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LETTER

Boosting nearest-neighbour to long-range integrable spin chains

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Abstract. We present an integrability-preserving recursion relation for the explicit construction of long-range spin chain Hamiltonians. These chains are generalizations of the Haldane–Shastry and Inozemtsev models and they play an important role in recent advances in string/gauge duality. The method is based on arbitrary nearest-neighbour integrable spin chains and it sheds light on the moduli space of deformation parameters. We also derive the closed chain asymptotic Bethe equations.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz), spin chains, ladders and planes (theory)

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1. Introduction and overview

Integrable spin chains constitute a fascinating topic of theoretical physics: their Hilbert space grows exponentially with the length of the chain and the Hamiltonian eigenstates are usually in a linear combination of almost all states in a canonical basis. Nevertheless, owing to integrability, the eigenstates can be determined efficiently by solving a system of algebraic equations, the so-called Bethe equations [1], whose number of unknowns typically grows linearly with the length.

The best-studied integrable spin chains are the *nearest-neighbour* chains whose Hamiltonians act on pairs of spins at adjacent sites. The prime example in this class is the Heisenberg model [2]. The only widely known examples of spin chains with interactions of well-separated spin sites, so-called *long-range* chains, are the Haldane–Shastry and the Inozemtsev chains [3, 4].

The discovery and investigation of integrable structures in planar maximally supersymmetric gauge theory in four spacetime dimensions [5]–[7] (see [8] for a review) introduced a novel exciting long-range chain. Its interactions are more general than those on which the Haldane–Shastry and Inozemtsev models are based: they involve more than two spins at a time. A subsequent study [9] has revealed a large class of long-range models. However in all of these models complete integrability was merely shown to be plausible and it has not yet been proven.

In this letter we provide a proof of existence for a very large class of *integrable* long-range spin chains including those proposed in [9]. Specifically, we present a recursion relation for explicitly constructing long-range Hamiltonians which manifestly preserves integrability.

2. Integrable long-range chains

We start by reviewing the notion of perturbatively long-range integrable spin chain models [6].

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General definition

A perturbatively long-range integrable spin chain is defined as a deformation of an infinitely long homogeneous nearest-neighbour integrable spin chain. This means that the model has a set of local homogeneous commuting charges $\mathcal{Q}_r(\lambda)$, $r \geq 2$,

$$[\mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda)] = 0, \tag{1}$$

which are deformations of the commuting charges $\mathcal{Q}_r^{\mathrm{NN}}$ of the nearest-neighbour model at $\lambda = 0$. For simplicity we shall identify the Hamiltonian $\mathcal{H}$ with the lowest charge $\mathcal{Q}_2$. The charges are composed from local operators

$$\mathcal{Q}_r(\lambda) = \sum_k c_{r,k}(\lambda)\, \mathcal{L}_k, \tag{2}$$

where the $\mathcal{L}_k$ form a basis of operators acting *locally* and *homogeneously* on the chain

$$\mathcal{L}_k := \sum_a \mathcal{L}_k(a). \tag{3}$$

Here and in the following $\mathcal{L}_k(a)$ is some operator which acts on several consecutive spin sites starting with site a. The number of interacting sites is called the range $[\mathcal{L}_k]$ of the operator $\mathcal{L}_k$. The coefficients $c_{r,k}(\lambda)$ are defined as series expansions around $\lambda = 0$ such that the range of $\mathcal{Q}_r$ grows at most by one step per order in λ, i.e.

$$c_{r,k}(\lambda) = \mathcal{O}(\lambda^{[\mathcal{L}_k]-r}). \tag{4}$$

In other words, $\mathcal{Q}_r(\lambda)$ at order λ^ℓ must consist of operators $\mathcal{L}_k$ of range $[\mathcal{L}_k]$ at most $r+\ell$.

