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# A NEW CRITERION FOR SECULAR INSTABILITY OF RAPIDLY ROTATING STARS*

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## ABSTRACT

A new variational criterion sufficient for secular instability of rapidly rotating stars to bar modes driven by gravitational radiation reaction is derived. Application to disk models indicates that differentially rotating stars can be stable beyond where they should be unstable according to the tensor virial method. We show why the tensor virial method is invalid.
*Subject headings:* gravitation — instabilities — stars: rotation

## I. INTRODUCTION

Secular instability of (Newtonian) rapidly rotating stars due to the action of viscous dissipation and/or gravitational radiation reaction has received considerable attention over the last several years. Much of the work has focused on the presumably dominant instability to two-armed bar modes in differentially rotating stars (e.g., Ostriker and Tassoul 1969; Ostriker and Bodenheimer 1973), since astrophysically realistic equations of state result in loss of centrifugal equilibrium at the equator before the onset of a secular bar instability when the rotation is uniform. However, as emphasized by Hunter (1977), when viscosity is the dominant dissipative mechanism a differentially rotating star will change its rotation law, presumably toward uniform rotation over the bulk of the star (Durisen 1973), at least as rapidly as any barlike deformation is likely to grow. The question of secular instability to bar modes for differentially rotating stars is well posed only if gravitational radiation reaction or something else which has no effect on the axisymmetric equilibrium is the dominant dissipative mechanism.

The usual test for secular instability has been to search for a zero-frequency bar "mode" in a tensor virial approximation to the hydrodynamic equations (Tassoul and Ostriker 1968) along a sequence of differentially rotating configurations of increasing angular momentum. Ostriker and Tassoul (1969) have claimed that the tensor virial method locates a bifurcation to a Dedekind-like sequence of stationary bar configurations and the onset of secular instability *exactly*, at a value of $t$, the ratio of rotational kinetic energy $T$ to the negative of the gravitational potential energy $W$, very close to 0.138 for a wide variety of models (Ostriker and Bodenheimer 1973). More recently, Friedman and Schutz (1975*a*) and Hunter (1977) have argued that the tensor virial method, because of its relationship to a variational principle with a specific trial function, can provide only a *sufficient* condition for the onset of secular instability, and that the true bifurcation point at marginal secular instability to gravitational radiation reaction is at $t = t_c \leq t_v$, where $t_v$ is the tensor virial estimate.

In this *Letter* we report on new results which show that the real situation is quite different. The tensor virial criterion is neither necessary nor sufficient for the onset of secular instability along a sequence of differentially rotating models, and therefore provides no information. We have derived a revised variational criterion which is a sufficient condition for secular instability to gravitational radiation reaction. Numerical calculations using this revised criterion on two-armed modes indicate that the actual value of $t_c$ is typically *greater* than $t_v$, perhaps by as much as 20% or so in some cases. Mass limits on differentially rotating white dwarfs (Ostriker and Tassoul 1969; Durisen 1975) and binding energy limits for supermassive stars (Wilson 1972) based on tensor virial results are somewhat more strict than really justified. On the other hand, secular instability to multiarmed modes (Friedman and Schutz 1977*b*) may act to lower previous estimates of mass limits on rotating neutron stars (e.g., Shapiro and Lightman 1976).

We explain the new methods to the extent necessary for their astrophysical application. Full technical details of the Newtonian derivations will be given separately (Friedman and Schutz 1977*a*,*b*), as will additional numerical results (Bardeen, in preparation).

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## II. THE VARIATIONAL FORMALISM

Consider an axisymmetric rotating star in equilibrium. We work in cylindrical coordinates $r$, $\phi$, $z$. The density $\rho$ and pressure $p$ are related by an equation of state $p = p(\rho, s)$, where $s$ is the specific entropy. The fluid rotates with an angular velocity $\Omega(r, z)$ about the $z$-axis. Now perturb the star by displacing each fluid element from its equilibrium position $r_0$, $\phi_0 = \phi_c + \Omega_0 t$, $z_0$, with $\Omega_0 = \Omega(r_0, z_0)$, to a new position $r = r_0 + \xi^r(r_0, \phi_0, z_0, t)$, $\phi = \phi_0 + \xi^\phi$, $z = z_0 + \xi^z$. Keep the mass and entropy of the fluid element constant.

