

Superconformal Indices of $\mathcal{N} = 4$ SYM Field Theories

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Abstract. Superconformal indices (SCIs) of $4d$ $\mathcal{N} = 4$ SYM theories with simple gauge groups are described in terms of elliptic hypergeometric integrals. For F_4, E_6, E_7, E_8 gauge groups this yields first examples of integrals of such type. S -duality transformation for G_2 and F_4 SCIs is equivalent to a change of integration variables. Equality of SCIs for $SP(2N)$ and $SO(2N+1)$ group theories is proved in several important special cases. Reduction of SCIs to partition functions of $3d$ $\mathcal{N} = 2$ SYM theories with one matter field in the adjoint representation is investigated, corresponding $3d$ dual partners are found, and some new related hyperbolic beta integrals are conjectured.

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1. Introduction

The problem of electric-magnetic duality for non-abelian gauge theories was raised by Goddard et al. [1] (see also [2]). Its consideration in the context of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensional space-time is a quite old area of research [3]. This duality (called also S -duality) states the equivalence of the theory with an “electric” gauge group G_c to a similar theory with a “magnetic” gauge group G_c^\vee . Let G_c be a simply laced Lie group. This means that its Dynkin diagram contains only simple links, and therefore all roots of the corresponding Lie algebra have the same length, which is true for $SU(N), SO(2N), E_6, E_7,$ and E_8 groups. Then, $G_c^\vee = G_c$ and the S -duality transformation maps the complex coupling constant $\tau = \theta/2\pi + 4\pi i/g^2$ to $-1/\tau$. Taken together with the symmetry transformation $\tau \rightarrow \tau + 1$, the S -duality becomes equivalent to the $SL(2, \mathbb{Z})$ -group of modular transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \quad (1)$$

For the non-simply laced gauge groups, the S -duality acts as $\tau \rightarrow -1/m\tau$, where m is the ratio of the lengths-squared of long and short roots of the corresponding root system. One has $m=2$ for $SO(2N+1)$ and $SP(2N)$ group theories dual to each other [1]. For F_4 and G_2 groups one has $m=2$ and $m=3$, respectively; corresponding theories were discussed in [4] from the algebraic point of view and the S -duality transformation of their moduli spaces was described.

Here we discuss a new test of $\mathcal{N}=4$ SYM field theory dualities based on the superconformal indices (SCIs) suggested by Kinney et al. in [5] (for the definition of indices in $\mathcal{N}=1$ theories, see [6,7]). $\mathcal{N}=4$ SYM theory has the $PSU(2,2|4)$ space-time symmetry group generated by J_a, \bar{J}_a , $a=1,2,3$, representing $SU(2)$ subgroups (Lorentz rotations), $P_\mu, Q_{i,\alpha}, \bar{Q}_{i,\dot{\alpha}}$ (supertranslations) with $\mu=0,1,2,3$, $i=1,2,3,4$ and $\alpha, \dot{\alpha}=1,2$, $K_\mu, S_{i,\alpha}, \bar{S}_{i,\dot{\alpha}}$ (special superconformal transformations), and H (dilations) whose state eigenvalues are given by conformal dimensions [8]. As to the $SU(4)_R$ R -symmetry subgroup, we mention only its commuting maximal torus generators R_1, R_2, R_3 . For a distinguished pair of supercharges, say, $Q := Q_{1,1}$ and $Q^\dagger := S_{1,1}$, in appropriate normalization one has

$$\{Q, Q^\dagger\} = H - 2J_3 - 2 \sum_{k=1}^3 \left(1 - \frac{k}{4}\right) R_k =: \Delta. \quad (2)$$

In this case SCI is defined as the following gauge-invariant trace

$$I(t, y, v, w) = \text{Tr} \left((-1)^{\mathcal{F}} t^{2(H+J_3)} y^{2\bar{J}_3} v^{R_2} w^{R_3} e^{-\beta \Delta} \right), \quad (3)$$

where \mathcal{F} is the fermion number operator and t, y, v, w, g_a, β are group parameters (chemical potentials). The trace is effectively taken over the space of zero modes of the operator Δ (the space of BPS states [9]), because relation (2) is preserved by operators used in (3); the contributions from other states cancel together with the dependence on β . In comparison to $\mathcal{N}=1, 2$ theories, all fields of $\mathcal{N}=4$ SYM theory lie in the adjoint representation of G_c , i.e. only the adjoint representation characters enter SCIs.

The $U(N)$ -gauge group SCI has the following matrix integral form [5]

$$I(t, y, v, w) = \int_{G_c} [dU] \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} f(t^m, y^m, v^m, w^m) \text{Tr}(U^\dagger)^m \text{Tr} U^m \right), \quad (4)$$

where $[dU]$ is the invariant measure and $f(t, y, v, w) \text{Tr} U^\dagger \text{Tr} U$ is the so-called single-particle states index with

$$f(t, y, v, w) = \frac{t^2(v+1/w+w/v) - t^3(y+1/y) - t^4(w+1/v+v/w) + 2t^6}{(1-t^3y)(1-t^3/y)}.$$

As shown in [10] (see there the discussion following formula (5.33)), this expression can be obtained from the superconformal group character or partition function for $\mathcal{N}=4$ theories by imposing the shortening condition for the multiplets.

The SCI technique has found many applications in supersymmetric field theories. Römelsberger conjectured [6,7] that SCIs of the Seiberg dual $\mathcal{N}=1$ SYM theories coincide. Dolan and Osborn explicitly confirmed this conjecture for a number of examples [11]. It appeared that SCIs are expressed in terms of elliptic hypergeometric integrals whose theory was developed earlier in [12,13] (see also [14] for a general survey). Equality of indices in dual theories happened to be equivalent either to exact computability of elliptic beta integrals discovered in [12] or to nontrivial Weyl group symmetry transformations for higher order elliptic hypergeometric functions [13,15]. In a series of papers [16–19] we applied this technique to analyzing all previously found Seiberg dualities. We suggested also many new such dualities on the basis of known identities for elliptic hypergeometric integrals and showed that known nontrivial duality checks are satisfied for them. As a payback to mathematics, it happened that many old dualities lead to new, still unproven highly nontrivial relations for integrals.

This line of thoughts was further developed in beautiful papers by Gadde et al. [20,21]. In [22], a particular one dimensional elliptic hypergeometric integral was shown to have $W(F_4)$ Weyl group of symmetry, which follows from the elliptic beta integral evaluation formula [12]. It was used in [20] for confirming S -duality for $\mathcal{N}=2$ SYM theory with $SU(2)$ gauge group and four hypermultiplets and for ensuring associativity of the operator algebra of $2d$ topological field theories behind that duality. The SCI for a E_6 SCFT theory was constructed in [21] from the index of $\mathcal{N}=2$ SYM theory with $G_c = SU(3)$ and six hypermultiplets and a new test of the Argyres–Seiberg duality was suggested.

Here we construct $\mathcal{N}=4$ SCIs for all simple gauge groups, show their S -duality invariance for G_2 and F_4 cases, and give new mathematical arguments supporting equality of SCIs for $SP(2N)$ and $SO(2N+1)$ theories conjectured in [20]. All $\mathcal{N}=4$ indices degenerate in a specific limit to orthogonality measures for the Macdonald polynomials and admit thus exact evaluations. Another limit leads to computable $3d$ partition functions described by the hyperbolic beta integrals.

2. Duality of $SO(2N+1)$ and $SP(2N)$ $\mathcal{N}=4$ SYM Theories

SCIs for $SP(2N)$ and $SO(2N+1)$ $\mathcal{N}=4$ SYM theories were described in [20] and discussed briefly in the simplest case in [17]. Here we prove equality of these SCIs in several important limiting cases.

In all $\mathcal{N}=4$ theories the single-particle index is

$$\frac{1}{(1-p)(1-q)} \left(\sum_{k=1}^3 s_k - pq \sum_{k=1}^3 s_k^{-1} - p - q + 2pq \right) \chi_{\text{adj}}(z), \tag{5}$$

where $\chi_{\text{adj}}(z)$ is the character of the adjoint representation of the corresponding gauge group (see the Appendix). For convenience, we have denoted

$$s_1 = t^2 v, \quad s_2 = t^2 w^{-1}, \quad s_3 = t^2 w v^{-1}, \quad p = t^3 y, \quad q = t^3 y^{-1}.$$

Using explicit expressions of the group invariant measures, SCIs can be written as particular elliptic hypergeometric integrals [14]. So, $SP(2N)$ -electric theory index gets the following form

$$I_E = \chi_N \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{k=1}^3 \Gamma(s_k z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \quad (6)$$

and for $SO(2N + 1)$ -magnetic theory one has

$$I_M = \chi_N \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\prod_{k=1}^3 \Gamma(s_k y_i^{\pm 1} y_j^{\pm 1}; p, q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{k=1}^3 \Gamma(s_k y_j^{\pm 1}; p, q)}{\Gamma(y_j^{\pm 1}; p, q)} \frac{dy_j}{2\pi i y_j}, \quad (7)$$

where $|s_k| < 1, k = 1, 2, 3$. For $|s_k| \geq 1$ the indices are defined as analytical continuations of the expressions (6) and (7). Here \mathbb{T} denotes the unit circle with positive orientation and we use conventions $\Gamma(a, b; p, q) := \Gamma(a; p, q)\Gamma(b; p, q), \Gamma(az^{\pm 1}; p, q) := \Gamma(az; p, q)\Gamma(az^{-1}; p, q)$, where

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \quad |p|, |q| < 1,$$

is the elliptic gamma function. The coefficient in front of the integrals is

$$\chi_N = \frac{(p; p)_{\infty}^N (q; q)_{\infty}^N}{2^N N!} \prod_{k=1}^3 \Gamma^N(s_k; p, q),$$

with $(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - a q^k)$. The constraint $\prod_{k=1}^3 s_k = pq$ plays the role of the balancing condition for integrals.

