Gravitational radiation and the validity of the far-zone quadrupole formula in the Newtonian limit of general relativity

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We examine the gravitational radiation emitted by a sequence of spacetimes whose near-zone Newtonian limit we have previously studied. The spacetimes are defined by initial data which scale in a Newtonian fashion: the density as ϵ^2, velocity as ϵ, pressure as ϵ^4, where ϵ is the sequence parameter. We asymptotically approximate the metric at an event which, as $\epsilon \to 0$, remains a fixed number of gravitational wavelengths distant from the system and a fixed number of wave periods to the future of the initial hypersurface. We show that the radiation behaves like that of linearized theory in a Minkowski spacetime, since the mass of the metric vanishes as $\epsilon \to 0$. We call this Minkowski far-zone limiting manifold FM; it is a boundary of the sequence of spacetimes, in which the radiation carries an energy flux given asymptotically by the usual far-zone quadrupole formula (the Landau-Lifshitz formula), as measured both by the Isaacson average stress-energy tensor in FM or by the Bondi flux on $\mathscr{I}^+_{\mathrm{FM}}$. This proves that the quadrupole formula is an asymptotic approximation to general relativity. We study the relation between $\mathscr{I}^+_\epsilon$, the sequence of null infinities of the individual manifolds, and $\mathscr{I}^+_{\mathrm{FM}}$; and we examine the gauge invariance of FM under certain gauge transformations. We also discuss the relation of this calculation with similar ones in the framework of matched asymptotic expansions and others based on the characteristic initial-value problem.

I. INTRODUCTION

The most striking difference between the Newtonian and Einsteinian theories of gravity is that Newton's theory does not admit gravitational waves and Einstein's does.[1] Nevertheless, Newton's theory is a limiting case of Einstein's, so there are solutions of Einstein's equations which are quasi-Newtonian: they obey Newton's equations to some high accuracy. As relativistic solutions, they generally emit gravitational waves, for which there is no description in Newtonian terms, even approximately. The study of this radiation is therefore one of the most delicate aspects of treating the Newtonian limit of general relativity. In two previous papers,[2,3] we have demonstrated that the Newtonian and post-Newtonian hierarchy[4,5] are in fact asymptotic approximations to a well-defined sequence of solutions of Einstein's field equations. This proof included the order at which radiation-reaction effects were first seen in the system's equations of motion, thus demonstrating that the near-zone "quadrupole formula"[5,6] is an asymptotic approximation to general relativity. We turn in this paper to the study of the radiation emitted by the same sequence of relativistic systems as they approach their Newtonian limit. Instead of working with null infinity ($\mathscr{I}^\pm$) of any spacetime in the sequence, we define a limiting four-dimensional "far-zone manifold" (FM) of the sequence, in which the outgoing radiation is asymptotic to a solution of linearized theory.[7,8] It is then easy to deduce from this that the far-field quadrupole formula for the outgoing wave energy flux is also an asymptotic approximation to general relativity.

The far zone and its relation to the source has been studied in a large number of other investigations, most successfully by variants of the method of matched asymptotic expansions.[6,9] All of these have found results compatible with ours, and in particular, supporting the quadrupole formula. What is new about the work we describe here is that we (i) demonstrate that the far-zone formulas are truly asymptotic approximations to a well-defined sequence of radiating relativistic solutions; (ii) derive the far-zone metric by the use of uniformly approximated retarded integrals, rather than by matching; and (iii) give a clear geometrical picture of the relation of the Newtonian far zone to the sequence of manifolds, showing in particular that it is not $\mathscr{I}^+$ for any manifold. A novel method of studying radiation in the Newtonian manifold has been carried out by Winicour and collaborators,[10] based on the characteristic initial-value problem. It has much in common with our work, and we shall study the relationship between the two later in this paper.

The sequence of relativistic solutions we study is what we have called[2] a regular, asymptotically Newtonian sequence. It is defined by initial data having the Newtonian scaling property: $v^i \sim \epsilon$, $\rho \sim \epsilon^2$, $p \sim \epsilon^4$, where ϵ is the sequence's parameter. Because the velocities approach zero, the system's characteristic time scales get longer as ϵ^{-1}, so a map from one solution to another at fixed x^i and

$$\tau = \epsilon t \tag{1.1}$$

will join events at similar stages of dynamical evolution, at least in the limit.[11] For this reason we call τ the *dynamical time.* The post-Newtonian approximations are asymptotic to the regular, asymptotically Newtonian sequence as $\epsilon \to 0$ for fixed x^i and τ.

We shall call the map given by Eq. (1.1) the *near-zone map,* for reasons that will be clear in the next paragraph. Figures 1 and 2 show two different ways of looking at it. If we regard the one-parameter sequence of manifolds as a (trivial) fiber bundle over the base space R^1 parameterized by ϵ, then Fig. 2 shows two different $\epsilon=0$ boundaries of this bundle: Minkowski spacetime OM (the fiber $\epsilon=0$, reached by the map x^i=const, t=const); and Cartan spacetime NM (the near-zone limit at x^i=const, τ=const, which has[2] the degenerate metric and regular connection of Cartan's geometrical description of Newtonian gravity[12]). The manifold NM is the near-zone manifold, in which the Newtonian dynamics takes place.

Any gravitational radiation emitted by the system will have a characteristic period given by some dynamical time interval $\Delta\tau$, which therefore scales as ϵ^{-1} in t as $\epsilon \to 0$. The wavelength of the radiation will therefore also go as ϵ^{-1} in the x^i coordinates. This means that any point at fixed x^i will find itself less than one wavelength distant from the system for sufficiently small ϵ. That is why we have called the map at fixed x^i the near-zone map.

In order to study gravitational radiation, we have to stay in the far, or wave, zone. We therefore define another scaled coordinate,

$$\eta^{i'} = \epsilon x^i . \tag{1.2}$$

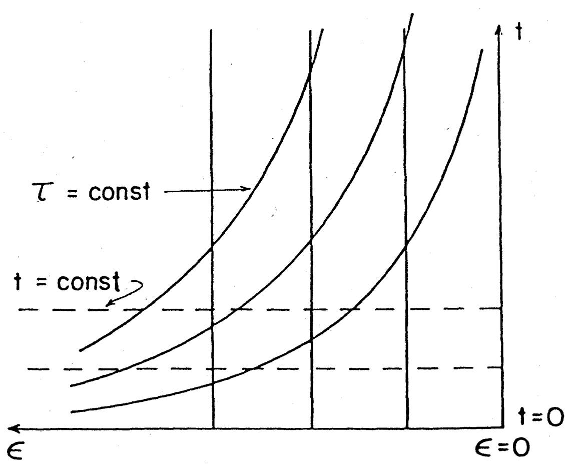

FIG. 1. The sequence of four-dimensional spacetime manifolds forms a five-dimensional manifold, two dimensions of which are illustrated here: the parameter ϵ and the time coordinate t, which as $\epsilon \to 0$ becomes proper time. The near-zone map from one spacetime to another fixes x^i and $\tau=\epsilon t$ and therefore follows the hyperbolas. It never reaches the $\epsilon=0$ spacetime, which is Minkowski spacetime. For $t<0$ the picture would be reflected through $t=0$; but for reasons discussed in Ref. 8 we do not consider $t<0$.

