A universal inequality between angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter

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Abstract. We prove that for sub-extremal axisymmetric and stationary black holes with arbitrary surrounding matter the inequality $8\pi|J| < A$ holds, where J is the angular momentum and A the horizon area of the black hole.

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1. Introduction

A well-known property of the Kerr solution, describing a single rotating black hole in vacuum, is given by

$$|p_J| \leq 1 \quad \text{with} \quad p_J := \frac{8\pi J}{A}, \tag{1}$$

where J and A denote the angular momentum and the horizon area of the black hole respectively. Equality in (1) holds if and only if the Kerr black hole is *extreme*. As was shown in [1], the equation $|p_J| = 1$ is even true more generally for axisymmetric and stationary black holes with surrounding matter in the degenerate limit (i.e. for vanishing surface gravity κ). Moreover, it was also conjectured in [1] that $|p_J| \leq 1$ still holds if the black hole is surrounded by matter. In this paper we prove this conjecture‡.

We start by requiring that a physically relevant non-degenerate black hole be characterized through the existence of trapped surfaces (i.e. surfaces with negative expansion $\theta_{(l)}$ of outgoing null geodesics) in an interior vicinity of the event horizon. That is, in the terminology of [3], we concentrate on *sub-extremal* black holes. In the following we show that such surfaces cannot exist for $|p_J| \geq 1$, provided that an appropriate functional I (to be defined below) cannot fall below 1. In turn, this can be proved by means of methods from the calculus of variations.

‡ Note that in [1] a more general conjecture, incorporating the black hole's electric charge Q, was formulated. Here we prove this conjecture for the pure Einstein field, i.e. for $Q = 0$, and vanishing cosmological constant $\Lambda = 0$. (It should be noted that, for $\Lambda \neq 0$, the inequality $|p_J| \leq 1$ can be violated. An example is the Kerr-(A)dS family of black holes, see [3].)

2. Coordinate systems and Einstein equations

We introduce suitable coordinates and metric functions by adopting our notation from [1], which is based on [2]. In spherical coordinates (r, θ, φ, t), an exterior vacuum vicinity of the event horizon§ can be described by the line element

$$ds^2 = e^{2\mu}(dr^2 + r^2 d\theta^2) + r^2 B^2 e^{-2\nu} \sin^2\theta \, (d\varphi - \omega dt)^2 - e^{2\nu} dt^2. \qquad (2)$$

The function B must be a solution of $\nabla \cdot (r \sin\theta \, \nabla B) = 0$, with $B r \sin\theta = 0$ at the event horizon $\mathcal{H}$. Here ∇ is the nabla operator in three-dimensional flat space. In order to fix the coordinates we chose the particular solution $B = 1 - r_h^2/r^2$, where $r_h = \text{constant} > 0$. In this manner we obtain coordinates in which $\mathcal{H}$ is located at the coordinate sphere $r = r_h$. We now decompose the potential ν in the form $\nu = u + \ln B$, thereby obtaining three regular metric functions μ, u and ω, depending on r and θ only.

Following Bardeen [2], we introduce the new metric potentials $\hat{\mu} = r^2 e^{2\mu}$, $\hat{u} = r^2 e^{-2u}$, which are positive and regular functions of r and $\cos\theta$, as well as the new radial coordinate $R = \frac{1}{2}\left(r + \frac{r_h^2}{r}\right)$. We arrive at the Boyer-Lindquist type line element

$$ds^2 = \hat{\mu}\left(\frac{dR^2}{R^2 - r_h^2} + d\theta^2\right) + \hat{u} \sin^2\theta \, (d\varphi - \omega dt)^2 - \frac{4}{\hat{u}}(R^2 - r_h^2) dt^2, \qquad (3)$$

which is singular at $\mathcal{H}$ $(R = r_h)$.

