formulas (see (3-4)):

$$\begin{aligned} s_w^{(1)} &\mapsto t_0^{(1)}; \quad \dots; \quad s_w^{(r_0)} \mapsto t_0^{(r_0)}; \quad s_w^{(r_0+1)}, s_1^{(1)} \mapsto t_1^{(1)}; \quad \dots; \quad s_w^{(r_0+r_1)}, s_1^{(r_1)} \mapsto t_1^{(r_1)}; \\ s_w^{(r_0+r_1+1)}, s_1^{(r_1+1)}, s_1^{(r_1+2)} &\mapsto t_2^{(1)}; \quad \dots, s_w^{(r_0+r_1+r_2)}, s_1^{(r_1+2r_2-1)}, s_1^{(r_1+2r_2)} \mapsto t_2^{(r_2)}; \\ s_w^{(r_0+r_1+r_2+1)}, s_1^{(r_1+2r_2+1)}, s_1^{(r_1+2r_2+2)}, s_1^{(r_1+2r_2+3)} &\mapsto t_3^{(1)} \dots \end{aligned} \tag{4-3}$$

One has

$$\begin{aligned} \varphi_{\bar{r}}^\vee(p_k(s_w^{(1)}, \dots, s_w^{(L)})) &= \sum_{i=0}^{\zeta} p_k(t_i^{(1)}, \dots, t_i^{(r_i)}), \\ \varphi_{\bar{r}}^\vee(p_k(s_1^{(1)}, \dots, s_1^{(a)})) &= \sum_{i=0}^{\zeta} i p_k(t_i^{(1)}, \dots, t_i^{(r_i)}). \end{aligned}$$

Lemma 3.17 with  $m = 1$  implies:

**Proposition 4.6.** *For any  $\bar{r} \in \mathcal{R}(a, L)$ ,*

$$\varphi_{\bar{r}}^\vee \left( \bigcap_{\bar{r}' \in \mathcal{R}(a, L), \bar{r}' \prec \bar{r}} \ker(\varphi_{\bar{r}'}^\vee) \right) \supset \prod_{0 \leq \alpha < \beta \leq \zeta, 1 \leq i \leq r_\alpha, 1 \leq j \leq r_\beta} (t_\alpha^{(i)} - t_\beta^{(j)})^{\beta-\alpha-1} \Lambda_{\bar{r}}(t). \tag{4-4}$$

Thus we reobtain the following theorem from [Dumanski and Feigin 2023].

**Theorem 4.7.** *For any  $\bar{r}$  grad-graded vector spaces  $\text{gr}_{<} J_{\text{red}}^{\infty}(R([0, \zeta]))[\bar{r}]$  and  $(\widehat{J_{\text{red}}^{\infty}(R([0, \zeta]))})[\bar{r}]$  are naturally isomorphic. One has the isomorphism of graded rings*

$$J_{\text{red}}^{\infty}(R([0, \zeta])) \simeq \overline{J_{\text{red}}^{\infty}(R([0, \zeta]))}.$$

**Corollary 4.8.** *The action of  $\mathcal{A}_L$  is free on  $J_{\text{red}}^{\infty}(R([0, \zeta]))[L]^*$ .*

*Proof.* This follows from Theorem 3.14. □

**4.2. Two-dimensional case.** Let  $f : [0, \eta] \rightarrow \mathbb{R}_{\geq 0}$  be a convex function such that for any integer  $i \in [0, \eta]$ :  $\zeta_i := f(i) \in \mathbb{N}$ . In this subsection we consider the case of the two-dimensional polygon obtained as the convex hull of the segment  $[0, \eta] \times \{0\}$  and all points  $(i, f(i))$ ,  $i \in [0, \eta]$ . In other words, we consider a positive integer  $\eta$  and let  $\zeta_0, \dots, \zeta_{\eta}$  be a tuple of nonnegative integers such that  $\zeta_0 - \zeta_1 \leq \zeta_1 - \zeta_2 \leq \dots \leq \zeta_{\eta-1} - \zeta_{\eta}$ . Then the integer points of the corresponding convex lattice polygon are  $(i, j)$ ,  $0 \leq i \leq \eta$ ,  $0 \leq j \leq \zeta_i$ .

**Example 4.9.** The case

$$\zeta = \begin{cases} \zeta(\alpha) = \alpha/b & \text{if } \alpha \in [0, b], \\ \zeta(\alpha) = 1 & \text{if } \alpha \in [b, \eta], \end{cases} \quad 0 \leq b \leq \eta,$$

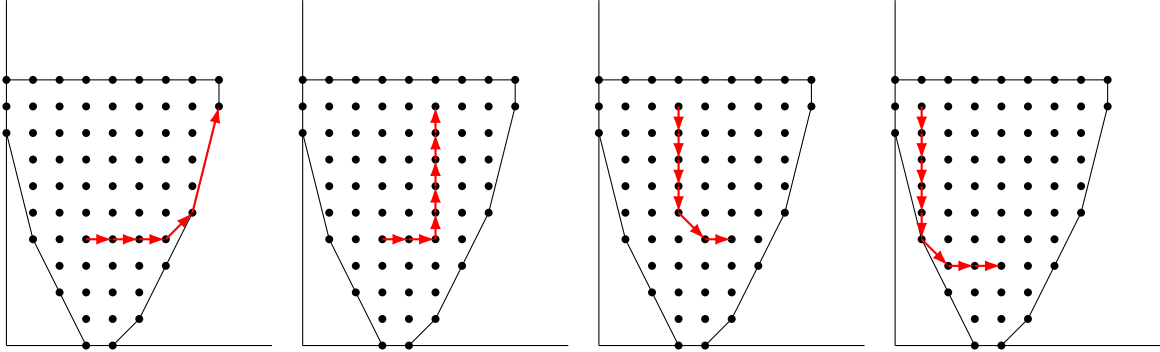
corresponds to Hirzebruch surface.

**Example 4.10.** The case  $\zeta(\alpha) = \alpha$ ,  $\alpha \in [0, \eta]$  corresponds to Veronese embedding of  $\mathbb{P}^2$ .

It will be convenient for us to reflect this polygon across the horizontal line  $j = \zeta_{\max}/2$ , where  $\zeta_{\max} = \max\{\zeta_i \mid i = 0, \dots, \eta\}$ . The integer points of the reflected polytope  $P$  are  $(i, j)$  with  $0 \leq i \leq \eta$  and  $\zeta_{\max} - \zeta_i \leq j \leq \zeta_{\max}$ . An example of such a reflected polygon  $P$  is seen in the Figure 1.

Consider two distinct integer points  $\bar{\alpha} = (\alpha_1, \alpha_2), \bar{\beta} = (\beta_1, \beta_2) \in P$ . We assume that  $\alpha_1 \leq \beta_1$  and if  $\alpha_1 = \beta_1$ , then  $\alpha_2 \leq \beta_2$ . We will now define a certain polygonal curve with integer vertices connecting these two points. Denote the vertices of this curve by  $(i_0, j_0), \dots, (i_{k+1}, j_{k+1})$ . We require the following to hold:

- $(i_0, j_0) = \bar{\alpha}$  and  $(i_{k+1}, j_{k+1}) = \bar{\beta}$ .
- The sequence  $i_0, \dots, i_{k+1}$  is weakly increasing.
- The sequence  $j_0, \dots, j_{k+1}$  is weakly monotonic, it increases (weakly) if  $\alpha_2 \leq \beta_2$  and decreases (weakly) if  $\alpha_2 \geq \beta_2$ .
- For every  $1 \leq \ell \leq k+1$  we either have  $i_{\ell} - i_{\ell-1} = 1$  or  $i_{\ell} = i_{\ell-1}$  and  $|j_{\ell} - j_{\ell-1}| = 1$ .
- The curve is concave in the sense that a segment connecting two points of the curve cannot contain points lying below the curve.
- All  $(i_{\ell}, j_{\ell}) \in P$ .



**Figure 1.** Concave polygonal curves with integer vertices in  $P$ .

This provides a concave polygonal curve with integer vertices in  $P$  which is directed either towards the top-right or the bottom-right. However, these conditions still leave plenty of degrees of freedom, the curve we choose can be described as follows (modulo one exceptional case which will be discussed below). The curve contains as many vertices as possible on the horizontal line  $i = \min(\alpha_2, \beta_2)$  (i.e., that which contains the lower endpoint) and as many vertices as possible on the vertical line which contains the higher endpoint (i.e., either  $j = \alpha_1$  or  $j = \beta_1$  depending on  $\max(\alpha_2, \beta_2)$ ). All intermediate vertices (if any) are chosen as vertices of  $P$ . In other words, for  $1 \leq \ell \leq k + 1$  let  $e_\ell = (i_\ell, j_\ell) - (i_{\ell-1}, j_{\ell-1})$  be the vector corresponding to the  $\ell$ -th segment, explicit formulas for these vectors can be given as follows.

Suppose  $\alpha_2 \leq \beta_2$ , i.e., the curve is directed towards the top-right (Figure 1, first and second image). Choose the maximal  $u \leq \beta_1$  for which  $(\alpha_1, u) \in P$ .

$$\begin{aligned} e_1 &= \dots = e_{u-\alpha_1} = (1, 0), \\ e_{u-\alpha_1+1} &= (1, \zeta_{\max} - \zeta_{u+1} - \alpha_2), \\ e_{u-\alpha_1+2} &= (1, \zeta_{u+2} - \zeta_{u+1}), \quad \dots, \quad e_{\beta_1-\alpha_1} = (1, \zeta_{\beta_1-1} - \zeta_{\beta_1}), \\ e_{\beta_1-\alpha_1+1} &= \dots = e_{k+1} = (0, 1). \end{aligned}$$

Now suppose  $\alpha_2 > \beta_2$  and choose the minimal  $u \geq \alpha_1$  for which  $(u, \beta_2) \in P$ . First suppose that  $u = \alpha_1$  (so that  $(\alpha_1, \beta_2) \in P$ , see Figure 1, third image), this is the exceptional case mentioned above, note vector  $e_{\alpha_2-\beta_2}$ . Here we set

$$e_1 = \dots = e_{\alpha_2-\beta_2-1} = (0, -1), \quad e_{\alpha_2-\beta_2} = (1, -1), \quad e_{\alpha_2-\beta_2+1} = \dots = e_{k+1} = (1, 0).$$

Finally, suppose  $(\alpha_1, \beta_2) \notin P$  (Figure 1, fourth image). We proceed similarly to the case of  $\alpha_2 < \beta_2$ :

$$\begin{aligned} e_1 &= \dots = e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})} = (0, -1), \\ e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+1} &= (1, \zeta_{\alpha_1} - \zeta_{\alpha_1+1}), \quad \dots, \quad e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+u-1-\alpha_1} = (1, \zeta_{u-2} - \zeta_{u-1}), \\ e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+u-\alpha_1} &= (1, \zeta_{\max} - \zeta_{u-1} - \beta_2), \\ e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+u-\alpha_1+1} &= \dots = e_{k+1} = (1, 0). \end{aligned}$$

We will use the notation  $k(\bar{\alpha}, \bar{\beta}) = k(\bar{\beta}, \bar{\alpha}) = k$  and  $e_\ell(\bar{\alpha}, \bar{\beta}) = e_\ell$ . Note the following two properties of these vectors, both immediate from concavity and monotonicity:

- For any subset  $I \subset \{1, \dots, k(\bar{\alpha}, \bar{\beta}) + 1\}$  one has  $\bar{\alpha} + \sum_{j \in I} e_j(\bar{\alpha}, \bar{\beta}) \in P$ .
- $\bar{\alpha} + \sum_{j=1}^{k(\bar{\alpha}, \bar{\beta})+1} e_j(\bar{\alpha}, \bar{\beta}) = \bar{\beta}$  and there is no proper nonempty  $I \subset \{1, \dots, k(\bar{\alpha}, \bar{\beta}) + 1\}$  such that  $\bar{\alpha} + \sum_{j \in I} e_j(\bar{\alpha}, \bar{\beta}) \in \{\bar{\alpha}, \bar{\beta}\}$ .

