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# A universal inequality between the angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter

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**Abstract**
We prove that for sub-extremal axisymmetric and stationary black holes with arbitrary surrounding matter the inequality $8\pi|J| < A$ holds, where $J$ is the angular momentum and $A$ the horizon area of the black hole.

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## 1. Introduction

A well-known property of the Kerr solution, describing a single rotating black hole in vacuum, is given by

$$|p_J| \leqslant 1 \qquad \text{with} \quad p_J := \frac{8\pi J}{A}, \tag{1}$$

where $J$ and $A$ denote the angular momentum and the horizon area of the black hole respectively. Equality in (1) holds if and only if the Kerr black hole is *extreme*. As was shown in [1], the equation $|p_J| = 1$ is even true more generally for axisymmetric and stationary black holes with surrounding matter in the degenerate limit (i.e. for vanishing surface gravity $\kappa$). Moreover, it was also conjectured in [1] that $|p_J| \leqslant 1$ still holds if the black hole is surrounded by matter. In this paper we prove this conjecture[1].

We start by requiring that a physically relevant non-degenerate black hole be characterized through the existence of trapped surfaces (i.e. surfaces with a negative expansion $\theta_{(l)}$ of outgoing null geodesics) in every sufficiently small interior vicinity of the event horizon. That is, in the terminology of [3], we concentrate on *sub-extremal* black holes. In the following we show that such surfaces cannot exist for $|p_J| \geqslant 1$, provided that an appropriate functional $I$

[1] Note that in [1] a more general conjecture, incorporating the black hole's electric charge $Q$, was formulated. Here we prove this conjecture for the pure Einstein field, i.e. for $Q = 0$, and vanishing cosmological constant $\Lambda = 0$. (It should be noted that, for $\Lambda \neq 0$, the inequality $|p_J| \leqslant 1$ can be violated. An example is the Kerr–(A)dS family of black holes, see [3].)

(to be defined below) cannot fall below 1. In turn, this can be proved by means of methods from the calculus of variations.

## 2. Coordinate systems and Einstein equations

We introduce suitable coordinates and metric functions by adopting our notation from [1], which is based on [2]. In spherical coordinates $(r, \theta, \varphi, t)$, an exterior vacuum vicinity of the event horizon[2] can be described by the line element

$$\mathrm{d}s^2 = \mathrm{e}^{2\mu}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + r^2 B^2\,\mathrm{e}^{-2\nu}\sin^2\theta(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2 - \mathrm{e}^{2\nu}\,\mathrm{d}t^2. \tag{2}$$

The function $B$ must be a solution of $\nabla\cdot(r\sin\theta\,\nabla B) = 0$, with $Br\sin\theta = 0$ at the event horizon $\mathcal{H}$. Here $\nabla$ is the nabla operator in a three-dimensional flat space. In order to fix the coordinates we chose the particular solution $B = 1 - r_\mathrm{h}^2/r^2$, where $r_\mathrm{h} = \text{constant} > 0$. In this manner, we obtain coordinates in which $\mathcal{H}$ is located at the coordinate sphere $r = r_\mathrm{h}$. We now decompose the potential $\nu$ in the form $\nu = u + \ln B$, thereby obtaining three regular metric functions $\mu$, $u$ and $\omega$, depending on $r$ and $\theta$ only.

Following Bardeen [2], we introduce the new metric potentials $\hat\mu = r^2\,\mathrm{e}^{2\mu}$, $\hat u = r^2\,\mathrm{e}^{-2u}$, which are positive and regular functions of $r$ and $\cos\theta$, as well as the new radial coordinate $R = \frac{1}{2}\left(r + \frac{r_\mathrm{h}^2}{r}\right)$. We arrive at the Boyer–Lindquist-type line element

$$\mathrm{d}s^2 = \hat\mu\left(\frac{\mathrm{d}R^2}{R^2 - r_\mathrm{h}^2} + \mathrm{d}\theta^2\right) + \hat u\sin^2\theta(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2 - \frac{4}{\hat u}\left(R^2 - r_\mathrm{h}^2\right)\mathrm{d}t^2, \tag{3}$$

which is singular at $\mathcal{H}$ $(R = r_\mathrm{h})$.