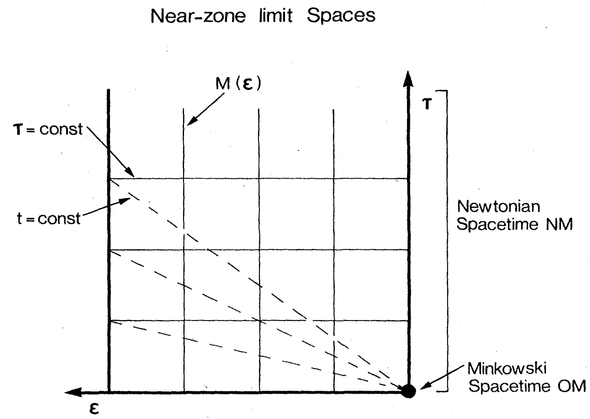

FIG. 2. The same sequence as in Fig. 1 is now displayed using τ rather than t as a vertical coordinate. The near-zone map is horizontal, and its limit for $\epsilon=0$ is the Cartan or near-zone manifold NM. The t=const map converges on $\tau=0$ for $\epsilon=0$: the whole $\epsilon=0$ spacetime OM of Fig. 1 is a single point here. This point is in no sense a limit point of NM, since both are four-dimensional manifolds. (In a similar way, $\mathscr{I}^0$ is not a limit point of $\mathscr{I}^+$, even though it may look like one, in a conformal diagram.) (This figure is reproduced with permission from Ref. 21.)

The map between solutions at fixed $\eta^{i'}$ and τ is called the *far-zone map,* because it ensures that if the event $(\eta^{i'},\tau)$ is, say, about 10 wavelengths from the source in one solution then it will remain 10 wavelengths away as $\epsilon \to 0$. We shall study the solutions at fixed $(\eta^{i'},\tau)$ in order to develop an asymptotic approximation to the field an observer would see who is located a certain number of wavelengths from the relativistic source. (Primes on indices will be used to denote the coordinates $\eta^{i'}$ and τ.) This linkage of the limit $r \to \infty$ with the limit $\epsilon \to 0$, which has been used in the matched-asymptotic-expansion work as well,[9] is at the heart of our method.

The far-zone map $(x^i,t) \to (x^i/\epsilon, t/\epsilon)$ defines a four-dimensional congruence through the fiber bundle of solutions, and because the map is a simple scaling by ϵ, it is possible to associate with its $\epsilon \to 0$ limit a flat Minkowski manifold FM (the far-zone manifold) in which the far-field waves are solutions of linearized theory to the order required for studying their wave fluxes. The manifold FM appears to be essentially the same as the four-dimensional manifold that Wincour and co-workers[10] have introduced in their study of the Newtonian limit of the characteristic initial-value problem. Figures 3 and 4 illustrate the relation between FM and NM.

The plan of the paper is as follows. Section II calculates the metric in the far zone, showing in particular, that its dominant source is the integral over the near-zone distribution of stress energy. The result is a metric essentially identical to that of radiation in linearized theory. In Sec. III we define the far-zone manifold FM as the Minkowskian manifold in which this linear radiation appears; we show the relation between it and the asymptotic infinities $\mathscr{I}^+_\epsilon$ of the individual manifolds of the sequence; we calculate the outgoing energy flux by both the Isaacson

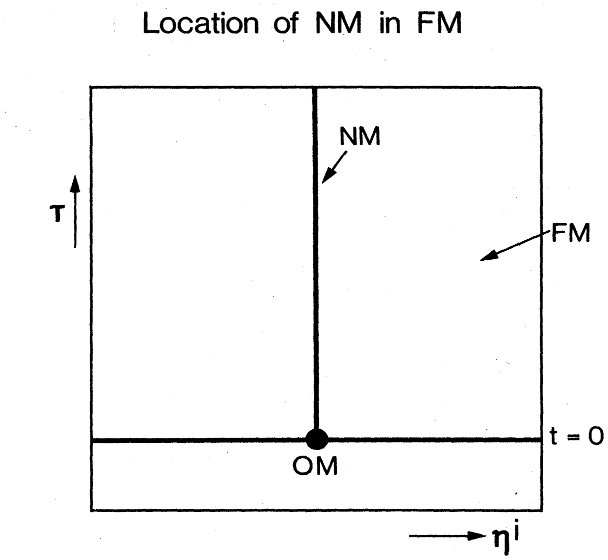

FIG. 3. The far-zone map at fixed $\eta^{i'}=\epsilon x^i$ and $\tau=\epsilon t$ limits to FM. The near zone shrinks in this picture as $\epsilon\rightarrow 0$, so that NM is a line embedded at the spatial origin of FM. (This figure is reproduced with permission from Ref. 21.)

and the Bondi methods in FM and establish the validity of the far-zone quadrupole formula there; and we show that FM is invariant under gauge transformations within the Lorentz gauge that leave the near-zone Newtonian limit unchanged. (The general problem of formulating a coordinate-independent definition of FM is not addressed here.) Section IV provides a discussion of these results.

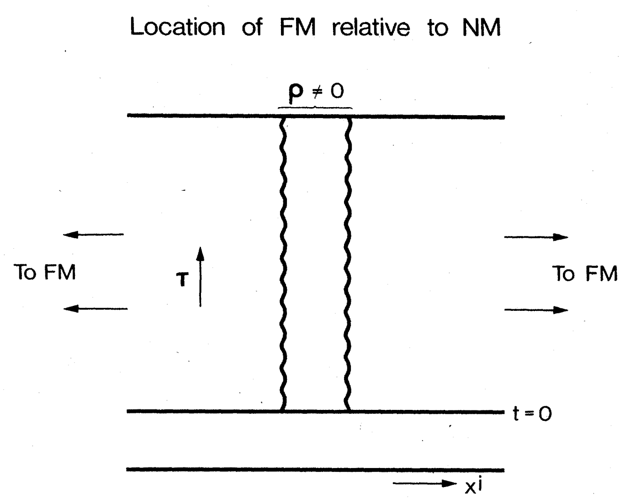

FIG. 4. The picture in Fig. 3 is here shown from the point of view of the near-zone map, in which the limit to FM takes points to spatial infinity. (This figure is reproduced with permission from Ref. 21.)

