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LINEAR PULSATIONS AND STABILITY OF DIFFERENTIALLY ROTATING STELLAR MODELS. II. GENERAL-RELATIVISTIC ANALYSIS*

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ABSTRACT

The author has previously given an Eulerian velocity-potential variational principle for relativistic perfect-fluid hydrodynamics. The second variation of the principle is here used as the Lagrangian density governing the evolution of small perturbations of fully relativistic, differentially rotating stellar models. Noether's theorem is used to construct a globally conserved angular-momentum density, whose integral over a spacelike hypersurface is the second-order correction to the star's total angular momentum. From the Hamiltonian is constructed a globally conserved energy density, whose integral is the second-order correction to the star's active gravitational mass. By Liapunov's second theorem, positive-definiteness of the energy density guarantees stability of the star. In the Newtonian limit and in the special case of relativistic radial pulsations, this is equivalent to stability criteria already known. Means are discussed whereby the general criterion might be made more suitable for practical applications.

I. INTRODUCTION AND SUMMARY

The importance of general relativity to so many astrophysical problems makes an analysis of the stability of relativistic systems very desirable. In the Newtonian regime the theory of the stability of perfect-fluid stellar models against small dynamical perturbations is well established (cf. Schutz 1972 [preceding paper], hereafter referred to as Paper I; see also the references cited therein). The corresponding relativistic analysis, however, is complicated by two factors: the existence of 10 components of the gravitational field, and the emission of gravitational radiation by the pulsating star.

Only for radial pulsations of spherical systems has a fully relativistic dynamical stability analysis been performed: by Chandrasekhar (1964) for relativistic stars; and by Ipser and Thorne (1968), Ipser (1969), and Fackerell (1970) for relativistic clusters of stars. In addition, Chandrasekhar (1965*a*, *b*) has analyzed the nonradial pulsations of stars in the post-Newtonian approximation, which excludes gravitational radiation. Chandrasekhar and Friedman (1971) have also recently investigated criteria for the existence of zero-frequency modes in rigidly rotating stars, where radiation is also negligible. Their work should prove applicable to stellar models that become unstable through zero-frequency oscillations. The equations governing arbitrary nonradial pulsations of fully relativistic nonrotating stars were derived by Thorne and Campolattaro (1967) (see also Ipser and Thorne 1972). They are so complicated, however, that—although they have yielded information about convection (Islam 1970) and about the emission of and damping by gravitational radiation (Thorne 1969; Ipser 1971)—they have so far given us no information about dynamical stability.

The existence of 10 perturbed metric functions instead of just one perturbed gravitational potential is an algebraic complication. It means that in general there will be many coupled equations, which will rarely possess a solution in closed form. It means

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that relativistic stability analyses will probably have to rely more heavily upon numerical calculations than the corresponding Newtonian analyses do.

The complication of gravitational radiation is more fundamental. It means that realistic pulsations will always have complex frequencies; that normal modes will be replaced by "resonances" of finite width; that self-adjoint equations (standing-wave boundary conditions) will not describe realistic systems; and that a single stability criterion that is both necessary *and* sufficient is probably not to be hoped for. It is possible to look for necessary conditions for stability by examining standing-wave modes in the zero-frequency limit. This is the approach of Chandrasekhar and Friedman (1971). But such approaches neglect damping by gravitational radiation, so they may not pinpoint the onset of instability accurately. It is therefore useful to have sufficient conditions for stability as well.

In Paper I we showed that all known Newtonian dynamical stability criteria could be derived from the velocity-potential variational principle of Seliger and Whitham (1968). That variational principle can be extended to general relativity (Schutz 1970; see also Schmid 1970*a*, *b* for an independent derivation of the special relativistic version). In this paper we show that methods similar to those we used in Paper I lead us in general relativity to a sufficient condition for the stability of arbitrary pulsations of fully relativistic, differentially rotating stellar models.

We could presumably also derive our criterion from the variational principle of Taub (1954), or from any of the many other relativistic perfect-fluid variational principles. Taub (1969) in fact derived Chandrasekhar's (1964) stability criterion for radial pulsations using a method very similar to the one we use here, but starting from a different variational principle. We have elected to start with the velocity-potential variational principle because it is an Eulerian principle: it does not require us to deal explicitly with "fluid elements" or "particle paths."