In these coordinates, the vacuum Einstein equations read as follows[3]:

$$\left(R^2 - r_\mathrm{h}^2\right)\tilde u_{,RR} + 2R\tilde u_{,R} + \tilde u_{,\theta\theta} + \tilde u_{,\theta}\cot\theta = 1 - \frac{\hat u^2}{8}\sin^2\theta\left(\omega_{,R}^2 + \frac{\omega_{,\theta}^2}{R^2 - r_\mathrm{h}^2}\right), \tag{4}$$

$$\left(R^2 - r_\mathrm{h}^2\right)(\omega_{,RR} + 4\omega_{,R}\tilde u_{,R}) + \omega_{,\theta\theta} + \omega_{,\theta}(3\cot\theta + 4\tilde u_{,\theta}) = 0, \tag{5}$$

$$\begin{aligned}\left(R^2 - r_\mathrm{h}^2\right)\tilde\mu_{,RR} + R\tilde\mu_{,R} + \tilde\mu_{,\theta\theta} = {}& \frac{\hat u^2}{16}\sin^2\theta\left(\omega_{,R}^2 + \frac{\omega_{,\theta}^2}{R^2 - r_\mathrm{h}^2}\right)\\ &+ R\tilde u_{,R} - \left(R^2 - r_\mathrm{h}^2\right)\tilde u_{,R}^2 - \tilde u_{,\theta}(\tilde u_{,\theta} + \cot\theta),\end{aligned} \tag{6}$$

where $\tilde u := \frac{1}{2}\ln\left(r_\mathrm{h}^{-2}\hat u\right)$ and $\tilde\mu := \frac{1}{2}\ln\left(r_\mathrm{h}^{-2}\hat\mu\right)$. At the horizon, the metric potentials obey the boundary conditions [2]

$$\mathcal{H}\!:\quad \omega = \text{constant} = \omega_\mathrm{h}, \qquad \frac{2r_\mathrm{h}}{\sqrt{\hat\mu\hat u}} = \text{constant} = \kappa, \tag{7}$$

with the horizon angular velocity $\omega_\mathrm{h}$ and the surface gravity $\kappa$. On the horizon's north and south pole ($R = r_\mathrm{h}$ and $\sin\theta = 0$), the following regularity conditions hold:

$$\hat\mu(R = r_\mathrm{h}, \theta = 0) = \hat u(R = r_\mathrm{h}, \theta = 0) = \hat\mu(R = r_\mathrm{h}, \theta = \pi) = \hat u(R = r_\mathrm{h}, \theta = \pi) = \frac{2r_\mathrm{h}}{\kappa}. \tag{8}$$

[2] For a stationary spacetime, the immediate vicinity of a black hole event horizon must be vacuum, see e.g. [2].

[3] Throughout this paper we consider pure gravity, i.e. no electromagnetic fields, as well as vanishing cosmological constant, $\Lambda = 0$.

## 3. Necessary condition for the existence of trapped surfaces

A crucial quantity for the following discussion is the expansion $\theta_{(l)} = h^{ab}\nabla_a l_b$ of outgoing null rays for 2-surfaces $\mathcal{S}$ in an interior vicinity of the horizon, where $h$ is the interior metric of $\mathcal{S}$ and $l$ is the vector field describing outgoing null rays. In order to analyze $\theta_{(l)}$ inside the black hole, we introduce horizon-penetrating coordinates $(R, \theta, \tilde{\varphi}, \tilde{t})$, in which the metric is *regular* at the horizon $\mathcal{H}$:

$$\mathrm{d}\tilde{t} = \mathrm{d}t + \frac{T(R)}{\Delta}\,\mathrm{d}R, \qquad \mathrm{d}\tilde{\varphi} = \mathrm{d}\varphi + \frac{\Phi(R)}{\Delta}\,\mathrm{d}R, \qquad \Delta := 4\big(R^2 - r_{\mathrm{h}}^2\big). \tag{9}$$

The free functions $T$ and $\Phi$ are chosen in such a way that

$$a(R,\theta) := \frac{4\hat{\mu}\hat{u} - T^2}{\Delta}, \qquad b(R,\theta) := \frac{\Phi - \omega T}{\Delta} \tag{10}$$

are regular at $R = r_{\mathrm{h}}$. Furthermore, we require $a > 0$ in order to guarantee that $\tilde{t} = \text{constant}$ is a *spacelike* surface. As a consequence, $T$ has to obey the conditions

$$T = 2\sqrt{\hat{\mu}\hat{u}} = \frac{4r_{\mathrm{h}}}{\kappa} \quad \text{for} \quad R = r_{\mathrm{h}}, \qquad T > 2\sqrt{\hat{\mu}\hat{u}} \quad \text{for} \quad R < r_{\mathrm{h}}. \tag{11}$$