II. CALCULATION OF THE FAR-ZONE METRIC

A. Formulation of the problem

The notation and formulation of the equations are the same as in paper I. We adopt the harmonic gauge,

$$\bar{h}^{\mu\nu}{}_{,\nu}=0\ , \tag{2.1}$$

in which the field equations take the form

$$\Box\bar{h}^{\mu\nu}=-16\pi\Lambda^{\mu\nu}\ , \tag{2.2}$$

where we have

$$\Lambda^{\mu\nu}=(-g)(T^{\mu\nu}+t^{\mu\nu}_{LL})+\chi^{\mu\alpha\nu\beta}{}_{,\alpha\beta}\ . \tag{2.3}$$

(The reader should refer to paper I for unfamiliar symbols.) The formal solution of Eq. (2.2) for initial data on the $t=0$ hypersurface is

$$\bar{h}^{\mu\nu}(t,x^k;\epsilon)=4\int_{C(t,x^k;\epsilon)}|x^j-y^j|^{-1}\times\Lambda^{\mu\nu}(\tau-\epsilon|x^j-y^j|,y^i;\epsilon)d^3y+\bar{h}^{\mu\nu}_H\ , \tag{2.4}$$

where $C(t,x^k;\epsilon)$ is the past coordinate light cone of the event (t,x^k) *truncated* at $t=0$ in the manifold given by ϵ (see Fig. 5), and $\bar{h}^{\mu\nu}_H$ is the solution of the homogeneous equation

$$\Box\bar{h}^{\mu\nu}_H=0 \tag{2.5}$$

which evolves from the given initial data. Equation (2.4) gives, of course, only an implicit solution for $\bar{h}^{\mu\nu}$, but as in the near-zone calculation it can be solved iteratively.

Figure 5 shows that if the field point (t,x^i) is in the far zone, the integral over C will include contributions from both the far and near zones. Since the approximations for the integrands are different in these regions, we split the integral into two parts explicitly. We take the origin of our coordinates to be inside the near zone, and we define the boundary of the near and far zones to be at a fixed ra-

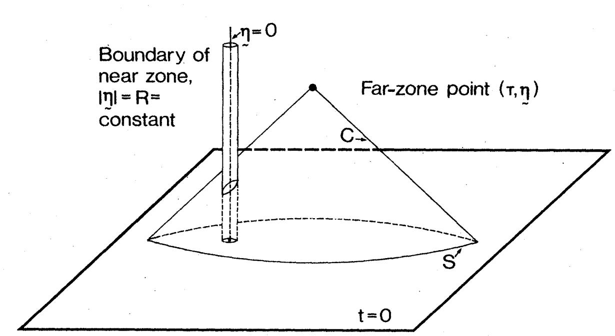

FIG. 5. Data set at $t=0$ determine the field at a point $(\eta^{i'},\tau)$ by the solution of the homogeneous equation plus an integral over the past light-cone of $(\eta^{i'},\tau)$ truncated at $t=0$. This light cone goes through the material system, so we use separate asymptotic approximations to its integrand in the region inside and outside the tube of radius $\eta=R$ shown here.

dius $|\eta^{i'}|=R$ in far-zone coordinates, i.e., at a radius $|x^i|=R/\epsilon$ in near-zone coordinates. The value of R is arbitrary: our results will not depend upon it.[13] After transforming Eq. (2.2) to far-zone coordinates (primed indices), we define the two inhomogeneous pieces of $\bar{h}^{\mu\nu}$:

$$\bar{h}_F^{\mu'\nu'}(\tau,\eta^{i'};\epsilon)=4\epsilon^{-2}\int_{C-C\cap \mathrm{NZ}}|\eta^{j'}-\xi^{j'}|^{-1} \times\Lambda_F^{\mu'\nu'}(\tau-|\eta^{j'}-\xi^{i'}|,\xi^{i'};\epsilon)d^3\xi\ , \tag{2.6}$$

and

$$\bar{h}_N^{\mu'\nu'}(\tau,\eta i';\epsilon)=4\epsilon^{3}\int_{C\cap \mathrm{NZ}}|\eta^{j'}-\epsilon y^{j}|^{-1} \times\Lambda_N^{\mu\nu}(\tau-|\eta^{j'}-\epsilon y^{j}|,y^{i};\epsilon)d^3y\ . \tag{2.7}$$

Then we have for the far-zone metric

$$\bar{h}^{\mu'\nu'}=\bar{h}_N^{\mu'\nu'}+\bar{h}_F^{\mu'\nu'}+\bar{h}_H^{\mu'\nu'}\ . \tag{2.8}$$

Here we denote by NZ the region $|\eta^{i'}|<R$, i.e., the near zone, and we define the far- and near-zone source functionals by, respectively,

$$\Lambda_F^{\mu'\nu'}(\tau,\xi^{i'};\epsilon)=\epsilon^2\Lambda^{\mu\nu}(\tau/\epsilon,\xi^{i'}/\epsilon;\epsilon) \tag{2.9}$$

and

$$\Lambda_N^{\mu\nu}(\tau,y^i;\epsilon)=\Lambda^{\mu\nu}(\tau/\epsilon,y^i;\epsilon)\ . \tag{2.10}$$

Explicit factors of ϵ in Eqs. (2.6)—(2.9) come from changing from unprimed to primed indicates and from the factor of $|x^j-y^j|$ in Eq. (2.4).

We shall next show that $\bar{h}_F^{\mu'\nu'}$ and $\bar{h}_H^{\mu'\nu'}$ are negligible compared to $\bar{h}_N^{\mu'\nu'}$ in Eq. (2.8), provided we are only interested in the lowest-order radiation terms.

B. Neglecting $\bar{h}_H^{\mu'\nu'}$ and $\bar{h}_F^{\mu'\nu'}$

First consider $\bar{h}_F^{\mu'\nu'}$. In the unscaled coordinates (t,x^i), the limit to $\epsilon=0$ of the components $g^{\mu\nu}$ of the metric is $\eta^{\mu\nu}$ along *any* curve through the fiber bundle, since the $\epsilon=0$ fiber is Minkowski spacetime. (The metric of NM is degenerate because it is the limit of the components of $g^{\mu\nu}$ in the partly scaled coordinates τ and x^i.) Therefore in far-zone coordinates $(\tau,\eta^{i'})$ the components are asymptotic to $\epsilon^2\eta^{\mu\nu}$. Since $\bar{h}^{\mu\nu}$ is the perturbation in the metric, and since we have assumed (paper I) smoothness as $\epsilon\to 0$, it follows that $\bar{h}^{\mu'\nu'}$ must be smaller than this:

$$\bar{h}^{\mu'\nu'}=o(\epsilon^2)\ . \tag{2.11}$$

Inspection of $t_{LL}^{\mu'\nu'}$ in the far zone shows that its components are of order $|\bar{h}^{\mu'\nu'}|^2$, so that by Eq. (2.6) we have $\bar{h}_F^{\mu'\nu'}\sim\epsilon^{-2}|\bar{h}^{\mu'\nu'}|^2$, or

$$\bar{h}_F^{\mu'\nu'}=o(\bar{h}^{\mu'\nu'})\ , \tag{2.12}$$

i.e., that $\bar{h}_F^{\mu'\nu'}$ is smaller than $\bar{h}^{\mu'\nu'}$. We will see that the dominant component of $\bar{h}^{\mu'\nu'}$ is $\bar{h}_N^{\tau\tau}$ and is of order ϵ^5, so that (by the argument above) $\bar{h}_F^{\mu'\nu'}$ does not contribute until order ϵ^8, which is beyond the order we shall need for calculating fluxes in the far zone. It is this fact which makes the dominant far-zone radiation identical to that of

