Future asymptotics of tilted Bianchi type II cosmologies

Sigbjørn Hervik[1], Woei Chet Lim[2], Patrik Sandin[3] and Claes Uggla[3]

[1] Faculty of Science and Technology, University of Stavanger, N-4036 Stavanger, Norway
[2] Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany
[3] Department of Physics, University of Karlstad, S-65188 Karlstad, Sweden

E-mail: sigbjorn.hervik@uis.no, wclim@aei.mpg.de, patrik.sandin@kau.se and claes.uggla@kau.se

Abstract

In this paper we study the future asymptotics of spatially homogeneous Bianchi type II cosmologies with a tilted perfect fluid with a linear equation of state. By means of Hamiltonian methods we first find a monotone function for a special tilted case, which subsequently allows us to construct a new set of monotone functions for the general tilted type II cosmologies. In the context of a new partially gauge-invariant dynamical system, this then leads to a proof for a theorem that for the first time gives a complete description of the future asymptotic states of the general tilted Bianchi type II models. The generality of our arguments suggests how one can produce monotone functions that are useful for determining the asymptotics of other tilted perfect fluid cosmologies, as well as for other sources.

PACS numbers: 04.20.−q, 04.20.Ha, 98.80.Jk

1. Introduction

Spatially homogeneous anisotropic perfect fluid models have been successfully studied during the last decades using a dynamical system approach. The book [1] summarizes most of the presently known results about the so-called non-tilted perfect fluid cosmologies, while the more general 'tilted' perfect fluid models have been primarily investigated more recently [2–15].

In all of the papers investigating tilted models, the analysis has relied on techniques from dynamical system theory. In particular, most of the results concern the identification of fixed points and a subsequent linear stability analysis of these points. In order to get a grip on the global aspects of the solutions, an effective tool is the use of monotone functions. Unfortunately such functions are hard to find, and in most of the previous works on tilted models the monotone functions were obtained by brute force, trial and error and luck. It

would therefore be desirable to have a more systematic method to seek and find monotone functions.

For non-tilted spatially homogeneous perfect fluid models, virtually all known results crucially rely on the existence of conserved quantities and monotone functions. These have turned out to be connected to the existence of certain symmetries, intimately associated with conservation laws such as the preservation of the number of particles in a fluid element, and the so-called scale-automorphism group [16]. Although not necessary, the symmetries and associated structures were, to a large extent, found by means of Hamiltonian techniques, see chapter 10 in [1] and [16]. One aim of this paper is to illustrate that one can generalize methods that previously have been applied to non-tilted models to tilted ones. To do so, we will consider an example—the tilted Bianchi type II models.

The tilted perfect fluid Bianchi type II models have been analyzed before as a dynamical system in [6]. In that paper, as well as in the present one, the perfect fluid was assumed to obey a linear equation of state characterized by $\tilde{p}/\tilde{\rho} = w = $ const, where $\tilde{p}$ and $\tilde{\rho}$ are the pressure and energy density with respect to the rest frame of the fluid, respectively; special cases of interest are dust, $w = 0$, radiation, $w = \frac{1}{3}$, a stiff fluid, $w = 1$, while a cosmological constant Λ can formally be regarded as a perfect fluid with $w = -1$. In this paper, however, we will consider the range $-1 < w < 1$.

In [6] the fixed points of the system were found and their linear stability properties were studied. It was observed that the future stability of the fixed points depended on the equation of state parameter w, but that a future stable fixed point existed for all w in the range $-1 < w < 1$ under consideration. On the basis of the linear analysis and numerical computations it was conjectured that the future linearly stable fixed points were attractors in the full state space, i.e. that all orbits, except possibly a set of measure zero, approach the stable fixed points asymptotically toward the future. This conjecture was corroborated by means of monotone functions for some ranges of w, but the complete picture was not fully substantiated, mainly because of the lack of a set of sufficiently restrictive monotone functions. In this paper we generalize methods that previously have been applied to the non-tilted models and find a set of new monotone functions that are sufficiently restrictive to determine the future asymptotics of all tilted solutions, thus filling in the missing gaps in the conclusions drawn in [6]. Our results as regards the general tilted Bianchi type II models are collected in theorem 3.1, and show that *all* orbits in the general tilted case approach the stable fixed points, i.e. these fixed points are not only locally stable but also globally stable. In section 4 we discuss the possible use of our monotone functions in the context of the initial mixmaster singularity, a much more formidable and physically interesting problem than the future asymptotics.

The outline of the paper is as follows. In section 2 we give a new partially gauge-invariant dynamical system for the general tilted Bianchi type II models, derived in appendix A where we also introduce some of our definitions and the relations needed for producing monotone functions. Then, based on the structure of monotone functions obtained for more special models by means of Hamiltonian methods, presented in appendix B, we obtain the new monotone functions in section 3, which lead to theorem 3.1. Finally, in section 4 we conclude with a discussion about the possible use of the monotone functions in the context of the initial oscillatory regime, and the structure of the monotone functions, why they exist, and why one can hope to expect similar structures in other models.

2. Dynamical system description of tilted Bianchi type II cosmologies

In [17] the so-called conformally Hubble-normalized orthonormal frame equations are given in full generality. These are in turn specialized to the spatially homogeneous Bianchi case in

appendix A.3 and then to the presently studied Bianchi type II models with a general tilted perfect fluid with a linear equation of state in appendix A.4, where we also introduce a new set of variables, invariant under frame rotations in the 23-plane. This yields the following state vector and dynamical system.

State vector:

$$\mathbf{S} = (\Sigma_+, \bar{\Sigma}, \tilde{\Sigma}^2, \check{\Sigma}^2, \Omega_k, v^2), \tag{1}$$

where we treat the squared quantities $\tilde{\Sigma}^2$, $\check{\Sigma}^2$, and v^2 as variables, but where we have refrained from giving them new names.

Evolution equations:

$$\Sigma'_+ = -(2 - q)\Sigma_+ - 3\check{\Sigma}^2 + 4\Omega_k + \tfrac{1}{2}(1 + w)G_+^{-1}v^2\Omega, \tag{2a}$$

$$\bar{\Sigma}' = -(2 - q)\bar{\Sigma} - 2\sqrt{3}\tilde{\Sigma}^2 + \sqrt{3}\check{\Sigma}^2 + \tfrac{\sqrt{3}}{2}(1 + w)G_+^{-1}v^2\Omega, \tag{2b}$$

$$(\tilde{\Sigma}^2)' = -2(2 - q - 2\sqrt{3}\bar{\Sigma})\tilde{\Sigma}^2, \tag{2c}$$

$$(\check{\Sigma}^2)' = -2[2 - q - 3\Sigma_+ + \sqrt{3}\bar{\Sigma}]\check{\Sigma}^2, \tag{2d}$$

$$\Omega'_k = 2(q - 4\Sigma_+)\Omega_k, \tag{2e}$$

$$(v^2)' = 2G_-^{-1}(1 - v^2)[3w - 1 - \Sigma_+ - \sqrt{3}\bar{\Sigma}]v^2. \tag{2f}$$

Constraint equation:

$$f(\mathbf{S}) = 4\check{\Sigma}^2\Omega_k - (1 + w)^2 G_+^{-2}v^2\Omega^2 = 0. \tag{2g}$$

The variables $\Sigma_+, \bar{\Sigma}, \tilde{\Sigma}^2, \check{\Sigma}^2$ describe the shear of the congruence normal to the homogeneous hypersurfaces relative to the overall Hubble expansion, while Ω_k is proportional to the Hubble-normalized scalar spatial curvature, and v is the speed of the matter relative to the rest space defined by the homogeneous hypersurfaces. In the above equations, w is the equation of state parameter, and the density parameter Ω is given by the Gauss constraint

$$\Omega = 1 - \Sigma^2 - \Omega_k, \tag{3}$$

where

$$\Sigma^2 = \Sigma_+^2 + \bar{\Sigma}^2 + \tilde{\Sigma}^2 + \check{\Sigma}^2. \tag{4}$$

The condition $\Omega \geqslant 0$ in combination with (3) yields $0 \leqslant \Sigma_+^2 + \bar{\Sigma}^2 + \tilde{\Sigma}^2 + \check{\Sigma}^2 + \Omega_k \leqslant 1$. The deceleration parameter q is given by

$$q = 2\Sigma^2 + \tfrac{1}{2}G_+^{-1}[1 + 3w + (1 - w)v^2]\Omega, \tag{5}$$

while

$$G_\pm = 1 \pm wv^2; \tag{6}$$

finally, $'$ denotes differentiation with respect to a dimensionless time parameter τ, determined by $d\tau = H \, dt$, where t is the clock time along the congruence normal to the spatially homogeneous hypersurfaces.