We will also write  $\kappa(\alpha, \beta)$  for the number of vertical segments, i.e., we set

$$\kappa(\alpha, \beta) = k(\alpha, \beta) + 1 - (\beta_1 - \alpha_1) = \begin{cases} \beta_2 - \alpha_2 & \text{if } \alpha_2 \leq \beta_2 \text{ and } (\beta_1, \alpha_2) \in P, \\ \beta_2 - (\zeta_{\max} - \zeta_{\beta_1}) & \text{if } \alpha_2 \leq \beta_2 \text{ and } (\beta_1, \alpha_2) \notin P, \\ \alpha_2 - \beta_2 - 1 & \text{if } \alpha_2 > \beta_2 \text{ and } (\alpha_1, \beta_2) \in P, \\ \alpha_2 - (\zeta_{\max} - \zeta_{\alpha_1}) & \text{if } \alpha_2 > \beta_2 \text{ and } (\alpha_1, \beta_2) \notin P. \end{cases} \quad (4-5)$$

**Proposition 4.11.** *The coefficients of the following series are zero in the ring  $J_{\text{red}}^\infty(R(P))$ :*

$$W'_{\bar{\alpha}, \bar{\beta}, k'} = \sum_{I \subset \{1, \dots, k(\bar{\alpha}, \bar{\beta})\}} (-1)^{|I|} \frac{\partial^{k'} Y_{\bar{\alpha} + \sum_{j \in I} e_j(\bar{\alpha}, \bar{\beta})}(s)}{\partial s^{k'}} Y_{\bar{\beta} - \sum_{j \in I} e_j(\bar{\alpha}, \bar{\beta})}(s), \quad k' = 0, \dots, k(\bar{\alpha}, \bar{\beta}) - 1. \quad (4-6)$$

*Proof.* We define the map  $[0, 1]^{k(\bar{\alpha}, \bar{\beta})+1} \rightarrow P$ :

$$(x_1, \dots, x_{k(\bar{\alpha}, \bar{\beta})+1}) \mapsto \bar{\alpha} + x_1 e_1(\bar{\alpha}, \bar{\beta}) + \dots + x_{k(\bar{\alpha}, \bar{\beta})+1} e_{k(\bar{\alpha}, \bar{\beta})+1}(\bar{\alpha}, \bar{\beta}).$$

This map of polytopes defines the map of rings  $\eta : R(\mathcal{B}_{k(\bar{\alpha}, \bar{\beta})+1}) \rightarrow R(P)$ . Applying the corresponding arc map to  $W_{k(\bar{\alpha}, \bar{\beta})+1, k'}$  (1-14) we have

$$J^\infty(\eta)(W_{k(\bar{\alpha}, \bar{\beta})+1, k'}) = W'_{\bar{\alpha}, \bar{\beta}, k'}.$$

Thus the element  $W'_{\bar{\alpha}, \bar{\beta}, k'}$  is nilpotent. This completes the proof. □

We define a lexicographic order on points  $\bar{\alpha} \in P$ :

$$\bar{\alpha} < \bar{\beta} \quad \text{if } \alpha_2 < \beta_2 \text{ or } \alpha_2 = \beta_2 \text{ and } \alpha_1 < \beta_1.$$

We define the following monomial order on the monomials in the variables  $X_{\bar{\alpha}}, \bar{\alpha} \in P$ . Let  $X^{\bar{r}} := \prod_{\bar{\alpha} \in P} X_{\bar{\alpha}}^{r_{\bar{\alpha}}}$ .

We define  $r_i := \sum_j r_{(i,j)}$ ,  $L(r) := \sum_{\bar{\alpha} \in P} r_{\bar{\alpha}}$ . If

$$(L(\bar{r}), r_0, r_1, \dots, r_\eta) > (L(\bar{r}'), r'_0, r'_1, \dots, r'_\eta)$$

in the standard lexicographic order, then  $X^{\bar{r}} > X^{\bar{r}'}$ . For  $\bar{r} = (r_{\bar{\alpha}})_{\bar{\alpha} \in P}$ ,  $\bar{r}' = (r'_{\bar{\alpha}})_{\bar{\alpha} \in P}$  such that

$$(L(\bar{r}), r_0, r_1, \dots, r_\eta) = (L(\bar{r}'), r'_0, r'_1, \dots, r'_\eta)$$

the order  $<$  is lexicographic: we consider the minimal  $\alpha$  with respect  $<$  for which  $r_{\bar{\alpha}} \neq r'_{\bar{\alpha}}$  and set  $X^{\bar{r}} < X^{\bar{r}'}$  if  $r_{\bar{\alpha}} < r'_{\bar{\alpha}}$ .

As before, we have an induced order on the graded parts of the ring  $J^\infty(\mathbb{k}[X_\alpha])$  and a filtration on the ring  $J^\infty(R(P))$ . Let  $W'_{\bar{\alpha}, \bar{\beta}, k'}(X)$  be defined as the right-hand side of (4-6) with  $Y_\bullet$  replaced by  $X_\bullet$ .

**Lemma 4.12.** *The initial part in  $\prec W'_{\bar{\alpha}, \bar{\beta}, k'}(X)$  is equal to*

$$\frac{\partial^{k'} X_{\bar{\alpha}}(s)}{\partial s^{k'}} X_{\bar{\beta}}(s).$$

*Proof.* We need to prove that  $X_{\bar{\alpha} + \sum_{j \in I} e_j(\bar{\alpha}, \bar{\beta})} X_{\bar{\beta} - \sum_{j \in I} e_j(\bar{\alpha}, \bar{\beta})} \prec X_{\bar{\alpha}} X_{\bar{\beta}}$ . This is clear from the definition of the order.  $\square$

**Corollary 4.13.** *The dual of the graded component  $\text{gr}_{\prec} J_{\text{red}}^{\infty} R(P)[\bar{r}]$  has a natural homogeneous  $\mathfrak{h}[s]$ -equivariant embedding into the space*

$$\prod_{\bar{\alpha} \prec \bar{\beta} \in P} \prod_{1 \leq i \leq r_{\bar{\alpha}}, 1 \leq j \leq r_{\bar{\beta}}} (t_{\bar{\alpha}}^{(i)} - t_{\bar{\beta}}^{(j)})^{k(\bar{\alpha}, \bar{\beta})} \Lambda_{\bar{r}}(\mathbf{t}).$$

The rest of this section is dedicated to the proof of the following theorem.

**Theorem 4.14.** *The coefficients of the series  $W'_{\bar{\alpha}, \bar{\beta}, k'}(X)$  with distinct  $\bar{\alpha}, \bar{\beta} \in P$  and  $k' \in [0, k(\bar{\alpha}, \bar{\beta}) - 1]$  generate the ideal of relations in  $J_{\text{red}}^{\infty}(R(P))$ . Initial parts in  $\prec W'_{\bar{\alpha}, \bar{\beta}, k'}(X)$  generate the corresponding initial ideal.*

*Proof.* As before we study the dual map for the inclusion  $\varphi_{\bar{r}} : A_{\bar{r}} \hookrightarrow J^{\infty}(\mathbb{k}[z_1, z_2, w])[a, L]$ , where  $\bar{a} = \sum_{\bar{\alpha} \in P} r_{\bar{\alpha}} \bar{\alpha}$ ,  $L = L(\bar{r})$ . Recall the notation  $\bar{r} \in \mathcal{R}(\bar{a}, L)$ . The image of the map  $\varphi_{\bar{r}}^{\vee} : \Lambda_{a_1, a_2, L}(\mathbf{s}) \rightarrow \Lambda_{\bar{r}}(\mathbf{t})$ ,

$$\begin{aligned} s_1^{(\alpha_1^1(j-1)+1)}, \dots, s_1^{(\alpha_1^1(j-1)+\alpha_1^1)}, s_2^{(\alpha_2^1(j-1)+1)}, \dots, s_2^{(\alpha_2^1(j-1)+\alpha_2^1)}, s_w^{(j)} &\mapsto t_{\bar{\alpha}^1}^{(j)}, \quad j = 1, \dots, r_{\bar{\alpha}^1}; \\ s_1^{(r_{\bar{\alpha}^1} \alpha_1^1 + \alpha_1^2(j-1)+1)}, \dots, s_1^{(r_{\bar{\alpha}^1} \alpha_1^1 + \alpha_1^2(j-1)+\alpha_1^2)}, s_2^{(r_{\bar{\alpha}^1} \alpha_2^1 + \alpha_2^2(j-1)+1)}, \dots, \\ & s_2^{(r_{\bar{\alpha}^1} \alpha_2^1 + \alpha_2^2(j-1)+\alpha_2^2)}, s_w^{(r_{\bar{\alpha}^1} + j)} &\mapsto t_{\bar{\alpha}^2}^{(j)}, \quad j = 1, \dots, r_{\bar{\alpha}^2}; \\ &\dots \end{aligned}$$

is isomorphic to the dual space of  $A_{\bar{r}}$ . The dual space to  $\sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'} / \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'}$  is isomorphic to the space  $\varphi_{\bar{r}}^{\vee} \left( \bigcap_{\bar{r}' \in \mathcal{R}(\bar{a}, L), \bar{r}' \prec \bar{r}} \ker(\varphi_{\bar{r}'}^{\vee}) \right)$ . Our goal is to compute this space.

We consider the set of variables  $u_i^{(j)}$ ,  $i = 0, \dots, \eta$ ,  $j = 1, \dots, r_i$ . Consider the map

$$\varphi_{(r_0, \dots, r_{\eta})}^{\vee} \otimes \text{id} : \Lambda_{a_1, L}(\mathbf{s}) \otimes \Lambda_{a_2}(\mathbf{s}_2) = \Lambda_{a_1, a_2, L}(\mathbf{s}) \rightarrow \Lambda_{(r_0, \dots, r_{\eta}, a_2)}(\mathbf{u}, \mathbf{s}_2)$$

(see Section 4.1):

$$\begin{aligned} s_w^{(j)} &\mapsto u_0^{(j)}, \quad j = 1, \dots, r_0; \\ s_1^{(j)}, s_w^{(r_0+j)} &\mapsto u_1^{(r_0+j)}, \quad j = 1, \dots, r_1; \\ s_1^{(r_1+2j-1)}, s_1^{(r_1+2j)}, s_w^{(r_0+r_1+j)} &\mapsto u_1^{(r_0+r_1+j)}, \quad j = 1, \dots, r_2; \dots; \quad s_2^{(i)} \mapsto s_2^{(i)}. \end{aligned}$$

We have an order on vectors  $\text{pr}_2(\bar{r}) := (r_0, \dots, r_{\eta})$  as in the previous section. By definition,

- if  $\bar{r}' \prec \bar{r}$ , then  $\text{pr}_2(\bar{r}') \leq \text{pr}_2(\bar{r})$ ;
- the map  $\varphi_{\bar{r}}^{\vee}$  is right divisible by  $\varphi_{\text{pr}_2(\bar{r})}^{\vee}$ .