The plan of the paper is as follows. In § II we derive the Lagrangian governing arbitrary perturbations of arbitrary flows of a relativistic perfect fluid. This Lagrangian is the second variation of the Lagrangian for the velocity-potential variational principle of Schutz (1970). In § III we specialize the unperturbed state to that of an axially symmetric, differentially rotating star. From Noether's theorem we construct the conserved angular-momentum density of the perturbations (including the gravitational waves), and from the Hamiltonian we construct the conserved energy density. Both are quadratic in the perturbations.

We obtain the following results: (i) The total angular momentum and energy (integrals of the densities over a spacelike hypersurface of the unperturbed spacetime) are unique and gauge-invariant. (ii) If the star is stable, and if the "unperturbed" star is defined to be the star that is left behind after the pulsations have damped out, then all first-order contributions to the total angular momentum and energy vanish. (iii) If the star is stable, the total angular momentum and energy are the second-order corrections to the total angular momentum and active gravitational mass of the star. (iv) The gravitational-wave parts of the densities of energy and angular momentum become, in the short-wavelength approximation, the appropriate components of the Isaacson (1968) stress-energy tensor for gravitational radiation. (v) In the case of the nonrotating unperturbed star, the energy density reduces in the Newtonian limit to the energy density derived in Paper I.

In § IV we prove that a sufficient condition for stability is that the total energy be positive-definite. Unfortunately, as the energy contains contributions from gravitational radiation, it is not yet in its most practical form for astrophysical applications. A more practical form would be an integral of purely fluid quantities over just the star's interior. We therefore discuss what procedures are most likely to succeed in reducing the stability criterion to such a form. We conclude § IV by demonstrating that our sufficient condition for stability reduces for the case of radial pulsations to the necessary and sufficient condition of Chandrasekhar (1964).

II. PERTURBATIONS OF AN ARBITRARY FLOW

a) The Velocity-Potential Variational Principle

As in Paper I, we begin from the Eulerian velocity-potential variational principle, the general-relativistic version of which was obtained by Schutz (1970). (We follow the notation and conventions of Schutz 1970 throughout. In particular, Greek indices run from 0 to 3, while Latin indices run from 1 to 3. The metric signature is $+2$.)

The four-velocity has the representation

$$U_\nu = \mu^{-1}(\psi_{,\nu} + \alpha\beta_{,\nu} - S\theta_{,\nu}) . \tag{1}$$

(We find it convenient to deal with $\psi \equiv \phi + \theta S$ rather than with ϕ, which was used by Schutz 1970. This is the only way in which our conventions differ from those of that paper.) In equation (1), S is the specific entropy and μ the specific enthalpy (including rest mass),

$$\mu = 1 + \Pi + p/\rho_0 = (\rho + p)/\rho_0 ; \tag{2}$$

Π is the specific internal energy, p the pressure, ρ the density of total mass-energy, and ρ_0 the rest-mass density (number density of baryons times rest mass of one baryon), all as measured in a locally comoving inertial frame.

The velocity potentials obey the equations of evolution

$$U^\nu\psi_{,\nu} = -\mu + TS , \tag{3a}$$

$$U^\nu\alpha_{,\nu} = 0 , \tag{3b}$$

$$U^\nu\beta_{,\nu} = 0 , \tag{3c}$$

$$U^\nu S_{,\nu} = 0 , \tag{3d}$$

$$U^\nu\theta_{,\nu} = T , \tag{3e}$$

where T is the temperature. Note that equations (1), (3a), (3c), and (3e) imply

$$U^\nu U_\nu = -1 . \tag{4}$$

Supplemented by an equation of state,

$$p = p(\mu, S) , \tag{5}$$

and the equation of continuity,

$$(\rho_0 U^\nu)_{;\nu} = 0 , \tag{6}$$

equations (1) and (3) are completely equivalent to the usual hydrodynamical equations: equations (4), (5), (6), and

$$T^{\mu\nu}{}_{;\nu} = 0 , \tag{7}$$

with

$$T^{\mu\nu} = \rho_0\mu U^\mu U^\nu + pg^{\mu\nu} . \tag{8}$$

Equations (3) and (6) plus the Einstein field equations follow from a variational principle whose action is

$$I = \int (R + 16\pi p)(-g)^{1/2}d^4x , \tag{9}$$

where R is the scalar curvature of spacetime (we set $c = G = 1$). The curvature is varied with respect to $g^{\sigma\nu}$ in the usual manner. The pressure is taken to be a function of μ

and S through the equation of state; its variation is found from the first law of thermodynamics:

$$dp = \rho_0 d\mu - \rho_0 T dS \,. \tag{10}$$

The independent variables of the principle are ψ, α, β, θ, S, and $g^{\sigma\nu}$. Equations (1) and (4) combine to give μ as a function of these variables:

$$\mu^2 = -g^{\sigma\nu}(\psi_{,\sigma} + \alpha\beta_{,\sigma} - S\theta_{,\sigma})(\psi_{,\nu} + \alpha\beta_{,\nu} - S\theta_{,\nu}) \,. \tag{11}$$