The S -duality hypothesis leads thus to the conjecture $I_E = I_M$, or

$$\int_{\mathbb{T}^N} \Delta_E(\underline{z}, \underline{s}) \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} = \int_{\mathbb{T}^N} \Delta_M(\underline{y}, \underline{s}) \prod_{j=1}^N \frac{dy_j}{2\pi i y_j}, \quad (8)$$

where the kernels $\Delta_E(\underline{z}, \underline{s})$ and $\Delta_M(\underline{y}, \underline{s})$ are read from integrals (6) and (7). Denoting $\rho(\underline{z}, \underline{y}, \underline{s}) = \Delta_E(\underline{z}, \underline{s}) / \Delta_M(\underline{y}, \underline{s})$, we have verified that this function represents the so-called totally elliptic hypergeometric term [17,23]. This is a rather rich mathematical statement giving strong evidence on the validity of the stated equality of integrals. It means that all the functions

$$h_i^{(z)} = \frac{\rho(\dots q z_i \dots, \underline{y}, \underline{s})}{\rho(\underline{z}, \underline{y}, \underline{s})}, \quad h_i^{(y)} = \frac{\rho(\underline{z}, \dots q y_i \dots, \underline{s})}{\rho(\underline{z}, \underline{y}, \underline{s})}, \quad i = 1, \dots, N,$$

$$h_{kl}^{(s)} = \frac{\rho(\underline{z}, \underline{y}, \dots q s_k, \dots, q^{-1} s_l \dots)}{\rho(\underline{z}, \underline{y}, \underline{s})}, \quad k, l = 1, 2, 3, \quad k \neq l,$$

are elliptic functions of all their arguments z_i, y_i, s_k , and q . For instance,

$$\begin{aligned} h_i^{(z)}(\underline{z}, \underline{y}, \underline{s}; q; p) &= h_i^{(z)}(\dots pz_j \dots, \underline{y}, \underline{s}; q; p) = h_i^{(z)}(\underline{z}, \dots py_j \dots, \underline{s}; q; p) \\ &= h_i^{(z)}(\underline{z}, \underline{y}, \dots ps_k \dots p^{-1}s_l; q; p) = h_i^{(z)}(\underline{z}, \underline{y}, \dots ps_l \dots; pq; p), \end{aligned}$$

where $k, l = 1, 2, 3$. This test is passed by all known integral identities, though it is not sufficient for their validity. For further consequences of the total ellipticity and various technical details of such computations, see [14,17,23].

Now we list various special cases when the equality $I_E = I_M$ can be verified rigorously. For low ranks of the gauge group, it follows from the change of integration variables associated with the affine transformation of the corresponding root system [20]. The electric SCI is obtained from the magnetic one after the substitution $y = z^2$ for $N = 1$, and $y_1 = z_1 z_2$ and $y_2 = z_1 / z_2$ for $N = 2$.

The limit $s_k \rightarrow 1$. Suppose that one of the parameters, say, s_1 goes to 1. Then elliptic gamma functions of the integrand denominators are cancelled and no singularities appear on the integration contour. Because now $s_2 s_3 = pq$, and $\Gamma(a, b; p, q) = 1$ for $ab = pq$, the integrands are actually equal to 1. However, the factor χ_N is divergent in this limit. As a result, we have $\lim_{s_1 \rightarrow 1} I_E / I_M = 1$.

Reduction $p = q = 0$. Consider the limit $p \rightarrow 0$. For fixed z , the limit $p \rightarrow 0$ and further limit $q \rightarrow 0$ simplifies the elliptic gamma function to

$$\Gamma(z; p, q) \underset{p \rightarrow 0}{=} \frac{1}{(z; q)_\infty} \underset{q \rightarrow 0}{=} \frac{1}{1 - z}.$$

Because of the balancing condition for integrals, all parameters cannot be kept fixed. The simplest possibility consists in fixing $s_{1,2}$ and setting $s_3 = pq / s_1 s_2$. Then integral (6) reduces to

$$\begin{aligned} I_E^{p=0}(s_1, s_2 \text{ fixed}) &= \frac{(q; q)_\infty^N (s_1 s_2; q)_\infty^N}{2^N N! (s_1, s_2; q)_\infty^N} \\ &\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}, s_1 s_2 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}, s_2 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{(z_j^{\pm 2}, s_1 s_2 z_j^{\pm 2}; q)_\infty}{(s_1 z_j^{\pm 2}, s_2 z_j^{\pm 2}; q)_\infty} \frac{dz_j}{2\pi i z_j}, \end{aligned} \quad (9)$$

where $(a, b; q)_\infty := (a; q)_\infty (b; q)_\infty$. Integral (7) reduces to

$$\begin{aligned} I_M^{p=0}(s_1, s_2 \text{ fixed}) &= \frac{(q; q)_\infty^N (s_1 s_2; q)_\infty^N}{2^N N! (s_1, s_2; q)_\infty^N} \\ &\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(y_i^{\pm 1} y_j^{\pm 1}, s_1 s_2 y_i^{\pm 1} y_j^{\pm 1}; q)_\infty}{(s_1 y_i^{\pm 1} y_j^{\pm 1}, s_2 y_i^{\pm 1} y_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{(y_j^{\pm 1}, s_1 s_2 y_j^{\pm 1}; q)_\infty}{(s_1 y_j^{\pm 1}, s_2 y_j^{\pm 1}; q)_\infty} \frac{dy_j}{2\pi i y_j}. \end{aligned} \quad (10)$$

For $q = 0$ the integrands have only a finite number of poles and the integrals can be evaluated by computing the residues. However, we did not find a simple way of performing these computations for arbitrary N and have verified equality of the resulting $p = q = 0$ SCIs only for $N = 3$.

One can tie the limit $p, q \rightarrow 0$ to a very natural choice of the fugacities v, w in (3) equal to 1. After fixing $s_k = (pq)^{1/3}, k = 1, 2, 3$, the limit $p, q \rightarrow 0$ strongly simplifies the integrals (set $q = s_1 = s_2 = 0$ in (9) and (10)). Then the SCIs can be evaluated exactly using two different special cases of the Selberg integral, description of which we skip for brevity, yielding $I_E = I_M = 1$.

A $p = 0, q \rightarrow 1$ limit. Let us set in (9), (10) $s_1 = q^\alpha, s_2 = q^\beta$ and consider the limit $q \rightarrow 1$ for fixed α and β . Known asymptotic formulas

$$\lim_{q \rightarrow 1} \frac{(q^\alpha z; q)_\infty}{(q^\beta z; q)_\infty} = (1 - z)^{\beta - \alpha}, \quad \lim_{q \rightarrow 1} \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1 - x} = \Gamma(x),$$

where $\Gamma(x)$ is the Euler gamma function, show that both integrands become equal to 1 and the leading asymptotics for SCIs is determined by the integral prefactors

$$I_{E,M}^{p=0, q \rightarrow 1}(s_1 = q^\alpha, s_2 = q^\beta) = \frac{1}{2^N N!} \left(\frac{\Gamma(\alpha)\Gamma(\beta)}{(1 - q)\Gamma(\alpha + \beta)} \right)^N (1 + o(1)).$$

A $p = 0, s_2 = 0$ limit. Let us set now in (9) $s_2 = 0$, which yields

$$I_{SP(2N)}^{p=s_2=0} = \frac{1}{2^N N!} \frac{(q; q)_\infty^N}{(s_1; q)_\infty^N} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{(z_j^{\pm 2}; q)_\infty}{(s_1 z_j^{\pm 2}; q)_\infty} \frac{dz_j}{2\pi i z_j}. \quad (11)$$