In these coordinates, the vacuum Einstein equations read as follows:‖

$$(R^2 - r_h^2)\tilde{u}_{,RR} + 2R\tilde{u}_{,R} + \tilde{u}_{,\theta\theta} + \tilde{u}_{,\theta} \cot\theta = 1 - \frac{\hat{u}^2}{8} \sin^2\theta \left(\omega_{,R}^2 + \frac{\omega_{,\theta}^2}{R^2 - r_h^2}\right), \qquad (4)$$

$$(R^2 - r_h^2)(\omega_{,RR} + 4\omega_{,R}\tilde{u}_{,R}) + \omega_{,\theta\theta} + \omega_{,\theta}(3\cot\theta + 4\tilde{u}_{,\theta}) = 0, \qquad (5)$$

$$\begin{aligned}(R^2 - r_h^2)\tilde{\mu}_{,RR} + R\tilde{\mu}_{,R} + \tilde{\mu}_{,\theta\theta} = {} & \frac{\hat{u}^2}{16} \sin^2\theta \left(\omega_{,R}^2 + \frac{\omega_{,\theta}^2}{R^2 - r_h^2}\right) + R\tilde{u}_{,R} \\ & - (R^2 - r_h^2)\tilde{u}_{,R}^2 - \tilde{u}_{,\theta}(\tilde{u}_{,\theta} + \cot\theta),\end{aligned} \qquad (6)$$

where $\tilde{u} := \frac{1}{2}\ln\left(r_h^{-2}\hat{u}\right)$ and $\tilde{\mu} := \frac{1}{2}\ln\left(r_h^{-2}\hat{\mu}\right)$. At the horizon, the metric potentials obey the boundary conditions [2]

$$\mathcal{H}: \quad \omega = \text{constant} = \omega_h, \quad \frac{2r_h}{\sqrt{\hat{\mu}\hat{u}}} = \text{constant} = \kappa, \qquad (7)$$

with the horizon angular velocity ω_h and the surface gravity κ. On the horizon's north and south pole ($R = r_h$ and $\sin\theta = 0$), the following regularity conditions hold:

$$\hat{\mu}(R = r_h, \theta = 0) = \hat{u}(R = r_h, \theta = 0) = \hat{\mu}(R = r_h, \theta = \pi) = \hat{u}(R = r_h, \theta = \pi) = \frac{2r_h}{\kappa}. \quad (8)$$

§ For a stationary space-time, the immediate vicinity of a black hole event horizon must be vacuum, see e. g. [2].

‖ Throughout this paper we consider pure gravity, i.e. no electromagnetic fields, as well as vanishing cosmological constant, $\Lambda = 0$.

3. Necessary condition for the existence of trapped surfaces

A crucial quantity for the following discussion is the expansion $\theta_{(l)} = h^{ab}\nabla_a l_b$ of outgoing null rays for two-surfaces $\mathcal{S}$ in an interior vicinity of the horizon, where h is the interior metric of $\mathcal{S}$ and l is the vector field describing outgoing null rays. In order to analyze $\theta_{(l)}$ inside the black hole, we introduce horizon-penetrating coordinates $(R, \theta, \tilde{\varphi}, \tilde{t}\,)$, in which the metric is *regular* at the horizon $\mathcal{H}$:

$$\mathrm{d}\tilde{t} = \mathrm{d}t + \frac{T(R)}{\Delta}\mathrm{d}R, \quad \mathrm{d}\tilde{\varphi} = \mathrm{d}\varphi + \frac{\Phi(R)}{\Delta}\mathrm{d}R, \quad \Delta := 4(R^2 - r_\mathrm{h}^2). \tag{9}$$

The free functions T and Φ are chosen in such a way that

$$a(R,\theta) := \frac{4\hat{\mu}\hat{u} - T^2}{\Delta}, \quad b(R,\theta) := \frac{\Phi - \omega T}{\Delta} \tag{10}$$

are regular at $R = r_\mathrm{h}$. Furthermore, we require $a > 0$ in order to guarantee that $\tilde{t} = \text{constant}$ is a *spacelike* surface. As a consequence, T has to obey the conditions

$$T = 2\sqrt{\hat{\mu}\hat{u}} = \frac{4r_\mathrm{h}}{\kappa} \quad \text{for } R = r_\mathrm{h}, \qquad T > 2\sqrt{\hat{\mu}\hat{u}} \quad \text{for } R < r_\mathrm{h}. \tag{11}$$

