Supersymmetric Dualities beyond the Conformal Window

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Using the superconformal (SC) indices techniques, we construct Seiberg type dualities for $\mathcal{N}=1$ supersymmetric field theories outside the conformal windows. These theories are physically distinguished by the presence of chiral superfields with small or negative R charges.

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Introduction.—Some of 4D $\mathcal{N}=1$ supersymmetric gauge field theories are related by the Seiberg duality [1]. A full list of presently known dualities of such type for simple gauge groups $G_c = SU(N)$, $SP(2N)$, G_2 is given in [2]. Remarkably, many of the listed dualities are new. Their discovery is based on the interplay between superconformal (SC) indices of [3–5] and the theory of elliptic hypergeometric integrals formulated in [6,7] (see also [8]).

The $SU(2,2|1)$ space-time symmetry group is generated by J_i, $\bar{J}_i$ [$SU(2)$ subgroups generators, or Lorentz rotations], P_μ, Q_α, $\bar{Q}_\alpha$ (supertranslations), K_μ, S_α, $\bar{S}_\alpha$ (special superconformal transformations), H (dilations) and R [$U(1)_R$-rotations]. For a distinguished pair of supercharges, say, $Q=\bar{Q}_1$ and $Q^\dagger = -\bar{S}_1$, one has

$$\{Q, Q^\dagger\} = 2\mathcal{H}, \qquad \mathcal{H} = H - 2\bar{J}_3 - 3R/2, \tag{1}$$

and the SC index is defined by the matrix integral

$$I(p,q,f_k) = \int_{G_c} d\mu(g)\, \mathrm{Tr}\Big[(-1)^F p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} e^{\sum_a g_a G^a} e^{\sum_k f_k F^k} e^{-\beta\mathcal{H}}\Big], \qquad \mathcal{R} = H - R/2, \tag{2}$$

where $d\mu(g)$ is the G_c-invariant measure and F is the fermion number operator. Operators G^a and F^k are the gauge and flavor group generators; p, q, g_a, f_k, β are group parameters (chemical potentials). The trace is taken over the whole space of states, but, because the operators used in (2) preserve relation (1), only the zero modes of the operator $\mathcal{H}$ contribute to the trace (hence, formally there is no dependence on β).

The key idea of Römelsberger [5] on the equality of SC indices (2) for the Seiberg dual theories was realized first by Dolan and Osborn for a number of examples [9]. These equalities are expressed in terms of the exact computability of elliptic beta integrals discovered in [6] or nontrivial symmetry transformations for higher order elliptic hypergeometric functions on root systems [7,10].

In addition to the description of new $\mathcal{N}=1$ dualities from known identities for integrals, another important result of [2] consisted in the formulation of new mathematical conjectures for integral identities following from known dualities. There are also examples when both the dualities and corresponding relations for integrals (indices) are new. The power of the theory of elliptic hypergeometric integrals in application to the SC indices techniques was demonstrated also in recent papers by Gadde *et al* [11,12].

Here we focus on some physical consequences following from the considerations of [2]. Namely, we concentrate on implications for the conformal windows introduced in [1,13]. In the original Seiberg work [1] it was shown that the corresponding $G_c = SU(N)$ SQCD duality has distinguished properties if the number of colors N and the number of chiral superfields (flavors) N_f satisfy

$$3N/2 < N_f < 3N. \tag{3}$$

This conformal window guarantees that both dual theories have asymptotic freedom and represent interacting SC theories at the IR fixed points. For $SP(2N)$ gauge groups with N_f flavors the conformal window is [13]

$$3(N+1)/2 < N_f < 3(N+1). \tag{4}$$

After some time it started to be believed that these conformal windows serve as the general necessary conditions for the existence of dualities between interacting gauge theories. Our goal is to describe some multiple dualities which do not fit this expectation.

Equality of SC indices of dual theories is a new nontrivial indication on the validity of Seiberg dualities. Earlier there were only the following justifying arguments [1].

1. The 't Hooft anomaly matching conditions. They were conjectured in [2] to be a consequence of the so-called total ellipticity condition for the elliptic hypergeometric integrals [8] describing SC indices.

2. Matching reduction of the number of flavors $N_f \to N_f - 1$. Integrating out k-th flavor quarks by the mass term $M_k^k Q_k \tilde{Q}^k$ in the original theory results in Higgsing the

magnetic theory gauge group with a reduction of the additional meson fields. From the elliptic hypergeometric integrals point of view this is realized by restricting in a special way a pair of parameters ($s_k t_k = pq$) which reduces the indices appropriately.

3. Matching of the moduli spaces and gauge invariant operators in dual theories. Perhaps, this information is hidden in the topological meaning of SC indices.