The perturbed hydrodynamic equations are written

$$A(\dot{\xi}) + B(\xi) + C(\dot{\xi}) = 0\,, \tag{1}$$

where $\dot{\xi} = \partial\xi/\partial t$. The vector operators $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are defined by

$$A_i(\xi) = \rho g_{ij}\xi^j\,, \tag{2}$$

where $\xi^{1,2,3} = \xi^{r,\phi,z}$ and $g_{ij}$ is the metric tensor,

$$B_i(\xi) = 2\rho\Omega(g_{ij}\xi^j{}_{,\phi} + \epsilon_{i3j}\xi^j)\,, \tag{3}$$

where $\epsilon_{ijk}$ is the antisymmetric tensor with nonzero components $r(-r)$ for $ijk$ even (odd) permutations of 123, and

$$C_i(\xi) = \rho\Omega^2 g_{ij}\xi^j{}_{,\phi\phi} + 2\rho\Omega^2\epsilon_{i3j}\xi^j{}_{,\phi} - \nabla_i(\gamma p\nabla_k\xi^k) + (\nabla_i p)\rho^{-1}\nabla_k(\rho\xi^k) - \nabla_i(\xi^k\nabla_k p) + \delta_i{}^1\rho r\xi^k\nabla_k(\Omega^2) + \rho\nabla_i(\delta_\xi\Phi)\,. \tag{4}$$

Here $\nabla_k$ is the covariant derivative operator, $\gamma = (\partial \ln p/\partial \ln \rho)_s$, and $\delta_\xi\Phi$ is the Eulerian perturbation in the gravitational potential induced by the Eulerian perturbation in the density

$$\delta_\xi\rho = -\nabla_k(\rho\xi^k)\,. \tag{5}$$

The operators $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are, respectively, Hermitian, anti-Hermitian, and Hermitian in that, for instance,

$$[\textstyle\int(\eta^k)^* A_k(\xi)dV]^* = \int(\xi^k)^* A_k(\eta)dV\,. \tag{6}$$

Equation (1) can be derived by varying the action

$$I = \tfrac{1}{2}\textstyle\int[(\dot{\xi}^k)^* A_k(\dot{\xi}) + (\dot{\xi}^k)^* B_k(\xi) - (\xi^k)^* C_k(\xi)]dVdt\,; \tag{7}$$

this is just the Lynden-Bell and Ostriker (1967) variational principle. From the time-independence and axial symmetry of the operators follows the existence of a dynamically conserved energy functional or canonical energy

$$E_c = \tfrac{1}{2}\textstyle\int[(\dot{\xi}^k)^* A_k(\dot{\xi}) + (\xi^k)^* C_k(\xi)]dV \tag{8}$$

and a dynamically conserved canonical angular momentum

$$J_c = -\mathrm{Re}\textstyle\int[(\dot{\xi}^k)^* A_k(\xi_{,\phi}) - \tfrac{1}{2}(\xi^k)^* B_k(\xi_{,\phi})]dV\,. \tag{9}$$

## III. PREVIOUS ARGUMENTS

If the only dissipative mechanism is gravitational radiation reaction, Friedman and Schutz (1975*b*) have shown that an energy functional which becomes the Newtonian $E_c$ in the nonrelativistic limit can only decrease as the result of radiation of gravitational waves by the perturbation. The rate of decrease of $E_c$ for a (nearly) Newtonian star is equal to the rate that physical energy is lost in gravitational waves.

They consider a sequence of, in general, differentially rotating models with increasing angular momentum. The normalization-independent quantity $E_c/\int(\xi^k)^*\xi_k dV$ is presumed positive-definite for nonaxisymmetric displacements at the slowly rotating end of the sequence. If its lower bound decreases smoothly through zero at some point along the sequence, then at this point a dynamical mode passes through zero frequency. Beyond this point the dynamical mode has $E_c < 0$; and if (as any nonspherical mode should) it does generate gravitational waves, it is secularly unstable in that it grows without bound in the linear theory on a time scale set by the coupling to the radiation.