Fundamental $\mathfrak{gl}(N)$ chain

The existence of interesting non-trivial integrable long-range models was suggested by the construction in [6]. In fact, the hyperbolic Inozemtsev chain [4] can be understood as one particular example [10]. However, a general survey [9] of long-range chains with $\mathfrak{gl}(N)$ symmetry and spins transforming in the fundamental representation has revealed a much larger moduli space: the starting point was the ansatz (2) and (4) for the charges $\mathcal{Q}_r(\lambda)$, $r = 2, 3$, up to order $\mathcal{O}(\lambda^4)$. The coefficients $c_{r,k}(\lambda)$ were then constrained by demanding commutativity (1). The resulting first few charges at the leading few orders read

$$\begin{aligned}
\mathcal{Q}_2 &= [1] - [2,1] + \alpha_3(\lambda)(-3[1] + 4[2,1] - [3,2,1]) + \mathcal{O}(\lambda^2),\\
\mathcal{Q}_3 &= \frac{\mathrm{i}}{2}([3,1,2] - [2,3,1]) + \frac{\mathrm{i}}{2}\alpha_3(\lambda)(6[2,3,1] - 6[3,1,2] + [4,1,3,2]\\
&\qquad + [4,2,1,3] - [2,4,3,1] - [3,2,4,1]) + \mathcal{O}(\lambda^2),\\
\mathcal{Q}_4 &= \frac{1}{3}(-[1] + 2[2,1] - [3,2,1] + [2,3,4,1] - [2,4,1,3] - [3,1,4,2] + [4,1,2,3]) + \mathcal{O}(\lambda).
\end{aligned} \tag{5}$$

The symbols $\mathcal{L}_k = [\cdots]$ represent local homogeneous interactions such that $\mathcal{L}_k(a)$ in (3) is the indicated permutation of consecutive spins; cf [11]. For example, $[2,1]$ represents the nearest-neighbour permutation $\sum_a P_{a,a+1}$. The commuting charges turned out to depend on a set of parameters α_r, $\beta_{r,s}$, $\gamma_{r,s}$ and ϵ_k whose individual roles can be identified in the resulting Bethe ansatz. The asymptotic Bethe equations for this model [9] are a special case of the form presented at the end of this letter.

Clearly this construction is sufficient neither to prove integrability at a certain perturbative order nor to show that the deformation can be continued to higher orders without having to spoil integrability. The first problem was overcome in [11] by showing that $\mathfrak{gl}(N)$ symmetry extends to a Yangian algebra. A perturbative Yangian generator $\mathcal{Y}(\lambda)$ was constructed, shown to commute with $\mathcal{Q}_2(\lambda)$ and to satisfy the Serre relations of the Yangian.

It is the aim of the present letter to overcome both problems, namely to show that the long-range integrable model can be constructed to all orders (and how).

3. General construction

In the following we shall present the construction of long-range integrable spin chains from an arbitrary conventional integrable spin chain.

Generating equation

Consider a one-parameter family of charges $\mathcal{Q}_r(\lambda)$ which obeys the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathcal{Q}_r(\lambda) = \mathrm{i}[\mathcal{X}(\lambda), \mathcal{Q}_r(\lambda)]. \tag{6}$$

Here $\mathcal{X}$ is some operator with well-defined commutation relations with $\mathcal{Q}_r$ at all λ. The differential equation guarantees that the algebra of the $\mathcal{Q}_r$ is independent of λ

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,[\mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda)] = \mathrm{i}[\mathcal{X}(\lambda), [\mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda)]]. \tag{7}$$

In particular, if the algebra of charges is Abelian anywhere, e.g. at $\lambda = 0$, it is Abelian everywhere

$$[\mathcal{Q}_r(0), \mathcal{Q}_s(0)] = 0 \quad \Longrightarrow \quad [\mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda)] = 0. \tag{8}$$

Given a conventional nearest-neighbour integrable system $\mathcal{Q}_r^{\mathrm{NN}}$ and some operator $\mathcal{X}(\lambda)$, the differential equation (6) defines an integrable deformation $\mathcal{Q}_r(\lambda)$ of $\mathcal{Q}_r(0) = \mathcal{Q}_r^{\mathrm{NN}}$, at least as a formal series in λ.

The integrable charges $\mathcal{Q}_r(\lambda)$ of the long-range model discussed above are *local* and *homogeneous*. Consequently, if (6) is to describe the above model we have to make sure that the equation violates neither of these properties. More explicitly, $[\mathcal{X}, \mathcal{Q}_r]$ must be local and homogeneous for all λ. In the following we shall discuss suitable choices for $\mathcal{X}$.

Local operators

Obviously, the commutator of any two local operators is again local. Thus any local operator is admissible as a deformation and we can set

$$\mathcal{X}(\lambda) = \sum_k \epsilon_k(\lambda)\,\mathcal{L}_k + \cdots, \tag{9}$$

where the $\mathcal{L}_k$ form a basis of local operators. This deformation changes eigenstates only locally and has no impact on the spectrum. The ϵ's are thus unphysical. Note that for a correct enumeration of deformation degrees of freedom one has to take into account that the local charges $\mathcal{Q}_r(\lambda)$ generate trivial deformations.

