

# GRAVITATIONAL RADIATION

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## 1 INTRODUCTION

Gravitational radiation is today one of the most exciting areas of classical general relativity in which to work. After the early decades of doubt over its physical reality, relativists came in the late 1950's and early 1960's to a general consensus about its reality and its broad physical effects. But the delicacy of some of the approximations needed to simplify Einstein's equations enough to study the waves has since led occasionally to passionate controversies over certain details of the broad picture. We are very far from having a complete understanding of this subject, and the prospect that observations of radiation may be made in the next decade makes the study even more interesting.

The first three sections of the review presented here try to give an idea of the breadth of modern research on the subject. This is clearly not possible in any depth in a short paper, and I have not been able to refer to as much of the interesting work now going on as I would have liked. I have also only described in any detail those subjects which I need for subsequent development, particularly in the final section. This is devoted to my own most immediate interest in the field, the description of radiation in the Newtonian limit.

The plan of the paper is as follows. Section 2 studies small-amplitude waves in vacuum, both in linearized theory and in perturbations of nonlinear solutions, as well as in asymptotically flat spacetimes (near null infinity). Section 3 reviews work on large-amplitude waves. Section 4 discusses the interaction of waves with matter, including the quadrupole formulas of linearized theory and a short review of current gravitational-wave detectors. Section 5 describes how to formulate the Newtonian limit and extend the quadrupole formulas to self-gravitating weak-field systems. My conventions follow those of Misner et al (1973).

## 2 SMALL AMPLITUDE WAVES IN VACUUM

### 2.1 Linearized Theory

The simplest approximation in which to study gravitational radiation is linearized theory, by which we mean first-order perturbations of flat spacetime. If we let the metric components be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (1)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric and  $|h_{\mu\nu}| \ll 1$ , then we can pretend  $h_{\mu\nu}$  is a tensor on Minkowski spacetime, defining for example

$$h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}.$$

Then because  $g^{\mu\nu}$  is inverse to  $g_{\mu\nu}$  we have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2).$$

It is convenient to define

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h = -\delta(\sqrt{-g} g^{\mu\nu}) \quad (2)$$

where  $\delta$  denotes the first-order perturbation operator and

$$h = \eta^{\alpha\beta} h_{\alpha\beta}.$$

The metric components depend upon the coordinates, of course, and two types of coordinate transformations are permissible which preserve the condition  $|h_{\mu\nu}| \ll 1$ . The first is a 'background Lorentz transformation'

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu}$$

where  $\Lambda^{\mu'}_{\nu}$  is the constant matrix of a Lorentz transformation. The second is a small coordinate transformation

$$x^{\mu'} = x^{\mu} + \xi^{\mu}(x^{\alpha}).$$

This is called a gauge transformation, and to first order in  $\xi^{\mu}$  the components of a tensor  $T$  change by the amount  $\mathcal{L}_{\xi} T$ , where  $\mathcal{L}_{\xi}$  is the Lie derivative. For the metric this means that

$$g_{\mu'\nu'} = \eta_{\mu'\nu'} + h_{\mu'\nu'} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{L}_{\xi} \eta_{\mu\nu}$$

or the metric perturbation changes to

$$h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_{\xi} \eta_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}. \quad (3)$$

As long as  $|\xi_{\mu,\nu}| \ll 1$  the gauge transformation keeps us within linearized theory.

The field equations of linearized theory are derived in most serious textbooks, such as Misner *et al* (1973) or Schutz (1984a). If we adopt the gauge condition (Lorentz or de Donder gauge)

$$\bar{h}^{\mu\nu}_{,\nu} = 0 \quad (4)$$

then the vacuum field equations become

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then the vacuum field equations become

$$\square \bar{h}^{\mu\nu} = 0, \quad (5)$$

where  $\square$  denotes the flat-space wave operator  $\nabla^2 - \partial_t^2$ . A solution of (5) of the form

$$\bar{h}^{\mu\nu}(x^\alpha) = A^{\mu\nu} \exp(i k_\alpha x^\alpha), \quad k_\alpha k^\alpha = 0,$$

is called a plane gravitational wave. The constants  $A^{\mu\nu}$  are not all independent. First, the gauge condition (4) requires

$$A^{\mu\nu} k_\nu = 0.$$

Second, we have additional gauge freedom in that any vector  $\xi_\mu = b_\mu \exp(i k_\alpha x^\alpha)$  preserves condition (4) and transforms  $A^{\mu\nu}$  into  $A^{\mu\nu} + i b^\mu k^\nu + i b^\nu k^\mu - i b^\alpha k_\alpha \eta^{\mu\nu}$ . It is conventional to choose  $b^\mu$  such that  $A^\mu_\mu = 0$  and  $A^{\mu\nu} U_\nu = 0$  for some timelike vector  $U^\mu$ . Such a gauge is called transverse-traceless (TT), and in it  $A^{\mu\nu}$  has only two independent components, which correspond to the two polarizations.

Naturally, any physical observables must be gauge-invariant. In particular, the Riemann tensor, which to first order is

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} (h_{\mu\beta, \nu\alpha} + h_{\nu\alpha, \mu\beta} - h_{\nu\beta, \mu\alpha} - h_{\mu\alpha, \nu\beta}),$$

is invariant because its value for flat space is zero: the analogue of Eq. (3) would give  $R_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \xi \xi(0)$ . The Riemann tensor is the fundamental physical quantity of linearized theory: all the observable effects of the theory may be expressed in terms of it. Nevertheless, physicists are accustomed to dealing with other quantities, such as energy, which have great heuristic value even if they are not strictly necessary in the formulation of the theory. Since  $h_{\mu\nu}$  behaves like a free field on flat spacetime, it is natural to try to construct its stress-energy tensor and to ask what physical meaning can be attached to it.

One way of doing this is to construct the Lagrangian of linearized theory,

$$L \equiv \frac{1}{16\pi} (-h)^{\frac{1}{2}} R = \frac{1}{64\pi} (2h^{\beta\mu, \alpha} h_{\alpha\mu, \beta} - h^{\alpha\mu, \beta} h_{\alpha\mu, \beta} - 2h_{, \alpha} h^{\alpha\beta}_{, \beta} + h_{, \alpha} h^{\alpha}_{, \alpha}) + \text{div} + O(h^3),$$

where 'div' stands for a total divergence we have extracted. A formal stress-energy tensor may be defined from this by

$$\begin{aligned} T^{\mu\nu} &= 2 \partial L / \partial \eta_{\mu\nu} \\ &= \frac{1}{32\pi} [h_{\alpha\beta}^{(\mu} h^{\nu) \alpha\beta} - 4h^{\alpha\beta, (\mu} h^{\nu) \alpha}_{, \beta} + 4h^{(\mu}_{[\alpha, \beta]} h^{\nu) \alpha, \beta} \\ &\quad + 2h^{\mu\nu, \alpha} h^{\beta}_{\alpha, \beta} + 2h^{(\mu} h^{\nu) \alpha}_{, \alpha} + 2h^{\alpha(\mu, \nu)} h_{, \alpha} \\ &\quad - 2h^{\mu\nu, \alpha} h_{, \alpha} - h^{\mu}_{, \alpha} h^{\alpha, \nu}]. \end{aligned} \quad (6)$$

According to very general theorems (Schutz & Sorkin 1977), when the field equations are satisfied, this  $T^{\mu\nu}$  will differ from any other 'canonical'  $T^{\mu\nu}$  derived from the same Lagrangian only by terms of the form  $\partial_\alpha M^{\mu[\nu\alpha]}$ , which can change the localization of energy density but not its total integral over a spacelike hypersurface. This is characteristic of field theories and occurs in electromagnetism as well.

