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INFRARED FINITE OBSERVABLES IN $\mathcal{N}=8$ SUPERGRAVITY

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Abstract

Using the algorithm of constructing the IR finite observables suggested in [20] and discussed in details in [21], we consider construction of such observables in $\mathcal{N}=8$ SUGRA in NLO of PT. In general, contrary to the amplitudes defined in the presence of some IR regulator, such observables do not reveal any simple structure.

1 Introduction

In the last decade remarkable progress in understanding the structure of the amplitudes in supersymmetric gauge theories has been achieved [1]. Due to development of the so-called unitarity cut technique [2, 3], the three- and four-loop results for the four-point amplitudes became available for $\mathcal{N}=4$ super Yang-Mils theory (SYM) [4]. The application of the same technique in $\mathcal{N}=8$ supergravity (SUGRA), first used in [5, 6, 7], made it possible the computation of the three- [8] and four-loop [9] four-point amplitudes in $\mathcal{N}=8$ SUGRA. We want to stress that obtaining such results, using the standard diagram technique (component or superspace) seems practically impossible due to extreme complexity of such computations.

This results initiated once again the discussion of possible ultraviolet (UV) finiteness of the $\mathcal{N}=8$ SUGRA [10] and motivated the search of possible constraints on the UV divergences in $\mathcal{N}=8$ SUGRA from various points of view [11, 12, 13, 14] and of the presence of the symmetry group $E_{7(7)}$ in the theory [15, 16].

The answers for the amplitudes, which can be obtained from the unitarity cut technique, are usually represented as a sum of the master scalar integrals, defined in D dimensions with fixed coefficients. This provides the possibility to analyze the dependence (UV behavior) of the amplitude on the dimensionality of space-time D. Such an analysis up to four loops for the four-point amplitudes in $\mathcal{N}=8$ SUGRA revealed the same UV behavior of the amplitudes as in $\mathcal{N}=4$ SYM. This result, as claimed by the authors of [10, 17], does not follow from the properties of $\mathcal{N}=8$ SUSY and cannot be obtained, at least in some obvious way, from the previous analysis based on different versions of superspace technique (see, however, [18]). This makes it possible, once again, to suggest that $\mathcal{N}=8$ SUGRA amplitudes (the S-matrix) are [10, 17] UV finite as in the case of $\mathcal{N}=4$ SYM.

The expressions obtained by means of the unitarity cut technique in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA usually have a very "simple" form. This, and possible UV finiteness of the $\mathcal{N}=8$ SUGRA, allows one to make a conjecture, that the amplitudes (the S-matrix) in $\mathcal{N}=8$ SUGRA are the "simplest" among all $D=4$-dimensional QFTs with maximum spin ≤ 2[19].

However, despite the compact and simple form and the UV finiteness, the amplitudes in $\mathcal{N}=8$ SUGRA (as in any $D=4$ dimensional QFT with massless fields) are still, strictly speaking, ill defined in $D=4$ (without some infrared (IR) regulator) due to the presence of the IR divergences.

In the case of Yang-Mills (YM) theories (and QCD) in such cases the objects of physical interest are not the amplitudes themselves but the so-called IR finite observables which are build from the amplitudes, but possess no dependence on the IR regulator. In [20, 21] the authors suggested the example of such observables for $\mathcal{N}=4$ SYM. In contrast to relatively simple answers for several first orders of PT for the four- and five-point amplitudes [22], the IR finite observables considered in [20, 21], in the first nontrivial order of PT (NLO) has no simple structure and resemble the complicated (prior to properly applied simplifications) expressions for the six-point amplitudes in the second order of PT [23].

It seems natural to consider the similar IR finite observables in $\mathcal{N}=8$ SUGRA. It is this kind of observables that have a clear physical meaning. In the current article we discuss the construction of such observables in $\mathcal{N} = 8$ SUGRA using the same methods as in [20, 21].

