

## LAGRANGIAN PERTURBATION THEORY OF NONRELATIVISTIC FLUIDS\*

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### ABSTRACT

In this paper the conventional description of adiabatic perturbations of stationary fluids in terms of a Lagrangian displacement is reexamined, to take account of certain difficulties that have been overlooked in other treatments. A class of displacements—called trivials—that leave the physical variables unchanged is identified; these define “gauge” transformations of the initial data in the Lagrangian picture. The conserved canonical energy  $E_c$  (Hamiltonian) and angular momentum  $J_c$  (in the case of axisymmetric unperturbed fluids) associated with the dynamical equations are shown not to be invariant under these gauge transformations. Since  $E_c$  has formed the basis of previous criteria for secular stability of stars, it is necessary to eliminate the gauge freedom in order to regain a meaningful criterion. To this end a conserved inner product (the symplectic structure) is introduced and used to define a dynamically invariant class of “canonical” displacements orthogonal to the trivials. In general, canonical displacements obey the extra restriction that the Lagrangian change in  $\rho^{-1}\nabla_s \cdot \nabla \times V$  vanish; in fluids with uniform entropy  $s$  they obey the more restrictive condition that the Lagrangian change in  $\rho^{-1}\nabla \times V$  vanish. Restricting consideration to canonical displacements guarantees that  $E_c$  and  $J_c$  will be invariant under any residual gauge freedom. For nonaxisymmetric perturbations of axisymmetric fluids, to every physical perturbation corresponds a unique canonical displacement.

In an appendix the relationship between  $E_c$  and  $J_c$  and the (gauge-invariant) second-order changes in the total energy and angular momentum of the fluid is derived. Another appendix, dealing with uniformly rotating fluids, reexpresses the gauge-invariant combination  $E_c - \Omega J_c$  in terms of Eulerian changes in the fluid variables. A subsequent paper applies these results to the study of secular instability in stars.

*Subject headings:* hydrodynamics — instabilities — stars: rotation

### I. INTRODUCTION

This and a subsequent paper (Friedman and Schutz 1978, hereafter Paper II) are concerned with fluid perturbation theory, and with secular instability of rotating Newtonian stars. Although the subjects have received the attention of various authors in recent years (Clement 1964; Lynden-Bell and Ostriker 1967; Tassoul and Ostriker 1968; Chandrasekhar 1970; Chandrasekhar and Lebovitz 1968, 1973; Friedman and Schutz 1975; Hunter 1977), several points have been consistently overlooked, and they substantially alter the previously accepted picture. First, stellar perturbations have conventionally been described in terms of a Lagrangian displacement; but Schutz and Sorkin (1977) have recently pointed out the existence of a class of trivial displacements which leave the physical variables unchanged, but which do not leave invariant the functional that governs secular instability to radiation reaction. One must therefore eliminate the trivial displacements in order to regain a meaningful stability criterion.<sup>1</sup> (A special kind of trivial had been pointed out by Lynden-Bell and Ostriker (1967), but these are harmless in that the criterion is unaffected by them.) A second previously unsuspected result is the presence of a generic instability of rotating stars to radiation reaction. That is, we find that in all rotating stars<sup>2</sup> gravitational radiation reaction will excite an instability or marginal instability in nonaxisymmetric modes of the form  $e^{im\phi}$  with  $m$  sufficiently large. As has been pointed out, for example, by Hunter (1977), the criterion for secular instability to viscosity is not identical to that governing instability to radiation reaction. Because the proof of generic instability to radiation reaction does not hold in the case of viscosity, one expects on the basis of our results that stellar models which are unstable against radiation reaction can nevertheless be stable against viscous dissipation. In fact, the work of Lindblom and Detweiler

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<sup>1</sup> The elimination of displacements that do not change the boundary of uniform-density ellipsoids (e.g., Chandrasekhar 1969) is similar in character, although this class of displacements is larger than the trivials.

<sup>2</sup> Technically, we consider rotating axisymmetric perfect fluids having equations of state of the form  $p = p(\rho, s)$ .

(1977) suggests that in cases where the viscous and gravitational radiation time scales are comparable, viscosity may damp out the generic instability.

Physical implications of this work, in particular the modification of instability points obtained by Ostriker, Tassoul, Bodenheimer *et al.*, using tensor virial methods, have already been discussed in a *Letter* (Bardeen *et al.* 1977).

In this first paper we develop a formalism for perturbations of a stationary Newtonian fluid. In § II, a description of fluid perturbations in terms of a Lagrangian displacement is introduced. The class of trivial displacements is identified, and an explicit form obtained for the generic trivial. Section III deals with the formal structure of the perturbation equations. The canonical energy and angular momentum are introduced, together with two related dynamically conserved inner products. We find that the canonical energy and angular momentum do not vanish on trivial displacements. They therefore cannot be identified with the second-order changes in the system's energy and angular momentum, and following Bardeen (1975), we obtain the explicit relation between the canonical and physical conserved quantities. (The details of derivations in this section are in two appendices.) In the final section (IV), the inner products defined in § III are used to characterize a dynamically invariant class of canonical displacements orthogonal to the trivials. In Paper II the formalism developed here is used to treat secular instability of Newtonian stars.

## II. TRIVIAL LAGRANGIAN DISPLACEMENTS

We will consider the perturbations of a stationary perfect fluid with density  $\rho$ , pressure  $p$ , entropy per baryon  $s$ , and fluid velocity  $v^t$ . The equilibrium configuration is to be a solution  $(\rho, p, v^t, s)$  to the equations

$$p = p(\rho, s), \quad (1a)$$

$$\nabla_t(\rho v^t) = 0, \quad (1b)$$

$$v^t \nabla_t s = 0, \quad (1c)$$

and

$$v^j \nabla_j v_i + \frac{1}{\rho} \nabla_i p + \nabla_t \Phi = 0, \quad (1d)$$

where  $\Phi$  is the gravitational potential, defined by the equation

$$\nabla^2 \Phi = 4\pi G \rho. \quad (2)$$

Small perturbations of such a star can be treated in either of two ways. The first is a "macroscopic" point of view: one simply considers changes in fluid variables at a particular point in space. These perturbations, written  $\delta\rho$ ,  $\delta p$ ,  $\delta v^t$ ,  $\delta s$ , are the Eulerian changes. The other approach is "microscopic": one defines a "Lagrangian displacement" vector field  $\xi^t$ , which connects fluid elements in the equilibrium with corresponding ones in the perturbed configuration. One then defines the Lagrangian change in any fluid variable as the change with respect to a frame dragged by  $\xi^t$ . Formally this means that the Lagrangian change  $\Delta Q$  in a quantity  $Q$  is related to the Eulerian change by

$$\Delta Q = \delta Q + \mathcal{L}_\xi Q, \quad (3)$$

where the Lie derivative  $\mathcal{L}_\xi$  has the meaning

$$\mathcal{L}_\xi f \equiv \xi^t \nabla_t f \quad (4a)$$

for scalars  $f$ ,

$$\mathcal{L}_\xi v^t = \xi^j \nabla_j v^t - v^j \nabla_j \xi^t \quad (4b)$$

for contravariant vector fields  $v^t$ , and

$$\mathcal{L}_\xi v_t = \xi^j \nabla_j v_t + v_j \nabla_t \xi^j \quad (4c)$$

for covariant vector fields  $v_t$  (see, e.g., Yano 1955). The Lie derivative with respect to  $\xi^t$  has an easy interpretation: in a coordinate system in which  $\xi^t$  is one of the coordinate basis vectors, the Lie derivative is the ordinary partial derivative with respect to that coordinate. Equations (4) simply enable one to compute it in any coordinate system.

Defined in this way, the Lagrangian change in a vector (or tensor) measures the change in its components with respect to a "Lagrangian frame": the frame (or coordinate system) is embedded in the fluid and dragged along with the fluid by the perturbation. This definition was introduced by Taub (1969) for relativistic fluids and has been used in that context by Carter (1973), by us (Friedman and Schutz 1975), and by Schutz and Sorkin (1977) who studied the resulting formalism in some detail. The definition agrees with that of Chandrasekhar *et al.* and of Lynden-Bell and Ostriker (1967) only for scalars (the latter authors use  $\Delta = \delta + \xi \cdot \nabla$ ). The form of  $\Delta$  given in (3)

is mathematically somewhat more natural,<sup>3</sup> and it simplifies the formalism developed here. In particular, such key results as equations (59) and (70) below take more complicated forms (and might have been more difficult to derive) using the older definition of  $\Delta$ .

The Lagrangian change in the fluid velocity,

$$\Delta v^i = \partial_i \xi^i, \tag{5}$$

follows from the fact that a fluid element with trajectory  $c^i(t)$  in the unperturbed fluid has the perturbed trajectory  $c^i(t) + \xi^i[c(t), t]$ . Conservation of mass is expressed by

$$\Delta \rho = -\rho \nabla_i \xi^i. \tag{6}$$

We suppose that the perturbation is adiabatic, so that

$$\Delta s = 0; \tag{7}$$

and it then follows from the equation of state (1a) that

$$\frac{\Delta p}{p} = \gamma \frac{\Delta \rho}{\rho}, \tag{8}$$

where  $\gamma = (\rho/p)(\partial p/\partial \rho)_s$  is the adiabatic index. Equations (5)–(8) can also be written in terms of Eulerian perturbations, by means of equation (3). We have

$$\delta \rho = -\nabla_i(\rho \xi^i), \tag{9}$$

$$\delta s = -\xi^i \nabla_i s, \tag{10}$$

$$\delta v^i = \partial_i \xi^i + v^j \nabla_j \xi^i - \xi^j \nabla_j v^i, \tag{11}$$

and

$$\delta p = -\gamma p \nabla_i \xi^i - \xi^i \nabla_i p. \tag{12}$$

These expressions are identical to those used by Chandrasekhar *et al.* The Eulerian charge in the gravitational potential  $\delta \Phi$  is then given by

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho = -4\pi G \nabla_j(\rho \xi^j). \tag{13}$$

In this way one writes all variables of the perturbed configuration in terms of the displacement  $\xi^i$ . The perturbed equation of motion for the fluid,

$$\rho \Delta \left[ (\partial_i + v^j \nabla_j) v_i + \frac{1}{\rho} \nabla_i p + \nabla_i \Phi \right] = 0, \tag{14}$$

then takes the form

$$\begin{aligned} 0 &= \rho \partial_i^2 \xi_i + 2\rho v^j \nabla_j \partial_i \xi_i + \rho (v^j \nabla_j)^2 \xi_i - \nabla_i(\gamma p \nabla_j \xi^j) + \nabla_i p \nabla_j \xi^j - \nabla_j p \nabla_i \xi^j + \rho \xi^j \nabla_j \nabla_i \Phi + \rho \nabla_i \delta \Phi \\ &\equiv A_j^i \partial_i^2 \xi^j + B_j^i \partial_i \xi^j + C_j^i \xi^j. \end{aligned} \tag{15}$$

By introducing a Lagrangian displacement, one automatically satisfies the linearized conservation equations

$$\delta(\partial_i s + v^j \nabla_j s) = 0 \tag{16}$$

and

$$\delta[\partial_i \rho + \nabla_j(\rho v^j)] = 0. \tag{17}$$

As a result, one acquires an unconstrained action for the dynamical equation (14) (see § III) and a related Hamiltonian formalism useful for stability theory. Moreover, *any* initial perturbation ( $\delta \rho, \delta p, \delta v^i, \delta s$ ) can be characterized by some displacement  $\xi^i$  via equations (9)–(12), provided that the total mass and entropy of the configurations are unchanged.<sup>4</sup> (In the case of isentropic fluids, one is restricted to isentropic perturbations.)