$SU(N)$ gauge group.—

(A) $SU(2N)$ gauge group with $N_f = 4$.—The starting electric theory has $G_c = SU(2N)$ and the matter fields content $4f + 4\bar{f} + T_A + \bar{T}_A$, where f and T_A denote the fundamental and absolutely antisymmetric tensor representations of G_c (the bar means conjugate representations). The flavor group for $N > 2$ is $SU(4) \times SU(4) \times U(1)_1 \times U(1)_2 \times U(1)_B$. The SC index is given by the following integral [2]

$$I_E = \kappa_N \int_{\mathbb{T}^{2N-1}} \prod_{1 \le i < j \le 2N} \frac{\Gamma(U z_i z_j, V z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \times \prod_{j=1}^{2N} \prod_{k=1}^{4} \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}, \quad (5)$$

where $\prod_{j=1}^{2N} z_j = 1$, $\mathbb{T}$ is the unit circle with positive orientation, $|U|, |V|, |s_k|, |t_k| < 1$, and $(UV)^{2N-2} \prod_{k=1}^{4} s_k t_k = (pq)^2$. We use conventions $\Gamma(a, b; p, q) \equiv \Gamma(a; p, q) \times \Gamma(b; p, q)$, $\Gamma(a z^{\pm 1}; p, q) \equiv \Gamma(az; p, q)\Gamma(az^{-1}; p, q)$, where

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \qquad |p|, \qquad |q| < 1,$$

is the elliptic gamma function. Finally,

$$\kappa_N = \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!}$$

with $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - a q^k)$. The parameters U, V, s_k, t_k are related to f_k in (2) and z_j replace g_a.

In [2] we described three magnetic duals for this model (one of which was found earlier in [14]). Equality of the corresponding SC indices is not proven yet, though their $N_f = 3$ simplifications do coincide, as follows from the identities established in [7]. The dualities beyond the conformal window of interest emerge after some "reduction" of these theories. Namely, we restrict the parameters U and V in (5) by the constraint $UV = pq$. Now $\prod_{k=1}^{4} s_k t_k = (pq)^{4-2N}$ and some of the parameters have modulus bigger than 1. In this case it is necessary to use the analytical continuation of integral (5) reached by passing from $\mathbb{T}$ to a contour separating sequences of integrand's poles converging to zero from their reciprocals. Because of the inversion formula $\Gamma(z, pqz^{-1}; p, q) = 1$, the parameters U and V disappear completely from the electric SC index. As a result, it becomes equal to the index of the theory without the fields T_A and $\bar{T}_A$ and global $U(1)_1 \times U(1)_2$ symmetry, which coincides with the Seiberg electric theory with $N_f = 4$ [1]. The type I A_N-elliptic beta integral evaluation [8] shows that for $N > 1$ the reduced SC index is equal to zero.

TABLE I. First $SU(2N)$ dual theory, where $m = 0, \ldots, N-2$.

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_B$	$U(1)_R$
q	f	f	1	0	-1	$-\frac{1}{2}(N-2)$
$\tilde{q}$	$\bar{f}$	1	f	0	1	$-\frac{1}{2}(N-2)$
H_m	1	T_A	1	-1	2	$2m - N + 3$
G	1	T_A	1	$N-1$	2	1
$\tilde{H}_m$	1	1	T_A	1	-2	$2m - N + 3$
$\tilde{G}$	1	1	T_A	$1-N$	-2	1

The dual magnetic theories are reduced in a similar way. We substitute into the magnetic indices described in [2] $U = \sqrt{pq}x$, $V = \sqrt{pq}x^{-1}$, where x is the chemical potential of the $U(1)_1$-group, and interpret them as the indices of reduced theories. The fields content and some of the R-charges of the resulting theories differ from the original ones. As a result, we find the following set of dualities. First magnetic theory is described in Table I In all our tables the first column contains symbols of the fields and the second—the gauge group representations. For $U(1)$ groups we give corresponding hypercharges. We skip also the vector superfield and its duals (adjoint representations of G_c and singlets of the flavor groups).

The global symmetry and field content of the second magnetic theory is the same as in Seiberg's dual theory with $N_f = 4$ (see Table II), but the gauge group is now $SU(2N)$ instead of $SU(N_f - 2N)$. The most complicated is the third magnetic theory (see Table III). SC indices of all these magnetic duals vanish for $N > 1$, which coincides with the electric index. For $N = 1$ we come to the family of dualities considered in detail in [15].

(B) $SU(N)$ gauge group with $N_f = N + 2$.—The electric part of the next set of dualities coincides with the Seiberg theory for $N_f = N + 2$ and arbitrary N. Its canonical magnetic dual has $G_c = SU(2)$, and it is IR free for $N > 4$ [1].