We now give a brief description of the invariant subset, see table 1, and fixed points of the system (2), which is analogous to the analysis given by Hewitt *et al* [6]. Note that although our system is invariant under frame rotations in the 23-plane it is not invariant under all rotations. Hence there exist multiple representations of solutions. For further comments on this, see [6].

Table 1. Invariant sets of the state space. The last column indicates if a subset is part of the boundary of the state space of the general tilted Bianchi type II models or if it is an interior subset.

	Name	Restrictions	Dimension	Interior/boundary
(i)	Non-tilted non-vacuum Bianchi type II	$v^2 = \check{\Sigma}^2 = 0$	4	Boundary
(ii)	Non-tilted non-vacuum Bianchi type I	$v^2 = \Omega_k = 0$	4	Boundary
(iii)	Vacuum Bianchi type II	$\check{\Sigma}^2 = \Omega = 0$	4	Boundary
(iv)	Vacuum Bianchi type I (Kasner)	$\Omega_k = \Omega = 0$	4	Boundary
(v)	Extreme tilt	$v^2 = 1$	4	Boundary
(vi)	Orthogonally transitive Bianchi type II	$\tilde{\Sigma}^2 = 0$	4	Interior

Fixed points:

 (i) The flat Friedmann solution, F: $-1 < w < 1$,

$$\Sigma_+ = \bar{\Sigma} = \tilde{\Sigma}^2 = \check{\Sigma}^2 = \Omega_k = v^2 = 0.$$

 (ii) The Collins–Stewart solution [19], CS: $-\frac{1}{3} < w < 1$,

$$\Sigma_+ = \tfrac{1}{8}(3w + 1), \qquad \bar{\Sigma} = \tilde{\Sigma}^2 = \check{\Sigma}^2 = v^2 = 0, \qquad \Omega_k = \tfrac{3}{64}(3w + 1)(1 - w).$$

 (iii) Hewitt's solution [3], H: $\frac{3}{7} < w < 1$,

$$\Sigma_+ = \frac{1}{8}(3w + 1), \qquad \bar{\Sigma} = \frac{\sqrt{3}}{8}(7w - 3), \qquad \tilde{\Sigma}^2 = 0,$$

$$\check{\Sigma}^2 = \frac{3(1 - w)(11w + 1)(7w - 3)}{16(17w - 1)},$$

$$\Omega_k = \frac{3(1 - w)(5w + 1)(3w - 1)}{4(17w - 1)}, \qquad v^2 = \frac{(3w - 1)(7w - 3)}{(11w + 1)(5w + 1)}.$$

 (iv) Hewitt *et al*'s 1-parameter set of solutions [6][4], HL: $w = \frac{5}{9}, \quad 0 < b = \text{const} < 1$,

$$\Sigma_+ = \frac{1}{3}, \qquad \bar{\Sigma} = \frac{1}{3\sqrt{3}}, \qquad \tilde{\Sigma}^2 = \frac{4}{27}b, \qquad \check{\Sigma}^2 = \frac{4(4b + 1)(8 - 3b)}{513},$$

$$\Omega_k = \frac{(2b + 1)(17 - 8b)}{171}, \qquad v^2 = \frac{3(4b + 1)(2b + 1)}{(17 - 8b)(8 - 3b)}.$$

 (v) Hewitt *et al*'s extreme tilted point [6], H_{ET}: $-1 < w < 1$,

$$\Sigma_+ = \frac{1}{3}, \qquad \bar{\Sigma} = \frac{1}{3\sqrt{3}}, \qquad \tilde{\Sigma}^2 = \frac{4}{27}, \qquad \check{\Sigma}^2 = \frac{100}{513}, \qquad \Omega_k = \frac{3}{19}, \quad v^2 = 1.$$

The system also admits the following fixed point sets: *the Kasner circle* K°, for which $v^2 = 0$, *the Kasner lines* $KL_\pm$, for which $v^2 = \text{const}$, and *the extremely tilted Kasner circle* K_{ET}°, for which $v^2 = 1$. These subsets reside on the Bianchi type I vacuum boundary, i.e. $\Sigma^2 = 1$, with $\tilde{\Sigma} = \check{\Sigma} = 0$, see [6]; however, since these fixed points do not play a prominent role in this paper we refrain from giving them explicitly.

Remark. Below we will refer to the relevant fixed point values for Σ_+ and $\bar{\Sigma}$ by Σ_{+0} and $\bar{\Sigma}_0$, respectively.

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[4] There is a misprint in [6] where the square root on the b in the Σ_1 expression of the line of fixed points, HL, has disappeared.

3. Future asymptotes in tilted Bianchi type II cosmology

In what follows certain monotone functions will play a crucial role. Based on our results in appendix A and appendix B, we hence begin by deriving them.

3.1. Monotone functions

There are several auxiliary equations that are useful in the context of monotone functions, see appendix A.

Auxiliary equations:

$$\Omega' = \left[2q - (1 + 3w) + (1 + w)(3w - 1 - \Sigma_+ - \sqrt{3}\bar{\Sigma})G_+^{-1}v^2\right]\Omega, \tag{7a}$$

$$Q' = -[2(1 - q) + \Sigma_+ + \sqrt{3}\bar{\Sigma}]Q, \tag{7b}$$

$$\Psi' = [2q - (1 + 3w)]\Psi, \tag{7c}$$

where

$$Q = (1 + w)G_+^{-1}v\,\Omega, \qquad \Psi = \Gamma^{-(1-w)}G_+^{-1}\Omega, \tag{8}$$

and Γ is given by equation (A.3).

   Since

$$2q - (1 + 3w) = 4\Sigma^2 - (1 + 3w)(1 - \Omega) + (1 - 3w)(1 + w)G_+^{-1}v^2\Omega \geqslant 0,$$
$$\text{if} \quad -1 < w \leqslant -1/3, \tag{9}$$

as follows from (A.19), Ψ is a monotonically increasing function when $-1 < w \leqslant -1/3$; henceforth we denote Ψ in this interval of the equation of state parameter by M_{F}.

   Before we continue, let us introduce some notation:

$$\phi_* = 1 - \Sigma_{+0}\Sigma_+ - \bar{\Sigma}_0\bar{\Sigma}, \tag{10a}$$

$$\varphi_* = [\Sigma_{+0}(\Sigma_+ - \Sigma_{+0}) + \bar{\Sigma}_0(\bar{\Sigma} - \bar{\Sigma}_0)]^2, \tag{10b}$$

$$\bar{\varphi}_* = [\bar{\Sigma}_0(\Sigma_+ - \Sigma_{+0}) - \Sigma_{+0}(\bar{\Sigma} - \bar{\Sigma}_0)]^2, \tag{10c}$$

where the subscript $_*$ henceforth denotes a specific fixed point, while Σ_{+0} and $\bar{\Sigma}_0$ are the associated fixed point values for Σ_+ and $\bar{\Sigma}$, respectively. In the following it is important that $\phi_* > 0$, which can be seen as follows:

$$\phi_* = \tfrac{1}{2}\left[1 - \Sigma_{+0}^2 - \bar{\Sigma}_0^2 + 1 - \Sigma_+^2 - \bar{\Sigma}^2 + (\Sigma_+ - \Sigma_{+0})^2 + (\bar{\Sigma} - \bar{\Sigma}_0)^2\right]$$
$$> \tfrac{1}{2}\left[1 - \Sigma_{+0}^2 - \bar{\Sigma}_0^2\right] > 0, \tag{11}$$

where we have used the Gauss constraint (3) and $\Omega > 0$, $\Omega_{\mathrm{k}} > 0$.

   For non-tilted perfect fluid models

$$M_{\mathrm{CS}:v^2=0} = \phi_{\mathrm{CS}}^{-2}\Omega_k^m\Omega^{1-m} = \frac{\Omega_k^m\Omega^{1-m}}{(1 - \Sigma_{+0}\Sigma_+)^2};$$
$$m = \frac{3(1 - w)\Sigma_{+0}}{8(1 - \Sigma_{+0}^2)}, \quad \Sigma_{+0} = \frac{1}{8}(1 + 3w), \quad \bar{\Sigma}_0 = 0, \tag{12}$$

is a monotonically increasing function. However, it is of interest to generalize this function by replacing Ω with Ψ, which is equal to Ω in the non-tilted case, i.e.

$$M_{\mathrm{CS}} = \phi_{\mathrm{CS}}^{-2}\Omega_k^m\Psi^{1-m}, \tag{13}$$

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which leads to the following time derivative in the present fully tilted state space:

$$(\ln M_{\rm CS})' = 3\phi_{\rm CS}^{-1}\left[(1-w)\left(\frac{\varphi_{\rm CS}}{(1-\Sigma_{+0}^2)\Sigma_{+0}^2} + \frac{\bar{\varphi}_{\rm CS}}{\Sigma_{+0}^2} + \tilde{\Sigma}^2\right)\right.$$
$$\left. + \frac{1}{8}(3-7w)\left(2\check{\Sigma}^2 + (1+w)G_+^{-1}v^2\Omega\right)\right], \tag{14}$$

and hence $M_{\rm CS}$ is monotonically increasing when $-1/3 < w \leqslant 3/7$.

