Mon. Not. R. astr. Soc. (1972) **159**, *Short Communication.*

LIMITS ON GRAVITATIONAL RADIATION FROM TWO GRAVITATIONALLY BOUND BLACK HOLES

G. W. Gibbons and Bernard F. Schutz

(Communicated by M. J. Rees)

(Received 1972 August 15)

SUMMARY

Upper bounds are computed for the energy emitted in gravitational radiation by two black holes falling into one another from rest at a range of separations. It is shown that before the ratio of total mass to proper separation exceeds 0·759 an event horizon must have formed around both.

INTRODUCTION

In a previous paper (Gibbons 1972a), one of us combined Hawking's (1972) recent theorem (that the area of the event horizon must increase in time) with a study of the time-symmetric initial value problem to obtain upper bounds on the energy available for gravitational radiation during the evolution of systems more general than those originally considered by Hawking. In this paper we apply these methods to a class of time-symmetric initial value surfaces first given by Misner (Misner (1960, 1963), Wheeler (1961)), all of which can be given various topologies, including that of a wormhole. Our aims are:

1. To obtain upper limits on the efficiency for radiating gravitational waves when the initial system can be regarded as two separate non-rotating black holes with finite separation (i.e. gravitationally bound); and
2. To obtain some criterion for deciding when the initial system must be regarded as a single distorted non-rotating black hole.

We find that the upper limit on the efficiency decreases with decreasing initial separation of the black holes, being always less than the 29·3 per cent obtained by Hawking. We also find that the initial system must be regarded as a single black hole by the time the ratio of total mass to proper separation has increased to 0·759.

THE INITIAL VALUE SURFACE

The solution given by Misner (1960, 1963) has the following form. Let μ, θ, ϕ be bispherical coordinates for Euclidean space. The two poles of the system are a distance $2a$ apart. Then ϕ is a cylindrical angle about the axis joining the poles ($0 \leq \phi < 2\pi$). The lines θ = constant, ϕ = constant are arcs of circles with end points on the poles, θ being π minus half the angle subtended by the arc ($0 < \theta \leq \pi$, with $\theta = \pi$ the axis). The lines μ = constant, ϕ = constant are the orthogonal trajectories of these circles and form a system of coaxial circles. As $\mu \rightarrow \pm\infty$, the circles shrink down onto the poles; $\mu = 0$ is the plane perpendicularly bisecting the axis. The black holes are constructed by removing the spheres

$|\mu| > \mu_0$, calling their boundaries T_1 and T_2, and giving the initial hypersurface S the metric

$$ds^2 = a^2\Phi^4\{d\mu^2+d\theta^2+\sin^2\theta\, d\phi^2\},$$

$$\Phi = \sum_{n=-\infty}^{n=+\infty} [\cosh(\mu+2n\mu_0)-\cos\theta]^{-1/2}.$$

This metric is conformally flat and a is an overall constant length scale. Notice that Φ is a periodic function of μ with period $2\mu_0$.

If corresponding points on T_1 and T_2 are identified, a 'wormhole' results. If S is joined to a similar surface S' at the spheres we obtain two sheets joined by two Einstein–Rosen bridges. Many other topologies are possible. Here we are only concerned with the exterior region given above. The hypersurface S is asymptotically flat with total mass (Misner 1960)

$$M = a\sum_{n=1}^{n=\infty} 4\,(\sinh n\mu_0)^{-1}.$$

The least proper distance between the two spheres, passing through the external region, is (Lindquist 1963)

$$L = 2a\left[1+2\sum_{n=1}^{n=\infty} n\mu_0\,(\sinh n\mu_0)^{-1}\right].$$

In the limit $\mu_0 \to \infty$ we obtain ordinary Euclidean space. Asymptotically we have

$$M \to 4a/\sinh\mu_0,$$
$$L \to 2a(1+\mu_0\, e^{-\mu_0}).$$

Thus the two black holes shrink to zero mass at a distance $2a$ apart. As μ_0 decreases, both M and L increase such that M/L increases.

The spheres T_1 and T_2 are both minimal surfaces and marginally trapped surfaces and must lie within the event horizon (Gibbons 1972a). For close separations we expect a third minimal surface T_3 to enclose both T_1 and T_2. This must lie within the event horizon. Fig. 1 illustrates the three qualitatively different situations that are obtained as μ_0 decreases from ∞.

In both types I and II of Fig. 1, the lack of a third minimal surface means that the area of the horizon, $A(\partial B)$, must exceed the sum of the areas of T_1 and T_2.* Thus we have

$$A(\partial B) > A(T_1)+A(T_2) = 2A(T_1).$$

Knowing $A(T_1)$ gives a lower bound for $A(\partial B)$.

For type III we have

$$A(\partial B) > A(T_3) < A(T_1)+A(T_2).$$

In this case we cannot obtain a bound on $A(\partial B)$ this way.