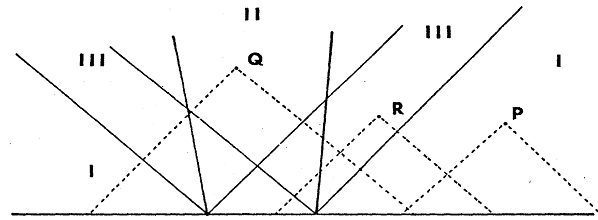

FIG. 6. When solving the homogeneous equation for initial data which are generated by the constraint equations using matter data of compact support, the future of $t=0$ is divided into three regions as shown. The far-zone limit studies radiation in region II. A full discussion of the solution in these regions is in Ref. 8.

linearized theory. We shall neglect $\bar{h}_F^{\mu'\nu'}$ from now on.

For the homogeneous field $\bar{h}_H^{\mu'\nu'}$ one must distinguish the spatial components $\bar{h}_H^{i'j'}$ from the temporal ones $\bar{h}_H^{\tau\mu'}$. Initial data for $\bar{h}_H^{i'j'}$ are averaged in our picture (paper II) in such a way that their mean value is zero, so the mean $\bar{h}_H^{i'j'}$ is zero as well. The initial data for $\bar{h}_H^{\tau\mu'}$, on the other hand, came from solving the initial value constraints. The dominant initial data are solutions of Poisson's equation with sources $(T^{\tau\mu'})$ of compact support, in which case $\bar{h}_H^{\tau\mu'}$ has different behavior in three different regions of spacetime[8] (Fig. 6). We are interested in the outgoing radiation in region II, where this piece of $\bar{h}_H^{\tau\mu'}$ is zero. The largest contribution to $\bar{h}_H^{\tau\tau}$ from initial data of noncompact support is of order ϵ^8 at fixed x^i and τ (paper II), and since this falls off as $|x^i|^{-1}$ it contributes a term of order ϵ^9 to the far-zone limit of $\bar{h}_H^{\tau\tau}$, and is therefore negligible here. Similar considerations also eliminate $\bar{h}_H^{\tau i'}$. We shall therefore consider only $\bar{h}_N^{\mu'\nu'}$ from now on.

C. Calculating $\bar{h}_N^{\mu'\nu'}$

the asymptotic approximation to Eq. (2.7) is developed as in our previous papers by differentiating with respect to ϵ at fixed τ, $\eta^{i'}$, and y^j. Consider first $\bar{h}_N^{\tau\tau}$. Since Λ_N^{tt} is of order ϵ^2, Eq. (2.7) shows that $\bar{h}_N^{\tau\tau}$ is of order ϵ^5. Moreover, since Λ_N^{tt} is of compact support at lowest order [just $\epsilon^2\,{}_2\rho(\tau,y^i)$], the upper limit on $|y^j|$ is finite and fixed in Eq. (2.7), so retardation of the integrand across the near zone (which we may call differential retardation) may be neglected. If we define the far-zone retarded time

$$u=\tau-\eta,\quad \eta\equiv|\eta^{i'}|\ , \tag{2.13}$$

then we obtain

$${}_5\bar{h}_N^{\tau\tau}=\frac{1}{5!}\left[\frac{\partial^5}{\partial\epsilon^5}\bar{h}_N^{\tau\tau}\right]_{\epsilon=0}=4\,{}_2M/\eta\ , \tag{2.14}$$

where

$${}_2M=\int {}_2\rho(u,y^j)d^3y \tag{2.15}$$

is independent of u by the near-zone continuity equation. So the dominant piece of $\bar{h}^{\tau\tau}$ far away is just the Newtonian potential. Its order, ϵ^5, is composed of ϵ^2 from the Newtonian mass, ϵ^2 from the conversion of indices from t to τ, and ϵ from the replacement of radial distance $|x^i|$ by η/ϵ.

The next derivative of $\bar{h}^{\tau\tau}_N$ still involves only ${}_2\Lambda^{tt}_N$, since the next higher order in Λ^{tt}_N is ϵ^4. We find contributions from differential retardation [differentiating the ϵ in the τ argument of $\Lambda^{\mu\nu}_N$ in Eq. (2.7)] and from the Newtonian dipole moment:

$$ {}_6\bar{h}^{\tau\tau}_N = 4\,{}_3P_i n^i/\eta + 4\,{}_2D_i n^i/\eta^2 \ , \tag{2.16} $$

with the definitions

$$ n^i = \eta^{i'}/\eta = x^i/|x^j| \ , \tag{2.17} $$

$$ {}_3P_i = \int {}_2\rho(u,y^j)\,{}_1v_i(u,y^j)d^3y \ , \tag{2.18} $$

$$ {}_2D_i(u) = \int {}_2\rho(u,y^j)y_i d^3y \ . \tag{2.19} $$

Again, the near-zone dynamical equations ensure that ${}_3P_i$ is constant and ${}_2D_i$ is at most a linear function of u. These terms are exactly the same as they would be in linearized theory, since they come from ${}_2\rho(\tau,y^i)$ alone.

We shall calculate $\bar{h}^{\tau\tau}_N$ to one more order. Contributions from moments and differential retardation of ${}_2\rho$ are again the same as in linearized theory. But now we also have a contribution from ${}_4\Lambda^{tt}_N$, which is not compact (it includes the Newtonian field energy). By Lemma 2 (ii) of paper I we can neglect the differential retardation of ${}_4\Lambda^{tt}_N$ [which is the same as ${}_6\Lambda^{\tau\tau}$ in Eq. (4.27) of paper I]. Moreover, the upper limit of integration in Eq. (2.7) above is $|y^i| = R/\epsilon$; its ϵ dependence must in principle be taken into account, but at this order R/ϵ is set to ∞. The result is

$$ {}_7\bar{h}^{\tau\tau}_N = 4\,{}_4M/\eta + 2\,{}_2I_{ij,uu}n^in^j/\eta + 6\,{}_2{I\!\!\!-}_{ij,u}n^in^j/\eta^2 + 6\,{}_2{I\!\!\!-}_{ij}n^in^j/\eta^3 \tag{2.20} $$

with

$$ {}_4M = \int {}_4\Lambda^{tt}_N(u,y^i)d^3y \ , \tag{2.21} $$

$$ {}_2{I\!\!\!-}_{ij}(u) = \int {}_2\rho(u,y^k)y_iy_j d^3y \ , \tag{2.22} $$

$$ {}_2{I\!\!\!-}_{ij} = {}_2I_{ij} - \tfrac{1}{3}\delta_{ij}\,{}_2I^k{}_k \ . \tag{2.23} $$

Clearly ${}_4M$ is the post-Newtonian contribution to the mass at infinity, and it is conserved. The other terms are the usual Newtonian quadrupole terms.