It is not true, however, that a physical perturbation uniquely determines a displacement  $\xi^i$ . In fact, there is a

<sup>3</sup> This is essentially because the reference frame is "Lagrangian," whereas the older definition refers a Lagrangian perturbation to an Eulerian frame and thereby complicates the mathematical description. The fundamental element underlying Lagrangian perturbation theory is a map between the perturbed and unperturbed stars (see, for example, Appendix B). The Lie derivative is defined only by this map; the derivative  $\xi \cdot \nabla$  requires in addition the Euclidean structure of the space.

<sup>4</sup> These are natural restrictions in the following sense: any perturbation that changes the total mass and entropy can be viewed as a perturbation of a nearby equilibrium having mass  $M + \delta M$  and entropy  $S + \delta S$ . The corresponding perturbation of this new equilibrium leaves its mass and entropy unchanged.

class of trivial displacements  $\eta^i$  for which the corresponding Eulerian changes in  $p$ ,  $\rho$ ,  $v^i$ , and  $s$  all vanish. Two displacements  $\xi^i$  and  $\zeta^i$  then correspond to the same physical perturbation if (and only if) they differ by a trivial displacement,

$$\hat{\xi}^i = \xi^i + \eta^i, \tag{18}$$

where  $\eta^i$  satisfies the equations that result from setting the left-hand sides of equations (9)–(12) to zero, namely

$$\nabla_i(\rho\eta^i) = 0, \tag{19}$$

$$\eta^i \nabla_i s = 0, \tag{20}$$

and

$$\partial_i \eta^i + v^j \nabla_j \eta^i - \eta^j \nabla_j v^i = 0. \tag{21}$$

(Note that when  $\delta\rho$  and  $\delta s$  vanish,  $\delta p$  vanishes as well by virtue of the equation of state (1), or, equivalently, by eq. [12]).

The general solution to equations (19)–(21) is

$$\eta^i = \frac{1}{\rho} \epsilon^{ijkl} \nabla_j s \nabla_k f, \tag{22a}$$

where the function  $f$  is a scalar, constant along fluid trajectories,

$$(\partial_t + \mathcal{L}_v)f = 0, \tag{22b}$$

but otherwise arbitrary. That equation (22) provides the general trivial can be seen in the following way. Equation (20) implies that the vector  $\eta^i$  lies in surfaces of constant  $s$ . Equation (19) can therefore be written in the manner

$$D_i(\rho\chi^{-1}\eta^i) = 0, \tag{23}$$

where  $\chi \equiv (\nabla_i s \nabla^i s)^{1/2}$  and where  $D_i$  is the covariant derivative operator in the  $s$ -constant surfaces. Equation (23) implies that

$$\rho\eta^i = \epsilon^{ijkl} \nabla_j s \nabla_k f \tag{24}$$

(in the language of differential forms,  $\rho\chi^{-1}\eta^i$  is a closed 1-form on  $s = \text{constant}$  surfaces and all closed 1-forms are exact on surfaces with spherical topology). Finally, to see that the time dependence of  $\eta^i$  is necessarily that given by equation (22), it suffices—by uniqueness of solutions to ordinary differential equations—to show that  $\eta^i$  satisfies equation (21), which can be written in the form

$$\partial_t \eta^i + \mathcal{L}_v \eta^i = 0. \tag{25}$$

Now

$$\mathcal{L}_v \epsilon^{ijkl} = -\nabla_i v^l \epsilon^{ijkl}, \tag{26}$$

so that

$$\mathcal{L}_v \left( \frac{1}{\rho} \epsilon^{ijkl} \right) = -\frac{1}{\rho^2} \nabla_i(\rho v^l) \epsilon^{ijkl} = 0. \tag{27}$$

Moreover,

$$\mathcal{L}_v s = v^i \nabla_i s = 0, \tag{28}$$

and therefore

$$\mathcal{L}_v \left( \frac{1}{\rho} \epsilon^{ijkl} \nabla_j s \nabla_k f \right) = \frac{1}{\rho} \epsilon^{ijkl} \nabla_j s \nabla_k \mathcal{L}_v f,$$

where we have used the fact that Lie derivatives commute with ordinary derivatives on scalars. Thus

$$\begin{aligned} (\partial_t + \mathcal{L}_v)\eta^i &= \frac{1}{\rho} \epsilon^{ijkl} \nabla_j s \nabla_k (\partial_t + \mathcal{L}_v)f \\ &= 0. \end{aligned} \tag{29}$$

The trivial displacements are permutations of fluid elements in surfaces of constant entropy that preserve the volume of each fluid element. They amount to a relabeling of particles, and the time evolution of equation (22) means that the initial relabeling is carried along by the unperturbed motion of the star. In the special case of an

isentropic fluid, equation (25) is automatically satisfied, and the trivial displacements comprise a larger class of vector fields having the form

$$\eta^i = \frac{1}{\rho} \epsilon^{ijk} \nabla_j \zeta_k, \tag{30}$$

where the time dependence of the vector  $\zeta_k$ , like that of the function  $f$  above, is given by

$$(\partial_t + \mathbf{v} \cdot \nabla) \zeta_i = 0. \tag{31}$$

III. FORMAL PROPERTIES OF THE EQUATIONS AND OF THE CONSERVED QUANTITIES

a) Symmetry of the Operators and Existence of a Variational Principle

The following identities show that the operators  $A^i_j$ ,  $B^i_j$ , and  $C^i_j$  of equation (15) are, respectively, symmetric, antisymmetric, and symmetric on any vector fields  $\xi^i$  and  $\eta^i$ :

$$\eta_i A^i_j \xi^j = \rho \eta_i \xi^i, \tag{32}$$

$$\eta_i B^i_j \xi^j = \rho(\eta_i v^j \nabla_j \xi^i - \xi_i v^j \nabla_j \eta^i) + \nabla_j(\rho v^j \eta_i \xi^i), \tag{33}$$

$$\begin{aligned} n_i C^i_j \xi^j = & -\rho v \cdot \nabla \eta_i v \cdot \nabla \xi^i + \gamma p \nabla \cdot \eta \nabla \cdot \xi + \nabla \cdot \eta \xi \cdot \nabla p + \nabla \cdot \xi \eta \cdot \nabla p + \eta^i \xi^j (\nabla_i \nabla_j p + \rho \nabla_i \nabla_j \Phi) \\ & - \frac{1}{4\pi G} \nabla \delta_n \Phi \cdot \nabla \delta_\xi \Phi + \nabla_i \left[ \rho v^i \eta_j (v \cdot \nabla) \xi^j - \gamma p \eta^i \nabla \cdot \xi - \eta^i \xi \cdot \nabla p + \delta_\xi \Phi \left( \frac{1}{4\pi G} \nabla^i \delta_n \Phi + \rho \eta^i \right) \right], \end{aligned} \tag{34}$$

where  $\delta_\xi$  and  $\delta_\eta$  are the changes due to  $\xi$  and  $\eta$ . It follows (cf. Lynden-Bell and Ostriker 1967) that equation (15) can be derived from a variational principle whose action is

$$I = \int \mathcal{L} dV = \frac{1}{2} \int [\xi_i A^i_j \xi^j + \xi_i B^i_j \xi^j - \xi_i C^i_j \xi^j] dV. \tag{35}$$

The existence of an action of this sort was apparently first discovered by Chandrasekhar (1961). Conversely, the fact that variations of  $\xi$  in (35) are unconstrained guarantees the symmetry properties (32)–(34) (Kulsrud 1968; Friedman and Schutz 1975), and these account for the power of the Lagrangian framework in analyzing stability, as we shall see below.

It may seem strange that an unconstrained action principle exists for the perturbations in the Lagrangian framework, since the unperturbed fluid equations do not follow from an unconstrained action principle. However, the discussion of the exact fluid variational principle in Schutz and Sorkin (1977) makes it clear that there should be no surprise. The constraints on the exact action uniquely limit variations in  $\rho$ ,  $s$ , and  $v^i$  to those that arise from a deformation of the paths of fluid elements (generated by an arbitrary vector field  $\xi^i$ ) in which the entropy and mass of each fluid element remain unchanged. The Lagrangian framework deals with perturbations that are in precisely this class, so the constraints that the perturbation's action principle inherits from the full one are automatically fulfilled. In this way the existence of an unconstrained action and the resulting symmetry properties (32)–(34) are natural consequences of adopting the Lagrangian point of view.

We shall find it convenient to discuss arbitrary complex solutions to equation (15). We use the conventional inner product notation

$$\langle \eta^i, \xi_i \rangle = \int (\eta^i)^* \xi_i dV,$$

where the asterisk denotes complex conjugation. On complex functions the operators  $A^i_j$  and  $C^i_j$  are Hermitian, while  $B^i_j$  is anti-Hermitian:

$$\langle \eta, A\xi \rangle = \langle \xi, A\eta \rangle^* \tag{36}$$

and similarly for  $C$ ;

$$\langle \eta, B\xi \rangle = -\langle \xi, B\eta \rangle^*. \tag{37}$$

(We drop indices on vectors and operators where there is no chance of confusion.)

b) The Symplectic Structure and Conserved Quantities

Associated with any Lagrangian system is a symplectic structure, a dynamically conserved antisymmetric product involving the configuration space variables and their conjugate momenta. In our case, configuration space is the set of vector fields  $\xi^i$ , and the momentum conjugate to  $\xi^i$  is

$$\frac{\partial \mathcal{L}}{\partial \xi^i} = A^i_j \xi^j + \frac{1}{2} B^i_j \xi^j = \rho \xi^i + \rho v^j \nabla_j \xi^i. \tag{38}$$

The symplectic structure, which we will denote by  $W$ , has the form

$$W(\eta, \xi) = \langle \eta, A\dot{\xi} + \frac{1}{2}B\xi \rangle - \langle A\dot{\eta} + \frac{1}{2}B\eta, \xi \rangle. \tag{39}$$

The product  $W$  may be regarded as acting on the space of all pairs  $(\xi^i, \dot{\xi}^i)$  of vector fields at a given time, the initial data sets for the dynamical equation (15). If  $\xi^i$  and  $\eta^i$  evolve as solutions to (15), this inner product will be conserved:

$$\begin{aligned} \partial_t W(\eta, \xi) &= \langle \dot{\eta}, A\dot{\xi} + \frac{1}{2}B\xi \rangle + \langle \eta, A\ddot{\xi} + \frac{1}{2}B\dot{\xi} \rangle - \langle A\dot{\eta} + \frac{1}{2}B\eta, \dot{\xi} \rangle - \langle A\ddot{\eta} + \frac{1}{2}B\dot{\eta}, \xi \rangle \\ &= \langle \dot{\eta}, A\dot{\xi} + \frac{1}{2}B\xi \rangle + \langle \eta, -\frac{1}{2}B\dot{\xi} - C\xi \rangle - \langle A\dot{\eta} + \frac{1}{2}B\eta, \dot{\xi} \rangle - \langle -\frac{1}{2}B\dot{\eta} - C\eta, \xi \rangle \\ &= (\langle \dot{\eta}, A\dot{\xi} \rangle - \langle A\dot{\eta}, \dot{\xi} \rangle) + \frac{1}{2}(\langle \dot{\eta}, B\xi \rangle + \langle B\dot{\eta}, \xi \rangle) - \frac{1}{2}(\langle \eta, B\dot{\xi} \rangle + \langle B\eta, \dot{\xi} \rangle) + (\langle C\eta, \xi \rangle - \langle \eta, C\xi \rangle) \\ &= 0, \end{aligned} \tag{40}$$

where equation (15) and the symmetry properties (36)–(37) have been used.