2 Inclusive cross-sections as IR finite observables

One of the possible IR finite observables are properly constructed perturbative inclusive cross-sections. Such observables naturally appear in the parton model of perturbative QCD [24, 25, 26, 27, 28] and their construction is based on the Kinoshita-Lee-Nauenberg (KLN) theorem [29]. In such inclusive cross-sections, if the dimensional regularization is used, the IR divergences appears as $1/\epsilon$ poles. Cancellation of this infrared divergences coming from the loops includes two main ingredients: emission of additional soft real quanta and redefinition of the asymptotic states resulting in the splitting terms governed by the Altarelli-Parisi kernels. The latter ones take care of the collinear divergences and are absorbed into the probability distributions of initial and final particles over the fraction of momenta $q_i(z)$. The appearance of such distributions can be heuristically understood in the following fashion: in massless QFT a particle with momentum p is indistinguishable from the jet of particles with the same overall momentum and quantum numbers flying parallell. In the first none trivial order of PT in dimensional regularization the momentum distribution $q_i(z)$ can be represented as

$$q_i(z, \frac{Q_f^2}{\mu^2}) \quad = \quad \delta(1-z) + \frac{g^2}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\sum_j P_{ij}(z), \tag{1}$$

where $P_{ij}(z)$ are the so-called splitting functions. They define the probability that the particle i emits the collinear particle j with a fraction of momentum z. Here g is the coupling constant, μ is the mass parameter of dimensional regularization. Q_f is usually called the factorization scale and can be interpreted as the width of a jet of collinear particles. The contributions from $P_{ij}(z)$ to the cross-section sometimes are called the collinear counterterms.

Schematically, the class of IR finite observables discussed above, which are the inclusive cross-sections, can be written as

$$\begin{aligned} d\sigma^{incl}_{obs} \quad &= \quad \sum_{n=2}^{\infty}\int_0^1 dz_1 q_1(z_1, \frac{Q_f^2}{\mu^2})\int_0^1 dz_2 q_2(z_2, \frac{Q_f^2}{\mu^2})\prod_{i=1}^{n}\int_0^1 dx_i q_i(x_i, \frac{Q_f^2}{\mu^2}) \times \\ &\times d\sigma^{2\to n}(z_1p_1, z_2p_2, ...)S_n(\{z\},\{x\}) = g^4\sum_{L=0}^{\infty}\left(\frac{g^2}{16\pi^2}\right)^L d\sigma_L^{Finite}(s,t,u,Q_f^2), \end{aligned} \tag{2}$$

where $\mathcal{S}_n(\{z\},\{x\})$ are the so-called measurement functions, which define what inclusive cross-section (total, differential distribution, etc.) is considered. Note that not for any choice of $\mathcal{S}_n(\{z\},\{x\})$ one gets the IR finite result (see [20, 21] for discussion).

3 Computation of inclusive cross-section in $\mathcal{N}=8$ SUGRA

Our aim is to evaluate the inclusive differential polarized cross-section in NLO in the week coupling limit in planar $\mathcal{N}=8$ SUGRA in analytical form and to trace the cancelation of the IR divergences.

Consider the inclusive cross-section for the scattering of polarized gravitons in $\mathcal{N}=8$ SUGRA in the first nontrivial order of PT and follow the algorithm discussed in [20, 21]. We assume that all the gravitons have a fixed helicity $(+)$ or $(-)$ and consider all of them to be outgoing ones. The incoming graviton has the opposite momentum and helicity. Then all the tree amplitudes with helicity configurations $(+...+)$ or $(-+...+)$ are identically zero. The first nonzero tree level graviton amplitudes, similar to the YM case, are those with helicity configurations $(--+...+)$. They are called the MHV (or anti-MHV in the case $(++-...-)$) amplitudes. We restrict ourselves in our consideration to the class of the MHV amplitudes (the MHV channel) only. Consideration of the anti-MHV channel can be done in the similar fashion. Note, however, that such a restriction is possible only in the leading order. In higher orders of PT in order to achieve the IR finiteness, in general, one has to consider all type of the amplitudes since they are mixed.

Consider the process of 2×2 graviton scattering where all incoming and all outgoing gravitons have positive helicity. In our notations this corresponds to the amplitude with helicity configuration $(--++)$. This amplitude is the MHV and anti-MHV at the same time. So one has to take it with the weight $1/2$ for considering the MHV channel only.