Varying ψ, α, β, θ, S, and $g^{\sigma\nu}$ gives, respectively, equations (6), (3c), (3b), (3d), (3e), and the field equations

$$R_{\sigma\nu} - \tfrac{1}{2}Rg_{\sigma\nu} = 8\pi T_{\sigma\nu} \,, \tag{12}$$

with $T_{\sigma\nu}$ from equation (8). Equation (3a) follows from the rest of equations (3) and equation (11); it is not an independent Euler-Lagrange equation.

b) Gauge Freedom in the Perturbations

A perturbation in the fluid's motion perturbs the geometry of spacetime. If the perturbation is small, it is reasonable to separate it from the "background" unperturbed spacetime and to treat it as a field on the background geometry. We therefore define $h^{\sigma\nu}$ to be the (Eulerian) perturbation in $g^{\sigma\nu}$, and $g_{(B)}{}^{\sigma\nu}$ to be the background unperturbed metric:

$$g^{\sigma\nu} \text{ (perturbed spacetime)} = g_{(B)}{}^{\sigma\nu} + h^{\sigma\nu} \,. \tag{13}$$

Now $h^{\sigma\nu}$ is a tensor on the background spacetime. We can therefore raise and lower its indices with $g_{(B)}$; e.g.,

$$h^{\sigma}{}_{\lambda} \equiv h^{\sigma\nu} g_{(B)\nu\lambda} \,.$$

Our definition of $h^{\sigma\nu}$ is at slight variance with the usual usage, where $h_{\sigma\nu}$ is taken to be the perturbation in $g_{\sigma\nu}$. Here we have

$$h_{\sigma\nu} = h^{\alpha\beta} g_{(B)\alpha\sigma} g_{(B)\beta\nu} = -\delta g_{\sigma\nu} + O(h^2) \,. \tag{14}$$

The "background" geometry is a fiction, however. Because the real spacetime possesses fine structure that is absent from the "background," there is no unique way to identify points in real spacetime with points in the background; thus, there is no unique way to define $h^{\sigma\nu}$ from equation (13). If η^σ generates a point transformation in the perturbed spacetime that is small (i.e., a change in the identification of points between the fictitious background and the real perturbed spacetime that is on the order of the scale of the "fine structure" of the real spacetime), then $h^{\sigma\nu}$ undergoes the change

$$h^{\sigma\nu} \rightarrow h^{\sigma\nu} + \pounds_\eta g_{(B)}{}^{\sigma\nu} = h^{\sigma\nu} - \eta^{\sigma;\nu} - \eta^{\nu;\sigma} \,. \tag{15}$$

Here $\pounds_\eta$ is the Lie derivative along η^σ, and semicolons (throughout this paper) denote derivatives covariant with respect to the *unperturbed* spacetime.

Under the same point transformation the perturbations in the velocity potentials must also change. For example, we define $\delta\psi$, the Eulerian change in ψ, by the equation

$$\psi \text{ (perturbed spacetime)} = \psi_{(B)} + \delta\psi \,. \tag{16}$$

Then $\delta\psi$ changes by

$$\delta\psi \rightarrow \delta\psi + \pounds_\eta \psi_{(B)} = \delta\psi + \psi_{(B),\sigma}\eta^\sigma \,. \tag{17}$$

Similarly, all functions of the perturbed velocity potentials change: e.g.,

$$\delta U_\nu \rightarrow \delta U_\nu + \pounds_\eta U_{(B)\nu} = \delta U_\nu + U_{(B)\nu;\sigma}\eta^\sigma + U_{(B)\sigma}\eta^\sigma{}_{;\nu} \,. \tag{18}$$

Equations (15) and (17) together are called a *gauge transformation.* Most of our expressions—such as the energy density in the pulsations—will not be gauge-invariant.

Nevertheless, we will see that physically measurable quantities, such as the *total* energy, are gauge-invariant.