Where electromagnetism and linearized theory differ is that the electromagnetic analogy of Eq. (6),

$$T_{EM}^{\mu\nu} = \frac{1}{4\pi} (F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}),$$

is electromagnetic-gauge-invariant whereas Eq. (6) is not invariant under the gauge transformations of Eq. (3). In fact we find

$$T^{\mu\nu} = T^{\mu\nu} + \partial_\alpha N^{\mu\nu\alpha} \quad (7)$$

where  $N^{\mu\nu\alpha}$  is a function of  $\xi^\beta$  and  $h_{\lambda\sigma}$  but is not antisymmetric on  $\nu$  and  $\alpha$ . This means that a gauge transformation not only changes the localization of energy but also its total value:

$$\int T^{\mu\nu} d^3x = \int T^{\mu\nu} d^3x + \int N^{\mu\nu\alpha}{}_{,0} d^3x. \quad (8)$$

Energy in linearized theory is still conserved, but it has no unique zero-point.

Under certain circumstances both the localization and the gauge ambiguities may be reduced. If  $h_{\mu\nu}$  is periodic in time then we may average  $T^{\mu\nu}$  over one period. Since the extra term in Eq. (8) averages to zero for all gauge changes which preserve the periodicity of  $h_{\mu\nu}$ , the mean value  $\int \langle T^{\mu\nu} \rangle d^3x$  is gauge-invariant in this restricted sense. More generally, if  $h_{\mu\nu}$  is primarily composed of spatial and temporal frequencies larger than some  $k$  then the average of  $T^{\mu\nu}$  over a space-time region of dimension  $L$  will be invariant under gauge and localization changes to order  $T^{\mu\nu}/kL$  (Misner, et al 1973). This average is called the Brill-Hartle average  $\langle T^{\mu\nu} \rangle_{BH}$ , and Isaacson (1968a,b) has shown that at the next order of perturbation theory this is the effective stress-energy tensor that generates the second-order, long length-scale metric perturbation. All pseudotensor methods of generating alternative expressions for  $T^{\mu\nu}$  are Brill-Hartle equivalent to Eq. (6). The expression for  $T^{\mu\nu}$  in the TT gauge is therefore of general interest in this sense:

$$T^{\mu\nu} = \frac{1}{32\pi} A^{\alpha\beta} A^{\star}_{\alpha\beta} k^\mu k^\nu.$$

Finally, we should mention that one can define not just a conserved energy and momentum from  $T^{\mu\nu}$  but also an angular momentum density

$$J^{\alpha\beta\mu} = x^\alpha T^{\beta\mu} - x^\beta T^{\alpha\mu} \quad (9)$$

where  $x^\alpha$  is the coordinate of our nearly-Lorentz system. Brill-Hartle

averaging of  $J^{\alpha\beta\mu}$  over some region removes its ambiguities only if the intrinsic (spin) angular momentum of the waves in the region is small compared to their orbital angular momentum about the origin of coordinates, so that  $x^\alpha$  may be regarded as constant in Eq. (9). This is not a very interesting case, since it does not allow us to study the angular momentum carried away by waves from a radiating source. We can do better by restricting the gauge. It can be shown that if we are in Lorentz gauge, Eq. (4), and we make a gauge transformation that stays in such a gauge, then the quantity  $N^{\mu\nu\alpha}$  in Eq. (7) is itself a divergence,  $N^{\mu\nu\alpha} = M^{\mu\nu\alpha\beta}_{,\beta}$ . It then follows that

$$J^{\alpha\beta\mu} = J^{\alpha\beta\mu} + \partial_\nu (x^\alpha N^{\beta\mu\nu} - x^\beta N^{\alpha\mu\nu} + M^{\alpha\mu\beta\nu} - M^{\beta\mu\alpha\nu}).$$

Thus, the total angular momentum,  $\int J^{ij0} d^3x$ , will be Brill-Hartle invariant within Lorentz gauge. We will find this result useful in our later discussion of gravitational radiation in the Newtonian limit.

## 2.2 Waves on curved backgrounds

Much of linearized theory can be generalized to the case of small perturbations of a curved spacetime, where

$$g_{\mu\nu} = g_{\mu\nu}^B + h_{\mu\nu}$$

and the 'background' (or unperturbed) metric  $g_{\mu\nu}^B$  dominates the perturbation  $h_{\mu\nu}$ . Then we can treat  $h_{\mu\nu}$  as a tensor field on the manifold whose metric is  $g_{\mu\nu}^B$ . Equations (2)-(4) generalize with  $\eta_{\mu\nu}$  replaced by  $g_{\mu\nu}^B$  and with derivatives made covariant with respect to  $g_{\mu\nu}^B$ . If  $g_{\mu\nu}^B$  is a vacuum metric ( $R_{\mu\nu}^B = 0$ ) then the wave equation becomes, in Lorentz gauge,

$$\nabla^\alpha \nabla_\alpha \bar{h}_{\mu\nu} + 2 R_{\alpha\mu\beta\nu}^B \bar{h}^{\alpha\beta} = 0. \quad (10)$$

Moreover, if we consider waves with wavelengths much shorter than the background radius of curvature, so that  $|\nabla_\alpha \nabla_\beta \bar{h}_{\mu\nu}| \gg |R_{\alpha\mu\beta\nu}^B \bar{h}_{\lambda\sigma}|$ , then Eq. (10) admits a 'geometrical optics' approximation, in which gravitational waves propagate on null geodesics, parallel-transporting their polarization tensors (Misner, et al 1973). Within this short-wavelength approximation the Brill-Hartle averaging technique again produces approximately gauge-invariant stress-energy tensors for the waves. Isaacson (1968a,b) has shown that this serves as an effective stress-energy tensor which curves the background, so that to second order

$$G_{\mu\nu}^B = 8\pi \langle T_{\mu\nu} \rangle_{BH}.$$

## 2.3 Waves in asymptotically flat regions

None of the above techniques is suitable for strong waves in highly curved regions. One of the most important advances in our

understanding of such waves was the realization by Bondi, Penrose, and collaborators that if such regions were parts of asymptotically flat spacetimes, then sufficiently far from such regions the waves would be weak and it would be possible to make gauge-invariant global definitions of energy and momentum and their fluxes. Although the subject is too large to give a detailed review of here (see Walker 1983, Geroch 1976 and references therein), there are a few aspects of it which need to be made clear for our subsequent discussion of radiation in the Newtonian limit.

In order to go 'sufficiently far' from strong-field regions and still stay with the radiation, we must move out along radial null geodesics. In the Schwarzschild metric these are lines of constant

$$u = t - r_* = t - r - 2M \ln(r/2M - 1), \quad (11)$$

and in other spacetimes we can expect this to be correct to some order. If we let each geodesic have an affine parameter  $v$ , then we want to describe the radiation as  $v \rightarrow \infty$  for fixed  $u$ . The basic idea is to make a (singular) coordinate transformation that maps the whole manifold onto a finite region in a way that preserves the individuality of the outgoing radial null geodesics. The method devised by Penrose is to define a manifold  $\hat{M}$  whose metric  $\hat{g}_{\alpha\beta}$  is

$$\hat{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$$

where  $\Omega$  is a non-zero,  $C^\infty$  function which approaches zero as  $v \rightarrow \infty$ . This means that the limit  $v \rightarrow \infty$  in  $\hat{M}$  is a limit to a finite distance. If the limiting boundary of  $\hat{M}$  which we approach in this manner has topology  $R \times S^2$  (for the coordinates  $u, \theta, \phi$ ), then we call the boundary  $\mathcal{J}^+$ , future null infinity. With any sphere  $u = \text{const}$  of  $\mathcal{J}^+$  it is possible to associate a mass and linear momentum of the metric, and from the shear of the light cones  $u = \text{const}$  as  $v \rightarrow \infty$  one can define an energy-momentum flux on  $\mathcal{J}^+$ . These are called the Bondi mass, momentum and flux. The energy flux is non-negative and it has recently been shown that the Bondi mass is positive-definite (Horowitz & Perry 1982, Ludvigsen & Vickers 1982, Walker 1983). The Bondi mass of a stationary metric equals its gravitational mass as measured by distant orbits. The Penrose definition of asymptotic flatness seems not to be overly restrictive: Friedrich (1983) has recently given an existence theorem for the construction of general spacetimes from data in and near  $\mathcal{J}^+$ .