We thus obtain the regular line element¶

$$\begin{aligned}\mathrm{d}s^2 = {} & \left(\frac{a}{\hat{u}} + \hat{u}b^2\sin^2\theta\right)\mathrm{d}R^2 - \frac{\Delta}{\hat{u}}\mathrm{d}\tilde{t}^{\,2} + \hat{u}\sin^2\theta(\mathrm{d}\tilde{\varphi} - \omega\mathrm{d}\tilde{t}\,)^2 + \hat{\mu}\mathrm{d}\theta^2 \\ & + 2\left(\frac{T}{\hat{u}} + \omega\hat{u}b\sin^2\theta\right)\mathrm{d}R\mathrm{d}\tilde{t} - 2\hat{u}b\sin^2\theta\,\mathrm{d}R\mathrm{d}\tilde{\varphi}.\end{aligned} \tag{12}$$

In these coordinates, we calculate the expansion $\theta_{(l)}$ for a surface $\mathcal{S}$ in a small interior neighborhood of the horizon, described by

$$\mathcal{S}: \quad R = r_\mathrm{h} - \varepsilon\hat{r}(\theta), \quad \theta \in [0,\pi], \quad \tilde{\varphi} \in [0, 2\pi), \quad \tilde{t} = \text{constant}, \tag{13}$$

where $\varepsilon > 0$, $\hat{r} > 0$ and $\varepsilon\hat{r} \ll r_\mathrm{h}$. We obtain

$$\theta_{(l)}(\theta) = \frac{\hat{T}}{2\sqrt{2\hat{\mu}a}\sin\theta}\left[\frac{(\hat{r}_{,\theta}\sin\theta)_{,\theta}}{\hat{r}r_\mathrm{h}} - \frac{(\hat{\mu}\hat{u})_{,R}}{2\hat{\mu}\hat{u}}\sin\theta\right]\bigg|_{\mathcal{H}}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{14}$$

where we have used that T is of the form $T = 2\sqrt{\hat{\mu}\hat{u}}|_{\mathcal{H}} + \varepsilon\hat{T}$ with $\hat{T} = \hat{T}(\theta) > 0$ on $\mathcal{S}$; see (11). [Note that $\theta_{(l)} = 0$ on the horizon ($\varepsilon = 0$).]

Following Booth and Fairhurst [3], we study the criterion $\delta_{\bar{n}}\theta_{(\bar{l})} < 0$ (in their notation) for the existence of trapped surfaces, i.e. we characterize a physically relevant, sub-extremal black hole through a negative variation of the expansion on the horizon in direction of an ingoing null field $\bar{n}$. In our formulation, this is equivalent to the existence of a surface $\mathcal{S}$ with negative expansion $\theta_{(l)}$. We first show the following lemma.

Lemma 3.1. *A necessary condition for the existence of trapped surfaces in the interior vicinity of the event horizon of an axisymmetric and stationary black hole is*

$$\int_0^\pi (\hat{\mu}\hat{u})_{,R}|_{\mathcal{H}}\sin\theta\,\mathrm{d}\theta > 0. \tag{15}$$

¶ Note that for the Kerr solution with mass M and angular momentum J, we obtain Kerr type coordinates (where the slices $\tilde{t} = \text{constant}$ are spacelike) by choosing $T = 4M(M + 2R)$ and $\Phi = \text{constant} = 2J/M$.

Proof. Let the surface $\mathcal{S}$, defined in (13) for sufficiently small ε, be trapped. Then $\theta_{(l)}$ is negative everywhere on $\mathcal{S}$, and so is the term in square brackets in (14) for all $\theta \in [0,\pi]$. Thus, the integral of this term along the horizon $\mathcal{H}$ is negative. Integrating by parts, and using that $\hat\mu\hat u = \text{constant} > 0$ on $\mathcal{H}$ [see (7)], yields

$$\frac{1}{r_\mathrm{h}}\int\limits_0^\pi \frac{\hat r_{,\theta}^2}{\hat r^2}\sin\theta\,\mathrm{d}\theta - \frac{1}{2\hat\mu\hat u}\int\limits_0^\pi (\hat\mu\hat u)_{,R}\sin\theta\,\mathrm{d}\theta < 0. \tag{16}$$