We will use the symplectic structure in § IV to eliminate the trivial displacements by picking out a class of physical displacements orthogonal to the trivials with respect to  $W$ . In the meantime it gives us an elegant way to introduce the canonical conserved quantities that arise via Noether's theorem from symmetries of the equilibrium configuration. In particular, because the background is stationary, the operators  $A$ ,  $B$ , and  $C$  which appear in the dynamical equation (15) are time independent. Thus  $\partial_t \xi^i(x, t)$  satisfies the dynamical equation when  $\xi^i(x, t)$  is a solution, and it follows that the product  $H(\xi, \eta)$ , defined by

$$H(\xi, \eta) = W(\partial_t \xi, \eta), \tag{41}$$

is also conserved. By virtue of the symmetry properties (32) and (33),  $H$  is symmetric in its two arguments (Hermitian when complex solutions are considered). The canonical energy is defined by

$$E_c(\xi) = \frac{1}{2}H(\xi, \xi) = \frac{1}{2}W(\dot{\xi}, \xi). \tag{42}$$

We have

$$\begin{aligned} E_c(\xi) &= \frac{1}{2}W(\dot{\xi}, \xi) = \frac{1}{2}\langle \dot{\xi}, A\dot{\xi} + \frac{1}{2}B\xi \rangle - \frac{1}{2}\langle A\dot{\xi} + \frac{1}{2}B\dot{\xi}, \xi \rangle \\ &= \frac{1}{2}\langle \dot{\xi}, A\dot{\xi} + \frac{1}{2}B\xi \rangle - \frac{1}{2}\langle -\frac{1}{2}B\dot{\xi} - C\xi, \xi \rangle = \frac{1}{2}\langle \dot{\xi}, A\dot{\xi} \rangle + \frac{1}{2}\langle \xi, C\xi \rangle, \end{aligned} \tag{43}$$

where equation (15) and the fact that  $B$  is anti-Hermitian were used. Equation (43) is the form of the more common (but equivalent) way of defining the canonical energy, namely,

$$E_c = \int \left( \dot{\xi}^i \frac{\partial \mathcal{L}}{\partial \dot{\xi}^i} - \mathcal{L} \right) dV. \tag{44}$$

Explicitly,

$$\begin{aligned} E_c(\xi) &= \frac{1}{2} \int \left[ \rho |\dot{\xi}|^2 - \rho |v \cdot \nabla \xi|^2 + \gamma p |\nabla \cdot \xi|^2 + \xi^* \cdot \nabla p \nabla \cdot \xi + \xi \cdot \nabla p \nabla \cdot \xi^* \right. \\ &\quad \left. + \xi^{i*} \xi^j (\nabla_i \nabla_j p + \rho \nabla_i \nabla_j \Phi) - \frac{1}{4\pi G} |\nabla \delta \Phi|^2 \right] dV. \end{aligned} \tag{45}$$

Similarly, when the equilibrium configuration is axisymmetric, the operators  $A$ ,  $B$ , and  $C$  commute with  $\partial_\phi$  (with  $\xi_\phi$ , where  $\phi$  is the rotational symmetry vector). Thus one obtains another symmetric conserved product,

$$J(\xi, \eta) = W(\partial_\phi \xi, \eta); \tag{46}$$

and the associated canonical angular momentum is

$$J_c(\xi) = -\frac{1}{2}W(\partial_\phi \xi, \xi) = -\int \partial_\phi \xi^i \frac{\partial \mathcal{L}}{\partial \dot{\xi}^i} dV. \tag{47}$$

We have

$$J_c(\xi) = -\text{Re} \langle \partial_\phi \xi, A\dot{\xi} + \frac{1}{2}B\xi \rangle, \tag{48}$$

or

$$J_c(\xi) = -\text{Re} \int \rho \partial_\phi \xi^{i*} (\dot{\xi}_i + v \cdot \nabla \xi_i) dV. \tag{49}$$

The values of  $E_c$  and  $J_c$  for complex  $\xi^i$  are the sum of the corresponding quantities for the real and imaginary parts of  $\xi^i$ . In particular a normal mode,  $\xi^i = \xi^i(x)e^{i\sigma t}$ , has canonical energy

$$E_c = \sigma [(\text{Re } \sigma) \langle \xi, A\xi \rangle - \frac{1}{2} \langle \xi, iB\xi \rangle]. \tag{50}$$

Since  $iB$  is Hermitian, the product  $\langle \xi, iB\xi \rangle$  is real; and since  $E_c$  is real by equation (43), it follows that either  $\sigma$  is real or  $E_c$  vanishes. This simply confirms the fact that if  $E_c$  is to be conserved for a mode which grows (or dies) exponentially, it must vanish. Similarly, when the background is axisymmetric, a normal mode  $\xi^t$  has the form  $\xi^t = \xi^t(r, \theta)e^{i(m\phi + \sigma t)}$ , and we find

$$J_c = -m(\text{Re } \sigma \langle \xi, A\xi \rangle - \frac{1}{2} \langle \xi, iB\xi \rangle). \tag{51}$$

Thus  $J_c$  also vanishes if  $\sigma$  is not real. For real frequency modes we have the result

$$E_c/J_c = -\sigma/m \equiv \omega_p, \tag{52}$$

where  $\omega_p$  is the pattern speed of the mode. This relation follows only from the Hermiticity properties of the operators  $A$ ,  $B$ , and  $C$  and is generally true for linear Lagrangian systems.

*c) Gauge Dependence of the Canonical Energy*

If  $\eta^i$  is a trivial displacement, then the Lagrangian displacements  $\xi^i$  and  $\xi^i + \eta^i$  correspond to the same physical perturbation. Unfortunately, the canonical energy  $E_c$  is not invariant under such gauge transformations: in general

$$E_c(\xi + \eta) \neq E_c(\xi). \tag{53}$$

To prove this, one need only show that  $E_c(\eta) \neq 0$  for some trivial  $\eta^i$ . In Appendix A we find that the canonical energy of a trivial displacement has the form

$$E_c(\eta) = -\frac{1}{2} \int \rho \eta^i \mathcal{L}_v \eta^j (\nabla_i v_j - \nabla_j v_i) dV. \tag{54a}$$

When the displacement  $\eta^i$  is written in terms of a generating function  $f$  as in equation (22), the expression (54a) for  $E_c$  becomes

$$E_c(\eta) = \frac{1}{2} \int \alpha \epsilon^{ijk} \nabla_i s \nabla_j (\mathcal{L}_v f) \nabla_k f dV, \tag{54b}$$

where the scalar  $\alpha$  is defined by

$$\alpha = \frac{1}{\rho} \epsilon^{ijk} \nabla_i s \nabla_j v_k. \tag{55}$$

In particular, if the unperturbed star is axisymmetric and rotating with angular velocity  $\Omega$ , and if the generating function  $f$  has the form  $f = g(\varpi, z) \cos m\phi$  at  $t = 0$ , then

$$E_c(\eta) = \frac{1}{2} \pi m^2 \int \alpha [s_{,\varpi}(g^2 \Omega)_{,z} - s_{,z}(g^2 \Omega)_{,\varpi}] d\varpi dz. \tag{56}$$

Now a necessary condition for stability against convection is that (Fricke 1971),

$$\alpha \equiv \frac{1}{\varpi \rho} [s_{,\varpi}(\varpi^2 \Omega)_{,z} - s_{,z}(\varpi^2 \Omega)_{,\varpi}] \neq 0. \tag{57}$$

Thus, unless the equilibrium is everywhere marginally unstable to convection, one can give  $E_c$  any value at all, by suitably choosing the function  $g$ .

*d) Complete Expressions for the Energy and Angular Momentum through Second Order*

At first sight the noninvariance of  $E_c$  under a gauge transformation is surprising. Previous work seems universally to have assumed that  $E_c$  was the second-order change in energy of the star due to the perturbation.<sup>5</sup> But this energy depends only on physical quantities, and hence should be invariant under a gauge change. Therefore, the canonical energy is not the full energy at second order.

In principle, the second-order part of the energy is the sum of two pieces: one piece quadratic in the first-order changes of the fluid variables  $p$ ,  $\rho$ ,  $v^i$ , and  $s$ , and another linear in the second-order perturbations of these quantities. In many physical theories (e.g., vacuum electromagnetism, vacuum general relativity) the piece linear in the second-order perturbations of the field variables vanishes identically; the reason is that it has the same functional form as the first-order change in the energy, which vanishes identically: the energy of stationary solutions is an extremum. For fluid dynamics, however, the energy of a stationary solution is not an extremum against all perturbations; this

<sup>5</sup> See Schutz (1972b) for a "proof" of this in the relativistic context. His equation (80) is incorrect because step (b) of his argument fails:  $E_c$  and  $J_c$  do not necessarily vanish if the physical perturbation is zero. The existence of the trivials was not suspected at that time.

has been discussed in detail from a field-theoretic viewpoint by Schutz and Sorkin (1977). In particular, a first-order Lagrangian change in the velocity leads to a first-order change in the (kinetic) energy of a solution:

$$\delta E = \int dV \rho v^i \Delta v_i. \tag{58}$$

In Appendix B, we calculate the change in energy and angular momentum accurate to second order in the perturbation. We find for the change in energy the expression

$$\delta E = \int dV \rho v^i \Delta v_i + E_c, \tag{59}$$

where now  $\delta$  and  $\Delta$  are complete to second order. Thus the energy at this order includes in addition to  $E_c$  a term involving the *Lagrangian* change in  $v_i$  and which (like  $E_c$ ) is gauge-dependent. Because  $\delta E$  is a physical property of the configuration, it is independent of gauge; and because it is the change in energy, it is conserved. Then conservation of  $E_c$  implies that the term

$$\int dV \rho v^i \Delta v_i$$

must also be conserved. This additional conserved quantity can be related to the circulation in the fluid, at least when the unperturbed flow is circular:

$$\int dV \rho v^i \Delta v_i = \int d^2 x \rho \Omega \oint dx^i \Delta v_i = \frac{1}{2\pi} \int dM \Omega \Delta \oint v_i dx^i, \tag{60}$$

where  $dM$  is the mass of the ring around which the circulation integral is calculated. This was in fact the form in which the correction to  $E_c$  was first found by J. Bardeen (private communication, 1975).

The analogous expression for the second-order change in the angular momentum is

$$\delta J = \int dV \rho \phi^i \Delta v_i + J_c, \tag{61}$$

where  $\delta$  and  $\Delta$  are complete to second order. Again, neither term is gauge-invariant but their sum is, and both terms are separately conserved.

It is remarkable that for *uniformly rotating* stars a gauge-invariant expression quadratic in  $\xi^i$  does exist (Bardeen *et al.* 1977). If  $v^i = \Omega \phi^i$  for  $\Omega$  constant, then inspection of (59) and (61) shows that

$$\delta E - \Omega \delta J = E_c - \Omega J_c. \tag{62}$$

Since the left-hand side is gauge-invariant, so is the right. This quantity has a simple interpretation: it is  $E_{c,R}$ , the canonical energy of the perturbation with respect to the frame rotating with the star. This is easily seen by applying the definition (42) in the rotating frame. If we denote by  $t'$  the time coordinate of a rotating observer (whose angular coordinate is  $\phi' = \phi - \Omega t$ ), then  $t' = t$  and

$$\partial_{t'} = \partial_t + \Omega \partial_\phi. \tag{63}$$

Thus

$$E_{c,R} = \frac{1}{2} W(\partial_{t'} \xi, \xi) = \frac{1}{2} W(\partial_t \xi, \xi) + \Omega \frac{1}{2} W(\partial_\phi \xi, \xi) = E_c - \Omega J_c. \tag{64}$$

In the rotating frame, the fluid is not moving; since gauge problems arise only in moving fluids, it is not surprising that  $E_{c,R}$  is gauge-invariant.

