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# A generalized scaling function for AdS/CFT

**Lisa Freyhult**[1,2], **Adam Rej**[1] **and Matthias Staudacher**[1,3]

[1] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany
[2] Department of Theoretical Physics, Uppsala University, SE-751 08 Uppsala, Sweden
[3] Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge CB3 0EH, UK
E-mail: lisa.freyhult@aei.mpg.de, arej@aei.mpg.de and matthias@aei.mpg.de

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**Abstract.** We study a refined large spin limit for twist operators in the $\mathfrak{sl}(2)$ sector of AdS/CFT. We derive a novel non-perturbative equation for the generalized two-parameter scaling function associated with this limit, and analyze it for weak coupling. It is expected to smoothly interpolate between weakly coupled gauge theory and string theory for strong coupling.

**Keywords:** integrable quantum field theory, integrable spin chains (vertex models), quantum integrability (Bethe ansatz)

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## Contents



## 1. Introduction, main result, and open problem

The perhaps most interesting subset of all local composite quantum operators of planar $\mathcal{N} = 4$ supersymmetric gauge theory is formed by the sector of $\mathfrak{sl}(2)$ twist operators. The reason is that these bear many similarities with the twist operators of QCD. They may be symbolically written as

$$\mathrm{Tr}(\mathcal{D}^M \mathcal{Z}^L) + \cdots, \tag{1.1}$$

which is a shorthand notation for intricate linear superpositions of all states where the $M$ covariant derivatives $\mathcal{D}$ act in all possible ways on the $L$ complex scalar fields $\mathcal{Z}$. Here $L$ is an $\mathfrak{su}(4)$ $R$-charge, frequently denoted as $J$ in the literature, and $M$ is a Lorentz spin, often called $S$. Our labeling refers to the magnetic spin chain picture of these operators, where $L$ is the length of the chain, and $M$ is the 'magnon number'. The twist of an operator is defined as the classical dimension minus its Lorentz spin, so the length $L$ equals the twist in the case of (1.1).

$\mathcal{N} = 4$ gauge theory is a superconformal field theory. Therefore proper superpositions of the operators (1.1) must carry a definite charge $\Delta$ under dilatations. It generically



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is, in contradistinction to the $R$-charge $L$ and the Lorentz charge $M$, coupling constant dependent: $\Delta = \Delta(g)$. Its anomalous part $\gamma(g)$ is defined as[4]

$$\Delta(g) = M + L + \gamma(g), \tag{1.2}$$

where $M + L$ is the classical dimension of the operators (1.1). In the case of the operators (1.1) $\gamma(g)$ behaves in a very interesting way as the spin $M$ gets large at fixed twist $L$. It grows logarithmically with $M$ at all orders of the coupling constant $g$ defined as

$$g^2 = \frac{g_{\text{YM}}^2 N}{8\pi^2} = \frac{\lambda}{16\pi^2}, \tag{1.3}$$

where $\lambda$ is the 't Hooft coupling. The prefactor of the logarithm is a function of $g$. We call it the *universal scaling function* $f(g)$:

$$\Delta - M - L = \gamma(g) = f(g) \log M + \cdots. \tag{1.4}$$

This behavior is a special case of so-called Sudakov scaling; see [1]. In the twist $L = 2$ case it equals *twice* the cusp anomalous dimension of light-like Wilson loops [2]. The independence or 'universality' of the function $f(g)$ as regards the twist $L$, with $L$ arbitrary but finite, or even $L \to \infty$ as long as $L \ll \log M$, was first pointed out at one-loop order in [3], and conjectured to hold at arbitrary loop order in [6]. It would be very interesting to rigorously prove that the twist-2 ($L = 2, M \to \infty$), twist-$L$ ($L$ fixed, $M \to \infty$), and universal ($L, M \to \infty, L \ll \log M$) scaling functions $f(g)$ of the operators (1.1) indeed all coincide for arbitrary values of $g$: $f(g) = f^{(2)}(g) = f^{(3)}(g) = \cdots = f^{(L)}(g) = \cdots = f^{(\infty)}(g)$.

On the basis of the conjectured all-loop integrability of planar $\mathcal{N} = 4$ theory [4], the weak coupling expansion of $f(g)$ is known from the solution of an integral equation obtained from the asymptotic Bethe ansatz for these operators [5]–[7]. It agrees to four orders[5] with field theory [8]:

$$f(g) = 8g^2 - \tfrac{8}{3}\pi^2 g^4 + \tfrac{88}{45}\pi^4 g^6 - 16\left(\tfrac{73}{630}\pi^6 + 4\zeta(3)^2\right) g^8 \pm \cdots. \tag{1.5}$$

Testing the Bethe ansatz to five orders in field theory might not be out of reach [9].

The strong coupling expansion may also be obtained from the same integral equation [7] which generates the small $g$ expansion (1.5). After the initial studies [10, 11], an impressive analytical expansion method to any desired order was worked out in [12]. The starting point of this systematic approach was an important decoupling method discovered by Eden [13]; see also [10]. The series starts as

$$f(g) = 4g - \frac{3\log 2}{\pi} - \frac{K}{4\pi^2}\frac{1}{g} - \cdots, \tag{1.6}$$

where $K = \beta(2)$ is Catalan's constant. The first two terms on the rhs agree, respectively, with the classical and one-loop [14, 15] results from semi-classical string theory, and the

[4] The anomalous dimension $\gamma(g)$ is related to the energy $E(g)$ of the integrable long range spin chain describing the operators (1.1) through $\gamma(g) = 2\,g^2\,E(g)$. It should not be confused with the energy of string states which equals $\Delta(g)$ via the AdS/CFT correspondence.

[5] To be precise, the four-loop field theory result of [8] agrees numerically with the analytic Bethe ansatz prediction in (1.5) to 0.001%. An analytic proof would be most welcome.



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third term is the two-loop correction very recently obtained in [17, 18]. It would be very interesting to also check the three-loop term in string theory[6].

So it appears that $f(g)$ is the first example of an exactly known, via the solution of a linear integral equation [7], function which smoothly *interpolates* between a gauge theory and a string theory observable in the AdS/CFT system. A natural question is whether further interesting examples may be found, and whether the function $f(g)$ may be generalized. A major obstacle is the fact that we currently only know the *asymptotic* spectrum of the planar $\mathcal{N} = 4$ model, as was recently unequivocally established in [19]. Important clues come from both taking a closer look at the scaling law (1.4) in the one-loop gauge theory [3], and intriguing string theory results [20, 21] generalizing the expansion (1.6). Put together, these suggest that at weak coupling an interesting generalized scaling limit might exist, where[7]

$$M \to \infty, \qquad L \to \infty, \qquad \text{with } j := \frac{L}{\log M} = \text{fixed}. \tag{1.7}$$

We will prove in this paper that this is indeed the case, first at one-loop order, and then beyond. More precisely, we will show that a *generalized* scaling function $f(g, j)$ exists to all orders in perturbation theory

$$\Delta - M - L = \gamma(g) = f(g, j) \log M + \cdots, \tag{1.8}$$

where $f(g, 0) = f(g)$ in (1.4). This extends the one-loop results in [3], and the all-loop result at $j = 0$ of [6, 7]. The latter is possible since in the limit (1.7) $L \to \infty$, and we may therefore use the asymptotic Bethe ansatz methodology of [5]–[7].

The final result of our analysis, presented in detail in the ensuing sections, is the following integral equation:

$$\hat{\sigma}(t) = \frac{t}{\mathrm{e}^t - 1}\left(\hat{\mathcal{K}}(t, 0) - 4\int_0^\infty \mathrm{d}t' \hat{\mathcal{K}}(t, t')\hat{\sigma}(t')\right). \tag{1.9}$$

It is essentially identical in form to the 'ES' (no dressing phase) [6] and 'BES' (with proper dressing phase) [7] equations. The kernel corresponding to the generalized scaling limit is quite involved as it contains various contributions. It reads

$$\begin{aligned}\hat{\mathcal{K}}(t, t') = g^2\hat{K}(2gt, 2gt') + \hat{K}_{\mathrm{h}}(t, t'; a) - \frac{J_0(2gt)}{t}\frac{\sin at'}{2\pi t'}\mathrm{e}^{t'/2} \\ - 4g^2\int_0^\infty \mathrm{d}t'' t'' \hat{K}(2gt, 2gt'')\hat{K}_{\mathrm{h}}(t'', t'; a).\end{aligned} \tag{1.10}$$

Here

$$\hat{K}(t, t') = \hat{K}_0(t, t') + \hat{K}_1(t, t') + \hat{K}_{\mathrm{d}}(t, t') \tag{1.11}$$

[6] However, one important fact to keep in mind is that the weak coupling expansion (1.5) has a finite radius of convergence, while the strong coupling series (1.6) is asymptotic, and, apparently, not even Borel-summable [12]. So (1.6) follows from knowing all terms in (1.5), but, conversely, knowing all terms of the string expansion (1.6) does not allow one to reconstruct the gauge-theoretic perturbation series (1.5) without further input. Unfortunately, it is currently not even known what the nature of this input might be.

[7] The variable $j$ was first explicitly introduced (up to a factor of 1/2) in equation (3.1) of [21].



