

KLT and new relations for $\mathcal{N} = 8$ SUGRA and $\mathcal{N} = 4$ SYM

Bo Feng^{a,c} and Song He^{b,c}

^a*Center of Mathematical Science, Zhejiang University, Hangzhou, China*

^b*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Golm, Germany*

^c*Kavli Institute for Theoretical Physics China, CAS, Beijing 100190, China*

E-mail: b.feng@cms.zju.edu.cn, songhe@aei.mpg.de

ABSTRACT: In this short note, we prove the supersymmetric Kawai-Lewellen-Tye (KLT) relations between $\mathcal{N} = 8$ supergravity (SUGRA) and $\mathcal{N} = 4$ super Yang-Mills (SYM) tree-level amplitudes in the frame of S-matrix program, especially we do not use string theory or the explicit Lagrangian form of corresponding theories. Our supersymmetric KLT relations naturally unify the non-supersymmetric KLT relations and newly discovered gauge theory identities and produce more identities for amplitudes involving scalars and fermions. We point out also that these newly discovered identities can be used to reduce helicity basis from $(n - 3)!$ further down.

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Contents

1	Introduction	1
2	BCFW proof of KLT relations for $\mathcal{N} = 8$ SUGRA	3
2.1	Super-KLT relations with manifest $(n - 2)!$ permutation symmetries	4
3	New relations among amplitudes in $\mathcal{N} = 4$ SYM	7
4	Conclusion and discussions	10

1 Introduction

S-matrix program [1–3] is a program to study properties of quantum field theory based on some general principles, like the Lorentz invariance, Locality, Causality, Gauge symmetry as well as Analytic property. Because it does not use specific information like Lagrangian, result obtained by this method is quite general. Also exactly because its generality with very few assumptions, study along this line is very challenging.

There are three important progresses in the frame of S-matrix program worth to mention. The first one is the unitarity-cut method proposed in [4], where on-shell tree amplitudes have been used for calculations of loop amplitudes without drawing many many Feynman diagrams. The second one is the BCFW on-shell recursion relations [5, 6]. The derivation of BCFW relations beautifully demonstrates the idea of S-matrix program. It relies only on basic analytic properties of the tree-level amplitude: the single pole structure and the factorization property when a propagator reaches its mass-shell. Then using the familiar complex analysis the whole amplitude can be uniquely fixed if there are no boundary contributions.¹

Based on the BCFW recursion relations, the third beautiful work along the line of S-matrix program is given by Benincasa and Cachazo [10]. In the paper, by assuming the applicability of BCFW recursion relations for four-particle amplitudes, they have easily re-derived many well-known (but difficult to prove) fundamental facts in S-matrix program, such as the non-Abelian structure for gauge theory and all matters couple to gravity with same coupling constant. Given the non-Abelian structure, the applicability of BCFW recursion relations has been proved, using purely S-matrix argument, for gauge theory amplitudes with arbitrary number of particles [11, 12].

¹The boundary behavior is one important subject to study. In [7], background field method has been applied to the study. In [8, 9], the situation with nonzero boundary contributions has also been discussed. It will be interesting to study the boundary behavior in the frame of S-matrix program.

One important observation in [10] is that three-point amplitude (on-shell) of any theory is uniquely fixed by the Lorentz invariance and the spin of external particles. For example, the three-gluon amplitudes are given by

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|1\rangle}, \quad A_3(1^+, 2^+, 3^-) = \frac{[1|2]^3}{[2|3] [3|1]}, \quad (1.1)$$

while the three-graviton amplitude is given by

$$M_3(1, 2, 3) = A_3(1, 2, 3)^2. \quad (1.2)$$

Another result determined by Lorentz invariance and spin symmetry is

$$A_3(1, 2, 3^+)A_3(1, 2, 3^-) = 0. \quad (1.3)$$

It is worth to emphasize that the vanishing of (1.3) is because for three massless particles, on-shell condition requires either $\langle 1|2\rangle \sim \langle 2|3\rangle \sim \langle 3|1\rangle = 0$ or $[1|2] \sim [2|3] \sim [3|1] = 0$. For three gluons, no matter which helicity configuration it is, we will always have $\sum_{i=1,2,3} h_i \neq 0$, thus amplitude will contain either $\langle | \rangle$ or $[|]$ depending on the sum of helicities to be negative or positive. If the sum of helicities is zero $\sum_{i=1,2,3} h_i = 0$, the situation is very tricky and systematic exploration of this particular case is still missing.

