Supersymmetric dualities beyond the conformal window

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Using the superconformal indices techniques, we construct Seiberg type dualities for $\mathcal{N}=1$ supersymmetric field theories outside the conformal windows. These theories are physically distinguished by the presence of fields with small or negative R-charges for chiral superfields.

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I. INTRODUCTION

Some of 4D $\mathcal{N}=1$ supersymmetric gauge field theories are known to be related by the Seiberg duality [1]. A full list of presently known dualities of such type for simple gauge groups $G_c = SU(N), SP(2N), G_2$ is given in [2]. Remarkably, many of the listed dualities are *new*. Their discovery is based on the interplay between superconformal indices of [3–5] and the theory of elliptic hypergeometric integrals formulated in [6, 7] (see also [8] for a general survey).

The $SU(2,2|1)$ space-time symmetry group is generated by $J_i, \overline{J}_i$ ($SU(2)$ subgroups generators, or Lorentz rotations), $P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha}$ (supertranslations), $K_\mu, S^\alpha, \overline{S}^{\dot\alpha}$ (special superconformal transformations), H (dilations) and R ($U(1)_R$-rotations). For a distinguished pair of supercharges, say, $Q=\overline{Q}_1$ and $Q^\dagger = -\overline{S}^1$, one has

$$\{Q, Q^\dagger\} = 2\mathcal{H}, \quad \mathcal{H} = H - 2\overline{J}_3 - 3R/2, \qquad (1)$$

and the superconformal index is defined by the matrix integral

$$I(p,q,f_k) = \int_{G_c} d\mu(g)\, Tr\Big((-1)^{\mathrm{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} \times e^{\sum_a g_a G^a} e^{\sum_k f_k F^k} e^{-\beta\mathcal{H}}\Big), \quad \mathcal{R} = H - R/2, \qquad (2)$$

where $d\mu(g)$ is the G_c-invariant measure and F is the fermion number operator. Operators G^a and F^k are the gauge and flavor group generators; p, q, g_a, f_k, β are group parameters (chemical potentials). The trace is taken over the whole space of states, but, because the operators standing in (2) preserve relation (1), only the zero modes of the operator $\mathcal{H}$ contribute to the trace (hence, formally there is no dependence on β).

The key idea of Römelsberger [5] on the equality of superconformal indices (2) for the Seiberg dual theories was realized first by Dolan and Osborn for a number of examples [9]. It appears that these equalities are expressed in terms of the exact computability of elliptic beta integrals discovered in [6] or nontrivial symmetry transformations for higher order elliptic hypergeometric functions on various root systems [7, 10].

In addition to the description of new $\mathcal{N}=1$ dualities from known identities for integrals, another important result of [2] consisted in the formulation of new mathematical conjectures for integrals' identities following from known dualities. There are also examples when both the dualities and corresponding relations for integrals (indices) are new. The power of the theory of elliptic hypergeometric integrals in application to the superconformal indices techniques was demonstrated also in recent papers by Gadde et al [11, 12], where some of the $\mathcal{N}=2$ superconformal dualities have been considered.

Here we focus on some physical consequences following from the considerations of [2]. Namely, we concentrate on implications for the conformal windows introduced in [1, 13]. In the original Seiberg work [1] it was shown that the corresponding $\mathcal{N}=1$ SQCD duality has distinguished properties if the number of colors $N_c \equiv N$ of the gauge group $SU(N)$ and the number of chiral superfields (flavors) N_f satisfy the following inequalities

$$3N/2 < N_f < 3N. \qquad (3)$$

This conformal window guarantees that both dual theories have asymptotic freedom and represent interacting superconformal theories at the infrared fixed points. For $SP(2N)$ gauge groups with N_f flavors the conformal window is [13]

$$3(N+1)/2 < N_f < 3(N+1). \qquad (4)$$

After some time it started to be believed that these conformal windows serve as the general necessary conditions for the existence of dualities between interacting gauge theories. The primary goal of this paper is to show that this is not the case.

Equality of superconformal indices of dual theories is a new non-trivial indication on the validity of Seiberg duality conjectures. Earlier there were only three following justifying arguments [1].

1. The 't Hooft anomaly matching conditions. They were conjectured in [2] to be a consequence of the so-called total ellipticity for the elliptic hypergeometric integrals [8] describing superconformal indices.