This integral can be evaluated exactly using the multivariable extension of the Askey–Wilson integral (or particular q -Selberg integral serving as the orthogonality measure for Koornwinder polynomials) found in [24]

$$\begin{aligned} & \frac{1}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(b z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{(z_j^{\pm 2}; q)_\infty}{\prod_{i=1}^4 (a_i z_j^{\pm 1}; q)_\infty} \frac{dz_j}{2\pi i z_j} \\ &= \prod_{j=1}^N \left(\frac{(t; q)_\infty (b^{N+j-2} a_1 a_2 a_3 a_4; q)_\infty}{(b^j; q)_\infty (q; q)_\infty} \prod_{1 \leq i < k \leq 4} \frac{1}{(b^{j-1} a_i a_k; q)_\infty} \right), \end{aligned} \quad (12)$$

where $|b|, |a_i| < 1$. This formula reduces to our case after the substitutions

$$b = s_1, \quad a_{1,2} = \pm \sqrt{s_1}, \quad a_{3,4} = \pm \sqrt{q s_1}.$$

The same limit applied to (10) leads to the integral

$$I_{SO(2N+1)}^{p=s_2=0} = \frac{1}{2^N N!} \frac{(q; q)_\infty^N}{(s_1; q)_\infty^N} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{(z_j^{\pm 1}; q)_\infty}{(s_1 z_j^{\pm 1}; q)_\infty} \frac{dz_j}{2\pi i z_j}, \quad (13)$$

which is obtained from (12) after setting

$$b = s_1, \quad a_1 = s_1, \quad a_2 = -1, \quad a_{3,4} = \pm \sqrt{q}.$$

Corresponding computations on the right-hand side of (12) yield

$$I_{SP(2N)}^{p=s_2=0} = I_{SO(2N+1)}^{p=s_2=0} = \prod_{j=0}^{N-1} \frac{(qs_1^{2j+1}; q)_\infty}{(s_1^{2j+2}; q)_\infty}. \quad (14)$$

Equality of indices established earlier in the limit $s_k = (pq)^{\frac{1}{3}} \rightarrow 0$, $k = 1, 2, 3$, is a special case of relation (14) obtained after fixing $s_1 = q = 0$.

The integrals in (14) were computed under the assumption that $|s_1| < 1$, but for finite N we can analytically continue SCIs to arbitrary values of s_1 as meromorphic functions using the right-hand side expression. For $|s_1| < 1$, the limit $N \rightarrow \infty$ yields a ratio of double infinite products appearing in the elliptic gamma function with $p = s_1^2$. From the physical point of view this limit is relevant for testing the AdS/CFT correspondence. In [25], it was suggested to consider the maximal angular momentum limit for indices $t \rightarrow 0$, $y \rightarrow \infty$ with $t^3 y$ fixed, which corresponds to $q \rightarrow 0$ with fixed p . Due to the symmetry between p and q this is similar to our limit $p = s_2 = 0$, but we have the additional free parameter s_1 absent in [25].

The hyperbolic limit. Let us study the hyperbolic limit [26,27] of elliptic hypergeometric integrals (6) and (7). First we parametrize the variables as

$$p = e^{2\pi i v \omega_1}, \quad q = e^{2\pi i v \omega_2}, \quad s_i = e^{2\pi i v \alpha_i}, \quad i = 1, 2, 3,$$

where $\sum_{i=1}^3 \alpha_i = \omega_1 + \omega_2$ (the balancing condition), and then take the limit $v \rightarrow 0$. To simplify the integrals we use the Ruijsenaars limit

$$\Gamma(e^{2\pi i r z}; e^{2\pi i r \omega_1}, e^{2\pi i r \omega_2}) \underset{r \rightarrow 0}{=} e^{-\pi i(2z - \omega_1 - \omega_2)/12r\omega_1\omega_2} \gamma^{(2)}(z; \omega_1, \omega_2), \quad (15)$$

where

$$\gamma^{(2)}(u; \omega_1, \omega_2) = e^{-\frac{\pi i}{2} B_{2,2}(u; \omega_1, \omega_2)} \frac{(e^{2\pi i(u - \omega_2)/\omega_1}; e^{-2\pi i \omega_2/\omega_1})_\infty}{(e^{2\pi i u/\omega_2}; e^{2\pi i \omega_1/\omega_2})_\infty} \quad (16)$$

is the hyperbolic gamma function and $B_{2,2}(u; \omega)$ is the second order Bernoulli polynomial,

$$B_{2,2}(u; \omega) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

The following conventions are used below $\gamma^{(2)}(a, b; \omega) := \gamma^{(2)}(a; \omega) \gamma^{(2)}(b; \omega)$ and $\gamma^{(2)}(a \pm u; \omega) := \gamma^{(2)}(a + u; \omega) \gamma^{(2)}(a - u; \omega)$.

We skip the general expressions for hyperbolic integrals arising in this limit and present only the result appearing after taking the additional limit $\alpha_2 \rightarrow \infty$ (which

mimics altogether the previously considered limit $p=0, s_2=0$):

$$I_{SP(2N)}^{h, \alpha_2 \rightarrow \infty} = \xi_N \int_{-\infty}^{i\infty} \prod_{1 \leq i < j \leq N} \frac{\gamma^{(2)}(\alpha_1 \pm u_i \pm u_j; \omega)}{\gamma^{(2)}(\pm u_i \pm u_j; \omega)} \prod_{j=1}^N \frac{\gamma^{(2)}(\alpha_1 \pm 2u_j; \omega)}{\gamma^{(2)}(\pm 2u_j; \omega)} du_j, \quad (17)$$

$$I_{SO(2N+1)}^{h, \alpha_2 \rightarrow \infty} = \xi_N \int_{-\infty}^{i\infty} \prod_{1 \leq i < j \leq N} \frac{\gamma^{(2)}(\alpha_1 \pm u_i \pm u_j; \omega)}{\gamma^{(2)}(\pm u_i \pm u_j; \omega)} \prod_{j=1}^N \frac{\gamma^{(2)}(\alpha_1 \pm u_j; \omega)}{\gamma^{(2)}(\pm u_j; \omega)} du_j, \quad (18)$$

where $\xi_N = \gamma^{(2)}(\alpha_1; \omega)^N / N! (2i\sqrt{\omega_1\omega_2})^N$ and we dropped the common multiplier $\exp\{\frac{\pi i}{2\omega_1\omega_2}(\alpha_1^2 + 2\alpha_1\alpha_2 - \alpha_1(\omega_1 + \omega_2))(2N^2 + N)\}$. To obtain these expressions we used the inversion relation $\gamma^{(2)}(z, \omega_1 + \omega_2 - z; \omega) = 1$ and the asymptotic formulas

$$\begin{aligned} \lim_{u \rightarrow \infty} e^{\frac{\pi i}{2} B_{2,2}(u; \omega)} \gamma^{(2)}(u; \omega) &= 1, & \text{for } \arg \omega_1 < \arg u < \arg \omega_2 + \pi, \\ \lim_{u \rightarrow \infty} e^{-\frac{\pi i}{2} B_{2,2}(u; \omega)} \gamma^{(2)}(u; \omega) &= 1, & \text{for } \arg \omega_1 - \pi < \arg u < \arg \omega_2. \end{aligned} \quad (19)$$

The following hyperbolic analog of the Selberg integral was computed in [26] (for $N=1$, see [28]):

$$\begin{aligned} & \frac{1}{2^N N!} \int_{-\infty}^{i\infty} \prod_{1 \leq i < k \leq N} \frac{\gamma^{(2)}(\tau \pm u_i \pm u_k; \omega)}{\gamma^{(2)}(\pm u_i \pm u_k; \omega)} \prod_{j=1}^N \frac{\prod_{i=1}^4 \gamma^{(2)}(\mu_i \pm u_j; \omega)}{\gamma^{(2)}(\pm 2u_j; \omega)} \frac{du_j}{i\sqrt{\omega_1\omega_2}} \\ &= \prod_{j=1}^N \frac{\gamma^{(2)}(j\tau; \omega)}{\gamma^{(2)}(\tau; \omega)} \prod_{j=0}^{N-1} \frac{\prod_{1 \leq i < k \leq 4} \gamma^{(2)}(j\tau + \mu_i + \mu_k; \omega)}{\gamma^{(2)}((2N-2-j)\tau + \sum_{i=1}^4 \mu_i; \omega)}, \end{aligned} \quad (20)$$

where the Mellin–Barnes integration contour separates sequences of integrand poles going to infinity. One can obtain integral (17) from (20) after the substitutions

$$\tau = \alpha_1, \quad \mu_1 = \frac{1}{2}\alpha_1, \quad \mu_2 = \frac{1}{2}(\alpha_1 + \omega_1), \quad \mu_3 = \frac{1}{2}(\alpha_1 + \omega_2), \quad \mu_4 = \frac{1}{2}(\alpha_1 + \omega_1 + \omega_2),$$

and integral (18) after the substitutions

$$\tau = \alpha_1, \quad \mu_1 = \alpha_1, \quad \mu_2 = \frac{1}{2}\omega_1, \quad \mu_3 = \frac{1}{2}\omega_2, \quad \mu_4 = \frac{1}{2}(\omega_1 + \omega_2)$$

and application of the duplication formula $\gamma^{(2)}(2z; \omega) = \gamma^{(2)}(z, z + \omega_1/2, z + \omega_2/2, z + (\omega_1 + \omega_2)/2; \omega)$. Direct computations show that

$$I_{SP(2N)}^{h, \alpha_2 \rightarrow \infty} = I_{SO(2N+1)}^{h, \alpha_2 \rightarrow \infty} = \prod_{j=0}^{N-1} \frac{\gamma^{(2)}((2j+2)\alpha_1; \omega)}{\gamma^{(2)}((2j+1)\alpha_1 + \omega_1 + \omega_2; \omega)}. \quad (21)$$

Relations (14) and (21) provide the best available SCI justifications of the duality of $\mathcal{N}=4$ SYM field theories with $SP(2N)$ and $SO(2N+1)$ gauge groups.