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is the kernel of the 'BES' equation, where

$$\hat{K}_0(t,t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2}, \tag{1.12}$$

$$\hat{K}_1(t,t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2}, \tag{1.13}$$

and the kernel encoding the effects of the dressing phase is given by the convolution

$$\hat{K}_{\mathrm{d}}(t,t') = 8g^2 \int_0^\infty \mathrm{d}t'' \hat{K}_1(t,2gt'') \frac{t''}{\mathrm{e}^{t''} - 1} \hat{K}_0(2gt'',t'). \tag{1.14}$$

For a possible mechanism generating this type of convolution structure see [22]. The novel contributions generated by a non-vanishing $j$ are encoded in the kernel

$$\hat{K}_{\mathrm{h}}(t,t';a) = \frac{1}{2\pi t} \mathrm{e}^{-t/2} \frac{t\cos(at')\sin(at) - t'\cos(at)\sin(at')}{t^2 - t'^2} \mathrm{e}^{t'/2}, \tag{1.15}$$

as well as the explicit, rightmost term of the first line of (1.10), and the further convolution in the second line of that equation. The index h of $\hat{K}_{\mathrm{h}}(t,t';a)$ stands for 'hole'; its meaning will become clear below. The corrections of the refined limit depend on a 'gap' parameter $a$ whose interpretation will also be explained. Its relation to $j$ is fixed by the constraint

$$j = \frac{4a}{\pi} - \frac{16}{\pi} \int_0^\infty \mathrm{d}t \hat{\sigma}(t) \mathrm{e}^{t/2} \frac{\sin at}{t}. \tag{1.16}$$

Lastly, the generalized scaling function of (1.8) is given by

$$f(g,j) = j + 16\hat{\sigma}(0). \tag{1.17}$$

It is determined by first solving the integral equation (1.9) with the kernel (1.10) for the fluctuation density $\hat{\sigma}(t) = \hat{\sigma}(t;g,a)$ as a function of $g$ and $a$. Then $a$ is found as a function of $j$ by inverting the relation (1.16), i.e. by computing $a(j)$. This then yields $\hat{\sigma}(t) = \hat{\sigma}(t;g,a(j))$ as a function of $g$ and $j$, and the generalized scaling function $f(g,j)$ is finally obtained by evaluating the latter at $t = 0$, see (1.17).

As in [6, 7], in practice it appears impossible to produce a closed-form solution of the equation (1.9). In fact, we did not even find an explicit solution at one-loop order, i.e. for $g = 0$. It is however possible to solve it iteratively in a double expansion in small $g$ and small $j$. Excitingly, the function obtained appears to be 'bi-analytic', i.e. analytic in $g$ around $g = 0$ at arbitrary finite values of $j$, and vice versa. We therefore believe that our equations actually hold for *arbitrary* values of $g$ and $j$. The beginning of this double expansion may be found in (4.17), (4.19), (4.20), (4.21), which we have displayed by giving the four-loop result of the functions $f_1(g) \cdots f_4(g)$ defined through

$$f(g,j) = f(g) + \sum_{n=1}^{\infty} f_n(g) j^n. \tag{1.18}$$

Our truncation at four-loop order, $\mathcal{O}(g^8)$ and $\mathcal{O}(j^4)$, is due to space limitations, and one easily generates many more orders in $g^2$ and $j$ if needed. A curious fact is the absence of any terms of $\mathcal{O}(j^2)$, i.e. the function $f_2(g)$ is zero. We will come back to this point shortly.

A very interesting question is how $f(g,j)$ behaves at strong coupling. Indeed we would like to make contact with the results already known from string theory [15, 16, 3, 20, 21, 23].



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A potential problem is that in the semi-classical computations pioneered by Frolov and Tseytlin [15] the coupling constant $g$ is intricately entangled with the, respectively, $AdS_5$ and $S^5$ charges $M$ and $L$. In [3] the strong coupling limit of the dimension $\Delta$ of the operators (1.1) was predicted from the results of [15, 16] on the energy of a folded string soliton (see also the discussion in [20, 21, 23, 24]). The prediction reads

$$\Delta_{\text{classical}} = M + L\sqrt{1+z^2} + \cdots, \tag{1.19}$$

where[8]

$$M \to \infty, L \to \infty, g \to \infty, \qquad \text{with } M \gg L \qquad \text{and fixing } \frac{M}{g}, \frac{L}{g}, z := 4g\frac{\log M/\Lambda}{L}, \tag{1.20}$$

and $\Lambda$ is some scale[9]. The result (1.19) was derived in [24] from the asymptotic Bethe ansatz [5, 7] with the approximate strong coupling (AFS [25] only) dressing phase. While this constitutes an important check of the Bethe ansatz method, this had to work in the sense that the dressing phase [25] was extracted from the integrable structure of classical string theory [26, 27]. What is missing is a derivation from a solution of the exact ansatz [5, 7] which interpolates between weak and strong coupling. Now it is tempting to identify, in view of (1.7),

$$z = \frac{4g}{j}. \tag{1.21}$$

Certainly the condition $M \gg L$ with $M, L \to \infty$ is satisfied in the weak coupling limiting procedure (1.7). The more questionable assumptions in semi-classical string theory, as far as extrapolating weak coupling results is concerned, are the fixing of $M/g$, $L/g$ (see also footnotes 6 and 9). Proceeding under this caveat we could then rewrite (1.19) as

$$\Delta_{\text{classical}} - M - L = \left(4g\sqrt{1+\left(\frac{j}{4g}\right)^2} - j\right)\log M + \cdots. \tag{1.22}$$

If we now also expand in small $j$ we find

$$\Delta_{\text{classical}} - M - L = \left(4g - j + \frac{j^2}{8g} + \mathcal{O}(j^4)\right)\log M + \cdots. \tag{1.23}$$

The leading term $4g$ agrees with the first term in (1.6). The one-loop string correction to (1.19) was computed in [20]:

$$\begin{aligned}\Delta_{1-\text{loop}} = \frac{L}{\sqrt{\lambda}}\frac{1}{\sqrt{1+z^2}}\Big\{ & z\sqrt{1+z^2} - (1+2z^2)\log\left[z+\sqrt{1+z^2}\right] \\ & - z^2 + 2(1+z^2)\log(1+z^2) - (1+2z^2)\log\left[\sqrt{1+2z^2}\right]\Big\}.\end{aligned} \tag{1.24}$$

[8] The contemporaneously appearing work [23] uses the notation $\ell = 1/z$ and $\Lambda = 4\pi g$.

[9] The scale $\Lambda$ actually being used in the string theory calculations in [15, 16, 3, 20, 21, 23] seems to be somewhat unclear. Is the proper scale (1) $\Lambda = 4\pi g$ or (2) $\Lambda = L$ or (3) $\Lambda = 1$? Since these calculations start from fixing $M/g$ and $L/g$ it would seem that they require either (1) or (2). At weak coupling we definitely have (3), as we are proving to all orders in this paper. Understanding the crossover of scales as one moves from weak to strong coupling, or vice versa, should be very interesting.

Taking $z \to \infty$ it produces the second term on the rhs of (1.6). If we were to again expand in small $j$ via (1.21) we would find $j^2 \log j$ terms. The result (1.24) was fully derived in [24] from the asymptotic Bethe ansatz [7, 5] with the approximate strong coupling (AFS + HL [28]) dressing phase. Once again, this is an important cross-check on the consistency of the extraction of the one-loop correction of the dressing phase from one-loop string theory [28, 29]—see also the very recent derivation [30]—but does not answer the question of how the dimensions of the gauge theory states in (1.1) 'flow' to the energies of string theory states as the coupling increases.

An interesting insight into the structure of further quantum corrections, i.e. two-loop order and higher, to (1.19), (1.24) was obtained in [21] in the limit $z \to \infty$. In a paper contemporaneous with ours [23], an impressive direct two-loop string calculation, in this limit, is performed which agrees with the results of [21]. However, Roiban and Tseytlin argue in [23] that after resumming infinitely many terms of the form $j^2 \log^k j$ all terms of the form $j^2$ might vanish. They furthermore noticed that some initial support for these considerations is provided by a fascinating and curious by-product of our derivation: the function $f_2(g)$ in the expansion1.18 is exactly zero! This suggests that extrapolation between the result at small $g$ and the result at large $g$ might indeed work out.

Therefore an exciting open problem not addressed in this paper is how to now solve our equations at strong coupling $g \to \infty$ in order to see whether any of the above string results are reproduced, and whether the extrapolation works out. In fact, our derivation does not assume $j$ to be small, so we are hopeful that under the identification (1.21) the full strong coupling expansion of $f(g, j)$, i.e. (1.19), (1.24) and all further corrections, in generalization of the beautiful expansion of [12] at $j = 0$, will be obtained. As already mentioned this is however not assured, as we might run into an order-of-limits problem, namely (1.7) versus (1.20). It would also be important to gain an understanding how the states corresponding to the generalized scaling function fit into the general classification of classical integrable curves [27, 31] and their quantum fluctuations. It should be very interesting to see how the parameter $j$ in (1.7) relates to the 'filling fractions' of the classical curve.

This paper is organized as follows. In section 2 we extend the study in [3] and take a close look at the fine structure of the large spin $M$ anomalous dimensions of (1.1) at one-loop order. We derive our results using both traditional techniques and more sophisticated ones involving so-called non-linear integral (or also 'Destri–DeVega') equations; see [32] and references therein. In section 3 we generalize the methodology of the non-linear integral equations to all orders in the coupling constant and compute some novel finite size $\mathcal{O}(M^0)$ corrections to the scaling behavior (1.4). In section 4 we extend our one-loop results to all loop orders, prove the existence of the novel generalized scaling function in (1.8), and derive the above equations determining it.