We thus obtain the regular line element[4]

$$\mathrm{d}s^2 = \left(\frac{a}{\hat{u}} + \hat{u}b^2\sin^2\theta\right)\mathrm{d}R^2 - \frac{\Delta}{\hat{u}}\,\mathrm{d}\tilde{t}^2 + \hat{u}\sin^2\theta(\mathrm{d}\tilde{\varphi} - \omega\,\mathrm{d}\tilde{t})^2 + \hat{\mu}\,\mathrm{d}\theta^2$$
$$+2\left(\frac{T}{\hat{u}} + \omega\hat{u}b\sin^2\theta\right)\mathrm{d}R\,\mathrm{d}\tilde{t} - 2\hat{u}b\sin^2\theta\,\mathrm{d}R\,\mathrm{d}\tilde{\varphi}. \tag{12}$$

In these coordinates, we calculate the expansion $\theta_{(l)}$ for a surface $\mathcal{S}$ in a small interior neighborhood of the horizon, described by

$$\mathcal{S}: \quad R = r_{\mathrm{h}} - \varepsilon\hat{r}(\theta), \qquad \theta \in [0,\pi], \qquad \tilde{\varphi} \in [0, 2\pi), \qquad \tilde{t} = \text{constant}, \tag{13}$$

where $\varepsilon > 0$, $\hat{r} > 0$ and $\varepsilon\hat{r} \ll r_{\mathrm{h}}$. We obtain

$$\theta_{(l)}(\theta) = \frac{\hat{T}}{2\sqrt{2\hat{\mu}a}\sin\theta}\left[\frac{(\hat{r}_{,\theta}\sin\theta)_{,\theta}}{\hat{r}r_{\mathrm{h}}} - \frac{(\hat{\mu}\hat{u})_{,R}}{2\hat{\mu}\hat{u}}\sin\theta\right]\Bigg|_{\mathcal{H}}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{14}$$

where we have used that $T$ is of the form $T = 2\sqrt{\hat{\mu}\hat{u}}|_{\mathcal{H}} + \varepsilon\hat{T}$ with $\hat{T} = \hat{T}(\theta) > 0$ on $\mathcal{S}$; see (11). (Note that $\theta_{(l)} = 0$ on the horizon ($\varepsilon = 0$).)

Following Booth and Fairhurst [3], we study the criterion $\delta_{\bar{n}}\theta_{(\bar{l})} < 0$ (in their notation) for the existence of trapped surfaces, i.e. we characterize a physically relevant, sub-extremal black hole through a negative variation of the expansion on the horizon in the direction of an ingoing null field $\bar{n}$. In our formulation, this is equivalent to the existence of a surface $\mathcal{S}$ with a negative expansion $\theta_{(l)}$. We first show the following lemma.

**Lemma 3.1.** *A necessary condition for the existence of trapped surfaces in the interior vicinity of the event horizon of an axisymmetric and stationary black hole is*

$$\int_0^\pi (\hat{\mu}\hat{u})_{,R}\Big|_{\mathcal{H}}\sin\theta\,\mathrm{d}\theta > 0. \tag{15}$$

**Proof.** Let the surface $\mathcal{S}$, defined in (13) for sufficiently small $\varepsilon$, be trapped. Then $\theta_{(l)}$ is negative everywhere on $\mathcal{S}$, and so is the term in square brackets in (14) for all $\theta \in [0, \pi]$.

[4] Note that for the Kerr solution with the mass $M$ and the angular momentum $J$, we obtain Kerr-type coordinates (where the slices $\tilde{t} = \text{constant}$ are spacelike) by choosing $T = 4M(M + 2R)$ and $\Phi = \text{constant} = 2J/M$.

Thus, the integral of this term along the horizon $\mathcal{H}$ is negative. Integrating by parts, and using that $\hat{\mu}\hat{u} = \text{constant} > 0$ on $\mathcal{H}$ (see (7)), yields

$$\frac{1}{r_\mathrm{h}}\int_0^\pi \frac{\hat{r}_{,\theta}^2}{\hat{r}^2}\sin\theta\,\mathrm{d}\theta - \frac{1}{2\hat{\mu}\hat{u}}\int_0^\pi (\hat{\mu}\hat{u})_{,R}\sin\theta\,\mathrm{d}\theta < 0. \tag{16}$$