A simple argument for secular instability whenever $E_c$ is negative for some nonaxisymmetric trial displacement $\xi$ has been given by Hunter (1977). Hunter proves that $E_c/\int(\xi^k)^*\xi_k dV$ is bounded from below. If $E_c < 0$ initially, it and the displacement can never return to zero. The radiation will presumably decrease $E_c$ indefinitely, and the magnitude of $\xi$ as measured by $\int(\xi^k)^*\xi_k dV$ will increase without bound in the linear theory.

These arguments are incorrect. For almost *any* rotating star there are trial displacements with angular dependence $e^{im\phi}$ that make $E_c < 0$ for every nonzero value of $m$. The physical perturbations these displacements represent can die away in time even though the displacement and its canonical energy must always remain nonzero.

## IV. TRIVIAL DISPLACEMENTS

Schutz and Sorkin (1977) pointed out the existence of a class of displacements we shall call "trivial," for which the *Eulerian* perturbations in density, pressure, and velocity vanish. These displacements just relabel the fluid elements within the star. They represent a kind of "gauge" freedom in the description of the perturbations. Such quantities as the change in total physical energy $\Delta E$ and the change in total physical angular momentum $\Delta J$ are "gauge"-invariant, but the canonical energy and angular momentum $E_c$ and $J_c$ are not. It is always possible to choose a "gauge" in which $\Delta E = E_c$ and $\Delta J = J_c$, but without such a restriction on $\xi$ an initial $E_c < 0$ cannot by itself imply any instability. If the initial $\Delta E > 0$ for all $\xi$, the gravitational radiation of a finite energy $\Delta E$ removes the physical perturbations.

A trivial displacement $\eta$ satisfies

$$\delta\rho = -\nabla_k(\rho\eta^k) = 0\,, \tag{10}$$

$$\delta s = -\eta^k\nabla_k s = 0\,, \tag{11}$$

and

$$\delta v^i = \dot{\eta}^i + \Omega\eta^i{}_{,\phi} - \delta^i{}_2\eta^k\nabla_k\Omega = 0 \tag{12}$$

The general solution to these equations is

$$\eta^i = \frac{1}{\rho}\epsilon^{ijk}\nabla_j h\nabla_k f\,, \tag{13}$$

where $f$ is an arbitrary function of $r$, $\phi_c = \phi - \Omega t$, and $z$. In any region of the star where $\nabla s \neq 0$, $h$ must be proportional to $s(r, z)$; otherwise, except for continuity requirements, $h$ is an arbitrary function of $r$ and $z$.

For simplicity we only evaluate $E_c$ for a barotropic star, so $\Omega = \Omega(r)$. Consider a trivial displacement with $f = (\hat{f}r, z)e^{im(\phi-\Omega t)}$ and $\hat{f}$ and $h$ real. Then

$$E_c = \frac{1}{2}\int dV\left(2\Omega + r\frac{d\Omega}{dr}\right)\frac{m^2\hat{f}}{r\rho}h_{,z}\left(2\Omega(h_{,z}\hat{f}_{,r} - h_{,r}\hat{f}_{,z}) + \frac{d\Omega}{dr}\hat{f}h_{,z}\right). \tag{14}$$

Even if $\nabla s \neq 0$ everywhere, so $h = s$, the appropriate choice of $f$ can make $E_c$ have any value. The barotropic condition excludes the possibility that $s_{,z} = 0$ everywhere with $\nabla s \neq 0$. Only a toroidal "star" with $2\Omega + r d\Omega/dr = 0$ everywhere can have $E_c$ identically zero for all trivial displacements.