Since the first integral is non-negative, we immediately obtain (15). □

4. Calculation of p_J

Following the notation of [1], we express angular momentum J and horizon area A of the black hole by

$$J = \frac{1}{8\pi}\oint\limits_{\mathcal{H}} m^{a;b}\mathrm{d}S_{ab} = -\frac{1}{16}\int\limits_0^\pi \hat u^2\omega_{,R}|_{\mathcal{H}}\sin^3\theta\,\mathrm{d}\theta = -\frac{r_\mathrm{h}^2}{4}\int\limits_{-1}^1 V\mathrm{e}^{2U}(1-x^2)\mathrm{d}x, \tag{17}$$

$$A = 2\pi\int\limits_0^\pi \sqrt{\hat\mu\hat u}|_{\mathcal{H}}\sin\theta\,\mathrm{d}\theta = 4\pi\sqrt{\hat\mu\hat u}|_{\mathcal{H}} = 4\pi r_\mathrm{h}^2\mathrm{e}^{2U(1)}, \tag{18}$$

where $x := \cos\theta$, $U(x) := \left[\frac{1}{2}\ln\left(r_\mathrm{h}^{-2}\hat u\right)\right]\big|_{\mathcal{H}}$, $V(x) := \left[\frac{1}{4}\,\hat u\,\omega_{,R}\right]\big|_{\mathcal{H}}$, and m^a is the Killing vector corresponding to axisymmetry. Note that for this formulation we have used conditions (7) and (8). As a consequence, p_J takes the form

$$p_J \equiv \frac{8\pi J}{A} = -\frac{1}{2}\mathrm{e}^{-2U(1)}\int\limits_{-1}^1 V\mathrm{e}^{2U}(1-x^2)\mathrm{d}x. \tag{19}$$

5. Reformulation in terms of a variational problem

In order to prove the inequality in question, we show that for $|p_J| \geq 1$ Eq. (15) is violated, i.e. that there are no trapped surfaces in the interior vicinity of the horizon.

Using (4) and (6), the integrand in (15) can be expressed

$$(\hat\mu\hat u)_{,R}|_{\mathcal{H}} = \frac{2}{r_\mathrm{h}}\hat u^2|_{\theta=0}\left[1 - \frac{1}{16}\omega_{,R}^2\hat u^2\sin^2\theta - \frac{\hat u_{,\theta}}{2\hat u}\left(\frac{\hat u_{,\theta}}{2\hat u} + 2\cot\theta\right)\right]\Big|_{\mathcal{H}}. \tag{20}$$

Hence, we can write (15) in terms of U, V and x as follows:

$$\frac{1}{2}\int\limits_{-1}^1 \left[(V^2+U'^2)(1-x^2) - 2xU'\right]\mathrm{d}x < 1 \quad \text{with} \quad U' := \frac{\mathrm{d}U}{\mathrm{d}x}. \tag{21}$$

From (19), the violation of (15) for $|p_J| \geq 1$ can thus be formulated as the following implication, to be valid for any regular functions U, V defined on $[-1,1]$ and satisfying the condition $U(-1) = U(1)$ [which follows from (8)]:

$$\left|\int\limits_{-1}^1 V\mathrm{e}^{2U}(1-x^2)\mathrm{d}x\right| \geq 2\mathrm{e}^{2U(1)} \quad \Rightarrow \quad \frac{1}{2}\int\limits_{-1}^1 \left[(V^2+U'^2)(1-x^2) - 2xU'\right]\mathrm{d}x \geq 1. \tag{22}$$

We now show that the validity of this implication holds provided that an appropriate functional I (defined below) cannot fall below 1.

Applying the Cauchy-Schwarz inequality to the first inequality in (22) we obtain

$$4\mathrm{e}^{4U(1)} \le \left(\int_{-1}^{1} V\mathrm{e}^{2U}(1-x^2)\mathrm{d}x\right)^2 \le \int_{-1}^{1} V^2(1-x^2)\mathrm{d}x \int_{-1}^{1} \mathrm{e}^{4U}(1-x^2)\mathrm{d}x. \quad (23)$$

Given this inequality, we replace the term $\int V^2(1-x^2)\mathrm{d}x$ in the second inequality in (22) and see that

$$I[U] := \frac{1}{2}\int_{-1}^{1} \left[U'^2(x)(1-x^2) - 2xU'(x)\right]\mathrm{d}x + \frac{2\mathrm{e}^{4U(1)}}{\int_{-1}^{1} \mathrm{e}^{4U(x)}(1-x^2)\mathrm{d}x} \ge 1 \quad (24)$$

is as a sufficient condition for the validity of the implication, with the functional $I : W^{1,2}(-1,1) \to \mathbb{R}$ defined on the Sobolev space $W^{1,2}(-1,1)$. Thus we have shown the following.