The tree amplitude under consideration written in helicity spinor formalism [30] has the form

$$\mathcal{M}_4^{tree}(1^-,2^-,3^+,4^+) = (16\pi G_N)i\ \langle 1\,2\rangle^8\ \frac{[1\,2]}{\langle 3\,4\rangle\ N(4)}, \tag{3}$$

where

$$N(n) \equiv \prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \langle i\,j\rangle\,. \tag{4}$$

Here the notation, now standard, for the spinor inner product has been used:

$$\epsilon^{ab}\lambda_a^{(i)}\lambda_b^{(j)} = \langle\lambda^{(i)}\lambda^{(j)}\rangle \doteq \langle ij\rangle, \quad \epsilon^{\dot a\dot b}\bar\lambda_{\dot a}^{(i)}\bar\lambda_{\dot b}^{(j)} = [\bar\lambda^{(i)}\bar\lambda^{(j)}] \doteq [ij]. \tag{5}$$

Under the complex conjugation one has

$$(\langle ij\rangle)^* = [ij], \tag{6}$$

and also

$$\langle ij\rangle[ij] = s_{ij}, \tag{7}$$

where $s_{ij}=(p_i+p_j)^2$, and $p_i,\ p_j$ are some on-shell momenta which correspond to the i-th and j-th particles.

Then the differential cross-section $d\sigma(g^-g^- \to g^+g^+)/d\Omega$ can be written as

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)_0^{(--++)} = \frac{1}{E^2}\int d\phi_2 |\mathcal{M}_4^{tree|}|^2 \mathcal{S}_2, \tag{8}$$

where $d\phi_2$ is the two particle phase space, $\mathcal{S}_n$ is the measurement function which, in our case, has the form

$$\mathcal{S}_2 = \delta_{+,h_3}\delta^{D-2}(\Omega - \Omega_{13}), \tag{9}$$

$d\Omega_{13} = d\phi_{13} dcos(\theta_{13})$, θ_{13} is the scattering angle of a particle with momentum p_3 with respect to the particle with momentum p_1 in the center of mass (c.m.) frame, δ_{+,h_3} corresponds to the fact that particle with momentum p_3 has the positive helicity.

From now on the dimensional regularization (reduction) will be used. In the c.m. frame the Born contribution to the cross-section can then be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_0^{(--++)} = \frac{1}{E^2}\frac{\alpha_{Gr}^2 s^6}{t^2u^2}\left(\frac{\mu^2}{s}\right)^\epsilon = \frac{(\alpha_{Gr}E^2)^2}{E^2}\left(\frac{\mu^2}{s}\right)^\epsilon \frac{16}{(1-c^2)^2}, \tag{10}$$

where s, t, u are the standard Mandelstam variables, E is the total energy of initial particles in the c.m. frame, $c = \cos\theta_{13}$ is the cosine of the scattering angle of the third particle, μ and ϵ are the parameters of dimensional regularization, and $\alpha_{Gr} = G_N/4\pi$. In the c.m. frame for the Mandelstam variables one has

$$s = E^2, \quad t = -E^2/2(1-c), \quad u = -E^2/2(1+c).$$

3.1 Virtual contribution

Consider now the one-loop correction to the Born contribution. The corresponding amplitude has the form:

$$\begin{aligned} \mathcal{M}_4^{\mathcal{N}=8}(1^-,2^-,3^+,4^+) &= (16\pi G_N)^2 \frac{1}{4}\langle 1\,2\rangle^8 \\ &\times \Big[h(1,\{2\},3)h(3,\{4\},1)Tr^2[1234]\, I_4^{1234} + \text{perms}\Big], \end{aligned} \tag{11}$$

where $Tr[i_1i_2i_3i_4] \equiv Tr[\hat{k}_{i_1}\hat{k}_{i_2}\hat{k}_{i_3}\hat{k}_{i_4}]$ and

$$h(a,\{1\},b) = \frac{1}{\langle a\,1\rangle^2\langle 1\,b\rangle^2}.$$

The summation goes over all possible permutations and the scalar integral I_4^{1234} is equal to

$$I_4^{1234}(s,t) = -\frac{2}{st}\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left(\frac{1}{\epsilon^2}\Big(\left(\frac{\mu}{s}\right)^\epsilon + \left(\frac{\mu}{-t}\right)^\epsilon\Big) + \frac{1}{2}\log^2\left(\frac{s}{-t}\right) + \frac{\pi^2}{2}\right) + \mathcal{O}(\epsilon).$$

So the contribution to the cross-section in c.m. frame has the form

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{virt}^{(--++)} &= \frac{(\alpha_{Gr}E^2)^3}{\pi E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{64}{(1-c^2)^2}\left[\frac{1}{\epsilon}\Big((1+c)\log(\frac{1+c}{2})\right. \\ &\left. +(1-c)\log(\frac{1-c}{2})\Big) + 2\log(\frac{1+c}{2})\log(\frac{1-c}{2})\right]. \end{aligned} \tag{12}$$

It should be stressed that in this order of PT the UV divergences in $\mathcal{N}=8$ are absent and all the divergences have the IR nature. Note also the absence of the $1/\epsilon^2$ pole which in this case cancels due to permutations in (11) contrary to the gauge theories in $D=4$ where it is usually present. As one can see from (12), despite the extremely complicated intermediate expressions appearing in diagrammatical computations, the final result for the four-point one-loop MHV graviton amplitude in $\mathcal{N}=8$ has a relatively "simple" form.

