Hamiltonian Theory of a Relativistic Perfect Fluid*

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The velocity-potential version of the hydrodynamics of a relativistic perfect fluid is put into Hamiltonian form by applying Dirac's method to the version's degenerate Lagrangian. There is only one independent momentum, and the Hamiltonian density is $-T_0{}^0(-g^{00})^{-1/2}$. The Einstein equations for a perfect fluid are then put into Hamiltonian form by analog with Arnowitt, Deser, and Misner's vacuum Einstein equations. The Hamiltonian density splits into two pieces, which are the coordinate densities of energy and momentum of the fluid relative to an observer at rest on the hypersurface of constant coordinate time.

INTRODUCTION

The velocity-potential version of perfect-fluid hydrodynamics as formulated by Seliger and Whitham,[1] generalized to relativity by Schutz,[2] and independently discovered by Schmid,[3] can be regarded as a nonlinear relativistic field theory for five coupled scalar fields, whose Lagrangian density is simply the pressure of the fluid. The theory is degenerate: Not all the generalized momenta are independent, so they cannot be solved for the generalized velocities. In this paper we use Dirac's[4] algorithm for degenerate theories to cast the equations of perfect-fluid hydrodynamics into Hamiltonian form, whose Hamiltonian density is the energy density of the fluid. We then match the theory to the Arnowitt-Deser-Misner[5,6] (hereafter referred to as ADM) canonical theory for the vacuum gravitational field.

The independent variables of the theory are the velocity potentials: five scalar fields ϕ, α, β, θ, and S. Here S is the entropy per baryon, while the others have less obvious interpretations.[2,3(a)] The fluid's four-velocity is a combination of the potentials and their gradients,[7]

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}), \tag{1}$$

where μ is the specific enthalpy of the fluid,

$$\mu = (\rho + p)/\rho_0. \tag{2}$$

(Here ρ_0 is the rest-mass density, ρ is the density of total mass-energy, and p is the pressure.) Through the equation of state,

$$p = p(\mu, S), \tag{3}$$

all thermodynamic quantities are expressed in terms of S (one of the velocity potentials) and μ. In its turn, μ is a function of all the velocity potentials through the equation

$$\mu^2 = -g^{\sigma\nu}(\phi_{,\sigma} + \alpha\beta_{,\sigma} + \theta S_{,\sigma})(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}), \tag{4}$$

which is just the normalization constraint on the four-velocity.

The dynamical field equations are five coupled nonlinear first-order equations,

$$U^\nu \phi_{,\nu} = -\mu, \tag{5a}$$

$$U^\nu \alpha_{,\nu} = 0, \tag{5b}$$

$$U^\nu \beta_{,\nu} = 0, \tag{5c}$$

$$U^\nu \theta_{,\nu} = T, \tag{5d}$$

$$U^\nu S_{,\nu} = 0, \tag{5e}$$

(where T is the temperature) plus one nonlinear second-order equation,

$$(\rho_0 U^\nu)_{;\nu} = 0. \tag{6}$$

There are really only two independent equations among the three Eqs. (5a), (5c), and (5e) because of Eq. (4), so that there are five independent equations altogether.

These equations follow from extremizing the action,

$$I = \int p\sqrt{-g}\, d^4x. \tag{7}$$

First-order changes in p are computed from the equation

$$\delta p = \rho_0 \delta\mu - \rho_0 T \delta S,$$

which expresses the first law of thermodynamics. Equation (4) is used to obtain $\delta\mu$ in terms of the independent variations of ϕ, α, β, θ, and S.

When one formulates these equations in terms of a Hamiltonian, one singles out the time coordinate for special attention, thereby destroying the equations' four-dimensional symmetry. In what follows we will therefore use the ADM notation appropriate to such a (3+1)-dimensional split of spacetime: The four-dimensional metric ${}^4g_{\alpha\beta}$ is replaced by the three-dimensional metric $g_{ij} = {}^4g_{ij}$ (whose inverse is $g^{ij} \neq {}^4g^{ij}$), by the lapse function $N = (-{}^4g^{00})^{-1/2}$, and by the shift functions $N_i = {}^4g_{0i}$.

Derivatives covariant with respect to g_{ij} are denoted by ∇_i or by a subscripted slash (e.g., $h_{ij|k}$). Dots (e.g., $\dot{h}_{ij}$) denote partial derivatives in time.

The action (7) becomes

$$I=\int pNg^{1/2}\,d^3x\,dt\,,$$

so the Lagrangian density of the fluid is $L=pNg^{1/2}$. In all but the last section of this paper, we will treat the metric ${}^4g_{\alpha\beta}$ as a constant, not as part of the dynamics of the fluid. It will suffice until then to take as the fluid Lagrangian density

$$L=pN\,, \tag{8}$$

so that the action can be written in the standard way,

$$I=\int Ld(\text{three-volume})\,dt\,.$$

CONSTRAINTS ON THE MOMENTA

Let q_a stand for the five fields ϕ, α, β, θ, and S. The momenta conjugate to q_a are

$$p^a\equiv\frac{\partial L}{\partial\dot{q}_a}=\frac{\partial pN}{\partial\dot{q}_a}\,. \tag{9}$$