Lemma 5.1. *The inequality $|p_J| < 1$ for any sub-extremal axisymmetric and stationary black hole with surrounding matter holds provided that the inequality*

$$I[U] \ge 1 \quad (25)$$

is true for all $U \in W^{1,2}(-1,1)$ with $U(-1) = U(1)$.

This reformulation has led us to the following *variational problem*: Calculate the minimum of the functional I and show that it is not below 1.

6. Complete solution of the variational problem

Consider the functional

$$I_\varepsilon[U] := \frac{1}{2}\int_{-1+\varepsilon}^{1-\varepsilon} \left[U'^2(x)(1-x^2) - 2xU'(x)\right]\mathrm{d}x + \frac{2\mathrm{e}^{4U(1-\varepsilon)}}{\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U(x)}(1-x^2)\mathrm{d}x} \quad (26)$$

on $W^{1,2}(-1+\varepsilon, 1-\varepsilon)$, where $0 < \varepsilon \ll 1$ is a fixed real number. We use techniques from the calculus of variations to show that there exists a minimizer U_ε for I_ε in a suitable class with sufficiently large value $I_\varepsilon[U_\varepsilon]$. Following this investigation, we take the limit $\varepsilon \to 0$ and see that the claim of lemma 5.1 follows.

We now show the following statements:

(i) The functional I_ε is *well-defined on the Sobolev space* $\overset{\varepsilon}{W}{}^{1,2} := W^{1,2}(-1+\varepsilon, 1-\varepsilon)$ of functions U defined almost everywhere on $(-1+\varepsilon, 1-\varepsilon)$. This is due to the well-known proposition below.

Proposition 6.1 (Thm 2.2 in Buttazzo-Giaquinta-Hildebrandt [4])**.** *On any bounded interval $J \subseteq \mathbb{R}$, $W^{1,2}(J) \hookrightarrow C^0(\overline{J})$ compactly. Moreover, the fundamental theorem of calculus holds in $\overset{\varepsilon}{W}{}^{1,2}$.*

Here, we use the adapted inner product $\int_{-1+\varepsilon}^{1-\varepsilon} UV \,\mathrm{d}\mu + \int_{-1+\varepsilon}^{1-\varepsilon} U'V' \,\mathrm{d}\mu$ for $U, V \in \overset{\varepsilon}{W}{}^{1,2}$, where $\mathrm{d}\mu(x) := (1-x^2)\,\mathrm{d}x$, which is equivalent to the ordinary one and thus makes $\overset{\varepsilon}{W}{}^{1,2}$ a Hilbert space.

For ease of notation set $X^\varepsilon := \{U \in \overset{\varepsilon}{W}{}^{1,2} \mid U(1-\varepsilon) = 0\}$. Note that the functional I_ε is *invariant under addition of constants*. We will use this in order to restrict our attention to the Hilbert subspace X^ε on which $(U,V) := \int_{-1+\varepsilon}^{1-\varepsilon} U'V' \,\mathrm{d}\mu$ is an equivalent inner product inducing the norm $\|U\| := \left(\int_{-1+\varepsilon}^{1-\varepsilon} U'^{\,2} \,\mathrm{d}\mu\right)^{1/2}$.

(ii) I_ε is *bounded from below.* Using $0 \leq \left(\frac{x}{\sqrt{1-x^2}} - U'(x)\sqrt{1-x^2}\right)^2 = \frac{x^2}{1-x^2} - 2xU'(x) + U'^{\,2}(x)(1-x^2)$ we conclude that $I_\varepsilon[U] \geq -\frac{1}{2}\int_{-1+\varepsilon}^{1-\varepsilon} \frac{x^2}{1-x^2}\,\mathrm{d}x =: C(\varepsilon) > -\infty$.