The same calculations may be made for the other components of $\bar{h}^{\alpha'\beta'}_N$. The results up to order ϵ^7 are

$$ {}_6\bar{h}^{\tau i'} = 4\,{}_3P^i/\eta \ , \tag{2.24} $$

$$ {}_7\bar{h}^{\tau i'} = 2\,{}_2I^i{}_{j,uu}n^j/\eta + ({}_3M^{ij} + {}_2I^i{}_{j,u})n^j/\eta^2 \ , \tag{2.25} $$

$$ {}_7\bar{h}^{i'j'} = 2\,{}_2I^{ij}{}_{,uu}/\eta \ , \tag{2.26} $$

where

$$ {}_3M^{ij} = \int {}_2\rho({}_1v^iy^j - {}_1v^jy^i)d^3y \tag{2.27} $$

is the (conserved) Newtonian angular momentum. Again these are all known from linearized theory.

If we were to calculate the field $\bar{h}^{\alpha'\beta'}$ of linearized theory in the same fashion as we have approached the nonlinear theory, that is, by taking initial data of the same form and expanding at fixed far-zone coordinates τ and $\eta^{i'}$, we would have found exactly the same as our expressions (2.14)–(2.27), with the single exception that ${}_4M$ would have been zero. Its presence here shows us that the post-Newtonian mass is of the same order as the radiation terms in the far zone. But this mass term plays no role in the dynamical part of the metric (the radiation), so we see that the radiation is identical to that of linearized theory. Note also that terms in, say, M^2/r^2 that one finds in the far-zone field of a manifold of fixed mass do not appear here because they are simply of higher order: a factor of ϵ^3 higher order than the ${}_2M/\eta$ term in Eq. (2.14), for example. (Two factors of ϵ come from the extra factor of M and one from converting the extra r^{-1} to η^{-1}.) This is a result of linking the $\epsilon \to 0$ and $r \to \infty$ limits.

The simplicity of our expressions would disappear if we studied the radiation approaching $\mathscr{I}^+$ in a single one of our sequence of manifolds, where we would have to use the correct curved-space null cones rather than our flat-space retarded coordinate u, and where we would not be able to express the asymptotic radiation field in terms of simple integrals over the source. By linking the limit $r \to \infty$ with the limit $\epsilon \to 0$, we have arrived at a picture of radiation on the Newtonian limit which is simpler than the one we would find in any single manifold.

III. THE FAR-FIELD QUADRUPOLE FORMULA

A. The far-zone manifold FM

We have identified a preferred four-dimensional congruence through the five-dimensional sequence of manifolds, defined by fixing $\eta^{i'} = \epsilon x^i$ and $\tau = \epsilon t$ as ϵ goes to zero. The Lie-dragged components $g^{\mu'\nu'}$ of the metric along this congruence asymptotically approach $\epsilon^2\eta^{\mu'\nu'}$, degenerating because of the expansion of the coordinates. If we apply a trivial conformal transformation to this metric, defining

$$ \underline{g}^{\mu'\nu'} = \epsilon^{-2}g^{\mu'\nu'} \tag{3.1} $$

for all ϵ, then $\underline{g}$ will asymptotically approach the Minkowski metric. We define the congruence with the limit of this conformally rescaled manifold to be the far-zone manifold FM. In it we may regard the full metric $\underline{g}^{\mu'\nu'}(\epsilon)$ as a sequence of metrics having the asymptotic expansion obtained from Eqs. (2.14)–(2.27) by conformal rescaling. If in addition we choose a Lorentz frame in which ${}_2P^i = {}_2D^i = 0$, then we have

$$ \underline{\bar{h}}^{\tau\tau} = 4\epsilon^3\,{}_2M\eta^{-1} + \epsilon^5(4\,{}_4M\eta^{-1} + 2\,{}_2I^{ij}{}_{,uu}n_in_j\eta^{-1} + 6\,{}_2{I\!\!\!-}^{ij}{}_{,u}n_in_j\eta^{-2} + 6\,{}_2{I\!\!\!-}^{ij}n_in_j\eta^{-3}) + O(\epsilon^6) \ , \tag{3.2} $$

$$ \underline{\bar{h}}^{\tau i'} = \epsilon^5[2\,{}_2I^{ij}{}_{,uu}n_j\eta^{-1} + ({}_3M^{ij} + {}_2I^{ij}{}_{,u})n_j\eta^{-2}] + O(\epsilon^6) \ , \tag{3.3} $$

$$ \underline{\bar{h}}^{i'j'} = 2\epsilon^5\,{}_2I^{ij}{}_{,uu}\eta^{-1} + O(\epsilon^6) \ . \tag{3.4} $$

It is useful to think of FM as a conformally rescaled boundary of the five-dimensional sequence of spacetimes, much as $\mathscr{I}^+$ is a boundary of a single spacetime. We can get some idea of the relation of $\mathscr{I}^+_\epsilon$ of each individual manifold M_ϵ of our sequence to FM by considering a simple special case: a sequence of stationary metrics whose masses scale as ours do, namely, as $\epsilon^2 m$ for a positive constant m. (This could include a sequence of exact Schwarzschild solutions.) Since our limit takes us far from the source region as $\epsilon \rightarrow 0$, we can take the weak-field form of the metric on M_ϵ for sufficiently small ϵ, which is

$$ds^2 = -(1-2\epsilon^2 m/r)dt^2 + (1+2\epsilon^2 m/r)(dx^2+dy^2+dz^2) \ .$$

(We do not assume M_ϵ is weak-field everywhere: only that this form obtains for large r.) This is in the harmonic gauge. It may be compactified at null infinity by defining

$$r_* = r + 2\epsilon^2 m \ln(r - 2\epsilon^2 m) \qquad (3.5)$$

and introducing the coordinates

$$u_\epsilon = \tan^{-1}(t - r_*), \quad v_\epsilon = \tan^{-1}(t + r_*) \ . \qquad (3.6)$$