They are explicitly

$$\begin{aligned}
&p^\phi=-\rho_0U^0N\,,\\
&p^\alpha=p^\theta=0\,,\\
&p^\beta=\alpha p^\phi\,,\\
&p^S=\theta p^\phi\,.
\end{aligned} \tag{10}$$

Since only one momentum is independent, there are four constraints on the momenta (the Dirac φ equations),

$$\begin{aligned}
&\varphi_1=p^\alpha=0\,,\\
&\varphi_2=p^\theta=0\,,\\
&\varphi_3=p^\beta-\alpha p^\phi=0\,,\\
&\varphi_4=p^S-\theta p^\phi=0\,.
\end{aligned} \tag{11}$$

There are no arbitrary functions of time in velocity-potential hydrodynamics: What gauge freedom exists lies only in the choice of *initial values* for the potentials. Consequently, we do not expect any of these φ's to be first-class: None of them has a vanishing Poisson bracket [see Eq. (16)] with all the others.

That there is only one independent momentum is surprising. One might expect at least three (for the spatial components of velocity), if not more. The mathematical reason seems to be that, of all the field equations, only Eq. (6) is second order in time derivatives. Equation (6) is obtained by varying ϕ in the Lagrangian, and p^ϕ is the only independent momentum.

The physical reason (if one exists) that there is only one independent momentum is not clear. It would be a mistake to conclude that a perfect fluid has only one dynamical "degree of freedom": that such constraints as zero viscosity and conservation of entropy have wiped out the other degrees of freedom. The relationship between independent momenta and degrees of freedom is not well understood. In the velocity-potential representation one must specify six independent functions on an initial Cauchy hypersurface in order to determine the future evolution of the perfect fluid.[2] This indicates the existence of three dynamical degrees of freedom.

What seems to be the case here is that two of the three second-order dynamical equations (one for each component of velocity) have been replaced by four first-order equations [the four independent equations among Eqs. (5)]. Hidden among the four potentials α, β, θ, and S are two dynamical variables and their momenta. Since all four are treated as coordinates here, they appear to have no independent momenta among them.

There are some tantalizing suggestions that this may be just the hint of a deeper canonical relationship among the potentials. Seliger and Whitham[1] show that one can modify the formalism slightly and introduce a function $\mathcal{H}$ such that $d\alpha/d\tau=\partial\mathcal{H}/\partial\beta$ and $d\beta/d\tau=-\partial\mathcal{H}/\partial\alpha$. Moreover, Schmid[3(a)] points out that ϕ obeys the relativistic Hamilton-Jacobi equation

$$-g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}+\epsilon^2=\mu^2\,,$$

where

$$\epsilon^2=g^{\alpha\beta}(\alpha\beta_{,\alpha}+\theta S_{,\alpha})(\alpha\beta_{,\beta}+\theta S_{,\beta})$$

is positive definite because the vector $\alpha\beta_{,\alpha}+\theta S_{,\alpha}$ is spacelike (it is orthogonal to U^α). We have nothing more to add to these considerations here, so we return to the Dirac method.

THE HAMILTONIAN AND THE EQUATIONS OF MOTION

The Hamiltonian density is defined in the conventional way,

$$H=\sum_a p^a\dot{q}_a-L \tag{12a}$$

$$=p^\phi(\dot{\phi}+\alpha\dot{\beta}+\theta\dot{S})-pN \tag{12b}$$

$$=-T^0{}_0N\,. \tag{12c}$$

Although $\dot{\phi}$, $\dot{\beta}$, and $\dot{S}$ appear explicitly in H, we still have $(\partial H/\partial\dot{q}_a)_{p,q}=0$, so that we can differentiate H with respect to p^a and q_a while holding $\dot{q}_a$ constant.

Because of the φ equations one cannot solve for all the $\dot{q}_a$'s in terms of p^a's. Instead, one introduces additional variables λ_a (which Dirac[4] calls

u_a) in place of the $\dot{q}_a$'s. If one varies Eq. (12a) with respect to q_a and p^a, the λ_a's serve as Lagrange multipliers which ensure that variations in the q_a's and p^a's maintain the φ equations. Then one gets

$$\dot{q}_a = \frac{\partial H}{\partial p^a} + \lambda_m \frac{\partial \varphi_m}{\partial p^a}, \tag{13}$$

$$-\frac{\partial L}{\partial q_a} = \frac{\partial H}{\partial q_a} + \lambda_m \frac{\partial \varphi_m}{\partial q_a}. \tag{14}$$

(A sum on m from 1 to 4 is implied here and throughout.) For the perfect fluid, Eqs. (13) can be solved for the λ_m's to give

$$\lambda_1 = \dot{\alpha}, \quad \lambda_2 = \dot{\theta}, \quad \lambda_3 = \dot{\beta}, \quad \lambda_4 = \dot{S}. \tag{15}$$

Thus in this case the λ's are self-consistent: Equation (14) implies nothing new. So the Hamiltonian variables now are $p^\phi, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \phi, \alpha, \beta, \theta, S$.