More precisely we define the map  $\tilde{\psi}_{\bar{r}} : \Lambda_{(r_0, \dots, r_\eta, a_2)}(\mathbf{u}, \mathbf{s}_2) \rightarrow \Lambda_{\bar{r}}(\mathbf{t})$ ,

$$u_i^{(j)}, s_2^{((\zeta_{\max} - \zeta_i)(j-1)+1)}, \dots, s_2^{((\zeta_{\max} - \zeta_i)j)} \mapsto t_{i, (\zeta_{\max} - \zeta_i)}^{(j)}, \quad j = 1, \dots, r_{i, \zeta_{\max} - \zeta_i};$$

$$u_i^{(r_{i, \zeta_{\max} - \zeta_i} + j)}, s_2^{(r_{i, \zeta_{\max} - \zeta_i}(\zeta_{\max} - \zeta_i) + (\zeta_{\max} - \zeta_i + 1)(j-1)+1)}, \dots, \\ s_2^{(r_{i, \zeta_{\max} - \zeta_i}(\zeta_{\max} - \zeta_i) + (\zeta_{\max} - \zeta_i + 1)j)} \mapsto t_{i, (\zeta_{\max} - \zeta_i + 1)}^{(j)}, \quad j = 1, \dots, r_{i, \zeta_{\max} - \zeta_i + 1}; \dots$$

Then

$$\varphi_{\bar{r}}^\vee = \tilde{\psi}_{\bar{r}} \circ (\varphi_{\text{pr}_2(\bar{r})}^\vee \otimes \text{id}).$$

Thus we have

$$\begin{aligned} & \varphi_{\text{pr}_2(\bar{r})}^\vee \otimes \text{id} \left( \bigcap_{\bar{r}' \prec \bar{r}} \ker \varphi_{\bar{r}'}^\vee \right) \\ & \supset \prod_{1 \leq i < j \leq \eta} \prod_{1 \leq \ell \leq r_i, 1 \leq \ell' \leq r_j} (u_i^{(\ell)} - u_j^{(\ell')})^{j-i-1} \Lambda_{(r_0, \dots, r_\eta, a_2)}(\mathbf{u}, \mathbf{s}_2) \cap \bigcap_{\bar{r}' \prec \bar{r}, \text{pr}_2(\bar{r}') = (r_0, \dots, r_\eta)} \ker(\tilde{\psi}_{\bar{r}'}) \\ & \supset \prod_{1 \leq i < j \leq \eta} \prod_{1 \leq \ell \leq r_i, 1 \leq \ell' \leq r_j} (u_i^{(\ell)} - u_j^{(\ell')})^{j-i-1} \left( \bigcap_{\bar{r}' \prec \bar{r}, \text{pr}_2(\bar{r}') = (r_0, \dots, r_\eta)} \ker(\tilde{\psi}_{\bar{r}'}) \right). \end{aligned}$$

Therefore

$$\varphi_{\bar{r}}^\vee \left( \bigcap_{\bar{r}' \prec \bar{r}} \ker \varphi_{\bar{r}'}^\vee \right) \supset \tilde{\psi}_{\bar{r}} \left( \prod_{\substack{0 \leq a < b \leq \eta, \\ 1 \leq \ell \leq r_a, 1 \leq \ell' \leq r_b}} (u_a^{(\ell)} - u_b^{(\ell')})^{b-a-1} \right) \tilde{\psi}_{\bar{r}}(\Lambda_{(r_0, \dots, r_\eta, a_2)}(\mathbf{u}, \mathbf{s}_2)).$$

Consider now the set of  $\bar{r}' \prec \bar{r}$  such that  $\text{pr}_2(\bar{r}') = \bar{\rho}$ , where  $\bar{\rho} = \text{pr}_2(\bar{r})$ . One has

$$\varphi_{\bar{r}}^\vee \left( \bigcap_{\bar{r}' \prec \bar{r}, \text{pr}_2(\bar{r}') = \bar{\rho}} \ker \varphi_{\bar{r}'}^\vee \right) \supset \tilde{\psi}_{\bar{r}} \left( \prod_{\substack{0 \leq a < b \leq \eta, \\ 1 \leq i \leq r_a, 1 \leq j \leq r_b}} (u_a^{(i)} - u_b^{(j)})^{b-a-1} \right) \tilde{\psi}_{\bar{r}} \left( \bigcap_{\bar{r}' \prec \bar{r}, \text{pr}_2(\bar{r}') = \bar{\rho}} \ker(\tilde{\psi}_{\bar{r}'}) \right). \quad (4-7)$$

Let us study the two factors in (4-7) separately. For the first factor one has

$$\tilde{\psi}_{\bar{r}} \left( \prod_{\substack{0 \leq a < b \leq \eta, \\ 1 \leq i \leq r_a, 1 \leq j \leq r_b}} (u_a^{(i)} - u_b^{(j)})^{b-a-1} \right) = \prod_{\bar{\alpha}, \bar{\beta} \in P, \alpha_1 < \beta_1} \prod_{1 \leq \ell \leq r_{\bar{\alpha}}, 1 \leq \ell' \leq r_{\bar{\beta}}} (t_{\bar{\alpha}}^{(\ell)} - t_{\bar{\beta}}^{(\ell')})^{\beta_1 - \alpha_1 - 1}.$$

Now let us compute the second factor. We define the map

$$\begin{aligned} g_{\bar{\rho}} : \Lambda_{(\bar{\rho}, a_2)}(\mathbf{u}, \mathbf{s}_2) & \rightarrow \Lambda_{(\bar{\rho}, a_2 - \sum_{i=0}^{\eta} r_i(\zeta_{\max} - \zeta_i))}(\mathbf{u}, \mathbf{s}_2), \\ s_2^{(\sum_{i'=0}^{i-1} r_{i'}(\zeta_{\max} - \zeta_{i'}) + (\ell-1)(\zeta_{\max} - \zeta_i) + 1)}, \dots, s_2^{(\sum_{i'=0}^{i-1} r_{i'}(\zeta_{\max} - \zeta_{i'}) + \ell(\zeta_{\max} - \zeta_i))}, & u_i^{(\ell)} \mapsto u_i^{(\ell)}, \\ s_2^{(\ell)} \mapsto s_2^{(\ell - \sum_{i=0}^{\eta} r_i(\zeta_{\max} - \zeta_i))}, & \ell \geq \sum_{i=0}^{\eta} r_i(\zeta_{\max} - \zeta_i). \end{aligned}$$

Then

$$\tilde{\psi}_{\bar{r}} = \psi_{d(\bar{r})} \circ g_{\bar{\rho}},$$



where  $d(\bar{r}) = d(\bar{r})_{i,j}$ ,  $i = 0, \dots, \eta$ ,  $j = 0, \dots, \zeta_i$ ,  $d(\bar{r})_{i,j} = r_{i,j+\zeta_{\max}-\zeta_i}$  and  $\psi_{d(\bar{r})}$  is defined in the following way (it differs from the  $\psi_{d(\bar{r})}$  defined in Section 3.4 only by a change of the subscripts):

$$\begin{aligned}
 u_i^{(j)} &\mapsto t_{i,(\zeta_{\max}-\zeta_i)}^{(j)}, & j = 1, \dots, r_{i,\zeta_{\max}-\zeta_i}; \\
 u_i^{(r_{i,\zeta_{\max}-\zeta_i}+j)}, s_2^{(j)} &\mapsto t_{i,(\zeta_{\max}-\zeta_i+1)}^{(j)}, & j = 1, \dots, r_{i,\zeta_{\max}-\zeta_i+1}; \\
 u_i^{(r_{i,\zeta_{\max}-\zeta_i}+r_{i,\zeta_{\max}-\zeta_i+1}+j)}, s_2^{(r_{i,\zeta_{\max}-\zeta_i+1}+2j-1)}, s_2^{(r_{i,\zeta_{\max}-\zeta_i+1}+2j)} &\mapsto t_{i,(\zeta_{\max}-\zeta_i+2)}^{(j)}, & j = 1, \dots, r_{i,\zeta_{\max}-\zeta_i+2}; \dots
 \end{aligned}$$

The map  $g_{\bar{\rho}}$  is surjective. Thus

$$\tilde{\psi}_{\bar{r}} \left( \bigcap_{\bar{r}' < \bar{r}, \text{pr}_2(\bar{r}') = \bar{\rho}} \ker(\tilde{\psi}_{\bar{r}'}) \right) = \psi_{d(\bar{r})} \left( \bigcap_{\bar{r}' > \bar{r}, \text{pr}_2(\bar{r}') = \bar{\rho}} \ker(\psi_{d(\bar{r}')} ) \right).$$

Note that the order on vectors  $\bar{r}$  gives the lexicographic order on vectors  $d(\bar{r})$  with respect to the following order on coordinates:  $(i, j) < (i', j')$  if and only if  $j + (\zeta_{\max} - \zeta_i) < j' + (\zeta_{\max} - \zeta_{i'})$  or  $j + (\zeta_{\max} - \zeta_i) = j' + (\zeta_{\max} - \zeta_{i'})$  and  $i < i'$ . So using Lemma 3.17 we have

$$\psi_{d(\bar{r})} \left( \bigcap_{\bar{r}' < \bar{r}, \text{pr}_2(\bar{r}') = \bar{\rho}} \ker(\psi_{d(\bar{r}')} ) \right) \supset \prod_{\bar{\alpha} < \bar{\beta}} \prod_{1 \leq \ell \leq r_{\bar{\alpha}}, 1 \leq \ell' \leq r_{\bar{\beta}}} (t_{\bar{\alpha}}^{(\ell)} - t_{\bar{\beta}}^{(\ell')})^{s(\bar{\alpha}, \bar{\beta})} \Lambda_{\bar{r}}(\mathbf{t})$$

(see (4-5)). Multiplying the two factors of (4-7) we obtain

$$\varphi_{\bar{r}}^{\vee} \left( \bigcap_{\bar{r}' < \bar{r}} \ker(\varphi_{\bar{r}'}) \right) \supset \prod_{\bar{\alpha} < \bar{\beta}} \prod_{1 \leq \ell \leq r_{\bar{\alpha}}, 1 \leq \ell' \leq r_{\bar{\beta}}} (t_{\bar{\alpha}}^{(\ell)} - t_{\bar{\beta}}^{(\ell')})^{k(\bar{\alpha}, \bar{\beta})} \Lambda_{\bar{r}}(\mathbf{t}).$$

Therefore, due to Corollary 4.13 these spaces are equal. Thus in  $W'_{\bar{\alpha}, \bar{\beta}, k'}(X)$  generate the defining ideal of  $\text{gr}_{<} J_{\text{red}}^{\infty}(R(P))$ . This completes the proof of the theorem.  $\square$

**Corollary 4.15.** *The action of  $\mathcal{A}_L$  is free on  $J_{\text{red}}^{\infty}(R(P))[L]^*$ .*

**4.3. Higher-dimensional case.** In this subsection we study a family of polytopes such that the (dual of the) reduced arc rings of the corresponding toric varieties admit a free action of the polynomial algebras.

Consider a convex lattice polytope  $P \subset \mathbb{R}^n$ . Let  $\bar{\alpha}^1, \dots, \bar{\alpha}^m$  be the tuple of integer points of  $P$ ,  $<$  be a monomial order on  $\mathbb{k}[X_{\bar{\alpha}^1}, \dots, X_{\bar{\alpha}^m}]$ ,  $\gamma(\bar{\alpha}^i, \bar{\alpha}^j) = \gamma(\bar{\alpha}^i, \bar{\alpha}^j)$  with  $1 \leq i, j \leq m$  be a collection of non-negative integers. Assume that for any  $i \neq j$  there exists a set of vectors  $e_1(\bar{\alpha}^i, \bar{\alpha}^j), \dots, e_{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1}(\bar{\alpha}^i, \bar{\alpha}^j)$  with the following properties:

- $\bar{\alpha}^i + \sum_{\ell=1}^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1} e_{\ell}(\bar{\alpha}^i, \bar{\alpha}^j) = \bar{\alpha}^j$ .
- For any  $J \subset \{1, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1\}$ :  $\bar{\alpha}^i + \sum_{\ell \in J} e_{\ell}(\bar{\alpha}^i, \bar{\alpha}^j) \in P$ .
- For any nonempty proper subset  $J \subset \{1, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1\}$ ,

$$X_{\bar{\alpha}^i} X_{\bar{\alpha}^j} < X_{\bar{\alpha}^i + \sum_{\ell \in J} e_{\ell}(\bar{\alpha}^i, \bar{\alpha}^j)} X_{\bar{\alpha}^j - \sum_{\ell \in J} e_{\ell}(\bar{\alpha}^i, \bar{\alpha}^j)}.$$

- $e_{\ell}(\bar{\alpha}^j, \bar{\alpha}^i) = -e_{\ell}(\bar{\alpha}^i, \bar{\alpha}^j)$ .

We call  $(P, <, \gamma, \{e_{\ell}\})$  a *cube generating data*.

We define the linear maps of polytopes  $\eta_{\bar{\alpha}^i, \bar{\alpha}^j} : [0, 1]^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1} \rightarrow P$ ,

$$\eta_{\bar{\alpha}^i, \bar{\alpha}^j} \left( \sum_{\ell=1}^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1} x_\ell e_\ell \right) = \bar{\alpha}^i + \sum_{\ell=1}^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1} x_\ell e_\ell(\bar{\alpha}^i, \bar{\alpha}^j), \tag{4-8}$$

where  $\{e_\ell\}$  is the basis vectors of  $\mathbb{R}^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1}$ ,  $0 \leq x_\ell \leq 1$ . The corresponding map of rings

$$R([0, 1]^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1}) \rightarrow R(P)$$

will also be denoted by  $\eta_{\bar{\alpha}^i, \bar{\alpha}^j}$ .