Discuss now a physical interpretation of integrals (17), (18) and their exact evaluation (21). In [29] it was shown that the hyperbolic limit of $4d$ $\mathcal{N}=1$ SCIs leads to partition functions of $3d$ $\mathcal{N}=2$ SYM and CS theories constructed in [30–32] following [33]. Our hyperbolic integrals describe partition functions of $3d$ $\mathcal{N}=2$ SYM theories with $SP(2N)$ and $SO(2N+1)$ gauge groups containing one chiral superfield in the adjoint representation with the $U(1)_A$ -group hypercharge 1. First, these $3d$ theories are dual to each other and, second, they share the same confining phase described by a Wess–Zumino type model with $2N$ chiral fields with the $U(1)_A$ -hypercharges $2k, -2k+1, k=1, \dots, N$, and zero R -charges, whose partition function is given by expression (21). Taking $\alpha_1=(\omega_1+\omega_2)/2$ in (17) and (18) one obtains partition functions for pure $3d$ $\mathcal{N}=4$ SYM theories. As follows from the exact evaluation (21), these partition functions vanish indicating thus to the spontaneous supersymmetry breaking [34].

As to the hyperbolic integrals obtained from SCIs for arbitrary α_1 and α_2 , they describe partition functions of $3d$ $\mathcal{N}=2$ SYM theories with 3 chiral superfields in the adjoint representation. The constraint $\alpha_1=(\omega_1+\omega_2)/2$ leads to partition functions of $3d$ $\mathcal{N}=4$ SYM theories with one hypermultiplet in the adjoint representation. In these cases, $3d$ theories with $SP(2N)$ and $SO(2N+1)$ gauge group are dual to each other in the same way as the parent $4d$ $\mathcal{N}=4$ models. A similar situation holds for all other cases considered below.

3. G_2 Gauge Group

We consider now the S -duality conjecture for $\mathcal{N}=4$ SYM theory with the gauge group G_2 . This group has two maximal torus variables z_1 and z_2 , but it is convenient to introduce the third variable $z_3=z_1^{-1}z_2^{-1}$ (see the Appendix). Then the electric SCI takes the form

$$I_E = \kappa_2 \int_{\mathbb{T}^2} \prod_{1 \leq i < j \leq 3} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^2 \frac{dz_j}{2\pi i z_j}, \tag{22}$$

where $|s_k| < 1, k=1, 2, 3$, and

$$\kappa_2 = \frac{(p; p)_\infty^2 (q; q)_\infty^2}{2^{23}} \prod_{k=1}^3 \Gamma^2(s_k; p, q).$$

In the magnetic theory one has

$$I_M = \kappa_2 \int_{\mathbb{T}^2} \prod_{1 \leq i < j \leq 3} \frac{\prod_{k=1}^3 \Gamma(s_k (y_i y_j)^{\pm 3}, s_k (y_i y_j^{-1})^{\pm 1}; p, q)}{\Gamma((y_i y_j)^{\pm 3}, (y_i y_j^{-1})^{\pm 1}; p, q)} \prod_{j=1}^2 \frac{dy_j}{2\pi i y_j}, \tag{23}$$

where $y_1 y_2 y_3 = 1$ (we are indebted to S. Razamat for pointing to a misprint in our initial expression for this integral).

The S -duality hypothesis assumes the equality of these elliptic hypergeometric integrals, $I_E = I_M$. Remarkably, this identity can be easily established by the following change of the integration variables

$$y_1 = (z_2 z_3^2)^{1/3}, \quad y_2 = (z_3 z_1^2)^{1/3}, \quad y_3 = (z_1 z_2^2)^{1/3},$$

associated with the rotation of the G_2 root system [4]. The SCI test confirms thus the S -duality in this case.

Application of the limit $p = s_2 = 0$ reduces integral (22) to

$$I_{G_2}^{p=s_2=0} = \frac{1}{2^2 3} \frac{(q; q)_\infty^2}{(s_1; q)_\infty^2} \int_{\mathbb{T}^2} \prod_{1 \leq i < j \leq 3} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^2 \frac{dz_j}{2\pi i z_j}, \tag{24}$$

where $z_1 z_2 z_3 = 1$. This integral admits exact evaluation [35]

$$I_{G_2}^{p=s_2=0} = \frac{(q s_1, q s_1^5; q)_\infty}{(s_1^2, s_1^6; q)_\infty}. \tag{25}$$

4. F_4 Gauge Group

Consider the S -duality for $\mathcal{N} = 4$ SYM theory with the gauge group F_4 [1–4]. The electric SCI has the following form

$$I_E = \kappa_4 \int_{\mathbb{T}^4} \prod_{1 \leq i < j \leq 4} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 2} z_j^{\pm 2}; p, q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \prod_{j=1}^4 \frac{\prod_{k=1}^3 \Gamma(s_k z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \\ \times \frac{\prod_{k=1}^3 \Gamma(s_k z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1}; p, q)}{\Gamma(z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1}; p, q)} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \tag{26}$$

where $|s_k| < 1$, $k = 1, 2, 3$, and

$$\kappa_4 = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^7 3^2} \prod_{k=1}^3 \Gamma^4(s_k; p, q).$$

In the derivation of this expression we used the F_4 group adjoint representation character which is obtained from the expression given in the Appendix after the replacement $z_i \rightarrow z_i^2$.

Using similar prescription for the magnetic theory, we find

$$I_M = \kappa_4 \int_{\mathbb{T}^4} \prod_{1 \leq i < j \leq 4} \frac{\prod_{k=1}^3 \Gamma(s_k y_i^{\pm 1} y_j^{\pm 1}; p, q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1}; p, q)} \prod_{j=1}^4 \frac{\prod_{k=1}^3 \Gamma(s_k y_j^{\pm 2}; p, q)}{\Gamma(y_j^{\pm 2}; p, q)} \\ \times \frac{\prod_{k=1}^3 \Gamma(s_k y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1}; p, q)}{\Gamma(y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1}; p, q)} \prod_{j=1}^4 \frac{dy_j}{2\pi i y_j}. \tag{27}$$

These are the first examples of multiple elliptic hypergeometric integrals defined for the F_4 root system (in [22] the integrals were defined on the $SU(2)$ group and the Weyl group $W(F_4)$ was acting in the parameter space).

The S -duality conjecture suggests the transformation formula $I_E = I_M$. Again, as suggested to us by S. Razamat, this identity is easily established by the change of variables

$$y_1 = z_1 z_2, \quad y_2 = z_1 / z_2, \quad y_3 = z_3 z_4, \quad y_4 = z_3 / z_4,$$

associated with the rotation of the F_4 root system [4]. We see thus validity of the SCI test for this S -duality.

The limit $p = s_2 = 0$ reduces integral (26) to

$$\begin{aligned} I_{F_4}^{p=s_2=0} &= \frac{1}{273^2} \frac{(q; q)_\infty^4}{(s_1; q)_\infty^4} \int_{\mathbb{T}^4} \prod_{1 \leq i < j \leq 4} \frac{(z_i^{\pm 2} z_j^{\pm 2}; q)_\infty}{(s_1 z_i^{\pm 2} z_j^{\pm 2}; q)_\infty} \prod_{j=1}^4 \frac{(z_j^{\pm 2}; q)_\infty}{(s_1 z_j^{\pm 2}; q)_\infty} \\ &\quad \times \frac{(z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1}; q)_\infty}{(s_1 z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1}; q)_\infty} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \end{aligned} \quad (28)$$

which admits exact evaluation [35]

$$I_{F_4}^{p=s_2=0} = \frac{(q s_1, q s_1^5, q s_1^7, q s_1^{11}; q)_\infty}{(s_1^2, s_1^6, s_1^8, s_1^{12}; q)_\infty}. \quad (29)$$

5. $SU(N)$ and $SO(2N)$ Gauge Groups

Consider now SCIs for self-dual $\mathcal{N}=4$ SYM theories with $SU(N)$ and $SO(2N)$ gauge groups [1]. The $SU(N)$ theory SCI is

$$I_{SU(N)} = \chi_N \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{-1} z_j, s_k z_i z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (30)$$

where $\prod_{j=1}^N z_j = 1$, parameters s_k satisfy the constraints $|s_k| < 1$, $k = 1, 2, 3$, and

$$\chi_N = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \prod_{k=1}^3 \Gamma^{N-1}(s_k; p, q).$$

The limit $p = 0, s_2 = 0$ reduces integral (30) to

$$I_{SU(N)}^{p=s_2=0} = \frac{1}{N!} \frac{(q; q)_\infty^{N-1}}{(s_1; q)_\infty^{N-1}} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{(z_i^{-1} z_j, z_i z_j^{-1}; q)_\infty}{(s_1 z_i^{-1} z_j, s_1 z_i z_j^{-1}; q)_\infty} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (31)$$

where $\prod_{j=1}^N z_j = 1$, which admits exact evaluation [35]

$$I_{SU(N)}^{p=s_2=0} = \prod_{j=1}^{N-1} \frac{(qs_1^j; q)_\infty}{(s_1^{j+1}; q)_\infty}. \tag{32}$$

For $N \rightarrow \infty$ this index equals to $(s_1; q)_\infty / (s_1; s_1)_\infty$, which coincides with the reduced form of $N \rightarrow \infty$ asymptotics (after passing from $U(N)$ to $SU(N)$ gauge group) found in [5] from the AdS/CFT correspondence.