Since the first integral is non-negative, we immediately obtain (15). □

## 4. Calculation of $p_J$

Following the notation of [1], we express the angular momentum $J$ and the horizon area $A$ of the black hole by

$$J = \frac{1}{8\pi}\oint_{\mathcal{H}} m^{a;b}\,\mathrm{d}S_{ab} = -\frac{1}{16}\int_0^\pi \hat{u}^2\omega_{,R}\Big|_{\mathcal{H}}\sin^3\theta\,\mathrm{d}\theta = -\frac{r_\mathrm{h}^2}{4}\int_{-1}^1 V\,\mathrm{e}^{2U}(1-x^2)\,\mathrm{d}x, \tag{17}$$

$$A = 2\pi\int_0^\pi \sqrt{\hat{\mu}\hat{u}}\Big|_{\mathcal{H}}\sin\theta\,\mathrm{d}\theta = 4\pi\sqrt{\hat{\mu}\hat{u}}|_{\mathcal{H}} = 4\pi r_\mathrm{h}^2\,\mathrm{e}^{2U(1)}, \tag{18}$$

where $x := \cos\theta$, $U(x) := \left[\frac{1}{2}\ln\left(r_\mathrm{h}^{-2}\hat{u}\right)\right]\big|_{\mathcal{H}}$, $V(x) := \left[\frac{1}{4}\hat{u}\omega_{,R}\right]\big|_{\mathcal{H}}$ and $m^a$ is the Killing vector corresponding to axisymmetry. Note that for this formulation we have used conditions (7) and (8). As a consequence, $p_J$ takes the form

$$p_J \equiv \frac{8\pi J}{A} = -\frac{1}{2}\mathrm{e}^{-2U(1)}\int_{-1}^1 V\,\mathrm{e}^{2U}(1-x^2)\,\mathrm{d}x. \tag{19}$$

## 5. Reformulation in terms of a variational problem

In order to prove the inequality in question, we show that for $|p_J| \geqslant 1$ equation (15) is violated, i.e. that there are no trapped surfaces in the interior vicinity of the horizon.

Using (4) and (6), the integrand in (15) can be expressed as

$$(\hat{\mu}\hat{u})_{,R}|_{\mathcal{H}} = \frac{2}{r_\mathrm{h}}\hat{u}^2\Big|_{\theta=0}\left[1 - \frac{1}{16}\omega_{,R}^2\hat{u}^2\sin^2\theta - \frac{\hat{u}_{,\theta}}{2\hat{u}}\left(\frac{\hat{u}_{,\theta}}{2\hat{u}} + 2\cot\theta\right)\right]\Big|_{\mathcal{H}}. \tag{20}$$

Hence, we can write (15) in terms of $U$, $V$ and $x$ as follows:

$$\frac{1}{2}\int_{-1}^1 [(V^2 + U'^2)(1-x^2) - 2xU']\,\mathrm{d}x < 1 \qquad \text{with} \quad U' := \frac{\mathrm{d}U}{\mathrm{d}x}. \tag{21}$$

From (19), the violation of (15) for $|p_J| \geqslant 1$ can thus be formulated as the following implication, to be valid for any regular functions $U$, $V$ defined on $[-1, 1]$ and satisfying the condition $U(-1) = U(1)$ [which follows from (8)]:

$$\left|\int_{-1}^1 V\,\mathrm{e}^{2U}(1-x^2)\,\mathrm{d}x\right| \geqslant 2\,\mathrm{e}^{2U(1)} \quad \Rightarrow \quad \frac{1}{2}\int_{-1}^1 [(V^2 + U'^2)(1-x^2) - 2xU']\,\mathrm{d}x \geqslant 1. \tag{22}$$

We now show that the validity of this implication holds provided that an appropriate functional $I$ (defined below) cannot fall below 1.

Applying the Cauchy–Schwarz inequality to the first inequality in (22) we obtain

$$4\,\mathrm{e}^{4U(1)} \leqslant \left(\int_{-1}^1 V\,\mathrm{e}^{2U}(1-x^2)\,\mathrm{d}x\right)^2 \leqslant \int_{-1}^1 V^2(1-x^2)\,\mathrm{d}x\int_{-1}^1 \mathrm{e}^{4U}(1-x^2)\,\mathrm{d}x. \tag{23}$$

Given this inequality, we replace the term $\int V^2(1-x^2)\,\mathrm{d}x$ in the second inequality in (22) and see that

$$I[U] := \frac{1}{2}\int_{-1}^{1}[U'^2(x)(1-x^2) - 2xU'(x)]\,\mathrm{d}x + \frac{2\,\mathrm{e}^{4U(1)}}{\int_{-1}^{1}\mathrm{e}^{4U(x)}(1-x^2)\,\mathrm{d}x} \geqslant 1 \tag{24}$$

is as a sufficient condition for the validity of the implication, with the functional $I\colon W^{1,2}(-1,1) \to \mathbb{R}$ defined on the Sobolev space $W^{1,2}(-1,1)$. Thus we have shown the following.