## 2. One-loop theory

### 2.1. Magnons and holes

The one-loop diagonalization problem of the operators (1.1) is equivalent to that of an integrable spin chain with $\mathfrak{sl}(2)$ symmetry. This was first discovered in [33, 34] and more specifically in the $\mathcal{N} = 4$ context, extending the discovery of [35], in [36]. The allows us



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to apply the Bethe ansatz, which then leads to the following one-loop Bethe equations:

$$\left(\frac{u_k+\mathrm{i}/2}{u_k-\mathrm{i}/2}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{u_k-u_j-\mathrm{i}}{u_k-u_j+\mathrm{i}}, \tag{2.1}$$

where $L$ is the length (=twist in this case) and $M$ is the number of magnons; see (1.1). The cyclicity constraint and the one-loop anomalous dimension $\gamma_1$ (see (1.2)) are

$$\prod_{k=1}^{M}\frac{u_k+\mathrm{i}/2}{u_k-\mathrm{i}/2}=1 \qquad \text{and} \qquad \gamma_1=\left.\frac{\gamma(g)}{g^2}\right|_{g=0}=2\sum_{k=1}^{M}\frac{1}{u_k^2+1/4}. \tag{2.2}$$

With the help of the Baxter function

$$Q(u)=\prod_{k=1}^{M}(u-u_k) \tag{2.3}$$

one can write down an off-shell version of these equations:

$$\left(u+\frac{\mathrm{i}}{2}\right)^L Q(u+\mathrm{i})+\left(u-\frac{\mathrm{i}}{2}\right)^L Q(u-\mathrm{i})=t(u)Q(u), \tag{2.4}$$

where

$$t(u)=2u^L+\sum_{i=2}^{L}q_i u^{L-i} \tag{2.5}$$

is the transfer matrix given in terms of the charges. The ground state for arbitrary $L$ and $M$ is unique and thus the corresponding charges are fixed. Clearly setting $u=u_k$ in equation (2.4) brings us back to (2.1). However one of the advantages of (2.4) is the possibility of identification of solutions $u=u_{\mathrm{h}}^{(k)}$ complementary to (2.1) [3]. They are found as the zeros of the transfer matrix, i.e. from $t(u)=0$ and describe 'holes'. We thus have

$$t(u)=2\prod_{k=1}^{L}(u-u_{\mathrm{h}}^{(k)}). \tag{2.6}$$

We can intuitively think of the hole roots as rapidities describing the motion of the $\mathcal{Z}$-particles in the spin chain interpretation of the operators (1.1). For a general value of $L$ the equation (2.6) has $L$ solutions and thus there are $L$ holes. One can prove that for any state all magnon roots $u_k$ and all hole roots $u_{\mathrm{h}}^{(k)}$ are real. It is possible to find the $q_2$ charge analytically by matching the three highest powers of $u$ in the Baxter equation (2.4):

$$q_2=-\tfrac{1}{4}L(L-1)-LM-M(M-1). \tag{2.7}$$

Because $q_2$ and all higher charges explicitly depend on $M$ the roots of $t(u)$ will also, generically, depend on $M$. One can argue, however, that for the ground state, and in the case $L\ll M$, two of them are special; see [3]: their magnitude is larger than those of any other (hole or magnon) Bethe roots, and scales with $M$ as the magnon number $M$ gets large; see [3]. To identify these roots one recalls that the mode numbers for magnons for



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the ground states, when $L \ll M$, are given by [6]

$$n_k = k + \frac{L-3}{2}\mathrm{sgn}(k) \qquad \text{for } k = \pm 1, \pm 2, \ldots, \pm \frac{M}{2}. \tag{2.8}$$

The absolute value of the roots grows monotonically with $|n_k|$. It follows from (2.8) that the rapidities of the magnons and holes are parity invariant. Among the holes there are two 'universal holes' which occupy the highest allowed mode numbers

$$n_{\mathrm{h}}^{u,1} = \frac{L+M-1}{2} \qquad n_{\mathrm{h}}^{u,2} = -\frac{L+M-1}{2}. \tag{2.9}$$

The corresponding hole roots are precisely the ones that scale with $M$. The remaining holes fill the gap in the mode numbers of magnons

$$n_{\mathrm{h}}^{r} \in \left\{ -\frac{L-3}{2}, \ldots, \frac{L-3}{2} \right\}. \tag{2.10}$$

For the ground state, when $L \ll M$, the magnitudes of the roots are thus ordered as

$$|u_{\mathrm{h}}^{(1,2)}| > |u_k| > u_{\mathrm{h}}^{(j)} \qquad (j \neq 1, 2). \tag{2.11}$$

### 2.2. The counting function and the NLIE

A nice way to exploit the existence of the hidden hole degrees of freedom employs the so-called *counting function*; see [32] and references therein. It is defined as

$$Z(u) = L\phi\left(u, \frac{1}{2}\right) + \sum_{k=1}^{M} \phi(u - u_k, 1) \qquad \text{where } \phi(u, \xi) = \mathrm{i}\log\left(\frac{\mathrm{i}\xi + u}{\mathrm{i}\xi - u}\right). \tag{2.12}$$

Its name stems from the fact that, as one immediately sees from the definition (2.12), $Z(\pm\infty) = \pm\pi(L+M)$ while the Bethe equations for the magnons and holes may be, respectively, expressed as

$$Z(u_j) = \pi(2n_j + \delta - 1) \qquad j = 1, \ldots, M, \tag{2.13}$$

$$Z(u_{\mathrm{h}}^{(k)}) = \pi(2n_{\mathrm{h}}^{(k)} + \delta - 1) \qquad k = 1, \ldots, L, \tag{2.14}$$

where

$$\delta = L + M \qquad \mathrm{mod}\, 2. \tag{2.15}$$

So $Z(u)$ is a smooth function which yields the corresponding mode number (times $\pi$) whenever $u$ equals a hole or magnon root. The mode numbers clearly 'label' or 'count' the solutions of the Bethe equations, and the counting function smoothly interpolates between them.

To write down the one-loop non-linear integral equation, we recall [32] that for an arbitrary function $f(u)$, which is analytic within a strip around the real axis, the following identity holds:

$$\sum_{k=1}^{M} f(u_k) + \sum_{j=1}^{L} f(u_{\mathrm{h}}^{(j)}) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} f'(u) Z(u) + \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{\pi} f'(u) \mathrm{Im}\log[1 + (-1)^{\delta} \mathrm{e}^{\mathrm{i}Z(u+\mathrm{i}0)}]. \tag{2.16}$$

Applying this identity to $Z(u)$ and adapting the steps of [32] to the present case, we find[10]

$$Z(u) = \mathrm{i}L\log\frac{\Gamma(1/2+\mathrm{i}u)}{\Gamma(1/2-\mathrm{i}u)} + \sum_{j=1}^{L}\mathrm{i}\log\frac{\Gamma(-\mathrm{i}(u-u_{\mathrm{h}}^{(j)}))}{\Gamma(\mathrm{i}(u-u_{\mathrm{h}}^{(j)}))}$$
$$+ \lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}\frac{\mathrm{d}v}{\pi}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}u}\log\frac{\Gamma(-\mathrm{i}(u-v))}{\Gamma(\mathrm{i}(u-v))}\mathrm{Im}\log\left[1+(-1)^{\delta}\mathrm{e}^{\mathrm{i}Z(v+\mathrm{i}0)}\right]. \tag{2.17}$$

The identity (2.16) may also be used to express all conserved charges in terms of the counting function. The first charge (the momentum), however, needs to be regularized:

$$P = \lim_{\alpha\to\infty}\left(-\int_{-\alpha}^{\alpha}\frac{\mathrm{d}u}{2\pi}p'(u)Z(u) - \sum_{j=1}^{L}p(u_{\mathrm{h}}^{(j)})\int_{-\alpha}^{\alpha}\frac{\mathrm{d}u}{\pi}p'(u)\mathrm{Im}\log\left[1+(-1)^{\delta}\mathrm{e}^{\mathrm{i}Z(u+\mathrm{i}0)}\right]\right). \tag{2.18}$$

In the above formula $p(u)$ denotes the momentum of a single particle

$$p(u) = \frac{1}{\mathrm{i}}\log\frac{u+\mathrm{i}/2}{u-\mathrm{i}/2}. \tag{2.19}$$

Due to antisymmetry of $Z(u)$ and $p(u)$ one easily finds

$$P = 0. \tag{2.20}$$

Similarly, the one-loop anomalous dimension $\gamma_1$ (see (2.2)) may be rewritten as

$$\gamma_1 = 4\gamma_{\mathrm{E}}L + 2\sum_{j=1}^{L}\{\psi(1/2+\mathrm{i}u_{\mathrm{h}}^{(j)}) + \psi(1/2-\mathrm{i}u_{\mathrm{h}}^{(j)})\}$$
$$+ 2\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{\pi}\mathrm{i}\frac{\mathrm{d}^2}{\mathrm{d}v^2}\left(\log\frac{\Gamma(1/2+\mathrm{i}v)}{\Gamma(1/2-\mathrm{i}v)}\right)\mathrm{Im}\log\left[1+(-1)^{\delta}\mathrm{e}^{\mathrm{i}Z(v+\mathrm{i}0)}\right], \tag{2.21}$$

where $\gamma_{\mathrm{E}}$ is Euler's constant.

Note that the NLIE (2.17) in conjunction with the Bethe equations for the hole roots (2.14) is fully equivalent, for the ground state, to the algebraic Bethe equation (2.1) for arbitrary finite values of $M$ and $L$. (The generalization to the case of excited states is fairly straightforward but will not be discussed in this paper.) Likewise, the expressions for the one-loop anomalous dimension $\gamma_1$ given in (2.2) and (2.21) are equivalent.