## V. PHYSICAL DISPLACEMENTS

It is possible to restrict one's choice of trial displacements to a special class of "physical" displacements $\zeta$ which in a certain sense (Friedman and Schutz 1977*a*) are "orthogonal" to all trivial displacements. The physical displacements $\zeta$ satisfy the constraint

$$\epsilon^{ijk}\nabla_i h\nabla_j[\dot{\zeta}_k + \Omega(\zeta_\phi)_{,k} + \epsilon_{k3l}\Omega\zeta^l] = 0\,, \tag{15}$$

where $h$ is an arbitrary function of $r$, $\phi$, $z$ in a region where $\nabla s = 0$; otherwise $h = s$. Equivalent to equation (15) is the requirement that the Lagrangian perturbation in the circulation vanish,

$$\Delta \oint_c v_i dx^i = 0\,, \tag{16}$$

for any closed curve $c$ lying in a constant-entropy surface. The condition (15) or (16) does not constrain the physical Eulerian perturbations; it really is just a choice of "gauge" in the description of the perturbations.

Explicit calculation shows that the physical changes in energy and angular momentum are, to second order,

$$\Delta E = \frac{1}{2\pi}\int \rho\Omega\left[\Delta \oint_{c'} v_i dx^i\right]dV + E_c \tag{17}$$

and

$$\Delta J = \frac{1}{2\pi}\int \rho\left[\Delta \oint_{c'} v_i dx^i\right]dv + J_c\,, \tag{18}$$

where $c'$ is the curve which in the unperturbed star is a circle at constant $r$ and $z$ around the axis of symmetry and the integration is over such rings. Since the $c'$ are special cases, by axisymmetry, of the curves $c$ in equation (16), we see that if the constraint of equation (16) is applied through *second* order in $\zeta$ the physical displacements automatically give $\Delta E = E_c$ and $\Delta J = J_c$. Only the first-order constraint of equation (15) is needed to evaluate $E_c$ and $J_c$ and therefore $\Delta E$ and $\Delta J$ once this assumption is made.

Since gravitational radiation reaction acts through a modification of the Newtonian gravitational potential (see

Misner, Thorne, and Wheeler 1973), the actual displacement developing as the result of any gravitational-radiation-reaction-driven secular instability must satisfy equation (16) to all orders in $\zeta$. If $\Delta E = E_c$ initially, $\Delta E = E_c$ throughout the time development of the secular perturbation.

### VI. CONSEQUENCES FOR THE TENSOR VIRIAL METHOD

The tensor virial method (Tassoul and Ostriker 1968) identifies the point of marginal secular instability along a sequence of models of increasing Ostriker parameter $t$ with the existence of a zero-frequency "mode" of the tensor virial moments of the hydrodynamic equations when the displacements are *constrained* to obey

$$\xi^r = re^{i(2\phi-\omega t)},\ \xi^\phi = ie^{i(2\phi-\omega t)},\ \xi^z = 0\ . \tag{19}$$

At this point the canonical energy $E_c$ vanishes for a zero-frequency trial displacement of the form (19). One concludes that $E_c < 0$ for some trial displacements beyond this point along the sequence and probably even before, but we have seen that this can imply secular instability only if the displacement is a "physical" displacement, so $E_c = \Delta E$.

When the star is barotropic, so $\Omega = \Omega(r)$, equation (15) is satisfied for a zero-frequency tensor virial displacement if and only if the rotation is uniform, $\Omega_{,r} = 0$, and probably the restriction to barotropic models is unnecessary. The true point of marginal secular instability may be either before or after the point indicated by the tensor virial method along a sequence of differentially rotating stellar models.

### VII. A VALID SECULAR INSTABILITY CRITERION

At the point of marginal secular instability along a sequence of models the minimum value of $E_c/\int(\xi^k)^*\xi_k dV$ for all *physical* displacements $\xi$ is zero; once there are physical displacements with $E_c < 0$, gravitational radiation reaction will drive a genuine instability. A nontrivial nonaxisymmetric displacement will radiate unless it is zero frequency, in which case $E_c = 0$ and the perturbed configuration is unreachable from an initial physical perturbation with $E_c < 0$. This argument applies separately for each angular eigenvalue $m \geq 2$.