In appendix B we derive a monotone function for the tilted orthogonally transitive case $\tilde{\Sigma}^2 = 0$ which can be written as

$$M_{\rm H} = \phi_{\rm H}^{-(3+13w)}(\check{\Sigma}^2)^{\frac{7w-3}{2}}\Omega_{\rm k}^{3w-1}\Psi^4. \tag{15}$$

It sometimes turns out to be the case that a monotone function for a given state space is also monotone in a more general state space in which the original is embedded in, at least for a limited range of the equation of state parameter, see e.g. [11, 16]. We hence compute the time derivative for $M_{\rm H}$ in the full tilted case; this gives us

$$(\ln M_{\rm H})' = \phi_{\rm H}^{-1}\left[\frac{49(16\varphi_{\rm H} + 3(1-w)(3+13w)\bar{\varphi}_{\rm H})}{8 + \frac{3}{8}(91w-31)(7w-3)} + \frac{3}{4}(5-9w)(13w+3)\tilde{\Sigma}^2\right], \tag{16}$$

and hence $M_{\rm H}$ is monotonically increasing when $3/7 < w \leqslant 5/9$.

The above monotone functions all have the form

$$M_* = \phi_*^{-\beta}(\tilde{\Sigma}^2)^{\alpha_1}(\check{\Sigma}^2)^{\alpha_2}\Omega_{\rm k}^{\alpha_3}(\Psi)^{\alpha_4}, \tag{17}$$

where $\beta = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \geqslant 0$. For the individual cases we have

$$M_{\rm F}: \quad \alpha_1 = \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = 1, \quad \beta = 2, \tag{18}$$

$$M_{\rm CS}: \quad \alpha_1 = \alpha_2 = 0, \quad \alpha_3 = m, \quad \alpha_4 = 1-m, \quad \beta = 2, \tag{19}$$

$$M_{\rm H}: \quad \alpha_1 = 0, \quad \alpha_2 = \frac{1}{2}(7w-3), \quad \alpha_3 = 3w-1, \quad \alpha_4 = 4, \quad \beta = 3+13w. \tag{20}$$

Let us assume the form (17) in order to find a monotone function for the range $5/9 < w < 1$. We obtain

$$M_{\rm H_{ET}} = \phi_{\rm H_{ET}}^{-46}(\tilde{\Sigma}^2)^4(\check{\Sigma}^2)^{10}\Omega_{\rm k}^9, \tag{21}$$

and hence $\alpha_1 = 4$, $\alpha_2 = 10$, $\alpha_3 = 9$, $\alpha_4 = 0$, $\beta = 46$, which leads to

$$\left(\ln M_{\rm H_{ET}}\right)' = \phi_{\rm H_{ET}}^{-1}\left[243\varphi_{\rm H_{ET}} + 207\bar{\varphi}_{\rm H_{ET}} + \frac{23(9w-5)}{3}G_+^{-1}(1-v^2)\Omega\right], \tag{22}$$

and thus $M_{\rm H_{ET}}$ is monotonically increasing when $5/9 < w < 1$.

3.2. Future asymptotic limits

Let us denote the invariant set for the general tilted Bianchi type II case, for which $(1-v^2)\,v^2\,\tilde{\Sigma}^2\,\check{\Sigma}^2\,\Omega\,\Omega_{\rm k} \neq 0$, by $\mathcal{S}_{\rm Gen}$, while we denote the invariant set for the orthogonally transitive case, for which $(1-v^2)\,v^2\,\check{\Sigma}^2\,\Omega\,\Omega_{\rm k} \neq 0$, by $\mathcal{S}_{\rm OT}$.

A local analysis, as in [6], reveals that the state space **S** has fixed points as *local sinks* according to table 2, which leads to the following bifurcation diagram:

$$\text{F} \xrightarrow{w=-1/3} \text{CS} \xrightarrow{w=3/7} \text{H} \xrightarrow{w=5/9,b=0} \text{HL} \xrightarrow{w=5/9,b=1} \text{H}_{ET}, \tag{23}$$

where the parameter b was introduced in solution (iv) above.

Table 2. Sinks for $\mathcal{S}_{\text{Gen}}$.

Range of w	Sink
$-1 < w \leqslant -1/3$	F
$-1/3 < w \leqslant 3/7$	CS
$3/7 < w < 5/9$	H
$w = 5/9$	HL
$5/9 < w < 1$	H_{ET}

It was conjectured in [6] that the above local sinks were future attractors, i.e. that all orbits, except a set of measure zero, asymptotically approach these sinks. Using the above monotone functions we, in the following theorem, show that *all* orbits that belong to $\mathcal{S}_{\text{Gen}}$ end up asymptotically at the above sinks, i.e. the local sinks are *globally* stable. The theorem uses the concept of the *ω-limit set* of a point, which we first define (see [1, 23], and references therein).

Definition. *For a dynamical system on a state space* **X***, the ω-limit set $\omega(\mathbf{x})$ of a point $\mathbf{x} \in \mathbf{X}$ is defined as the set of all accumulation points toward the future (i.e. as $\tau \to \infty$) of the orbit $\gamma(\tau)$ through* **x***.*

Theorem 3.1. *For all $\mathbf{x} \in \mathcal{S}_{\text{Gen}}$*

$$\omega(\mathbf{x}) \begin{cases} = \text{F} & -1 < w \leqslant -1/3 \\ = \text{CS} & -1/3 < w \leqslant 3/7 \\ = \text{H} & 3/7 < w < 5/9 \\ \subset \text{HL} & w = 5/9 \\ = H_{\text{ET}} & 5/9 < w < 1. \end{cases}$$

The proof makes use of the *monotonicity principle* [1, 18], which gives information about the global asymptotic behavior of solutions of a dynamical system. It is stated as follows: let ϕ_τ be a flow on $\mathbb{R}^n$ with **X** being an invariant set. Furthermore, let M be a $\mathcal{C}^1$ function $M : \mathbf{X} \to \mathbb{R}$. Then if M is increasing on orbits, then for all $\mathbf{x} \in \mathbf{X}$

$$\omega(\mathbf{x}) \subseteq \left\{ \mathbf{s} \in \overline{\mathbf{X}} \backslash \mathbf{X} \mid \lim_{\mathbf{y} \to \mathbf{s}} M(\mathbf{y}) \neq \inf_{\mathbf{X}} M \right\}. \tag{24}$$

Proof. We make use of the fact that $\mathcal{S}_{\text{Gen}}$ is a relatively compact set and hence that every orbit in $\mathcal{S}_{\text{Gen}}$ has an ω-limit point in $\overline{\mathcal{S}}_{\text{Gen}}$. Moreover, the ω-limit set of every orbit in $\mathcal{S}_{\text{Gen}}$ must be an invariant set. In every case $\inf_{\mathcal{S}_{\text{Gen}}} M_* = 0$, and hence we only have to investigate the set where $(\ln M_*)' = 0$. It turns out that in all cases the invariant set associated with $(\ln M_*)' = 0$ is precisely the pertinent fixed point(s), which thus is the ω-limit of every orbit in $\mathcal{S}_{\text{Gen}}$. Hence the proof, and the situations, is virtually identical to that of the non-tilted Bianchi type II perfect fluid case given in [1] on p 151. For the cases $-1 < w \leqslant -1/3$, $-1/3 < w \leqslant 3/7$, $3/7 < w < 5/9$ and $w = 5/9$, $5/9 < w < 1$, one uses M_{F}, M_{CS}, M_{H}, and $M_{H_{\text{ET}}}$, respectively. $\qquad\square$

Remark. It follows that the isolated fixed points F, CS, H, H_{ET} attract a 4-parameter set of solutions, as does the line HL. However, in this case it follows from the reduction theorem, see e.g. [20], that each point on the line attracts a 3-parameter set since the line is transversally hyperbolic. For a discussion about the physical interpretation of the future asymptotic limits given in theorem 3.1, see [6].