### 2.3. Magnon density

If the number of magnon roots $M$ gets large we may expect, for the ground state, that they form a dense distribution on the union of two intervals $[-b, -a]$ and $[a, b]$ on the real axis. This allows us to introduce a distribution density $\rho_{\mathrm{m}}(u)$; see section 3.2. of [6] for

[10] Due to superficial divergencies one needs to apply (2.16) to $Z'(u)$ and then to integrate the resulting equation twice. The constants of integration are fixed by antisymmetry of $Z(u)$ and the condition

$$\lim_{u\to\infty} Z'(u) = 0.$$



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further details. It then follows from (2.8) and (2.13) that

$$\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}u}Z(u) = 2\pi\rho_{\mathrm{m}}(u) + 2\pi\frac{L-2}{M}\delta(u) + \mathcal{O}\left(\frac{1}{M^2}\right), \qquad \text{with } 2\int_a^b \mathrm{d}u\rho_{\mathrm{m}}(u) = 1, \tag{2.22}$$

where the $\delta$-function stems from the gap in the center of the magnon mode numbers (2.8). Using this relation one can rewrite (2.12) as

$$2\pi\rho_{\mathrm{m}}(u) + 2\pi\frac{L-2}{M}\delta(u) - \frac{L}{M}\frac{1}{u^2+1/4} - 2\left(\int_{-b}^{-a}\mathrm{d}v + \int_a^b \mathrm{d}v\right)\frac{\rho_{\mathrm{m}}(v)}{(u-v)^2+1} = 0, \tag{2.23}$$

where $u \in [-b,-a] \cup [a,b]$. If there is a gap $2a > 0$ we therefore may drop the term involving the $\delta$-function in (2.23). In principle, if interpreted appropriately, this equation should hold for large $M$ and *arbitrary*, small or large, $L$.

If in addition $L$ stays finite (but arbitrary) we can apply the scaling procedure $\bar{u} = u/M$ of [6], as in this case the gap $2a$ closes ($a \to 0$), and in addition $b \to M/2$. Then the non-singular integral equation (2.23 turns into a singular integral equation, and with $\bar{\rho}_0(\bar{u}) = M\rho_{\mathrm{m}}(u)$ we find

$$-4\pi\delta(\bar{u}) - 2\fint_{-1/2}^{-1/2}\mathrm{d}\bar{u}'\frac{\bar{\rho}_0(\bar{u}')}{(\bar{u}-\bar{u}')^2} = 0. \tag{2.24}$$

The solution is the (singular) density

$$\bar{\rho}_0(\bar{u}) = \frac{1}{\pi}\log\frac{1+\sqrt{1-4\bar{u}^2}}{1-\sqrt{1-4\bar{u}^2}} = \frac{2}{\pi}\arctan h(\sqrt{1-4\bar{u}^2}), \tag{2.25}$$

first derived in [37]. It should be considered as a distribution (in the mathematical sense) rather than as a regular function. The reason is that the expression for the one-loop anomalous dimension in (2.2) formally turns into $4\pi\int \mathrm{d}\bar{u}\bar{\rho}_{\mathrm{m}}(\bar{u})\delta(\bar{u}) = 4\pi\rho_{\mathrm{m}}(0) = \infty$. However, a more careful analysis [6] of the multiplication of the distributions $\rho_{\mathrm{m}}(\bar{u})$ and $\delta(\bar{u})$ leads to

$$\gamma_1 = 8\log M + \mathcal{O}(M^0). \tag{2.26}$$

If instead $L \to \infty$ along with $M \to \infty$ such that $\beta = M/L$ is kept finite, the gap $2a$ does not close. We may then drop the $\delta$-function term in (2.23) and obtain after rescaling $\bar{u} = u/M$ with $\bar{a} = a/M$, $\bar{b} = b/M$

$$-\frac{1}{\beta}\frac{1}{\bar{u}^2} - 2\left(\fint_{-\bar{b}}^{-\bar{a}}\mathrm{d}\bar{v} + \fint_{\bar{a}}^{\bar{b}}\mathrm{d}\bar{v}\right)\frac{1}{(\bar{u}-\bar{v})^2}\bar{\rho}_{\mathrm{m}}(\bar{v}) = 0, \tag{2.27}$$

which is essentially (up to a rescaling of $\bar{u}$ by $\beta$) the derivative $\mathrm{d}/\mathrm{d}\bar{u}$ of the singular two-cut integral equation first derived in [16]. The original equation is easily reconstructed by integrating both sides of (2.27) w.r.t. $\bar{u}$, with a constant of integration of $2\pi/\beta$ on the right-hand side. The explicit solution for the density $\bar{\rho}_{\mathrm{m}}(\bar{v})$ along with $\bar{a}, \bar{b}$ was also given in [16]. When $\beta \to \infty$ the gap $2\bar{a}$ disappears and the limiting distribution (2.25) is recovered. However, this procedure does *not* reproduce the correct behavior of the anomalous dimension of the previous large $M$ limit at fixed $L$, i.e. (2.26); instead, one



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finds

$$\gamma_1 = \frac{8}{L}\log^2\frac{M}{L} + \cdots. \tag{2.28}$$

See also the discussion in [6]. We notice that large $M$ analysis is quite subtle if the gap $2a$ is very small but non-vanishing.

In fact, there is a very interesting perturbation on the scaling behavior (2.26) first noticed in [3]. Let us understand this effect through a more refined analysis of (2.22), (2.23). It is convenient to split the density $\rho_{\mathrm{m}}(u)$ into the singular, leading piece $\rho_0(u)$ and a fluctuation correction $\tilde{\sigma}(u)$: $\rho_{\mathrm{m}}(u) = \rho_0(u)+\tilde{\sigma}(u)$ where $\rho_0(u) = 1/M\,\bar{\rho}_0(u/M)$; see (2.25). The trick is to now add

$$2\int_{-a}^{a}\mathrm{d}v\frac{\rho_0(v)}{(u-v)^2+1} = \frac{4\log M}{\pi M}\left(\arctan(u+a)-\arctan(u-a)\right)+\mathcal{O}(M^0) \tag{2.29}$$

to (2.23), and to subsequently extend the domain of validity of the equation to the entire real axis, after replacing on the lhs $\rho_{\mathrm{m}}$ by $\rho_{\mathrm{m}}+\rho_{\mathrm{h}}$. We then see that $\tilde{\sigma}(u)$ scales as $\log M/M$ and we should therefore define, in analogy with [6], a *fluctuation density* $\sigma(u)$ through

$$\rho_{\mathrm{m}}(u)+\rho_{\mathrm{h}}(u) = \rho_0(u) - \frac{8\log M}{M}\sigma(u). \tag{2.30}$$

It satisfies

$$2\pi\sigma(u) - \frac{1}{2\pi}\left(\arctan(u+a)-\arctan(u-a)\right) + \frac{j}{8}\frac{1}{u^2+1/4} - 2\left(\int_{-\infty}^{-a}\mathrm{d}v + \int_{a}^{\infty}\mathrm{d}v\right)\frac{\sigma(v)}{(u-v)^2+1} = 0. \tag{2.31}$$

This integral equation fully determines the fluctuation density $\sigma(u)$, as the edge parameter $a$ may be determined from the normalization condition $(\int_{-\infty}^{-a}+\int_a^{\infty})\mathrm{d}u\ \rho_{\mathrm{m}}(u) = 1$, which implies

$$j = \frac{4a}{\pi} - 8\int_{-a}^{a}\mathrm{d}u\sigma(u). \tag{2.32}$$

The one-loop anomalous dimension is then given from (2.2) by

$$\frac{\gamma_1(j)}{\log M} = 8 - \frac{16}{\pi}\arctan 2a - 16\left(\int_{-\infty}^{-a}\mathrm{d}u + \int_{a}^{\infty}\mathrm{d}u\right)\frac{\sigma(u)}{u^2+1/4}. \tag{2.33}$$

### 2.4. Fourier space equation

It is very instructive to change from $u$-space to Fourier space. After rewriting (2.31) as

$$\sigma(u) = \frac{1}{4\pi^2}\left(\arctan(u+a)-\arctan(u-a)\right) - \frac{j}{16\pi}\frac{1}{u^2+1/4} + \int_{-\infty}^{\infty}\frac{\mathrm{d}v}{\pi}\frac{\sigma(v)}{1+(u-v)^2} - \int_{-a}^{a}\frac{\mathrm{d}v}{\pi}\frac{\sigma(v)}{1+(u-v)^2} \tag{2.34}$$



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and Fourier transforming[11]

$$\hat{\sigma}(t) = \mathrm{e}^{-t/2} \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}tu} \sigma(u) \tag{2.35}$$

one obtains

$$\hat{\sigma}(t) = \frac{t}{\mathrm{e}^t - 1} \left( \hat{K}_{\mathrm{h}}(t, 0; a) - \frac{j}{8t} - 4 \int_0^{\infty} \mathrm{d}t' \hat{K}_{\mathrm{h}}(t, t'; a) \hat{\sigma}(t') \right), \tag{2.36}$$

where the kernel is given by

$$\hat{K}_{\mathrm{h}}(t, t'; a) = \frac{\mathrm{e}^{(t'-t)/2}}{4\pi t} \int_{-a}^{a} \mathrm{d}u \cos(tu) \cos(t'u), \tag{2.37}$$

which leads to the expression (1.15) in the introduction. Likewise, Fourier transforming the normalization condition (2.32) yields the relation (1.16) between the physical parameter $j$ and the fluctuation density $\hat{\sigma}(t)$ in Fourier space stated already in the introduction. The one-loop anomalous dimension is then given by

$$\frac{\gamma_1(j)}{\log M} = 8 \left[ 1 - \frac{2}{\pi} \arctan 2a - 4 \int_0^{\infty} \mathrm{d}t \left( \hat{\sigma}(t) - 4t \int_0^{\infty} \mathrm{d}t' \hat{K}_{\mathrm{h}}(t, t'; a) \hat{\sigma}(t') \right) \right]. \tag{2.38}$$