2. Matching reduction of the number of flavors $N_f \to N_f - 1$. Integrating out k-th flavor quarks by the mass

term ${M_k}^k Q_k \widetilde{Q}^k$ in the original theory results in Higgsing the magnetic theory gauge group with a reduction of the additional meson fields. From the elliptic hypergeometric integrals point of view this is realized by restricting in a special way a pair of parameters ($s_k t_k = pq$) which reduces the indices appropriately.

3. Matching of the moduli spaces and the gauge invariant operators in dual theories. There is no clear moduli space description in the superconformal indices. Perhaps, it enters only through quantum numbers fixed by superpotentials.

II. DUALITIES OUTSIDE THE CONFORMAL WINDOW FOR $SU(N)$ GAUGE GROUP

A. $SU(2N)$ gauge group with $N_f = 4$

The starting electric theory has $G_c = SU(2N)$ and the matter fields content $4f + 4\overline{f} + T_A + \overline{T}_A$, where f and T_A denote the fundamental and absolutely antisymmetric tensor representations of G_c (the bar means conjugate representations). The flavor group for $N > 2$ is $SU(4) \times SU(4) \times U(1)_1 \times U(1)_2 \times U(1)_3$. The superconformal index is given by the following integral [2]

$$I_E = \kappa_N \int_{\mathbb{T}^{2N-1}} \prod_{1 \le i < j \le 2N} \frac{\Gamma(U z_i z_j, V z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \times \prod_{j=1}^{2N} \prod_{k=1}^{4} \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi \mathrm{i} z_j}, \quad (5)$$

where $\prod_{j=1}^{2N} z_j = 1$, $\mathbb{T}$ is the unit circle with positive orientation, $|U|, |V|, |s_k|, |t_k| < 1$, and $(UV)^{2N-2} \prod_{k=1}^4 s_k t_k = (pq)^2$. We use conventions $\Gamma(a, b; p, q) \equiv \Gamma(a; p, q)\Gamma(b; p, q)$, $\Gamma(az^{\pm 1}; p, q) \equiv \Gamma(az; p, q)\Gamma(az^{-1}; p, q)$, where

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \quad |p|, |q| < 1,$$

is the elliptic gamma function. Finally,

$$\kappa_N = \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!}$$

with $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k)$. The parameters U, V, s_k, t_k are related to f_k in (2) and z_j replace g_a.

In [2] we described three magnetic duals for this model (one of which was found earlier in [14]). The dualities beyond the conformal window of interest are obtained after some "reduction" of these theories. To obtain them we restrict the parameters U and V in (5) by the constraint

$$UV = pq. \quad (6)$$

Now $\prod_{k=1}^4 s_k t_k = (pq)^{4-2N}$ and some of the parameters have modulus bigger than 1. In this case it is necessary to use the analytical continuation of integral (5) reached by passing from $\mathbb{T}$ to a contour separating sequences of integrand's poles converging to zero from their reciprocals. The parameters U and V disappear then completely from the electric superconformal index, so that it starts to coincide with the index of the theory without the fields T_A and $\overline{T}_A$ and global $U(1)_1$ and $U(1)_3$ symmetries. After such a "decoupling", the electric theory coincides exactly with the Seiberg SQCD with $N_f = 4$ [1]:

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_2$	$U(1)_R$
Q	f	f	1	1	$-\frac{1}{2}(N-2)$
$\widetilde{Q}$	$\overline{f}$	1	f	-1	$-\frac{1}{2}(N-2)$
V	adj	1	1	0	1

In all our tables the first column contains symbols of the fields and the second — the gauge group representations. For $U(1)$ groups we give corresponding hypercharges.

The dual magnetic theories are reduced in a similar way. Namely, we substitute $U = \sqrt{pq}x, V = \sqrt{pq}x^{-1}$ into the corresponding magnetic indices described in [2] and reinterpret the latter as the indices of reduced theories. The fields content and some of the R-charges of the resulting theories differ from the original ones. Some of the indices depend on the parameter x reflecting presence of the additional global $U(1)_1$-group. As a result, we find the following set of dualities.

The first magnetic theory is described in the table below

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
q	f	f	1	0	-1	$-\frac{1}{2}(N-2)$
$\widetilde{q}$	$\overline{f}$	1	f	0	1	$-\frac{1}{2}(N-2)$
H_m	1	T_A	1	-1	2	$2m-N+3$
G	1	T_A	1	$N-1$	2	1
$\widetilde{H}_m$	1	1	T_A	1	-2	$2m-N+3$
$\widetilde{G}$	1	1	T_A	$1-N$	-2	1
$\widetilde{V}$	adj	1	1	0	0	1

where $m = 0, \ldots, N-2$.