Then $\mathscr{I}_\epsilon^+$ is the limit $v_\epsilon \rightarrow \pi/2$ for u_ϵ constant. We can similarly compactify FM by introducing

$$U = \tan^{-1}(\tau - \eta), \quad V = \tan^{-1}(\tau + \eta) \ .$$

Then we have the relations

$$\begin{aligned} U &= \tan^{-1}[\epsilon \tan u_\epsilon + 2\epsilon^3 m \ln(r - 2\epsilon^2 m)] \ , \\ V &= \tan^{-1}[\epsilon \tan v_\epsilon - 2\epsilon^3 m \ln(r - 2\epsilon^2 m)] \ , \end{aligned} \qquad (3.7)$$

where we regard r as an implicit function of u_ϵ and v_ϵ through Eqs. (3.5) and (3.6). Then if we fix u_ϵ and set $v_\epsilon = \pi/2$, we see that as $\epsilon \rightarrow 0$ this point of $\mathscr{I}_\epsilon^+$ approaches the point $V = \pi/2$, $U = 0$ on $\mathscr{I}_{\rm FM}^+$. Thus, all of $\mathscr{I}_\epsilon^+$ is mapped to a single point of $\mathscr{I}_{\rm FM}^+$ (see Fig. 7). This should not be surprising: a finite interval of time in FM corresponds to an infinite amount of time in M_ϵ as $\epsilon \rightarrow 0$, because FM uses τ as a time coordinate.

This mismatch between proper time t in the physical

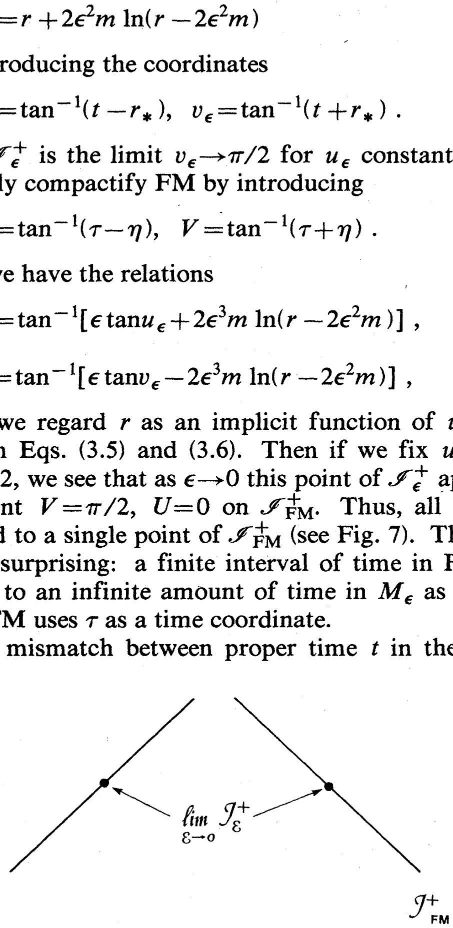

FIG. 7. Future null infinity of FM, $\mathscr{I}_{\rm FM}^+$, is shown here. All points of $\mathscr{I}_\epsilon^+$ of any individual spacetime will limit to a single cut of $\mathscr{I}_{\rm FM}^+$ as shown. This cut is the intersection of $\mathscr{I}_{\rm FM}^+$ with the image under the far-zone map of the light cones of the origin ($x^i=0$, $t=0$) of all the $\epsilon \neq 0$ manifolds. All the light cones of M_ϵ for $\epsilon \neq 0$ map to this cone in FM because finite intervals of t become zero intervals of τ in the limit. (This is the same reason that OM is a "point" in Fig. 2.) This singular embedding can be eliminated if we apply a conformal transformation to each M_ϵ before attaching $\mathscr{I}_\epsilon^+$ to it. This spreads out $\mathscr{I}_\epsilon^+$ and makes the limit to $\mathscr{I}_{\rm FM}^+$ regular.

manifolds M_ϵ and the dynamical time τ which governs NM and FM suggests that it might be more convenient to apply the conformal transformation ϵ^2 to the metric of each manifold M_ϵ and then to regard the conformally transformed metrics as the physical ones of our sequence. This is certainly possible, since under a *constant* conformal transformation the new metric is also a solution of Einstein's equations. In this view, the coordinates τ and η^i become Lorentzian as $\epsilon \rightarrow 0$, the bodies of the system have physical dimensions which shrink as ϵ, and their velocities also shrink so that their characteristic time scales remain constant. By the virial theorem we have the mass $M \sim v^2 R$ shrinking as ϵ^3. In this limit the near zone has fixed size (but gets arbitrarily large relative to the size of the system), and the far zone is simply the limit at fixed *physical* coordinates τ and η^i. Then there is a *regular* map from $\mathscr{I}_\epsilon^+$ to $\mathscr{I}_{\rm FM}^+$. This appears to be the sort of sequence constructed by Winicour and co-workers[10] in their characteristic-initial-value formulation of the Newtonian limit. It is well adapted for discussing radiation but less convenient for the discussion of the motion of the bodies.

B. The outgoing energy flux

In order to answer any experimental question, the distant observer of the radiation needs only to know its amplitude $\bar{h}^{\mu\nu}$ and the gauge. Our expressions (3.2)–(3.4) form an asymptotic approximation for any observer in the far zone, and so suffice to describe any experiment on the radiation. We could, therefore, stop the calculation at this point. Nevertheless, the concept of energy has had considerable power in physics, and not unnaturally much attention has been focused on it in the present problem. In the near zone, the demonstration (as in paper II) that the near-zone quadrupole formula correctly gives the rate of change of the Newtonian energy immediately allows one to calculate various observables about the system, such as the rate of change of a binary's orbital period. It would be possible, but more difficult, to calculate these directly from a knowledge of the metric through radiation-reaction order. Similarly, in the far zone, it would be helpful to know that the energy in the waves bore some relation to the loss of energy by the system, so that observation of the system's near-zone energy loss would immediately imply an amplitude for waves in the far zone, independently of knowing any other details of the system's dynamics. Considerations of this sort must ultimately provide the physical justification for definitions of a far-zone energy flux. It follows, therefore, that it does not really matter very much which far-zone definition one takes for energy flux, as long as it has a clear relation to the amplitude of the waves on the one hand, and it can be related to (is preferably equal to) the loss of near-zone energy on the other. We shall consider two definitions here, the Isaacson average stress-energy tensor[14] and the Bondi flux[15] on $\mathscr{I}_{\rm FM}^+$.