**Lemma 5.1.** *The inequality* $|p_J| < 1$ *for any sub-extremal axisymmetric and stationary black hole with surrounding matter holds provided that the inequality*

$$I[U] \geqslant 1 \tag{25}$$

*is true for all* $U \in W^{1,2}(-1,1)$ *with* $U(-1) = U(1)$.

This reformulation has led us to the following *variational problem*: calculate the minimum of the functional $I$ and show that it is not below 1.

## 6. Complete solution of the variational problem

Consider the functional

$$I_\varepsilon[U] := \frac{1}{2}\int_{-1+\varepsilon}^{1-\varepsilon}[U'^2(x)(1-x^2) - 2xU'(x)]\,\mathrm{d}x + \frac{2\,\mathrm{e}^{4U(1-\varepsilon)}}{\int_{-1+\varepsilon}^{1-\varepsilon}\mathrm{e}^{4U(x)}(1-x^2)\,\mathrm{d}x} \tag{26}$$

on $W^{1,2}(-1+\varepsilon, 1-\varepsilon)$, where $0 < \varepsilon \ll 1$ is a fixed real number. We use techniques from the calculus of variations to show that there exists a minimizer $U_\varepsilon$ for $I_\varepsilon$ in a suitable class with sufficiently large value $I_\varepsilon[U_\varepsilon]$. Following this investigation, we take the limit $\varepsilon \to 0$ and see that the claim of lemma 5.1 follows.

We now show the following statements:

(i) The functional $I_\varepsilon$ is *well defined on the Sobolev space* $\overset{\varepsilon}{W}{}^{1,2} := W^{1,2}(-1+\varepsilon, 1-\varepsilon)$ of functions $U$ defined almost everywhere on $(-1+\varepsilon, 1-\varepsilon)$. This is due to the well-known proposition below.

**Proposition 6.1** (theorem 2.2 in Buttazzo–Giaquinta–Hildebrandt [4])**.** *On any bounded interval* $J \subseteq \mathbb{R}$, $W^{1,2}(J) \hookrightarrow C^0(\overline{J})$ *compactly. Moreover, the fundamental theorem of calculus holds in* $\overset{\varepsilon}{W}{}^{1,2}$.

Here, we use the adapted inner product $\int_{-1+\varepsilon}^{1-\varepsilon} UV\,\mathrm{d}\mu + \int_{-1+\varepsilon}^{1-\varepsilon} U'V'\,\mathrm{d}\mu$ for $U, V \in \overset{\varepsilon}{W}{}^{1,2}$, where $\mathrm{d}\mu(x) := (1-x^2)\,\mathrm{d}x$, which is equivalent to the ordinary one and thus makes $\overset{\varepsilon}{W}{}^{1,2}$ a Hilbert space.

For ease of notation set $X^\varepsilon := \{U \in \overset{\varepsilon}{W}{}^{1,2} | U(1-\varepsilon) = 0\}$. Note that the functional $I_\varepsilon$ is *invariant under addition of constants*. We will use this in order to restrict our attention to the Hilbert subspace $X^\varepsilon$ on which $(U, V) := \int_{-1+\varepsilon}^{1-\varepsilon} U'V'\,\mathrm{d}\mu$ is an equivalent inner product inducing the norm $\|U\| := \left(\int_{-1+\varepsilon}^{1-\varepsilon} U'^2\,\mathrm{d}\mu\right)^{1/2}$.

(ii) $I_\varepsilon$ is *bounded from below*. Using $0 \leqslant \left(\frac{x}{\sqrt{1-x^2}} - U'(x)\sqrt{1-x^2}\right)^2 = \frac{x^2}{1-x^2} - 2xU'(x) + U'^2(x)(1-x^2)$ we conclude that $I_\varepsilon[U] \geqslant -\frac{1}{2}\int_{-1+\varepsilon}^{1-\varepsilon}\frac{x^2}{1-x^2}\,\mathrm{d}x =: C(\varepsilon) > -\infty$.