SCI for the $SO(2N)$ theory has the form

$$I_{SO(2N)} = \chi_N \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\prod_{k=1}^3 \Gamma(skz_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi iz_j}, \tag{33}$$

where $|s_k| < 1$, $k = 1, 2, 3$, and

$$\chi_N = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^{N-1} N!} \prod_{k=1}^3 \Gamma^N(s_k; p, q).$$

Note that for $N = 1$ the SCI is equal to χ_1 .

Taking the ratio of integral kernel to itself with different integration variables in (30) and (33) one gets totally elliptic hypergeometric terms. However, consequences of this statement are much less informative than in the cases with nontrivial symmetry transformations for integrals.

The limit $p = 0, s_2 = 0$ reduces (33) to the integral

$$I_{SO(2N)}^{p=s_2=0} = \frac{1}{2^{N-1} N!} \frac{(q; q)_\infty^N}{(s_1; q)_\infty^N} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{dz_j}{2\pi iz_j}, \tag{34}$$

with exact evaluation [35]

$$I_{SO(2N)}^{p=s_2=0} = \frac{(qs_1^{N-1}; q)_\infty}{(s_1^N; q)_\infty} \prod_{j=0}^{N-2} \frac{(qs_1^{2j+1}; q)_\infty}{(s_1^{2j+2}; q)_\infty}. \tag{35}$$

In the same way as for $SP(2N)$ and $SO(2N + 1)$ SYM theories, this case can be obtained from the q -Selberg integral (12) using special parameter values

$$b = s_1, \quad a_{1,2} = \pm 1, \quad a_{3,4} = \pm \sqrt{q}.$$

Consider now the hyperbolic degeneration of (30) and (33) joint with the $\alpha_2 \rightarrow \infty$ limit similar to $SP(2N)$ and $SO(2N + 1)$ SCIs. For $SU(N)$ -SCI we obtain, after dropping the multiplier $\exp\{\frac{\pi i}{2\omega_1\omega_2}(\alpha_1^2 + 2\alpha_1\alpha_2 - \alpha_1(\omega_1 + \omega_2))(N^2 - 1)\}$,

$$I_{SU(N)}^{h, \alpha_2 \rightarrow \infty} = \frac{\gamma^{(2)}(\alpha_1; \omega)^{N-1}}{N!(i\sqrt{\omega_1\omega_2})^{N-1}} \int_{-i\infty}^{i\infty} \prod_{1 \leq i < j \leq N} \frac{\gamma^{(2)}(\alpha_1 \pm (u_i - u_j); \omega)}{\gamma^{(2)}(\pm(u_i - u_j); \omega)} \prod_{j=1}^{N-1} du_j, \tag{36}$$

where $\sum_{j=1}^N u_j = 0$. In the analysis of convergency of this integral there are two extremal options when integration variables go to infinity: in the first case $u_j = iR + v_j$, $j = 1, \dots, N-1$, and $u_N = -(N-1)iR - \sum_{j=1}^{N-1} v_j$, where $R \rightarrow +\infty$, and the integrand behaves as $\exp(2\pi N(N-1)\alpha_1 R/\omega_1\omega_2)$. In the second case, $u_1 = iR$, $\Im(u_j) \ll R$, $j = 2, \dots, N-1$, and $u_N = -iR - \sum_{j=2}^{N-1} u_j$, $R \rightarrow +\infty$, and the integrand behaves as $\exp(2\pi N\alpha_1 R/\omega_1\omega_2)$. In both cases, for $\Re(\alpha_1/\omega_1\omega_2) < 0$ the integrand is exponentially suppressed and has no singularities on the integration contour.

To our knowledge integral (36) cannot be obtained as a limit of known hyperbolic beta integrals. Formally it is related to the limit $\sum_{i=1}^4 \mu_i + (N-1)\tau - \omega_1 - \omega_2 \rightarrow 0$ in formula (20), which is not uniform. Therefore we have separately computed this integral for $N=2, 3$ by showing that the sum of residues for poles on the left-hand side of the integration contours is proportional to the product of sums of residues of two trigonometric integrals (32) with bases $q = e^{2\pi i\omega_1/\omega_2}$ and $\tilde{q} = e^{-2\pi i\omega_2/\omega_1}$, $|q| < 1$, which yields

$$I_{SU(N)}^{h,\alpha_2 \rightarrow \infty} = \prod_{j=1}^{N-1} \frac{\gamma^{(2)}((j+1)\alpha_1; \omega)}{\gamma^{(2)}(j\alpha_1 + \omega_1 + \omega_2; \omega)}. \tag{37}$$

For $N=4$ this integral coincides with the $SO(6)$ -integral given below. Note that formula (37) defines a hyperbolic analogue of the orthogonality measure normalization for Macdonald polynomials on A_{N-1} root system (31), (32) (for arbitrary N we consider it as a conjecture).

The hyperbolic limit for SCI of $SO(2N)$ -theory ($N > 1$) yields, after dropping the multiplier $\exp\{\frac{\pi i}{2\omega_1\omega_2}(\alpha_1^2 + 2\alpha_1\alpha_2 - \alpha_1(\omega_1 + \omega_2))(2N^2 - N)\}$,

$$I_{SO(2N)}^{h,\alpha_2 \rightarrow \infty} = \xi_N \int_{-i\infty}^{i\infty} \prod_{1 \leq i < j \leq N} \frac{\gamma^{(2)}(\alpha_1 \pm u_i \pm u_j; \omega)}{\gamma^{(2)}(\pm u_i \pm u_j; \omega)} \prod_{j=1}^N du_j. \tag{38}$$

This integral is obtained from (20) after the substitutions

$$\tau = \alpha_1, \quad \mu_1 = 0, \quad \mu_2 = \frac{1}{2}\omega_1, \quad \mu_3 = \frac{1}{2}\omega_2, \quad \mu_4 = \frac{1}{2}(\omega_1 + \omega_2),$$

which leads to the evaluation

$$I_{SO(2N)}^{h,\alpha_2 \rightarrow \infty} = \frac{\gamma^{(2)}(N\alpha_1; \omega)}{\gamma^{(2)}((N-1)\alpha_1 + \omega_1 + \omega_2; \omega)} \prod_{j=0}^{N-2} \frac{\gamma^{(2)}(2(j+1)\alpha_1; \omega)}{\gamma^{(2)}((2j+1)\alpha_1 + \omega_1 + \omega_2; \omega)}. \tag{39}$$

Again, one can see that expressions (36) and (37), (38) and (39) describe partition functions of $3d$ $\mathcal{N}=2$ SYM theories with one chiral matter superfield in the adjoint representation of the respective $SU(N)$ and $SO(2N)$ gauge groups and their dual confining partners. Substitution $\alpha_1 = (\omega_1 + \omega_2)/2$ in these expressions leads to vanishing partition functions of $3d$ $\mathcal{N}=4$ pure SYM theories.

6. Exceptional Gauge Groups E_6 , E_7 , and E_8

For the E_6 gauge group theory we have the SCI

$$\begin{aligned}
 I_{E_6} = & \kappa_6 \int_{\mathbb{T}^6} \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j} \prod_{1 \leq i < j \leq 5} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 2} z_j^{\pm 2}; p, q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \frac{\prod_{k=1}^3 \Gamma(s_k (z_6^3 Z)^{\pm 1}; p, q)}{\Gamma((z_6^3 Z)^{\pm 1}; p, q)} \\
 & \times \prod_{1 \leq i < j \leq 5} \frac{\prod_{k=1}^3 \Gamma(s_k (z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; p, q)}{\Gamma((z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; p, q)} \prod_{i=1}^5 \frac{\prod_{k=1}^3 \Gamma(s_k (z_6^{-3} z_i^2 Z)^{\pm 1}; p, q)}{\Gamma((z_6^{-3} z_i^2 Z)^{\pm 1}; p, q)},
 \end{aligned} \tag{40}$$

where for convenience we denoted $Z = (z_1 z_2 z_3 z_4 z_5)^{-1}$ and

$$\kappa_6 = \frac{(p; p)_\infty^6 (q; q)_\infty^6}{2^7 3^4 5} \prod_{k=1}^3 \Gamma^6(s_k; p, q).$$

The combinatorial factors appearing here are the same as, for example, the ones given in [35]. Similar to the F_4 -group case, we took the adjoint representation character given in the Appendix and replaced in it $z_j \rightarrow z_j^2$ (the same was done for the E_7 and E_8 group cases considered below).