The idea of [10] has intrigued several important works in last few months. Starting from the antisymmetry of (1.1) plus the validity of BCFW recursion relations, four important properties of color-ordered gluon amplitudes have been proved in the frame of S-matrix program in [13]. They are color-ordered reversed relations, U(1)-decoupling relations, Kleiss-Kuijff relations [14] and Bern-Carrasco-Johansson(BCJ) relations [15].² In their proof, there are no need for those inputs, such as Lagrangian definition and string theory. In other words, it is possible to have a deformed Yang-Mills Lagrangian and same results will hold as long as the BCFW recursion relations can be applied.³

Using similar ideas, based only on observations (1.2) and (1.3), as well as the validity of BCFW recursion relations for graviton amplitudes [7, 20], new form of Kawai-Lewellen-Tye type relations [21] and new gauge amplitude identities have been found and proved in [22, 23]. Again, in the proof, there is no need for input, such as Einstein Lagrangian or string theory. The BCJ relations are used in the proof, but since BCJ relations have been proved in the frame of S-matrix, results obtained in [22, 23] are nicely fit in the S-matrix program.

In this short note, we will generalize the result in [22, 23] to the case of $\mathcal{N} = 8$ SUGRA. Especially we will show that the KLT relations and the new gauge theory identities can be unified into the $\mathcal{N} = 8$ KLT relations. With this unified form we can get even more identities involving scalars and fermions. We want to emphasize that although the $\mathcal{N} = 8$ KLT relations have been discussed in [24], our study in this note is in the frame of S-matrix program, i.e., we do not use the string theory to derive the $\mathcal{N} = 8$ KLT relations. Instead, we start from the three point amplitude of $\mathcal{N} = 8$ SUGRA and use BCFW recursion

²The BCJ relations have also been proved in string theory [16, 17].

³The BCJ relations have also been generalized to include the matter and to the $\mathcal{N} = 4$ supersymmetric theory in [18, 19] along the same line.

relations to derive and prove general $\mathcal{N} = 8$ KLT relations purely from the point of view of field theory.

The plan of the note is the following. In section two, we write down and prove the supersymmetric KLT relations between $\mathcal{N} = 8$ SUGRA and $\mathcal{N} = 4$ SYM. In section three, we discuss how the new supersymmetric KLT relations can be used to produce many new identities for helicity amplitudes. In section four, we give a brief summary and some discussions.

2 BCFW proof of KLT relations for $\mathcal{N} = 8$ SUGRA

One basic property of supersymmetric field theory is that different fields are grouped into a supermultiplet. With such grouping, the type and the helicity of fields are represented by the expansion of a superfield in terms of supersymmetry Grassmann variables η^A where $A = 1, \dots, \mathcal{N}$ and \mathcal{N} is the number of total supersymmetries, thus there is $SU(\mathcal{N})$ R -symmetry with $2^{\mathcal{N}}$ on-shell states. For example, the on-shell $\mathcal{N} = 4$ superfield is given by [25, 26]

$$\Phi(p, \eta^a) = G^+(p) + \eta^a F_a^+(p) + \frac{1}{2} \eta^a \eta^b S_{ab}(p) + \frac{1}{3!} \epsilon_{abcd} \eta^b \eta^c \eta^d F_-^a(p) + \frac{1}{4!} \epsilon_{abcd} \eta^a \eta^b \eta^c \eta^d G_-(p), \quad (2.1)$$

where $a, b, c, d = 1, 2, 3, 4$ and it contains following $2^4 = 16$ components: one positive-helicity gluon G^+ , four positive-helicity fermions F_a^+ , six scalars S_{ab} which satisfy self-duality condition $S^{ab} = \epsilon^{abcd} S_{cd}/2$, four negative-helicity fermions F_-^a , and finally one negative-helicity gluon G_- . Similarly all 256 helicity states in $\mathcal{N} = 8$ SUGRA are unified in a superfield, which depend on Grassmann variables η^A with the $SU(8)$ R -symmetry index $A = 1, \dots, 8$.

It is a well-known fact that states of $\mathcal{N} = 8$ theory can be written as the square of states of $\mathcal{N} = 4$ theory. In other words, the $SU(8)$ R -index $A = 1, 2, \dots, 8$ can be split into two $SU(4)$ R -index, $\tilde{a} = 1, 2, 3, 4$ and $a = 5, 6, 7, 8$ (see for example [24]). This square structure is most transparent in string theory where closed-string vertex is the product of left- and right-hand open-string vertices. Using string theory, accurate relations of tree-level scattering amplitudes between gravitons and gluons are given in [21]. The KLT relations express the superamplitude $\mathcal{M}_n(\{p_i, \eta_i^A\})$ with total $n!$ symmetry in terms of product of two color-ordered superamplitudes $\mathcal{A}_n(\sigma|\{p_i, \eta_i^a\})$ and $\tilde{\mathcal{A}}_n(\sigma|\{p_i, \eta_i^{\tilde{a}}\})$. Using these relations, explicit mapping of states has also been given in [24].