The power of the Dirac approach is that the Poisson-bracket version of Hamilton's equations,

$$\dot{q} = [q, H],$$

$$\dot{p} = [p, H],$$

can easily be generalized to the degenerate case. Before applying this to fluids, however, we must define a Poisson bracket for fields in a curved three-dimensional space. The conventional definition from particle dynamics,

$$[A, B] = \sum_a \left(\frac{\partial A}{\partial q_a}\frac{\partial B}{\partial p^a} - \frac{\partial A}{\partial p^a}\frac{\partial B}{\partial q_a}\right),$$

is not sufficient when A and B are functions of the spatial derivatives of the fields q_a and p^a. In the Appendix we generalize this definition to fields. For the perfect fluid (five scalar fields) the result is

$$[A, B] = \sum_{a=1}^{5}\left(\frac{\partial A}{\partial q_a}\frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a}\frac{\delta B}{\delta q_a} + \frac{\partial A}{\partial q_{a,i}}\frac{\partial}{\partial x^i}\frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a{}_{,i}}\frac{\partial}{\partial x^i}\frac{\delta B}{\delta q_a} + \frac{\partial A}{\partial q_{a,i|j}}\nabla_j\frac{\partial}{\partial x^i}\frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a{}_{,i|j}}\nabla_j\frac{\partial}{\partial x^i}\frac{\delta B}{\delta q_a} + \cdots \right), \tag{16}$$

where A and B are any functions of q_a, p^a, and their spatial derivatives (of any order), and where $\delta B/\delta q_a$ is the *spatial* variational derivative

$$\frac{\delta B}{\delta q_a} = \frac{\partial B}{\partial q_a} - \nabla_i \frac{\partial B}{\partial q_{a,i}} + \nabla_j \nabla_i \frac{\partial B}{\partial q_{a,i|j}} - \cdots. \tag{17}$$

In the Poisson bracket all q's and p's are treated as if they were independent: The φ equations are used only *after* the Poisson bracket has been computed.

Dirac[8] shows that the time derivative of any function f of the q's, p's, and their spatial derivatives (and possibly explicitly of time) can be expressed in the form

$$\dot{f} = [f, H] + [f, \lambda_m \varphi_m] + \left(\frac{\partial f}{\partial t}\right)_{p,q}. \tag{18}$$

In the second term, one is to regard λ_m as independent of q_a and p^a, but dependent upon position. For example, one contribution to that term will be from a term like

$$\frac{\delta \lambda_m \varphi_m}{\delta p^a} = \frac{\partial \lambda_m \varphi_m}{\partial p^a} - \nabla_i \frac{\partial \lambda_m \varphi_m}{\partial p^a{}_{|i}} + \cdots$$

$$= \lambda_m \frac{\partial \varphi_m}{\partial p^a} - \nabla_i\left(\lambda_m \frac{\partial \varphi_m}{\partial p^a{}_{|i}}\right) + \cdots.$$

In Eq. (18) one must treat H as a function only of the *independent* momenta [cf. Eq. (12b)]. Contributions to $\dot{f}$ from the other momenta come from the φ brackets. Equation (18) can be stated concisely as

$$\dot{f} = [f, H'] + \frac{\partial f}{\partial t} \tag{19}$$

by defining a generalized Hamiltonian,

$$H' = H + \lambda_m \varphi_m. \tag{20}$$

Because the φ's are all zero, H' is numerically equal to H.

The equations of motion are a special case of Eq. (19),

$$\begin{aligned} \dot{p}^\phi &= (\rho_0 N U^i)_{|i}, \\ \dot{p}^\alpha &= -\rho_0 U^\nu \beta_{,\nu} N, \\ \dot{p}^\beta &= (\rho_0 U^i \alpha N)_{|i}, \\ \dot{p}^\theta &= -\rho_0 U^\nu S_{,\nu} N, \\ \dot{p}^S &= (\rho_0 U^i \theta N)_{|i} - \rho_0 T N. \end{aligned} \tag{21}$$

The first is the continuity equation, Eq. (6). Upon application of the φ equations and the continuity equation, we see that the remaining four equations are the four independent velocity-potential equations among Eqs. (5).

One must also demand that the φ equations be maintained in time, i.e., that

$$[\varphi_m, H'] = 0. \tag{22}$$

These equations are just the four independent velocity-potential equations, Eqs. (5): There are no Dirac χ equations; i.e., there are no equations from Eq. (22) that involve p's and q's without λ's or $\dot{q}$'s, which would thus be constraints like the φ equations. For example, φ_1 – the constraint on p^α – is preserved at zero by the equation $U^\nu \beta_{,\nu}=0$, which is obtained from the original variational principle by varying α. This equation can be rearranged to read

$$\lambda_3 = N^i \beta_{,i} + \frac{\rho_0 N}{\mu p^\phi} g^{ij}\beta_{,i}(\phi_{,j}+\alpha\beta_{,j}+\theta S_{,j}) . \qquad (23)$$

This is not really solved for λ_3 in terms of p^ϕ and q_a because ρ_0 and μ on the right-hand side implicitly depend on all the λ's. Nevertheless, all the λ's do have *unique* solutions (through the velocity-potential equations) in terms of p^ϕ and q_a. This means that there are no first-class φ equations (as we guessed earlier) and no arbitrary functions of time in the solutions.