For any  $k \leq \gamma(\bar{\alpha}^i, \bar{\alpha}^j) - 1$ ,

$$J^\infty(\eta_{\bar{\alpha}^i, \bar{\alpha}^j})(W_{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1, k})$$

is nilpotent. Therefore the coefficients of the series

$$W_{\bar{\alpha}^i, \bar{\alpha}^j, k}(X) := \sum_{I \subset \{2, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1\}} (-1)^{|I|} X_{\bar{\alpha}^i + \sum_{\ell \in I \cup \{1\}} e_\ell(\bar{\alpha}^i, \bar{\alpha}^j)}(s) \frac{\partial^k X_{\bar{\alpha}^j - \sum_{\ell \in I \cup \{1\}} e_\ell(\bar{\alpha}^i, \bar{\alpha}^j)}(s)}{\partial s^k}$$

belong to the defining ideal of  $J_{\text{red}}^\infty(R(P))$  for  $k \leq \gamma(\bar{\alpha}^i, \bar{\alpha}^j) - 1$ . By construction we have

$$\text{in}_{<} W_{\bar{\alpha}^i, \bar{\alpha}^j, k}(X) = X_{\bar{\alpha}^i}(s) \frac{\partial^k X_{\bar{\alpha}^j}(s)}{\partial s^k}. \tag{4-9}$$

Recall that in view of Proposition 1.28 the coefficients of (4-9) belong to the defining ideal of the associated graded ring  $\text{gr}_{<} J_{\text{red}}^\infty(R(P))$ . Suppose that, moreover, they generate this ideal. In this case we call the data  $(P, <, \gamma, \{e_\ell\})$  a *strict cube generating data*.

**Proposition 4.16.** *Consider a strict cube generating data  $(P, <, \gamma(\bar{\alpha}^i, \bar{\alpha}^j), \{e_\ell(\bar{\alpha}^i, \bar{\alpha}^j)\})$ . Then for any  $\bar{\rho} \in \mathbb{N}^m$  we have the following isomorphism of grad-graded spaces:*

$$\text{gr}_{<} J_{\text{red}}^\infty(R(P))[\bar{\rho}]^* \simeq \prod_{i < j} \prod_{\substack{1 \leq \ell \leq \rho_i \\ 1 \leq \ell' \leq \rho_j}} (t_i^{(\ell)} - t_j^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \Lambda_{\bar{\rho}}(\mathbf{t}).$$

*Proof.* This follows from Proposition 2.8. □

**Corollary 4.17.** *For  $L \in \mathbb{N}$  the action of  $\mathcal{A}_L$  is free on  $J_{\text{red}}^\infty(R(P))[L]^*$ .*

*Proof.* This follows from Theorem 3.14. □

Consider a convex function  $\zeta : P \rightarrow \mathbb{R}_{\geq 0}$  with  $\zeta(\bar{\alpha}^i) = \zeta_i \in \mathbb{Z}$ , set  $\zeta_{\max} = \max\{\zeta_i\}$ . Denote by  $P^\zeta$  the polytope in  $\mathbb{R}^{n+1}$  obtained as the convex hull of  $P \times \zeta_{\max}$  and all  $\bar{\alpha}^i \times (\zeta_{\max} - \zeta_i)$ . In what follows we construct a cube generating data for  $P^\zeta$  starting from a cube generating data for  $P$  satisfying the following condition. For any  $\bar{\alpha}^i, \bar{\alpha}^j \in P$  such that  $\zeta_i \leq \zeta_j$ , there exist nonnegative integers  $f_1(\bar{\alpha}^i, \bar{\alpha}^j), \dots, f_{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1}(\bar{\alpha}^i, \bar{\alpha}^j)$  such that

- $\zeta_i + \sum_{\ell=1}^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1} f_\ell(\bar{\alpha}^i, \bar{\alpha}^j) = \zeta_j$ ;
- for any  $J \subset \{1, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1\}$ :  $\zeta_i + \sum_{\ell \in J} f_\ell(\bar{\alpha}^i, \bar{\alpha}^j) \leq \zeta(\bar{\alpha}^i + \sum_{\ell \in J} e_\ell(\bar{\alpha}^i, \bar{\alpha}^j))$ .

**Example 4.18.** Consider the cube generating data  $(P, \prec, \gamma, \{e_\ell\})$ . Assume that for any  $\bar{\alpha}^i, \bar{\alpha}^j$  the  $\ell$ -th coordinate of every  $e_p(\bar{\alpha}^i, \bar{\alpha}^j)$  is either nonnegative or nonpositive. Then any convex function of the form  $\zeta(\bar{\alpha}) = \zeta(\alpha_\ell)$ , i.e., depending only on the  $\ell$ -th coordinate admits such a set of integers  $f_p(\bar{\alpha}^i, \bar{\alpha}^j)$ . Indeed, without loss of generality we may assume that  $\zeta(\bar{\alpha}^i) \geq \zeta(\bar{\alpha}^j)$ . We may consider an enumeration of vectors  $e_p(\bar{\alpha}^i, \bar{\alpha}^j)$  such that the absolute value of the  $\ell$ -th coordinate is weakly increasing. Let  $x$  be the smallest number such that  $\zeta(\bar{\alpha}^i + \sum_{p=1}^x e_p(\bar{\alpha}^i, \bar{\alpha}^j)) < \zeta(\bar{\alpha}^i)$ . We put

$$f_q(\bar{\alpha}^i, \bar{\alpha}^j) = \begin{cases} 0 & \text{if } 1 \leq q < x, \\ \zeta(\bar{\alpha}^i + \sum_{p=1}^x e_p(\bar{\alpha}^i, \bar{\alpha}^j)) - \zeta(\bar{\alpha}^i) & \text{if } q = x, \\ \zeta(\bar{\alpha}^i + \sum_{p=1}^q e_p(\bar{\alpha}^i, \bar{\alpha}^j)) - \zeta(\bar{\alpha}^i + \sum_{p=1}^{q-1} e_p(\bar{\alpha}^i, \bar{\alpha}^j)) & \text{if } x < q \leq \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1. \end{cases}$$

Note that the numbers  $f_q(\bar{\alpha}^i, \bar{\alpha}^j)$  are all either nonnegative or nonpositive.

For  $\bar{r} = (r_{(\bar{\alpha}^i, a)})_{(\bar{\alpha}^i, a) \in P^\zeta}$  we define  $\text{pr}_{n+1}(\bar{r}) = (\text{pr}_{n+1}(\bar{r}))_{\bar{\alpha}^i}$ ,  $i = 1, \dots, m$  by the formula

$$(\text{pr}_{n+1}(\bar{r}))_{\bar{\alpha}^i} = \sum_{a=\zeta_{\max}-\zeta_i}^{\zeta_{\max}} r_{(\bar{\alpha}^i, a)}.$$

For  $(\bar{\alpha}^i, a), (\bar{\alpha}^j, b) \in P^\zeta \cap \mathbb{Z}^{n+1}$  we let

$$(\bar{\alpha}^i, a) < (\bar{\alpha}^j, b) \quad \text{if } a < b \text{ or } a = b \text{ and } i < j. \tag{4-10}$$

We define an order  $\prec^\zeta$  on monomials in  $\mathbb{k}[X_{(\bar{\alpha}^i, a)}]$ ,  $(\bar{\alpha}^i, a) \in P^\zeta \cap \mathbb{Z}^{n+1}$  in the following way. For  $r, r' \in \mathbb{N}^{P^\zeta \cap \mathbb{Z}^{n+1}}$  with  $\text{pr}_{n+1}(\bar{r}) < \text{pr}_{n+1}(\bar{r}')$  we set  $\bar{r} \prec^\zeta \bar{r}'$  and  $X^{\bar{r}} \prec^\zeta X^{\bar{r}'}$ . If  $\text{pr}_{n+1}(\bar{r}) = \text{pr}_{n+1}(\bar{r}')$ , then  $\prec^\zeta$  compares  $r$  to  $r'$  and  $X^{\bar{r}}$  to  $X^{\bar{r}'}$  lexicographically with respect to the order (4-10).

Suppose that  $(\bar{\alpha}^i, a) \leq (\bar{\alpha}^j, b)$ . We set

$$\gamma((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) = \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + \kappa((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)),$$

where

$$\kappa((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) = \begin{cases} b - (\zeta_{\max} - \zeta_j) & \text{if } \zeta_{\max} - \zeta_j > a, \\ b - a & \text{if } \zeta_{\max} - \zeta_j \leq a \text{ and } i \leq j, \\ b - a - 1 & \text{if } \zeta_{\max} - \zeta_j \leq a \text{ and } i > j. \end{cases} \tag{4-11}$$

We construct the vectors  $e_\ell((\bar{\alpha}^i, a), (\bar{\alpha}^j, b))$  in the following way.

Suppose that  $\zeta_{\max} - \zeta_j > a$ . Let  $y$  be the number such that

$$\sum_{\ell=1}^{y-1} f_\ell(\bar{\alpha}^i, \bar{\alpha}^j) < a - (\zeta_{\max} - \zeta_i) \leq \sum_{\ell=1}^y f_\ell(\bar{\alpha}^i, \bar{\alpha}^j).$$

Then

$$e_\ell((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) = \begin{cases} (e_\ell(\bar{\alpha}^i, \bar{\alpha}^j), 0) & \text{if } \ell = 1, \dots, y-1, \\ (e_y(\bar{\alpha}^i, \bar{\alpha}^j), f_y(\bar{\alpha}^i, \bar{\alpha}^j) - (\zeta_{\max} - \zeta_i)) & \text{if } \ell = y, \\ (e_\ell(\bar{\alpha}^i, \bar{\alpha}^j), f_\ell(\bar{\alpha}^i, \bar{\alpha}^j)), & \text{if } \ell = y+1, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1, \\ (0, 1), & \text{if } \ell = \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 2, \dots, \gamma((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) + 1. \end{cases}$$

If  $a \geq \zeta_{\max} - \zeta_j$  and  $i \leq j$ , then

$$e_\ell((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) = \begin{cases} (e_\ell(\bar{\alpha}^i, \bar{\alpha}^j), 0) & \text{if } \ell = 1, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1, \\ (0, 1) & \text{if } \ell = \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 2, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + b - a + 1. \end{cases}$$

If  $a \geq \zeta_{\max} - \zeta_j$  and  $i > j$ , then

$$e_\ell((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) = \begin{cases} (e_\ell(\bar{\alpha}^i, \bar{\alpha}^j), 0) & \text{if } \ell = 1, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j), \\ (e_{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)+1}, 1) & \text{if } \ell = \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 1, \\ (0, 1) & \text{if } \ell = \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + 2, \dots, \gamma(\bar{\alpha}^i, \bar{\alpha}^j) + b - a. \end{cases}$$

For  $a > b$  we put  $e_\ell((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) = -e_\ell((\bar{\alpha}^j, b), (\bar{\alpha}^i, a))$ . The following lemma follows directly from the definitions.

**Lemma 4.19.**  $(P^\zeta, \prec^\zeta, \gamma, \{e_\ell\})$  satisfy the conditions for cube generating data.

We consider the maps

$$\eta_{(\bar{\alpha}^i, a), (\bar{\alpha}^j, b)} : [0, 1]^{\gamma((\bar{\alpha}^i, a), (\bar{\alpha}^j, b)) + 1} \rightarrow P^\zeta$$

corresponding to this data. By construction and Proposition 2.7 we have

$$(\text{gr}_{\prec} J_{\text{red}}^\infty(R(P^\zeta))[\bar{r}])^* \hookrightarrow \prod_{(\bar{\alpha}^i, a) < (\bar{\alpha}^j, b)} \prod_{\substack{1 \leq \ell \leq r_{(\bar{\alpha}^i, a)}, \\ 1 \leq \ell' \leq r_{(\bar{\alpha}^j, b)}}} (t_{(\bar{\alpha}^i, a)}^{(\ell)} - t_{(\bar{\alpha}^j, b)}^{(\ell')})^{\gamma((\bar{\alpha}^i, a), (\bar{\alpha}^j, b))} \Lambda_{\bar{r}}(\mathbf{t}). \quad (4-12)$$

We now show that this inclusion is, in fact, an equality.

**Theorem 4.20.** Assume that  $(P, \prec, \gamma, \{e_\ell\})$  is a strict cube generating data. Then  $(P^\zeta, \prec^\zeta, \gamma, \{e_\ell\})$  is a strict cube generating data.