The energy  $E_{c,R}$  has been discussed by Howard and Siegmund (1969) in connection with the stability of rotating flows. Their treatment is entirely Eulerian, and they find the expression for  $E_{c,R}$  in terms of Eulerian perturbations; this must exist because  $E_{c,R}$  is gauge-invariant. Appendix C expands on this briefly.

#### IV. CANONICAL DISPLACEMENTS

The description of fluid perturbations in terms of a Lagrangian displacement involves, as we have seen, a gauge freedom associated with the trivial displacements: the same physical perturbation can be described by more than that one Lagrangian displacement. This ambiguity is particularly unsatisfactory for stability theory, where, as we shall see in Paper II, one would like to test stability by asking whether the canonical energy  $E_c$  is positive definite. To make sense of such a criterion, the functional  $E_c$  must have the same value for two displacements that describe the same physical perturbation, and, as was pointed out in § IIIc, this will not be the case if one allows the full class of trivial displacements.

We would therefore like to restrict in some way the class of displacements—to pick out a subclass of “canonical” displacements. The subclass should be large enough to include all physical perturbations, small enough that  $E_c$  is



invariant under any remaining gauge freedom, and should be such that the time evolution of a canonical displacement is canonical. That is, one requires of a class of "canonical" displacements that

- i) the class be dynamically invariant,
- ii) to each physical perturbation correspond at least one "canonical" displacement, and
- iii) if  $\xi^i$  and  $\xi^{i'}$  are two "canonical" displacements corresponding to the same physical perturbation,  $E_c(\xi) = E_c(\xi')$ .

One can often do somewhat better than this, in the sense that it is ordinarily possible to find a dynamically preserved class small enough that to each physical perturbation corresponds a *unique* displacement in the class. On the other hand, it is apparently not always possible to satisfy (ii), and in such cases restricting consideration to the "canonical" displacements may then be a genuine restriction.

Each of the conserved inner products  $W$ ,  $H$ , and  $J$  introduced in § IIIb can be used to define a natural class of displacements satisfying (i)–(iii) above (except in special cases) by requiring that a canonical displacement be orthogonal to all trivial displacements with respect to the inner product. That is, a displacement  $\xi^i$  will be canonical with respect to the inner product  $W$  if  $W(\xi, \eta) = 0$  for all trivial  $\eta$ . In fact, the three inner products provide the *same* substance of displacements orthogonal to the trivials, and we conjecture that the class obtained in this way is the only subspace satisfying conditions (i)–(iii).<sup>6</sup> However, uniqueness of the subspace is not required for the stability criterion obtained in Paper II.

a) *Fluids with Two-Parameter Equations of State*

Consider the condition that a displacement be orthogonal to all trivials,  $\eta^i$ , with respect to  $W$ . We have

$$W(\eta, \xi) = \int \rho[\eta^i(\dot{\xi}_i + v^j \nabla_j \xi_i) - \xi^i(\dot{\eta}_i + v^j \nabla_j \eta_i)] dV = \int \rho \eta^i (\dot{\xi}_i + v^j \nabla_j \xi_i + v^j \nabla_i \xi_j) dV, \tag{65}$$

where we have used equation (21), valid for a trivial displacement  $\eta^i$  and integrated by parts to obtain (65). Substituting now for  $\eta^i$  the explicit formula (22) for a generic trivial, we have

$$W(\eta, \xi) = \int \epsilon^{ijk} \nabla_j \nabla_k f (\dot{\xi}_i + v^j \nabla_j \xi_i + v^j \nabla_i \xi_j) dV, \tag{66}$$

whence  $W(\eta, \xi) = 0$  for all trivial  $\eta$  if and only if

$$\int f \epsilon^{ijk} \nabla_j \nabla_k (\dot{\xi}_i + v^j \nabla_j \xi_i + v^j \nabla_i \xi_j) dV = 0, \tag{67}$$

for all functions  $f$ . Thus  $\xi^i$  is orthogonal to the trivials if and only if

$$\epsilon^{ijk} \nabla_i \nabla_j (\dot{\xi}_k + v^l \nabla_l \xi_k + v^l \nabla_k \xi_l) = 0. \tag{68}$$

Now the expression in parentheses in equation (68) can be written in the form

$$\dot{\xi}_i + v^j \nabla_j \xi_i + v^j \nabla_i \xi_j = g_{ij} \dot{\xi}^j + v^l \mathcal{L}_\xi g_{lj} = g_{ij} \Delta_\xi v^j + v^l \Delta_\xi g_{lj} = \Delta_\xi v_i, \tag{69}$$

where the definition (3) of  $\Delta_\xi$  and the relation  $\mathcal{L}_\xi g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i$  have been used. Equation (68) then has the form

$$\epsilon^{ijk} \nabla_i \nabla_j \Delta_\xi v_k = 0. \tag{70}$$

In other words, the Lagrangian change in the vorticity,

$$\Delta_\xi (\nabla_i v_j - \nabla_j v_i) = \nabla_i \Delta_\xi v_j - \nabla_j \Delta_\xi v_i, \tag{71}$$

is a tensor orthogonal to the constant entropy surfaces. (We have made use in eq. [71] of the fact that Lie derivatives commute with exterior derivatives, that, e.g.,  $\mathcal{L}_\xi (\nabla_i A_j - \nabla_j A_i) = \nabla_i \mathcal{L}_\xi A_j - \nabla_j \mathcal{L}_\xi A_i$ ). An equivalent statement is that the circulation in surfaces of constant entropy be unchanged:

$$\Delta \oint_c v_i dx^i = 0, \tag{72}$$

where  $c$  is any curve lying in a constant-entropy surface. In the special case of isentropic stars (see below), all curves lies in surfaces of constant entropy and the canonical displacements conserve all components of the vorticity.

A final equivalent form of the orthogonality condition is

$$\Delta_\xi \alpha \equiv \Delta_\xi \left( \frac{1}{\rho} \epsilon^{ijk} \nabla_i \nabla_j v_k \right) = 0, \tag{73}$$

<sup>6</sup> Strictly speaking, we are concerned here with displacements of the form  $\xi^i = \xi^i(r, \theta) e^{im\phi}$  with  $m \neq 0$ .

which follows from the relations

$$\Delta_z \epsilon^{ijk} = -\nabla \cdot \xi \epsilon^{ijk}, \quad \Delta_z \rho = -\rho \nabla \cdot \xi, \quad \text{and} \quad \Delta_z s = 0.$$

The quantity  $\alpha$  that appears in equation (73) was previously introduced in IIIc. In a rotating (nonisentropic) fluid,  $\alpha$  is conserved:

$$(\partial_t + v^j \nabla_j) \alpha = 0; \tag{74}$$

and the gauge condition can be interpreted as a requirement that the perturbation also conserve  $\alpha$  for each fluid element. In the general unperturbed flow, exactly three quantities are constant on a fluid element: its mass, its entropy, and the value of  $\alpha$ . Conservation of  $\alpha$  is called Ertel's theorem (Ertel 1942a, b).<sup>7</sup>

Having explicitly characterized the space of displacements orthogonal (in  $W$ ) to the trivials, we can show that the spaces arising from the products  $H$  and  $J$  are identical and that the canonical displacements so defined satisfy (i)–(iii) above. We will specialize our considerations to the case where the unperturbed configuration is axisymmetric without meridional circulation, and first deal only with perturbations having angular behavior  $e^{im\phi}$  for  $m \neq 0$ .

The fact that  $W$ ,  $H$ , and  $J$  define the same class of orthogonal displacements follows from the defining equations (41) and (46) for  $H$  and  $J$  together with the fact that if  $\eta^i$  is trivial (and nonaxisymmetric), so is  $\partial_t \eta^i$ ,  $\partial_\phi \eta^i$ ,  $\int \eta^i dt$ , and  $\int^\phi \eta^i d\phi$ . Condition (iii) follows from equation (42) for the canonical energy  $E_c$  together with the fact that orthogonality to all trivials with respect to  $W$  implies orthogonality with respect to  $H$ . That is, if  $\xi^i$  and  $\xi^i$  correspond to the same physical perturbation, their difference,  $\eta^i = \xi^i - \xi^i$  is a trivial displacement. If  $\xi^i$  and  $\xi^i$  are both orthogonal to all trivials,  $\eta^i$  is itself orthogonal to all trivials, including itself. Thus we have

$$E_c(\xi) = \frac{1}{2} H(\xi + \eta, \xi + \eta) = \frac{1}{2} H(\xi, \xi) + H(\xi, \eta) + \frac{1}{2} H(\eta, \eta) = \frac{1}{2} H(\xi, \xi) = E_c(\xi). \tag{75}$$

Finally, consider (ii), the question of whether there is at least one canonical displacement corresponding to each physical perturbation. That is, given any displacement  $\xi^i$ , we want to find a  $\xi^i$  orthogonal to the trivials and with  $\xi^i - \xi^i$  trivial. Let  $\eta^i = \xi^i - \xi^i$ . The displacement  $\xi^i$  will be orthogonal to the trivials if

$$\Delta_z \alpha = 0. \tag{76}$$

We have

$$0 = \Delta_z \alpha = \Delta_z \alpha + \Delta_\eta \alpha = \Delta_z \alpha + \mathcal{L}_\eta \alpha = \Delta_z \alpha + \eta^j \nabla_j \alpha. \tag{77}$$

Writing the trivial displacement  $\eta^i$  in the form (22), we have for  $f$  the equation

$$\frac{1}{\rho} \epsilon^{ijk} \nabla_i \alpha \nabla_j s \nabla_k f = -\Delta_z \alpha. \tag{78}$$

In cylindrical coordinates  $(\phi, \varpi, z)$ , with  $\varpi$  the distance from the axis, its solution is

$$\partial_\phi f = -\frac{\varpi \rho \Delta_z \alpha}{\alpha_{,\varpi s,z} - \alpha_{,z s,\varpi}}. \tag{79}$$

Thus one can always find such an  $f$  (and therefore a canonical  $\xi^i$ ) provided that  $\nabla \alpha \times \nabla s \neq 0$  away from the axis of symmetry. Finally, it is not difficult to show that when such a  $\xi^i$  orthogonal to the trivials exists, it is unique. For if  $\xi$  and  $\zeta$  are both canonical and represent the same physical perturbation,  $\eta = \zeta - \xi$  is a trivial orthogonal to all trivials. Thus

$$0 = \Delta_\eta \alpha = \frac{1}{\rho} \epsilon^{ijk} \nabla_i \alpha \nabla_j s \nabla_k f;$$

and, except at points where  $\nabla \alpha \times \nabla s = 0$ , the equation implies  $\partial_\phi f = 0$ . Assuming that the locus of such points is of measure zero in the equilibrium fluid, we have  $\partial_\phi f = 0$  everywhere, implying that  $f = 0$ . Thus  $\zeta = \xi$ , and the canonical displacements are uniquely determined in the nonaxisymmetric case.