COUPLING TO GRAVITY

Until now we have treated the metric tensor $g_{\alpha\beta}$ as a constant because we were interested in the canonical theory of the fluid. The fluid is, however, coupled to the gravitational field, and one ought to treat the full dynamical system, fluid plus field.

The Hamiltonian density of the free gravitational field is[5,6]

$$H_G = NR^0 + N_i R^i , \qquad (24)$$

with[5]

$$R^i = -2\pi^{ij}{}_{|j} , \qquad (25)$$

$$R^0 = -g^{1/2}[{}^3R + g^{-1}(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})] , \qquad (26)$$

and

$$\pi^{ij} = N g^{1/2}({}^4\Gamma^0{}_{kl} - g_{kl}\,{}^4\Gamma^0{}_{mn}g^{mn})g^{ik}g^{jl} . \qquad (27)$$

Here ${}^3R$ is the scalar curvature of the hypersurface, and π^{ij} is the momentum canonical to g_{ij}. Since the Lagrangian density of the fluid, $pNg^{1/2}$, does not depend upon time derivatives of the metric, the full Hamiltonian is

$$\mathcal{H} = H_G + 16\pi H' g^{1/2} . \qquad (28)$$

Note that H_G splits into two pieces, with R^0 and R^i independent of N and N_i. Dirac[6] shows that this will also be true of the Hamiltonian density for *any* field. In our case, we split up $H'g^{1/2}$ in two steps: (i) Differentiate with respect to ${}^4g_{\mu\nu}$ while holding p_a and q_a constant,

$$\frac{\partial H' g^{1/2}}{\partial\, {}^4g_{\mu\nu}} = -\frac{\partial p(-{}^4g^{1/2})}{\partial\, {}^4g_{\mu\nu}} = -\tfrac{1}{2}T^{\mu\nu}(-{}^4g)^{1/2} ,$$

and (ii) convert derivatives with respect to ${}^4g_{\mu\nu}$ to derivatives with respect to N, N_i, g_{ij} with the formulas given by Schutz.[9] We obtain

$$\frac{\partial H' g^{1/2}}{\partial N_i} = -g^{1/2}N(T^{0i}+N^iT^{00})$$

$$= -g^{1/2}Ng^{ij}T^0{}_j \qquad (29)$$

$$= g^{1/2}p^\phi g^{ij}(\phi_{,j}+\alpha\beta_{,j}+\theta S_{,j}) , \qquad (30)$$

and

$$\frac{\partial H' g^{1/2}}{\partial N} = g^{1/2}N^2T^{00}$$

$$= g^{1/2}N^2\,{}^4g^{0\mu}T^0{}_\mu \qquad (31)$$

$$= \frac{H}{N}g^{1/2} - \frac{N_i}{N}\frac{\partial H' g^{1/2}}{\partial N_i} . \qquad (32)$$

Equation (32) implies

$$H' = N\frac{\partial H'}{\partial N} + N_i\frac{\partial H'}{\partial N_i} + \lambda_m\varphi_m . \qquad (33)$$

Since $\partial(H'g^{1/2})/\partial N_i$ is manifestly independent of N and N_i, differentiation of Eq. (33) shows that $\partial(H'g^{1/2})/\partial N$ is also independent of N and N_i.

The two pieces of H' have straightforward physical interpretations, as is shown by Schutz.[9] Let $\eta^\alpha = -N\,{}^4g^{0\alpha}$ be the unit normal to the spacelike hypersurface. Then the two pieces of $H'g^{1/2}$ are

$$\frac{\partial H' g^{1/2}}{\partial N} = g^{1/2}\eta^\alpha\eta^\beta T_{\alpha\beta} \equiv \mathcal{E} \qquad (34)$$

and

$$\frac{\partial H' g^{1/2}}{\partial N_i} = g^{1/2}g^{ij}\eta^\alpha T_{\alpha j} \equiv \mathcal{P}^i . \qquad (35)$$

They are, respectively, the coordinate densities of energy and momentum measured by an observer at rest in the hypersurface.

By analogy with Eq. (16) we may define a general Poisson bracket for any two functions of π^{ij}, g_{ij}, q_a, p^a, and their spatial derivatives (but not of N or N_i, which are arbitrary functions that contain coordinate information but have no dynamical content),

$$[A,B] = \sum_{i,j}\left(\frac{\partial A}{\partial g_{ij}}\frac{\delta B}{\delta\pi^{ij}} - \frac{\partial A}{\partial\pi^{ij}}\frac{\delta B}{\delta g_{ij}} + \frac{\partial A}{\partial g_{ij|k}}\nabla_k\frac{\delta B}{\delta\pi^{ij}} - \frac{\partial A}{\partial\pi^{ij}{}_{|k}}\nabla_k\frac{\delta B}{\delta g_{ij}} + \cdots\right)$$
$$+ \sum_a\left(\frac{\partial A}{\partial q_a}\frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a}\frac{\delta B}{\delta q_a} + \frac{\partial A}{\partial q_{a,i}}\frac{\partial}{\partial x^i}\frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a{}_{,i}}\frac{\partial}{\partial x^i}\frac{\delta B}{\delta q_a} + \cdots\right) . \qquad (36)$$