(iii) Applying the Cauchy–Schwarz inequality to $\int_{-1+\varepsilon}^{1-\varepsilon} xU'(x)\,\mathrm{d}x$, we obtain that $I_\varepsilon[U] \geqslant \frac{1}{2}\|U\|^2 + 2C(\varepsilon)\|U\|$ for any $U \in X^\varepsilon$ with $C(\varepsilon)$ as in (ii). Hence, for every $P \in \mathbb{R}$ there exists a $Q_P \in \mathbb{R}$ such that $I_\varepsilon[U] \geqslant P$ whenever $\|U\| \geqslant Q_P$. This is equivalent to *coercivity of the functional* $I_\varepsilon$ *with respect to the weak topology on* $X^\varepsilon$.

(iv) The functional $I_\varepsilon$ is *sequentially lower semi-continuous (lsc) with respect to the weak topology in* $X^\varepsilon$. Recall that lower semi-continuity is additive and that the first terms can be dealt with by standard theory (see, e.g., [6]). For the last term, we use proposition 6.1 to deduce that $U_k \to U$ in $C^0([-1+\varepsilon, 1-\varepsilon])$. Whence there exists a uniform bound $D > 0$ of $\{U_k\}$ so that by Lipschitz continuity of the exponential map on $[-4D, 4D]$ with Lipschitz constant $L$, we have

$$\left|\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U_k}\,\mathrm{d}\mu - \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\,\mathrm{d}\mu\right| \leqslant 4L\int_{-1+\varepsilon}^{1-\varepsilon} |U_k - U|\,\mathrm{d}\mu \leqslant 8L\|U_k - U\|_{C^0([-1+\varepsilon,1-\varepsilon])} \stackrel{k\to\infty}{\longrightarrow} 0. \tag{27}$$

The last term thus being lsc, we have shown $I_\varepsilon$ to be lsc.

We can now show the *existence of a global minimizer for* $I_\varepsilon$ *in a suitable class*: as we have seen in (ii), $I_\varepsilon$ is bounded from below on $X^\varepsilon$. Hence for $a \in \mathbb{R}$ and $c > 0$ we can choose a minimizing sequence $\{U_k\}$ in the class $\mathcal{K}_{a,c}^\varepsilon := \left\{U \in X^\varepsilon \,\middle|\, \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\,\mathrm{d}\mu = c, U(-1+\varepsilon) = a\right\}$, with values tending to the infimum $i_{a,c}^\varepsilon$ of $I_\varepsilon$ in $\mathcal{K}_{a,c}^\varepsilon$. By coercivity, $\{U_k\}$ is bounded and we can extract a weakly converging subsequence with the limit $U_{a,c}^\varepsilon \in X^\varepsilon$ by Hilbert space techniques (theorem of Eberlein–Shmulyan [6]).

The class $\mathcal{K}_{a,c}^\varepsilon$ is weakly sequentially closed by proposition 6.1, which can be shown as in (iv). Whence by (iv), $U_{a,c}^\varepsilon \in \mathcal{K}_{a,c}^\varepsilon$ satisfies $I_\varepsilon\big[U_{a,c}^\varepsilon\big] = i_{a,c}^\varepsilon$.

Set $\stackrel{\varepsilon}{W}{}^{1,2}_{a,0} := \{U \in \stackrel{\varepsilon}{W}{}^{1,2} \mid U(-1+\varepsilon) = a, U(1-\varepsilon) = 0\} \subset X^\varepsilon$ for any $a \in \mathbb{R}$. By the theory of Lagrange multipliers, each minimizer of $I_\varepsilon$ in the class $\mathcal{K}_{a,c}^\varepsilon$ is a critical point of the functional

$$J_{\varepsilon,c} : \stackrel{\varepsilon}{W}{}^{1,2}_{a,0} \to \mathbb{R} : U \mapsto \frac{1}{2}\int_{-1+\varepsilon}^{1-\varepsilon} [U'^2(x)(1-x^2) - 2xU'(x)]\,\mathrm{d}x + \frac{\lambda}{2}\left(\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\,\mathrm{d}\mu - c\right) \tag{28}$$

for some $\lambda \in \mathbb{R}$, which is well defined and sufficiently smooth by proposition 6.1. In other words, there is $\lambda := \lambda_{a,c}^\varepsilon \in \mathbb{R}$ such that $U := U_{a,c}^\varepsilon \in \mathcal{K}_{a,c}^\varepsilon$ satisfies