Boost charges

Consider *boost operators* defined by

$$\mathcal{B}[\mathcal{L}_k] := \sum_a a\,\mathcal{L}_k(a). \tag{10}$$

In contradistinction to the $\mathcal{L}_k$ defined in (3) a boost acts locally, but inhomogeneously along the chain. Recall [12] that the boost $\mathcal{B}[\mathcal{Q}_2]$ can be used to generate all the higher charges of a conventional integrable system through the recursive relation $\mathrm{i}[\mathcal{B}[\mathcal{Q}_2], \mathcal{Q}_r] \simeq -r\,\mathcal{Q}_{r+1}$.

In general, the commutator of some boost operator with some local operator is again a boost operator, $[\mathcal{B}[\mathcal{L}_k], \mathcal{L}_l] = \mathcal{B}[\mathcal{L}_m]$. However, if the underlying local operators commute, the commutator becomes homogeneous

$$[\mathcal{L}_k, \mathcal{L}_l] = 0 \quad \Longrightarrow \quad [\mathcal{B}[\mathcal{L}_k], \mathcal{L}_l] = \mathcal{L}_m. \tag{11}$$

Consequently the boosts of commuting charges are admissible as deformations

$$\mathcal{X}(\lambda) = \cdots + \sum_{r=3}^{\infty} \tilde{\alpha}_r(\lambda)\,\mathcal{B}[\mathcal{Q}_r(\lambda)] + \cdots. \tag{12}$$

Note that the definition of $\mathcal{B}[\mathcal{L}_k]$ is ambiguous modulo local operators $\mathcal{L}_l$. This is not troublesome because we have already accounted for all local operators in (9), i.e. the unphysical parameters ϵ_k can absorb the ambiguity.

Bi-local charges

Next, consider bi-local operators

$$[\mathcal{L}_k|\mathcal{L}_l] := \sum_{a\le b} \tfrac{1}{2}(1 - \tfrac{1}{2}\delta_{a,b})\{\mathcal{L}_k(a), \mathcal{L}_l(b)\}. \tag{13}$$

The commutation properties of bi-local operators $[\mathcal{L}_k|\mathcal{L}_l]$ are reminiscent of those of boost operators discussed above. A commutator of a bi-local with a local operator yields a bi-local operator in general. However, for commuting charges it remains local

$$[\mathcal{L}_{k,l}, \mathcal{L}_m] = 0 \quad \Longrightarrow \quad [[\mathcal{L}_k|\mathcal{L}_l], \mathcal{L}_m] = \mathcal{L}_n. \tag{14}$$

Therefore bi-local combinations of the charges are admissible as deformations

$$\mathcal{X}(\lambda) = \cdots + \sum_{s>r=2}^{\infty} \tilde{\beta}_{r,s}(\lambda)\,[\mathcal{Q}_r(\lambda)|\mathcal{Q}_s(\lambda)]. \tag{15}$$

Note that also bi-local operators are uniquely defined only modulo local operators which is acceptable due to (9).

Shifts

The above operators exhaust all admissible operators that we can think of. In fact they almost agree with the proposed moduli space of the Bethe ansatz [9]. The only missing parameters correspond to taking linear combinations of the charges. We introduce them by adding another term to the equation (6) which obviously does not spoil integrability

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathcal{Q}_r(\lambda) = \mathrm{i}[\mathcal{X}(\lambda), \mathcal{Q}_r(\lambda)] + \sum_{s=2}^{\infty} \tilde{\gamma}_{r,s}(\lambda)\,\mathcal{Q}_s(\lambda). \tag{16}$$

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Yangian generators

The Yangian generators are deformed in the same way as the integrable charges

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{Y}(\lambda) = \mathrm{i}[\mathcal{X}(\lambda), \mathcal{Y}(\lambda)]. \tag{17}$$

This guarantees that the algebra among the Yangian generators and the integrable charges is the same for all values of λ. In particular it is the same as for the conventional integrable chain at $\lambda = 0$ in line with the results [11] on the Yangian of $\mathfrak{gl}(N)$ long-range chains.