Because the displacements are constrained by equation (15), the old argument for a zero-frequency mode at the point of marginal secular instability is no longer valid. Whenever a zero-frequency dynamical mode does exist, $E_c$ is still stationary, but not necessarily a minimum, for arbitrary physical variations about the eigenfunction.

In spite of the lack of proof, we believe that a zero-frequency dynamical mode whose eigenfunction is qualitatively similar to the tensor-virial trial displacement very probably does mark the exact point of marginal secular instability to two-armed perturbations. "Qualitatively similar" means no nodes in the $r$-dependence or $z$-dependence of $\xi^r$ and $\xi^\phi$ to go with the $e^{2i\phi}$ angular-dependence. This generalizes to $m$-armed modes.

The search for a zero-frequency dynamical mode is numerically complex for a rotating star, so it seems useful to correct the tensor virial approach by suggesting a modified trial displacement which does satisfy equation (15) for differentially rotating stars. From now on we consider only displacements with axial eigenvalue $m = 2$.

Numerical calculations of dynamical modes have been carried out (Bardeen, in preparation) for infinitesimally thin, differentially rotating disks in which the pressure acts only in the plane of the disk and is a prescribed function of the surface density. For a given surface density distribution, as the disk is "cooled" and the Ostriker parameter $t$ increased from zero, there is a first two-armed bar mode to pass through zero frequency, and it is qualitatively similar to the tensor virial displacement (19). For all disk configurations the tensor virial estimate of the point of marginal secular instability is $t_v = 0.125$ (see Hunter 1977). Interestingly, the zero-frequency mode presumably signifying the exact point of marginal secular instability to two-armed modes is at $t_c > 0.125$ when the angular velocity decreases from the center to the rim. In one example $t_c = 0.1344$ when the ratio of angular velocities is about 6.4.

In the few examples calculated, the eigenfunction of this zero-frequency mode very nearly has the same Eulerian surface density perturbation $\delta\sigma$ as that generated by the tensor-virial trial displacement (19). We therefore suggest that for barotropic stars near marginal secular instability to the two-armed mode, a good estimate of the minimum value of $E_c$ can be obtained using a trial displacement of the form

$$\xi^r = rA(r)e^{2i\phi},\ \xi^\phi = i(A(r) + B(r))e^{2i\phi},\ \xi^z = 0\ . \tag{20}$$

The functions $A(r)$ and $B(r)$ are chosen so that (1) the Eulerian perturbation in the surface density $\delta\sigma = \int\delta\rho dz$ is exactly the same as from the tensor virial displacement and (2) equation (15) is satisfied. They are solutions of

$$r\frac{dA}{dr} = (1 - A)\frac{r}{\sigma}\frac{d\sigma}{dr} + 2B\ , \tag{21}$$

$$r\frac{dB}{dr} = -4B - \frac{r}{\Omega}\frac{d\Omega}{dr}(A + 2B)\ , \tag{22}$$

with $B = 0$ at $r = 0$ and $A = 1$ at the equator, where $\sigma = 0$.

The error in the minimum of $E_c$ at marginal secular instability is quadratic in the deviation from the exact minimizing physical displacement. It certainly seems to be small for reasonable disk models and will hopefully prove to be small for perfect fluid stars as well. In any case, a *sufficient* condition for secular instability to two-armed modes is that $E_c < 0$ for the displacement of equations (20)–(22).

The evaluation of $E_c$ is reasonably straightforward except for solving the Poisson equation to find $\delta\Phi$ from $\delta\rho$. Explicit solution of the Poisson equation was avoided in the tensor virial method, but usually cannot be avoided here. The infinitesimally thin disks are exceptions, since the potential perturbation then depends only on $\delta\sigma$ and is the same as for the tensor virial displacement (19). We find

$$E_c = |W|(\tfrac{1}{2} - 4t) + \int 2\pi\sigma r^3 dr\left[\left(-\frac{r}{\Omega}\frac{d\Omega}{dr}\right)(1 - A^2) - 4\Omega^2 B^2\right], \tag{23}$$

where $W$ is the gravitational potential energy of the unperturbed disk.