The limit $p=0, s_2=0$ reduces (40) to the integral

$$\begin{aligned}
 I_{E_6}^{p=s_2=0} = & \frac{1}{2^7 3^4 5} \frac{(q; q)_\infty^6}{(s_1; q)_\infty^6} \int_{\mathbb{T}^6} \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j} \prod_{1 \leq i < j \leq 5} \frac{(z_i^{\pm 2} z_j^{\pm 2}; q)_\infty}{(s_1 z_i^{\pm 2} z_j^{\pm 2}; q)_\infty} \\
 & \times \frac{((z_6^3 Z)^{\pm 1}; q)_\infty}{(s_1 (z_6^3 Z)^{\pm 1}; q)_\infty} \prod_{1 \leq i < j \leq 5} \frac{((z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; q)_\infty}{(s_1 (z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; q)_\infty} \prod_{i=1}^5 \frac{((z_6^{-3} z_i^2 Z)^{\pm 1}; q)_\infty}{(s_1 (z_6^{-3} z_i^2 Z)^{\pm 1}; q)_\infty},
 \end{aligned} \tag{41}$$

which can be computed exactly [35],

$$I_{E_6}^{p=s_2=0} = \frac{(q s_1, q s_1^4, q s_1^5, q s_1^7, q s_1^8, q s_1^{11}; q)_\infty}{(s_1^2, s_1^5, s_1^6, s_1^8, s_1^9, s_1^{12}; q)_\infty}. \tag{42}$$

For $\mathcal{N}=4$ SYM theory with the E_7 gauge group the SCI has the form

$$\begin{aligned}
 I_{E_7} = & \kappa_7 \int_{\mathbb{T}^7} \prod_{j=1}^6 \frac{\prod_{k=1}^3 \Gamma(s_k z_7^{\pm 2} (z_j^2 Z)^{\pm 1}; p, q)}{\Gamma(z_7^{\pm 2} (z_j^2 Z)^{\pm 1}; p, q)} \prod_{1 \leq i < j \leq 6} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 2} z_j^{\pm 2}; p, q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 & \times \frac{\prod_{k=1}^3 \Gamma(s_k z_7^{\pm 4}; p, q)}{\Gamma(z_7^{\pm 4}; p, q)} \prod_{1 \leq i < j < l \leq 6} \frac{\prod_{k=1}^3 \Gamma(s_k z_7^{\pm 2} z_i^2 z_j^2 z_l^2 Z; p, q)}{\Gamma(z_7^{\pm 2} z_i^2 z_j^2 z_l^2 Z; p, q)} \prod_{j=1}^7 \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{43}$$

where we denoted $Z = (z_1 z_2 z_3 z_4 z_5 z_6)^{-1}$ and

$$\kappa_7 = \frac{(p; p)_\infty^7 (q; q)_\infty^7}{2^{10} 3^4 5 \cdot 7} \prod_{k=1}^3 \Gamma^7(s_k; p, q).$$

The limit $p=0, s_2=0$ reduces (43) to the integral

$$\begin{aligned} I_{E_7}^{p=s_2=0} &= \frac{1}{2^{10} 3^4 5 \cdot 7} \frac{(q; q)_\infty^7}{(s_1; q)_\infty^7} \int_{\mathbb{T}^7} \prod_{j=1}^6 \frac{(z_7^{\pm 2} (z_j^2 Z)^{\pm 1}; q)_\infty}{(s_1 z_7^{\pm 2} (z_j^2 Z)^{\pm 1}; q)_\infty} \prod_{1 \leq i < j \leq 6} \frac{(z_i^{\pm 2} z_j^{\pm 2}; q)_\infty}{(s_1 z_i^{\pm 2} z_j^{\pm 2}; q)_\infty} \\ &\quad \times \frac{(z_7^{\pm 4}; q)_\infty}{(s_1 z_7^{\pm 4}; q)_\infty} \prod_{1 \leq i < j < l \leq 6} \frac{(z_7^{\pm 2} z_i^2 z_j^2 z_l^2 Z; q)_\infty}{(s_1 z_7^{\pm 2} z_i^2 z_j^2 z_l^2 Z; q)_\infty} \prod_{j=1}^7 \frac{dz_j}{2\pi i z_j}, \end{aligned} \quad (44)$$

which can be evaluated exactly [35],

$$I_{E_7}^{p=s_2=0} = \frac{(qs_1, qs_1^5, qs_1^7, qs_1^9, qs_1^{11}, qs_1^{13}, qs_1^{17}; q)_\infty}{(s_1^2, s_1^6, s_1^8, s_1^{10}, s_1^{12}, s_1^{14}, s_1^{18}; q)_\infty}. \quad (45)$$

Finally, the largest exceptional gauge group E_8 theory has the SCI

$$\begin{aligned} I_{E_8} &= \kappa_8 \int_{\mathbb{T}^8} \prod_{j=1}^8 \frac{dz_j}{2\pi i z_j} \prod_{1 \leq i < j \leq 8} \frac{\prod_{k=1}^3 \Gamma(s_k (z_i^2 z_j^2 Z)^{\pm 1}; p, q)}{\Gamma((z_i^2 z_j^2 Z)^{\pm 1}; p, q)} \frac{\prod_{k=1}^3 \Gamma(s_k Z^{\pm 1}; p, q)}{\Gamma(Z^{\pm 1}; p, q)} \\ &\quad \times \prod_{1 \leq i < j \leq 8} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 2} z_j^{\pm 2}; p, q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \prod_{1 \leq i < j < l < m \leq 8} \frac{\prod_{k=1}^3 \Gamma(s_k (z_i^2 z_j^2 z_l^2 z_m^2 Z)^{\pm 1}; p, q)}{\Gamma((z_i^2 z_j^2 z_l^2 z_m^2 Z)^{\pm 1}; p, q)}, \end{aligned} \quad (46)$$

where $Z = (z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8)^{-1}$ and

$$\kappa_8 = \frac{(p; p)_\infty^8 (q; q)_\infty^8}{2^{14} 3^5 5^2 7} \prod_{k=1}^3 \Gamma^8(s_k; p, q).$$

Again, the limit $p=0, s_2=0$ reduces (46) to the integral

$$\begin{aligned} I_{E_8}^{p=s_2=0} &= \frac{1}{2^{14} 3^5 5^2 7} \frac{(q; q)_\infty^8}{(s_1; q)_\infty^8} \int_{\mathbb{T}^8} \prod_{j=1}^8 \frac{dz_j}{2\pi i z_j} \prod_{1 \leq i < j \leq 8} \frac{((z_i^2 z_j^2 Z)^{\pm 1}; q)_\infty}{(s_1 (z_i^2 z_j^2 Z)^{\pm 1}; q)_\infty} \\ &\quad \times \frac{(Z^{\pm 1}; q)_\infty}{(s_1 Z^{\pm 1}; q)_\infty} \prod_{1 \leq i < j \leq 8} \frac{(z_i^{\pm 2} z_j^{\pm 2}; q)_\infty}{(s_1 z_i^{\pm 2} z_j^{\pm 2}; q)_\infty} \prod_{1 \leq i < j < l < m \leq 8} \frac{((z_i^2 z_j^2 z_l^2 z_m^2 Z)^{\pm 1}; q)_\infty}{(s_1 (z_i^2 z_j^2 z_l^2 z_m^2 Z)^{\pm 1}; q)_\infty}, \end{aligned} \quad (47)$$

which can be evaluated exactly [35],

$$I_{E_8}^{p=s_2=0} = \frac{(qs_1, qs_1^7, qs_1^{11}, qs_1^{13}, qs_1^{17}, qs_1^{19}, qs_1^{23}, qs_1^{29}; q)_\infty}{(s_1^2, s_1^8, s_1^{12}, s_1^{14}, s_1^{18}, s_1^{20}, s_1^{24}, s_1^{30}; q)_\infty}. \quad (48)$$

In all three integrals (40), (43), and (46) we assumed the restrictions $|s_k| < 1, k = 1, 2, 3$. As expected, ratios of their kernels to themselves with different integration

variables yield totally elliptic hypergeometric terms. These integrals represent first known examples of elliptic hypergeometric integrals based on the exceptional root systems of E -type.