As the system evolves to form a final Schwarzschild black hole with final mass M_f, the area of the event horizon *increases* to a final value $16\pi M_f^2$. Thus

* To make the argument strictly rigorous for type II, one must put a topology on the space of surfaces homologous to ∂B and show that (1) the space is compact, and (2) the area is a continuous function on this space. There seems no reason why this should not be possible.

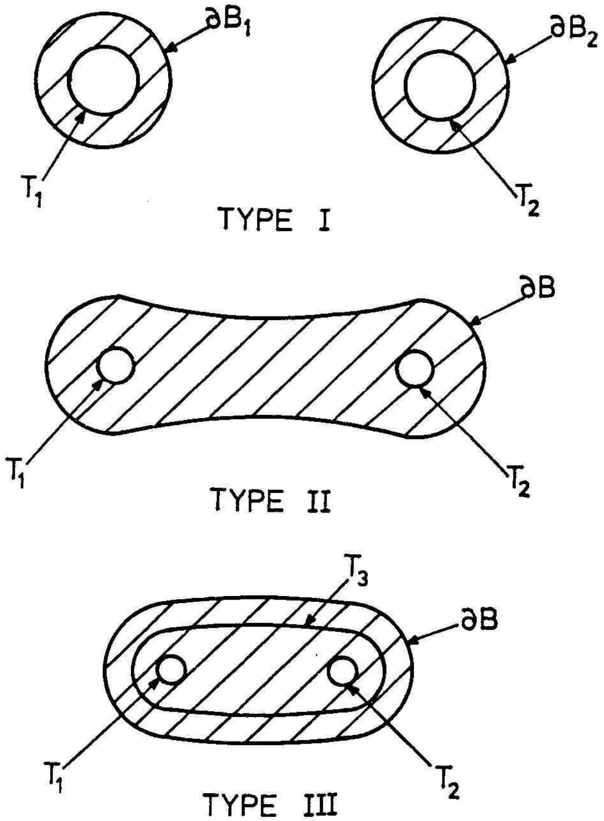

FIG. 1. *The three possible topologies for the exterior of* T_1 *and* T_2. *The intersection of the event horizon with* S *is* ∂B. *Shaded parts are the interiors of the black holes. In type I,* T_1 *and* T_2 *are enclosed by disconnected pieces of* ∂B. *(But note that if* T_1 *and* T_2 *are identified to form a wormhole, then by the precise definition of a black hole given by Hawking (1972) the shaded region is a single black hole.) In type II the minimal surfaces are close enough to be enclosed within a connected* ∂B, *but there does not exist a third minimal surface. In type III the third minimal surface,* T_3, *has formed inside* ∂B.

the efficiency for radiating gravitational waves,

$$\eta \equiv 1 - \frac{M_f}{M_i},$$

is bounded above by

$$\eta < 1 - \frac{1}{M}\sqrt{\frac{A(\partial B)}{16\pi}} < 1 - \frac{1}{M}\sqrt{\frac{A(T_1)}{8\pi}} \equiv \eta_0,$$

the last inequality holding only for types I and II.

The proper area of the sphere T_1 is given by

$$A(T_1) = 2\pi a^2 \int_0^\pi [\Phi(\mu = \mu_0, \theta)]^4 \sin\theta \, d\theta.$$

For large μ_0 this becomes

$$A(T_1) \to 128\,\pi a^2 (\cosh \mu_0)^{-2}.$$

From this one can deduce that asymptotically

$$\eta_0 \to 1 - \frac{1}{\sqrt{2}}\left(1 + \frac{1}{8}\frac{M^2}{L^2}\right).$$

NUMERICAL RESULTS AND DISCUSSION

We have computed M, L, A and η_0 for a range of the parameter μ_0. The results are shown in Table I and Fig. 2. Note that as M/L increases (holes becoming more tightly bound), η_0 falls from Hawking's maximum of 29·3 per cent to zero at $\mu_0 = 1{\cdot}16$, $M/L = 0{\cdot}759$. If this were a valid efficiency limit, then for larger M/L we would have $M_f > M_i$, an absurd result. We must conclude that the system has become of type III (or that possibly a naked singularity will develop)

TABLE I

Calculated values of mass (M), *proper separation* (L), *'relativity parameter'* (M/L), *area of* T_1 (A), *and upper limit on efficiency* (η_0), *for a range of* μ_0. *Computational accuracies are: for* M *and* L, ±0·05 *per cent; for* M/L, ±0·1 *per cent; for* A/2π, ±0·5 *per cent; for* η_0, ±0·003 *absolute. Negative* η_0 *indicates that* A(T_1) *no longer limits the efficiency, i.e. that the system is of type III.*