The Isaacson flux is a measure of the waves' mean contribution to the curvature of the spacetime. When averaged over a region of spacetime a few wavelengths in size it gives the Einstein tensor that a "coarse-grained" observer would measure. The terms that contribute to the flux contain time derivatives, so the static part of the metric (3.2)—(3.4) will not contribute. The time-dependent part may be put into TT gauge[16] to give

$$\underline{\bar{h}}^{(TT)}_{i'j'}=2\epsilon^5 P_{ikjl}\,{}_2\not{I}^{kl}{}_{,uu}/\eta+O(\epsilon^6)\ , \tag{3.8}$$

where

$$P_{ikjl}=P_{ik}P_{jl}-\tfrac{1}{2}P_{ij}P_{kl}\ , \tag{3.9}$$

$$P_{ik}=\delta_{ik}-n_i n_k\ . \tag{3.10}$$

In this gauge the Isaacson flux is

$$\mathscr{F}_{i'}=-\frac{1}{32\pi}\langle \underline{\bar{h}}^{(TT)}_{j'k',\tau}\underline{\bar{h}}^{(TT)j'k'}{}_{,i'}\rangle\ , \tag{3.11}$$

where angle brackets $\langle\ \rangle$ denote an average over a few wavelengths in space and time. For large η, the derivative with respect to $\eta^{i'}$ in Eq. (3.11) acts mainly on the argument $u=\tau-\eta$ of $\not{I}^{j'k'}$, giving

$$\mathscr{F}_{a'}=\frac{\epsilon^{10}}{8\pi}P_{minj}P^m{}_k{}^n{}_l\langle {}_2\dddot{\not{I}}^{ij}\,{}_2\dddot{\not{I}}^{kl}\rangle n_a\eta^{-2}+O(\epsilon^{11})\ , \tag{3.12}$$

where dots denote derivatives with respect to u. When integrated over a sphere this gives a total far-zone luminosity of

$$\underline{L}_{\rm GW}=\tfrac{1}{5}\epsilon^{10}\langle {}_2\dddot{\not{I}}_{jk}\,{}_2\dddot{\not{I}}^{jk}\rangle+O(\epsilon^{11})\ , \tag{3.13}$$

which is the far-zone quadrupole formula. It exactly balances the rate of loss of Newtonian energy (paper II).

The Bondi flux is an invariant characterization of the energy reaching $\mathscr{I}^+$. In our flat manifold FM, $\mathscr{I}^+_{\rm FM}$ is trivially the set of future projective end points of the outgoing null geodesics $u=$const. The Bondi mass is, of course, just $\epsilon^3{}_2M+\epsilon^5{}_4M+O(\epsilon^6)$. The Bondi flux is calculated from the asymptotic shear tensor of the null cones as they approach infinity. This is given by[17]

$$\underline{\sigma}_{i'j'}(u)=\lim_{\substack{\eta\to\infty\\ u={\rm const}}}\eta\underline{\bar{h}}^{(TT)}_{i'j'}\ . \tag{3.14}$$

The invariant flux is derived from the "news tensor"

$$\begin{aligned}\dot{\underline{\sigma}}_{i'j'}&=d\underline{\sigma}_{i'j'}/du\\ &=2\epsilon^5P_{ikjl}\,{}_2\dddot{\not{I}}^{kl}+O(\epsilon^6)\ .\end{aligned} \tag{3.15}$$

The luminosity measured at $\mathscr{I}^+$ is

$$\underline{\mathscr{L}}_{\rm GW}=\frac{1}{32\pi}\int \dot{\underline{\sigma}}_{i'j'}\dot{\underline{\sigma}}^{i'j'}d\Omega\ , \tag{3.16}$$

where $d\Omega$ is the element of solid angle. This will clearly give the same answer as Eq. (3.13).

C. Gauge invariance of FM

Our construction of the limiting manifold FM is tied to the coordinate system we have adopted; we shall not attempt here to give a coordinate-invariant description of this boundary. But we can at least examine the question of gauge transformations within the Lorentz gauge. Since we have not specialized the gauge to any particular Lorentz gauge, it is clear that the construction will go through in any such coordinate system in which the initial data take their assumed form. What will be the relationship of the far-field description of one such coordinate system to that of another? The answer, as we shall see, contains no surprises: provided we choose a gauge which does not alter the Newtonian order in the near-zone limit, then the effect on the far-zone limit is that of a gauge transformation of gravitational waves in linearized theory.

We consider an infinitesimal coordinate transformation generated by a vector $\xi^\mu(x^\alpha;\epsilon)$. To lowest order in ξ^μ and $\bar{h}^{\alpha\beta}$ the change in the metric is given by

$$\delta\bar{h}^{\mu\nu}=\eta^{\mu\beta}\xi^\nu{}_{,\beta}+\eta^{\nu\beta}\xi^\mu{}_{,\beta}-\eta^{\mu\nu}\xi^\beta{}_{,\beta}+O(h\xi)\ . \tag{3.17}$$

If the new metric is also to be in harmonic gauge then we have

$$\Box\xi^\mu=O(h\xi)\ . \tag{3.18}$$

This implies as well that

$$\Box\delta h^{\mu\nu}+O(h\xi)\ . \tag{3.19}$$

The gauge transformation is determined by Eq. (3.18) provided we give initial data for ξ^μ. These data are subject to the following asymptotic falloff restrictions in order to ensure the unique existence of solutions to the initial value constraints [paper II, Eq. (2.5)]:

$$\begin{aligned}&\delta\bar{h}^{ij}(\tau=0,x^i;\epsilon)=O(r^{-1})\ ,\\ &\delta\bar{h}^{ij}{}_{,\tau}(\tau=0,x^i;\epsilon)=O(r^{-2})\ .\end{aligned} \tag{3.20}$$

The constraints [paper I, Eq. (3.12)] then imply

$$\delta\bar{h}^{\tau\mu}(\tau=0,x^i;\epsilon)=O(r^{-1})\ . \tag{3.21}$$

We choose the initial data subject to the restriction that the Newtonian equations of motion are unchanged. This condition requires

$$\delta\Lambda^{\mu\nu}=O(\epsilon^5)\ ,$$

which implies

$$\delta\bar{h}^{\tau\tau}=O(\epsilon^5),\ \ \delta\bar{h}^{\tau i}=O(\epsilon^4),\ \ \delta\bar{h}^{ij}=O(\epsilon^3)\ . \tag{3.22}$$

In order that the harmonicity condition

$$\delta\bar{h}^{\mu\nu}{}_{,\nu}=0 \tag{3.23}$$

be preserved at all orders, this would require that ${}_3\delta\bar{h}^{ij}{}_{,j}=0$. In turn, Eq. (3.17) implies $\nabla_3{}^2\xi^i=0$, which has no regular solutions. At the next order, the constraints force ${}_4\delta\bar{h}^{\tau i}$ to vanish if ${}_4\delta\Lambda^{\tau i}$ vanishes, then the same argument as above leads to ${}_4\delta\bar{h}^{ij}$ vanishing. We conclude, therefore that in near-zone coordinates (τ,x^i),

$$\xi^\mu=O(\epsilon^5)\ . \tag{3.24}$$

This means that the $O(h\xi)$ terms in Eqs. (3.17)—(3.19) are of order ϵ^9 in the near zone.