Then the time derivative of any such function that does not depend *explicitly* on time is

$$\dot{A} = [A, \mathcal{H}] \quad (37)$$

$$= [A, NR^0 + 16\pi N\mathcal{E}] + [A, N_i R^i + 16\pi N_i \mathcal{P}^i] + [A, \lambda_m \varphi_m] . \quad (38)$$

In particular, the ADM form of the Einstein field equations follows by using g_{ij} and π^{ij} for A,

$$\dot{g}_{ij} = \dot{g}_{ij}(\text{vac}), \quad (39)$$

$$\dot{\pi}^{ij} = \dot{\pi}^{ij}(\text{vac}) + 8\pi N g^{1/2}(T^{ij} - N^i N^j T^{00}), \quad (40)$$

where (vac) indicates the terms that are there in the vacuum case (see ADM). These must be supplemented by the constraint equations that come from varying the Lagrangian density ($\mathcal{L} = -\mathcal{H} + \pi^{ij}\dot{g}_{ij} + \sum_a p^a \dot{q}_a$) with respect to N and N_i (which are not Hamiltonian variables),

$$R^0 + 16\pi\mathcal{E} = 0, \quad (41)$$

$$R^i + 16\pi\mathcal{P}^i = 0. \quad (42)$$

Equations (39)–(42) are identical to those derived by Schutz[9] for a general stress-energy tensor.

The constraints, Eqs. (41) and (42), must be maintained in time; i.e., we must have

$$[R^0 + 16\pi\mathcal{E}, \mathcal{H}] = 0 \quad (43)$$

and

$$[R^i + 16\pi\mathcal{P}^i, \mathcal{H}] = 0. \quad (44)$$

In the vacuum case these are the Bianchi identities. In our case the Bianchi identities reduce these to the equations of motion, $T^{\mu\nu}{}_{;\nu} = 0$. These four equations can be used to replace the four independent velocity-potential equations among Eqs. (5),[10] which themselves guaranteed the maintenance of the φ equations. Therefore, the full canonical set of equations is

$$\dot{g}_{ij} = [g_{ij}, \mathcal{H}], \quad (45a)$$

$$\dot{\pi}^{ij} = [\pi^{ij}, \mathcal{H}], \quad (45b)$$

$$\dot{\phi} = [\phi, \mathcal{H}], \quad (45c)$$

$$\dot{p}^{\phi} = [p^{\phi}, \mathcal{H}], \quad (45d)$$

with *either* the constraints (41) and (42) (maintained in time) *or* the φ equations (11) (also maintained in time).

CONCLUDING REMARKS

The direction of any further analysis of these equations must depend upon the application they are intended for. It would in principle be possible to reduce the 12 gravitational variables (π^{ij} and g_{ij}) to four in exactly the same manner as ADM. Solving the constraint equations would then involve the fluid variables ϕ and p^{ϕ}, but the coordinate conditions would be unaltered (as was pointed out by ADM).

Methods very similar to these have been used by the author to derive the Hamiltonian density and from it a conserved energy density for the pulsations of and gravitational radiation from a differentially rotating relativistic star. These results will be published elsewhere.

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APPENDIX: POISSON BRACKETS FOR FIELDS IN CURVED SPACES

For a system with n degrees of freedom, the Poisson bracket (P.b.) of two functions of p^a and q_a is

$$[A, B] = \sum_{a=1}^{n} \left(\frac{\partial A}{\partial q_a}\frac{\partial B}{\partial p^a} - \frac{\partial A}{\partial p^a}\frac{\partial B}{\partial q_a} \right). \quad (46)$$

A classical field has an infinite number of degrees of freedom, one (or more) for each point in space. Functions like A and B may be functions not only of the fields p^a and q_a, but also of their spatial derivatives. In this case, a simple definition like Eq. (46) above is not sufficient.

Let us suppose that the field variable is a vector field q_i with canonical momentum $p^i \equiv \partial L/\partial \dot{q}_i$. Our results can be extended in a straightforward manner to cases where the field is a higher-rank tensor or a scalar.

Because the field variables at different points are independent, we wish the P.b. of two functions to be nonzero only if they are evaluated at the same point. Accordingly we define the canonical P.b.'s,

$$[q_i(\vec{x}), q_j(\vec{x}')] = [p^i(\vec{x}), p^j(\vec{x}')] = 0, \quad (47a)$$

$$[q_i(\vec{x}), p^j(\vec{x}')] = -[p^j(\vec{x}'), q_i(\vec{x})] = -\Omega^{j'}{}_i \delta^3(\vec{x} - \vec{x}'). \quad (47b)$$

Here $\Omega^{j'}{}_i$ is the derivative of Synge's *world function*[11] $\Omega(\vec{x}, \vec{x}')$ with respect to x^i and x'^j, with the index j' raised by the metric at $\vec{x}'$. Because of the δ function the only properties of $\Omega^{j'}{}_i$ that we will need are[11] (1) its limit as $\vec{x}'$ approaches $\vec{x}$,

$$\lim_{\vec{x}' \to \vec{x}} \Omega^{j'}{}_i = -\delta^{j'}{}_i, \quad (48a)$$

and (2) the same limit of its covariant derivatives,

$$\lim_{\vec{x}'\to\vec{x}} \nabla_{k'}\Omega^{j'}{}_i = -\lim_{\vec{x}'\to\vec{x}} \nabla_k \Omega^{j'}{}_i , \tag{48b}$$

where $\nabla_{k'}$ is a covariant derivative at $\vec{x}'$ and acts only on primed indices, and vice versa for ∇_k.