As we have mentioned in the introduction, we will not use string theory to study the relations between gravity theory and Yang-Mill theory. Instead we will derive and prove their relations in the frame of S-matrix program. To do this, let us start with the on-shell three point function [27]

$$\mathcal{A}_3^{\text{MHV}} = \frac{\delta^{(8)}(\sum_i |i\rangle \eta_i^a)}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|1\rangle}, \quad \mathcal{A}_3^{\overline{\text{MHV}}} = \frac{\delta^{(4)}(\eta_1^a [2|3] + \eta_2^a [3|1] + \eta_3^a [1|2])}{[1|2] [2|3] [3|1]} \quad (2.2)$$

for gauge theory and

$$\mathcal{M}_3^{\text{MHV}} = \frac{\delta^{(16)}(\sum_i |i\rangle \eta_i^A)}{(\langle 1|2\rangle \langle 2|3\rangle \langle 3|1\rangle)^2}, \quad \mathcal{M}_3^{\overline{\text{MHV}}} = \frac{\delta^{(8)}(\eta_1^A [2|3] + \eta_2^A [3|1] + \eta_3^A [1|2])}{([1|2] [2|3] [3|1])^2} \quad (2.3)$$

for gravity. Eq. (2.2) and (2.3) are the supersymmetric generalizations of eq. (1.1) and (1.2). However, there is one important difference we want to emphasize. Supersymmetry does not only group different fields together, it fixes interactions to some level. The most severe constraints arise in the $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ SUGRA theory, where interactions are completely determined by supersymmetry. Thus supersymmetry adds the so called "selection rule" for non-vanishing scattering amplitudes, i.e., they must be $SU(\mathcal{N})$ R -symmetry invariant. In other words, comparing to the non-supersymmetric case, besides the familiar assumptions in our S-matrix frame, *for supersymmetric case we have added another assumption, the supersymmetric selection rule.*

Having eq. (2.2) and (2.3), we can write down the total amplitude

$$\mathcal{A}_3(1, 2, 3) = \mathcal{A}_3^{\text{MHV}}(1, 2, 3) + \mathcal{A}_3^{\overline{\text{MHV}}}(1, 2, 3) \quad (2.4)$$

for $\mathcal{N} = 4$ SYM theory and

$$\mathcal{M}_3(1, 2, 3) = \mathcal{M}_3^{\text{MHV}}(1, 2, 3) + \mathcal{M}_3^{\overline{\text{MHV}}}(1, 2, 3) \quad (2.5)$$

for $\mathcal{N} = 8$ SUGRA theory. One important result is that

$$\mathcal{M}_3(1, 2, 3) = \mathcal{A}_3(1, 2, 3)\tilde{\mathcal{A}}_3(1, 2, 3) \quad (2.6)$$

where we have used the state mapping in [24] that $SU(8)$ index $A = 1, \dots, 8$ split into $SU(4)$ index $\tilde{a} = 1, 2, 3, 4$ of $\tilde{\mathcal{A}}_n$ and $SU(4)$ index $a = 5, 6, 7, 8$ of \mathcal{A}_n respectively, as well as the supersymmetric generalization of (1.3)

$$\mathcal{A}_3^{\text{MHV}}(1, 2, 3)\mathcal{A}_3^{\overline{\text{MHV}}}(1, 2, 3) = 0. \quad (2.7)$$

It is worth to emphasize that eq. (2.6) unifies both KLT relations (1.2) as well as the vanishing identity (1.3) for three-point amplitudes in the S-matrix frame. Then using the BCFW on-shell recursion relations we will generalize this result to general n , thus unify the results presented in [22, 23]. Since most details can be found in [22, 23], our discussion will be brief.

2.1 Super-KLT relations with manifest $(n-2)!$ permutation symmetries

The super-KLT relations with manifest $(n-2)!$ permutation symmetries for $\mathcal{N} = 8$ SUGRA and $\mathcal{N} = 4$ SYM can be written as following

$$\mathcal{M}_n(\{p_i, \eta_i^A\}) = \frac{1}{s_{12\dots(n-1)}} \sum_{\gamma, \beta \in S_{n-2}} \tilde{\mathcal{A}}_n(n, \gamma, 1|\{p_i, \eta_i^{\tilde{a}}\}) \mathcal{S}[\gamma|\beta]_{p_1} \mathcal{A}_n(1, \beta, n|\{p_i, \eta_i^a\}). \quad (2.8)$$

where again we have split $SU(8)$ index $A = 1, \dots, 8$ into $SU(4)$ index $\tilde{a} = 1, 2, 3, 4$ of $\tilde{\mathcal{A}}_n$ and $SU(4)$ index $a = 5, 6, 7, 8$ of \mathcal{A}_n respectively. The kinematic invariants are defined as $s_K = (\sum_{i \in K} p_i)^2$ for any index set $K \subseteq \{1, \dots, n\}$, and the functional \mathcal{S} is defined as [22, 23]