*Proof.* In this proof we compute the space  $(\text{gr}_{\prec} J_{\text{red}}^\infty(R(P^\zeta))[\bar{r}])^*$  in the same way as in the proof of Theorem 4.14. As before we study the dual map for the inclusion

$$\varphi_{\bar{r}} : A_{\bar{r}} \hookrightarrow J^\infty(\mathbb{k}[z_1, z_2, \dots, z_{n+1}, w])[(\bar{a}, a_{n+1}), L],$$

where

$$(\bar{a}, a_{n+1}) = \sum_{(\bar{\alpha}, \alpha_{n+1}) \in P^\zeta} r_{(\bar{\alpha}, \alpha_{n+1})}(\bar{\alpha}, \alpha_{n+1}), \quad L = \sum_{(\bar{\alpha}, \alpha_{n+1}) \in P^\zeta} r_{(\bar{\alpha}, \alpha_{n+1})}.$$

The image of the map  $\varphi_{\bar{r}}^\vee : \Lambda_{(\bar{a}, a_{n+1}), L}(\mathbf{s}) \rightarrow \Lambda_{\bar{r}}(\mathbf{t})$ ,

$$\begin{aligned} s_1^{(\alpha_1^1(j-1)+1)}, \dots, s_1^{(\alpha_1^1(j-1)+\alpha_1^1)}, \dots, s_{n+1}^{((\zeta_{\max}-\zeta_1)(j-1)+1)}, \dots, \\ s_{n+1}^{((\zeta_{\max}-\zeta_1)j)}, s_w^{(j)} \mapsto t_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)}^{(j)}, \quad j = 1, \dots, r_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)}; \\ s_1^{(r_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)}\alpha_1^1 + \alpha_1^1(j-1)+1)}, \dots, s_{n+1}^{(r_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)}(\zeta_{\max}-\zeta_1) + (\zeta_{\max}-\zeta_1+1)j)}, \\ s_w^{(j)} \mapsto t_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)}^{(j)}, \quad j = r_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)} + 1, \dots, r_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1)} + r_{(\bar{\alpha}^1, \zeta_{\max}-\zeta_1-1)}; \\ \dots \end{aligned}$$

is isomorphic to the dual space of  $A_{\bar{r}}$ . Here for every integer point  $(\bar{\alpha}, \alpha_{n+1}) \in P^\zeta$  and  $1 \leq i \leq n + 1$  the map  $\varphi_{\bar{r}}^\vee$  takes  $r_{(\bar{\alpha}, \alpha_{n+1})} \alpha_i$  variables of the form  $s_i^*$  to variables of the form  $t_{(\bar{\alpha}, \alpha_{n+1})}^*$ , it also takes  $r_{(\bar{\alpha}, \alpha_{n+1})}$  of the  $s_w^*$  to the  $t_{(\bar{\alpha}, \alpha_{n+1})}^*$ . The dual space to  $A_{\bar{r}} / (A_{\bar{r}} \cap (\bigoplus_{r' \prec r} A_{\bar{r}}))$  is isomorphic to the space  $\varphi_{\bar{r}}^\vee(\bigcap_{r' \prec r} \ker(\varphi_{\bar{r}'}^\vee))$ . Our goal is to compute this space.

Denote  $\bar{\rho} := \text{pr}_{n+1}(\bar{r})$ . We consider the set of variables  $u_i^{(j)} = u_{\alpha_i}^{(j)}, i \in P, j = 1, \dots, \rho_i$ . Consider the map  $\varphi_{\bar{\rho}}^\vee \otimes \text{id} : \Lambda_{(\bar{\alpha}, \alpha_{n+1}), L}(\mathbf{s}) \rightarrow \Lambda_{(\bar{\rho}, \alpha_{n+1})}(\mathbf{u}, \mathbf{s}_{n+1})$ , i.e.,  $s_{n+1}^{(i)} \mapsto s_{n+1}^{(i)}$  and it is equal to  $\varphi_{\bar{\rho}}^\vee$  on  $s_1^{(i)}, \dots, s_n^{(i)}, s_w^{(i)}$  (but with  $t$  replaced by  $u$ ).

By definition,

- if  $\bar{r}' \prec \bar{r}$ , then  $\text{pr}_{n+1}(\bar{r}') \preceq \text{pr}_{n+1}(\bar{r})$ ;
- the map  $\varphi_{\bar{r}}^\vee$  is right divisible by  $\varphi_{\bar{\rho}}^\vee$ .

More precisely, we define the map  $\tilde{\psi}_{\bar{r}} : \Lambda_{(\bar{\rho}, \alpha_{n+1})}(\mathbf{u}, \mathbf{s}_{n+1}) \rightarrow \Lambda_{\bar{r}}(\mathbf{t})$ ,

$$\begin{aligned} u_i^{(j)}, s_{n+1}^{((\zeta_{\max} - \zeta_i)(j-1)+1)}, \dots, s_{n+1}^{((\zeta_{\max} - \zeta_i)j)} &\mapsto t_{(\bar{\alpha}^i, (\zeta_{\max} - \zeta_i))}^{(j)}, \quad j = 1, \dots, r_{i, \zeta_{\max} - \zeta_i}; \\ u_i^{(r_{i, \zeta_{\max} - \zeta_i} + j)}, s_{n+1}^{(r_{i, \zeta_{\max} - \zeta_i}(\zeta_{\max} - \zeta_i) + (\zeta_{\max} - \zeta_i + 1)(j-1) + 1)}, \dots, \\ & s_{n+1}^{(r_{i, \zeta_{\max} - \zeta_i}(\zeta_{\max} - \zeta_i) + (\zeta_{\max} - \zeta_i + 1)j)} \mapsto t_{(\bar{\alpha}^i, (\zeta_{\max} - \zeta_i + 1))}^{(j)}, \quad j = 1, \dots, r_{i, \zeta_{\max} - \zeta_i + 1}; \\ &\dots \end{aligned}$$

Here for every  $(\bar{\alpha}_i, \alpha_{n+1}) \in P^\zeta$  we have  $r_{(\bar{\alpha}_i, \alpha_{n+1})}$  variables of the form  $u_i^*$  and  $\alpha_{n+1}$  variables of the form  $s_{n+1}^*$  being mapped to variables of the form  $t_{(\bar{\alpha}_i, \alpha_{n+1})}^*$ . Then  $\varphi_{\bar{r}}^\vee = \tilde{\psi}_{\bar{r}} \circ (\varphi_{\bar{\rho}}^\vee \otimes \text{id})$ .

By the definition of strict cube generating data and Proposition 1.28, the right-hand sides of (4-9) generate the ring  $\text{gr}_{\prec} J_{\text{red}}^\infty(R(P))$ . A component of the latter ring is considered in Corollary 3.7 and Proposition 2.8 provides

$$\varphi_{\bar{\rho}}^\vee \otimes \text{id} \left( \bigcap_{\bar{\rho}' \prec \bar{\rho}} \ker(\varphi_{\bar{\rho}'}^\vee \otimes \text{id}) \right) = \prod_{1 \leq i < j \leq m} \prod_{1 \leq \ell \leq \rho_i, 1 \leq \ell' \leq \rho_j} (u_{\bar{\alpha}^i}^{(\ell)} - u_{\bar{\alpha}^j}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \Lambda_{(\rho, \alpha_{n+1})}(\mathbf{u}, \mathbf{s}_{n+1}).$$

Consequently,

$$\varphi_{\bar{\rho}}^\vee \otimes \text{id} \left( \bigcap_{\text{pr}_{n+1}(\bar{r}') \prec \bar{\rho}} \ker \varphi_{\bar{r}'}^\vee \right) \supset \prod_{1 \leq i < j \leq m} \prod_{1 \leq \ell \leq \rho_i, 1 \leq \ell' \leq \rho_j} (u_{\bar{\alpha}^i}^{(\ell)} - u_{\bar{\alpha}^j}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \Lambda_{(\rho, \alpha_{n+1})}(\mathbf{u}, \mathbf{s}_{n+1}).$$

Therefore,

$$\begin{aligned} \varphi_{\bar{\rho}}^\vee \otimes \text{id} \left( \bigcap_{\bar{r}' \prec \bar{r}} \ker \varphi_{\bar{r}'}^\vee \right) &\supset \prod_{1 \leq i < j \leq m} \prod_{1 \leq \ell \leq \rho_i, 1 \leq \ell' \leq \rho_j} (u_{\bar{\alpha}^i}^{(\ell)} - u_{\bar{\alpha}^j}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \Lambda_{(\rho, \alpha_{n+1})}(\mathbf{u}, \mathbf{s}_{n+1}) \cap \bigcap_{\bar{r}' \prec \bar{r}, \text{pr}_{n+1}(\bar{r}') = \bar{\rho}} \ker(\tilde{\psi}_{\bar{r}'}) \\ &\supset \prod_{1 \leq i < j \leq m} \prod_{1 \leq \ell \leq \rho_i, 1 \leq \ell' \leq \rho_j} (u_{\bar{\alpha}^i}^{(\ell)} - u_{\bar{\alpha}^j}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \left( \bigcap_{\bar{r}' \prec \bar{r}, \text{pr}_{n+1}(\bar{r}') = \bar{\rho}} \ker(\tilde{\psi}_{\bar{r}'}) \right). \end{aligned}$$

Hence

$$\varphi_{\bar{r}}^{\vee} \left( \bigcap_{\bar{r}' \prec^{\zeta} \bar{r}} \ker \varphi_{\bar{r}'}^{\vee} \right) \supset \tilde{\psi}_{\bar{r}} \left( \prod_{1 \leq i < j \leq m} \prod_{\substack{1 \leq \ell \leq \rho_i, \\ 1 \leq \ell' \leq \rho_j}} (u_{\bar{\alpha}_i}^{(\ell)} - u_{\bar{\alpha}_j}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \right) \tilde{\psi}_{\bar{r}} \left( \bigcap_{\bar{r}' \prec^{\zeta} \bar{r}, \text{pr}_{n+1}(\bar{r}') = \bar{\rho}} \ker(\tilde{\psi}_{\bar{r}'}) \right). \quad (4-13)$$

Let us study the two factors above separately. By definition of  $\tilde{\psi}_{\bar{r}}$  we have

$$\tilde{\psi}_{\bar{r}} \left( \prod_{1 \leq i < j \leq m} i \prod_{\substack{1 \leq \ell \leq \rho_i, \\ 1 \leq \ell' \leq \rho_j}} (u_{\bar{\alpha}_i}^{(\ell)} - u_{\bar{\alpha}_j}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)} \right) = \prod_{\substack{(\bar{\alpha}^i, \alpha_{n+1}^i), (\bar{\alpha}^j, \alpha_{n+1}^j) \in P^{\zeta} \\ \text{where } i < j}} \prod_{\substack{1 \leq \ell \leq r_{(\bar{\alpha}^i, \alpha_{n+1}^i)}, \\ 1 \leq \ell' \leq r_{(\bar{\alpha}^j, \alpha_{n+1}^j)}}} (t_{(\bar{\alpha}^i, \alpha_{n+1}^i)}^{(\ell)} - t_{(\bar{\alpha}^j, \alpha_{n+1}^j)}^{(\ell')})^{\gamma(\bar{\alpha}^i, \bar{\alpha}^j)}.$$

Define the map

$$\begin{aligned} g_{\bar{\rho}} : \Lambda_{(\bar{\rho}, a_{n+1})}(\mathbf{u}, \mathbf{s}_{n+1}) &\rightarrow \Lambda_{(\bar{\rho}, a_{n+1} - \sum_{\alpha_i \in P} \rho_i (\zeta_{\max} - \zeta_i))}(\mathbf{u}, \mathbf{s}_{n+1}), \\ s_{n+1}^{(\sum_{i'=1}^{i-1} \rho_{i'} (\zeta_{\max} - \zeta_{i'}) + (\ell-1)(\zeta_{\max} - \zeta_i) + 1)} &, \dots, s_{n+1}^{(\sum_{i'=1}^{i-1} \rho_{i'} (\zeta_{\max} - \zeta_{i'}) + \ell(\zeta_{\max} - \zeta_i))}, \quad u_i^{(\ell)} \mapsto u_i^{(\ell)}, \\ s_{n+1}^{(\ell)} &\mapsto s_{n+1}^{(\ell - \sum_{i=0}^m \rho_i (\zeta_{\max} - \zeta_i))}, \quad \ell \geq \sum_{\alpha_i \in P} \rho_i (\zeta_{\max} - \zeta_i). \end{aligned}$$