The global symmetry and the field content of the second magnetic theory is the same as in Seiberg's dual theory with $N_f = 4$, but the gauge group is now $SU(2N)$ instead of $SU(N_f - 2N)$:

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_2$	$U(1)_R$
q	f	$\overline{f}$	1	1	$-\frac{1}{2}(N-2)$
$\widetilde{q}$	$\overline{f}$	1	$\overline{f}$	-1	$-\frac{1}{2}(N-2)$
M_k	1	f	f	0	$2k-N+2$
$\widetilde{V}$	adj	1	1	0	1

where $k = 0, \ldots, N-1$.

The most complicated is the third magnetic theory

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
q	f	$\overline{f}$	1	0	-1	$-\frac{1}{2}(N-2)$
$\widetilde{q}$	$\overline{f}$	1	$\overline{f}$	0	1	$-\frac{1}{2}(N-2)$
M_k	1	f	f	0	0	$2k-N+2$
H_m	1	T_A	1	-1	2	$2m-N+3$
G	1	T_A	1	$N-1$	2	1
$\widetilde{H}_m$	1	1	T_A	1	-2	$2m-N+3$
$\widetilde{G}$	1	1	T_A	$1-N$	-2	1
$\widetilde{V}$	adj	1	1	0	0	1

where $k=0,\ldots,N-1$ and $m=0,\ldots,N-2$. For $N=1$ these three dualities are particular cases of the $SP(2)$ gauge group dualities considered in detail in [15].

B. $SU(N)$ gauge group with $N_f=N+2$

The electric part of the next set of dualities coincides with the Seiberg theory with $G_c=SU(N)$ for arbitrary N and $N_f=N+2$.

	$SU(N)$	$SU(N+2)$	$SU(N+2)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$\frac{2}{N+2}$
$\widetilde{Q}$	$\overline{f}$	1	f	-1	$\frac{2}{N+2}$
V	adj	1	1	0	1

The dual theories are characterized by a rather complicated structure of the flavor symmetry group $SU(K)\times SU(M)\times U(1)_1\times SU(K)\times SU(M)\times U(1)_2\times U(1)_B$, where $M=N+2-K$ and K is an arbitrary integer in the range $K=1,\ldots,N+1$. The dual gauge group is the same $G_c=SU(N)$ (self-duality). The field content is described in the table at the end of this section.

These dualities were derived in [2] (for $N=2$, see [15]) on the basis of symmetry transformations for elliptic hypergeometric integrals established by Rains in [10] (one of which was suggested earlier in [7]). Here we just stress that they lie beyond the conformal window (3) for $N>3$, since the left-hand side inequality is violated in this case. Surprisingly, for $N=3$ we obtain a new duality lying *inside* the conformal window.

	$SU(N)$	$SU(K)$	$SU(M)$	$U(1)_1$	$SU(K)$	$SU(M)$	$U(1)_2$	$U(1)_B$	$U(1)_R$
q_1	$\overline{f}$	f	1	$\frac{K(K-2)}{N}-K+M$	1	1	$\frac{MK}{N}$	$1-M$	$\frac{2}{N+2}$
q_2	f	1	f	$-\frac{K(K-2)}{N}$	1	1	$\frac{-MK}{N}$	$1-K$	$\frac{2}{N+2}$
q_3	f	1	1	$\frac{MK}{N}$	f	1	$\frac{K(K-2)}{N}-K+M$	$M-1$	$\frac{2}{N+2}$
q_4	$\overline{f}$	1	1	$-\frac{MK}{N}$	1	f	$-\frac{K(K-2)}{N}$	$K-1$	$\frac{2}{N+2}$
X_1	1	f	1	M	1	f	$-K$	0	$\frac{4}{N+2}$
X_2	1	1	f	$-K$	f	1	M	0	$\frac{4}{N+2}$
Y_1	1	$\overline{f}$	$\overline{f}$	$K-M$	1	1	0	N	$\frac{2N}{N+2}$
Y_2	1	1	1	0	$\overline{f}$	$\overline{f}$	$K-M$	$-N$	$\frac{2N}{N+2}$
$\widetilde{V}$	adj	1	1	0	1	1	0	0	1