In the remainder of this paper we will drop the "(B)" on the background quantities. Quantities such as $g_{\sigma\nu}$, U_ν, ψ, ... are understood to take their unperturbed values.

c) The Second Variation

In the Newtonian case (Paper I) we constructed the Lagrangian density for the perturbations from the second variation of the action, equation (9). The analogous calculations in the relativistic case are complicated by the perturbation in the geometry, so the details have been left to Appendix A. We treat the pressure and curvature parts of the action separately.

i) *Second Variation of the Fluid Lagrangian*

The fluid Lagrangian density is $p(-g)^{1/2}$. Its second variation is

$$\delta^2[p(-g)^{1/2}] = (\delta^2 p)(-g)^{1/2} + 2\delta p\delta[(-g)^{1/2}] + p\delta^2[(-g)^{1/2}] . \tag{19}$$

In Appendix A we show that

$$\begin{aligned}\delta^2 p = \delta\rho_0\delta\mu - \frac{\rho_0}{\mu}(\delta\mu)^2 - \delta(\rho_0 T)\delta S - 2\rho_0 U_\nu h^{\nu\sigma}\delta V_\sigma \\ - \frac{\rho_0}{\mu} g^{\nu\sigma}\delta V_\nu \delta V_\sigma - 2\rho_0 U^\nu(\delta\alpha\,\delta\beta_{,\nu} - \delta S\,\delta\theta_{,\nu}) ,\end{aligned} \tag{20}$$

where we let V_ν denote the Taub (1959) current vector

$$V_\nu = \mu U_\nu = \psi_{,\nu} + \alpha\beta_{,\nu} - S\theta_{,\nu} . \tag{21}$$

In equation (20) it is understood that $\delta\rho_0$ is a function (through the equation of state) of $\delta\mu$ and δS, and that $\delta\mu$ is a function of the independent perturbations ($\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS, $h^{\nu\sigma}$) through the perturbed version of equation (11):

$$\delta\mu = -\tfrac{1}{2}\mu h^{\nu\sigma}U_\nu U_\sigma - U^\sigma\delta V_\sigma . \tag{22}$$

From Appendix A we also have

$$\delta[(-g)^{1/2}] = -\tfrac{1}{2}h(-g)^{1/2} \tag{23a}$$

and

$$\delta^2[(-g)^{1/2}] = (\tfrac{1}{4}h^2 + \tfrac{1}{2}h^{\nu\sigma}h_{\nu\sigma})(-g)^{1/2} , \tag{23b}$$

where h is the trace of $h^{\nu\sigma}$:

$$h \equiv h^{\nu\sigma}g_{\nu\sigma} . \tag{24}$$

If we assemble all these terms and define

$$\mathcal{P} \equiv (-g)^{-1/2}\delta^2[p(-g)^{1/2}] ,$$

we have the fluid perturbations' Lagrangian density

$$\begin{aligned}\mathcal{P} = \delta\rho_0\delta\mu - \frac{\rho_0}{\mu}(\delta\mu)^2 - \delta(\rho_0 T)\delta S - \frac{\rho_0}{\mu}g^{\nu\sigma}\delta V_\nu\delta V_\sigma - 2\rho_0 U^\nu(\delta\alpha\,\delta\beta_{,\nu} - \delta S\,\delta\theta_{,\nu}) \\ - 2\rho_0 U_\nu h^{\nu\sigma}\delta V_\sigma - \delta p h + \tfrac{1}{4}p h^2 + \tfrac{1}{2}p h^{\nu\sigma}h_{\nu\sigma} .\end{aligned} \tag{25}$$

This is perfectly general: no assumptions have yet been made about the unperturbed spacetime.

ii) *Second Variation of the Curvature Lagrangian*

The Lagrangian density for the curvature is $R(-g)^{1/2}$. It is simplest to treat it the Palatini way: $R_{\alpha\beta}$ is a function only of the Christoffel symbols,

$$R_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\nu\mu}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha\mu} . \tag{26}$$

We define the perturbation in $\Gamma^{\mu}{}_{\alpha\beta}$ to be $\mathcal{S}^{\mu}{}_{\alpha\beta}$:

$$\Gamma^{\mu}{}_{\alpha\beta}(\text{perturbed}) = \Gamma^{\mu}{}_{\alpha\beta}(\text{background}) + \mathcal{S}^{\mu}{}_{\alpha\beta} . \tag{27}$$

It is well known that $\mathcal{S}^{\mu}{}_{\alpha\beta}$, being the difference between two affine connections, is a tensor on the background spacetime.