Here  $\zeta_{\max} - \zeta_i$  variables of the form  $s_{n+1}^{\bullet}$  are mapped to every variable of the form  $u_i^{\bullet}$  while the remaining  $s_{n+1}^{\bullet}$  are mapped to (distinct)  $s_{n+1}^{\bullet}$ . We have

$$\tilde{\psi}_{\bar{r}} = \psi_{d(\bar{r})} \circ g_{\bar{\rho}},$$

where  $d(\bar{r}) = (d(\bar{r})_{(\bar{\alpha}^i, a)})$ ,  $i = 0, \dots, m$ ,  $a = 0, \dots, \zeta_i$ ,  $d(\bar{r})_{(\bar{\alpha}^i, a)} = \bar{r}_{(\bar{\alpha}^i, a + \zeta_{\max} - \zeta_i)}$  and  $\psi_{d(\bar{r})}$  is defined in the following way (it differs from the  $\psi_{d(\bar{r})}$  defined in Section 3.4 only by a change of the subscripts):

$$\begin{aligned} u_{\bar{\alpha}^i}^{(j)} &\mapsto t_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i)}^{(j)}, \quad j = 1, \dots, r_{\bar{\alpha}^i, (\zeta_{\max} - \zeta_i)}; \\ u_{\bar{\alpha}^i}^{(r_{\bar{\alpha}^i, \zeta_{\max} - \zeta_i} + j)} &, s_{n+1}^{(j)} \mapsto t_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 1)}^{(j)}, \quad j = 1, \dots, r_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 1)}; \\ u_i^{(r_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i)} + r_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 1)} + j)} &, \\ s_{n+1}^{(r_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 1)} + 2j - 1)} &, s_{n+1}^{(r_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 1)} + 2j)} \mapsto t_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 2)}^{(j)}, \quad j = 1, \dots, r_{(\bar{\alpha}^i, \zeta_{\max} - \zeta_i + 2)}; \\ &\dots \end{aligned}$$

Here distinct variables of the form  $u_{\bar{\alpha}_i}^{\bullet}$  are mapped to distinct variables of the form  $t_{(\bar{\alpha}_i, a)}^{\bullet}$  and  $a - (\zeta_{\max} - \zeta_i)$  variables of the form  $s_{n+1}^{\bullet}$  are mapped to each variable  $t_{(\bar{\alpha}_i, a)}^{\bullet}$ . The map  $g_{\bar{\rho}}$  is surjective. Thus

$$\tilde{\psi}_{\bar{r}} \left( \bigcap_{\bar{r}' \prec^{\zeta} \bar{r}, \text{pr}_{n+1}(\bar{r}') = \bar{\rho}} \ker(\tilde{\psi}_{\bar{r}'}) \right) = \psi_{d(\bar{r})} \left( \bigcap_{\bar{r}' \prec^{\zeta} \bar{r}, \text{pr}_{n+1}(\bar{r}') = \bar{\rho}} \ker(\psi_{d(\bar{r}')}) \right).$$

Note that the order  $\prec^{\zeta}$  on vectors  $\bar{r}$  defines an order on the vectors  $d(\bar{r})$ . This order is lexicographic with respect to the following order on coordinates:  $(\bar{\alpha}^i, a) <_d (\bar{\alpha}^j, b)$  if and only if  $a + (\zeta_{\max} - \zeta_i) < b + (\zeta_{\max} - \zeta_j)$

or  $a + (\zeta_{\max} - \zeta_i) = b + (\zeta_{\max} - \zeta_j)$  and  $i < j$ . Thus using Lemma 3.17 we have

$$\psi_{d(\bar{r})} \left( \bigcap_{\bar{r}' < \bar{r}, \text{pr}_{n+1}(\bar{r}') = \bar{\rho}} \ker(\psi_{d(\bar{r}')} ) \right) \supset \prod_{\substack{(\bar{\alpha}^i, \alpha_{n+1}^i) < (\bar{\alpha}^j, \alpha_{n+1}^j) \\ \in P^\zeta}} \prod_{\substack{1 \leq \ell \leq r_{(\bar{\alpha}^i, \alpha_{n+1}^i)}, \\ 1 \leq \ell' \leq r_{(\bar{\alpha}^j, \alpha_{n+1}^j)}}} (t_{(\bar{\alpha}^i, \alpha_{n+1}^i)}^{(\ell)} - t_{(\bar{\alpha}^j, \alpha_{n+1}^j)}^{(\ell')})^{\chi((\bar{\alpha}^i, \alpha_{n+1}^i), (\bar{\alpha}^j, \alpha_{n+1}^j))} \Lambda_{\bar{r}}(\mathbf{t})$$

(see (4-11)). We obtain

$$\varphi_{\bar{r}}^\vee \left( \bigcap_{\bar{r}' < \bar{r}} \ker(\varphi_{\bar{r}'}^\vee) \right) \supset \prod_{\substack{(\bar{\alpha}^i, \alpha_{n+1}^i) < (\bar{\alpha}^j, \alpha_{n+1}^j) \\ \in P^\zeta}} \prod_{\substack{1 \leq \ell \leq r_{(\bar{\alpha}^i, \alpha_{n+1}^i)}, \\ 1 \leq \ell' \leq r_{(\bar{\alpha}^j, \alpha_{n+1}^j)}}} (t_{(\bar{\alpha}^i, \alpha_{n+1}^i)}^{(\ell)} - t_{(\bar{\alpha}^j, \alpha_{n+1}^j)}^{(\ell')})^{\gamma((\bar{\alpha}^i, \alpha_{n+1}^i), (\bar{\alpha}^j, \alpha_{n+1}^j))} \Lambda_{\bar{r}}(\mathbf{t}).$$

Therefore the map (4-12) is bijective. □

### 5. Examples

**5.1. Parallelepiped.** Consider the polytope  $P_{d_1, \dots, d_n} := (x_1, \dots, x_n) \in \mathbb{R}^n, 0 \leq x_i \leq d_i, d_1, \dots, d_n \in \mathbb{N}$ . Choose  $d_{n+1} \in \mathbb{N}$ , and define  $\zeta(\bar{\alpha}) = d_{n+1}, \bar{\alpha} \in P_{d_1, \dots, d_n}$ . Then we have

$$P_{d_1, \dots, d_n, d_{n+1}} = P_{d_1, \dots, d_n}^\zeta.$$

By induction we can construct a strict cube generating data for this polytope (in this example  $f_i(\bar{\alpha}, \bar{\beta}) = 0$  for all  $\bar{\alpha}, \bar{\beta}, i$ ). In particular, the action of  $\mathcal{A}_L$  on the  $L$ -th graded component of  $J_{\text{red}}^\infty(R(P_{d_1, \dots, d_n}))^*$  is free. Let us compute the graded dimension of this component.

We define an order on  $P_{d_1, \dots, d_n} \cap \mathbb{Z}^n$  by setting  $\bar{\alpha} <_n \bar{\beta}$  if for the largest  $k$  such that  $\alpha_k \neq \beta_k$  we have  $\alpha_k < \beta_k$ . Note that this order corresponds to the order (4-10), i.e.,  $<_{n+1} = <_n^\zeta$ . Let us compute  $\gamma(\bar{\alpha}, \bar{\beta})$  for  $\bar{\alpha}, \bar{\beta} \in P_{d_1, \dots, d_n}$ . First, by (4-11) for  $0 \leq i \leq n - 1$  and  $(\bar{\alpha}, \alpha_{i+1}) \leq_{i+1} (\bar{\beta}, \beta_{i+1}) \in P_{d_1, \dots, d_i}^{\zeta_i}$ , where  $\zeta_i \equiv d_{i+1}$ , we have

$$\chi((\bar{\alpha}, \alpha_{i+1}), (\bar{\beta}, \beta_{i+1})) = \begin{cases} \beta_{i+1} - \alpha_{i+1} & \text{if } \bar{\alpha} \leq_i \bar{\beta}, \\ \bar{\beta}_{i+1} - \bar{\alpha}_{i+1} - 1 & \text{if } \bar{\beta} <_i \bar{\alpha}. \end{cases} \tag{5-1}$$

Therefore for  $\bar{\alpha}, \bar{\beta} \in P_{d_1, \dots, d_n}$  we have

$$\gamma(\bar{\alpha}, \bar{\beta}) = \sum_{i=1}^n |\beta_i - \alpha_i| - S,$$

where  $S$  is the number of  $i$  for which the largest  $j < i$  with  $\alpha_j \neq \beta_j$  satisfies  $(\alpha_i - \beta_i)(\alpha_j - \beta_j) < 0$  (or no such  $j$  exists). We note that if all  $d_i = 1$ , then we recover Remark 1.32.

Note that the graded dimension of  $\Lambda_{\bar{r}}(\mathbf{s})$  is equal to

$$\prod_{\bar{\alpha} \in P_{d_1, \dots, d_n}} \prod_{\ell=1}^{r_{\bar{\alpha}}} \frac{1}{1 - q^\ell}.$$

Using Proposition 2.8 we can now write the graded dimension of  $J_{\text{red}}^\infty R(P_{d_1, \dots, d_n})[L]$  as

$$\sum_{\bar{r} \mid \sum_{\bar{\alpha} \in P_{d_1, \dots, d_n}} r_{\bar{\alpha}} = L} v^{\sum_{\bar{\alpha} \in P_{d_1, \dots, d_n}} r_{\bar{\alpha}} \bar{\alpha}} \prod_{\bar{\alpha} \in P_{d_1, \dots, d_n}} \prod_{\ell=1}^{r_{\bar{\alpha}}} \frac{1}{1-q^\ell} \prod_{\bar{\alpha} < \bar{\beta} \in P_{d_1, \dots, d_n}} q^{r_{\bar{\alpha}} r_{\bar{\beta}} \gamma(\bar{\alpha}, \bar{\beta})}.$$

Here variables  $v_1, \dots, v_n$  correspond to the  $n$ -dimensional torus action on  $R(P_{d_1, \dots, d_n})$  (and, subsequently, on  $J_{\text{red}}^\infty R(P_{d_1, \dots, d_n})$ ) while  $q$  corresponds to the grading grad.

**5.2. Simplex.** Consider the polytope

$$P_{n,d} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid d \geq x_n \geq \dots \geq x_1 \geq 0\}$$

for  $d \in \mathbb{N}$ . Consider  $\zeta((x_1, \dots, x_n)) = d - x_n$ . Then  $P_{n,d}^\zeta$  is equal to the polytope  $P_{n+1,d}$ . Thus by induction we can construct a strict cube generating data for all the polytopes  $P_{n,d}$  using the construction from Example 4.18. In particular, the action of  $\mathcal{A}_L$  on the  $L$ -th graded component of  $J_{\text{red}}^\infty R(P_{n,d})^*$  is free.