Corollary 3.2. *For all* $\mathbf{x} \in \mathcal{S}_{\mathrm{OT}}$

$$\omega(\mathbf{x}) = \begin{cases} \mathrm{F} & -1 < w \leqslant -1/3 \\ \mathrm{CS} & -1/3 < w \leqslant 3/7 \\ \mathrm{H} & 3/7 < w < 1. \end{cases}$$

Proof. This follows immediately from the previous proof, in combination with noting the form for $(\ln M_{\mathrm{H}})'$ when $\tilde{\Sigma}^2 = 0$. $\qquad\square$

We hence have established that the local bifurcation diagram

$$\mathrm{F} \xrightarrow{\ w=-1/3\ } \mathrm{CS} \xrightarrow{\ w=3/7\ } \mathrm{H} \tag{25}$$

from [6] reflects the global features of the solution space of $\mathcal{S}_{\mathrm{OT}}$.

4. Discussion

The goal of this paper was twofold: (i) to investigate the general Bianchi type II tilted perfect fluid models, and (ii) to use these models as an example for how one can produce monotone functions for tilted perfect fluid models with a non-diagonal metric.

As regards (i) we have found that there exists a collection of monotonically increasing functions that completely determine and describe the future asymptotics of tilted Bianchi type II models. Unfortunately the late stage regime of these models is not that physically interesting, but our results are of relevance as regards the intermediate regime of models with a cosmological constant, which is arguably more physically interesting. Nevertheless, there is no doubt that it is the initial mixmaster regime that is of most physical interest. We believe that our monotone functions are of use also for determining past asymptotics, but unfortunately they do not suffice to determine the initial asymptotic behavior. Nevertheless, we expect them to be crucial, perhaps even necessary, ingredients for a successful investigation of this regime. Hopefully they provide a structure that is similar to that in the recent proof of the Bianchi type XI attractor [22]. Even so, a proof about the initial singularity for the present models would require many other ideas as well, and is clearly outside the scope of this paper.

As regards (ii) we first note that all monotone functions take the form

$$M_* = \phi_*^{-\beta}(\tilde{\Sigma}^2)^{\alpha_1}(\check{\Sigma}^2)^{\alpha_2}\Omega_{\mathrm{k}}^{\alpha_3}(\Psi)^{\alpha_4}, \tag{26}$$

where $\beta = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \geqslant 0$. Moreover, the time derivatives of the monotone functions M_{CS}, M_{H}, $M_{\mathrm{H}_{\mathrm{ET}}}$, with nonzero $\Sigma_{+0}^2 + \bar{\Sigma}_0^2$ (and hence non-zero $\varphi_* + \bar{\varphi}_*$) all take the form

$$(\ln M_*)' = \phi_*^{-1}(A_*\varphi_* + B_*\bar{\varphi}_* + \mathrm{Inv}_*), \tag{27}$$

where A_* and B_* are constants, and Inv_* is a function that vanishes on one of the invariant sets in table 1. Outside these invariant sets the function Inv_* is positive only in a limited range of values for w. In the M_{CS} case $\mathrm{Inv}_{\mathrm{CS}}$ is zero for the non-tilted subset for which $v^2 = \check{\Sigma}^2 = 0$; in the M_{H} case $\mathrm{Inv}_{\mathrm{H}}$ is zero for the orthogonally transitive subset for which $\tilde{\Sigma}^2 = 0$; finally, in the $M_{\mathrm{H}_{\mathrm{ET}}}$ case $\mathrm{Inv}_{\mathrm{H}_{\mathrm{ET}}}$ is zero for the extreme tilt subset for which $v^2 = 1$. Clearly it is easier to find these monotone functions on these subsets first and then extend them to the larger state space, as we did in order to find the key monotone function M_{H} in appendix B. Moreover, we also first computed $M_{\mathrm{H}_{\mathrm{ET}}}$ for the extreme tilt subset $v^2 = 1$, although we did not use Hamiltonian methods in this case[5].

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[5] If one wants to use Hamiltonian methods to deal with extreme tilt, one first has to observe that these models have the same equations as those associated with a source that takes the form of a perfect fluid with a radiation equation of state $w = 1/3$, but with a null vector field replacing the time-like 4-velocity.

The importance of the hierarchical structure of Bianchi cosmology, where we have systems with boundaries on boundaries, has been emphasized before, see e.g. [21–23] and references therein. Here we see yet another context for this observation, which suggests that one should first try to find monotone functions for subsets and then attack the case one is really interested in. Hence one should first identify subsets for a given state space and write them on the form $Z_A = 0$ and then, if there exists a locally future stable fixed point on a given subset that admits subsets $Z_a = 0$, attempt to find monotone functions of the form[6]

$$M_* = \phi_*^{-\beta} \prod_a Z_a^{\alpha_a}. \tag{28}$$

Is there a deeper reason why monotone functions like this should exist? The analysis of the non-tilted case in [16] suggests that the existence of these monotone functions is related to the scale-automorphism group. In the tilted case this group can be viewed as consisting of an off-diagonal special automorphism group and a diagonal scale-automorphism group. The off-diagonal special automorphism gives rise to conserved momenta, if the underlying symmetry is not broken by source terms, and hence also to monotone functions, in a similar way as for the non-tilted models, see [16]. We have not discussed such monotone functions here since we did not need them for the future asymptotics, although they could be of help for the much more difficult past asymptotic behavior. However, the off-diagonal automorphisms also have other dynamical consequences. It is because of the off-diagonal automorphisms we only had diagonal shear degrees of freedom in ϕ_* in the present case, and it is because of this one would expect the present analysis also to be of relevance for other tilted models, and for other sources[7]. Moreover, in the present case we have encountered a hierarchy of source subsets that is typical. It is the increasing complexity of the source that breaks the vacuum symmetry group associated with the scale-automorphism group, creating a hierarchy of monotone functions associated with different source subsets. This is also what one encounters in the non-tilted case [16], but with an increasingly complex source this phenomenon seems to be even more pronounced! Hence a systematic attempt on the tilted models, or other sources, strongly suggests a deeper investigation into the dynamical consequences for the scale-automorphism group for the various relevant subsets. This is a quite ambitious task and we have therefore refrained from doing it here; instead we used the structures one can expect to arise from such an analysis without deriving all the details that completely determine all the monotone functions from the scale-automorphism group (for a hint of how this can be done, see the complete analysis of the class A non-tilted models from a Hamiltonian perspective in [16]) in order to see if this is likely to be a fruitful project. The answer seems to be yes.

Acknowledgments

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[6] Note that although the difficult part in (28) is the detailed structure of ϕ_*, the product part also appears in simpler monotone functions. This situation is somewhat similar to the one in [24], where special solution curves were used to obtain more general exact solutions, which again emphasizes the importance of considering special subsets first in order to obtain information about more general cases.

[7] The present methods extend previous work from diagonalizable models to non-diagonal models. Hence they may also be of relevance for non-diagonal class B models without any source! In particular we have in mind the generic exceptional Bianchi type $VI_{-1/9}$ vacuum models. There are good reasons for believing that there exists a monotone function of the present type for those models, and the construction of such a monotone function may very well be crucial if an attempt to determine these models' initial oscillatory behavior [25] is to be successful.

Appendix A. Definitions, relations and derivation of the dynamical system

Bianchi cosmologies with a tilted perfect fluid exhibit several structures. One set of structures arises from that spacetimes of spatially homogeneous cosmologies by definition are foliated by a geodesically parallel family of spatially homogeneous slices with a time-like unit normal vector n^a, see, e.g. [1, 26, 27] and references therein. Another set of structures arises from that the fluid has a 4-velocity $\tilde{u}^a \neq n^a$. Some structures naturally arise in a formulation with a time-independent left-invariant spatially homogeneous frame while other features are better captured in symmetry adapted orthonormal or conformally orthonormal frames. In this appendix we work out the relations between these structures and subsequently use them to derive monotone functions in appendix B and section 3. Moreover, these relations can also be used to obtain monotone functions for other tilted cosmologies. In addition we derive a new partially gauge-invariant dynamical system for Bianchi type II. Throughout we use units $c = 1$ and $8\pi G = 1$, where c is the speed of light and G is Newton's gravitational constant.