$$\mathcal{S}[i_1, \dots, i_m | j_1, \dots, j_m]_{p_1} = \prod_{t=1}^m \left(s_{i_t 1} + \sum_{q>t}^m \theta(i_t, i_q) s_{i_t i_q} \right), \quad (2.9)$$

where $\theta(i_a, i_b)$ is 1 if i_a appears after i_b in the sequence $\{j_1, \dots, j_m\}$, otherwise it is 0. The functional \mathcal{S} has some nice properties. For example,

$$\mathcal{S}[i_1, \dots, i_m | j_1, \dots, j_m] = \mathcal{S}[j_m, \dots, j_1 | i_m, \dots, i_1], \quad (2.10)$$

which ensures that eq. (2.8) is completely symmetric in \mathcal{A}_n and $\tilde{\mathcal{A}}_n$. More importantly, \mathcal{S} has the factorization,

$$\mathcal{S}[\gamma\sigma|\alpha\beta] = \mathcal{S}[\sigma|\alpha]\mathcal{S}_P[\gamma|\beta], \quad (2.11)$$

where $P = \sum_{i \in \{\sigma\}} p_i$ has been put on-shell, i.e., $P^2 = 0$.

Due to the appearance of on-shell singularity from $s_{12\dots(n-1)} = p_n^2$, eq. (2.8) is well-defined only after regularization. The details of the regularization and concrete examples of these relations, can be found in [22, 29].

Formula (2.8) is different from the well-known KLT formula presented in [21, 28] with only manifest $(n-3)!$ permutation symmetry. As shown in [29], there exist a family of more compact KLT relations with manifest $(n-3)!$ symmetric form, which do not need any regularization. The original ansatz in [28] is a special case of these relations, and since we do not bother to write down the most general form, we will present only following more symmetric form

$$\mathcal{M}_n(\{p_i, \eta_i^A\}) = \sum_{\gamma, \beta \in S_{n-3}} \tilde{\mathcal{A}}_n(n-1, n, \gamma, 1 | \{p_i, \eta_i^{\tilde{a}}\}) \mathcal{S}[\gamma|\beta]_{p_1} \mathcal{A}_n(1, \beta, n-1, n | \{p_i, \eta_i^a\}), \quad (2.12)$$

where γ and β are permutations of legs $2, \dots, n-2$. Note that both forms of KLT relations for superamplitudes are the same as those for pure gluon and graviton amplitudes.

As discussed in detail in [29], it is not easy to prove (2.12) without assumption of total symmetric property, i.e., the formula (2.12) is in fact $n!$ permutation symmetric (or at least $(n-2)!$ permutation symmetric), but the relation (2.8) is much easier to prove using supersymmetric BCFW recursion relations [25, 30] in $\mathcal{N} = 8$ SUGRA and $\mathcal{N} = 4$ SYM (we use \mathcal{M}_n to denote superamplitudes in both theories)

$$\mathcal{M}_n = \sum_{L,R} \int d^{\mathcal{N}} \eta \mathcal{M}_L(\hat{1}, \dots, \{-\hat{P}, \eta\}) \frac{1}{P^2} \mathcal{M}_R(\{\hat{P}, \eta\}, \dots, \hat{n}), \quad (2.13)$$

where as in [22], we have picked legs 1 and n to deform

$$\lambda_1(z) = \lambda_1 + z\lambda_n, \quad \tilde{\lambda}_n(z) = \tilde{\lambda}_n - z\tilde{\lambda}_1, \quad \eta_n(z) = \eta_n - z\eta_1. \quad (2.14)$$

It is worth to remind that for supersymmetric case [25], the choice of deformation does not depend on the helicity of 1 and n .

As expected, the proof follows exactly the same steps as that for pure graviton and gluon amplitudes [22, 23, 29]. The only difference is that we need to replace all helicity sums for graviton and gluon amplitudes by Grassmann integrations for superfields.

Now we prove both forms (2.8) and (2.12) of KLT relations by induction. The case $n = 3$ has been carefully discussed in previous paragraphs using only some general properties of supersymmetric field theory. Assuming both forms hold for any lower-point superamplitudes up to $n-1$ point, for the n -point case, we uses eq. (2.13) to expand both

superamplitudes \tilde{A}_n and A_n , on the R.H.S. of eq. (2.8) and eq. (2.12), in terms of lower-point superamplitudes. As discussed in [22, 29] there are two classes of contributions,

- The pole appears in only one of the amplitudes \tilde{A}_n and A_n .
- The pole appears in both amplitudes \tilde{A}_n and A_n .