7. Some Special $\mathcal{N}=1$ and $\mathcal{N}=2$ Dualities

Much attention is paid in this paper to supersymmetric theories with the exceptional gauge groups. Therefore we would like to describe one more duality example for such theories known to us. We take $\mathcal{N}=1$ SYM theory with E_6 gauge group and matter fields in the fundamental representation of $SU(6)$ flavor group and in 27-dimensional representation of E_6 .

This electric theory and its magnetic dual were suggested in [36,37] and validity of this duality was discussed further in [38]. The electric SCI is

$$\begin{aligned}
 I_E = & \kappa_6 \int_{\mathbb{T}_6} \prod_{1 \leq i < j \leq 5} \frac{\prod_{k=1}^6 \Gamma(s_k z_6^{-1} Z z_i^2 z_j^2; p, q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \frac{\prod_{k=1}^6 \Gamma(s_k z_6^{-4}, s_k z_6^{-1} Z; p, q)}{\Gamma((z_6^3 Z)^{\pm 1}; p, q)} \\
 & \times \prod_{1 \leq i < j \leq 5} \frac{1}{\Gamma((z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; p, q)} \prod_{i=1}^5 \frac{\prod_{k=1}^6 \Gamma(s_k z_6^2 z_i^{\pm 2}, s_k z_6^{-1} Z^{-1} z_i^{-2}; p, q)}{\Gamma((z_6^{-3} z_i^2 Z)^{\pm 1}; p, q)} \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j}, \tag{49}
 \end{aligned}$$

where $|s_k| < 1, k = 1, \dots, 6$, we denoted $Z = (z_1 z_2 z_3 z_4 z_5)^{-1}$ and

$$\kappa_6 = \frac{(p; p)_\infty^6 (q; q)_\infty^6}{2^7 3^4 5}.$$

The magnetic theory has chiral fields in the antifundamental representation of the flavor group and 27-dimensional representation of the gauge group. There are also singlet mesons given by the absolutely symmetric representation of the third rank of the flavor group. The magnetic SCI is

$$\begin{aligned}
 I_M = & \kappa_6 \prod_{j=1}^6 \Gamma(s_j^3; p, q) \prod_{i, j=1; i \neq j}^6 \Gamma(s_i s_j^2; p, q) \int_{\mathbb{T}_6} \prod_{1 \leq i < j \leq 5} \frac{1}{\Gamma((z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; p, q)} \\
 & \times \prod_{1 \leq i < j \leq 5} \frac{\prod_{k=1}^6 \Gamma(S^{\frac{1}{3}} s_k^{-1} z_6^{-1} Z z_i^2 z_j^2; p, q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \frac{\prod_{k=1}^6 \Gamma(S^{\frac{1}{3}} s_k^{-1} z_6^{-4}, S^{\frac{1}{3}} s_k^{-1} z_6^{-1} Z; p, q)}{\Gamma((z_6^3 Z)^{\pm 1}; p, q)} \\
 & \times \prod_{i=1}^5 \frac{\prod_{k=1}^6 \Gamma(S^{\frac{1}{3}} s_k^{-1} z_6^2 z_i^{\pm 2}, S^{\frac{1}{3}} s_k^{-1} z_6^{-1} Z^{-1} z_i^{-2}; p, q)}{\Gamma((z_6^{-3} z_i^2 Z)^{\pm 1}; p, q)} \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j}, \tag{50}
 \end{aligned}$$

where $|s_k| < 1, k = 1, \dots, 6$. The balancing condition for both elliptic hypergeometric integrals has the form $S = \prod_{i=1}^6 s_i = pq$.

We have checked that the ratio of these integral kernels yields a totally elliptic hypergeometric term, which is an important test suggesting that these dualities and the equality $I_E = I_M$ might be true. Interestingly, the limit $s_6 \rightarrow 1$ reduces the integrals to SCIs of peculiar E_6 and F_4 SYM theories dual to each other [37].

Finally, as an additional advertisement of the applications of the theory of elliptic hypergeometric integrals, we present SCI of a particular $\mathcal{N}=2$ quiver SYM theory described in [39]. Define

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^6 (q; q)_\infty^6}{8} \int_{\mathbb{T}} \frac{dx}{2\pi ix} \int_{\mathbb{T}} \frac{dy}{2\pi iy} \int_{\mathbb{T}^2} \prod_{j=1}^2 \frac{dz_j}{2\pi iz_j} \int_{\mathbb{T}} \frac{dr}{2\pi ir} \int_{\mathbb{T}} \frac{dw}{2\pi iw} \\
 &\times \frac{\Gamma(t^2 vx^{\pm 1}, t^2 vy^{\pm 2}, t^2 vz_1^{\pm 1} z_2^{\pm 1}, t^2 vr^{\pm 2}, t^2 vw^{\pm 1}; p, q)}{\Gamma(x^{\pm 1}, y^{\pm 2}, z_1^{\pm 1} z_2^{\pm 1}, r^{\pm 2}, w^{\pm 1}; p, q)} \\
 &\times \Gamma\left(\frac{t^2}{\sqrt{v}} y^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1}; p, q\right)^2 \Gamma\left(\frac{t^2}{\sqrt{v}} x^{\pm 1} y^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1} w^{\pm 1}; p, q\right) \\
 &\times \prod_{j=1}^2 \Gamma\left(\frac{t^2}{\sqrt{v}} y^{\pm 1} z_j^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1} z_j^{\pm 1}; p, q\right), \tag{51}
 \end{aligned}$$

where t is the same parameter as in $\mathcal{N}=4$ theories before and v is the chemical potential associated with some combination of the $U(2)_R$ -group commuting R -charges. Introducing the variables $\alpha^2 = z_1 z_2$, $\beta^2 = z_1/z_2$, $\gamma^2 = x$, and $\delta^2 = w$, one can rewrite this integral as

$$\begin{aligned}
 I_M &= \frac{(p; p)_\infty^6 (q; q)_\infty^6}{64} \int_{\mathbb{T}} \frac{d\gamma}{2\pi i\gamma} \int_{\mathbb{T}} \frac{dy}{2\pi iy} \int_{\mathbb{T}} \frac{d\alpha}{2\pi i\alpha} \int_{\mathbb{T}} \frac{d\beta}{2\pi i\beta} \int_{\mathbb{T}} \frac{dr}{2\pi ir} \int_{\mathbb{T}} \frac{d\delta}{2\pi i\delta} \\
 &\times \frac{\Gamma(t^2 v\gamma^{\pm 2}, t^2 vy^{\pm 2}, t^2 v\alpha^{\pm 2}, t^2 v\beta^{\pm 2}, t^2 vr^{\pm 2}, t^2 v\delta^{\pm 2}; p, q)}{\Gamma(\gamma^{\pm 2}, y^{\pm 2}, \alpha^{\pm 2}, \beta^{\pm 2}, r^{\pm 2}, \delta^{\pm 2}; p, q)} \\
 &\times \Gamma\left(\frac{t^2}{\sqrt{v}} \gamma^{\pm 1} \gamma^{\pm 1} y^{\pm 1}, \frac{t^2}{\sqrt{v}} \delta^{\pm 1} \delta^{\pm 1} r^{\pm 1}, \frac{t^2}{\sqrt{v}} y^{\pm 1} \alpha^{\pm 1} \beta^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1} \alpha^{\pm 1} \beta^{\pm 1}; p, q\right). \tag{52}
 \end{aligned}$$

The identity $I_E = I_M$ can be interpreted as the equality of SCIs for particular $\mathcal{N}=2$ SYM generalized quiver theories (although it does not correspond to an intrinsic electric-magnetic duality). The “electric” part is an $SO(3) \times SP(2) \times SO(4) \times SP(2) \times SO(3)$ $\mathcal{N}=2$ SYM quiver and the “magnetic” part is the same theory rewritten as an $SU(2)^6$ -quiver, as illustrated in Fig. 9 of [39].

8. Discussion

In this paper we have described SCIs for $\mathcal{N}=4$ SYM theories with simple gauge groups as elliptic hypergeometric integrals and analyzed some of their mathematical properties. For all classical simple gauge groups we have found particular limiting values of chemical potentials ($p \rightarrow 0$ followed by the $s_2 \rightarrow 0$ limit and the hyperbolic limit followed by the $\alpha_2 \rightarrow \infty$ limit) for which $\mathcal{N}=4$ indices are computable exactly. According to the general ideology [6, 7, 11, 17], exact computability of non-abelian gauge group SCIs is associated with the confinement in the dual phase of the theory, since it provides a group-theoretical representation of indices

without local gauge group symmetry. Therefore we conclude that there should exist some interesting supersymmetric field theories similar to the Wess–Zumino model whose SCIs are described by the right-hand sides of equalities (11), (13), (24), (28), (31), (34), (41), (44), and (47). The hyperbolic analogs of these relations describe equalities of $3d$ partition functions of particular dual $3d$ $\mathcal{N}=2$ and $\mathcal{N}=4$ SYM theories.