III. DUALITIES OUTSIDE THE CONFORMAL WINDOW FOR $SP(2N)$ GAUGE GROUP

We describe now dualities lying outside the conformal window (4). The starting electric theory has the gauge group $G_c=SP(2N)$, 8 flavors and a matter field X in the adjoint representation of G_c. As shown in [15], this theory has many dual partners (one of which was described earlier in [16]). The electric superconformal index has the form

$$I_E=\kappa_N\Gamma(t;p,q)^{N-1}\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{\Gamma(tz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\times\prod_{j=1}^N\frac{\prod_{k=1}^8\Gamma(t_kz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi\mathrm{i}z_j}, \qquad (7)$$

where $|t|,|t_k|<1$, $t^{2N-2}\prod_{k=1}^8t_k=(pq)^2$ and

$$\kappa_N = \frac{(p;p)_\infty^N(q;q)_\infty^N}{2^NN!}.$$

This integral has nice symmetry transformations described by the Weyl group of the exceptional root system E_7 [10] (for $N=1$ the key transformation was found in [7]). The transformed integrals coincide with the superconformal indices of dual magnetic theories [15].

Now we restrict the value of the t-parameter to $t=\sqrt{pq}$ and analytically continue function (7) by replacing $\mathbb{T}$ to a contour separating geometric sequences of integrand's poles converging to zero from their reciprocals. This leads to the "decoupling" of the field X from the electric theory, so that the same index is generated by the model

	$SP(2N)$	$SU(8)$	$U(1)_R$
Q	f	f	$-\frac{N-3}{4}$
V	adj	1	1

To obtain the dual description, we set $t = \sqrt{pq}$ in the magnetic superconformal indices described in [15] and reinterpret the resulting integrals as coming from different dual theories, similar to the $SU(2N)$ case described above. The first magnetic theory has the following fields content (note the change in the flavor group)

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	f	1	-1	$-\frac{N-3}{4}$
$\widetilde{q}$	f	1	f	1	$-\frac{N-3}{4}$
M_J	1	T_A	1	2	$J-\frac{N-3}{2}$
$\widetilde{M}_J$	1	1	T_A	-2	$J-\frac{N-3}{2}$
$\widetilde{V}$	adj	1	1	0	1

where $J = 0, \ldots, N-1$.

The second magnetic theory is

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	1	$-\frac{N-3}{4}$
$\widetilde{q}$	f	1	$\overline{f}$	-1	$-\frac{N-3}{4}$
M_J	1	f	f	0	$J-\frac{N-3}{2}$
$\widetilde{V}$	adj	1	1	0	1

And, finally, the third magnetic theory is

	$SP(2N)$	$SU(8)$	$U(1)_R$
q	f	$\overline{f}$	$-\frac{N-3}{4}$
M_J	1	T_A	$J-\frac{N-3}{2}$
$\widetilde{V}$	adj	1	1

The flavor symmetry here coincides with the electric one.

IV. CONCLUSION

For all dualities described in this paper we have checked validity of the 't Hooft anomaly matching conditions and the $N_f \to N_f - 1$ reductions. Equality of the superconformal indices for theories in Sect. IIA is not proven, but in the reduced case $N_f = 3$ these indices are equal due to the integral identities established in [7]. Equality of indices in general case for theories of Sects. IIB and III follows [2] from the identities proven rigorously in [10]. Note also that all the models described in our tables are asymptotically free and define interacting conformal field theories at the infrared fixed point.

We conclude that the notion of conformal windows should be used with care — it is applicable only to particular types of dualities. Our results raise a natural question on classification of all 4D theories dual to the original Seiberg "minimal" electric SQCD. It is necessary to analyze various infrared physics implications following from the dualities described above. In particular, this concerns the structure of superpotentials (see, e.g., some preliminary discussions in [17]). It would be interesting to understand which properties of the superconformal indices are responsible for the description of moduli spaces and natural choices of the superpotentials. Equalities of dual indices remain valid beyond the infrared fixed points. This and other mathematical properties of the corresponding relations raise the problem of establishing the full physical content hidden in them.

For superconformal field theories (e.g., for $\mathcal{N} = 1$ theories at the infrared fixed points) the dimension of the scalar component of the gauge invariant chiral (and antichiral) superfields are related to their R-charges as $\Delta = 3R/2$. For the meson field $M = Q\widetilde{Q}$ with $G_c = SU(N)$ the dimension is $\Delta[M] = \Delta[Q] + \Delta[\widetilde{Q}] = 3R = 3(1 - N/N_f)$. The conventional superconformal algebra wisdom on unitarity demands that $\Delta[M] \geq 1$, or $N_f \geq 3N/2$, which is clearly broken in our theories for $N > 4$ (for the theories in Sect. IIB the unitarity is intact for $N = 3, 4$). Therefore one has to find physical ways out of this obstacle either by modifying the infrared dynamics or by other means.

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