Similar to the discussions in [22], for eq. (2.8), when only A_n has the pole, we obtain

$$\begin{aligned} & \frac{(-)^n}{s_{\hat{1}2\dots(n-1)}} \sum_{\gamma,\sigma,\beta} \frac{\int d^4\eta \tilde{A}_{n-k+1}(\hat{n}, \gamma, \{-\hat{P}, \eta\}) \tilde{A}_k(\{\hat{P}, \eta\}, \sigma, \hat{1})}{s_{12\dots k}} \mathcal{S}[\gamma\sigma|\beta] A_n(\hat{1}, \beta, \hat{n}) \\ & \propto \sum_{\sigma} \tilde{A}_{k+1}(\{\eta, \hat{P}\}, \sigma, \hat{1}) \mathcal{S}[\sigma|\rho] = 0, \end{aligned} \tag{2.15}$$

where we have used the factorization property of \mathcal{S} , and ρ is the relative ordering of legs $2, \dots, k$ in β . Similarly the contribution when only \tilde{A}_n having the pole also vanishes.

When there are poles in both amplitudes of eq. (2.8), the result is,

$$\begin{aligned} & \frac{(-)^n}{s_{\hat{1}2\dots(n-1)}} \sum_{\gamma,\beta,\sigma,\alpha} \frac{\int d^4\eta \tilde{A}(\hat{n}, \gamma, \{-\hat{P}, \eta\}) \tilde{A}(\{\hat{P}, \eta\}, \sigma, \hat{1})}{s_{12\dots k}} \mathcal{S}[\gamma\sigma|\alpha\beta] \frac{\int d^4\eta A(\hat{1}, \alpha, \{-\hat{P}, \eta\}) A(\{\hat{P}, \eta\}, \beta, \hat{n})}{s_{\hat{1}2\dots k}} \\ & = \frac{(-)^n}{s_{12\dots k}} \int d^8\eta \sum_{\sigma,\alpha} \frac{\tilde{A}(\{-\hat{P}, \eta\}, \sigma, \hat{1}) \mathcal{S}[\sigma|\alpha] A(\hat{1}, \alpha, \{\hat{P}, \eta\})}{s_{\hat{1}2\dots k}} \sum_{\gamma,\beta} \frac{\tilde{A}(\hat{n}, \gamma, \{-\hat{P}, \eta\}) \mathcal{S}_{\hat{P}}[\gamma|\beta] A(\{\hat{P}, \eta\}, \beta, \hat{n})}{s_{\hat{P}k+1\dots(n-1)}} \\ & = \frac{\int d^8\eta \mathcal{M}_{k+1}(\hat{1}, \dots, k, \{-\hat{P}, \eta\}) \mathcal{M}_{n-k+1}(\{\hat{P}, \eta\}, k+1, \dots, \hat{n})}{s_{1\dots k}}, \end{aligned}$$

where in the first equality we have used $s_{\hat{1}2\dots k} = s_{\hat{P}k+1\dots(n-1)}$ and the factorization property of \mathcal{S} , and we have combined two $\mathcal{N} = 4$ Grassmann integrations over $\eta^{\hat{a}}$ and η^a into a single $\mathcal{N} = 8$ integration over η^A ; while in the second equality we have used the KLT relations for lower-point superamplitudes.

Therefore, by eq. (2.13) for $\mathcal{N} = 8$ SUGRA, we have proved the validity of n -point KLT relations, eq. (2.8). For manifestly $(n-3)!$ permutation symmetric KLT relations like eq. (2.12), as discussed in [29], one has to consider different pole structure, i.e., if we take $1, n$ to do the BCFW-deformation, then the pole includes both p_1, p_{n-1} will be very hard to prove. For this kind of difficult pole structures, the total $n!$ permutation symmetry (or at least the $(n-2)!$ permutation symmetry) has been assumed to avoid the direct proof of this kind of poles. Although this assumption is right from the point of view of string theory, as far as we know, there is no direct proof in field theory and it will be very interesting to do that, using, for example, the BCJ relations. Under this assumption, by simply using Grassmanian integrations instead of helicity sums, a similar proof for this form (2.12) of KLT relations is straightforward and we shall not repeat it here.

Note that in the case of pure gluon and graviton amplitudes [22, 29], there are the so-called mixed-helicity terms, which have to vanish for the use of usual BCFW recursion relations. In [23], new gluon amplitude relations were found which include the vanishing results for mixed-helicity terms. In our proof using superamplitudes, these mixed-helicity terms are unified with normal terms in the supersymmetric BCFW recursion relations

as we have demonstrated for the three-point superamplitude. As a result, we shall see immediately that eq. (2.8) and eq. (2.12) encode not only KLT relations in maximally-supersymmetric theories, but also new relations among amplitudes in $\mathcal{N} = 4$ SYM, which are generalizations of the gluon amplitude identities in [23].