A.1. Perfect fluids

Making a 3+1 split w.r.t. n^a of the total stress-energy tensor T_{ab}, and an associated irreducible spatial decomposition, yields

$$T_{ab} = \rho n_a n_b + 2q_{(a}n_{b)} + p h_{ab} + \pi_{ab}, \tag{A.1a}$$

$$\rho = n^a n^b T_{ab}, \qquad q_a = -h_a{}^b n^c T_{bc}, \qquad p = \tfrac{1}{3} h^{ab} T_{ab}, \qquad \pi_{ab} = h_{\langle a}{}^c h_{b\rangle}{}^d T_{cd}, \tag{A.1b}$$

where $h_{ab} = n_a n_b + g_{ab}$ and $A_{\langle ab\rangle} = h_a{}^c h_b{}^d A_{cd} - \tfrac{1}{3} h_{ab} h^{cd} A_{cd}$; ρ, p are the total energy density and total effective pressure, respectively, measured in the rest space of n^a. In this paper we consider a perfect fluid, which yields the stress-energy tensor:

$$T^{ab} = (\tilde{\rho} + \tilde{p})\tilde{u}^a \tilde{u}^b + \tilde{p} g^{ab}, \tag{A.2}$$

where $\tilde{\rho}$ and $\tilde{p}$ are the energy density and pressure, respectively, in the rest frame of the fluid, while $\tilde{u}^a$ is its 4-velocity; throughout we assume that $\tilde{\rho} \geqslant 0$. Making a 3+1 split with respect to n^a leads to

$$\tilde{u}^a = \Gamma(n^a + v^a); \qquad n_a v^a = 0, \qquad \Gamma = (1 - v^2)^{-1/2}, \tag{A.3}$$

where v^a is the 3-velocity of the fluid, also known as the tilt vector; this gives

$$\tilde{\rho} = \Gamma^{-2} G_+^{-1} \rho, \qquad q^a = (1+w) G_+^{-1} \rho\, v^a, \qquad p = w\rho + \tfrac{1}{3}(1 - 3w) q_a v^a, \qquad \pi_{ab} = q_{\langle a} v_{b\rangle}, \tag{A.4}$$

where $G_\pm = 1 \pm w\, v^2$, $w = \tilde{p}/\tilde{\rho}$.

A.2. Orthonormal frame equations

In Bianchi cosmology the metric can be written as

$$^4\mathbf{g} = -N^2(x^0)\, \mathrm{d}x^0 \otimes \mathrm{d}x^0 + g_{ij}(x^0)\, \hat{\omega}^i \otimes \hat{\omega}^j \qquad (i, j = 1, 2, 3), \tag{A.5}$$

where $\{\hat{\omega}^i\}$ is a time-independent left-invariant co-frame dual to a time-independent left-invariant spatial frame $\{\hat{e}_i\}$, see e.g. p 38 in [1] about the meaning of group-invariant frames. This frame is a basis of the Lie algebra with structure constants $\hat{C}^i{}_{jk}$, i.e.

$$[\hat{e}_i, \hat{e}_j] = \hat{C}^k{}_{ij} \hat{e}_k = \left(\epsilon_{ijm} \hat{n}^{mk} + 2\hat{a}_{[i} \delta_{j]}{}^k\right) \hat{e}_k \quad \text{or, equivalently,} \qquad \mathrm{d}\hat{\omega}^i = -\tfrac{1}{2} \hat{C}^i{}_{jk} \hat{\omega}^j \wedge \hat{\omega}^k, \tag{A.6}$$

where we have decomposed the structure constants $\hat{C}^i{}_{jk}$ as [28]

$$\hat{C}^i{}_{jk} = \epsilon_{jkm}\hat{n}^{mi} + \hat{a}_n\delta^{ni}_{jk}, \qquad \hat{a}_i = \tfrac{1}{2}\hat{C}^j{}_{ij}, \tag{A.7}$$

$\hat{a}_i\,\hat{a}_j = \tfrac{1}{2}h\epsilon_{ikm}\epsilon_{jpn}\,\hat{n}^{kp}\hat{n}^{mn}$, $\hat{a}^2 = \hat{a}_i\hat{a}^i = \tfrac{1}{2}h[(\hat{n}^i{}_i)^2 - \hat{n}^i{}_j\hat{n}^j{}_i]$, where h is a constant group invariant that is unaffected by frame choices. The Bianchi models are divided into two main classes: the class A models for which $\hat{a}_i = 0$, and the class B models for which $\hat{a}_i \neq 0$.

There exists a natural symmetry adapted orthonormal frame associated with (A.5),

$$\boldsymbol{e}_0 = N^{-1}\frac{\partial}{\partial x^0}, \qquad \boldsymbol{e}_\alpha = e_\alpha{}^i(x^0)\,\hat{e}_i \qquad \text{where} \quad \delta^{\alpha\beta}e_\alpha{}^i(x^0)e_\beta{}^j(x^0) = g^{ij}(x^0),$$
$$\alpha, \beta = 1, 2, 3, \tag{A.8}$$

where $g^{ij}(x^0)$ is the left-invariant contravariant spatial metric associated with $g_{ij}(x^0)$.

Since the unit normal to the spatial symmetry surfaces $\mathbf{n} = \boldsymbol{e}_0$ by definition is hypersurface forming, and since it is the tangent to a geodesic congruence due to spatial homogeneity, we obtain

$$[\boldsymbol{e}_0, \boldsymbol{e}_\alpha] = C^\beta{}_{0\alpha}\,\boldsymbol{e}_\beta = f_\alpha{}^\beta\,\boldsymbol{e}_\beta = -[H\,\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\Omega^\gamma]\boldsymbol{e}_\beta, \tag{A.9a}$$

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = C^\gamma{}_{\alpha\beta}\,\boldsymbol{e}_\gamma = [2a_{[\alpha}\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta}n^{\delta\gamma}]\boldsymbol{e}_\gamma, \tag{A.9b}$$

where H is the Hubble variable; $\sigma_{\alpha\beta}$ is the shear associated with $\mathbf{n}$; Ω^α is the Fermi rotation which describes how the spatial triad rotates with respect to a gyroscopically fixed so-called Fermi frame[8]. Relations (A.8) and (A.9) yield

$$H = \tfrac{1}{3}\big(g^{-\frac{1}{2}}\big)\boldsymbol{e}_0\big(g^{\frac{1}{2}}\big) = -\tfrac{1}{3}e^\alpha{}_i\boldsymbol{e}_0\big(e_\alpha{}^i\big), \qquad \sigma_{\alpha\beta} = -e^\gamma{}_i\delta_{\gamma\langle\alpha}\boldsymbol{e}_0\big(e_{\beta\rangle}{}^i\big) \tag{A.10a}$$

$$\Omega^\alpha = \tfrac{1}{2}\epsilon^\alpha{}_\beta{}^\gamma e^\beta{}_i\boldsymbol{e}_0\big(e_\gamma{}^i\big), \qquad n^{\alpha\beta} = g^{-\frac{1}{2}}e^\alpha{}_i e^\beta{}_j\hat{n}^{ij}, \qquad a_\alpha = e_\alpha{}^i\hat{a}_i, \tag{A.10b}$$

where g is the determinant of the spatial metric g_{ij}, and $e^\alpha{}_i$ is the inverse of $e_\alpha{}^i$, i.e.

$$g = \det(g_{ij}) = (\det(e^\alpha{}_i))^{-2} = (\det(e_\alpha{}^i))^{-2}, \qquad e^\alpha{}_i e_\beta{}^i = \delta^\alpha{}_\beta. \tag{A.11}$$

Note that the above relations between the orthonormal frame variables $n^{\alpha\beta}$ and a_α and the structure constants $\hat{n}^{ij}$ and $\hat{a}_i$, associated with the time-independent left-invariant spatial frame, are needed when one wants to derive monotone functions, as exemplified by equation (B.25) in appendix B.

By means of the decomposition associated with n^a in (A.1), the matter conservation equation $\nabla_a T^{ab} = 0$ for a perfect fluid with a linear equation of state yields

$$(\ln\rho)\dot{} = (1+w)G_+^{-1}[-3H + f_{\alpha\beta}v^\alpha v^\beta + 2a_\alpha v^\alpha], \tag{A.12a}$$

$$\dot{v} = G_-^{-1}(1-v^2)\big[3wH + f_{\alpha\beta}c^\alpha c^\beta - 2wa_\beta c^\beta v\big]v, \tag{A.12b}$$

$$\dot{c}_\alpha = [\delta_\alpha{}^\beta - c_\alpha c^\beta][f_\beta{}^\gamma c_\gamma - v(a_\beta + \epsilon_{\beta\gamma\delta}n^{\delta\zeta}c_\zeta c^\gamma)], \tag{A.12c}$$

where $f_{\alpha\beta}$ is defined in equation (A.9a), $a_\beta + \epsilon_{\beta\gamma\delta}\,n^{\delta\zeta}\,c_\zeta\,c^\gamma = C^\zeta{}_{\beta\gamma}\,c_\zeta\,c^\gamma$ and where instead of the 3-velocity v_α we have found it convenient to introduce $v = \sqrt{v_\alpha v^\alpha} \geqslant 0$ and the unit vector $c_\alpha = v_\alpha/v$ as variables; for the first introduction of the unit vector $c^a = v^a/v$, see [2]. To obtain a more compact notation, we also introduced $\dot{f} = (f)\dot{} = \boldsymbol{e}_0 f = N^{-1}\mathrm{d}f/\mathrm{d}x^0 = \mathrm{d}f/\mathrm{d}t$, where t is the clock time associated with the normal congruence of the spatial symmetry surfaces (i.e. $N = 1$ for this parameterization).