4. Range

The above definition of long-range chains (4) sets the bound $r + \ell$ to the range of interactions in $\mathcal{Q}_r$ at $\mathcal{O}(\lambda^\ell)$. A superficial consideration of the range of commutators in (6) shows that each power $\tilde{\alpha}_r$ increases the range of a charge $\mathcal{Q}$ by $r-1$. Likewise, $\tilde{\beta}_{r,s}$ naively increases the range by $r+s-2$. However, we observe [9] (cf (5)) that α_r merely increases the range by $r-2$ and $\beta_{r,s}$ by $s-1$.

Boost charges

Let us consider boost charges first, which superficially generate terms too long by one site. We observe that the contributions of leading range agree precisely with those in some conserved charge

$$\mathrm{i}(s-1)[\mathcal{B}[\mathcal{Q}_s(\lambda)], \mathcal{Q}_r(\lambda)] \simeq -(s+r-2)\mathcal{Q}_{s+r-1}. \tag{18}$$

Consequently we can reduce the range by one site by fixing $\tilde{\gamma}_{r,s}$ appropriately ($\tilde{\gamma}_{r,s} = 0$ for $s < r+2$)

$$\tilde{\gamma}_{r,s}(\lambda) = \frac{s-1}{s-r}\,\tilde{\alpha}_{s-r+1}(\lambda). \tag{19}$$

Furthermore, combinations of multiple $\tilde{\alpha}$'s lead to a range which is longer than expected. This problem can apparently be cured by choosing the $\tilde{\alpha}$'s as follows:

$$\sum_{r=3}^{\infty} \frac{\tilde{\alpha}_r(\lambda)}{(r-1)x^{r-2}} = \frac{\mathrm{d}u\,(x)}{\mathrm{d}\lambda}\bigg/\frac{\mathrm{d}u\,(x)}{\mathrm{d}x} = -\frac{\mathrm{d}x\,(u)}{\mathrm{d}\lambda}, \tag{20}$$

where $u(x)$ is a function of refined α's

$$u(x) = x + \sum_{r=3}^{\infty} \frac{\alpha_r(\lambda)}{x^{r-2}}, \qquad \alpha_r(\lambda) = \mathcal{O}(\lambda^{r-2}), \tag{21}$$

and $x(u) = u + \mathcal{O}(\lambda)$ is its inverse. At present, we have no good understanding of why the subtraction (19) and the function (20) and (21) reduce the range or how to prove these observations.

Bi-local charges

Similarly, the range of the terms due to bi-local charges depends on the definition of $\tilde{\beta}$'s. The correct choice seems to be (with $\beta_{r,s} = -\beta_{s,r}$)

$$\tilde{\beta}_{r,s}(\lambda) = 2\beta'_{r,s}(\lambda) + \sum_{r'=2}^{r-2} 2\,\beta_{r',s}(\lambda)\,\tilde{\gamma}_{r',r}(\lambda) + \sum_{s'=2}^{s-2} 2\,\beta_{r,s'}(\lambda)\,\tilde{\gamma}_{s',s}(\lambda). \tag{22}$$

Furthermore the regularization of bi-local operators in (13) apparently reduces the range as far as possible, i.e.

$$\beta_{r,s}(\lambda) = \mathcal{O}(\lambda^{s-1}). \tag{23}$$

5. Bethe ansatz

We would now like to apply the coordinate Bethe ansatz in order to derive the asymptotic Bethe equations.

Dispersion relations

First we prepare a one-magnon eigenstate $|p\rangle$ with definite momentum p along the chain in order to measure the dispersion relations of the charges. Applying the recursion relation (16) to the state yields an equation in the charge eigenvalues q_r

$$\frac{\mathrm{d}q_r}{\mathrm{d}\lambda} = -\sum_{s=3}^{\infty} \tilde{\alpha}_s q_s \frac{\mathrm{d}q_r}{\mathrm{d}p} + \sum_{s=2}^{\infty} \tilde{\gamma}_{r,s} q_s. \tag{24}$$

A solution with integration constant t reads (cf [13])

$$q_r(t,u) = \frac{\mathrm{i}}{r-1}\left(\frac{1}{x(u+\mathrm{i}t/2)^{r-1}} - \frac{1}{x(u-\mathrm{i}t/2)^{r-1}}\right). \tag{25}$$

The rapidity $u(p)$ is defined implicitly by

$$\exp(\mathrm{i}p(t,u)) = \frac{x(u+\mathrm{i}t/2)}{x(u-\mathrm{i}t/2)}. \tag{26}$$