The second variation of $R(-g)^{1/2}$ is

$$\begin{aligned}\delta^2[R(-g)^{1/2}] &= \delta^2[g^{\alpha\beta}R_{\alpha\beta}(\Gamma)(-g)^{1/2}] \\ &= 2h^{\alpha\beta}\delta R_{\alpha\beta}(-g)^{1/2} + 2h^{\alpha\beta}R_{\alpha\beta}\delta[(-g)^{1/2}] + g^{\alpha\beta}\delta^2 R_{\alpha\beta}(-g)^{1/2} \\ &\quad + 2g^{\alpha\beta}\delta R_{\alpha\beta}\delta[(-g)^{1/2}] + g^{\alpha\beta}R_{\alpha\beta}\delta^2[(-g)^{1/2}] .\end{aligned} \tag{28}$$

In Appendix A we show that

$$\begin{aligned}\mathcal{R} &\equiv (-g)^{-1/2}\delta^2[R(-g)^{1/2}] \\ &= 2\bar{h}^{\alpha\beta}(\mathcal{S}^{\mu}{}_{\alpha\beta;\mu} - \mathcal{S}^{\mu}{}_{\alpha\mu;\beta}) + 2g^{\alpha\beta}(\mathcal{S}^{\mu}{}_{\nu\mu}\mathcal{S}^{\nu}{}_{\alpha\beta} - \mathcal{S}^{\mu}{}_{\nu\beta}\mathcal{S}^{\nu}{}_{\alpha\mu}) \\ &\quad - hh^{\alpha\beta}R_{\alpha\beta} + R(\tfrac{1}{4}h^2 + \tfrac{1}{2}h_{\alpha\beta}h^{\alpha\beta}) ,\end{aligned} \tag{29}$$

where we have used the conventional abbreviation

$$\bar{h}^{\alpha\beta} \equiv h^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}h . \tag{30}$$

Again, we have not yet made any assumption about the background.

iii) *Varying the Perturbed Lagrangian*

The action for the perturbations is

$$I_2 = \int L_2(-g)^{1/2}d^4x = \int(\mathcal{R} + 16\pi\mathcal{P})(-g)^{1/2}d^4x . \tag{31}$$

Extremizing it with respect to $\mathcal{S}^{\mu}{}_{\alpha\beta}$ gives the equation

$$0 = \frac{\delta L_2}{\delta\mathcal{S}^{\mu}{}_{\alpha\beta}} = \delta^{\beta}{}_{\mu}[g^{\nu\sigma}\mathcal{S}^{\alpha}{}_{\nu\sigma} + \bar{h}^{\alpha\nu}{}_{;\nu}] + g^{\alpha\beta}\mathcal{S}^{\nu}{}_{\mu\nu} - \mathcal{S}^{\alpha}{}_{\nu\mu}g^{\nu\beta} - \mathcal{S}^{\beta}{}_{\mu\nu}g^{\alpha\nu} - \bar{h}^{\alpha\beta}{}_{;\mu} . \tag{32}$$

This is equivalent to

$$\mathcal{S}^{\mu}{}_{\alpha\beta} = -\tfrac{1}{2}(h^{\mu}{}_{\alpha;\beta} + h^{\mu}{}_{\beta;\alpha} - h_{\alpha\beta}{}^{;\mu}) \tag{33}$$

which is of course the correct expression for the perturbation of the Christoffel symbol. (Recall that eq. [14] is responsible for the overall minus sign in eq. [33].)

Extremizing I_2 with respect to $h^{\alpha\beta}$ gives the perturbed field equations:

$$\begin{aligned}\frac{\delta\mathcal{R}}{\delta h^{\alpha\beta}} &= 2(\mathcal{S}^{\mu}{}_{\alpha\beta;\mu} - \mathcal{S}^{\mu}{}_{\alpha\mu;\beta}) - g_{\alpha\beta}(\mathcal{S}^{\mu\nu}{}_{\nu;\mu} - \mathcal{S}^{\mu\nu}{}_{\mu;\nu}) \\ &\quad - R^{\mu\nu}(g_{\alpha\beta}h_{\mu\nu} - h_{\alpha\beta}g_{\mu\nu}) - h(R_{\alpha\beta} - \tfrac{1}{2}Rg_{\alpha\beta})\end{aligned} \tag{34a}$$

$$= 2(-g)^{-1/2}\delta[G_{\alpha\beta}(-g)^{1/2}] ; \tag{34b}$$

$$\frac{\delta \mathcal{P}}{\delta h^{\alpha\beta}} = -\mu U_\alpha U_\beta \delta\rho_0 + \rho_0 U_\alpha U_\beta \delta\mu - \rho_0 U_\alpha \delta V_\beta - \rho_0 U_\beta \delta V_\alpha$$