The δ function is normalized to proper volume,

$$\int \delta^3(\vec{y}) g^{1/2} d^3y = 1 , \tag{49a}$$

and has the usual property

$$\frac{\partial}{\partial x^i} \delta^3(\vec{x}-\vec{x}') = -\frac{\partial}{\partial x'^i}\delta^3(\vec{x}-\vec{x}') . \tag{49b}$$

Equation (48b) permits us to generalize Eq. (49b) to covariant differentiation,

$$\nabla_k[\Omega^{j'}{}_i \delta^3(\vec{x}-\vec{x}')] = -\nabla_{k'}[\Omega^{j'}{}_i\delta^3(\vec{x}-\vec{x}')] . \tag{50}$$

We define the differentiated canonical P.b.'s,

$$[q_i(\vec{x}), \nabla_{k'}p^j(\vec{x}')] = -\nabla_{k'}[\Omega^{j'}{}_i\delta^3(\vec{x}-\vec{x}')] , \tag{51a}$$

$$[\nabla_k q_i(\vec{x}), p^j(\vec{x}')] = -\nabla_k[\Omega^{j'}{}_i\delta^3(\vec{x}-\vec{x}')] , \tag{51b}$$

and so on for higher derivatives. *The Poisson bracket* $[\ ,\ ]$ *is the bilinear antisymmetric two-point differential operator whose domain is all* C^1 *functions of* p^i, q_i, *and their covariant derivatives and which obeys relations (47) and (51).*

By application of the chain rule we find

$$[A(\vec{x}), B(\vec{x}')] = \frac{\partial A}{\partial q_i}(\vec{x})[q_i(\vec{x}), p^j(\vec{x}')]\frac{\partial B}{\partial p^j}(\vec{x}')$$
$$+\frac{\partial A}{\partial p^i}(\vec{x})[p^i(\vec{x}), q_j(\vec{x}')]\frac{\partial B}{\partial q_j}(\vec{x}') + \frac{\partial A}{\partial q_{i|k}}(\vec{x})[q_{i|k}(\vec{x}), p^j(\vec{x}')]\frac{\partial B}{\partial p^j}(\vec{x}') + \cdots , \tag{52}$$

$$= \left[\frac{\partial A}{\partial q_i}(\vec{x})\frac{\partial B}{\partial p^i}(\vec{x}') - \frac{\partial A}{\partial p^i}(\vec{x})\frac{\partial B}{\partial q_i}(\vec{x}')\right]\delta^3(\vec{x}-\vec{x}')$$
$$-\frac{\partial A}{\partial q_{i|k}}(\vec{x})\frac{\partial B}{\partial p^j}(\vec{x}')\nabla_k[\Omega^{j'}{}_i\delta^3(\vec{x}-\vec{x}')] + \frac{\partial A}{\partial p^i{}_{|k}}(\vec{x})\frac{\partial B}{\partial q_j}(\vec{x}')\nabla_k[\Omega^i{}_{j'}\delta^3(\vec{x}-\vec{x}')]$$
$$-\frac{\partial A}{\partial q_i}(\vec{x})\frac{\partial B}{\partial p^j{}_{|k}}(\vec{x}')\nabla_{k'}[\Omega^{j'}{}_i\delta^3(\vec{x}-\vec{x}')] + \frac{\partial A}{\partial p^i}(\vec{x})\frac{\partial B}{\partial q_{j|k}}(\vec{x}')\nabla_{k'}[\Omega^i{}_{j'}\delta^3(\vec{x}-\vec{x}')] + \cdots . \tag{53}$$

This is the usual definition of a Poisson bracket in classical field theories. But for the purpose of practical calculations it is useful to obtain a one-point P.b. by integrating. *The left- (right-) integrated P.b. is the integral of the P.b. over all* $\vec{x}'$ ($\vec{x}$). We denote these by $_{\vec{x}}[\ ,\ ]$ and $[\ ,\ ]_{\vec{x}}$, respectively. Integrating Eq. (53) on $\vec{x}'$ and using Eq. (50) gives

$$_{\vec{x}}[A, B] = \int_{\text{all space}}[A(\vec{x}), B(\vec{x}')](g')^{1/2} d^3\vec{x}' \tag{54}$$

$$= \frac{\partial A}{\partial q_i}\frac{\delta B}{\delta p^i} - \frac{\partial A}{\partial p^i}\frac{\delta B}{\delta q_i}$$
$$+ \frac{\partial A}{\partial q_{i|k}}\nabla_k\frac{\delta B}{\delta p^i} - \frac{\partial A}{\partial p^i{}_{|k}}\nabla_k\frac{\delta B}{\delta q_i} + \cdots , \tag{55}$$

where $\delta B/\delta q_i$ is the variational derivative

$$\frac{\delta B}{\delta q_i} = \frac{\partial B}{\partial q_i} - \nabla_k\frac{\partial B}{\partial q_{i|k}} + \nabla_j\nabla_k\frac{\partial B}{\partial q_{i|k|j}} - \cdots . \tag{56}$$

Although one cannot generally integrate a tensor over a curved space, as we have done in Eq. (54), in this case the δ function limits the integration to only one point, so that the integral is unambiguous.