The existence of a mass and momentum on  $\mathcal{J}^+$  may be traced to its symmetry group. The set of coordinate transformations which preserve the induced metric on  $\mathcal{J}^+$  is called the Bondi-Metzner-Sachs (BMS) group. It resembles the Poincaré group, which is the semi-direct product of the Lorentz group and the 4-dimensional translation group. But the BMS group involves instead an infinite-dimensional (abelian) translation group, called the supertranslations. Fortunately, the BMS group contains a unique 4-dimensional normal subgroup, and the associated

elements of the BMS Lie algebra generate the Bondi energy and momentum. But the rest of the supertranslations make a unique definition of angular momentum impossible, although we have yet to hear the last word on this. The problems linearized theory has with angular momentum carry over to  $\mathcal{I}^+$ !

### 3 LARGE AMPLITUDE WAVES

Although detectable gravitational radiation will have very small amplitude, it may originate in regions of very strong fields and large wave amplitudes. These can be studied by means of exact solutions or numerically.

There are many vacuum solutions which behave like large-amplitude waves, travelling at the speed of light and exhibiting polarization properties. The majority are catalogued in Kramer *et al* (1981). There has been interesting recent work on soliton-like waves in what are basically inhomogeneous cosmologies (Carr & Verdaguer 1983). Unfortunately, there are no known asymptotically flat examples. This is understandable, because a time-dependent asymptotically flat metric can admit at most one Killing vector, that of axial symmetry, and exact solutions are easier to find in situations of greater symmetry. But the absence of such exact solutions means that we cannot yet study except by approximation methods three fundamental problems: (a) the strength of a wave in relation to its source, (b) the evolution of a strong near-field wave into a weak far-zone wave, and (c) the attachment of  $\mathcal{I}^+$  to the manifold in a radiation situation.

It is natural to look to numerical calculations for answers to questions about (a) and (b) above, but the large demands that general relativity with its ten metric potentials makes on computer storage and speed (Piran 1983) put a number of interesting problems out of reach at the moment. Nevertheless, many interesting problems are being studied, and there has been considerable work on technical questions such as suitable gauge conditions (e.g. Bardeen 1983). These have been reviewed recently by Piran (1983).

The first successful large-scale numerical study involving gravitational radiation was the calculation by Eppley and Smarr of the collision of two black holes (Smarr 1979). Starting from rest at moderate separation, the equal-mass holes fell together and merged behind a single horizon, converting about 0.07% of their mass into radiation. Perhaps the most remarkable result of the calculation was that the radiation generated by the collision was equal to that which would have been predicted by (i) using linear perturbation theory to calculate the radiation from a small mass  $m$  falling into a black hole of mass  $M$ , and then (ii) setting  $M$  equal to the sum of the black-hole masses and  $m$  to the reduced mass (Smarr 1979, Detweiler 1979). The nonlinearities in the collision of the holes seem not to have made much difference to the emitted radiation.



Interest has now shifted to the gravitational collapse of fluid bodies, as models of supernovae. Nakamura (1981, 1983) has shown that axially symmetric collapse does indeed produce a black hole, provided the initial angular-momentum distribution permits the collapse to proceed far enough. Besides those I have already mentioned, a number of other workers are pursuing numerical calculations, including Centrella, Miller, Mann, Stewart, and Wilson. This field will undoubtedly be one of the most important areas of our subject in the future, as hardware improvements make more interesting calculations possible and as gravitational wave observations begin to call for quantitatively reliable collapse calculations.

#### 4 INTERACTION OF WAVES WITH MATTER

##### 4.1 Generation of waves

In linearized theory, if we keep the lowest-order source term  $T_{\mu\nu}$  we obtain the generalization of Eq. (5):

$$\square \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}. \quad (12)$$

The gauge condition Eq. (4) implies the conservation law

$$T^{\mu\nu}_{, \nu} = 0, \quad (13)$$

which means that the source behaves as it would in special relativity, free from the effects of gravitational fields. This is because Eq. (12) implies that the field and source are of the same order, so any coupling between them would be of second order. Seen another way, the first-order conservation law must be Eq. (13) because the energy-momentum of the gravitational field is second-order, Eq. (6). Linearized theory is therefore consistent only for sources with negligible self-gravity.

The retarded solution of Eq. (12) is

$$\bar{h}^{\mu\nu}(\tilde{x}, t) = 4 \int T^{\mu\nu}(\tilde{y}, t - |\tilde{x} - \tilde{y}|) |\tilde{x} - \tilde{y}|^{-1} d^3y, \quad (14)$$

and it is easy to show that in the slow-motion limit and in the TT gauge far away the non-zero metric perturbations are

$$h^{\text{TT}}_{jk}(t, \tilde{x}) = \frac{2}{r} (\delta_{ij} \delta_{kl} - \frac{1}{3} \delta_{jk} \delta_{il}) \int T^{il}(t-r, \tilde{y}) d^3y + O(\frac{1}{r^2}),$$

where  $r = |\tilde{x}|$  (e.g. Schutz 1984a). The conservation law Eq. (13) implies that

$$\begin{aligned} \int T^{il}(t, \tilde{y}) d^3y &= \frac{1}{2} \frac{d^2}{dt^2} \int T^{00}(t, \tilde{y}) y^i y^l d^3y \\ &= \frac{1}{2} \frac{d^2}{dt^2} I^{il}(t), \end{aligned} \quad (15)$$

where  $I^{il}$  is the quadrupole tensor of the source. Defining the reduced or trace-free quadrupole tensor by

$$\bar{\mathbb{I}}^{il} = I^{il} - \frac{1}{3} \delta^{il} I^k_k,$$

we have

$$h_{jk}^{TT}(t, \mathbf{x}) = \frac{1}{r} \ddot{\mathbb{I}}_{jk}(t-r) + O\left(\frac{1}{r^2}\right)$$

and a corresponding luminosity from the integral of Eq. (6) over a sphere

$$L = \frac{1}{5} \ddot{\mathbb{I}}^{jk} \ddot{\mathbb{I}}_{jk}. \quad (16)$$

This is known as the Landau-Lifshitz formula or the far-field quadrupole formula. Our earlier discussion shows that, when averaged over a typical timescale of the source's motion, this luminosity is gauge and localization invariant.

It is possible, of course, to go beyond first order in linearized theory, to what are called post-linear approximations. At the next order the equation of motion of the matter will clearly be

$$T^{\mu\nu}_{,\nu} = -\Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} - \Gamma^{\nu}_{\alpha\nu} T^{\mu\alpha} \quad (17)$$

where  $\Gamma^{\mu}_{\alpha\nu}$  is computed from the first-order perturbation, Eq. (14). Again in the slow-motion limit it turns out (see e.g. Schutz 1980) that the part of the right-hand side of Eq. (17) that expresses the reaction to the emitted radiation is

$$F^j_{\text{react}} = -\frac{2}{5} x_k I^{(5)jk} T^{00}, \quad (18)$$

which causes  $\int T^{00} d^3x$  to decrease in time at a rate  $-L$  when averaged over a suitable timescale. This is known as the near-zone quadrupole formula.