(iii) Applying the Cauchy-Schwarz inequality to $\int_{-1+\varepsilon}^{1-\varepsilon} xU'(x)\mathrm{d}x$, we obtain that $I_\varepsilon[U] \geq \frac{1}{2}\|U\|^2 + 2C(\varepsilon)\|U\|$ for any $U \in X^\varepsilon$ with $C(\varepsilon)$ as in (ii). Hence, for every $P \in \mathbb{R}$ there exists a $Q_P \in \mathbb{R}$ such that $I_\varepsilon[U] \geq P$ whenever $\|U\| \geq Q_P$. This is equivalent to *coercivity of the functional I_ε with respect to the weak topology on X^ε.*

(iv) The functional I_ε is *sequentially lower semi-continuous (lsc) with respect to the weak topology in X^ε*. Recall that lower semi-continuity is additive and that the first terms can be dealt with by standard theory (see e.g. [6]). For the last term, we use proposition 6.1 to deduce that $U_k \to U$ in $C^0([-1+\varepsilon, 1-\varepsilon])$. Whence there exists a uniform bound $D > 0$ of $\{U_k\}$ so that by Lipschitz continuity of the exponential map on $[-4D, 4D]$ with Lipschitz constant L, we have

$$\left|\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U_k}\mathrm{d}\mu - \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\mathrm{d}\mu\right| \leq 4L\int_{-1+\varepsilon}^{1-\varepsilon} |U_k - U|\mathrm{d}\mu \\ \leq 8L\|U_k - U\|_{C^0([-1+\varepsilon,1-\varepsilon])} \overset{k\to\infty}{\longrightarrow} 0. \quad (27)$$

The last term thus being lsc, we have shown I_ε to be lsc.

We can now show *existence of a global minimizer for I_ε in a suitable class*: As we have seen in (ii), I_ε is bounded from below on X^ε. Hence for $a \in \mathbb{R}$ and $c > 0$ we can choose a minimizing sequence $\{U_k\}$ in the class $\mathcal{K}^\varepsilon_{a,c} := \{U \in X^\varepsilon \mid \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\,\mathrm{d}\mu = c,\ U(-1+\varepsilon) = a\}$, with values tending to the infimum $i^\varepsilon_{a,c}$ of I_ε in $\mathcal{K}^\varepsilon_{a,c}$. By coercivity, $\{U_k\}$ is bounded and we can extract a weakly converging subsequence with limit $U^\varepsilon_{a,c} \in X^\varepsilon$ by Hilbert space techniques (theorem of Eberlein-Shmulyan [6]).

The class $\mathcal{K}^\varepsilon_{a,c}$ is weakly sequentially closed by proposion 6.1, which can be shown as in (iv). Whence by (iv), $U^\varepsilon_{a,c} \in \mathcal{K}^\varepsilon_{a,c}$ satisfies $I_\varepsilon[U^\varepsilon_{a,c}] = i^\varepsilon_{a,c}$.

Set $\overset{\varepsilon}{W}{}^{1,2}_{a,0} := \{U \in \overset{\varepsilon}{W}{}^{1,2} \mid U(-1+\varepsilon) = a,\ U(1-\varepsilon) = 0\} \subset X^\varepsilon$ for any $a \in \mathbb{R}$. By the theory of Lagrange multipliers, each minimizer of I_ε in the class $\mathcal{K}^\varepsilon_{a,c}$ is a critical point of the functional

$$J_{\varepsilon,c} : \overset{\varepsilon}{W}{}^{1,2}_{a,0} \to \mathbb{R} : U \mapsto \frac{1}{2}\int_{-1+\varepsilon}^{1-\varepsilon} \left[U'^{\,2}(x)(1-x^2) - 2xU'(x)\right]\mathrm{d}x + \frac{\lambda}{2}\left(\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\mathrm{d}\mu - c\right) \quad (28)$$

for some $\lambda \in \mathbb{R}$, which is well-defined and sufficiently smooth by proposition 6.1. In other words, there is $\lambda := \lambda^{\varepsilon}_{a,c} \in \mathbb{R}$ such that $U := U^{\varepsilon}_{a,c} \in \mathcal{K}^{\varepsilon}_{a,c}$ satisfies

$$\int\limits_{-1+\varepsilon}^{1-\varepsilon} \left[U'(x)\varphi'(x)(1-x^2) - x\varphi'(x)\right]\mathrm{d}x + 2\lambda \int\limits_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U(x)}\varphi(x)(1-x^2)\mathrm{d}x = 0 \tag{29}$$