3 New relations among amplitudes in $\mathcal{N} = 4$ SYM

As mentioned before, the SYM superamplitudes $\tilde{\mathcal{A}}_n$ and \mathcal{A}_n can be expanded as polynomials of $\eta_i^{\tilde{a}}$ and η_i^a ($i = 1, \dots, n$) respectively, and the coefficients are component amplitudes with all possible external helicity states, or matter contents,

$$\mathcal{A}_n = \sum_{\{I_i\}} \prod_{i=1}^n \eta_i^{I_i} A_n(\{I_i\}), \tag{3.1}$$

where $I_i \subseteq \{5, 6, 7, 8\}$ are 16 possible powers of η_i^a , which are in one-to-one correspondence with 16 possible external states in eq. (2.1). For example, if $I_i = \{\{5, 6, 7, 8\}, \{5, 6, 7, 8\}, \emptyset, \emptyset\}$, then $A_4(\{I_i\}) = A(\{G_-, G_-, G^+, G^+\})$; for $I_i = \{\{5, 6\}, \emptyset, \{7, 8\}, \{5, 6, 7\}, \emptyset, \{8\}\}$, we have $A_6(\{I_i\}) = A(\{S_{56}, G^+, S_{78}, F_8^-, G^+, F_8^+\})$. Similarly, for $\tilde{\mathcal{A}}_n$, we have,

$$\tilde{\mathcal{A}}_n = \sum_{\{\tilde{I}_i\}} \prod_{i=1}^n \eta_i^{\tilde{I}_i} \tilde{A}_n(\{\tilde{I}_i\}), \tag{3.2}$$

where $\tilde{I}_i \subseteq \{1, 2, 3, 4\}$.

Plugging both expansions into eq. (2.8) and (2.12), it is straightforward to see that the L.H.S. can be expanded as a polynomial of η_i^A ,

$$\mathcal{M}_n = \sum_{\{J_i\}} \prod_{i=1}^n \eta_i^{J_i} M_n(\{J_i\}), \tag{3.3}$$

where $J_i = \tilde{I}_i \cup I_i \subseteq \{1, \dots, 8\}$ are 256 possible powers of η_i^A , corresponding to 256 external states in the $\mathcal{N} = 8$ multiplet. The component amplitudes $M_n(\{J_i\})$ satisfy the component KLT relations which directly follow from eq. (2.8)

$$M_n(\{J_i = \tilde{I}_i \cup I_i\}) = \frac{1}{s_{12\dots(n-1)}} \sum_{\gamma, \beta \in S_{n-2}} \tilde{A}_n(n, \gamma, 1|\{\tilde{I}_i\}) \mathcal{S}[\beta|\gamma] A_n(1, \beta, n|\{I_i\}), \tag{3.4}$$

or from eq. (2.12)

$$M_n(\{J_i = \tilde{I}_i \cup I_i\}) = \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_n(n-1, n, \gamma, 1|\{\tilde{I}_i\}) \mathcal{S}[\beta|\gamma] A_n(1, \beta, n-1, n|\{I_i\}). \tag{3.5}$$

However, there are more relations in eq. (3.4) and eq. (3.5) than the usual gravity KLT relations and their matter generalizations [21, 28, 31, 32], because $M_n(\{J_i\})$ can vanish even when both \tilde{A}_n and A_n are non-zero amplitudes. In this case, eq. (3.4) and eq. (3.5) represent new relations among amplitudes in $\mathcal{N} = 4$ SYM.

We now determine the sufficient and necessary conditions for the appearance of such relations. The key point is that the superamplitude must be invariant under the $SU(\mathcal{N})$ R -symmetry, as we have mentioned before. In $\mathcal{N} = 4$ SYM, $SU(4)$ R -symmetry puts a constraint on any coefficient of the η expansion: it is non-zero if and only if there are same numbers of η^1, η^2, η^3 and η^4 in that term and each of them appear at least 2 times and at most $n - 2$ times (the three-point amplitude is an exception). In other words, if we define the number of 1, 2, 3 and 4 in $\{\tilde{I}_i\}$ as n_1, n_2, n_3 and n_4 respectively, then the component amplitude $\tilde{A}_n(\{\tilde{I}_i\})$ is non-zero if and only if $2 \leq n_1 = n_2 = n_3 = n_4 = \tilde{k} \leq n - 2$. We denote this number by \tilde{k} , it is straightforward to see that $4\tilde{k}$ is the total degree of η and $n - 2\tilde{k}$ has the interpretation as the sum of helicities of n external states. Similarly $A_n(\{I_i\})$ is non-zero if and only if $2 \leq n_5 = n_6 = n_7 = n_8 \leq n - 2$, and we denote the number by k .