Our new magnetic dual theories have $G_c = SU(N)$ and the flavor symmetry group $SU(K) \times SU(M) \times U(1)_1 \times SU(K) \times SU(M) \times U(1)_2 \times U(1)_B$, where $M = N + 2 - K$ and $K = 1, \ldots, N + 1$. For the field content see Table IV.

TABLE II. Second $SU(2N)$ dual theory, where $m = 0, \ldots, N-2$.

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	$\bar{f}$	1	1	$-\frac{1}{2}(N-2)$
$\tilde{q}$	$\bar{f}$	1	$\bar{f}$	-1	$-\frac{1}{2}(N-2)$
M_k	1	f	f	0	$2k - N + 2$

TABLE III. Third $SU(2N)$ dual theory, where $k = 0, \dots, N-1$ and $m = 0, \dots, N-2$.

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_B$	$U(1)_R$
q	f	$\bar{f}$	1	0	-1	$-\frac{1}{2}(N-2)$
$\tilde{q}$	$\bar{f}$	1	$\bar{f}$	0	1	$-\frac{1}{2}(N-2)$
M_k	1	f	f	0	0	$2k-N+2$
H_m	1	T_A	1	-1	2	$2m-N+3$
G	1	T_A	1	$N-1$	2	1
$\tilde{H}_m$	1	1	T_A	1	-2	$2m-N+3$
$\tilde{G}$	1	1	T_A	$1-N$	-2	1

These dualities were derived in [2] (for $N = 2$, see [15]) from the equality of SC indices of the corresponding theories, which follows from the identities established by Rains [10] (for $K = 1$, see [7]). Here we just stress that they lie outside the conformal window (3) for $N > 3$, since the left-hand side inequality is violated. Surprisingly, for $N = 3$ we obtain a new duality lying *inside* the conformal window.

SP(2N) gauge group.—We describe now dualities lying outside the conformal window (4). The starting electric theory has $G_c = SP(2N)$ and the matter fields $8f + T_A$. As shown in [15], this theory has many dual partners (one of which was found earlier in [16]). The electric SC index has the form

$$I_E = \kappa_N \Gamma(t;p,q)^{N-1} \int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\Gamma(t z_i^{\pm1} z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1} z_j^{\pm1};p,q)} \times \prod_{j=1}^N \frac{\prod_{k=1}^8 \Gamma(t_k z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)} \frac{dz_j}{2\pi i z_j}, \qquad (6)$$

where $|t|, |t_k| < 1$, $t^{2N-2}\prod_{k=1}^8 t_k = (pq)^2$, and

$$\kappa_N = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!}.$$

This integral has nice symmetry transformations described by the Weyl group of the exceptional root system E_7 [10] (for $N = 1$, see [7]).

TABLE V. First $SP(2N)$ dual theory, where $J = 0, \dots, N-1$.

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	f	1	-1	$-\frac{N-3}{4}$
$\tilde{q}$	f	1	f	1	$-\frac{N-3}{4}$
M_J	1	T_A	1	2	$J-\frac{N-3}{2}$
$\tilde{M}_J$	1	1	T_A	-2	$J-\frac{N-3}{2}$

Now we restrict the t-parameter value to $t = \sqrt{pq}$ and analytically continue function (6) by replacing $\mathbb{T}$ to a contour separating geometric sequences of integrand's poles converging to zero from their reciprocals. This leads to the "decoupling" of the T_A-field from the electric theory, so that the same index is generated by the model with 8 quarks in fundamental representations of G_c and flavor group $SU(8)$ with the R-charge equal to $(3-N)/4$.

To obtain the dual description, we set $t = \sqrt{pq}$ in the magnetic SC indices [15] and interpret the resulting integrals as coming from different dual theories, similar to the $SU(2N)$ case described above. The field content of first magnetic theory is given in Table V (note the change of the flavor group) Second and third magnetic theories are described in Tables VI and VII. The third theory was found in [13], its flavor group coincides with the electric one. Note that SC indices of all four dual theories are equal to zero for $N > 2$, as follows from vanishing of the type I BC_N-elliptic beta integral for $N_f < N + 2$ [8].

Conclusion.—For all new dualities described in this paper we have checked validity of the 't Hooft anomaly matching conditions. As mentioned already, they pass also the new duality test by having equal SC indices.