We define an order on  $P_{n,d} \cap \mathbb{Z}^n$  by setting  $\bar{\alpha} <_n \bar{\beta}$  if for the largest  $k$  such that  $\alpha_k \neq \beta_k$  we have  $\alpha_k < \beta_k$ . Note that this order corresponds to the order (4-10), i.e.,  $<_{n+1} = <^\zeta$ . Let us compute  $\gamma(\bar{\alpha}, \bar{\beta})$  for  $\bar{\alpha}, \bar{\beta} \in P_{n,d}$ . First, by (4-11) for  $1 \leq i \leq n-1$  and  $(\bar{\alpha}, \alpha_{i+1}) \leq_{i+1} (\bar{\beta}, \beta_{i+1}) \in P_{i,d}$  we have

$$\kappa((\bar{\alpha}, \alpha_{i+1}), (\bar{\beta}, \beta_{i+1})) = \begin{cases} \beta_{i+1} - \beta_i & \text{if } \beta_i > \alpha_{i+1}, \\ \beta_{i+1} - \alpha_{i+1} & \text{if } \beta_i \leq \alpha_{i+1} \text{ and } \bar{\alpha} \leq \bar{\beta}, \\ \beta_{i+1} - \alpha_{i+1} - 1 & \text{if } \beta_i \leq \alpha_{i+1} \text{ and } \bar{\alpha} > \bar{\beta}. \end{cases} \quad (5-2)$$

Now, for  $\bar{\alpha}, \bar{\beta} \in P_{n,d}$  and  $i = 1, \dots, n-1$  we define

$$\kappa_i(\bar{\alpha}, \bar{\beta}) = \begin{cases} \beta_{i+1} - \beta_i & \text{if } \beta_i > \alpha_{i+1}, \\ \beta_{i+1} - \alpha_{i+1} & \text{if } \beta_i \leq \alpha_{i+1} \text{ and } (\alpha_1, \dots, \alpha_i) \leq (\beta_1, \dots, \beta_i), \\ \beta_{i+1} - \alpha_{i+1} - 1 & \text{if } \beta_i \leq \alpha_{i+1} \text{ and } (\alpha_1, \dots, \alpha_i) > (\beta_1, \dots, \beta_i), \end{cases} \quad (5-3)$$

and

$$\kappa_0(\bar{\alpha}, \bar{\beta}) = \begin{cases} |\alpha_1 - \beta_1| - 1 & \text{if } \alpha_1 \neq \beta_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then by induction we have

$$\gamma(\bar{\alpha}, \bar{\beta}) = \sum_{i=0}^{n-1} \kappa_i(\bar{\alpha}, \bar{\beta}).$$

Note that  $R(P_{n,d})$  is naturally embedded into  $R(P_{n,1})$  as a sum of homogeneous components and  $R(P_{n,1})$  is a polynomial algebra in  $n+1$  variables which is acted upon by  $\mathfrak{sl}_{n+1}(\mathbb{k})$ . Consider the restriction of this action to  $R(P_{n,d})$ . By Lemmas 1.23 and 1.16 we have an action of  $\mathfrak{sl}_{n+1}(\mathbb{k})[s]$  on  $J_{\text{red}}^\infty R(P_{n,d})$  and for any  $L \in \mathbb{N}$  the graded component  $J_{\text{red}}^\infty R(Q)[L]$  is preserved by this action. Due to [Dumanski and Feigin 2023] we have that  $J_{\text{red}}^\infty R(Q)[L]$  is isomorphic to the global Demazure module  $\mathbb{D}_{d, L\omega_1}$ . The graded dimension of  $J_{\text{red}}^\infty R(Q)[L]$  is equal to

$$\sum_{\bar{r} \mid \sum_{\bar{\alpha} \in P_{n,d}} r_{\bar{\alpha}} = L} v^{\sum_{\bar{\alpha} \in P_{n,d}} r_{\bar{\alpha}} \bar{\alpha}} \prod_{\bar{\alpha} \in P_{n,d}} \prod_{\ell=1}^{r_{\bar{\alpha}}} \frac{1}{1-q^\ell} \prod_{\bar{\alpha} < \bar{\beta} \in P_{n,d}} q^{r_{\bar{\alpha}} r_{\bar{\beta}} \gamma(\bar{\alpha}, \bar{\beta})}, \quad (5-4)$$

where the  $v_i$  correspond to coordinates in a maximal torus of  $\mathfrak{sl}_{n+1}(\mathbb{k})$ .



**Appendix: Veronese–Segre embeddings**

In this appendix we deal with a special kind of toric projective embeddings. We use an approach, completely different to the one in the main body of the paper, to study their (reduced) coordinate rings and compute their characters. This approach is based on the representation theory of the current algebras. It would be interesting to investigate the combinatorial identities, obtained by comparing the formulae for the graded characters of the form (5-4) with the ones of the form (A-3), (A-4).

For a semisimple Lie algebra  $\mathfrak{g}$  we denote by  $V_\lambda^{\mathfrak{g}}$  and  $D_{d,\lambda}^{\mathfrak{g}}$  the irreducible  $\mathfrak{g}$ -module of highest weight  $\lambda$  and the  $\mathfrak{g}[t]$  affine Demazure module of level  $d$  and highest weight  $d\lambda$  (see [Fourier and Littelmann 2007; Chari and Venkatesh 2015] for details on affine Demazure modules).

Let  $\mathfrak{g}_1, \dots, \mathfrak{g}_m$  be simple Lie algebras. To the algebra  $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$  one can associate the affine Kac–Moody Lie algebra  $\widehat{\mathfrak{g}}_1 \oplus \dots \oplus \widehat{\mathfrak{g}}_m$ . Consider the  $m$ -tuple  $(\Lambda_1, \dots, \Lambda_m)$ , where  $\Lambda_i$  is an affine integrable  $\widehat{\mathfrak{g}}_i$ -weight. Then  $L(\Lambda_1) \otimes \dots \otimes L(\Lambda_m)$  naturally is an irreducible  $\widehat{\mathfrak{g}}_1 \oplus \dots \oplus \widehat{\mathfrak{g}}_m$ -module, where  $L(\Lambda_i)$  is an irreducible integrable  $\widehat{\mathfrak{g}}_i$ -module.

Let  $D_{(d_1, \dots, d_m), (\lambda_1, \dots, \lambda_m)}^{\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m} \subset L(\Lambda_1) \otimes \dots \otimes L(\Lambda_m)$  be an affine Demazure module, where  $d_1, \dots, d_m$  are positive integers and  $\lambda_i$  is a dominant weight of a simple Lie algebra  $\mathfrak{g}_i$ . It is clear that

$$D_{(d_1, \dots, d_m), (\lambda_1, \dots, \lambda_m)}^{\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m} \simeq D_{d_1, \lambda_1}^{\mathfrak{g}_1} \otimes \dots \otimes D_{d_m, \lambda_m}^{\mathfrak{g}_m}.$$

In this appendix we deal with the arc space of the following composition of Veronese and Segre embeddings:

$$\begin{aligned} \mathbb{P}(\mathbb{k}^{n_1}) \times \dots \times \mathbb{P}(\mathbb{k}^{n_m}) &\hookrightarrow \mathbb{P}(S^{d_1} \mathbb{k}^{n_1}) \times \dots \times \mathbb{P}(S^{d_m} \mathbb{k}^{n_m}) \\ &\hookrightarrow \mathbb{P}((S^{d_1} \mathbb{k}^{n_1}) \otimes \dots \otimes (S^{d_m} \mathbb{k}^{n_m})). \end{aligned} \tag{A-1}$$

The image of this map is a projective toric variety, corresponding to the product of simplices.

Consider the semisimple group  $SL_{n_1} \times \dots \times SL_{n_m}$  and its flag variety  $\mathcal{B} = (SL_{n_1}/B_{n_1}) \times \dots \times (SL_{n_m}/B_{n_m})$ . We consider each space  $\mathbb{k}^{n_i}$ , appearing in (A-1) as the first fundamental  $\mathfrak{sl}_{n_i}$ -module, so  $\mathbb{k}^{n_i} \simeq V_{\omega_1}^{\mathfrak{sl}_{n_i}}$  and  $S^{d_i} \mathbb{k}^{n_i} \simeq V_{d_i \omega_1}^{\mathfrak{sl}_{n_i}}$ . Therefore the image of the map (A-1) is isomorphic to the image of the map

$$\mathcal{B} \rightarrow \mathbb{P}(V_{d_1 \omega_1}^{\mathfrak{sl}_{n_1}} \otimes \dots \otimes V_{d_m \omega_1}^{\mathfrak{sl}_{n_m}}).$$

In order to study the arc space of this image, one should replace the flag variety  $\mathcal{B}$  by the semi-infinite flag variety  $\mathcal{Q}$  of the semisimple group  $SL_{n_1} \times \dots \times SL_{n_m}$  (see [Finkelberg and Mirković 1999; Braverman and Finkelberg 2014; Kato 2018]) and study the homogeneous coordinate ring of the map

$$\mathcal{Q} \rightarrow \mathbb{P}((V_{d_1 \omega_1}^{\mathfrak{sl}_{n_1}} \otimes \dots \otimes V_{d_m \omega_1}^{\mathfrak{sl}_{n_m}})[[t]]).$$

Using Corollary 1.14 as well as [Dumanski and Feigin 2023, Proposition 4.1] applied to the algebra  $\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}$  and its weight  $(d_1 \omega_1, \dots, d_m \omega_1)$  we can describe the coordinate ring as

$$\Gamma = \bigoplus_{\ell \geq 0} (D_{(d_1, \dots, d_m), (\omega_1, \dots, \omega_1)}^{\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}}[t]^{\odot \ell})^*$$

(we use that  $V_{d_i \omega_1}^{\mathfrak{sl}_{n_i}} \simeq D_{d_i, \omega_1}^{\mathfrak{sl}_{n_i}}$ ). Here for a Lie algebra  $\mathfrak{a}$  and its cyclic module  $W$  with a cyclic vector  $w$  we denote by  $W^{\odot \ell}$  the Cartan component  $U(\mathfrak{a}) \cdot w^{\otimes \ell} \subset W^{\otimes \ell}$ . We are going to compute the character of the  $\ell$ -th homogeneous component  $\Gamma_\ell \subset \Gamma$ .

The modules of the form  $\Gamma_\ell = D_{(d_1, \dots, d_m), (\omega_1, \dots, \omega_1)}^{\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}}[t]^{\odot \ell}$  were studied in [Dumanski and Feigin 2023; Dumanski et al. 2021]. In particular, it was shown that  $\Gamma_\ell$  admits an action of the algebra  $\mathcal{A}_\ell$ , which in this case is isomorphic to the algebra of symmetric polynomials in  $\ell$  variables. It was also shown that the fiber of  $\Gamma_\ell$  with respect to this algebra at a generic point  $\mathbf{c} = (c_1, \dots, c_\ell)$  is isomorphic to

$$\Gamma_\ell \otimes_{\mathcal{A}_\ell} \mathbb{k}_{\mathbf{c}} \simeq \bigotimes_{i=1}^{\ell} D_{(d_1, \dots, d_m), (\omega_1, \dots, \omega_1)}^{\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}}(c_i), \tag{A-2}$$

where  $\mathbb{k}_{\mathbf{c}}$  is the quotient of  $\mathcal{A}_\ell$  by the maximal ideal, corresponding to  $\mathbf{c}$ . It was also proved that to show that  $\Gamma_\ell$  is free over  $\mathcal{A}_\ell$  it is sufficient to obtain the surjection

$$\Gamma_\ell \otimes_{\mathcal{A}_\ell} \mathbb{k}_0 \leftarrow D_{(d_1, \dots, d_m), (\ell \omega_1, \dots, \ell \omega_1)}^{\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}},$$

where the latter has the same dimension as the right-hand side of (A-2). But this was shown for restriction to any  $\mathfrak{sl}_{n_i}$  in [Dumanski and Feigin 2023, Proposition 3.2]. The surjectivity of the whole map follows. Therefore we proved the following:

**Lemma A.1.** *Module  $\Gamma_\ell$  is free over the algebra of symmetric polynomials in  $\ell$  variables  $\mathcal{A}_\ell$ . The fiber at 0 with respect to this algebra is isomorphic to*

$$D_{(d_1, \dots, d_m), (\ell \omega_1, \dots, \ell \omega_1)}^{\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}}.$$

Thus, we obtain the desired:

**Corollary A.2.** *The character of the  $\ell$ -th homogeneous component of the arc space of the projective embedding (A-1) is equal to*

$$\text{ch} \mathcal{A}_\ell \text{ch} D_{(d_1, \dots, d_m), (\ell \omega_1, \dots, \ell \omega_1)}^{\mathfrak{sl}_{n_1} \oplus \dots \oplus \mathfrak{sl}_{n_m}} = \frac{1}{(q)_\ell} \prod_{i=1}^m \text{ch} D_{d_i, \ell \omega_1}^{\mathfrak{sl}_{n_i}}. \tag{A-3}$$

The last factor in this expression was found in [Feigin et al. 2004, Theorem 2.11]. Namely, in the notation of [Feigin et al. 2004],

$$D_{d_i, \ell \omega_1}^{\mathfrak{sl}_{n_i}} = \underbrace{V_{d_i \omega_1}^{\mathfrak{sl}_{n_i}} * \dots * V_{d_i \omega_1}^{\mathfrak{sl}_{n_i}}}_{\ell} = V(\mathbf{n}_i, \mathbf{d}_i),$$

where

$$\mathbf{n}_i = \underbrace{(n_i, \dots, n_i)}_{\ell}, \quad \mathbf{d}_i = \underbrace{(d_i, \dots, d_i)}_{\ell},$$

and  $*$  denotes the fusion product. Now [Feigin et al. 2004, Formula (2.52)] gives that

$$\text{ch} D_{d_i, \ell \omega_1}^{\mathfrak{sl}_{n_i}} = \sum_{\substack{\lambda \in \mathbb{Z}^{n_i} \\ |\lambda| = d_i \ell}} e^{\lambda_2 \omega_1} \dots e^{\lambda_{n_i} \omega_{n_i-1}} \tilde{S}_{\lambda, (\ell d_i)}(q), \tag{A-4}$$

where  $\lambda = (\lambda_1, \dots, \lambda_{n_i})$ ,  $e^{\omega_i}$  stay for the  $\mathfrak{sl}_{n_i}$ -weights and  $\tilde{S}_{\lambda, \mu}$  is the  $q$ -supernomial coefficient, defined in [Schilling 2002, Formula (2.1)].