This integrated P.b. is the generalization of the simple P.b., Eq. (46), to which it reduces when neither A nor B depends upon derivatives of q_i and p^i. When such derivatives are involved, the left-integrated P.b. is the P.b. of A at the point $\vec{x}$ with the entire field B: Values of B at other points influence the bracket through the spatial derivatives of B at $\vec{x}$. Note also that the integrated P.b.'s are independent of any coordinate system.

The following interesting properties follow directly from the definition of the integrated brackets:

1. $_{\vec{x}}[A, B] = -[B, A]_{\vec{x}}$. (57a)

2. $_{\vec{x}}[A, B] = -{}_{\vec{x}}[B, A]$ (57b)

if and only if both A and B are independent of derivatives of q_i and p^i.

3. $$\int_{\text{all space}} {}_{\vec{x}}[A, B] g^{1/2} d^3x = \int_{\text{all space}}[A, B]_{\vec{x}} g^{1/2} d^3x \tag{57c}$$

if $_{\vec{x}}[A, B]$ is a scalar (if not, the integrals are undefined).

4. $\nabla_i\, {}_{\vec{x}}[A, B] = {}_{\vec{x}}[\nabla_i A, B]$, (57d)

$\nabla_i[A, B]_{\vec{x}} = [A, \nabla_i B]_{\vec{x}}$. (57e)

The integrated brackets fit into the Hamiltonian theory because the canonical equations are (for a system whose momenta are all independent)

$$\dot{q}_i = \delta H/\delta p^i \,, \tag{58a}$$

$$\dot{p}^i = \delta H/\delta q_i \,. \tag{58b}$$

They translate to (from now on we will use only the left-integrated brackets)

$$\dot{q}_i = {}_{\vec{x}}[q_i, H] \,, \tag{59a}$$

$$\dot{p}^i = {}_{\vec{x}}[p^i, H] \,. \tag{59b}$$

By property 4 above these imply

$$\dot{q}_{i|k} = {}_{\vec{x}}[q_{i|k}, H] \,, \tag{60a}$$

$$\dot{p}^i{}_{|k} = {}_{\vec{x}}[p^i{}_{|k}, H] \,, \tag{60b}$$

which in turn imply

$$\dot{A} = {}_{\vec{x}}[A, H] \tag{61}$$

for any function A (not necessarily a scalar) of q_i, p^i, and their spatial derivatives that does not explicitly depend on time.

Property 2 implies that in general $\dot{H} \neq 0$. This is to be expected: Energy can be transferred from point to point. We should only expect that

$$\int_{\text{all space}} \dot{H} g^{1/2} d^3x = 0 \,, \tag{62}$$

which is true because of properties 3 and 1. Thus, in general, there exists a canonical Poynting vector S^i such that

$$\dot{H} + \nabla_i S^i = 0 \,.$$

For the simple case where H depends on no derivatives of q_i and p^i higher than first order (which includes almost all physical systems), the Poynting vector is

$$S^i = \frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p^j{}_{|i}} - \frac{\partial H}{\partial p^j} \frac{\partial H}{\partial q_{j|i}} \,. \tag{63}$$

For a degenerate system (momenta not all independent) the equations of motion are almost as simple. Dirac[4] shows that for a system with a finite number of degrees of freedom,

$$\dot{q}_i = [q_i, H] + \lambda_m [q_i, \varphi_m] \,, \tag{64a}$$

$$\dot{p}^i = [p^i, H] + \lambda_m [p^i, \varphi_m] \,. \tag{64b}$$

For a degenerate field theory these become

$$\dot{q}_i = {}_{\vec{x}}[q_i, H] + {}_{\vec{x}}[q_i, \lambda_m \varphi_m] \,, \tag{65a}$$

$$\dot{p}^i = {}_{\vec{x}}[p^i, H] + {}_{\vec{x}}[p^i, \lambda_m \varphi_m] \,. \tag{65b}$$

In these equations λ_m appears inside the integrated bracket because it is generally a function of position. To compute a bracket that has λ_m inside, one treats λ_m as a function of $\vec{x}$ independent of p^i and q_i. For example, the variational derivative of Eq. (56) is

$$\frac{\delta \lambda_m \varphi_m}{\delta q_i} = \frac{\partial \lambda_m \varphi_m}{\partial q_i} - \nabla_k \frac{\partial \lambda_m \varphi_m}{\partial q_{i|k}} + \cdots$$
$$= \lambda_m \frac{\partial \varphi_m}{\partial q_i} - \nabla_k \left(\lambda_m \frac{\partial \varphi_m}{\partial q_{i|k}} \right) + \cdots \,. \tag{66}$$