3.2 Real emission

Following the algorithm for construction of the IR finite observables we consider the contribution to the cross-section from amplitudes with additional particles in the final state. For the fixed helicity of initial particles one has two types of the graviton amplitudes:

1. Three gravitons in the final state with positive helicity: $g^-g^- \to g^+g^+g^+$. This is the MHV amplitude;

2. Two gravitons in the final state with positive and one with negative helicity: $g^-g^- \to g^+g^+g^-$. This is the anti-MHV amplitude.

In the MHV channel only the first amplitude contributes. In this case there are three identical particles in the final state so we have to choose which particle we are detecting. This can be achieved, for example, by arranging the particles in according to their energy and selecting "the fastest one". It would correspond to the measurement function of the form

$$\mathcal{S}_3^{(--+++),1} = \delta_{+,h_3}\Theta(p_3^0 > p_4^0)\Theta(p_3^0 > p_5^0)\delta^{D-2}(\Omega - \Omega_{13}). \tag{13}$$

In practice it is more convenient to work with the measurement function written as [21]

$$\mathcal{S}_3(p_3,p_4,p_5) = \Theta(p_3^0 - \frac{1-\delta}{2}E)\delta^{D-2}(\Omega - \Omega_3), \tag{14}$$

where δ is an arbitrary parameter which can be fixed, for example, from the requirement that the detected particle is the fastest one (this corresponds to $\delta = 1/3$). We will leave the value of δ arbitrary to be convicted that the cancelation of the IR divergences occurs for arbitrary δ.

The amplitude for the process $g^-g^- \to g^+g^+g^+$ can be written as:

$$\mathcal{M}_5^{tree}(1^-,2^-,3^+,4^+,5^+) = (16\pi G_N)^{\frac{3}{2}} i\ \langle 1\,2\rangle^8 \frac{\epsilon(1,2,3,4)}{N(5)}, \tag{15}$$

where

$$\epsilon(i,j,m,n) \equiv 4i\epsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma \;=\; [i\,j]\,\langle j\,m\rangle\,[m\,n]\,\langle n\,i\rangle - \langle i\,j\rangle\,[j\,m]\,\langle m\,n\rangle\,[n\,i]\ .$$

Then, the cross-section is

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{Real} = \frac{1}{E^2}\int d\phi_3 |\mathcal{M}_5|^2 \mathcal{S}_3, \tag{16}$$

where $d\phi_3$ is three particle phase space. After the integration one has

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{Real} = \frac{(\alpha_{Gr}E^2)^3}{\pi E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{64}{(1-c^2)^2}\left[\frac{1}{\epsilon}\left((1+c)\log(\frac{1+c}{2})\right.\right. \tag{17}$$
$$\left.\left.+(1-c)\log(\frac{1-c}{2})\right) + \text{Finite part}(\delta, c)\right].$$

The finite part is a complicated polynomial function of log, $\log^2$ and Li_2 with the argument of the form $(1 \pm c)/2$ and in general have the same structure as in [21][1].

3.3 Collinear counterterms

Consider now the contributions from collinear counterterms. In the MHV channel the splitting function $P_{ij}(z)$ has the form (we use here slightly different notation for the splitting functions indicating explicitly all three particles like $P^{init}_{fin_1,fin_2}(z)$ to avoid confusion.)

$$P^{g^-}_{g^+g^+} = [\frac{1}{z(1-z)_+}]^2 = \frac{1}{z^2} + \frac{2}{z(1-z)_+} + \frac{1}{[(1-z)^2]_+}. \tag{18}$$

Here $1/[(1-z)^2]_+$ should be understood as

$$\int dz \frac{f(z)}{[(1-z)^2]_+} = \int dz \frac{f(z) - f(1) - f'(1)(z-1)}{(1-z)^2}.$$

The splitting function $P^{g^-}_{g^+g^+}$ can be obtained as the collinear limit of the corresponding amplitude [30]. Note also that in the case of $\mathcal{N} = 8$ SUGRA the probability distribution $q(z)$ does not receive radiative corrections in α_{Gr} except for the first order[6].