To our knowledge, hyperbolic beta integrals for exceptional groups were not considered in the literature. Analysing such exact integration formulas given in [14,26,27,29] and references therein, we conjecture that the hyperbolic analogs of all our exceptional gauge group q -beta integrals are obtained from them after the replacement of infinite products $(q^n s_k^m z_j^\ell; q)_\infty$ with m or $\ell \neq 0$ by $1/\gamma^{(2)}(n(\omega_1 + \omega_2) + m\alpha_k + \ell u_j; \omega)$, the measure elements $(q; q)_\infty dz_j / 2\pi i z_j$ by $du_j / i\sqrt{\omega_1 \omega_2}$, and \mathbb{T} by the Mellin–Barnes integration contours. From the physical point of view this is equivalent to the conjecture on the particular structure of confining phases of corresponding $3d$ $\mathcal{N}=2$ SYM theories with G_2, F_4, E_6, E_7, E_8 gauge groups and one matter field in the adjoint representation. For $\alpha_1 = (\omega_1 + \omega_2)/2$ this would yield vanishing partition functions for $3d$ $\mathcal{N}=4$ pure SYM theories.

One of the initial motivations for consideration of SCIs in [5] was an analysis of the AdS/CFT correspondence for $\mathcal{N}=4$ SYM theory with $U(N)$ gauge group which required consideration of the $N \rightarrow \infty$ limit. In this limit, the original index coming from the BPS states not forming long multiplets can be computed from the dual spectrum of gravitons appearing in the Type IIB supergravity compactified on $AdS_5 \times S^5$. It would be interesting to understand the meaning of the reduction $p \rightarrow 0$ from the AdS/CFT point of view on the level of graviton spectra. All our $p = s_2 = 0$ indices for gauge groups of rank N are well defined in the limit $N \rightarrow \infty$ for $|s_1| < 1$, being given by curious explicit infinite products. We expect that the $p = s_2 = 0$ limit corresponds to an essentially simplified picture for the corresponding gravitational duals for both finite and infinite N .

In [40–42], marginal deformations of SCFTs were studied and the importance of global symmetries for the conformal manifold (a manifold of coupling constants of the theory where it stays conformal) is shown. A β -deformation of the $\mathcal{N}=4$ SYM theory [43] is obtained by introduction of a marginal deformation of the superpotential $h\text{Tr}(e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2)$ breaking $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$ (h is the Yukawa coupling). The arbitrary parameter β may be complex and this does not spoil superconformal invariance of the theory [44]. The initial R -symmetry $SU(4)_R$ breaks to $U(1)_R$ with the additional global symmetry $U(1)_1 \times U(1)_2$ [43]. From the indices point of view the parameters v and w play now the role of chemical potentials for the latter global group. SCI for the β -deformed theory is the same as in the initial theory [5]. This means that these theories share essentially the same set of BPS states. In the conclusion of [17], we discussed appearance of the $SO(3)$ $\mathcal{N}=4$ SYM theory from an $\mathcal{N}=1$ model after a superpotential deformation, such that both theories share the same SCI. Actually, SCIs of all exactly marginally deformed theories coincide, only the interpretation of chemical

potentials is different, being tied to global groups of different meaning. Therefore these indices serve as invariants of the conformal manifold with their structure reflecting only a part of the global symmetries preserved by the superpotential.

As an example of different deformation of $\mathcal{N}=4$ theories we can mention the deformation to $\mathcal{N}=1$ SYM theory with two chiral superfields in the adjoint representation and an additional $U(1)$ global group (see [45] and references therein). This theory has an $SL(2, \mathbb{Z})$ group electric-magnetic duality inherited from $\mathcal{N}=4$ SYM theory in its infrared fixed point. At the level of SCIs such a deformation is realized in a very simple way, it is just necessary to fix, say, $s_3 = \sqrt{pq}$, which excludes this parameter completely from the integrals.

The q -beta integrals appearing from SCIs of all $\mathcal{N}=4$ SYM theories in the limit $p \rightarrow 0, s_2 \rightarrow 0$ determine orthogonality measures for special cases of the Koornwinder and Macdonald orthogonal polynomials (for E_6, E_7 , and E_8 root systems these measures are generic [35]). We come thus to a natural question on whether one can give a similar meaning to general elliptic hypergeometric integrals describing $\mathcal{N}=4$ SCIs and construct corresponding biorthogonal functions. The first example of such biorthogonal functions in the univariate case has been found in [13] and a particular $SP(2N)$ -group multivariable generalization of them has been constructed in [15]. For the exceptional root systems $\mathcal{N}=4$ SCIs define the only currently known integrals pretending to such a role.

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Appendix A: Characters of the Adjoint Representations

Here we list characters of the adjoint representations for simple Lie groups G depending on the maximal torus variables z_j .

For $SU(N)$ group one has N variables $z_j, \prod_{j=1}^N z_j = 1$, and

$$\chi_{SU(N), \text{adj}}(z_1, \dots, z_N) = \sum_{1 \leq i < j \leq N} (z_i z_j^{-1} + z_i^{-1} z_j) + N - 1.$$

For $SO(2N+1)$ group of rank N the character is (no constraints on z_j)

$$\chi_{SO(2N+1), \text{adj}}(z) = \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + \sum_{i=1}^N z_i^{\pm 1} + N,$$

where $z_i^{\pm 1} z_j^{\pm 1} := z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1}$ and $z_i^{\pm 1} := z_i + z_i^{-1}$.

For $SP(2N)$ and $SO(2N)$ groups of rank N the characters are

$$\begin{aligned} \chi_{SP(2N),\text{adj}}(z) &= \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + \sum_{i=1}^N z_i^{\pm 2} + N, \\ \chi_{SO(2N),\text{adj}}(z) &= \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + N. \end{aligned}$$

The character for the adjoint representation of G_2 group is a symmetric polynomial of two parameters z_1 and z_2 , but it is convenient to introduce the third variable using relation $z_1 z_2 z_3 = 1$. Then,

$$\chi_{G_2,\text{adj}}(z_1, z_2, z_3) = 2 + \sum_{1 \leq i < j \leq 3} z_i^{\pm 1} z_j^{\pm 1}.$$

The exceptional F_4 group has rank four and

$$\begin{aligned} \chi_{F_4,\text{adj}}(z_1, \dots, z_4) &= \sum_{i=1}^4 z_i^{\pm 1} + \sum_{1 \leq i < j \leq 4} z_i^{\pm 1} z_j^{\pm 1} \\ &\quad + (z_1^{1/2} + z_1^{-1/2})(z_2^{1/2} + z_2^{-1/2})(z_3^{1/2} + z_3^{-1/2})(z_4^{1/2} + z_4^{-1/2}) + 4. \end{aligned}$$

Description of the exceptional Lie groups $E_{6,7,8}$ can be found in [46]. The rank of the group E_6 is equal to six and

$$\begin{aligned} \chi_{E_6,\text{adj}}(z_1, \dots, z_6) &= 6 + \sum_{1 \leq i < j \leq 5} z_i^{\pm 1} z_j^{\pm 1} \\ &\quad + z_6^{3/2} \prod_{i=1}^5 z_i^{-1/2} \left(1 + \sum_{1 \leq i < j \leq 5} z_i z_j + \sum_{1 \leq i < j < k < l \leq 5} z_i z_j z_k z_l \right) \\ &\quad + z_6^{-3/2} \prod_{i=1}^5 z_i^{1/2} \left(1 + \sum_{1 \leq i < j \leq 5} (z_i z_j)^{-1} + \sum_{1 \leq i < j < k < l \leq 5} (z_i z_j z_k z_l)^{-1} \right). \end{aligned}$$

The rank of the group E_7 is equal to seven and the needed character is

$$\begin{aligned} \chi_{E_7,\text{adj}}(z_1, \dots, z_7) &= 7 + \sum_{1 \leq i < j \leq 6} z_i^{\pm 1} z_j^{\pm 1} + z_7^{\pm 2} \\ &\quad + (z_7 + z_7^{-1}) \left(\prod_{l=1}^6 z_l^{1/2} \sum_{i=1}^6 z_i^{-1} + \prod_{l=1}^6 z_l^{-1/2} \left(\sum_{i=1}^6 z_i + \sum_{1 \leq i < j < k \leq 6} z_i z_j z_k \right) \right). \end{aligned}$$

The group E_8 is the biggest exceptional Lie group, it has rank eight and

$$\begin{aligned} \chi_{E_8,\text{adj}}(z_1, \dots, z_8) &= 8 + \sum_{1 \leq i < j \leq 8} z_i^{\pm 1} z_j^{\pm 1} + \prod_{i=1}^8 z_i^{-1/2} \left(1 + \sum_{1 \leq i < j \leq 8} z_i z_j \right) \\ &\quad + \prod_{i=1}^8 z_i^{1/2} \left(1 + \sum_{1 \leq i < j \leq 8} (z_i z_j)^{-1} + \sum_{1 \leq i < j < k < l \leq 8} (z_i z_j z_k z_l)^{-1} \right). \end{aligned}$$

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