Although this energy balance is satisfying, the gauge problems surrounding energy in relativity suggest that one should remind oneself occasionally that energy is a secondary concept: all the physical predictions of the theory may be obtained from Eqs. (14) and (17) without any mention of energy.

The major shortcoming of linearized theory is the absence of self-gravitation at lowest order. This may be remedied either by beginning with the Newtonian approximation at first order or by perturbing not flat space but an existing curved metric. I will discuss the Newtonian alternative in section 5 below, but a few words on linear perturbation theory are appropriate here.

With a source Eq. (10) generalizes to

$$\nabla^{\alpha} \nabla_{\alpha} \bar{h}_{\mu\nu} + 2 R^B_{\alpha\mu\beta\nu} \bar{h}^{\alpha\beta} = -16\pi \delta T_{\mu\nu}, \quad (19)$$

where  $\delta T_{\mu\nu}$  is the first-order change in  $T_{\mu\nu}$  at a given coordinate

position (Eulerian change). At first order the equations of motion of the matter are

$$\nabla_{\alpha} \delta T^{\alpha\beta} = -T^{\mu\beta} \delta \Gamma^{\alpha}_{\mu\beta} - T^{\mu\alpha} \delta \Gamma^{\beta}_{\mu\beta}. \quad (20)$$

Thus, the metric perturbation generated in Eq. (19) acts back on its source  $\delta T^{\alpha\beta}$  via the perturbed Christoffel symbols in Eq. (20). Self-gravitational effects of  $\delta T^{\alpha\beta}$  can couple to  $\delta T^{\alpha\beta}$  at first order because  $T^{\alpha\beta}$  is non-zero at zero-order. While no general study of radiation-reaction seems yet to have been made in this context, there have been calculations in specific cases, particularly of the normal modes of pulsating relativistic stars, which have been reviewed by Detweiler (1979). In the limit of a weakly relativistic star, these results should be comparable with ones arrived at by the Newtonian approximation scheme. Such a comparison was made by Balbinski & Schutz (1982) with puzzlingly large discrepancies. More recent results (Lindblom & Detweiler 1983, Balbinski et al 1984) show closer agreement and suggest that the numerical errors in the earlier calculations reviewed by Detweiler (1979) were unexpectedly large.

The main limitation of perturbation theory is its restriction to small amplitudes, which excludes systems like binary stars and highly non-spherical collapses. The Newtonian alternative to be discussed later has no amplitude restrictions, but has a compensating restriction to weak internal gravity and slow motion.

#### 4.2 Detectors of waves

This review would not be complete without at least a mention of the intense activity now being directed toward the construction of laboratory detectors of gravitational radiation from astrophysical sources. Such detectors have negligible self-gravitation, so Eq. (17) fully describes their motion when  $\Gamma^{\mu}_{\alpha\beta}$  is calculated from the incoming wave field. The book edited by Deruelle & Piran (1983) contains a number of articles reviewing the current status of the major detectors.

It is not hard to derive estimates from Eq. (14) for the likely amplitude of waves incident on the Earth (see, e.g. Thorne 1983 or Schutz 1984b). A supernova probably converts no more than about  $0.1M_{\odot}$  into gravitational waves, and so a supernova in our own Galaxy might generate waves of amplitude  $h \lesssim 10^{-18}$ , while one in the Virgo cluster would give  $h \lesssim 10^{-21}$ , where  $h$  stands for the typical amplitude of  $h^{\text{TT}}_{ij}$ . The spectrum should be broadband, peaking around  $10^3$  Hz.

Current detectors, which are regarded as prototypes, are approaching the  $10^{-18}$  level, and coincidence experiments between some are planned. The goal of  $10^{-21}$  is some years away, but the amount of money being spent on such projects has been accelerating rapidly of late, particularly in the USA and Germany.

There are two main laboratory detectors, bars in the style pioneered in the 1960's by Weber (1960), and laser interferometers. Bars rely

generally on exciting their fundamental longitudinal mode of oscillation in resonance with the incident wave. Groups that are building bar detectors of special high-Q material and/or cryogenically cooled include Stanford/Louisiana, Rome/Frascati, Maryland, Rochester, Tokyo, Perth, Moscow, and elsewhere. See Blair (1983) for a review of their technical features. Properties of materials limit the length one can make a bar with a  $10^3$  Hz fundamental mode, and this is the most serious limitation on such detectors, since the strength of tidal gravitational forces is proportional to the size of the apparatus. The result is that a wave of  $h \sim 10^{-21}$  will typically deposit only a few percent of the energy of one phonon of oscillation. This means the detector (but not the wave) must be treated quantum-mechanically, and its state of vibration must be measured with the minimum of disturbance. The theory of how to do this has been studied by Caves *et al* (1980) and Braginsky (1983), but its practical implementation may be some years away.

If bar detectors cannot overcome this 'quantum limit' problem then laser interferometers may reach the magic  $10^{-21}$  first. Groups operating prototypes include Glasgow/Caltech, Munich, and MIT, and others are planned. Drever (1983) reviews their problems and principles. These measure by interferometry relative changes in the distances between a central mass and two others some distance away in perpendicular directions. The masses are freely suspended and act as free particles in the incident wave. Sensitivity may be increased by increasing the lengths of the arms and using multiple reflections along them. Current prototypes are of the order of 10m in size, but detectors of  $10^{-21}$  sensitivity having dimensions  $\geq 1$  km may be funded in the coming year, at least in the USA. Moreover, improvements in such things as mirror reflectivities and the maximum power of continuous-wave lasers could push sensitivities even deeper before these detectors encounter their 'quantum limit'. Ways of getting around this limit have been discussed by Caves (1981). They may be important, because it is not clear whether supernovae will in fact be as powerful emitters of waves as we have assumed.

Besides laboratory detectors, space-based detectors are at present being designed for the next generation of detectors. One such 'detector' system is already in operation: the signals transmitted between Earth and various inter-planetary spacecraft. By searching for anomalous time-delays in the round-trip transponder signal one can monitor the gravitational-wave background. Such detectors are optimum for wavelengths of about 1 AU, or frequencies in the milli-Hertz region, such as might come from the formation of supermassive black holes in galactic centers. These results have been reviewed by Hellings (1983).

## 5 RADIATION IN THE NEWTONIAN LIMIT

### 5.1 The near-zone quadrupole formula

Although Newton's equations are a weak-field limit of Einstein's equations, they differ from linearized theory in that they

do not admit radiation. This means that the twin problems of determining how much radiation leaves a nearly-Newtonian system (far-field radiation problem) and of assessing the reaction effects on the system itself (near-zone or radiation-reaction problem) are mathematically delicate. These questions are important observationally, because reaction effects seem to be important in the binary pulsar system (Taylor & Weisberg 1982, Boriakoff *et al* 1982), in cataclysmic variables (Paczynski & Sienkiewicz 1983), and in limiting the rotation rates of neutron stars (Friedman 1983); and also because we want to make inferences from observed gravitational waves (or observational upper limits in their amplitudes) about the behaviour of their sources. These questions are also important theoretically, since nearly-Newtonian systems are the only self-gravitating asymptotically flat systems in which we can study radiation analytically.

In linearized theory in the slow-motion limit, the far-zone wave luminosity, Eq. (16), and the near-zone reaction effects, Eq. (18), depend only on the quadrupole tensor,  $I_{jk}$  (defined in Eq. 15). This in turn depends only on the mass-density,  $\rho$ . Since the Newtonian limit is a slow-motion limit in which all other energies are small compared to the rest-mass  $\rho$ , one might guess that these formulae would apply in the Newtonian limit as well. This is in fact now widely accepted as true, but the difficulty of proving it generated a lively debate on the subject in the past decade. The initial demonstrations of the validity of Eqs. (16) and (18) in the Newtonian limit were given by Landau & Lifshitz (1962), Chandrasekhar & Esposito (1970), and Burke (1971). These and other derivations were critically examined by Ehlers *et al* (1976) and Walker & Will (1980a). The approach I will describe below is based on Futamase & Schutz (1983, 1984) and Futamase (1983), which contain references to other recent work.