---

[8] The sign in the definition of Ω^α is the same as in [17, 29], but opposite of that in [1].

When dealing with tilted perfect fluid models, several structures are most easily found by considering quantities associated with the rest frame of the fluid, and hence with $\tilde{u}^a$. In particular it is of interest to consider the particle number density $\tilde{n}$ and the chemical potential $\tilde{\mu}$, which, for a linear equation of state, can be defined as (see [27, 30] and references therein)

$$\tilde{\rho} = \tilde{n}^{1+w}, \qquad \tilde{\mu} = (1+w)\tilde{n}^w. \tag{A.13}$$

Defining

$$l = \tilde{n}\, g^{\frac{1}{2}}\, \Gamma, \tag{A.14}$$

yields the evolution equation $(\ln l)^{\cdot} = 2a_\alpha v^\alpha = 2(a_\alpha c^\alpha)\, v$, and hence l is a constant of the motion whenever $a_\alpha v^\alpha = 0$, e.g. for the class A perfect fluid models. Another quantity of interest is Taub's spatial circulation 1-form [31, 32] $t_a = \tilde{\mu}\, \tilde{u}_a$, whose spatial components can be written as $t_\alpha = \tilde{\mu}\, \Gamma\, v_\alpha = \tilde{\mu}\, \Gamma\, v\, c_\alpha$, with the norm $\tilde{\mu}\, \Gamma\, v$, which satisfies $(\ln \tilde{\mu}\Gamma v)^{\cdot} = f_{\alpha\beta} c^\alpha c^\beta$, which, together with (A.12c), yields $\dot{t}_\alpha = \tilde{\mu}\Gamma v(f_{\alpha\beta} - vC^\gamma{}_{\alpha\beta}c_\gamma)c^\beta$. This in turn gives $\hat{t}_i = e^\alpha{}_i t_\alpha$, where $e_0\, (e^\alpha{}_i) = -f_\beta{}^\alpha e^\beta{}_i$ obeys the equation

$$\dot{\hat{t}}_i = -(\tilde{\mu}\, \Gamma)^{-1}\, \hat{C}^k{}_{ij}\hat{t}_k\hat{t}^j, \tag{A.15}$$

which has a zero rhs for several models, e.g. Bianchi type II, thus leading to constants of the motion. Constants of the motion based on l and $\hat{t}_i$ play a critical role for deriving monotone functions for tilted perfect fluids, as exemplified by equation (B.25) in appendix B.

A.3. The Hubble-normalized dynamical system approach

In the conformal Hubble-normalized approach one factors out the Hubble variable H by means of a conformal transformation which yields dimensionless quantities [17, 33]. Note that the conformal Hubble-normalized approach reduces to the Hubble-normalized approach pioneered by Wainwright and Hsu [34] in a spatially homogeneous context, and that this formulation in turn owes a debt to Collins' insight about the importance of dimensionless variables [35]. In the spatially homogeneous case the Hubble-normalized approach amounts to the following:

$$
\begin{aligned}
(\Sigma_{\alpha\beta}, R^\alpha, N^{\alpha\beta}, A_\alpha) &= \frac{1}{H}(\sigma_{\alpha\beta}, \Omega^\alpha, n^{\alpha\beta}, a_\alpha), \\
(\Omega, P, Q_\alpha, \Pi_{\alpha\beta}) &= \frac{1}{3H^2}(\rho, p, q_\alpha, \pi_{\alpha\beta}),
\end{aligned}
\tag{A.16}
$$

where we have chosen to normalize the stress-energy quantities with $3H^2$ rather than H^2 in order to conform with the usual definition of Ω; in the perfect fluid case this results in

$$Q^\alpha = Q\, c^\alpha, \qquad P = w\Omega + \tfrac{1}{3}(1-3w)vQ, \qquad \Pi_{\alpha\beta} = vQc_{\langle\alpha}c_{\beta\rangle}, \tag{A.17}$$

where

$$Q = (1+w)G_+^{-1}v\Omega. \tag{A.18}$$

In addition to this we choose a new dimensionless time variable τ by means of the lapse choice $N = H^{-1}$. Since H is the only variable with dimension, its evolution equation, $H' = -(1+q)H$, which defines the deceleration parameter q decouples from the remaining equations for dimensional reasons; throughout a prime denotes $d/d\tau$. By means of one of Einstein's equations $G_{ab} = T_{ab}$—the Raychaudhuri equation, we obtain

$$q = 2\Sigma^2 + \tfrac{1}{2}(\Omega + 3P) = 2\Sigma^2 + \tfrac{1}{2}G_+^{-1}[1+3w+(1-w)v^2]\Omega, \qquad \Sigma^2 = \tfrac{1}{6}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta}, \tag{A.19}$$

where the last equality for q refers to the perfect fluid case. The remaining Einstein field equations together with the Jacobi identities yield the following set of equations, which we divide into evolution equations and constraints:

Evolution equations

$$\Sigma'_{\alpha\beta} = -(2-q)\Sigma_{\alpha\beta} - 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\Sigma_{\beta\rangle\gamma}R_\delta - {}^3\mathcal{R}_{\langle\alpha\beta\rangle} + 3\Pi_{\alpha\beta}, \tag{A.20a}$$

$$(N^{\alpha\beta})' = (3q\delta_\gamma{}^{(\alpha} - 2F_\gamma{}^{(\alpha})N^{\beta)\gamma}, \tag{A.20b}$$

$$A'_\alpha = F_\alpha{}^\beta A_\beta, \tag{A.20c}$$

$$\Omega' = (2q-1)\Omega - 3P + 2A_\alpha Q^\alpha - \Sigma_{\alpha\beta}\Pi^{\alpha\beta}, \tag{A.20d}$$

$$v' = -G_-^{-1}(1-v^2)\left[1 - 3w + \Sigma_{\alpha\beta}c^\alpha c^\beta + 2w(A_\beta c^\beta)v\right]v, \tag{A.20e}$$

$$c'_\alpha = \left[\delta_\alpha{}^\beta - c_\alpha c^\beta\right]\left[F_\beta{}^\gamma c_\gamma - v\left(A_\beta + \epsilon_\beta{}^\gamma{}_\delta N^{\delta\zeta}c_\zeta c_\gamma\right)\right]. \tag{A.20f}$$

Constraint equations

$$0 = 1 - \Sigma^2 - \Omega_{\rm k} - \Omega, \tag{A.21a}$$

$$0 = \left(3\delta_\alpha{}^\gamma A_\beta + \epsilon_\alpha{}^{\delta\gamma}N_{\delta\beta}\right)\Sigma^\beta{}_\gamma - 3Q_\alpha, \tag{A.21b}$$

$$0 = A_\beta N^\beta{}_\alpha, \tag{A.21c}$$

where $A^2 = A_\alpha A^\alpha$, and

$$F_\alpha{}^\beta = q\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta - \epsilon_\alpha{}^\beta{}_\gamma R^\gamma, \qquad \Omega_{\rm k} = \tfrac{1}{12}\mathcal{B}^\alpha{}_\alpha + A^2, \tag{A.22a}$$

$${}^3\mathcal{R}_{\langle\alpha\beta\rangle} = \mathcal{B}_{\langle\alpha\beta\rangle} - 2\epsilon^{\gamma\delta}{}_{\langle\alpha}N_{\beta\rangle\gamma}A_\delta, \qquad \mathcal{B}_{\alpha\beta} = 2N_{\alpha\gamma}N^\gamma{}_\beta - N^\gamma{}_\gamma N_{\alpha\beta}. \tag{A.22b}$$

The Gauss constraint (A.21a) and the Codazzi constraint (A.21b) figure prominently throughout.