The first and third magnetic duals of Sec. II(*A*) ["$SU(2N)$ gauge group with $N_f = 4$"] are rather unusual—they have the additional $U(1)_1$ group, which does not interact with the quarks and whose anomalies vanish. Vanishing of the indices of theories in Sec. II(*A*) for $N > 1$ and Sec. III ["$SP(2N)$ gauge group"] for $N > 2$ indicates that these models are similar to the Seiberg $SU(N)$ electric theory with $N_f \le N$ (e.g., they may have problems with the ground state). The $G_c = SP(4)$ case of Sec. III is

TABLE IV. $SU(N)$ magnetic theories with $N + 2$ flavors.

	$SU(N)$	$SU(K)$	$SU(M)$	$U(1)_1$	$SU(K)$	$SU(M)$	$U(1)_2$	$U(1)_B$	$U(1)_R$
q_1	$\bar{f}$	f	1	$\frac{K(K-2)}{N} - K + M$	1	1	$\frac{MK}{N}$	$1-M$	$\frac{2}{N+2}$
q_2	f	1	f	$-\frac{K(K-2)}{N}$	1	1	$\frac{-MK}{N}$	$1-K$	$\frac{2}{N+2}$
q_3	f	1	1	$\frac{MK}{N}$	f	1	$\frac{K(K-2)}{N} - K + M$	$M-1$	$\frac{2}{N+2}$
q_4	$\bar{f}$	1	1	$-\frac{MK}{N}$	1	f	$-\frac{K(K-2)}{N}$	$K-1$	$\frac{2}{N+2}$
X_1	1	f	1	M	1	f	$-K$	0	$\frac{4}{N+2}$
X_2	1	1	f	$-K$	f	1	M	0	$\frac{4}{N-2}$
Y_1	1	$\bar{f}$	$\bar{f}$	$K-M$	1	1	0	N	$\frac{2N}{N+2}$
Y_2	1	1	1	0	$\bar{f}$	$\bar{f}$	$K-M$	$-N$	$\frac{2N}{N+2}$

TABLE VI. Second $SP(2N)$ dual theory with 8 flavors.

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	$\bar{f}$	1	1	$-\frac{N-3}{4}$
$\tilde{q}$	f	1	$\bar{f}$	-1	$-\frac{N-3}{4}$
M_J	1	f	f	0	$J-\frac{N-2}{2}$

TABLE VII. Third $SP(2N)$ dual theory with 8 flavors.

	$SP(2N)$	$SU(8)$	$U(1)_R$
q	f	$\bar{f}$	$-\frac{N-3}{4}$
M_J	1	T_A	$J-\frac{N-3}{2}$

interesting as well. Corresponding electric theory is confining [13], which means that all our other dual theories (which were missed in [13]) also confine. Their SC indices obey $W(E_7)$ symmetry and can be evaluated explicitly [8], in difference from the $SP(2)$-group case [15].

As to the new dualities of Sec. II(B) ["$SU(N)$ gauge group with $N_f = N + 2$"], their origin is quite simple. The f and $\bar{f}$ representations of the dual $SU(2)$ gauge group are equivalent, and the corresponding flavor group gets enlarged from $SU(N_F) \times SU(N_F) \times U(1)_B$ to $SU(2N_F)$. Permuting corresponding character variables in an arbitrary way, one can construct "duals of duals" with $G_c = SU(N)$ in many different ways. Although this is a rather evident possibility, it was missed in the previous discussions of the Seiberg duality. We remark also that all the models described in our tables are asymptotically free and define interacting conformal field theories at the IR fixed point.

We conclude that the notion of conformal windows should be used with care—it is applicable only to particular types of dualities. Our results raise a natural question on classification of all 4D theories dual to the original Seiberg "minimal" electric SQCD. It is necessary to analyze various IR physics implications following from the described dualities. In particular, this concerns the structure of superpotentials (see, e.g., [17]). It would be interesting to understand which properties of the SC indices are responsible for the description of moduli spaces and natural choices of the superpotentials. Equalities of indices of dual theories remain valid away from the IR fixed points. This and other mathematical properties of SC indices raise the problem of establishing all physical information hidden in them.

For SC field theories (e.g., $\mathcal{N} = 1$ theories at the IR fixed points), the dimension of the scalar component of a gauge invariant chiral superfield is related to its R charge as $\Delta = 3R/2$. For the meson field $M = Q\tilde{Q}$ with $G_c = SU(N)$ the dimension is $\Delta[M] = \Delta[Q] + \Delta[\tilde{Q}] = 3R = 3(1 - N/N_f)$. The conventional SC algebra wisdom on unitarity demands that $\Delta[M] \geq 1$, or $N_f \geq 3N/2$, which is clearly broken in our theories for $N > 4$. Therefore one has to find physical ways out of this obstacle either by modifying the IR dynamics or by other means. The theories of Sec. II(B) are unitary for $N = 2, 3, 4$; the new $SU(3)$ duality satisfies thus all physical requirements and deserves further detailed investigation.

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