In linearized theory our solution for the field, Eq. (14), employed retarded potentials as a way of excluding extraneous incoming radiation. In nonlinear general relativity, such potentials do not exist, and it is tempting to replace them with the boundary condition that there be no radiation on  $\mathcal{I}^-$ , the past endpoints of incoming null geodesics (Ehlers *et al* 1976). Unfortunately, this transforms the radiation problem into a global one, for which rigorous calculations are difficult. There has been progress in this direction recently (Walker 1984), but our approach will be based on the initial-value problem on a spacelike hypersurface, and will consequently be more local. (An initial-value approach based on characteristic surfaces is given by Winicour 1983.)

Most textbooks (e.g. Misner *et al* 1973, Schutz 1984a) extract the Newtonian limit of general relativity in a physically reasonable but nonrigorous manner. Consider a system of mass  $M$ , typical size  $R$ , velocity  $v$ , density  $\rho$ , and pressure  $p$ . One wants a low-redshift limit, so we want a limit in which  $M/R$  approaches zero. If we keep the size of the system fixed, then we need  $M \rightarrow 0$ , and consequently  $\rho \rightarrow 0$ . But self-gravity must still be important, so by the virial

theorem we must have  $p/\rho$  and  $v^2$  of the same order as  $M/R$ . Thus we want  $v \rightarrow 0$ ,  $\rho = O(v^2)$ ,  $p = O(v^4)$ . These in turn imply the inequalities  $|T^{00}| \gg |T^{0i}| \gg |T^{ij}|$ . From the weak-field equations (12) we see that  $\bar{h}^{00}$  dominates and satisfies

$$\nabla^2 \bar{h}^{00} = -16\pi\rho, \quad (21)$$

from which we deduce that  $\bar{h}^{00} = -4\phi_N$ , where  $\phi_N$  is the Newtonian potential. But now a closer examination of the conservation law  $\nabla_\alpha T^{i\alpha} = 0$  shows that self-gravity is important. For a perfect fluid it becomes

$$\ddot{\rho} v^i + \rho v^j \nabla_j v^i + \nabla^i p + \rho \nabla^i (-\bar{h}^{00}/4) = 0, \quad (22)$$

which is the Newton-Euler equation. What such a derivation lacks, however, is precision: is  $\rho = O(v^2)$  everywhere in spacetime; is  $\bar{h}^{00}$  dominant over  $\bar{h}^{i\alpha}$  everywhere in spacetime; in going from  $\square$  in Eq. (12) to  $\nabla^2$  in Eq. (21), which of the many solutions of Eq. (12) (retarded, advanced, or a mixture) is  $\bar{h}^{00}$  of Eq. (21) taken to be the limit?

We can increase our precision if we understand why this particular limiting procedure gives a self-consistent set of equations in general relativity. Fundamentally, it is because the Newtonian equations, Eqs. (21) and (22) and the continuity equation (consequence of  $\nabla_\alpha T^{0\alpha} = 0$ ),

$$\dot{\rho} + \nabla_i (\rho v^i) = 0, \quad (23)$$

are invariant under the following scaling of their solutions:

$$\begin{aligned} \rho(x^i, t) &\rightarrow \varepsilon^2 \rho(x^i, \varepsilon t) \\ p(x^i, t) &\rightarrow \varepsilon^4 p(x^i, \varepsilon t) \\ v^j(x^i, t) &\rightarrow \varepsilon v^j(x^i, \varepsilon t) \\ \bar{h}^{00}(x^i, t) &\rightarrow \varepsilon^2 \bar{h}^{00}(x^i, \varepsilon t). \end{aligned} \quad (24)$$

The factors of  $\varepsilon$  in front are those we deduced above from virial-theorem arguments. The scaling of  $t$  is equally important, since as  $v^i \rightarrow 0$  ( $\varepsilon \rightarrow 0$ ) it takes longer for everything to happen. Given any solution  $\{\rho, p, v^i, \bar{h}^{00}\}$ , Eqs. (24) define a sequence of solutions in which, as  $\varepsilon \rightarrow 0$ , the field gets weaker, and the velocities lower, but the size remains the same. For a binary star system, for example, the masses would decrease, the orbits would remain the same, but the orbital period would increase.

The Newtonian limit is thus a limit of solutions of general relativity in which the scalings in Eqs. (24) are preserved as far as possible, getting better as  $\varepsilon \rightarrow 0$ . The following definition therefore seems natural (Futamase & Schutz 1983): a regular, asymptotically Newtonian sequence of solutions of Einstein's equations is one defined by the

following sequence of initial data:

$$\begin{aligned}
 \rho(t=0, x^i, \epsilon) &= \epsilon^2 a(x^i) \\
 p(t=0, x^i, \epsilon) &= \epsilon^4 b(x^i) \\
 v^j(t=0, x^i, \epsilon) &= \epsilon c^j(x^i) \\
 \bar{h}^{ij}(t=0, x^i, \epsilon) &= \bar{h}^{ij}_{,0}(t=0, x^i, \epsilon) = 0,
 \end{aligned} \tag{25}$$

where  $a$ ,  $b$ , and  $c^i$  are functions of compact support. The first three assert that the initial data scale exactly in the Newtonian manner. The fourth equation sets the free data for the gravitational field to zero. Initial values of  $\bar{h}^{00}$ ,  $\bar{h}^{0i}$ , and their time derivatives are determined by the constraint equations. This is not the only choice of initial data which will give a Newtonian limit, but it is probably the simplest. Setting the free-field data to zero does in fact lead to a retarded-type solution in the source after about one light-crossing time (see Schutz 1980), but it can nevertheless be relaxed considerably: Futamase (1983) shows that random initial data for  $\bar{h}^{ij}$  of order  $\epsilon^4$  do not affect our conclusions when the randomness is averaged over. Schutz (1980) has argued that such averaging provides a physically consistent statistical derivation of the irreversible effects of radiation reaction.

Of course, the nonlinearities of general relativity will break the exact scalings of Eqs. (25) for  $t > 0$ , but the existence of the limit suggests that only higher-order terms in  $\epsilon$  will appear. We expect an asymptotic expansion in  $\epsilon$  of the form

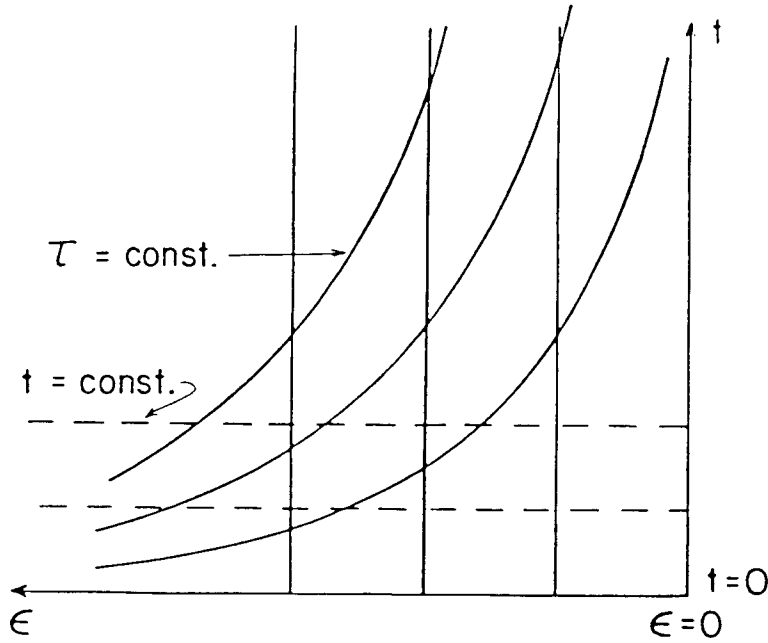
$$\rho(t, x^i, \epsilon) = \epsilon^2 f(t, x^i) + \epsilon^3 g(t, x^i) + \dots$$

We have so far not incorporated the scaling of time from Eq. (24), since we have discussed only initial data. If we define the Newtonian dynamical time

$$\tau = \epsilon t \tag{26}$$

then we expect our relativistic solutions for  $\epsilon$  near zero to be in a similar physical configuration (e.g. orbital phase) at events of constant  $x^i$  and  $\tau$ . We therefore define the post-Newtonian approximation to our regular, asymptotically Newtonian sequence to be the asymptotic expansion of the sequence of solutions in  $\epsilon$  at fixed  $x^i$  and  $\tau$ . This is illustrated in Fig. (1).