The contribution to the cross-section from the IR counterterms can be schematically written as

$$d\sigma^{spl,init}_{2\to 2} = \frac{\alpha_{Gr}}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q^2_f}\right)^{\epsilon}\int_0^1 dz P^{g^-}_{g^+g^+}(z) \sum_{i,j=1,2;\ i\neq j} d\sigma_{2\to 2}(zp_i, p_j, p_3, p_4)\mathcal{S}^{spl,init}_2(z), \tag{19}$$

$$d\sigma^{spl,fin}_{2\to 2} = \frac{\alpha_{Gr}}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q^2_f}\right)^{\epsilon} d\sigma_{2\to 2}(p_1, p_2, p_3, p_4)\int_0^1 dz P^{g^-}_{g^+g^+}(z)\mathcal{S}^{spl,fin}_2(z). \tag{20}$$

The measurement function in this case has the same form as in the case of the real emission, but now depends on the fraction of momentum z that restricts the integration over z

$$\mathcal{S}^{spl,1}_2(z) = \delta_{+,h_3}\delta^{D-2}(\Omega - \Omega_{13})\theta(z - z_{min}), \tag{21}$$

where z_{min} is equal to

$$z^{in}_{min} = \frac{(1-\delta)(1-c)}{1+\delta - c(1-\delta)}, \quad z^{fin}_{min} = (1-\delta) \tag{22}$$

[1] One can get it upon request from the authors

for the splitting function of the initial and final states, respectively. This conditions can be obtained from the requirement $p_3^0 > (1-\delta)E/2$ in the appropriate kinematics.

Integration over the phase space and over the fraction of momentum z gives:

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{InSplit} = \frac{(\alpha_{Gr}E^2)^3}{\pi E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{128}{(1-c^2)^2}\left[\frac{1}{\epsilon}\left(\frac{1-2\delta}{(\delta-1)\delta} - 2\log(1-\delta)\right.\right. \tag{23}$$
$$\left.\left. +2\log\delta - (1-c)\log\frac{1-c}{2} - (1+c)\log\frac{1+c}{2}\right) + \text{Finite part}(\delta, c)\right],$$

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{FnSplit} = \frac{(\alpha_{Gr}E^2)^3}{\pi E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{128}{(1-c^2)^2}\frac{1}{\epsilon}\left[\frac{2\delta-1}{(\delta-1)\delta} + 2\log(1-\delta) - 2\log(\delta)\right]. \tag{24}$$

4 IR finite observables in $\mathcal{N} = 8$ SUGRA

It is easy to see that the MHV part of inclusive cross-section defined in (2) as a sum of several contributions

$$A^{MHV} = \frac{1}{2}\left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--++)}_{0} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--++)}_{Virt} + \left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{Real} + \left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{InSplit} + \left(\frac{d\sigma}{d\Omega_{13}}\right)^{(--+++)}_{FnSplit}, \tag{25}$$

is IR finite, all the divergences cancel as expected. It should be stressed that this cancellation occurs for arbitrary δ. The analogous cancellation should take place in the anti-MHV channel. Note that A^{MHV} has no simple structure in contrast to the virtual correction to the cross-section $(d\sigma/d\Omega_{13})_0^{(--++)}$.

5 Conclusion

We have applied the algorithm for construction of IR finite observables in $\mathcal{N} = 8$ SUGRA and explicitly demonstrated the cancellation of the IR divergences in NLO of PT. One can see that such observables, in general, have no any simple structure in contrast to the amplitudes defined in the presence of some IR regulator. In [20, 21] a similar analysis has been performed for the $\mathcal{N} = 4$ SYM theory with the same conclusions. This makes us to conclude that this type of observables is not optimal in the sense of "simplicity" of the answer. It is also important to note the dependence of the answers on the factorization scale Q_f which breaks the conformal invariance. This dependence is a general feature of observables constructed on the base of eq.(2).

At the same time there are no "no-go" theorems which prohibit the existence of observables reflecting the rich symmetries of maximally supersymmetric YM or gravity theories. The search of such observables seems to be an interesting challenge, if one wants to construct physically meaningful expressions which reveal the possible integrability structure of the model.

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