for all $\varphi \in \overset{\varepsilon}{W}{}^{1,2}_{0,0}$. This can be restated to say that $U \in \overset{\varepsilon}{W}{}^{1,2}$ is a weak solution of

$$\begin{aligned} &-U''(x)(1-x^2) + 2xU'(x) + 1 + 2\lambda \mathrm{e}^{4U(x)}(1-x^2) = 0 \quad \forall x \in (-1+\varepsilon, 1-\varepsilon), \\ &U(-1+\varepsilon) = a, \quad U(1-\varepsilon) = 0, \quad \textstyle\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\mathrm{d}\mu = c. \end{aligned} \tag{30}$$

Any weak solution $U \in \overset{\varepsilon}{W}{}^{1,2}$ of (30) can be shown to be smooth and to satisfy equation (30) strongly: For all $\varphi \in \overset{\varepsilon}{W}{}^{1,2}_{0,0}$, we can rewrite (29) as

$$0 = \int\limits_{-1+\varepsilon}^{1-\varepsilon} \left[U'(x)(1-x^2) - x - 2\lambda \int_{-1+\varepsilon}^{x} \mathrm{e}^{4U}\mathrm{d}\mu\right] \varphi'(x)\,\mathrm{d}x, \tag{31}$$

where we used integration by parts and proposition 6.1. By the fundamental lemma of the calculus of variations, there is a constant $b \in \mathbb{R}$ such that

$$U'(x)(1-x^2) - x - 2\lambda \int_{-1+\varepsilon}^{x} \mathrm{e}^{4U}\mathrm{d}\mu = b \tag{32}$$

holds almost everywhere on $(-1+\varepsilon, 1-\varepsilon)$ as the integrand of (31) in square brackets is an L^1-function. Solving for U', we deduce the smoothness of U by a bootstrap argument (similar to p. 462 in [5]). Differentiating Eq. (32), we get strong validity of (30). In particular, $U^{\varepsilon}_{a,c}$ is a smooth classical solution of the Euler-Lagrange equation of $J_{\varepsilon,c}$.

Interestingly, there exists an integrating factor for Eq. (30) and it can be solved explicitly: For

$$F(x) := -(1-x^2)^2 U'^2(x) + 2x(1-x^2)U'(x) + \lambda \mathrm{e}^{4U(x)}(1-x^2)^2 - x^2 \tag{33}$$

we have

$$F'(x) = 2[x - (1-x^2)U'(x)]\left[U''(x)(1-x^2) - 2xU'(x) - 1 - 2\lambda \mathrm{e}^{4U(x)}(1-x^2)\right]. \tag{34}$$

Thus, it suffices to solve the first order equation $F =$ constant, which can be done with the substitution $W(x) := (1-x^2)^2 \mathrm{e}^{4U(x)}$.

The unique smooth solution turns out to be

$$W(x) = \varepsilon^2(2-\varepsilon)^2 \left(\frac{\mathrm{e}^{2\alpha\,\mathrm{artanh}(1-\varepsilon)} - \beta \mathrm{e}^{-2\alpha\,\mathrm{artanh}(1-\varepsilon)}}{\mathrm{e}^{2\alpha\,\mathrm{artanh}\,x} - \beta \mathrm{e}^{-2\alpha\,\mathrm{artanh}\,x}}\right)^2, \tag{35}$$

where the Lagrange multiplier λ was eliminated using the condition $U(1-\varepsilon) = 0$. The complex integration constants α and β are implicitly given in terms of a and c via $U(-1+\varepsilon) = a$ and $\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\mathrm{d}\mu = c$. The corresponding value of I_ε is

$$\begin{aligned} I_\varepsilon[U^{\varepsilon}_{a,c}] = {} & \frac{\alpha\beta}{2} \frac{(2-\varepsilon)^{4\alpha} - \varepsilon^{4\alpha}}{(1+\beta^2)[\varepsilon(2-\varepsilon)]^{2\alpha} - \beta[(2-\varepsilon)^{4\alpha} + \varepsilon^{4\alpha}]} + 1 - \varepsilon + \frac{1-\alpha^2}{2} \ln\frac{\varepsilon}{2-\varepsilon} \\ & + \frac{8\alpha[\varepsilon(2-\varepsilon)]^{2(\alpha-1)}}{[(2-\varepsilon)^{2\alpha} - \varepsilon^{2\alpha}]^2} \cdot \frac{(1+\beta^2)[\varepsilon(2-\varepsilon)]^{2\alpha} - \beta[(2-\varepsilon)^{4\alpha} + \varepsilon^{4\alpha}]}{(2-\varepsilon)^{4\alpha} - \varepsilon^{4\alpha}}. \end{aligned} \tag{36}$$