Figure 1. For each  $\epsilon$  we have drawn only the  $t$ -dimension of each solution vertically. Since the  $\epsilon=0$  manifold is Minkowski spacetime (see Eq. 25),  $t$  is a proper-time coordinate for small  $\epsilon$ . Lines of constant  $\tau$  are hyperbolae which connect points with similar physical configuration in different manifolds. As  $\epsilon \rightarrow 0$  these hyperbolae go to  $t = \infty$ , because weak-field solutions take longer times to evolve.



This approximation takes the form, e.g. for  $\rho$  (in an obvious notation)

$$\rho(\tau, x^i, \epsilon) = \epsilon^2 {}_2\rho(\tau, x^i) + \epsilon^3 {}_3\rho(\tau, x^i) + \dots \quad (27)$$

The first few terms ascend in powers of  $\epsilon$ , but beyond the order at which radiation reaction occurs one finds logarithmic terms in  $\epsilon$ .

The method used by Schutz and Futamase (1983) and Futamase (1983) may be briefly described as follows. Eq. (2) generalizes to

$$\bar{h}^{\mu\nu} = \eta^{\mu\nu} - (-g)^{\frac{1}{2}} g^{\mu\nu} \quad (28)$$

and with the same gauge condition (4) (now regarded as a full coordinate condition) the full field equations can be written in a form similar to Eq. (12),

$$\square \bar{h}^{\mu\nu} = -16\pi \Lambda^{\mu\nu} \quad (29)$$

$$\Lambda^{\mu\nu} = (-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}) + (16\pi)^{-1} (\bar{h}^{\mu\alpha} \bar{h}^{\nu\beta} - \bar{h}^{\mu\nu} \bar{h}^{\alpha\beta})_{,\alpha\beta} \quad (30)$$

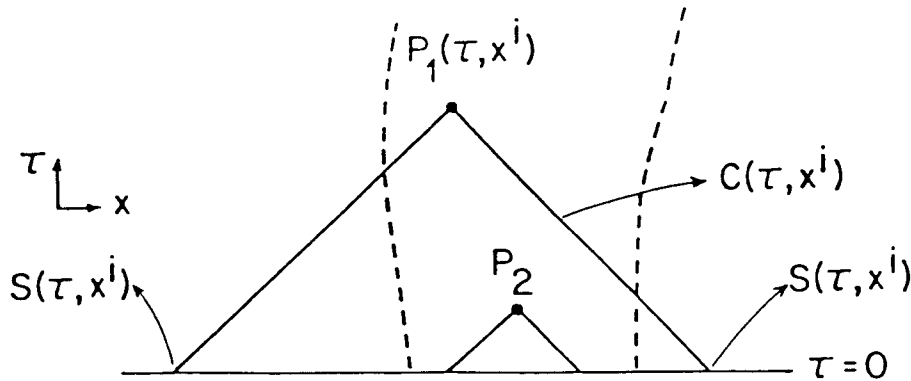


where  $t_{LL}^{\mu\nu}$  is the Landau-Lifshitz pseudotensor (Landau & Lifshitz 1962) and where  $\square$  is the flat-space wave operator in these coordinates. Equation (29) has the implicit solution given by Kirchoff's formula

$$\begin{aligned} \bar{h}^{\mu\nu}(\tau, x^j, \epsilon) = & 4 \int_{C(\tau, x^j, \epsilon)} \Lambda^{\mu\nu}(\tau - \epsilon r, y^j, \epsilon) r^{-1} d^3 y \\ & + \frac{\tau}{4\pi} \int_{S(\tau, x^j, \epsilon)} \bar{h}^{\mu\nu}_{,\tau}(\tau=0, y^j, \epsilon) d\Omega_y \\ & + \frac{1}{4\pi} \frac{\partial}{\partial \tau} \left\{ \tau \int_{S(\tau, x^j, \epsilon)} \bar{h}^{\mu\nu}(\tau=0, y^j, \epsilon) d\Omega_y \right\}, \end{aligned} \quad (31)$$

where  $r = |y^j - x^j|$ . This gives  $\bar{h}^{\mu\nu}$  at any  $\tau, x^j$  as an integral over the past coordinate-cone  $C(\tau, x^j, \epsilon)$  of the event  $(\tau, x^j)$  plus initial-data integrals from the intersection  $S(\tau, x^j, \epsilon)$  of  $C$  with the  $t=0$  hypersurface (see Fig. (2)).

Figure 2. The past coordinate-cone of  $P_1$  is  $C$  and intersects  $\tau=0$  at the sphere  $S$ . At early times ( $P_2$ ) the retarded integral does not cover the whole region where  $\rho \neq 0$  (dashed lines), but at later times ( $P_1$ ) it is essentially the usual retarded integral. As  $\epsilon \rightarrow 0$ , a point of fixed  $(\tau, x^i)$  moves upwards and the intersection  $S$  moves outwards.



When Eq. (31) is differentiated with respect to  $\epsilon$  at fixed  $\tau$  and  $x^i$  (after converting all  $t$ -indices on tensors to  $\tau$  indices), the various post-Newtonian approximations come out as successively higher derivatives. Although Eq. (31) is implicit in  $\bar{h}^{\mu\nu}$ , the right-hand side is so constructed that its  $n$ th derivative with respect to  $\epsilon$  at  $\epsilon=0$  depends only on lower-order derivatives of  $\bar{h}^{\mu\nu}$ . The successive derivatives may therefore be obtained explicitly and recursively. Using this method, Futamase (1983) showed that the post-Newtonian hierarchy of approximations (i) is genuinely asymptotic; (ii) is essentially identical to that of Chandrasekhar and colleagues up to and including

radiation-reaction order, with the modifications introduced by Anderson & Decanio (1975) and Kerlick (1980a,b), which removed some formally divergent terms; and (iii) apparently has no divergent terms in the approximation at any order in  $\epsilon$ . These results vindicate the use of the near-zone quadrupole formula, Eq. (18), in astrophysical situations.

### 5.2 The far-zone quadrupole formula

The above discussion is a near-zone discussion since we took  $x^i$  fixed as  $\epsilon \rightarrow 0$ . The wavelength of the waves emitted by the system will be proportional to the dynamical period of the system, which will scale as  $\epsilon^{-1}$ . Thus any fixed point  $x^i$  eventually becomes closer to the system than one wavelength as  $\epsilon \rightarrow 0$ . The key to a far-zone limit is to remain a fixed number of wavelengths from the system as  $\epsilon \rightarrow 0$ . We therefore introduce a far-zone spatial coordinate

$$\eta^I = \epsilon x^i \quad (32)$$

(capital-letter indices denoting the scaled coordinates), and construct the far-zone asymptotic approximation to, say,  $\bar{h}^{\alpha\beta}$  as an expansion in  $\epsilon$  at fixed  $\eta^I$  and  $\tau$ . This approach is described in Futamase & Schutz (1984).