Dressing phase

To understand how the S-matrix S depends on λ we consider two-magnon eigenstates

$$|u,u'\rangle = |u<u'\rangle + S(u,u')|u'<u\rangle + |\mathrm{local}\rangle. \tag{27}$$

The magnon momenta p, p' are implicitly defined through the rapidities u, u'. The recursion relation (16) implies the following dependence:

$$S(u,u') = \exp(-2\mathrm{i}\theta(u,u'))S_0(u,u'), \tag{28}$$

where $S_0(u,u')$ is the scattering matrix at $\lambda = 0$. The dressing phase takes the form proposed in [14, 9]

$$\theta(u,u') = \sum_{s>r=2}^{\infty} \beta_{r,s}(q_r(u)\,q_s(u') - q_r(u')\,q_s(u)). \tag{29}$$

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Bethe equations

On the basis of the above results for the dispersion relations and the scattering matrix we can write the deformed Bethe equations. The Bethe equations for an integrable spin chain based on an R-matrix with Yangian symmetry $\mathrm{Y}(\mathfrak{g})$ have been developed in [15]. The Lie (super)algebra $\mathfrak{g}$ of rank R is specified by the symmetric Cartan matrix C_{ab}, $a, b = 1, \ldots, R$. The closed spin chain consists of L identical Yangian modules with Dynkin labels t_a, $a = 1, \ldots, R$. Periodic eigenstates of the spin chain are described by the Bethe roots $u_{a,k}$, $k = 1, \ldots, K_a$, satisfying the Bethe equations

$$\exp(\mathrm{i}p(t_a, u_{a,k})L) = \prod_{\substack{b=1 \\ (b,j)\neq(a,k)}}^{R} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} + \mathrm{i}C_{ab}/2}{u_{a,k} - u_{b,j} - \mathrm{i}C_{ab}/2} \exp(2\mathrm{i}\theta(t_a, u_{a,k}; t_b, u_{b,j})). \tag{30}$$

The charge eigenvalues then take the form

$$\begin{aligned} \mathrm{e}^{\mathrm{i}P} &= \prod_{a=1}^{R} \prod_{k=1}^{K_a} \exp(\mathrm{i}p(t_a, u_{k,a})), \\ Q_r &= \sum_{a=1}^{R} \sum_{k=1}^{K_a} q_r(t_a, u_{a,k}). \end{aligned} \tag{31}$$

Note that these Bethe equations are merely *asymptotic* [16]: the charge eigenvalues Q_r are valid only up to terms of order $\mathcal{O}(\lambda^{L-r+1})$ for which the range of $\mathcal{Q}_r$ exceeds L and where it is thus not properly defined.

6. Conclusions and outlook

In this note we have presented a recursion relation for constructing integrable long-range spin chains from an arbitrary short-ranged model; cf [17] for further details. These models have appeared in the context of $\mathcal{N} = 4$ supersymmetric gauge theory, but their existence and all-orders consistency was largely conjectural so far.

The method applies to generic Lie (super)algebras and spin representations and it explains the set of allowed deformation parameters. The deformation parameters control the range of interactions and by taking suitable combinations of some parameters we were able to decrease the range systematically and in accordance with planar gauge theory. This observation, however, lacks a proof. Our construction method applies to infinitely long chains, but using the coordinate Bethe ansatz and earlier quantum algebra results we have derived asymptotic Bethe equations for closed chains.

Several aspects deserve further scrutiny: it would be important to have a better understanding of how and why the range is decreased for boost deformations. In particular, does the reduction apply to all algebras, to all representations and to quantum deformations? For example, we find [17] that it apparently applies to alternating spin chains such as the ones recently found [18] (see also [19]) in $\mathcal{N} = 6$ superconformal Chern–Simons theory [20].

The moduli space of open long-range integrable chains is slightly different [21]; e.g. a phase associated with the boundaries appears. Can such open chains including new degrees of freedom be constructed in a similar fashion?

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In our model the Lie algebra symmetry is manifest, whereas for $\mathcal{N}=4$ gauge theory only the compact part of $\mathfrak{psu}(2,2|4)$ acts canonically. The other generators are deformed much like the Hamiltonian, which is an inseparable part of the algebra. Recently a very similar recursion relation to ours has appeared for such systems [22] and it would be highly desirable to join the two structures to construct the complete $\mathfrak{psu}(2,2|4)$ representation.

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