In far-zone coordinates, Eq. (3.24) becomes

$$\xi^\tau=O(\epsilon^5),\ \ \xi^{i'}=O(\epsilon^6)\ . \tag{3.25}$$

From Eq. (3.17) in the far zone, remembering that $\eta^{\mu'\nu'}\sim\epsilon^2$, we find

$$\delta\overline{h}^{\mu'\nu'}=O(\epsilon^7)\ .$$

Removing the conformal factor gives

$$\delta\underline{\overline{h}}^{\mu'\nu'}=O(\epsilon^5)\ , \tag{3.26}$$

which is the order of the radiation terms. Thus, those parts of the far-zone metric which depend on Newtonian conserved quantities (mass, angular momentum) are unchanged, while those terms involving radiation are affected in the same manner as in linearized theory. Moreover, since the spatial component $\xi^{i'}$ is one order higher than ξ^{τ}, the change in the spatial metric is

$$_5\delta\underline{\overline{h}}^{i'j'}=-\delta^{ij}\xi^{\tau}{}_{,\tau}\ . \tag{3.27}$$

This drops out in the projection to find the TT part of the metric, so the Bondi flux is in fact gauge-invariant in this sense.

IV. CONCLUSIONS

We have been able to study the outgoing radiation in our regular, asymptotically Newtonian sequences, thereby linking our earlier proof of the near-zone quadrupole formula (paper II) to a proof that the far-zone quadrupole formula is also an asymptotic approximation to the radiation from our sequences. By introducing the far-zone manifold FM as a boundary of the five-dimensional manifold of our spacetime sequence, we have given a geometrical characterization of radiation in the Newtonian limit. Although our construction of FM is coordinate dependent, our investigation of its gauge invariance suggests that there is likely to be a coordinate-invariant way to define it. We will return to this point below.

Results similar to these have been obtained by other investigators. Those obtained by the method of matched asymptotic expansions[9] seem particularly close in spirit to the present calculation. Both methods divide the near zone from the far zone, adopting appropriate coordinate scalings in each. The matching method solves Einstein's differential equations locally in each zone, makes certain assumptions about appropriate forms for these solutions, and then matches the two at their boundary in order to determine various constants. Our method, by contrast, uses a single expression, Eq. (2.4), for the solution everywhere, simply developing asymptotic approximations for it in the different zones. This makes no assumptions other than the regularity of the initial-value problem and the form of the initial data, and therefore provides the rigorous underpinning of the matching methods.

Similarly, the characteristic initial-value approach[10] is even closer in spirit of our approach. Although at first sight it may look rather different from our methods of papers I and II, our present analysis shows it to be an approximation to a conformally related sequence of solutions. One puzzling aspect of the relation between the two approaches is not yet understood: we have complete freedom in our choice of wave initial data, but in the characteristic problem there seems to be no freedom at all.[10] This deserves further study.

A number of questions remain for the future. Not only would it be useful to find invariant characterizations of NM and FM (and therefore of sequences of solutions that have Newtonian limits), but there is also the question of the "coverage" of the sequence by NM or FM. As we discussed in paper I and elsewhere,[18] these limits can be uniform only for a finite interval of τ. For example, a relativistic binary will eventually destroy itself as the stars spiral together and collide, while its Newtonian approximation will not do so. So the Newtonian approximation can be uniform only for a limited time. Each finite time interval thus approximated produces manifolds NM and FM which contain systems of somewhat different total masses. It is not yet clear whether one can identify these manifolds in a sensible way. If one can, one might hope to develop this picture so that one could take a limit as the interval $\Delta\tau$ being approximated goes to zero, giving a continuously changing Newtonian approximation: the "osculating" Newtonian approximation, in the terminology of Walker and Will.[19]

In the following paper, one of us (T.F.) will make a different extension of the present method, which will allow the inclusion of compact relativistic bodies interacting by their Newtonian-type near-zone fields. This provides the final step in our demonstration that the quadrupole formulas may be used in the interpretation of the evolution of the binary pulsar system.[20]

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[1]*Any* theory of gravity compatible with special relativity will admit gravitational waves, because gravitational effects can only travel at finite velocities. See the explanation in, for example, B. F. Schutz, Am. J. Phys. **52**, 412 (1984).

[2]T. Futamase and B. F. Schutz, Phys. Rev. D **28**, 2363 (1983) (paper I).

[3]T. Futamase, Phys. Rev. D **28**, 2373 (1983) (paper II).

[4]S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965); S. Chandrasekhar and Y. Nutku, *ibid.* **158**, 55 (1969).

[5]S. Chandrasekhar and E. P. Esposito, Astrophys. J. **160**, 153 (1970).

[6]W. L. Burke, J. Math. Phys. **12**, 401 (1971).

[7]These solutions are derived in H. Stephani, *General Relativity* (Cambridge University Press, New York, 1982).

[8]Our approach to radiation reaction began with linearized

theory: B. F. Schutz, Phys. Rev. D **22**, 249 (1980).

[9]R. E. Kates, Ann. Phys. (N.Y.) **132**, 1 (1981); Phys. Rev. D **22**, 1871 (1980); J. L. Anderson, R. E. Kates, L. S. Kegeles, and R. G. Madonna, *ibid.* **25**, 2038 (1982); K. S. Thorne, in *Gravitational Radiation,* edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983).

[10]J. Winicour, J. Math. Phys. **24**, 1193 (1983); R. A. Isaacson, J. S. Welling, and J. Winicour, *ibid.* **24**, 1824 (1983); *Problems of Collapse and Numerical Relativity,* edited by M. Signore (Dordrecht, Reidel, 1984); report (unpublished).

[11]We use units in which $c = G = 1$. We adopt the conventions of C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (San Francisco, Freeman, 1973). In particular, latin indices denote spatial components, while greek indices run over all coordinates. We introduce other notations for components below.

[12]See Ref. 11 for a description and original references.

[13]This "matching radius" R is fixed in our calculations, but the fact that it may be changed without affecting the results probably is the key to a more rigorous formulation of the notion of a "matching region" in the matched asymptotic expansion approaches of Ref. 9.

[14]R. A. Isaacson, Phys. Rev. **166**, 1263 (1968); **166**, 1272 (1968).

[15]H. Bondi, M. G. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London **A269**, 21 (1962); R. K. Sachs, *ibid.* **A270**, 103 (1962).

[16]See Ref. 11.

[17]M. Walker in *Relativistic Astrophysics and Cosmology,* edited by X. Fustero and E. Verdaguer (World Scientific, Singapore, 1983).

[18]B. F. Schutz in *Exact Solutions in General Relativity,* edited by W. Dietz and C. Hoenselaers (Springer, Berlin, 1984).

[19]M. Walker and C. M. Will, Astrophys. J. **242**, L129 (1980).

[20]J. H. Taylor and J. M. Weisberg, Astrophys. J. **253**, 908 (1982); V. Boriakoff, D. C. Ferguson, M. P. Haugan, Y. Terzian, and S. Teukolsky, *ibid.* **261**, L101 (1982).

[21]B. F. Schutz in *Relativistic Astrophysics and Cosmology* (Ref. 17).