The same formal solution, Eq. (31), may be used again but the integration over  $C$  is delicate:  $C$  must be divided into a near-zone region and a far-zone region, the boundary being a sphere of fixed  $|\eta^I|$ . The result is that to leading order we have

$$\bar{h}^{\tau\tau}(\tau, \eta^J) = 4 \epsilon^5 {}_2M/|\eta^K| + \dots, \quad {}_2M = \int {}_2\rho(\tau, y^j) d^3y \quad (33)$$

$$\bar{h}^{\tau I}(\tau, \eta^J) = 4 \epsilon^6 {}_3P^I/|\eta^K| + \dots, \quad {}_3P^I = \int {}_2\rho(\tau, y^j) {}_1v^i(\tau, y^j) d^3y \quad (34)$$

$$\bar{h}^{IJ}(\tau, \eta^J) = 2 \epsilon^7 {}_2I^{IJ}_{,\tau\tau}(u)/|\eta^K| + \dots, \quad u = \tau - |\eta^K|,$$

$${}_2I^{IJ} = \int {}_2\rho(\tau, y^j) y^i y^j d^3y. \quad (35)$$

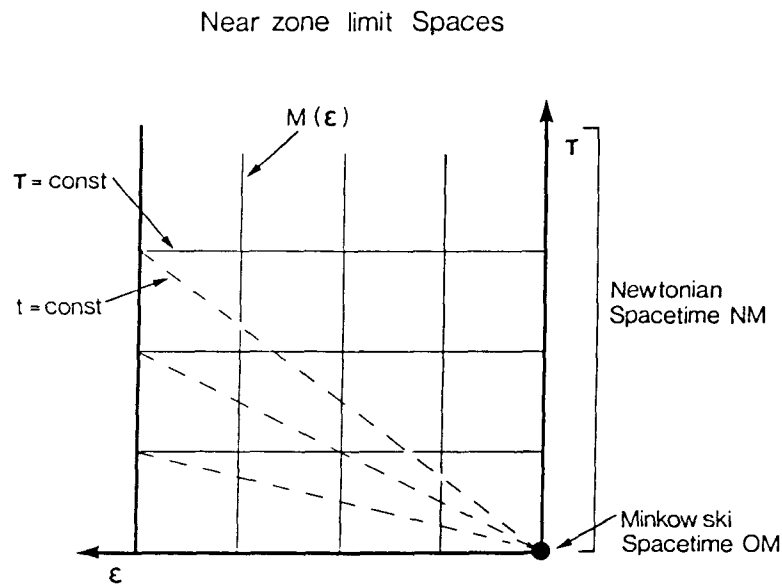
Notice that the coefficients  ${}_2M$ ,  ${}_3P^I$ , and  ${}_2I^{IJ}$  are all near-zone integrals over the Newtonian approximations to our variables. In fact, to order  $\epsilon^7$  in all components these are identical to the solution of Eq. (14) in linearized theory for  $T^{\mu\nu}$  obeying the relative Newtonian ordering of Eq. (25). The energy and angular momentum of these waves is therefore as well defined in the Newtonian limit as in linearized theory. (One can show that gauge transformations that preserve Eq. (4) and the near-zone Newtonian limit act in the far zone as linearized theory transformations of order  $\epsilon^7$ .) Moreover, our discussion of the linearized far-field quadrupole formula carries

over to this directly, verifying that the Landau-Lifshitz formula does indeed hold for self-gravitating systems.

### 5.3 The geometry of the near- and far-zone limits

It is helpful to regard our sequence of solutions  $M(\epsilon)$  of general relativity as a five-dimensional fiber bundle, with base space  $R^1$  (parametrized by  $\epsilon$ ) and fibers diffeomorphic to  $R^4$  (the spacetime manifolds). (See Schutz 1984c for a more complete description.) Then there are three interesting congruences of curves through the bundle, each parametrized by  $\epsilon$ : constant  $(t, x^i)$ , constant  $(\tau, x^i)$ , and constant  $(\tau, \eta^I)$ . These four-dimensional congruences each define different boundaries of the fiber bundle at  $\epsilon=0$ . As Fig. (1) shows, the curves of constant  $(t, x^i)$  limit to the  $\epsilon=0$  fiber, which is Minkowski spacetime. The near-zone limit is taken at fixed  $(\tau, x^i)$ , and is called NM (Fig. (3)).

Figure 3. We draw the same sequence as in Fig. (1), but now use  $\tau$  as the vertical time coordinate.

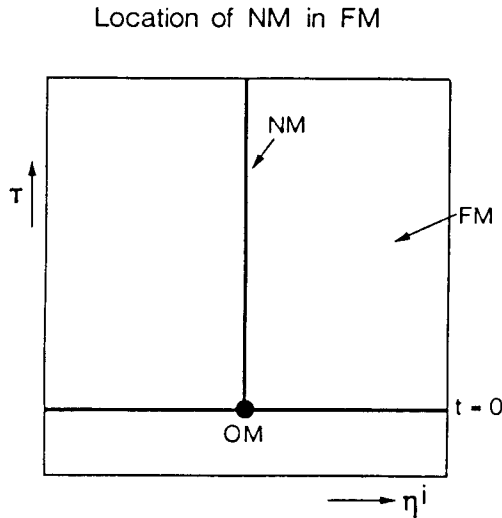


It has a degenerate metric ( $g^{\tau\tau} = \epsilon^2 g^{00} \rightarrow 0$ ) and a nonvanishing connection, which is the Cartan connection of the geometrical formulation of Newtonian gravity (Misner et al 1973). The far-zone limiting manifold FM is the  $\epsilon=0$  limit at fixed  $(\tau, \eta^I)$ . In these coordinates it has the metric

$$g^{AB} = \epsilon^2 \eta^{AB} \sim \epsilon^5 h^{AB} + \dots, \quad (36)$$

where  $g^{AB}$  may be deduced from Eqs. (33)-(35). By removing an overall factor of  $\varepsilon^2$  by a coordinate transformation we see that the manifold is flat to order  $\varepsilon^3$  and the outgoing waves are an  $\varepsilon^5$  perturbation, whose energy flux is of order  $\varepsilon^{10}$ . The relation between FM and NM is illustrated in Fig. (4).

Figure 4. As seen from FM, the near-zone manifold NM is squeezed to the origin  $\eta^I = 0$  for all  $\tau$ . The Minkowski manifold OM is squeezed to the point  $\eta^I = \tau = 0$ .



It is important to understand that FM is not the same as  $\mathcal{J}^+$ , which is a three-dimensional boundary of a single spacetime. The four-dimensional boundary FM seems naturally adapted to the radiation problem in the Newtonian limit, where we go not only to  $|x^i| = \infty$  but also to the mass  $M = 0$ . It is precisely the fact that  $M \rightarrow 0$  which makes the globally posed Newtonian limit, with boundary conditions on  $\mathcal{J}^-$  (whose relation to the manifold depends on  $M$ ), so difficult to solve. When one takes the two limits together, one obtains a regular and easily interpretable asymptotic expansion to the waves seen by an observer at a fixed number of wavelengths from the source.