$$\int_{-1+\varepsilon}^{1-\varepsilon} [U'(x)\varphi'(x)(1-x^2) - x\varphi'(x)]\,\mathrm{d}x + 2\lambda\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U(x)}\varphi(x)(1-x^2)\,\mathrm{d}x = 0 \tag{29}$$

for all $\varphi \in \stackrel{\varepsilon}{W}{}^{1,2}_{0,0}$. This can be restated to say that $U \in \stackrel{\varepsilon}{W}{}^{1,2}$ is a weak solution of

$$\begin{gathered} -U''(x)(1-x^2) + 2xU'(x) + 1 + 2\lambda\,\mathrm{e}^{4U(x)}(1-x^2) = 0 \qquad \forall x \in (-1+\varepsilon, 1-\varepsilon), \\ U(-1+\varepsilon) = a, \qquad U(1-\varepsilon) = 0, \qquad \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\,\mathrm{d}\mu = c. \end{gathered} \tag{30}$$

Any weak solution $U \in \stackrel{\varepsilon}{W}{}^{1,2}$ of (30) can be shown to be smooth and to satisfy equation (30) strongly: for all $\varphi \in \stackrel{\varepsilon}{W}{}^{1,2}_{0,0}$, we can rewrite (29) as

$$0 = \int_{-1+\varepsilon}^{1-\varepsilon} \left[U'(x)(1-x^2) - x - 2\lambda\int_{-1+\varepsilon}^{x} \mathrm{e}^{4U}\,\mathrm{d}\mu\right]\varphi'(x)\,\mathrm{d}x, \tag{31}$$

where we used integration by parts and proposition 6.1. By the fundamental lemma of the calculus of variations, there is a constant $b \in \mathbb{R}$ such that

$$U'(x)(1-x^2) - x - 2\lambda \int_{-1+\varepsilon}^{x} \mathrm{e}^{4U}\, \mathrm{d}\mu = b \tag{32}$$

holds almost everywhere on $(-1+\varepsilon, 1-\varepsilon)$ as the integrand of (31) in square brackets is an $L^1$-function. Solving for $U'$, we deduce the smoothness of $U$ by a bootstrap argument (similar to p 462 in [5]). Differentiating equation (32), we get strong validity of (30). In particular, $U_{a,c}^{\varepsilon}$ is a smooth classical solution of the Euler–Lagrange equation of $J_{\varepsilon,c}$.

Interestingly, there exists an integrating factor for equation (30) and it can be solved explicitly: for

$$F(x) := -(1-x^2)^2 U'^2(x) + 2x(1-x^2)U'(x) + \lambda\, \mathrm{e}^{4U(x)}(1-x^2)^2 - x^2 \tag{33}$$

we have

$$F'(x) = 2[x - (1-x^2)U'(x)][U''(x)(1-x^2) - 2xU'(x) - 1 - 2\lambda\, \mathrm{e}^{4U(x)}(1-x^2)]. \tag{34}$$

Thus, it suffices to solve the first-order equation $F =$ constant, which can be done with the substitution $W(x) := (1-x^2)^2\, \mathrm{e}^{4U(x)}$.

The unique smooth solution turns out to be

$$W(x) = \varepsilon^2(2-\varepsilon)^2 \left( \frac{\mathrm{e}^{2\alpha\, \mathrm{artanh}(1-\varepsilon)} - \beta\, \mathrm{e}^{-2\alpha\, \mathrm{artanh}(1-\varepsilon)}}{\mathrm{e}^{2\alpha\, \mathrm{artanh}\, x} - \beta\, \mathrm{e}^{-2\alpha\, \mathrm{artanh}\, x}} \right)^2, \tag{35}$$

where the Lagrange multiplier $\lambda$ was eliminated using the condition $U(1-\varepsilon) = 0$. The complex integration constants $\alpha$ and $\beta$ are implicitly given in terms of $a$ and $c$ via $U(-1+\varepsilon) = a$ and $\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\, \mathrm{d}\mu = c$. The corresponding value of $I_\varepsilon$ is

$$\begin{aligned} I_\varepsilon\big[U_{a,c}^{\varepsilon}\big] = \frac{\alpha\beta}{2} \frac{(2-\varepsilon)^{4\alpha} - \varepsilon^{4\alpha}}{(1+\beta^2)[\varepsilon(2-\varepsilon)]^{2\alpha} - \beta[(2-\varepsilon)^{4\alpha} + \varepsilon^{4\alpha}]} + 1 - \varepsilon + \frac{1-\alpha^2}{2} \ln \frac{\varepsilon}{2-\varepsilon} \\ + \frac{8\alpha[\varepsilon(2-\varepsilon)]^{2(\alpha-1)}}{[(2-\varepsilon)^{2\alpha} - \varepsilon^{2\alpha}]^2} \cdot \frac{(1+\beta^2)[\varepsilon(2-\varepsilon)]^{2\alpha} - \beta[(2-\varepsilon)^{4\alpha} + \varepsilon^{4\alpha}]}{(2-\varepsilon)^{4\alpha} - \varepsilon^{4\alpha}}. \end{aligned} \tag{36}$$