### 2.5. Hole density

The Bethe roots corresponding to the small holes lie inside some interval $[-c, c]$. In the 'thermodynamic' limit $L \to \infty$, where the number of small holes tends to infinity, their one-loop root distribution density $\rho_{\mathrm{h}}(u)$ is related to the counting function through

$$\frac{1}{L} \frac{\mathrm{d}}{\mathrm{d}u} Z(u) = 2\pi \rho_{\mathrm{h}}(u) + \mathcal{O}\left(\frac{1}{L}\right), \qquad \text{with} \int_{-c}^{c} \mathrm{d}u \rho_{\mathrm{h}}(u) = 1, \tag{2.39}$$

as one easily derives from (2.10) to (2.14). Using (2.17), we may then derive a non-linear integral equation for the distribution of holes

$$\begin{aligned} \rho_{\mathrm{h}}(u) = {} & \frac{1}{L} (\psi(\mathrm{i}(u - u_{\mathrm{h}}^{(1)})) + \psi(-\mathrm{i}(u - u_{\mathrm{h}}^{(1)}) + \psi(\mathrm{i}(u + u_{\mathrm{h}}^{(1)})) + \psi(-\mathrm{i}(u + u_{\mathrm{h}}^{(1)}))) \\ & + \frac{1}{L} \frac{\mathrm{d}}{\mathrm{d}u} \mathcal{I}(u) - \frac{1}{2\pi} \left( \psi\left(\frac{1}{2} + \mathrm{i}u\right) + \psi\left(\frac{1}{2} - \mathrm{i}u\right) \right) \\ & + \int_{-c}^{c} \frac{\mathrm{d}v}{2\pi} (\psi(\mathrm{i}(u - v)) + \psi(-\mathrm{i}(u - v))) \rho_{\mathrm{h}}(v), \end{aligned} \tag{2.40}$$

where the term $(1/L)(\mathrm{d}/\mathrm{d}u)\mathcal{I}(u)$ denotes the derivative of the last line in (2.17). The terms on the rhs of the first line of (2.40) are the contributions of the two large holes with rapidities $u_{\mathrm{h}}^{(1)}, u_{\mathrm{h}}^{(2)} = -u_{\mathrm{h}}^{(1)}$ (cf (2.11)), where we have also implicitly assumed $L \ll M$. Then the two rapid holes behave as[12] $u_{\mathrm{h}}^{(1,2)} \simeq \pm M/\sqrt{2}$, while the term $(1/L)(\mathrm{d}/\mathrm{d}u)\mathcal{I}(u)$ in (2.40) yields merely an additive $2 \log 2$; see the appendix for a discussion of this point.

[11] We have included a factor of $\mathrm{e}^{-t/2}$ in this definition for convenience. For all other Fourier transformed quantities in this paper, in particular all kernels $\hat{K}$, we do not include such a factor.

[12] Extensive numerical studies indicate that for the ground state at large $M$ all charges $q_i$ in (2.5) are small except $q_2$. Then one finds from $t(u) = 0$ and (2.7) $u_{\mathrm{h}}^{(1,2)} \simeq \pm M/\sqrt{2}$. See also [3].

The four terms on the rhs of the first line of (2.40) thus behave like $4\log M/\sqrt{2}$. Using (1.7) we thus derive a *linear* integral equation

$$\rho_{\mathrm{h}}(u)=\frac{2}{\pi j}-\frac{1}{2\pi}\left(\psi\left(\frac{1}{2}+\mathrm{i}u\right)+\psi\left(\frac{1}{2}-\mathrm{i}u\right)\right)+\int_{-c}^{c}\frac{\mathrm{d}v}{2\pi}(\psi(\mathrm{i}(u-v)) \\ +\psi(-\mathrm{i}(u-v)))\rho_{\mathrm{h}}(v). \qquad (2.41)$$

One then finds the generalized one-loop scaling function (cf (1.8)) from (2.21):

$$\frac{\gamma_1(j)}{\log M}=8+2j\int_{-c}^{c}\mathrm{d}u\rho_{\mathrm{h}}(u)\left(\psi\left(\frac{1}{2}+\mathrm{i}u\right)+\psi\left(\frac{1}{2}-\mathrm{i}u\right)-2\psi(1)\right). \qquad (2.42)$$

In order to easily generate the series expansion of (2.42) in powers of $j$, defined in (1.7), it is useful to rescale $u$ and define

$$\bar{u}=\frac{u}{c} \qquad \text{and} \qquad \bar{\rho}_{\mathrm{h}}(\bar{u})=jc\rho_{\mathrm{h}}(u). \qquad (2.43)$$

Defining the non-singular kernel

$$K(\bar{u},\bar{v})=\frac{c}{2\pi}\left(\psi(\mathrm{i}c(\bar{u}-\bar{v}))+\psi(-\mathrm{i}c(\bar{u}-\bar{v}))-\psi\left(\frac{1}{2}+ic\bar{u}\right)-\psi\left(\frac{1}{2}-ic\bar{u}\right)\right), \qquad (2.44)$$

the integral equation (2.41) becomes

$$\bar{\rho}_{\mathrm{h}}(\bar{u})=\frac{2}{\pi}c+\int_{-1}^{1}\mathrm{d}\bar{v}K(\bar{u},\bar{v})\bar{\rho}_{\mathrm{h}}(\bar{v}). \qquad (2.45)$$

It is of Fredholm type and may be immediately expanded in the small parameter $c$ and iteratively solved as a power series in $c$. The relation to the parameter $j$ is then determined through the normalization condition in (2.39) which becomes

$$j=\int_{-1}^{1}\mathrm{d}v\bar{\rho}_{\mathrm{h}}(\bar{u}). \qquad (2.46)$$

This yields $j$ as a series in $c$. The generalized one-loop scaling function (2.42) becomes

$$\frac{\gamma_1(j)}{\log M}=8+2\int_{-1}^{1}\mathrm{d}\bar{u}\bar{\rho}_{\mathrm{h}}(\bar{u})\left(\psi\left(\frac{1}{2}+\mathrm{i}c\bar{u}\right)+\psi\left(\frac{1}{2}-\mathrm{i}c\bar{u}\right)-2\psi(1)\right). \qquad (2.47)$$

This yields the one-loop scaling function as a series in $c$. Inverting the series (2.46) and substituting into the expansion of (2.47) gives the desired series of the scaling function in terms of $j$. It starts out as

$$\frac{\gamma_1(j)}{\log M}=8-8j\log 2+\frac{7}{12}j^3\pi^2\zeta(3)-\frac{7}{6}j^4\pi^2\log 2\zeta(3) \\ +2j^5\left(\tfrac{7}{8}\pi^2\log^2 2\zeta(3)-\tfrac{31}{640}\pi^4\zeta(5)\right)+\mathcal{O}(j^6). \qquad (2.48)$$

Note that by analytic continuation the density of the holes is related to $\sigma(u)$ via

$$j\rho_{\mathrm{h}}(u)=\frac{2}{\pi}-8\sigma(u) \qquad u\in(-c,c), \qquad (2.49)$$

which may be rewritten as

$$j\rho_{\rm h}(u) = \frac{2}{\pi} - \frac{8}{\pi}\int_0^\infty {\rm d}t\hat\sigma(t){\rm e}^{t/2}\cos tu. \tag{2.50}$$

The preceding derivation proceeds from the counting function; cf (2.39). We will closely follow this procedure in section 3, where we will treat the higher loop case. It should be noted, however, that our solution (2.45), (2.46), (2.47) may also be immediately recovered by Fourier analyzing the results of the previous section 2.4. The reader should multiply (2.36) with ${\rm e}^{t/2}\cos tu$, integrate in $t$ over the positive real semi-axis and use the integral representation of the kernel (2.37). Subsequently rewriting $j$ in terms of $\hat\sigma(t)$ with the help of (1.16) and finally using the relation (2.50) it is straightforward to derive (2.41). We thus conclude that

$$a = c. \tag{2.51}$$

This equation tells us that the gap $[-a, a]$ in the distribution of magnon roots is densely filled by the (small) hole roots.