#### 5.4 Secular changes in nearly-Newtonian systems

If we regard the near-zone approximation as an asymptotic approximation to the motions in a particular fiber of Fig. (1), say for  $\varepsilon = 1$ , then this can be a uniform approximation only for a finite time, since ultimately the relativistic system can change drastically, for example by binary stars spiralling together. One therefore makes different approximations to the  $\varepsilon = 1$  system, each valid for successive finite periods of  $\tau$ . This is exactly what the observers of, say, the

binary pulsar system do when they report that its period is changing (Taylor & Weisberg 1982; Boriakoff *et al* 1982): they fit Newtonian or first-post-Newtonian orbits to a several-month sequence of data, obtain a best-fit period, and then repeat the process for later periods of data. Mathematically, it seems best to idealize this in terms of an osculating Newtonian orbit (Walker & Will 1980b), which is defined at any time as that orbit which the system would have if it evolved from its configuration at time  $t$  by the Newtonian equations. This defines a continuously changing Newtonian approximation, and in particular a continuously changing period. Again, a careful formulation of the problem shows that the observable, secular change in the period is obtained by using the reaction force, Eq. (18), in (22) as a supplement in the Newtonian equations of motion (Futamase 1983; Schutz 1984c,d).

### 5.5 Astrophysical results

The quadrupole formulas can be used to gain some feeling for the likely magnitude and effects of gravitational radiation in a number of situations. Besides its explanation of secular effects in the binary pulsar system and in cataclysmic binaries (Paczynski & Sienkiewicz 1983), radiation has been shown to trigger certain instabilities in rapidly rotating stars (Chandrasekhar 1970; Friedman & Schutz 1978; Friedman 1983) and to counteract other destabilizing effects of viscosity in rapidly rotating stars (Lindblom & Detweiler 1977). By making severe simplifications of the hydrodynamics, it is possible to use the far-zone quadrupole formula to calculate the radiation that might be emitted in a supernova collapse (Saenz & Shapiro 1978, 1979, 1981 and Detweiler & Lindblom 1981). This and other interesting work is reviewed by Eardley (1983).

### References

- Anderson, J.L. & Decanio, T.C. (1975). *Gen. Rel. and Grav.*, 6, 197.  
 Balbinski, E.F.L., Detweiler, S.L., Lindblom, L. & Schutz, B.F. (1984).  
 In Preparation.  
 Balbinski, E.F.L. & Schutz, B.F. (1982). *Mon. Not. R. astr. Soc.*,  
200, 43P.  
 Bardeen, J.M. (1983). In Deruelle and Piran, p.433.  
 Blair, D.G. (1983). In Deruelle and Piran, p.339.  
 Boriakoff, V., Ferguson, D.C., Haugan, M.P., Terzian, Y., &  
 Teukolsky, S. (1982). *Astrophys. J.*, 261, L101.  
 Braginsky, V.B. (1983). In Deruelle and Piran, p.387.  
 Burke, W. (1971). *J. Math. Phys.*, 12, 401.  
 Carr, B.J. & Verdaguer, E. (1984). *Phys. Rev. D*, 28, 2995.  
 Caves, C.M., Thorne, K.S., Drever, R.W.P., Sandberg, V.D., &  
 Zimmerman, M. (1980). *Rev. Mod. Phys.*, 52, 341.  
 Caves, C.M. (1981). *Phys. Rev. D*, 23, 1693.  
 Chandrasekhar, S. (1970). *Phys. Rev. Lett.*, 24, 611.  
 Chandrasekhar, S. & Esposito, P. (1970). *Astrophys. J.*, 160, 153.

- Deruelle, N. & Piran, T. (1983). Gravitational Radiation. Amsterdam: North-Holland.
- Detweiler, S.L. (1979). *In* Sources of Gravitational Radiation, ed. L. Smarr, p.211. Cambridge: Cambridge University Press.
- Detweiler, S.L. & Lindblom, L. (1981). *Astrophys. J.*, 250, 739.
- Drever, R.W.P. (1983). *In* Deruelle & Piran, p.321.
- Eardley, D.M. (1983). *In* Deruelle & Piran, p.257.
- Ehlers, J., Rosenblum, A., Goldberg, J.N., & Havas, P. (1976). *Astrophys. J.*, 208, L77.
- Friedman, J.L. (1983). *Phys. Rev. Lett.*, 51, 11.
- Friedman, J.L. & Schutz, B.F. (1978). *Astrophys. J.*, 222, 281.
- Friedrich, H. (1983). *In* Deruelle & Piran, p.399.
- Fustero, X. & Verdaguer, E. (1984). Relativistic Astrophysics and Cosmology. World Scientific.
- Futamase, T. (1983). *Phys. Rev. D*, 28, 2373.
- Futamase, T. & Schutz, B.F. (1983). *Phys. Rev. D*, 28, 2363.
- Futamase, T. & Schutz, B.F. (1984). *In* Preparation.
- Geroch, R. (1976). *In* Asymptotic Structure of Space-Time, eds. F.P. Esposito & L.Witten. New York: Plenum, p.1.
- Hellings, R.W. (1983). *In* Deruelle & Piran, p.485.
- Horowitz, G.T. & Perry, M.J. (1982). *Phys. Rev. Lett.*, 48, 371.
- Isaacson, R.A. (1968a). *Phys. Rev.*, 166, 1263.
- Isaacson, R.A. (1968b). *Phys. Rev.*, 166, 1272.
- Kerlick, G.D. (1980a). *Gen. Rel. and Grav.*, 12, 467.
- Kerlick, G.D. (1980b). *Gen. Rel. and Grav.*, 12, 521.
- Kramer, D., Stephani, H., MacCallum, M. & Herlt, E. (1981). Exact Solutions of Einstein's Field Equations. Cambridge: Cambridge University Press.
- Landau, L.D. & Lifshitz, E.M. (1962). The Classical Theory of Fields (2nd ed.). Oxford: Pergamon.
- Lindblom, L. & Detweiler, S.L. (1977). *Astrophys. J.*, 211, 565.
- Lindblom, L. & Detweiler, S.L. (1983). *Astrophys. J. Suppl.*, 53, 73.
- Ludvigsen, M. & Vickers, J.A.G. (1982). *J. Phys. A.*, 15, L67.
- Misner, C.W., Thorne, K.S., & Wheeler, J.A. (1973). Gravitation. San Francisco: Freeman.
- Paczynski, B. & Sienkiewicz, R. (1983). *Astrophys. J.*, 268, 825.
- Piran, T. (1983). *In* Deruelle & Piran, p.203.
- Saenz, R.A. & Shapiro, S.L. (1978). *Astrophys. J.*, 221, 286.
- Saenz, R.A. & Shapiro, S.L. (1979). *Astrophys. J.*, 229, 1107.
- Saenz, R.A. & Shapiro, S.L. (1981). *Astrophys. J.*, 244, 1033.
- Schutz, B.F. (1980). *Phys. Rev. D*, 22, 249.
- Schutz, B.F. (1984a). A First Course in General Relativity. Cambridge: Cambridge University Press.
- Schutz, B.F. (1984b). *Am. J. Phys.* (in press).
- Schutz, B.F. (1984c). *In* Fustero & Verdaguer, p.35.
- Schutz, B.F. (1984d). *Mon. Not. R. astr. Soc.* (in press).
- Schutz, B.F. & Sorkin, R. (1977). *Ann. Phys. (NY)*, 107, 1.
- Smarr, L. (1979). *In* Sources of Gravitational Radiation, ed. L. Smarr. Cambridge: Cambridge University Press.
- Taylor, J.H. & Weisberg, J.M. (1982). *Astrophys. J.*, 253, 908.
- Thorne, K.S. (1983). *In* Deruelle & Piran, p.1.

- Walker, M. (1983). In Deruelle & Piran, p.145.  
Walker, M. (1984). In Fustero & Verdaguer.  
Walker, M. & Will, C.M. (1980a). Phys. Rev. Lett., 45, 1741.  
Walker, M. & Will, C.M. (1980b). Astrophys. J., 242, L129.  
Weber, J. (1960). Phys. Rev., 117, 307.  
Winicour, J. (1983). J. Math. Phys., 24, 1193.