When deriving monotone functions in section 3.1, we make use of the Bianchi type II specializations, see equation (7), of the following equations:

$$Q' = -\left[2 - q - F_{\alpha\beta}c^\alpha c^\beta + 2(A_\alpha c^\alpha)v\right]Q, \tag{A.23a}$$

$$(\ln\Psi)' = 2q - (1+3w) + 2(1+w)(A_\alpha c^\alpha)v, \qquad \Psi = \Gamma^{-(1-w)}G_+^{-1}\Omega, \tag{A.23b}$$

where Ψ is intimately connected with $\tilde{n}$ and l, since Ψ is obtained by taking $\tilde{n}\,\Gamma = g^{-1/2}l$ and raising the rhs to the power $1 + w$ and normalizing with H.[9]

A.4. Bianchi type II

For the Bianchi type II models we have $A_\alpha = 0$, and in addition we can choose a spatial frame e_α to be an eigenframe of the matrix $N_{\alpha\beta}$, with $N_{11} \neq 0$, while otherwise $N_{\alpha\beta} = 0$. It follows that equation (A.20b) yields $R_2 = \Sigma_{31}$, $R_3 = -\Sigma_{12}$, and that the Codazzi constraint (A.21b) implies

$$v_1 = 0 = c_1. \tag{A.24}$$

Inserting the conditions of equation (A.24) into equation (A.20f) gives the following relation:

$$0 = \Sigma_{12}c_2 + \Sigma_{31}c_3 = -R_3 c_2 + R_2 c_3 \qquad \Leftrightarrow \qquad \epsilon_{AB}R^A c^B = 0, \tag{A.25}$$

---

[9] This quantity has appeared before in the literature, e.g. in [6], where $\Gamma^{-(1-w)}G_+^{-1}$ has been denoted by β.

where $A, B = 2, 3$ and ϵ_{AB} is the two-dimensional permutation that has $\epsilon_{23} = 1$ (hence $c_A c^A = 1$). It follows from $\epsilon_{AB} R^A c^B = 0$ that $R_A \propto c_A \propto Q_A$, where the last relation holds when $\Omega \neq 0$; hence Σ_{12} and Σ_{31} are linearly dependent and can be replaced by a single variable.

We have the freedom to rotate in the 23-plane, which is expressed in the field equations as the freedom to choose R_1. To obtain a set of variables that are invariant under such rotations we introduce the following new shear variables:

$$\Sigma_+ = \tfrac{1}{2}\Sigma_A{}^A = -\tfrac{1}{2}\Sigma_{11}, \qquad \bar{\Sigma} = \tfrac{1}{\sqrt{3}}(\Sigma_{AB} - \Sigma_+\delta_{AB})\big(c^A c^B - \tfrac{1}{2}\delta^{AB}\big), \qquad (A.26a)$$

$$\tilde{\Sigma} = \tfrac{1}{\sqrt{3}}(\Sigma_{AB} - \Sigma_+\delta_{AB})\epsilon^B{}_C\big(c^A c^C - \tfrac{1}{2}\delta^{AC}\big), \qquad \check{\Sigma}^2 = \tfrac{1}{3}\big(\Sigma_{12}^2 + \Sigma_{31}^2\big). \qquad (A.26b)$$

Hewitt *et al* [6] make use of the freedom to rotate in the 23-plane to set $c_2 = 0$, which yields that $c_3 = 1$ and $R_2 = 0 = \Sigma_{31}$. This leads to a correspondence between the variables $\Sigma_-, \Sigma_1, \Sigma_3$ in [6] to our variables, when setting $c_2 = 0$, according to $\bar{\Sigma} = -\Sigma_-, \check{\Sigma}^2 = \Sigma_3^2$, $\tilde{\Sigma}^2 = \Sigma_1^2$.

Because of the existence of discrete symmetries one can simplify the analysis, e.g. the eigenvalue analysis, by introducing the following state vector $\mathbf{S} = (\Sigma_+, \bar{\Sigma}, \tilde{\Sigma}^2, \check{\Sigma}^2, \Omega_k, v^2)$, where $\Omega_k = N_{11}^2/12$, and where Ω can be obtained in terms of $\mathbf{S}$ via the Gauss constraint (A.21a). By inserting these restrictions and definitions into (A.20) and (A.21), we obtain the dynamical system given in section 2.

Appendix B. Hamiltonian considerations and derivation of monotone functions

B.1. Hamiltonian considerations

The scalar Hamiltonian is given by

$$\tilde{\mathcal{H}} = 2N g^{\frac{1}{2}} n^a n^b (G_{ab} - T_{ab}) = 2\tilde{N} g\, n^a n^b (G_{ab} - T_{ab}) = \tilde{N}\mathcal{H} = \tilde{N}(T + U_g + U_f), \qquad (B.1)$$

where we have defined $\tilde{N} = Ng^{-1/2}$, and where $(T, U_g, U_f) = 6 g H^2(-1+\Sigma^2, \Omega_k, \Omega)$. By means of (A.4), (A.13) and (A.14), this yields

$$U_f = 2g\rho = 2l^{1+w} g^{(1-w)/2} \Gamma^{1-w} G_+, \qquad (B.2)$$

while T and U_g depend on the model and the metric representation.

In [36] it was shown that the tilted orthogonally transitive Bianchi models exhibit a so-called time-like homothetic Jacobi symmetry. It was later realized that such symmetries are related to the existence of monotone functions (Uggla in chapter 10 in [1], and [16]). Unfortunately the analysis in [36] is quite cumbersome and we will therefore make a new derivation of the 'homothetic structure' and from this derive a monotone function. To do so, we need to connect the time-independent spatially homogeneous frame in (A.5) with the orthonormal frame. This is done in two steps: (i) diagonalization by means of the off-diagonal special automorphism group, and (ii) normalization by means of diagonal scaling. We hence write $e_\alpha{}^i$ in the transformation (A.8), i.e. $\mathbf{e}_\alpha = e_\alpha{}^i(x^0)\hat{\mathbf{e}}_i$, as $e_\alpha{}^i = (D^{-1})_\alpha{}^j (S^{-1})_j{}^i$, or $e^\alpha{}_i = D^\alpha{}_j S^j{}_i$, where $S^j{}_i$ is a special automorphism transformation that hence leaves $\hat{a}_i$ and $\hat{n}^{ij}$ unaffected. In addition we define the new metric variables $\beta^1, \beta^2, \beta^3$ via the matrix $D^\alpha{}_j$ so that

$$(D^{-1})_\alpha{}^j = \begin{pmatrix} \exp(-\beta^1) & 0 & 0 \\ 0 & \exp(-\beta^2) & 0 \\ 0 & 0 & \exp(-\beta^3) \end{pmatrix}, \qquad (B.3)$$

where $\beta^\alpha = \beta^\alpha(x^0)$; hence $g^{1/2} = \exp(\beta^1 + \beta^2 + \beta^3)$.

To obtain the Hamiltonian for the orthogonally tilted transitive Bianchi models we choose a spatially homogeneous frame so that the line element can be written as (A.5) with all structure constants being zero except $\hat{n}^{11} = \hat{n}_1$. In the orthogonally transitive case we can specify the spatially homogeneous frame so that g_{ij} in (A.5) have one off-diagonal component, g_{12}, and so that the perfect fluid velocity has a single non-zero left-invariant frame component, $\hat{v}_3$. Hence we follow [36] and write[10]

$$S^\alpha{}_j = \begin{pmatrix} 1 & -\sqrt{2}\hat{n}_1\theta^3(x^0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.4}$$

It follows from (A.10*b*) that

$$n^{11} = \exp(\beta^1 - \beta^2 - \beta^3)\hat{n}_1. \tag{B.5}$$

Due to that θ_3 is associated with the off-diagonal special automorphism group, the associated momentum, which is proportional to σ_{12}, is conserved [36]. Furthermore, note that (A.15) yields the constant of the motion $\hat{t}_3 = $ const. This is consistent with the Codazzi constraint, also known as the Hamiltonian momentum constraint, which linearly relates these two constants to each other. The constants $\hat{n}_1$, l, $\hat{t}_3$ allow us to write $T + U_g + U_f$ in (B.1) so that

$$\mathcal{H} = T + U_g + U_f = T_d + U_c + U_g + U_f, \tag{B.6}$$

where U_c is the so-called centrifugal potential, which is proportional to the Σ_{12}^2 term in Σ^2, where, furthermore, U_c, U_g and U_f are all expressible in terms of β^α, and no other time-dependent quantities. Expressing $\tilde{N}T_d$ in terms of $\dot{\beta}^\alpha$ or the associated momenta π_α, see [16] and also [37], by means of (A.10) and the transformations in this subsection leads to

$$\tilde{N}T_d = 2\tilde{N}^{-1}\mathcal{G}_{\gamma\delta}\dot{\beta}^\gamma\dot{\beta}^\delta = \tfrac{1}{4}\tilde{N}\mathcal{G}^{\gamma\delta}\pi_\gamma\pi_\delta, \tag{B.7}$$

where $\mathcal{G}_{\gamma\delta}$ is known as the minisuperspace metric for the diagonal degrees of freedom, which, together with its inverse $\mathcal{G}^{\alpha\beta}$, is given by