## 3. All-loop theory

### 3.1. The asymptotic non-linear integral equation (NLIE)

Let us now extend the one-loop results of the last section to the higher loop case. We will use the asymptotic Bethe ansatz for AdS/CFT, based on the $S$-matrix approach [38]. In the $\mathfrak{sl}(2)$ subsector the asymptotic all-loop Bethe equations [5, 7] read

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^M \frac{u_k-u_j-{\rm i}}{u_k-u_j+{\rm i}}\left(\frac{1-g^2/x_k^+x_j^-}{1-g^2/x_k^-x_j^+}\right)^2 {\rm e}^{2{\rm i}\theta(u_k,u_j)}. \tag{3.1}$$

We define the all-loop asymptotic counting function as

$$\begin{aligned} Z(u) = {}& {\rm i}L\log\frac{x({\rm i}/2+u)}{x({\rm i}/2-u)} + {\rm i}\sum_{k=1}^M \log\frac{{\rm i}+u-u_k}{{\rm i}-(u-u_k)} \\ & - 2{\rm i}\sum_{k=1}^M \log\frac{1+g^2/(x({\rm i}/2+u)x({\rm i}/2-u_k))}{1+g^2/(x({\rm i}/2-u)x({\rm i}/2+u_k))} + \sum_{k=1}^M \theta(u,u_k). \end{aligned} \tag{3.2}$$

As in the one-loop case, one finds the corresponding non-linear integral equation

$$\begin{aligned} Z(u) = {}& {\rm i}L\log\frac{x({\rm i}/2+u)}{x({\rm i}/2-u)} + \int_{-\infty}^{\infty}\frac{{\rm d}v}{2\pi}\phi'(u-v,1)Z(v) \\ & - \sum_{j=1}^L \phi(u-u_{\rm h}^{(j)},1) - \int_{-\infty}^{\infty}\frac{{\rm d}v}{\pi}\phi'(u-v,1){\rm Im}\log\left[1+(-1)^\delta\,{\rm e}^{{\rm i}Z(v+{\rm i}0)}\right] \\ & + \int_{-\infty}^{\infty}\frac{{\rm d}v}{2\pi}\left(2{\rm i}\frac{\rm d}{{\rm d}v}\log\frac{1+g^2/(x({\rm i}/2+u)x({\rm i}/2-v))}{1+g^2/(x({\rm i}/2-u)x({\rm i}/2+v))} - \theta(u,v)\right)Z(v) \\ & + \sum_{j=1}^L\left(2{\rm i}\log\frac{1+g^2/(x({\rm i}/2+u)x({\rm i}/2-u_{\rm h}^{(j)}))}{1+g^2/(x({\rm i}/2-u)x({\rm i}/2+u_{\rm h}^{(j)}))} - \theta(u,u_{\rm h}^{(j)})\right) \end{aligned}$$



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$$- \int_{-\infty}^{\infty} \frac{\mathrm{d}v}{\pi} \left( 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}v} \log \frac{1 + g^2/(x(\mathrm{i}/2 + u)x(\mathrm{i}/2 - v))}{1 + g^2/(x(\mathrm{i}/2 - u)x_(\mathrm{i}/2 + v))} - \theta(u,v) \right)$$
$$\times \operatorname{Im}\log\left[1 + (-1)^\delta \, \mathrm{e}^{\mathrm{i}Z(v+\mathrm{i}0)}\right]. \tag{3.3}$$

The counting function defined in (3.2) satisfies a relation similar to (2.22), but with the all-loop density on the rhs.

### 3.2. The NLIE in Fourier space

In Fourier $t$-space equation (3.3) becomes

$$\hat{Z}(t) = \frac{2\pi L \mathrm{e}^{t/2}}{\mathrm{i}t(\mathrm{e}^t - 1)} J_0(2gt) - \sum_{j=1}^{L} \frac{2\pi \cos\left(t u_{\mathrm{h}}^{(j)}\right)}{\mathrm{i}t(\mathrm{e}^t - 1)} - \frac{2}{\mathrm{e}^t - 1}\hat{\mathcal{L}}(t)$$
$$+ 8g^2 \frac{\mathrm{e}^{t/2}}{\mathrm{e}^t - 1} \int_0^\infty \mathrm{d}t' \mathrm{e}^{-t'/2} \hat{K}(2gt, 2gt') \left( t'\hat{\mathcal{L}}(t') + \frac{\pi}{\mathrm{i}} \sum_{j=1}^{L} \cos(t' u_{\mathrm{h}}^{(j)}) \right)$$
$$- 4g^2 \frac{\mathrm{e}^{t/2}}{\mathrm{e}^t - 1} \int_0^\infty \mathrm{d}t' \mathrm{e}^{-t'/2} t' \hat{K}(2gt, 2gt') \hat{Z}(t'), \tag{3.4}$$

where $\hat{\mathcal{L}}(t)$ denotes the Fourier transform of the 'Im log' term. Note that the $\hat{Z}(t)$ has a first order pole at $t = 0$. This is in accordance with (3.2), since the Fourier transform of this expression must be understood in the principal value sense. Note that we have not made any approximations. Therefore (3.4) is still fully equivalent to the original set of discrete asymptotic equations, (3.1).

### 3.3. Large parameter integrals

Let us now investigate the effects of taking the large $M$ limit with $L \ll M$. It will be important to understand the large $M$ expansion of integrals of the form

$$f(M) = \int_0^\infty \mathrm{d}x h(x) \sin(u(M)x), \tag{3.5}$$

where $h(x)$ is a smooth integrable function on $[0, \infty)$ and $u(M) \to \infty$ when $M \to \infty$. We first note that because of the relation to the Fourier transform (Plancherel's theorem) $\lim_{M\to\infty} f(M) = 0$. Since $f(M)$ is meromorphic and vanishes at infinity we have

$$f(M) = \sum_{j=0}^{\infty} \frac{c_j}{u(M)^{1+j}}. \tag{3.6}$$

To find $c_0$ it is sufficient to note that

$$c_0 = \lim_{M\to\infty} u(M) f(M) = \lim_{M\to\infty} \int_0^\infty \mathrm{d}x h(x) \left( -\frac{\mathrm{d}}{\mathrm{d}x} \cos(u(M)x) \right) = h(0), \tag{3.7}$$

since the integral after a partial integration vanishes again. By subsequent integrations by parts one finds that

$$c_n = \lim_{M\to\infty} u(M)^{n+1} \left( f(M) - \sum_{j=1}^{n-1} \frac{c_j}{u(M)^{1+j}} \right) = (-1)^{n/2} h^{(n)}(0) \qquad \text{for even } n. \tag{3.8}$$

The odd $c_n$ coefficients vanish, as follows from (3.5).

### 3.4. The leading order equation

To derive from (3.4) an equation reproducing the leading contribution to the scaling function in the limit where $M \to \infty$ and $L$ is kept fixed, it is sufficient to observe, on the basis of the results of the previous subsection, that upon iterating (3.4) only terms of the form

$$\frac{2\pi \mathrm{e}^{t/2}}{\mathrm{i}t(\mathrm{e}^t - 1)} - \frac{2\pi \cos\left(t u_{\mathrm{h}}^{(1,2)}\right)}{\mathrm{i}t(\mathrm{e}^t - 1)}, \tag{3.9}$$

where $u_{\mathrm{h}}^{(1,2)} \simeq \pm\sqrt{\frac{1}{2}q_2} \simeq \pm M/\sqrt{2}$ represent the universal holes, will give the leading (logarithmic) contribution. This is because we have

$$u_{\mathrm{h}}^{(j)} \simeq 0 \qquad j = 3, \ldots, L \tag{3.10}$$

at leading order, and the terms involving $\hat{\mathcal{L}}(t)$ do not contribute at this order; see the appendix. Thus the leading all-loop equation reads

$$\hat{Z}(t) = \frac{4\pi \mathrm{e}^{t/2}}{\mathrm{i}t(\mathrm{e}^t - 1)} - \frac{4\pi \cos(t u_{\mathrm{h}}^{(1)})}{\mathrm{i}t(\mathrm{e}^t - 1)} - 4g^2 \frac{\mathrm{e}^{t/2}}{\mathrm{e}^t - 1} \int_0^\infty \mathrm{d}t' \mathrm{e}^{-t'/2} t' \hat{K}(2gt, 2gt') \hat{Z}(t'). \tag{3.11}$$

Upon subtracting the one-loop part of this equation

$$\hat{Z}(t) = \hat{Z}_0(t) + \delta \hat{Z}_{\mathrm{BES}}(t) \tag{3.12}$$

and identifying $\delta \hat{Z}(t)$ with the fluctuation density

$$\delta \hat{Z}_{\mathrm{BES}}(t) = 16\pi \mathrm{i} g^2 \mathrm{e}^{t/2} \frac{\hat{\sigma}_{\mathrm{BES}}(t)}{t} \log(M) \tag{3.13}$$

one rederives the equation of [7]:

$$\hat{\sigma}_{\mathrm{BES}}(t) = \frac{t}{\mathrm{e}^t - 1} \left( \hat{K}(2gt, 0) - 4g^2 \int_0^\infty \mathrm{d}t' \hat{K}(2gt, 2gt') \hat{\sigma}_{\mathrm{BES}}(t') \right). \tag{3.14}$$

### 3.5. Subleading corrections to the twist operator dimensions

The large $M$ expansion of the anomalous dimensions of twist operators is expected to have the following form:

$$\gamma = f(g) \log M + f_{\mathrm{sl}}(g, L) + \mathcal{O}\left( \frac{1}{(\log M)^2} \right), \tag{3.15}$$

where $f_{\mathrm{sl}}(g, L)$ denotes the subleading effects of $\mathcal{O}(M^0)$. These are easily obtained from (3.4), and we may compute $f_{\mathrm{sl}}(g, L)$ to arbitrary order of perturbation theory:

$$\begin{aligned} f_{\mathrm{sl}}(g, L) = {} & (\gamma - (L-2)\log 2) f(g) - 8(7 - 2L)\zeta(3) g^4 \\ & + 8\left( \frac{4 - L}{3} \pi^2 \zeta(3) + (62 - 21L)\zeta(5) \right) g^6 \\ & - \frac{8}{15}((13 - 3L)\pi^4 \zeta(3) + 5(32 - 11L)\pi^2 \zeta(5) + 75(127 - 46L)\zeta(7)) g^8 \\ & \pm \cdots. \end{aligned} \tag{3.16}$$

Notice that the ‘universality’, i.e. independence of $L$ of the scaling function $f(g)$, is lost when one computes these $\mathcal{O}(M^0)$ terms. They contain $L$-independent terms and terms linear in $L$.