<sup>7</sup> By means of the symplectic structure we have related conservation of circulation to the trivial displacements. This relation can be understood in terms of Noether's theorem in the following way. First, note that for any (not necessarily canonical) displacement  $\xi^i$ , the product  $W(\xi, \eta)$  is conserved for all trivial  $\eta^i$ ; and this conservation law is simply Ertel's theorem for the perturbed flow:

$$(\partial_t + v^i \nabla_i) \Delta_i \alpha = 0.$$

Now each trivial displacement  $\eta^i$  generates a one-parameter family of symmetries of the set of solutions  $\xi^i$  to the dynamical equations, namely the transformation  $\lambda^i \rightarrow \xi^i + \lambda \eta^i$ . These transformations change the Lagrangian density by a pure divergence and are therefore what Trautman (1967), in his extension of Noether's theorem, calls generalized invariant transformations. The corresponding conserved quantity turns out simply to be the product  $W(\xi, \eta)$ . Thus Ertel's theorem for the perturbed fluid arises via Noether's theorem as the conservation law associated with the trivial displacements. Furthermore, when one treats the exact flow as a Lagrangian system, trivial displacements again play a role as generators of symmetry transformations and the associated conservation law is again Ertel's theorem.

We turn now to axisymmetric perturbations of axisymmetric stars. A trivial displacement is generated by a function  $f$  which, by equation (22) and axisymmetry, must be independent of  $\phi$  and  $t$ . Therefore the trivial displacement  $\eta$  can have only a  $\phi$ -component. From (73) it then follows that  $\eta$  is orthogonal to all trivials, since  $\Delta_{\eta}\alpha = \mathbf{f}_{\eta}\alpha = \eta^{\phi}\partial\alpha/\partial\phi = 0$ . This means that  $E_c$  and  $J_c$  are invariant under the whole class of axisymmetric gauge transformations. We can conclude two things: first, that there is no real need to eliminate trivials in this case; and second, and conversely, that there is no unique way to define the trivial part of a given Lagrangian displacement. On the other hand, it is certainly possible to find displacements  $\xi$  for which  $\Delta_z\alpha \neq 0$ . These must clearly differ in a physical way from those for which  $\Delta_z\alpha = 0$ , since all the trivials are in the latter class.

In order to resolve this difficulty, it is helpful to understand another problem with axisymmetric perturbations. It makes sense to discuss  $E_c$  and  $J_c$  only if the first-order changes in  $E$  and  $J$  vanish for the given perturbations. For nonaxisymmetric perturbations this is automatic, since the integral over the entire star of any function linear in the perturbation vanishes. For axisymmetric perturbations one must impose the extra condition

$$\Delta_z v_{\phi} = 0 \tag{80}$$

in order to guarantee this (cf. eq. [58]). This condition also turns out to guarantee  $\Delta_z\alpha = 0$ , for the following reason. Consider equation (70). If the perturbation is axisymmetric, then in coordinates  $(\phi, \varpi, z)$ ,  $\Delta_z v_k$  must not depend on  $\phi$ . This means that the two derivatives  $\nabla_i$  and  $\nabla_j$  (which are really only partial derivatives) must be chosen from  $\nabla_{\varpi}$  and  $\nabla_z$ , forcing the index  $k$  to take the value  $\phi$ . Then condition (80) guarantees satisfaction of (70) and hence of (73). This shows that the only perturbations which do not satisfy our gauge condition are those which change the angular momentum and energy of the star to first order. These have been excluded in all previous treatments of axisymmetric stability (see, e.g., Lynden-Bell and Ostriker 1967 and Chandrasekhar and Lebovitz 1968) by the imposition of (80).

*b) Isentropic Fluids*

An isentropic fluid has an equation of state of the form  $p = p(\rho)$ , and its adiabatic perturbations satisfy

$$\delta p = p'(\rho)\delta\rho. \tag{81}$$

It was pointed out in § II that the trivial displacements  $\eta^i$  of an isentropic fluid are proportional to curls of vector fields  $\zeta_i$ ,

$$\eta^i = \frac{1}{\rho} \epsilon^{ijk} \nabla_j \zeta_k, \tag{82}$$

where  $\zeta_i$  has the time dependence given by equation (31).

The requirement that a displacement  $\xi^i$  be orthogonal to all trivials is again given by

$$0 = W(\eta, \xi) = \int \rho \eta^i (\xi_i + v^j \nabla_j \xi_i + v^j \nabla_i \xi_j) dV = \int \rho \eta^i \Delta_z v_i dV \tag{83}$$

(see eqs. [65] and [69]). Now, however,  $\eta^i$  has the general form (82), and equation (83) becomes

$$0 = \int \epsilon^{ijk} \Delta_z v_i \nabla_j \zeta_k. \tag{84}$$

This will hold for all  $\zeta_i$  when

$$\epsilon^{ijk} \nabla_j \Delta_z v_k = 0. \tag{85a}$$

Equivalent forms of the condition are

$$\Delta_z (\nabla_j v_k - \nabla_k v_j) = 0, \tag{85b}$$

$$\Delta_z \left( \frac{1}{\rho} \epsilon^{ijk} \nabla_j v_k \right) = 0, \tag{85c}$$

and

$$\Delta \oint_c v_i dx^i = 0, \tag{85d}$$

where  $c$  is any closed curve lying in the fluid. These express the requirement that the perturbation preserves the vorticity of each fluid element.

To show that to each physical perturbation corresponds some  $\xi^i$  preserving vorticity (condition [iii] for the isentropic case), we have to solve the equation analogous to (78), namely,

$$\nabla_{[i} \mathbf{f}_{\eta]v_j} = -\nabla_{[i} \Delta_z v_{j]} \equiv \frac{1}{2} q_{ij}, \tag{86}$$

together with the requirement that  $\eta^i$  be trivial,

$$\nabla_j(\rho\eta^j) = 0. \tag{87}$$

For an axisymmetric, differentially rotating star having angular velocity  $\Omega$ , isentropy implies  $\Omega = \Omega(\varpi)$ . In cylindrical coordinates, equations (86) and (87) take the form

$$(\varpi^2\Omega)' \partial_z \eta^\phi = q_{\varpi z}, \tag{88}$$

$$(\varpi^2\Omega)' \partial_z \eta^\varpi = q_{z\phi}, \tag{89}$$

$$\partial_\varpi[(\varpi^2\Omega)' \eta^\varpi] + \partial_\phi[(\varpi^2\Omega)' \eta^\phi] = q_{\varpi\phi}, \tag{90}$$

and

$$\frac{1}{\varpi} \partial_\varpi(\varpi\rho\eta^\varpi) + \partial_z(\rho\eta^z) + \partial_\phi(\rho\eta^\phi) = 0. \tag{91}$$

Equations (88) and (89) imply

$$\eta^\phi = \frac{1}{(\varpi^2\Omega)'} \left[ \int_{-z_b(\varpi)}^z q_{\varpi z} dz' + \zeta^\phi(\varpi, \phi) \right], \tag{92}$$

$$\eta^\varpi = \frac{1}{(\varpi^2\Omega)'} \left[ \int_{-z_b(\varpi)}^z q_{z\phi} dz' + \zeta^\varpi(\varpi, \phi) \right], \tag{93}$$

where  $z_b(\varpi)$  is the value of  $z$  at the boundary of the equilibrium star and  $\zeta = (\zeta^\phi, \zeta^\varpi, 0)$  is a vector field independent of  $z$ . Making use of the identity

$$\partial_{[z} q_{\varpi\phi]} = 0, \tag{94}$$

we find that in order to satisfy equation (90),  $\zeta^i$  must be a solution to

$$\partial_\varpi \zeta^\varpi + \partial_\phi \zeta^\phi = q_{\phi\varpi}(\varpi, z = -z_b(\varpi), \phi). \tag{95}$$

This relation fixes  $\zeta^i$  up to

$$\zeta^\varpi \rightarrow \zeta^\varpi + \partial_\phi g, \quad \zeta^\phi \rightarrow \zeta^\phi - \partial_\varpi g, \tag{96}$$

where  $g$  is an arbitrary function of  $\varpi$  and  $\phi$  ( $g$  must vanish on the axis in order that  $\eta^i$  be regular there). Finally, from equation (91) we have

$$\eta^z = -\frac{1}{\rho} \int_{-z_b}^z dz' \left[ \frac{1}{\varpi} \partial_\varpi(\varpi\rho\eta^\varpi) + \partial_\phi(\rho\eta^\phi) \right]. \tag{97}$$

In order that  $\eta^z$  be defined at  $z = z_b(\varpi)$ , where  $\rho = 0$ , the freedom in  $\zeta^i$  given in equation (96) must be further restricted to guarantee that

$$\int_{-z_b}^{z_b} dz' \left[ \frac{1}{\varpi} \partial_\varpi(\varpi\rho\eta^\varpi) + \partial_\phi(\rho\eta^\phi) \right] = 0. \tag{98}$$

This can be done, but it leaves some ambiguity in  $\zeta^i$  corresponding to functions  $g$  in (96) which satisfy

$$\int_{-z_b}^{z_b} dz' \frac{1}{\varpi} \partial_\varpi \left| \frac{\varpi\rho}{(\varpi^2\Omega)'} \right| \partial_\phi g = 0. \tag{99}$$

When  $\zeta^i$  is smooth, with angular dependence  $e^{im\phi}$ ,  $m \neq 0$ , and when  $(\varpi^2\Omega)'$  is nonzero away from the axis, equations (92), (93), and (97) provide a smooth trivial displacement  $\eta^i$  for which  $\xi^i - \eta^i$  is orthogonal to the trivials. Now the requirement of local stability against convection is precisely that  $\varpi^2\Omega$ —the angular momentum per unit mass—increases outward, that  $(\varpi^2\Omega)' > 0$  away from the axis of symmetry (Goldreich and Schubert 1967). Thus for isentropic stars condition (iii) is satisfied: to each physical perturbation corresponds at least one canonical displacement. The canonical displacements are not quite unique, as we have seen, but the remaining ambiguity is harmless in that it leaves the canonical energy and angular momentum invariant.

Using canonical displacements for isentropic stars is natural in the sense that isentropic fluids conserve vorticity, and, in addition, the canonical displacements are precisely those which preserve the vorticity of each fluid element. In the context of stability theory, J. Bardeen (1975) and C. Hunter (1977) have previously advocated using the condition. It has apparently not been recognized, however, that this involved no restriction on the corresponding physical perturbations.