We now study the limit $\varepsilon \to 0$. For any $U \in W^{1,2}(-1,1)$, set $a_\varepsilon := U(-1+\varepsilon) - U(1-\varepsilon)$, $c_\varepsilon := \int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U - 4U(1-\varepsilon)}\, \mathrm{d}\mu$, $c := \int_{-1}^{1} \mathrm{e}^{4U}\, \mathrm{d}\mu$. Then $U|_{(-1+\varepsilon,1-\varepsilon)} - U(1-\varepsilon) \in \mathcal{K}_{a_\varepsilon,c_\varepsilon}^{\varepsilon}$ and thus $I_\varepsilon[U] \geqslant i_{a_\varepsilon,c_\varepsilon}^{\varepsilon}$. As we have seen, there is a minimizer $U_{a_\varepsilon,c_\varepsilon}^{\varepsilon}$ of $I_\varepsilon$ in $\mathcal{K}_{a_\varepsilon,c_\varepsilon}^{\varepsilon}$ and we obtain $I_\varepsilon[U] \geqslant I_\varepsilon\big[U_{a_\varepsilon,c_\varepsilon}^{\varepsilon}\big]$. As discussed above, this minimizer is smooth and solves the Euler–Lagrange equation of the functional $J_{\varepsilon,c_\varepsilon}$; the unique smooth solution of this equation is given by (35). For $\varepsilon \to 0$ we have $\alpha_\varepsilon \to 1$ (otherwise $I_\varepsilon$ diverges) and $\beta_\varepsilon \to -1$ (a consequence of $a_\varepsilon \to 0$ by (25)). Using these relations, we obtain $I_\varepsilon\big[U_{a_\varepsilon,c_\varepsilon}^{\varepsilon}\big] \geqslant C_{a_\varepsilon}^{\varepsilon}$ with $C_{a_\varepsilon}^{\varepsilon} \to 1$ as $\varepsilon \to 0$. By definition of $I$ and $I_\varepsilon$ we see that

$$\begin{aligned} |I(U) - I_\varepsilon(U)| \leqslant \frac{1}{2} \int_{1-\varepsilon \leqslant |x| < 1} |U'^2(x)(1-x^2) - 2xU'(x)|\, \mathrm{d}x \\ + 2 \left| \frac{\mathrm{e}^{4U(1)}}{\int_{-1}^{1} \mathrm{e}^{4U}\, \mathrm{d}\mu} - \frac{\mathrm{e}^{4U(1-\varepsilon)}}{\int_{-1+\varepsilon}^{1-\varepsilon} \mathrm{e}^{4U}\, \mathrm{d}\mu} \right|, \end{aligned} \tag{37}$$

where the first term tends to 0 and the denominators of the latter tend to each other as $\varepsilon \to 0$ (the integrands are $L^1$-functions, theorem of bounded convergence). The numerators of the latter term converge to each other by proposition 6.1. Thus, $I[U] \geqslant 1$.[5]

[5] Interestingly, equality in both inequalities of (22) (and hence $I(U) = 1$) is achieved only for the horizon functions $U$ and $V$ of a *degenerate* black hole (e.g. extreme Kerr), cf [1].

Applying lemma 5.1, we have shown $|p_J| < 1$ for *sub-extremal* black holes. Together with the results about *extremal* black holes [1][6] we arrive at the following.

**Theorem 6.2.** *Consider spacetimes with pure gravity (no electromagnetic fields) and vanishing cosmological constant. Then, for every axisymmetric and stationary sub-extremal black hole with arbitrary surrounding matter we have that* $8\pi|J| < A$. *The equality* $8\pi|J| = A$ *holds if the black hole is degenerate (extremal).*

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[6] With the presented *unique* smooth solution (35) of the differential equation (30), the assumption of equatorial symmetry as well as that of the existence of a continuous sequence in the proof presented in [1] can be abandoned, cf equation (35) in [1].