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}; \qquad \mathcal{G}^{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \tag{B.8}$$

The centrifugal potential $U_c = 2\,g\,H^2\,\Sigma_{12}^2 = 2\,g\,\sigma_{12}^2$, re-expressed via the Codazzi constraint in terms of $\hat{t}_3$, yields

$$U_c \propto \exp[-2(\beta^1 - \beta^2)]. \tag{B.9}$$

By means of $U_g = 6\,g\,H^2\,\Omega_k$ and (B.5) we find that

$$U_g \propto \exp(4\beta^1). \tag{B.10}$$

We can write $\tilde{u}^a\,\tilde{u}_a = -1$ as $0 = 1 - \Gamma^2 + \tilde{\mu}^{-2}t_\alpha\,t^\alpha$, and by defining

$$F = (1+w)^{-2}l^{-2w}g^w t_\alpha t^\alpha = (1+w)^{-2}l^{-2w}g^w g^{ij}\hat{t}_i\hat{t}_j, \tag{B.11}$$

we obtain

$$0 = 1 - \Gamma^2 + F\,\Gamma^{2w}, \tag{B.12}$$

which allows one to, in general, implicitly express Γ^2 in terms of F, i.e. $\Gamma^2 = \Gamma^2(F)$. In the present case we find that $F = (1+w)^{-2}\,l^{-2w}\,g^w\,g^{33}\,\hat{t}_3^2$, which yields

$$F = (1+w)^{-2}\,l^{-2w}\,\hat{t}_3^2\,\exp[w(\beta^1 + \beta^2) - (1-w)\beta^3]. \tag{B.13}$$

It follows that

$$U_f \propto \exp[(1-w)(\beta^1 + \beta^2 + \beta^3)]\phi, \tag{B.14}$$

where $\phi := \Gamma^{1-w}G_+$ is a function of the particular combination $w(\beta^1 + \beta^2) - (1-w)\beta^3$ only.

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[10] There is a typographical error in equation (2.61) in [36]; the exponent should be -1 and not $-1/2$ of $(\hat{m}^{(3)})$ in the expression for s_3.

B.2. Derivation of monotone functions

Based on the Hamiltonian for the diagonal degrees of freedom for the orthogonally transitive type II case in this subsection we derive a monotone function that is of key importance for understanding the dynamics. In chapter 10 in [1] and in [16] it is shown that monotone functions are associated with 'homothetic' symmetries of the potential, i.e. we require that there exists a vector $\mathbf{c} = c^\alpha \partial_{\beta^\alpha}$ such that $\mathbf{c}U = \mathbf{c}(U_\mathrm{c} + U_\mathrm{g} + U_\mathrm{f}) = r\,U$, where r is a constant. For this to be possible we require (i) that $\mathbf{c}\phi = 0$, and hence that

$$\mathbf{c}(w(\beta^1 + \beta^2) - (1-w)\beta^3) = w(c^1 + c^2) - (1-w)c^3 = 0, \tag{B.15}$$

and (ii) $\mathbf{c}U_\mathrm{c} = rU_\mathrm{c}$, $\mathbf{c}U_\mathrm{g} = rU_\mathrm{g}$ and $\mathbf{c}U_\mathrm{f} = r\,U_\mathrm{g}$, which due to condition (i) yields $\mathbf{c}\exp[(1-w)(\beta^1 + \beta^2 + \beta^3)] = r\,\exp[(1-w)(\beta^1 + \beta^2 + \beta^3)]$. This leads to

$$-2(c^1 - c^2) = r, \qquad 4c^1 = r, \qquad (1-w)(c^1 + c^2 + c^3) = r, \tag{B.16}$$

which yields $c^2 = 3c^1$, $c^3 = 4c^1 w/(1-w)$; w.l.g we can choose $c^1 = 1 - w$, which gives

$$(c_1, c_2, c_3) = (1-w, 3(1-w), 4w), \qquad r = 4(1-w). \tag{B.17}$$

The causal character of this vector with respect to the metric $\mathcal{G}_{\alpha\beta}$ is crucial [16]; we obtain

$$\mathcal{G}_{\gamma\delta}c^\gamma c^\delta = -2(1-w)(3+13w), \tag{B.18}$$

which is time-like if $-3/13 < w < 1$. The above properties of the potential U imply that the model satisfies the criteria given in [16] for admitting a monotone function given by[11]

$$M \propto (c^\alpha \pi_\alpha)^2 \exp\left[-r\frac{\mathcal{G}_{\gamma\delta}c^\gamma \beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma c^\delta}\right]. \tag{B.19}$$

We now have to express M in the dynamical system variables $\mathbf{S}$. We do so in two steps by first using $\Sigma_{\alpha\alpha} = \Sigma_\alpha$, and then in the second step we go over to the presently used dynamical system variables via the current gauge fixing (see the previous discussion about the correspondence between our variables and those used in [6]). Let us define

$$(V_\mathrm{g},\ V_\mathrm{f}) = \exp\left[-r\frac{\mathcal{G}_{\gamma\delta}c^\gamma \beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma c^\delta}\right](U_\mathrm{g}, U_\mathrm{f}); \tag{B.20}$$

then

$$\exp\left[-r\frac{\mathcal{G}_{\gamma\delta}c^\gamma \beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma c^\delta}\right]\mathcal{H} = \exp\left[-r\frac{\mathcal{G}_{\gamma\delta}c^\gamma \beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma c^\delta}\right]T + V_\mathrm{g} + V_\mathrm{f} = 0, \tag{B.21}$$

which yields

$$\exp\left[-r\frac{\mathcal{G}_{\gamma\delta}c^\gamma \beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma c^\delta}\right] = -\left(\frac{V_\mathrm{g} + V_\mathrm{f}}{T}\right) \propto \frac{V_\mathrm{g} + V_\mathrm{f}}{\pi_0^2(1 - \Sigma^2)}, \tag{B.22}$$

and hence

$$M \propto \left(\frac{c^\alpha \pi_\alpha}{\pi_0}\right)^2 \frac{V_\mathrm{g} + V_\mathrm{f}}{(1 - \Sigma^2)} = \left(\frac{c^\alpha \pi_\alpha}{\pi_0}\right)^2 V_\mathrm{f}\Omega^{-1}, \tag{B.23}$$

since $V_\mathrm{g}/V_\mathrm{f} = \Omega_\mathrm{k}/\Omega$ and $1 - \Sigma^2 = \Omega_\mathrm{k} + \Omega$.

It follows from our definitions that $\pi_\alpha = \frac{1}{6}\pi_0(2 - \Sigma_\alpha)$, $\pi_0 = \pi_1 + \pi_2 + \pi_3$, see also [16, 21]. In combination with (B.17) we find that this yields

$$\frac{c^\alpha \pi_\alpha}{\pi_0} \propto 1 - \frac{1}{8}(1-w)\Sigma_1 - \frac{3}{8}(1-w)\Sigma_2 - \frac{1}{2}w\Sigma_3 = 1 - \frac{\sqrt{3}}{8}(7w-3)\bar{\Sigma} - \frac{1}{8}(1+3w)\Sigma_+.$$
$$\tag{B.24}$$

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[11] This is actually the square of the monotone function in [16], but we find this form more convenient in the present context.

Next we need to solve for V_f in terms of the state space variables $(\Sigma_+, \bar{\Sigma}, \check{\Sigma}^2, \Omega_k, v^2)$. This can be done with the help of the constants of motion $\hat{n}_1$, l, $\hat{t}_3$ through equations (B.20), (B.13), (B.12), (B.5) and (B.2), using the form (B.17) of the homothetic vector **c**. After some algebra one finds that

$$V_f \propto \left[(v^2)^{3-7w} \Gamma^{8(1-w)} G_+^{2(1+7w)} \left(\frac{\Omega_k}{\Omega} \right)^{-(1-w)} \right]^{1/(3+13w)}. \tag{B.25}$$

Taking $M^{-(3+13w)/2}$ and replacing Ω with Ψ via (8), and v^2 through equation (2*g*), yields the monotone function M_H, which we give in the main text. Analogous, but much simpler, Hamiltonian methods were used to find the monotone function M_{CS} for the non-tilted Bianchi type II models, which is also given in the main text.

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