### Acknowledgements

The authors are indebted to Alexander Popkovich for useful discussions and his ability to keep calm in any situation. Ie. Makedonskyi is grateful to the Max Planck Institute for Mathematics for help in leaving Russia. The work was partially supported by the Basic Research Program at the HSE University.

### References

- [Ait El Manssour and Pogudin 2024] R. Ait El Manssour and G. Pogudin, “Multiplicity structure of the arc space of a fat point”, *Algebra Number Theory* **18**:5 (2024), 947–967. MR Zbl
- [Anderson and Stapledon 2013] D. Anderson and A. Stapledon, “Arc spaces and equivariant cohomology”, *Transform. Groups* **18**:4 (2013), 931–969. MR Zbl
- [Arakawa and Moreau 2021] T. Arakawa and A. Moreau, “Arc spaces and chiral symplectic cores”, *Publ. Res. Inst. Math. Sci.* **57**:3–4 (2021), 795–829. MR Zbl
- [Bai et al. 2020] Y. Bai, E. Gorsky, and O. Kivinen, “Quadratic ideals and Rogers–Ramanujan recursions”, *Ramanujan J.* **52**:1 (2020), 67–89. MR Zbl
- [Boulier et al. 1995] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot, “Representation for the radical of a finitely generated differential ideal”, pp. 158–166 in *ISSAC '95: Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation* (Montreal, 1995), ACM, New York, 1995. Zbl
- [Bourqui and Haiech 2021] D. Bourqui and M. Haiech, “On the nilpotent functions at a non-degenerate arc”, *Manuscripta Math.* **165**:1–2 (2021), 227–238. MR Zbl
- [Bourqui and Sebag 2019a] D. Bourqui and J. Sebag, “Finite formal model of toric singularities”, *J. Math. Soc. Japan* **71**:3 (2019), 805–829. MR Zbl
- [Bourqui and Sebag 2019b] D. Bourqui and J. Sebag, “The radical of the differential ideal generated by  $XY$  in the ring of two variable differential polynomials is not differentially finitely generated”, *J. Commut. Algebra* **11**:2 (2019), 155–162. MR Zbl
- [Braverman and Finkelberg 2014] A. Braverman and M. Finkelberg, “Weyl modules and  $q$ -Whittaker functions”, *Math. Ann.* **359**:1–2 (2014), 45–59. MR Zbl
- [Bruscek et al. 2013] C. Bruscek, H. Mourtada, and J. Schepers, “Arc spaces and the Rogers–Ramanujan identities”, *Ramanujan J.* **30**:1 (2013), 9–38. MR Zbl
- [Chambert-Loir et al. 2018] A. Chambert-Loir, J. Nicaise, and J. Sebag, *Motivic integration*, Progr. Math. **325**, Birkhäuser, New York, 2018. MR Zbl
- [Chari and Venkatesh 2015] V. Chari and R. Venkatesh, “Demazure modules, fusion products and  $Q$ -systems”, *Comm. Math. Phys.* **333**:2 (2015), 799–830. MR Zbl
- [Cox et al. 2011] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Grad. Stud. Math. **124**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl
- [Dumanski and Feigin 2023] I. Dumanski and E. Feigin, “Reduced arc schemes for Veronese embeddings and global Demazure modules”, *Commun. Contemp. Math.* **25**:8 (2023), art. id. 2250034. MR Zbl
- [Dumanski et al. 2021] I. Dumanski, E. Feigin, and M. Finkelberg, “Beilinson–Drinfeld Schubert varieties and global Demazure modules”, *Forum Math. Sigma* **9** (2021), art. id. e42. MR Zbl
- [Ein and Mustață 2009] L. Ein and M. Mustață, “Jet schemes and singularities”, pp. 505–546 in *Algebraic geometry, II* (Seattle, WA, 2005), Proc. Sympos. Pure Math. **80**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl
- [Feigin and Makedonskyi 2019] E. Feigin and I. Makedonskyi, “Vertex algebras and coordinate rings of semi-infinite flags”, *Comm. Math. Phys.* **369**:1 (2019), 221–244. MR Zbl
- [Feigin and Makedonskyi 2020] E. Feigin and I. Makedonskyi, “Semi-infinite Plücker relations and Weyl modules”, *Int. Math. Res. Not.* **2020**:14 (2020), 4357–4394. MR Zbl
- [Feigin et al. 2004] B. Feigin, M. Jimbo, R. Kedem, S. Loktev, and T. Miwa, “Spaces of coinvariants and fusion product, II:  $\widehat{\mathfrak{sl}}_2$  character formulas in terms of Kostka polynomials”, *J. Algebra* **279**:1 (2004), 147–179. MR Zbl

- [de Fernex and Docampo 2020] T. de Fernex and R. Docampo, “Differentials on the arc space”, *Duke Math. J.* **169**:2 (2020), 353–396. MR Zbl
- [Finkelberg and Mirković 1999] M. Finkelberg and I. Mirković, “Semi-infinite flags, I: Case of global curve  $\mathbb{P}^1$ ”, pp. 81–112 in *Differential topology, infinite-dimensional Lie algebras, and applications*, Amer. Math. Soc. Transl. Ser. 2 **194**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl
- [Fourier and Littelmann 2007] G. Fourier and P. Littelmann, “Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions”, *Adv. Math.* **211**:2 (2007), 566–593. MR Zbl
- [Frenkel 2007] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge Stud. Adv. Math. **103**, Cambridge Univ. Press, 2007. MR Zbl
- [Gorsky 2013] E. Gorsky, “Arc spaces and DAHA representations”, *Selecta Math. (N.S.)* **19**:1 (2013), 125–140. MR Zbl
- [Ion 2003] B. Ion, “Nonsymmetric Macdonald polynomials and Demazure characters”, *Duke Math. J.* **116**:2 (2003), 299–318. MR Zbl
- [Ishii 2004] S. Ishii, “The arc space of a toric variety”, *J. Algebra* **278**:2 (2004), 666–683. MR Zbl
- [Ishii 2007] S. Ishii, “Jet schemes, arc spaces and the Nash problem”, *C. R. Math. Acad. Sci. Soc. R. Can.* **29**:1 (2007), 1–21. MR Zbl
- [Ishii and Kollár 2003] S. Ishii and J. Kollár, “The Nash problem on arc families of singularities”, *Duke Math. J.* **120**:3 (2003), 601–620. MR Zbl
- [Kato 2018] S. Kato, “Demazure character formula for semi-infinite flag varieties”, *Math. Ann.* **371**:3–4 (2018), 1769–1801. MR Zbl
- [Kaveh and Manon 2019] K. Kaveh and C. Manon, “Khovanskii bases, higher rank valuations, and tropical geometry”, *SIAM J. Appl. Algebra Geom.* **3**:2 (2019), 292–336. MR Zbl
- [Kpogon and Sebag 2017] K. Kpogon and J. Sebag, “Nilpotency in arc scheme of plane curves”, *Comm. Algebra* **45**:5 (2017), 2195–2221. MR Zbl
- [Linshaw and Song 2024] A. R. Linshaw and B. Song, “Standard monomials and invariant theory for arc spaces, I: General linear group”, *Commun. Contemp. Math.* **26**:4 (2024), art. id. 2350013. MR Zbl
- [Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, 1995. MR Zbl
- [Makedonskyi 2022] I. Makedonskyi, “Semi-infinite Plücker relations and arcs over toric degeneration”, *Math. Res. Lett.* **29**:5 (2022), 1499–1536. MR Zbl
- [Makhlin 2022] I. Makhlin, “Gelfand–Tsetlin degenerations of representations and flag varieties”, *Transform. Groups* **27**:2 (2022), 563–596. MR Zbl
- [Miller and Sturmfels 2005] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Grad. Texts in Math. **227**, Springer, 2005. MR Zbl
- [Morán Cañón and Sebag 2020] M. Morán Cañón and J. Sebag, “On the tangent space of a weighted homogeneous plane curve singularity”, *J. Korean Math. Soc.* **57**:1 (2020), 145–169. MR Zbl
- [Morán Cañón and Sebag 2022] M. Morán Cañón and J. Sebag, “Two algorithms for computing the general component of jet scheme and applications”, *J. Symbolic Comput.* **113** (2022), 74–96. MR Zbl
- [Mourtada 2011] H. Mourtada, “Jet schemes of toric surfaces”, *C. R. Math. Acad. Sci. Paris* **349**:9–10 (2011), 563–566. MR Zbl
- [Mourtada 2014] H. Mourtada, “Jet schemes of rational double point singularities”, pp. 373–388 in *Valuation theory in interaction*, Eur. Math. Soc., Zürich, 2014. MR Zbl
- [Mourtada 2023] H. Mourtada, “Jet schemes and their applications in singularities, toric resolutions and integer partitions”, pp. 211–249 in *Handbook of geometry and topology of singularities, IV*, Springer, 2023. MR Zbl
- [Mustață 2001] M. Mustață, “Jet schemes of locally complete intersection canonical singularities”, *Invent. Math.* **145**:3 (2001), 397–424. MR Zbl
- [Mustață 2007] M. Mustață, “Spaces of arcs in birational geometry”, 2007, available at [https://dept.math.lsa.umich.edu/~mmustata/lectures\\_arcs.pdf](https://dept.math.lsa.umich.edu/~mmustata/lectures_arcs.pdf).

- [Naoi 2012] K. Naoi, “Weyl modules, Demazure modules and finite crystals for non-simply laced type”, *Adv. Math.* **229**:2 (2012), 875–934. MR Zbl
- [Nash 1995] J. F. Nash, Jr., “Arc structure of singularities”, *Duke Math. J.* **81**:1 (1995), 31–38. MR Zbl
- [Reguera 2023] A. J. Reguera, “Arc spaces and wedge spaces for toric varieties”, *Ann. Inst. Fourier (Grenoble)* **73**:5 (2023), 2135–2183. MR Zbl
- [Schilling 2002] A. Schilling, “ $q$ -supernomial coefficients: from riggings to ribbons”, pp. 437–454 in *MathPhys Odyssey*, 2001, edited by M. Kashiwara and T. Miwa, Prog. Math. Phys. **23**, Birkhäuser, Boston, MA, 2002. MR Zbl
- [Sebag 2011] J. Sebag, “Arcs schemes, derivations and Lipman’s theorem”, *J. Algebra* **347** (2011), 173–183. MR Zbl
- [Sottile and Sturmfels 2001] F. Sottile and B. Sturmfels, “A sagbi basis for the quantum Grassmannian”, *J. Pure Appl. Algebra* **158**:2-3 (2001), 347–366. MR Zbl
- [Stacks] “The Stacks project”, electronic reference, available at <http://stacks.math.columbia.edu>.
- [Vakil 2017] R. Vakil, “The rising sea: foundations of algebraic geometry”, 2017, available at <https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf>. To be published by Princeton Univ. Press, 2025. Zbl

Communicated by Edward Frenkel

Received 2022-09-19    Revised 2024-02-29    Accepted 2024-04-29

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

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ANT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

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# Algebra & Number Theory

Volume 19 No. 2 2025

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Breuil–Mézard conjectures for central division algebras ANDREA DOTTO	213
Canonical integral models for Shimura varieties of toral type PATRICK DANIELS	247
The geometric Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate Galois representations ANA CARAIANI, MATTHEW EMERTON, TOBY GEE and DAVID SAVITT	287
On reduced arc spaces of toric varieties ILYA DUMANSKI, EVGENY FEIGIN, IEVGEN MAKEDONSKYI and IGOR MAKHLIN	313
Divisibility of character values of the symmetric group by prime powers SARAH PELUSE and KANNAN SOUNDARARAJAN	365
Index of coregularity zero log Calabi–Yau pairs STEFANO FILIPAZZI, MIRKO MAURI and JOAQUÍN MORAGA	383