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## 4. The generalized scaling function

### 4.1. Derivation

Let us now finally treat the novel scaling limit (1.7), i.e. we consider the limit $L, M \to \infty$ with $j = L/\log M$ kept fixed. In this limit, in contradistinction to section 3.4, also the $L-2$ remaining holes contribute. Although individual hole terms separately do not develop logarithmic behavior in $M$, their collective contribution will be proportional to $L = j\log M$. Furthermore, in this limit all terms involving $\hat{\mathcal{L}}(t)$ can be dropped; see the appendix. Thus (3.4) for the counting function $\hat{Z}(t)$ in Fourier space linearizes in this limit to the form

$$\hat{Z}(t) = \frac{2\pi L \mathrm{e}^{t/2}}{\mathrm{i}t(\mathrm{e}^t-1)} J_0(2gt) - \sum_{j=1}^{L} \frac{2\pi\cos\left(tu_{\mathrm{h}}^{(j)}\right)}{\mathrm{i}t(\mathrm{e}^t-1)} + 8\pi\, g^2 \frac{\mathrm{e}^{t/2}}{\mathrm{i}(\mathrm{e}^t-1)} \sum_{j=1}^{L-2} \int_0^\infty \mathrm{d}t' \mathrm{e}^{-t'/2} \hat{K}(2gt, 2gt') \cos(t' u_{\mathrm{h}}^{(j)}) - 4g^2 \frac{\mathrm{e}^{t/2}}{\mathrm{e}^t-1} \int_0^\infty \mathrm{d}t' \mathrm{e}^{-t'/2} t' \hat{K}(2gt, 2gt') \hat{Z}(t'). \tag{4.1}$$

Note that in the above formula only quantum corrections to $u_{\mathrm{h}}^{j}$ for $j = 3, \ldots, L$ need to be taken into account, since the corrections to the universal holes are, upon the iteration, subleading. Like in section 3.4 we strip off the one-loop part by defining

$$\hat{Z}(t) = \hat{Z}_0(t) + \delta\hat{Z}(t). \tag{4.2}$$

We relate $\delta\hat{Z}(t)$ to the fluctuation density $\hat{\sigma}(t)$ through

$$\delta\hat{Z}(t) = 16\pi \mathrm{i} \mathrm{e}^{t/2} \frac{\hat{\sigma}(t)}{t} \log M, \tag{4.3}$$

and derive to the desired order

$$\hat{\sigma}(t) = \frac{t}{\mathrm{e}^t-1}\left[ g^2 \hat{K}(2gt, 0) - \frac{j}{8}\frac{J_0(2gt)}{t} + \frac{1}{8\log M}\sum_{j=3}^{L} \frac{\mathrm{e}^{-t/2}\cos(tu_{\mathrm{h}}^{(j)})}{t} - \frac{g^2}{2}\frac{1}{\log M}\sum_{j=3}^{L}\int_0^\infty \mathrm{d}t' \hat{K}(2gt, 2gt') \mathrm{e}^{-t'/2}\cos(t' u_{\mathrm{h}}^{(j)}) - 4g^2 \int_0^\infty \mathrm{d}t' \hat{K}(2gt, 2gt') \hat{\sigma}(t') \right]. \tag{4.4}$$

The corresponding anomalous dimension can be easily shown to be given by

$$\gamma = 8g^2 \log M \left( 1 - \frac{1}{\log M} \sum_{j=3}^{L} \int_0^\infty \mathrm{d}t \frac{J_1(2gt)}{2gt} \mathrm{e}^{-t/2} \cos(tu_{\mathrm{h}}^{(j)}) - 8\int_0^\infty \mathrm{d}t \frac{J_1(2gt)}{2gt} \hat{\sigma}(t) \right). \tag{4.5}$$



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The distribution of the small holes is found from

$$Z(u_{\rm h}^j) = \pi(2n_{\rm h}^j + \delta - 1), \tag{4.6}$$

which in Fourier space reads

$$\frac{\rm i}{\pi}\int_0^\infty \sin(tu_{\rm h}^j)\hat{Z}(t) = \pi(2n_{\rm h}^j + \delta - 1). \tag{4.7}$$

Plugging (4.2) into (4.7) and observing that (see section 3.3)

$$\begin{aligned} F'(x,y) \equiv \int_0^\infty {\rm d}t \cos tx \frac{{\rm e}^{t/2} - \cos ty}{{\rm e}^t - 1} \\ = \tfrac{1}{4}(\psi({\rm i}(x-y)) + \psi(-{\rm i}(x-y)) + \psi({\rm i}(x+y)) + \psi(-{\rm i}(x+y)) \\ - 2\psi\left(\tfrac{1}{2} - {\rm i}x\right) - 2\psi\left(\tfrac{1}{2} + {\rm i}x\right)), \end{aligned} \tag{4.8}$$

one easily derives from (4.7)

$$2\pi n_{\rm h}^{(k)} = 4F(u_{\rm h}^{(k)}, u_{\rm h}^{(1)}) - 16\log M \int_0^\infty {\rm d}t \frac{\hat{\sigma}(t)}{t}{\rm e}^{t/2}\sin(tu_{\rm h}^{(k)}). \tag{4.9}$$

Introducing the density of holes $\rho_{\rm h}(u)$ it follows from (4.9) that

$$j\rho_{\rm h}(u) = \frac{2}{\pi\log M}F'(u, u_{\rm h}^{(1)}) - \frac{8}{\pi}\int_0^\infty {\rm d}t\hat{\sigma}(t){\rm e}^{t/2}\cos(tu). \tag{4.10}$$

Note that $(2/\pi)(1/M)F'(u, u_{\rm h}^{(1)})$ is at large values of $M$ essentially the Korchemsky density $\rho_0(u)$, i.e. (2.25) after scaling back $u = M\bar{u}$, $\rho_0(u) = 1/M\bar{\rho}_0(\bar{u})$, up to small corrections at the boundaries of the distribution of the roots. Since the small holes occupy a finite interval $(-a, a)$ one can safely take the large $M$ limit[13]

$$F'(u, u_{\rm h}^{(1)}) = \log M + \mathcal{O}(M^0) \qquad\qquad u \in (-a, a). \tag{4.11}$$

After replacing the sum in (4.4) by an integral and using the above density we find

$$\begin{aligned} \hat{\sigma}(t) = \frac{t}{{\rm e}^t - 1}\Bigg[-\frac{j}{8t}J_0(2gt) + \hat{K}_{\rm h}(t,0;a) - 4\int_0^\infty {\rm d}t'\hat{K}_{\rm h}(t,t';a)\hat{\sigma}(t') \\ + g^2\hat{K}(2gt,0) - 4g^2\int_0^\infty {\rm d}t'\hat{K}(2gt,2gt')\hat{\sigma}(t') \\ - 4g^2\int_0^\infty {\rm d}t' t'\hat{K}(2gt,2gt')\left(\hat{K}_{\rm h}(t',0;a) - 4\int_0^\infty {\rm d}t''\hat{K}_{\rm h}(t',t'')\hat{\sigma}(t'')\right)\Bigg], \end{aligned} \tag{4.12}$$

where $\hat{K}_{\rm h}(t,t';a)$ is the one-loop kernel given in (1.15). The endpoints can be obtained from the normalization condition

$$\int_{-a}^a {\rm d}u\rho_{\rm h}(u) = 1, \tag{4.13}$$

which implies (1.16). Inserting (1.16) into (4.12) we find the final integral equation (1.9) announced in the introduction. Likewise, the anomalous dimension (4.5) may be re-

[13] The magnon density is related to $\hat{\sigma}(t)$ by a similar formula:

$$\rho_m(u) = \frac{2}{\pi}\frac{1}{M}F'(u, u_{\rm h}^{(1)}) - \frac{8\log M}{\pi M}\int_0^\infty {\rm d}t\,\hat{\sigma}(t){\rm e}^{t/2}\cos(tu).$$

expressed and simplified as

$$\gamma = 8g^2 \log M \left[1 - 8\int_0^\infty \mathrm{d}t \frac{J_1(2gt)}{2gt} t\hat{K}_{\mathrm{h}}(t,0;a) - 8\int_0^\infty \mathrm{d}t\, \frac{J_1(2gt)}{2gt}\left(\hat{\sigma}(t) - 4t\int_0^\infty \mathrm{d}t'\hat{K}_{\mathrm{h}}(t,t';a)\hat{\sigma}(t')\right)\right] = 16\log M\left(\hat{\sigma}(0) + \frac{j}{16}\right). \tag{4.14}$$

This concludes our derivation of the equations determining the generalized scaling function $f(g,j)$ in (1.8). Let us now apply them to obtain the first few terms in the double expansion of this function in powers of $g$ and $j$.