We now study the limit $\varepsilon \to 0$. For any $U \in W^{1,2}(-1,1)$ set $a_\varepsilon := U(-1+\varepsilon) - U(1-\varepsilon)$, $c_\varepsilon := \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U-4U(1-\varepsilon)}\mathrm{d}\mu$, $c := \int_{-1}^{1} \mathrm{e}^{4U}\mathrm{d}\mu$. Then $U|_{(-1+\varepsilon,1-\varepsilon)} - U(1-\varepsilon) \in \mathcal{K}^\varepsilon_{a_\varepsilon,c_\varepsilon}$ and thus $I_\varepsilon[U] \geq i^\varepsilon_{a_\varepsilon,c_\varepsilon}$. As we have seen, there is a minimizer $U^\varepsilon_{a_\varepsilon,c_\varepsilon}$ of I_ε in $\mathcal{K}^\varepsilon_{a_\varepsilon,c_\varepsilon}$ and we obtain $I_\varepsilon[U] \geq I_\varepsilon[U^\varepsilon_{a_\varepsilon,c_\varepsilon}]$. As discussed above, this minimizer is smooth and solves the Euler-Lagrange equation of the functional $J_{\varepsilon,c_\varepsilon}$; the unique smooth solution of this equation is given by (35). For $\varepsilon \to 0$ we have $\alpha_\varepsilon \to 1$ (otherwise I_ε diverges) and $\beta_\varepsilon \to -1$ [a consequence of $a_\varepsilon \to 0$ by (25)]. Using these relations, we obtain $I_\varepsilon[U^\varepsilon_{a_\varepsilon,c_\varepsilon}] \geq C^\varepsilon_{a_\varepsilon}$ with $C^\varepsilon_{a_\varepsilon} \to 1$ as $\varepsilon \to 0$. By definition of I and I_ε we see that

$$|I(U) - I_\varepsilon(U)| \leq \frac{1}{2} \int\limits_{1-\varepsilon\leq|x|<1} |U'^2(x)(1-x^2) - 2xU'(x)|\,\mathrm{d}x + 2\left|\frac{\mathrm{e}^{4U(1)}}{\int_{-1}^{1}\mathrm{e}^{4U}\mathrm{d}\mu} - \frac{\mathrm{e}^{4U(1-\varepsilon)}}{\int_{-1+\varepsilon}^{1-\varepsilon}\mathrm{e}^{4U}\mathrm{d}\mu}\right|, \quad (37)$$

where the first term tends to 0 and the denominators of the latter tend to each other as $\varepsilon \to 0$ (the integrands are L^1-functions, theorem of bounded convergence). The numerators of the latter term converge to each other by proposition 6.1. Thus, $I[U] \geq 1$.+

Applying Lemma 5.1, we have shown $|p_J| < 1$ for *sub-extremal* black holes. Together with the results about *extremal* black holes [1]* we arrive at the following.

Theorem 6.2. *Consider space-times with pure gravity (no electromagnetic fields) and vanishing cosmological constant. Then, for every axisymmetric and stationary sub-extremal black hole with arbitrary surrounding matter we have that* $8\pi|J| < A$. *The equality* $8\pi|J| = A$ *holds if the black hole is degenerate (extremal).*

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+ Interestingly, equality in both inequalities of (22) [and hence $I(U) = 1$] is achieved only for the horizon functions U and V of a *degenerate* black hole (e.g. extreme Kerr), cf. [1].

* With the presented *unique* smooth solution (35) of the differential equation (30), the assumption of equatorial symmetry as well as that of the existence of a continuous sequence in the proof presented in [1] can be abandoned, cf. equation (35) in [1].