A large decrease in $\Omega$ from the symmetry axis to equator, particularly if the surface density gradient is relatively less steep (in that the ratio of centrifugal force to gravitational force decreases outward), results in a value of $A$ substantially less than one over most of the disk. Then the estimate of $t_c$ from equation (23) can be as much as 10%–25% larger than $t_r = 0.125$. An example is the Mestel disk (Mestel 1963), whose linear velocity of rotation is uniform all the way to $r = 0$. At $r = 0$, $A = \frac{1}{2}$ and $B = \frac{1}{4}$. A numerical solution of equations (21) and (22) gives $t_c \approx 0.148$.

## VIII. CONCLUSION

We have suggested a sufficient condition for secular instability to two-armed modes generated by gravitational radiation reaction which is valid for differentially rotating stars. The tensor-virial instability estimate, while a sufficient condition for uniformly rotating stars, is neither necessary nor sufficient for differentially rotating stars. Characteristic growth rates of the instability for various types of stars are discussed by Friedman and Schutz (1975*a*).

The basic formalism of §§ II–IV also applies to multi-armed displacements, with axial eigenvalue $m > 2$. Friedman and Schutz (1977*b*) have shown that *all* nonviscous rotating stars are secularly unstable via gravitational radiation reaction to some multiarmed modes, though only to those with $m \gg 2$ when the rotation is slow. However, the characteristic growth time of the instability scales as $(R/c)\,(Rc^2/GM)^{m+1}t^{-(m+1)}$, so the multiarmed instabilities are likely to be astrophysically important only for such objects as neutron stars with $Rc^2/GM$ not large compared with 1 and then only for $m = 3$ or 4.

The disk models indicate that $t_c$ can be as much as 20% greater than the tensor virial estimate; if this extrapolates to stars, the value of $t_c$ applicable to white dwarfs, say, should be in the range $0.14 \lesssim t_c \lesssim 0.17$. Multiarmed modes may make the effective $t_c$ for neutron stars somewhat smaller than this range; the precise values have yet to be calculated.

The evolution of a differentially rotating star when viscosity is important and it cannot accommodate its angular momentum with uniform rotation is a complicated question (see Durisen 1973). As long as the star is dynamically stable against nonaxisymmetric modes, it is not clear that a significant barlike distortion will ever form. If the star is or becomes uniformly rotating, the tensor virial criterion is sufficient for secular instability to viscous dissipation alone (see Hunter 1977; Friedman and Schutz 1977*b*). Interference between viscous dissipation and radiation reaction can suppress secular instability (Lindblom and Detweiler 1977).

## REFERENCES


Durisen, R. H. 1973, *Ap. J.*, **183**, 215.
———. 1975, *Ap. J.*, **199**, 179.
Friedman, J. L., and Schutz, B. F. 1975*a*, *Ap. J.* (*Letters*), **199**, L157.
———. 1975*b*, *Ap. J.*, **200**, 204.
———. 1977*a*, submitted to *Ap. J.*
———. 1977*b*, submitted to *Ap. J.*
Hunter, C. 1977, *Ap. J.*, **213**, 497.
Lindblom, L., and Detweiler, S. L. 1977, *Ap. J.*, **211**, 565.
Lynden-Bell, D., and Ostriker, J. P. 1967, *M.N.R.A.S.*, **136**, 293.
Mestel, L. 1963, *M.N.R.A.S.*, **126**, 553.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973, *Gravitation* (San Francisco: Freeman).
Ostriker, J. P., and Bodenheimer, P. 1973, *Ap. J.*, **180**, 171.
Ostriker, J. P., and Tassoul, J. L. 1969, *Ap. J.*, **155**, 987.
Schutz, B. F., and Sorkin, R. 1977, *Ann. Phys.*, in press.
Shapiro, S. L., and Lightman, A. P. 1976, *Ap. J.*, **207**, 263.
Tassoul, J. L., and Ostriker, J. P. 1968, *Ap. J.*, **154**, 613.
Wilson, J. R. 1972, *Ap. J.*, **176**, 195.



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