### 4.2. Weak coupling expansion

The equation (4.12) is solved iteratively with relative ease in a double-perturbative series in $g$ and the gap parameter $a$. As in the one-loop case in section 2 one then inverts (1.16) to obtain $a(j)$ as a power series in $j$. This then yields the fluctuation density $\hat{\sigma}(t)$ as a series in $g$ and $j$. It starts out as

$$\begin{aligned}\hat{\sigma}(t) = g^2\hat{\sigma}_{\mathrm{BES}}(t) + j\Bigg(&-\frac{1}{8}\frac{1}{\mathrm{e}^{t/2}+\mathrm{e}^t} + \frac{g^2}{8}\frac{t(t-4\log 2)}{\mathrm{e}^t-1} \\ &+ g^4\frac{t}{\mathrm{e}^t-1}\frac{1}{96}\left(-3t^3 - 4\pi^2 t + 16\pi^2\log 2 + 24t^2\log 2 + 96\zeta(3)\right) + \cdots\Bigg) \\ &+ j^2\times 0 + j^3\Bigg(-\frac{\pi^2}{1536}t^2\mathrm{e}^{-t}\mathrm{csch}(t/2) + \frac{g^2\pi^2}{384}\frac{t(14\zeta(3) - \pi^2 t\mathrm{e}^{-t/2})}{\mathrm{e}^t-1} \\ &+ \frac{g^4\pi^2}{2304}\frac{t}{\mathrm{e}^t-1}(3\pi^4 t - \pi^4 t\mathrm{e}^{-t/2} + 140\pi^2\zeta(3) - 42\zeta(3)t^2 - 2232\zeta(5)) + \ldots\Bigg) \\ &+ \cdots.\end{aligned} \tag{4.15}$$

The generalized scaling function at weak coupling is simply given via (1.17) by evaluating the fluctuation density at $t=0$. Let us define an infinite set of functions $\{f_n(g)\}$ as

$$f(g,j) = f(g) + \sum_{n=1}^{\infty} f_n(g) j^n. \tag{4.16}$$

The first one, $f_1(g)$, is

$$\begin{aligned}f_1(g) = -8g^2\log 2 + g^4\left(\frac{8}{3}\pi^2\log 2 + 16\zeta(3)\right) - g^6\left(\frac{88}{45}\pi^4\log 2 + \frac{8}{3}\pi^2\zeta(3) + 168\zeta(5)\right) \\ + g^8\left(\frac{584}{315}\pi^6\log 2 + \frac{8}{5}\pi^4\zeta(3) + 64\log 2\zeta(3)^2 + \frac{88}{3}\pi^2\zeta(5) + 1840\zeta(7)\right) + \cdots.\end{aligned} \tag{4.17}$$

Note that $f_1(g)$ is special as it can be obtained from (3.16) by keeping only terms proportional to $L$. To this order the hole momenta are set to zero. Only at orders higher than linear in $j$ does one need to take into account the 'dynamics' of the holes.

We then find for $f_1(g), \ldots, f_4(g)$

$$f_1(g) = -f(g)\log 2 + 16g^4\zeta(3) - g^6(\tfrac{8}{3}\pi^2\zeta(3) + 168\zeta(5)) + g^8(\tfrac{8}{5}\pi^4\zeta(3) + \tfrac{88}{3}\pi^2\zeta(5) + 1840\zeta(7)) + \cdots, \tag{4.18}$$

$$f_2(g) = 0, \tag{4.19}$$

$$\begin{aligned} f_3(g) = {} & \tfrac{7}{12}g^2\pi^2\zeta(3) + g^4(\tfrac{35}{36}\pi^4\zeta(3) - \tfrac{31}{2}\pi^2\zeta(5)) \\ & + g^6(-\tfrac{73}{540}\pi^6\zeta(3) - \tfrac{155}{6}\pi^4\zeta(5) + \tfrac{635}{2}\pi^2\zeta(7)) \\ & + g^8(\tfrac{7}{108}\pi^8\zeta(3) + \tfrac{182}{3}\pi^2\zeta(3)^3 + \tfrac{28}{15}\pi^6\zeta(5) + \tfrac{3175}{6}\pi^4\zeta(7) - \tfrac{17885}{3}\pi^2\zeta(9)) \\ & + \cdots, \end{aligned} \tag{4.20}$$

$$\begin{aligned} f_4(g) = {} & -\tfrac{7}{6}g^2\pi^2\log 2\zeta(3) + g^4\left(-\tfrac{77}{18}\pi^4\log 2\zeta(3) + \tfrac{49}{6}\pi^2\zeta(3)^2 + 31\pi^2\log 2\zeta(5)\right) \\ & + g^6\left(-\tfrac{767}{270}\pi^6\log 2\zeta(3) + \tfrac{385}{18}\pi^4\zeta(3)^2 + \tfrac{341}{3}\pi^4\log 2\zeta(5)\right. \\ & \left. - \tfrac{651}{2}\pi^2\zeta(3)\zeta(5) - 635\pi^2\log 2\zeta(7)\right) \\ & + g^8\left(\tfrac{307}{270}\pi^8\log 2\zeta(3) + \tfrac{91}{15}\pi^6\zeta(3)^2 - 252\pi^2\log 2\zeta(3)^3 + \tfrac{1184}{15}\pi^6\log 2\zeta(5)\right. \\ & - \tfrac{15011}{18}\pi^4\zeta(3)\zeta(5) + 2883\pi^2\zeta(5)^2 - \tfrac{6985}{3}\pi^4\log 2\zeta(7) \\ & \left. + \tfrac{17780}{3}\pi^2\zeta(3)\zeta(7) + \tfrac{35770}{3}\pi^2\log 2\zeta(9)\right) + \cdots. \end{aligned} \tag{4.21}$$

At fixed $j$, we observe a constant degree of transcendentality [39] of all terms contributing to a given order of perturbation theory in the coupling $g$. Interestingly, the converse is not true, as may already be seen from the one-loop result (2.48).

As was announced earlier, the function $f_2(g)$ is identically zero, indicating that all terms of order $j^2$ in the $j$-expansion of $f(g, j)$ are absent to all orders in the coupling constant $g$. This is easily proven directly from our equations. Some potentially related very interesting observations at strong coupling were made in [23]. Roiban and Tseytlin found some intriguing evidence that terms of the form $j^2 \log^k j$ might upon resummation indeed result in a vanishing $j^2$ contribution; cf also the discussion in the introduction.

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## Appendix. The non-linear term

In this appendix we will discuss the integrals involving the non-linear term. For simplicity we will confine ourselves to the one-loop case, where it is sufficient to consider

$$\mathcal{I}(u) = \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} \frac{\mathrm{d}v}{\pi} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}u} \log \frac{\Gamma(-\mathrm{i}(u-v))}{\Gamma(\mathrm{i}(u-v))} \mathrm{Im}\log\left[1+(-1)^{\delta}\mathrm{e}^{\mathrm{i}Z(v+\mathrm{i}0)}\right]. \tag{A.1}$$

We first note that the function

$$\mathcal{L}(u) = \mathrm{Im}\log\left[1+(-1)^{\delta}\mathrm{e}^{\mathrm{i}Z(u+\mathrm{i}0)}\right] \tag{A.2}$$

is smooth apart from at a finite number of points, namely when $u$ is equal to the magnon or the hole rapidity. A closer inspection reveals that

$$\mathcal{L}(u_i - \epsilon) = \pi, \qquad \mathcal{L}(u_i + \epsilon) = -\pi, \tag{A.3}$$

where $u_i$ denotes either a hole or a magnon rapidity. We will assume that the small holes and the magnons are densely distributed along the real axis, as this is the case for the limits discussed in this paper. It is easy to convince oneself that the integral (A.1) gets the dominant contribution from $(-\alpha, -M/2) \cup (M/2, \alpha)$. Because the small roots and magnons are, at large values of $M$, densely and symmetrically distributed on $(-M/2, M/2)$ this part of the integral contributes starting at $\mathcal{O}(1/M^2)$ only. Assuming $v \in (-\alpha, -M/2) \cup (\frac{M}{2}, \alpha)$ we may expand the integrand in a power series in $u$. Because of the antisymmetry of the counting function only odd powers of $u$ survive the integration. Thus we may write

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}u} \log \frac{\Gamma(-\mathrm{i}(u-v))}{\Gamma(\mathrm{i}(u-v))} &= \mathrm{i}(\psi_1(-\mathrm{i}v) - \psi_1(\mathrm{i}v))u - \frac{\mathrm{i}}{6}(\psi_3(-\mathrm{i}v) - \psi_3(\mathrm{i}v))u^3 \\ &\quad + \mathcal{O}(u^5) + \text{even terms in } v.\end{aligned} \tag{A.4}$$

On the other hand from the definition of the counting function we have

$$\mathcal{L}(v) = -\frac{L+2M}{2v} + \mathrm{O}\left(\frac{1}{v^3}\right) \qquad v > \frac{M}{\sqrt{2}}. \tag{A.5}$$

Plugging (A.4) and (A.5) into (A.1) we find

$$\mathcal{I}(u) = \xi u + \mathcal{O}\left(\frac{u^3}{M^2}\right). \tag{A.6}$$

To fix the constant $\xi$ it is necessary to extend the expansion in (A.5) to the whole interval $v \in (M/2, \infty)$. However there is a much simpler method. Since the above discussion is not sensitive to the value of $L$ we may set $L = 2$. Then we may compute the corresponding anomalous dimension plugging (2.17) together with (A.1) into (2.16). Comparison with the exact one-loop result $\gamma_1 = 8S_1(M)$ fixes $\xi$ to be

$$\xi = 2\log 2. \tag{A.7}$$

Numerically we have checked that the expansion (A.6) breaks only in a small neighborhood of $\pm M/2$. This suggests that the radius of convergence of (A.6) lies close to the edge of the magnon distribution.

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