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APPENDIX A

CANONICAL ENERGY OF A TRIVIAL DISPLACEMENT

The canonical energy defined in equation (42) is

$$E_c(\eta) = \frac{1}{2}W(\dot{\eta}, \eta) = \frac{1}{2} \int \rho[\dot{\eta}^i(\dot{\eta}_i + v \cdot \nabla \eta_i) - \eta^i(\ddot{\eta}_i + v \cdot \nabla \dot{\eta}_i)]dV. \tag{A1}$$

A trivial displacement has time dependence given by equation (21),

$$\dot{\eta}^i = -\mathcal{L}_v \eta^i;$$

then

$$\ddot{\eta}^i = (\mathcal{L}_v)^2 \eta^i, \tag{A2}$$

and we have

$$E_c(\eta) = \frac{1}{2} \int \rho[\mathcal{L}_v \eta^i g_{ij} \mathcal{L}_v \eta^j - \mathcal{L}_v \eta^i v \cdot \nabla \eta_i - \eta_i (\mathcal{L}_v)^2 \eta^i + \eta_i v \cdot \nabla \mathcal{L}_v \eta^i]dV. \tag{A3}$$

The third term in this expression for  $E_c$  can be integrated by parts in the manner

$$\begin{aligned} -\frac{1}{2} \int \rho \eta_i (\mathcal{L}_v)^2 \eta^i dV &= -\frac{1}{2} \int [\rho \mathcal{L}_v (\eta_i \mathcal{L}_v \eta^i) - \rho \mathcal{L}_v \eta_i \mathcal{L}_v \eta^i]dV \\ &= -\frac{1}{2} \int \rho v \cdot \nabla (\eta_i \mathcal{L}_v \eta^i) dV + \frac{1}{2} \int \rho \mathcal{L}_v \eta^i \mathcal{L}_v \eta_i dV = \frac{1}{2} \int \rho \mathcal{L}_v \eta^i \mathcal{L}_v \eta_i dV, \end{aligned} \tag{A4}$$

where the relation  $\nabla \cdot (\rho v) = 0$  was used. After another integration by parts (of the last term on the right in eq. [A3]), we obtain

$$\begin{aligned} E_c &= \frac{1}{2} \int \rho \mathcal{L}_v \eta^i (g_{ij} \mathcal{L}_v \eta^j + \mathcal{L}_v \eta_i - 2v \cdot \nabla \eta_i) dV \\ &= \frac{1}{2} \int \rho \mathcal{L}_v \eta^i \eta^j (\nabla_i v_j - \nabla_j v_i) dV = -\frac{1}{2} \int \rho \epsilon_{ijk} \omega^k \eta^i \mathcal{L}_v \eta^j dV, \end{aligned} \tag{A5}$$

where  $\omega^i$  is the vorticity vector,

$$\omega^i = \epsilon^{ijk} \nabla_j v_k. \tag{A6}$$

When the general form (22) of the trivial displacement,

$$\eta^i = \frac{1}{\rho} \epsilon^{ijk} \nabla_j \nabla_k f,$$

is inserted in equation (A5), the expression for  $E_c$  takes the form

$$E_c = -\frac{1}{2} \int \frac{1}{\rho} \epsilon_{ijk} \omega^i (\epsilon^{lmn} \nabla_m \nabla_n f) (\epsilon^{kpq} \nabla_p \nabla_q \mathcal{L}_v f) dV = -\frac{1}{2} \int \frac{1}{\rho} \omega^m \nabla_m \epsilon^{ijk} \nabla_i \nabla_j f \nabla_k \mathcal{L}_v f dV. \tag{A7}$$

In terms of the scalar  $\alpha$ ,

$$\alpha = \frac{1}{\rho} \epsilon^{ijk} \nabla_i \nabla_j v_k = \frac{1}{\rho} \omega^i \nabla_i s$$

introduced in the text,  $E_c$  can be written as

$$E_c = -\frac{1}{2} \int \alpha \epsilon^{ijk} \nabla_i \nabla_j f \nabla_k \mathcal{L}_v f dV. \tag{A8}$$

In particular, suppose the equilibrium is an axisymmetric fluid rotating with angular velocity  $\Omega$ . If  $f$  is a function of the form (at  $t = 0$ )

$$f = g(\varpi, z) \cos m\phi, \tag{A9}$$

We have

$$\begin{aligned}
 \dot{E}_c &= -\frac{1}{2} \int \alpha \epsilon^{ijk} \nabla_i s \nabla_j (g \cos m\phi) \nabla_k (-m\Omega g \sin m\phi) dV \\
 &= -\frac{1}{2} m^2 \int \alpha \epsilon^{ijk} \nabla_i s [-\nabla_j g \Omega g \nabla_k \phi \cos^2 m\phi + g \nabla_j \phi \nabla_k (\Omega g) \sin^2 m\phi] dV \\
 &= \frac{1}{2} \pi m^2 \int \alpha \epsilon^{ijk} \nabla_i s \nabla_j (g^2 \Omega) \nabla_k \phi \varpi d\varpi dz \\
 &= \frac{1}{2} \pi m^2 \int \alpha [s_{,\varpi} (g^2 \Omega)_{,z} - s_{,z} (g^2 \Omega)_{,\varpi}] d\varpi dz .
 \end{aligned}
 \tag{A10}$$

APPENDIX B

SECOND-ORDER PERTURBATION THEORY

Our aim in this Appendix is to calculate the change in physical energy and angular momentum of a perturbed equilibrium configuration accurate to second order in the perturbation.

Suppose first that  $(\rho, p, v^i, s)$  is a solution to the equilibrium equations (1) and that  $[\bar{\rho}(\lambda), \bar{p}(\lambda), \bar{v}^i(\lambda), \bar{s}(\lambda)]$  is a family of solutions to the exact time-dependent equations, (1a), (2), and

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla_i (\bar{\rho} \bar{v}^i) = 0 , \tag{B1a}$$

$$\frac{\partial \bar{s}}{\partial t} + \bar{v}^i \nabla_i \bar{s} = 0 , \tag{B1b}$$

$$(\partial_t + \bar{v}^j \nabla_j) \bar{v}_i + \frac{1}{\bar{\rho}} \nabla_i \bar{p} + \nabla_i \bar{\Phi} = 0 . \tag{B1c}$$

The family is to be smoothly parametrized by  $\lambda$  and to coincide at  $\lambda = 0$  with the equilibrium configuration:

$$\bar{\rho}(0) = \rho , \quad \bar{p}(0) = p , \quad \bar{v}^i(0) = v^i , \quad \bar{s}(0) = s . \tag{B2}$$

We will further suppose that the family of time-dependent solutions is such that all of its members can be reached by an adiabatic deformation of the stationary solution. That is, there is to be a family of maps  $\psi_\lambda(x, t)$  with the following characteristics:

i) If  $c(t)$  is the path of a fluid element in the equilibrium configuration  $\psi_\lambda[c(t), t]$  is a path in the perturbed configuration.

ii) The entropy of each fluid element is conserved:

$$\bar{s}_\lambda(\psi_\lambda(x, t), t) = s(x, t) .$$

iii) The mass of each fluid element is conserved:

$$\bar{\rho}(\psi_\lambda(x, t), t) J_{\psi_\lambda} = \rho(x, t) ,$$

where  $J_\psi$  is the Jacobian of the map  $\psi$ .

An exact Eulerian change in a quantity  $Q$  will be defined by

$$\delta_\lambda Q = \bar{Q}(\lambda) - Q ; \tag{B3}$$

the Eulerian change in the linearized theory is then the first-order part of  $\delta_\lambda Q$ , which one could identify with

$$\left. \frac{d}{d\lambda} \delta_\lambda Q \right|_{\lambda=0} .$$

Similarly, the second-order change in energy that we want to evaluate is

$$\left. \frac{d^2}{d\lambda^2} E[\bar{\rho}(\lambda), \bar{p}(\lambda), v^i(\lambda), s(\lambda)] \right|_{\lambda=0} .$$

In order to construct a generalization in this exact framework of the linear Lagrangian displacement formalism, we write

$$\psi_{\lambda}^i(x, t) = r^i + \xi_{\lambda}^i(x, t) \tag{B4}$$

where  $r^i$  is the radial vector from the origin to the point  $x$ . [In Cartesian coordinates  $x = (x^1, x^2, x^3)$ ,  $r^i = x^i$ .] For simplicity we take

$$\xi_{\lambda}^i(x, t) = \lambda \zeta^i(x, t), \tag{B5}$$

so that  $(d^2/d\lambda^2)\xi_{\lambda}^i = 0$ .<sup>8</sup> We will suppress the index  $\lambda$  from now on, writing  $O(\xi^2)$  to mean  $O(\lambda^2)$ . The Lagrangian change in a quantity  $Q$  will be defined by

$$\Delta Q = \psi^* \bar{Q} - Q, \tag{B6}$$

where  $\psi^*$  is the differential map. Its action on a tensor  $T^{i\dots j}_{k\dots l}$ , expressed in a given coordinate system, has the form

$$\psi^* T^{i\dots j}_{k\dots l} = \frac{\partial \psi^p}{\partial x^k}(x) \dots \frac{\partial \psi^q}{\partial x^l}(x) \frac{\partial \psi^{-1i}}{\partial x^m}[\psi(x)] \dots \frac{\partial \psi^{-1j}}{\partial x^n}[\psi(x)] T^{m\dots n}_{p\dots q}[\psi(x)]; \tag{B7}$$

on the tensor density  $g^{1/2}$ , where  $g$  is the determinant of the metric  $g_{ij}$ ,  $\psi^*$  acts in the manner

$$\psi^* g^{1/2} = J_{\psi} g^{1/2}. \tag{B8}$$

In effect,  $\Delta Q$  compares the components of  $Q$  at  $x$  with respect to a frame at  $x$  with the components of  $\bar{Q}$  at  $\psi(x)$  with respect to a frame dragged along by the fluid to  $\psi(x)$ . To first order in  $\xi^i$ ,  $\Delta$  agrees with the operator of the linear theory:

$$\Delta = \delta + \mathcal{L}_{\xi} + O(\xi^2). \tag{B9}$$

In terms of the Lagrangian operator  $\Delta$ , conservation of mass and entropy, (ii) and (iii) above, take the form

$$\Delta s = 0 \tag{B10}$$

and

$$\Delta(\rho g^{1/2}) = 0. \tag{B11}$$

The change in integrals over the entire fluid having the form

$$\int f \rho dV$$

has the property<sup>9</sup>

$$\delta \int f \rho dV = \int \Delta f \rho dV. \tag{B12}$$

*a) The First-Order Change in Energy*

Before doing the second-order calculation, it will be helpful to evaluate the change in energy to first order in  $\xi^i$  without assuming that  $\Delta s$  and  $\Delta(\rho g^{1/2})$  vanish. The total energy of a fluid has the form

$$E = T + U + W, \tag{B13}$$

where the kinetic energy is

$$T = \int \frac{1}{2} v^2 \rho dV, \tag{B14}$$

<sup>8</sup> No loss of generality is involved in making this ansatz: If a time-dependent solution is related to the equilibrium fluid by a map  $\psi(x)$ , it can be reached by the family of maps  $x \rightarrow x + \lambda[\psi(x) - x]$ .