It is well known that the number k represents a component amplitude belonging to the N^{k-2} MHV sector, and schematically we have an expansion in terms of k for the superamplitudes \mathcal{A}_n ,

$$\mathcal{A}_n = \sum_{k=2}^{n-2} A_n^k(\eta)^{4k} = A_n^{\text{MHV}}(\eta)^8 + A_n^{\text{NMHV}}(\eta)^{12} + \dots + A_n^{\overline{\text{MHV}}}(\eta)^{4n-8}, \quad (3.6)$$

where we have used the fact that N^{n-2} MHV sector is equivalent to $\overline{\text{MHV}}$ sector, and we have similar expansions for $\tilde{\mathcal{A}}_n$ in terms of \tilde{k} .

Since $J_i = \tilde{I}_i \cup I_i$, we know the numbers of 1, 2, ... 8 are exactly given by n_1, \dots, n_8 in \tilde{I}_i, I_i . Now given $n_1 = n_2 = n_3 = n_4 = \tilde{k}$, $n_5 = n_6 = n_7 = n_8 = k$ and $2 \leq \tilde{k}, k \leq n - 2$, we conclude from $SU(8)$ R -symmetry that the necessary and sufficient condition for $M_n(\{J_i\})$ to be non-zero is $\tilde{k} = k$. In other words, when one uses eq. (3.6) for SYM superamplitudes in eq. (2.8) and eq. (2.12), only those component amplitudes from the same sector $\tilde{k} = k$ give non-zero result, while all interference terms from $\tilde{k} \neq k$ vanish. Therefore, we have seen that there are non-trivial identities among amplitudes in $\mathcal{N} = 4$ SYM from, say, eq. (2.12)

$$0 = \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_n^{\tilde{k}}(n-1, n, \gamma, 1) S[\beta|\gamma] A_n^k(1, \beta, n-1, n), \quad (3.7)$$

for any $\tilde{k} \neq k$.

One immediate implication of eq. (3.7) is the identities for flipped-helicity gluon amplitudes [23]. There the number of positive(negative) helicity legs in gluon amplitude A_n which is changed to negative(positive) helicity legs in \tilde{A}_n is denoted by $n^+(n^-)$, and it was found that the L.H.S. of KLT relations using A_n and \tilde{A}_n vanishes if $n^+ \neq n^-$ (one particular example is the mixed-helicity terms mentioned before with $(n^+, n^-) = (1, 0)$ or $(0, 1)$). Now since $\tilde{k} = k + n^- - n^+$ for gluon amplitudes, one can see that new gauge theory identities found in [23] are nicely packed into eq. (3.7).

Of course there are more identities when one includes scalars and fermions. Whenever we have different values for sums of helicities in \tilde{A}_n and A_n , there is an identity among these amplitudes. For example,

$$0 = \sum_{\gamma, \beta \in S_3} \tilde{A}_6(5, 6, \gamma, 1 | \{S_{12}, G_-, F_3^+, F_4^+, G_-, G^+\}) S[\beta|\gamma] A_6(1, \beta, 5, 6 | \{S_{56}, G^+, S_{78}, F_-^8, G^+, F_8^+\}), \quad (3.8)$$

since the first one is an NMHV amplitude while the second is an MHV amplitude.

Generally speaking, let us fix the \tilde{k} and matter contents⁴ of the $N^{\tilde{k}-2}$ MHV amplitude $\tilde{A}_n^{\tilde{k}}$, then eq. (3.7) can be considered as relations among $(n-3)!$ color-ordered N^{k-2} MHV amplitudes A_n^k with any particular $k \neq \tilde{k}$ and matter contents. It is worth to notice that since the sum in eq. (3.7) is over $(n-3)!$ basis amplitudes, we obtain a linear relation among these amplitudes in the basis.

However, the above statement is not in contradiction with the statement that $(n-3)!$ is the number of minimal basis amplitudes for gauge theory amplitudes, because for amplitudes in the sector $k = \tilde{k}$, the same combination gives an amplitude in $\mathcal{N} = 8$ SUGRA which does not vanish. Therefore, unlike helicity-independent relations such as Kleiss-Kuijf relations and BCJ relations, these relations only hold for amplitudes in sectors with $k \neq \tilde{k}$. In other words, new gauge identities tell us that although the helicity-independent basis consists of $(n-3)!$ amplitudes, for a given helicity category, it is possible to reduce the number of basis amplitudes further. This can be used to speed up the calculations of cross sections for, for example, the LHC experiments.