μ_0	M	L	M/L	$A/2\pi$	η_0
0·9	6·167	6·486	0·9508	196·4	−0·137
0·95	5·621	6·199	0·9069	154·3	−0·106
1·0	5·141	5·938	0·8659	122·3	−0·076
1·25	3·425	4·952	0·6916	43·97	0·031
1·5	2·394	4·294	0·5576	18·47	0·102
1·75	1·728	3·824	0·4518	8·562	0·153
2·0	1·274	3·472	0·3669	4·277	0·188
2·25	0·9530	3·199	0·2979	2·237	0·215
2·5	0·7205	2·983	0·2416	1·216	0·234
2·75	0·5490	2·808	0·1955	0·6789	0·249
3·0	0·4206	2·666	0·1578	0·3867	0·260
5·0	0·05433	2·137	0·02543	0·005906	0·293

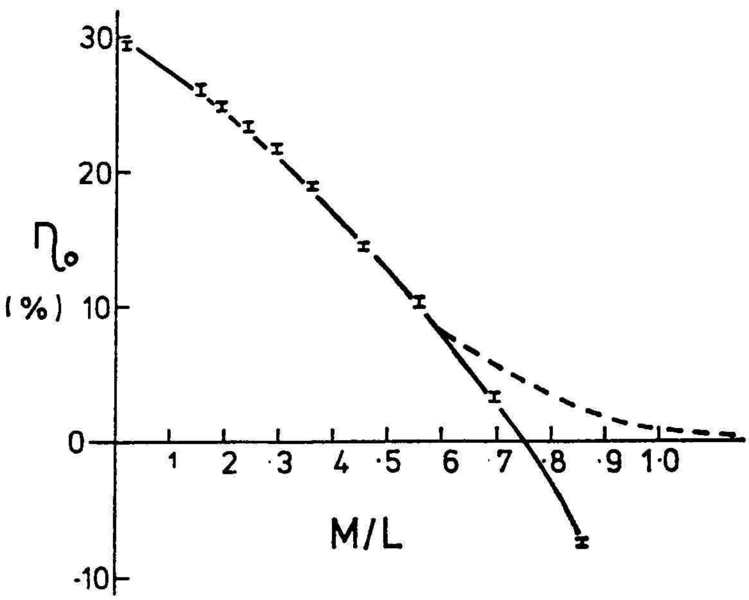

FIG. 2. *Plot of the limit on the efficiency computed using* A(T_1) (*solid line*) *against the parameter* M/L. *The dotted curve represents the conjectured behaviour of the efficiency for large* M/L. *Its point of departure from the solid line marks the boundary between types II and III. As this boundary has* not *been computed, no significance should be attached to the value of* M/L *or* η_0 *at which we indicate the junction of the two curves.*

before M/L reaches 0·759, invalidating the efficiency limit computed from $A(T_1)$. At the point of transition from type II to type III, we must have

$$A(T_3) = A(T_1)+A(T_2).$$

For larger M/L the real efficiency limit must branch off the curve of our calculations. Such a branch is shown dotted in Fig. 2. No significance should be attached to the values of M/L and η_0 at the branching point as they are not calculable by the simple methods used in this paper. As M/L gets very large the system looks more and more like a slightly distorted black hole, so the true efficiency limit should go asymptotically to zero. This view is supported by a second order perturbation calculation (Gibbons 1972b, to be published). Behaviour like this occurs in a similar system discussed by Brill & Lindquist (1963), though those authors did not compute areas.

For small M/L, η_0 depends only on $(M/L)^2$. The difference between η_0 and the Hawking limit of 29·3 per cent represents an energy unavailable for radiation, i.e. the difference in gravitational potential energy between the state at infinite separation and that at finite separation. Its dependence on $(M/L)^2$ might be thought to indicate an inverse cube law for the attraction between two widely separated black holes. This is not correct. The dependence on $(M/L)^2$ is only along a curve parameterized by μ_0 along which M and L cannot be varied independently. In fact, as $\mu_0 \to \infty$ and $M/L \to 0$, L becomes constant, so the change in potential energy results only from a change in the mass M. It is easy to show that the resulting change in efficiency (i.e. in the potential energy divided by M) is proportional to M^2, not M.

ACKNOWLEDGMENT

We thank S. W. Hawking for useful discussions. B. F. S. was an N.S.F. postdoctoral fellow while this work was performed.

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge

Present address:

Bernard F. Schutz, Department of Physics, Yale University, New Haven, Connecticut, U.S.A.

REFERENCES

Brill, D. & Lindquist, R., 1963. *Phys. Rev.*, **131**, 471.
Gibbons, G. W., 1972a. *Commun. M. Phys.*, **27**, 87.
Hawking, S. W., 1972. *Commun. M. Phys.*, **25**, 152.
Lindquist, R., 1963. *J. Math. Phys.*, **4**, 938.
Misner, C. W., 1960. *Phys. Rev.*, **118**, 1110.
Misner, C. W., 1963. *Ann. Phys.*, **24**, 102.
Wheeler, J. A., 1961. *Rev. mod. Phys.*, **33**, 63.