<sup>9</sup> This may be seen as follows:

$$\delta \int_V f \rho dV = \int_V \bar{f} \rho dV - \int_V f \rho dV.$$

Now

$$\int_V \bar{f} \rho dV = \int_{\psi^{-1}(V)} \psi^*(\bar{f} \rho g^{1/2}) d^3x = \int_{\psi^{-1}(V)} (\psi^* \bar{f}) \rho dV$$

when  $\Delta(\rho g^{1/2}) = 0$ . Taking for  $V$  the entire space, we recover (B12).

the internal energy

$$U = \int u\rho dV, \tag{B15}$$

(with  $u$  the specific internal energy), and the potential energy  $W$  is

$$W = \int \frac{1}{2}\Phi\rho dV. \tag{B16}$$

When  $\Delta(\rho g^{1/2})$  is nonzero, equation (B12) is replaced to first order by

$$\delta \int f\rho dV = \int \Delta f\rho dV + \int fg^{-1/2}\Delta(\rho g^{1/2})dV, \tag{B17}$$

and we have

$$\delta E = \int [\Delta(\frac{1}{2}v^2 + u + \frac{1}{2}\Phi)\rho + (\frac{1}{2}v^2 + u + \frac{1}{2}\Phi)g^{-1/2}\Delta(\rho g^{1/2})]dV. \tag{B18}$$

We want to evaluate

$$\Delta(\frac{1}{2}v^2 + u + \frac{1}{2}\Phi) = \frac{1}{2}\Delta v^i v_i + \frac{1}{2}v^i \Delta v_i + \Delta u + \frac{1}{2}\Delta\Phi. \tag{B19}$$

The Lagrangian change in the velocity field, as before, is

$$\Delta v^i = \partial_i \xi^i; \tag{B20}$$

and the corresponding change in the covariant field is

$$\Delta v_i = \Delta(g_{ij}v^j) = \Delta g_{ij}v^j + g_{ij}\Delta v^j = \partial_i \xi_i + v^j \nabla_i \xi_j + v^j \nabla_j \xi_i. \tag{B21}$$

The specific internal energy changes by

$$\Delta u = \left(\frac{\partial u}{\partial \rho}\right)_s \Delta \rho + \left(\frac{\partial u}{\partial s}\right)_\rho \Delta s = \frac{p}{\rho^2} \Delta \rho + T \Delta s; \tag{B22}$$

and, finally, the perturbed gravitational potential is

$$\Delta \Phi = \delta \Phi + \xi^j \nabla_j \Phi, \tag{B23}$$

with

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho. \tag{B24}$$

Substituting these expressions for the Lagrangian changes in  $\Delta v^i$ ,  $\Delta v_i$ ,  $\Delta u$ , and  $\Delta \Phi$  in equation (B18), we find after an integration by parts

$$\delta E = \int [(h + \frac{1}{2}v^2 + \Phi)g^{-1/2}\Delta(\rho g^{1/2}) + \rho T \Delta s + \rho v^i \Delta v_i + \xi^i (\nabla_i p + \rho \nabla_i \Phi + \rho v^j \nabla_j v_i)]dV, \tag{B25}$$

where  $h = u + p/\rho$  is the specific enthalpy. The last term in the integrand vanishes by the equation of equilibrium of the unperturbed star, and we are left with

$$\delta E = \int [(h + \frac{1}{2}v^2 + \Phi)g^{-1/2}\Delta(\rho g^{1/2}) + \rho T \Delta s + \rho v^i \Delta v_i]dV. \tag{B26}$$

This equation was obtained by Schutz and Sorkin (1977) by slightly less direct means.

It is clear from the expression for  $\delta E$  that the energy of a star is an extremum only against perturbations for which the Lagrangian changes in particle number, entropy, and a certain component of velocity either vanish or integrate to zero. Perturbations which do not satisfy these conditions can change the energy to first order; and this fact is closely related to the difference between  $E_c$  and the second-order change in energy. That is, to second order  $\delta E$  will, in general, have two contributions: terms linear in the fluid variables  $\Delta v_i$ ,  $\Delta \rho$ , and  $\Delta s$  occur in the combination (B26) as in the linearized theory, while a second group of terms, quadratic in the field variables, combine to give the canonical energy  $E_c$ . Thus  $E_c$  will be the second-order change in energy if  $\Delta(\rho g^{1/2})$ ,  $\Delta s$ , and  $v^i \Delta v_i$  vanish to second order in  $\xi^i$ .

*b) Second-Order Change in Energy and Angular Momentum*

We want now to calculate the change in energy and angular momentum to second order in the perturbation. We will assume that the mass and entropy of each fluid element are constant, and so we will find that  $\delta E - E_c$  involves



only  $v^i \Delta t_i$ . As a by-product will follow the fact that for uniformly rotating stars,  $E_c - \Omega J_c = \delta E - \Omega \delta J$ . Consequently  $E_c - \Omega J_c$  is an expression in  $\xi^i$  invariant under gauge transformations  $\xi^i \rightarrow \xi^i + \eta^i$ ,  $\eta^i$  trivial.

The conservation of the mass and entropy of each fluid element will allow us to express  $E$  in terms of the Lagrangian displacement  $\xi^i$ . We will first obtain expressions for  $\Delta g_{ij}$ ,  $\Delta \rho$ , and  $\Delta v^i$  in terms of  $\xi$ , which we will then use to calculate  $\delta T$ ,  $\delta U$ , and  $\delta W$ . In finding the Lagrangian change in the metric  $\Delta g_{ij}$ , it is simplest to work in Cartesian coordinates. Because  $\delta g_{ij} = 0$ , we have

$$\Delta g_{ij} = \psi^* g_{ij} - g_{ij}, \tag{B27}$$

$$\begin{aligned} \psi^* g_{ij} &= \partial_i(x^m + \xi^m) \partial_j(x^n + \xi^n) g_{mn}(x + \xi) \\ &= (\delta_i^m + \partial_i \xi^m)(\delta_j^n + \partial_j \xi^n) g_{mn}(x) = g_{ij} + \partial_i \xi_j + \partial_j \xi_i + \partial_i \xi^m \partial_j \xi_m. \end{aligned}$$

The corresponding covariant expression is then

$$\psi^* g_{ij} = g_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \xi^m \nabla_j \xi_m, \tag{B28}$$

and

$$\Delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \xi^m \nabla_j \xi_m. \tag{B29}$$

To express the change in density  $\Delta \rho$  in terms of the displacement vector, we use conservation of mass,

$$0 = \Delta(\rho g^{1/2}) = \Delta \rho g^{1/2} + \rho \Delta g^{1/2} + \Delta \rho \Delta g^{1/2}. \tag{B30}$$

From (B29) or (B8) it follows that

$$g^{-1/2} \Delta g^{1/2} = \nabla_i \xi^i + \frac{1}{2}(\nabla_i \xi^i \nabla_j \xi^j - \nabla_i \xi^j \nabla_j \xi^i) + O(\xi^3), \tag{B31}$$

and we find

$$\frac{\Delta \rho}{\rho} = -\nabla_i \xi^i + \frac{1}{2}(\nabla_i \xi^i \nabla_j \xi^j + \nabla_i \xi^j \nabla_j \xi^i) + O(\xi^3). \tag{B32}$$

The perturbed velocity field can be obtained from condition (i) above in the following way. Let  $c^i(t)$  be the trajectory of a fluid element in the equilibrium configuration. The trajectory of the corresponding fluid element in the perturbed configuration is

$$\bar{c}^i(t) = \psi^i[c(t), t] = c^i(t) + \xi^i[c(t), t]. \tag{B33}$$

The value of the unperturbed velocity field at  $c^i(t)$  is

$$v^i[c^i(t), t] = \frac{d}{dt} c^i(t);$$

and, similarly, the perturbed velocity field has at time  $t$  and position  $\bar{c}^i(t)$  the value

$$v^i(\bar{c}^i(t), t) = \frac{d}{dt} \bar{c}^i(t) = \frac{d}{dt} \{c^i(t) + \xi^i[c(t), t]\} = (v^i + \xi^i + v^j \partial_j \xi^i)[c(t), t], \tag{B34}$$

where  $\xi^i = \partial_i \xi^i$ . Now the Lagrangian change in the velocity field  $\Delta v^i$  is

$$\Delta v^i = \psi^* \bar{v}^i - v^i, \tag{B35}$$

or

$$\Delta v^i(x) = (\partial_j \psi^{-1j} \bar{v}^j)[\psi(x)] - v^i(x), \tag{B36}$$

where the time dependence is suppressed for simplicity of notation. We can obtain  $\psi^{-1i}$  iteratively by substituting  $\psi^{-1}(x)$  for  $x$  in the equation

$$x^i = \psi^i(x) - \xi^i(x)$$

to give

$$\psi^{-1i}(x) = x^i - \xi^i[\psi^{-1}(x)] = x^i - \xi^i\{x - \xi[\psi^{-1}(x)]\} = x^i - \xi^i(x) + \xi^j \partial_j \xi^i(x) + O(\xi^3). \tag{B37}$$

Then

$$\frac{\partial \psi^{-1i}}{\partial x^j} [\psi(x)] = \delta^i_j - \partial_j \xi^i(x) + \partial_j \xi^k \partial_k \xi^i + O(\xi^3). \tag{B38}$$

Finally, from (B34), (B36), and (B38), the expression for  $\Delta v^i$  has the covariant form,

$$\begin{aligned} \Delta v^i &= (\delta^i_j - \nabla_j \xi^i + \nabla_j \xi^k \nabla_k \xi^i)(v^j + \xi^j + v^l \nabla_l \xi^i) - v^i + O(\xi^3) \\ &= \xi^i - \xi^j \nabla_j \xi^i. \end{aligned} \tag{B39}$$

The corresponding change in the covariant vector is

$$\Delta v_i = \Delta(g_{ij}v^j) = g_{ij}\Delta v^j + \Delta g_{ij}v^j + \Delta g_{ij}\Delta v^j = \dot{\xi}_i + v^j(\nabla_i \xi_j + \nabla_j \xi_i) + \xi^j \nabla_i \xi_j + v^k \nabla_k \xi^i \nabla_i \xi_j, \tag{B40}$$

where the expression for  $\Delta g_{ij}$  and  $\Delta v^i$  given in equations (B29) and (B39) have been used.

The calculation of  $\delta T$ ,  $\delta U$ , and  $\delta W$  is now straightforward. We have

$$\delta T = \delta \int \frac{1}{2} \rho v^2 dV = \int \frac{1}{2} \rho \Delta v^2 dV, \tag{B41}$$

from equation (B12). Now

$$\begin{aligned} \Delta v^2 &= \Delta(g_{ij}v^i v^j) = 2v^i \Delta v_i - \Delta g_{ij}v^i v^j + g_{ij}\Delta v^i \Delta v^j + O(\xi^3) \\ &= 2v^i \Delta v_i - (\nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \xi^k \nabla_j \xi_k)v^i v^j + g_{ij}\dot{\xi}^i \dot{\xi}^j = 2v^i \Delta v_i - 2v^i v^j \nabla_i \xi_j - v^j \nabla_j \xi^i v^k \nabla_k \xi_i + \dot{\xi}^i \dot{\xi}_i. \end{aligned} \tag{B42}$$

Thus

$$\begin{aligned} \delta T &= \int [\rho(v^i \Delta v_i - v^i v^j \nabla_i \xi_j) + \frac{1}{2} \rho(\dot{\xi}^i \dot{\xi}_i - v^j \nabla_j \xi^i v^k \nabla_k \xi_i)] dV \\ &= \int [\rho v^i \Delta v_i + \rho \xi^i v^j \nabla_j v_i + \frac{1}{2} \rho(\dot{\xi}^i \dot{\xi}_i - v^j \nabla_j \xi^i v^k \nabla_k \xi_i)] dV \end{aligned} \tag{B43}$$

where  $\Delta v_i$  is given in terms of the displacement  $\xi^i$  by equation (B40).