Since for each given $N^{\tilde{k}-2}$ MHV amplitude $\tilde{A}_n^{\tilde{k}}$ with particular matter contents, eq. (3.7) gives a linear relation among $(n-3)!$ color-ordered N^{k-2} MHV amplitudes A_n^k , thus with different choices of \tilde{k} and matter contents for \tilde{A}_n , we get different linear equations for same basis set $A_n^{N^{k-2}MHV}$. In other words, for any fixed k and matter contents, eq. (3.7) represents a huge number of relations among these specific $(n-3)!$ amplitudes. Thus it would be very interesting to see how powerful the constraints from eq. (3.7) in reducing the number of independent color-ordered amplitudes.⁵

We use a simple example to demonstrate. For $n = 5$ and $k = 3$, there is only one non-trivial identity, since one can only choose $\tilde{k} = 2$ and in this MHV case all component amplitudes can be related to the gluon amplitude $\tilde{A}_5(\{G_-, G_-, G^+, G^+, G^+\})$. Amplitudes A_5^{NMHV} with any external states are also related to the googly amplitude $A_5^{\text{NMHV}}(G^+, G^+, G_-, G_-, G_-)$. Thus we have one relation for the two basis color-ordered amplitudes $A_5^{\text{NMHV}}(1, 2, 3, 4, 5)$ and $A_5^{\text{NMHV}}(1, 3, 2, 4, 5)$,

$$\left[\frac{s_{21}s_{31}}{\langle 45 \rangle \langle 52 \rangle \langle 23 \rangle \langle 31 \rangle \langle 14 \rangle} + \frac{(s_{21} + s_{32})s_{31}}{\langle 45 \rangle \langle 53 \rangle \langle 32 \rangle \langle 21 \rangle \langle 14 \rangle} \right] A^{\text{NMHV}}(1, 2, 3, 4, 5) + \left[\frac{(s_{31} + s_{32})s_{21}}{\langle 45 \rangle \langle 52 \rangle \langle 23 \rangle \langle 31 \rangle \langle 14 \rangle} + \frac{s_{31}s_{21}}{\langle 45 \rangle \langle 53 \rangle \langle 32 \rangle \langle 21 \rangle \langle 14 \rangle} \right] A^{\text{NMHV}}(1, 3, 2, 4, 5) = 0, \tag{3.9}$$

where we have used the expression of MHV amplitude. This relation reduces the number of independent color-ordered 5-point NMHV amplitudes to one.

For the special case $\tilde{k} = 2$, it is straightforward to derive these new relations among

⁴Given \tilde{k} , there are various $N^{\tilde{k}-2}$ MHV amplitudes $\tilde{A}_n^{\tilde{k}}$ differing from each other by the matter contents of n particles.

⁵Given any $\tilde{k} \neq k$, amplitudes with different external states are related by supersymmetric Ward identities. A basis for component amplitudes has been found in [33], from which $A_n^{\tilde{k}}$ with any matter contents can be obtained. We conjecture that for any $2 \leq \tilde{k} \leq n-2$ and $\tilde{k} \neq k$, there is one independent relation for each basis component amplitude found in [33].

non-MHV amplitudes explicitly. Since we have

$$0 = \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_n^{\text{MHV}}(n-1, n, \gamma, 1) S[\beta|\gamma] A_n^k(1, \beta, n-1, n) \tag{3.10}$$

for any $k > 2$. Plugging the expression of MHV amplitude and using BCJ relations recursively, we obtain [34]

$$0 = \sum_{\beta \in S_{n-3}} \prod_{i=1}^{n-3} [\beta_i | p_{\beta_{i+1}} + p_{\beta_{i+2}} + \dots + p_{\beta_{n-1}} | n] A_n^k(1, \beta, n-1, n), \tag{3.11}$$

where each β is a permutation of $2, \dots, n-2$ and β_i denotes its i -th element.

4 Conclusion and discussions

In this short note, we studied the supersymmetric version of KLT relations, including the $(n-3)!$ symmetric version [21, 28] and the newly discovered $(n-2)!$ symmetric version [22, 23], in the frame of S-matrix program. In this frame, we do not use string theory or the Lagrangian definition of field theory. Besides the well-known principles, we have added only following two assumptions: the validity of BCFW recursion relations and the supersymmetry.

Our main results are two formulae (2.8) and (2.12). The advantage of going to supersymmetric version is now we unified supersymmetric KLT relations [22] and the newly discovered gauge theory identities [23] into one frame and produced more vanishing identities involving the scalars and fermions. As we have discussed at the end of previous section, these new identities imply further reduction of number of helicity basis of various amplitudes. One obvious project is to study how many linearly independent relations we can obtain from these new identities.

The main reason that we can have a unified picture in the frame of S-matrix program is that one extra assumption, i.e., the supersymmetric selection rule, has been added. As we have emphasized in previous sections, the added supersymmetry fixed the interactions among scalars, fermions, gauge bosons and gravitons, thus it is a very strong extra condition. This is the price we need to pay for having the unified picture. Amazingly, for tree-level amplitudes of pure gluons or pure gravitons, results are the same with or without the supersymmetry. This is why we can lift the theory to the supersymmetric version and then infer their properties. In other words, *gluon and graviton know supersymmetry somehow*. For scalars and fermions, with or without supersymmetry will have huge differences, so their discussions will be more difficult.

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