Conservation laws for the Hamiltonian can be derived here too. They are especially simple in the case where H depends on no derivatives of p^i and only first derivatives of q_i, and where φ_m is independent of any derivatives. The equation maintaining the φ equations is

$$\dot{\varphi}_n = {}_{\vec{x}}[\varphi_n, H] + {}_{\vec{x}}[\varphi_n, \lambda_m \varphi_m]$$
$$= {}_{\vec{x}}[\varphi_n, H] + \lambda_m \, {}_{\vec{x}}[\varphi_n, \varphi_m] = 0 \,. \tag{67}$$

The time derivative of H is

$$\dot{H} = {}_{\vec{x}}[H, H] + {}_{\vec{x}}[H, \lambda_m \varphi_m] \,.$$

Using the properties of the integrated bracket, our assumptions about H and φ_m, and Eq. (67), we can show that this becomes

$$\dot{H} + \nabla_i S^i = 0 \,, \tag{68}$$

with

$$S^i = -\left(\frac{\partial H}{\partial p^j} + \lambda_m \frac{\partial \varphi_m}{\partial p^j} \right) \frac{\partial H}{\partial q_{j|i}} \,. \tag{69}$$

But by Eq. (65a) this is just

$$S^i = -\dot{q}_j \frac{\partial H}{\partial q_{j|i}} \,, \tag{70}$$

which is the canonical flux in the nondegenerate case as well.

In the body of this paper we will consistently use the left-integrated Poisson bracket, which we refer to simply as the Poisson bracket, denoted by [,].

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[6]A similar formulation of Einstein's equations, though not as complete in its discussion, can be found in P. A. M. Dirac, Proc. Roy. Soc. (London) A246, 333 (1958).

[7]We follow the notation of Schutz, Ref. 2. In particular, Greek indices run from 0 to 3, while Latin indices run from 1 to 3, with these exceptions: a and b label the velocity potentials and run from 1 to 5, while m and n label the Dirac φ equations and run from 1 to 4.

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New Equation of Motion for Classical Charged Particles*

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With the intuitive new ideas that (1) in classical electrodynamics, radiation reaction should be expressible by the external field and the charge's kinematics, (2) a charge experiences, in addition to the Lorentz forces, another "small" external force $e_1F^{\mu\lambda}\dot{u}_\lambda$ proportional to its acceleration, and (3) inertia plus radiation is balanced by these two external forces, we propose the new equation of motion,

$$m\dot{u}^\mu - (2e^3/3m)F^{\lambda\alpha}_{\text{ext}}\dot{u}_\lambda u_\alpha u^\mu = eF^{\mu\lambda}_{\text{ext}}u_\lambda + e_1F^{\mu\lambda}_{\text{ext}}\dot{u}_\lambda\ ,$$

where mass conservation requires $e_1 = 2e^3/3m$. (The particle's spin is not considered in this work.) This equation for a classical charge is free from all the well-known difficulties of the Lorentz-Dirac equation. It conserves energy and momentum in a modified form in which the energy-momentum tensor contains a part $t^{\mu\nu}(x)$ made of a new field-charge interaction $\phi^\mu(x)$, in addition to the conventional "local" part made of $F^{\mu\nu}_{\text{ret}}(x)$ and $F^{\mu\nu}_{\text{ext}}(x)$ only, and therefore it no longer satisfies the conventional "local" conservation laws. It predicts correct radiation damping, as demonstrated here by applying it to various cases of basic physical importance. Also, it implies that a massless particle follows a null geodesic and cannot interact with the electromagnetic field whether it be charged or not; this implication may add a new degree of freedom to the charge-conservation law.

I. INTRODUCTION

The equation of motion of a charged particle has been a subject of interest for many years.[1] The equation now generally accepted was obtained by Dirac by decomposing the energy-momentum tensor of the retarded self-field into a sum that renormalizes mass and a difference that gives reaction.[2] An explanation and rederivation based on an absorber mechanism was provided by Wheeler and Feynman.[3] However, as is well recognized, the Lorentz-Dirac equation has certain inherent difficulties. First, it involves the derivative of the acceleration and hence needs one extra condition, in addition to the Newtonian initial conditions, to determine the motion. Second, it gives runaway solutions which can be avoided only by artificially presenting a preacceleration.[4] Third, in certain cases it implies that the external energy supplied to the particle goes only into kinetic energy, and radiation is created from an "acceleration self-energy" which becomes more and more negative and is unphysical. It is the purpose of this work to obtain a new equation that is free from these difficulties, agrees with existing laboratory results, and predicts new phenomena that can distinguish the new equation from the old one and test its validity.

II. THE NEW EQUATION

By following the old idea of expressing the radiation reaction of a charged particle only by its kinematical quantities, it is not possible to construct an equation that includes reaction in a form simpler and more satisfactory than the Lorentz-Dirac one. However, in classical electrodynamics in an inertial frame[5] the only field that can accelerate a charged particle and make it radiate is the external electromagnetic field $F^{\mu\nu}_{\text{ext}}$. Accordingly, radiation reaction should be expressible by $F^{\mu\nu}_{\text{ext}}$