For  $U$  we have

$$\delta U = \delta \int \rho u dV = \int \rho \Delta u dV. \tag{B44}$$

Because  $\Delta s = 0$ ,

$$\Delta u = \left(\frac{\partial u}{\partial \rho}\right)_s \Delta \rho + \frac{1}{2} \left(\frac{\partial^2 u}{\partial \rho^2}\right)_s (\Delta \rho)^2 + O(\xi^3). \tag{B45}$$

But

$$\left(\frac{\partial u}{\partial \rho}\right)_s = \frac{p}{\rho^2}, \tag{B46}$$

and

$$\left(\frac{\partial^2 u}{\partial \rho^2}\right)_s = \frac{1}{\rho^2} \left(\frac{\partial p}{\partial \rho}\right)_s - \frac{2p}{\rho^3} = \frac{p}{\rho^3} (\gamma - 2), \tag{B47}$$

where  $\gamma$  is the adiabatic index. Substituting expression (B45) in equation (B44) for  $\delta U$  and integrating by parts, we find

$$\delta U = \int [\dot{\xi}^i \nabla_i p + \frac{1}{2} \gamma p (\nabla_i \xi^i)^2 + \xi^i \nabla_i p \nabla_j \xi^j + \frac{1}{2} \xi^i \xi^j \nabla_i \nabla_j p] dV. \tag{B48}$$

In calculating  $\delta W$  it is useful first to establish a few identities. From the exact relation

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho, \tag{B49}$$

it follows that

$$\int \delta \rho \Phi dV = \frac{1}{4\pi} \int \nabla^2 \delta \Phi \Phi dV = \int \delta \Phi \rho dV. \tag{B50}$$

Similarly,

$$\int \delta \rho \delta \Phi dV = \frac{-1}{4\pi G} \int \nabla_i \delta \Phi \nabla^i \delta \Phi dV. \tag{B51}$$

Now

$$\delta W = \delta \int \frac{1}{2} \rho \Phi dV = \int \frac{1}{2} \rho \Delta \Phi dV, \tag{B52}$$

as usual. We also have, however,

$$\delta W = \frac{1}{2} \int \delta(\rho \Phi) dV = \frac{1}{2} \int (\rho \delta \Phi + \delta \rho \Phi + \delta \rho \delta \Phi) dV = \int \rho \delta \Phi dV - \frac{1}{8\pi G} \int \nabla_i \delta \Phi \nabla^i \delta \Phi dV, \tag{B53}$$

where the identities (B50) and (B51) have been used. Together the two expressions (B52) and (B53) for  $\delta W$  imply

$$\delta W = \int \rho(\Delta\Phi - \delta\Phi)dV + \frac{1}{8\pi G} \int \nabla_i \delta\Phi \nabla^i \delta\Phi dV. \tag{B54}$$

Now

$$\begin{aligned} \Delta\Phi - \delta\Phi &= (\psi^*\bar{\Phi} - \Phi) - (\bar{\Phi} - \Phi) = \psi^*\bar{\Phi} - \bar{\Phi} = \psi^*\Phi - \Phi + \psi^*\delta\Phi - \delta\Phi \\ &= \xi^i \nabla_i \Phi + \xi^i \nabla_i \delta\Phi + \frac{1}{2} \xi^i \xi^j \nabla_i \nabla_j \Phi + O(\xi^3). \end{aligned} \tag{B55}$$

Thus

$$\delta W = \int \left[ \rho \xi^i \nabla_i \Phi + \rho \xi^i \nabla_i \delta\Phi + \frac{1}{8\pi G} \nabla_i \delta\Phi \nabla^i \delta\Phi + \frac{1}{2} \rho \xi^i \xi^j \nabla_i \nabla_j \Phi \right] dV. \tag{B56}$$

To first order, the Eulerian change in density is

$$\delta\rho = -\nabla_i(\rho\xi^i),$$

from which it follows that

$$\int \rho \xi^i \nabla_i \delta\Phi dV = \int \delta\rho \delta\Phi dV + O(\xi^3) = -\frac{1}{4\pi} \int \nabla_i \delta\Phi \nabla^i \delta\Phi dV, \tag{B57}$$

by equation (B51); and the expression for  $\delta W$  then takes the form

$$\delta W = \int \left( \rho \xi^i \nabla_i \Phi - \frac{1}{8\pi G} \nabla_i \delta\Phi \nabla^i \delta\Phi + \frac{1}{2} \rho \xi^i \xi^j \nabla_i \nabla_j \Phi \right) dV. \tag{B58}$$

Finally, combining equations (B43), (B48), and (B58) for  $\delta T$ ,  $\delta V$ , and  $\delta W$ , we obtain

$$\begin{aligned} \delta E &= \delta T + \delta U + \delta W \\ &= \int \left[ \rho \xi^i \left( v^j \nabla_j v_i + \frac{1}{\rho} \nabla_i p + \nabla_i \Phi \right) + \rho v^i \Delta v_i \right] dV \\ &\quad + \frac{1}{2} \int \left[ \rho \xi^i \xi_i - \rho v^j \nabla_j \xi^i v^k \nabla_k \xi_i + \gamma p (\nabla_i \xi^i)^2 + 2 \xi^i \nabla_i p \nabla_j \xi^j + \xi^i \xi^j (\nabla_i \nabla_j p + \rho \nabla_i \nabla_j \Phi) - \frac{1}{4\pi G} \nabla_i \delta\Phi \nabla^i \delta\Phi \right] dV. \end{aligned} \tag{B59}$$

The equilibrium of the unperturbed star implies that the first term in the integrand vanishes, and we have

$$\delta E = \int \rho v^i \Delta v_i dV + E_c, \tag{B60}$$

where  $E_c$  is the canonical energy given in equation (45). As anticipated, when the mass and entropy of each fluid element are conserved,  $\delta E$  differs from  $E_c$  only by a term involving the component of the change in velocity  $\Delta v_i$  along the unperturbed velocity field.

Finding the second-order change in angular momentum is much easier. We will write  $\phi^i$  for the rotational symmetry vector (in cylindrical or polar coordinates, it has the form  $\delta^i_\phi$ ). Then

$$J = \int \rho v_i \phi^i dV \tag{B61}$$

and

$$\delta J = \int \rho \Delta(v_i \phi^i) dV = \int \rho \phi^i \Delta v_i dV + \int \rho (v_i \Delta \phi^i + \Delta v_i \phi^i) dV. \tag{B62}$$

Because  $\delta\phi^i = 0$ ,

$$\Delta\phi^i = \psi^* \phi^i - \phi^i = \xi^j \nabla_j \phi^i - \phi^j \nabla_j \xi^i + (\phi^j \nabla_j \xi^k - \xi^j \nabla_j \phi^k) \nabla_k \phi^i + \frac{1}{2} \xi^i \xi^k \nabla_j \nabla_k \phi^i + O(\xi^3). \tag{B63}$$

It is easy to verify directly (and true of any Killing vector on flat space) that  $\nabla_j \nabla_k \phi^i = 0$ , whence—using the definition (4) of Lie derivative—we have

$$\Delta\phi^i = -\mathcal{L}_\Phi \xi^i + \mathcal{L}_\Phi \xi^k \nabla_k \xi^i + O(\xi^3). \tag{B64}$$

Then

$$\Delta\phi^i v_i + \Delta v_i \phi^i = -\mathcal{L}_\Phi \xi^i v_i - \mathcal{L}_\Phi \xi^i (\xi_i + v^j \nabla_j \xi^i); \tag{B65}$$

and from equation (B62),

$$\delta J = \int \rho \phi' \Delta v_i dV - \int [\rho \xi_\phi \xi^i (\xi_i + v^j \nabla_j \xi_i) + \rho \xi_\phi \xi^i v_i] dV. \tag{B66}$$

Now

$$\int \rho \xi_\phi \xi^i v_i dV = \int \xi_\phi (\rho \xi^i v_i) dV = 0, \tag{B67}$$

and we are left with

$$\delta J = \int \rho \phi' \Delta v_i dV - \int \rho \xi_\phi \xi^i (\xi_i + v^j \nabla_j \xi_i) dV = \int \rho \phi' \Delta v_i dV + J_c, \tag{B68}$$

where

$$J_c = - \int \rho \xi_\phi \xi^i (\xi_i + v^j \nabla_j \xi_i) dV \tag{B69}$$

is the canonical angular momentum given by equation (48).

As in equation (B60) for the energy, the difference between  $\delta J$  and  $J_c$  depends only on the  $\phi$ -component of  $\Delta v_i$ . In fact (as Bardeen first noted), when the unperturbed star rotates uniformly with angular velocity  $\Omega$  ( $v^i = \Omega \phi^i$ ), equations (B60) and (B68) imply

$$\delta E - \Omega \delta J = E_c - \Omega J_c.$$

Because  $\delta E$  and  $\delta J$  are physical quantities, the combination  $E_c - \Omega J_c \equiv E_{c,R}$  must be invariant under the gauge transformations  $\xi^i \rightarrow \xi^i + \eta^i$ ,  $\eta^i$  trivial. One therefore expects to be able to write  $E_{c,R}$  in terms of Eulerian changes in the fluid variables, and we do this in Appendix C.

### APPENDIX C

#### ENERGY IN THE ROTATING FRAME

The energy  $E_{c,R}$  may be found from equations (64), (45), and (B69):

$$E_{c,R} = \frac{1}{2} \int dV \left[ \rho |\partial \xi / \partial t'|^2 + \frac{1}{\rho} \delta \rho \delta p + (\nabla \cdot \xi) (\xi \cdot \mathcal{S}) - \frac{1}{4\pi G} |\nabla \delta \Phi|^2 \right], \tag{C1}$$

where

$$\mathcal{S} = \nabla p - \frac{\gamma p}{\rho} \nabla \rho = \left( \frac{\partial p}{\partial s} \right)_\rho \nabla s \tag{C2}$$

and where  $t'$  is the time-coordinate of the rotating frame. Our object is to express  $E_{c,R}$  in terms of the Eulerian perturbations. In this frame  $\delta v = \partial \xi / \partial t'$ , since the unperturbed velocity is zero, so only the third term in (C1) needs work. To do this we note that, because the star is uniformly rotating, the level surfaces of  $p$ ,  $\rho$ , and  $S$  all coincide. There are therefore relations inside the star

$$p = p(s), \quad \rho = \rho(s). \tag{C3}$$

Then the third term is

$$(\nabla \cdot \xi) \left( \frac{\partial p}{\partial s} \right)_\rho \xi \cdot \nabla s = \frac{1}{\rho} \delta s \left( \frac{\partial p}{\partial s} \right)_\rho (\delta \rho + \xi \cdot \nabla \rho) = \frac{1}{\rho} \delta s \left( \frac{\partial p}{\partial s} \right)_\rho \left[ \left( \frac{\partial \rho}{\partial p} \right)_s \delta p + \left( \frac{\partial \rho}{\partial s} \right)_p \delta s - \frac{d\rho}{ds} \delta s \right], \tag{C4}$$

where the last term inside the brackets is the derivative inside the star as given by (C3). This suffices to express  $E_{c,R}$  in terms of Eulerian perturbations, but the whole expression can be simplified considerably by using the identity

$$\left( \frac{\partial p}{\partial s} \right)_\rho \left( \frac{\partial s}{\partial \rho} \right)_p \left( \frac{\partial \rho}{\partial p} \right)_s = -1.$$

One combines the first term in (C4) with the second in (C1), and one combines the last two in (C4) with each other to get

$$E_{c,R} = \frac{1}{2} \int dV \left[ \rho |\delta v|^2 + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_s (\delta p)^2 + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial s} \right)_p \left( \frac{dp}{ds} \right) (\delta s)^2 - \frac{1}{4\pi G} |\nabla \delta \Phi|^2 \right]. \tag{C5}$$

This is equivalent to the energy integral defined by Howard and Siegmann (1969), apart from the  $\delta\Phi$  term which was absent from their treatment because they assumed an external gravitational field.

The Howard-Siegmann paper is partly concerned with geostrophic flows, which are time-independent local perturbations. In this connection they use the energy  $E_{c,R}$  as an inner product to define the "geostrophic part" of arbitrary initial data, a procedure which closely parallels our manner of defining the trivial part of initial data. Not surprisingly, this leads them to certain conditions involving Ertel's constant.

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