$$- \delta p g_{\alpha\beta} + p h_{\alpha\beta} + \tfrac{1}{2}(\rho_0 \mu U_\alpha U_\beta + p g_{\alpha\beta}) h \tag{35a}$$

$$= -(-g)^{-1/2}\delta[T_{\alpha\beta}(-g)^{1/2}]\,; \tag{35b}$$

$$\frac{\delta L_2}{\delta h^{\alpha\beta}} = 2(-g)^{-1/2}\delta[(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(-g)^{1/2}] = 0\,. \tag{36}$$

Extremizing I_2 with respect to $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, and δS gives, respectively, the perturbed versions of equations (6), (3c), (3b), (3d), and (3e). The perturbed version of equation (3a) follows from these and equation (22).

III. PERTURBATIONS OF DIFFERENTIALLY ROTATING STELLAR MODELS

In this section we specialize the Lagrangian density of § II to the case where the background is an axially symmetric, stationary stellar model. For the purpose of a stability analysis, this is hardly any restriction at all. A stability analysis would be very difficult if the unperturbed state were not stationary, and in general relativity—by contrast with Newtonian theory—it is very unlikely that nonaxially symmetric stationary configurations of perfect fluid can exist. (They would either emit gravitational waves or require anisotropic stresses for their support.)

Up to this point our analysis has followed closely that of Paper I. From now on it will be quite different, however, because of the complications introduced by gravitational radiation. In Newtonian theory, where the gravitational field has no dynamical freedom, we had little difficulty in reducing L_2 to a function only of ξ, the Lagrangian displacement of the fluid. We then derived the stability criterion directly from the reduced Lagrangian.

In the relativistic case there are two dynamical degrees of freedom in the gravitational field. In principle it would be possible to choose a gauge, to solve the perturbed initial value equations, and to be left with two dynamical gravitational variables (e.g., $h_{\mu\nu}{}^{TT}$, by analogy with Arnowitt, Deser, and Misner 1962, hereafter referred to as ADaM). Then L_2 could be expressed in terms of ξ and these two gravitational variables. Such a program would be very interesting, and it may well be necessary before a definitive solution of the stability problem is reached. We will discuss this in more detail later. However, there is a simpler way to obtain a stability criterion, and it requires no prior specialization of gauge. In this section we construct the conserved energy density and angular-momentum density of the pulsations and discuss some of their properties. In § IV we use the energy density as a Liapunov function whose positive-definiteness guarantees stability.

a) The Unperturbed Differentially Rotating Star

The asymptotically flat spacetime in which the star sits is characterized by two Killing vectors, $\vec{\xi}_{(t)}$ and $\vec{\xi}_{(\varphi)}$. The four-velocity of the fluid is some timelike normalized linear combination of these:

$$\vec{U} = [\vec{\xi}_{(t)} + \Omega\vec{\xi}_{(\varphi)}]/|\vec{\xi}_{(t)}\cdot\vec{\xi}_{(t)} + 2\Omega\vec{\xi}_{(t)}\cdot\vec{\xi}_{(\varphi)} + \Omega^2\vec{\xi}_{(\varphi)}\cdot\vec{\xi}_{(\varphi)}|^{1/2}\,. \tag{37}$$

This equation defines Ω: it is the angular velocity as seen from infinity.

We can introduce coordinates t and φ such that $\vec{\xi}_{(t)} = \partial/\partial t$ and $\vec{\xi}_{(\varphi)} = \partial/\partial\varphi$, and two other coordinates y^A ($A = 1, 2$) such that the line element takes the form (cf. Carter 1969 or review by Thorne 1971)

$$ds^2 = g_{00}dt^2 + 2g_{0\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{AB}dy^A dy^B\,. \tag{38}$$

However, we will not always want to specialize our coordinates this far; in this section we will usually work with three arbitrary spatial coordinates x^i and with the line element

$$ds^2 = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j . \tag{39}$$

It is understood, of course, that all $g_{\alpha\beta}$ and all other *physically measurable* unperturbed quantities are independent of t and φ. (The velocity potentials are *not* all independent of t and φ, but their physically measurable combinations, such as U_ν, *are* independent of t and φ.)

The relativistic velocity potentials for this case are similar to the Newtonian potentials:

$$S = \text{arbitrary function independent of } t \text{ and } \varphi , \tag{40a}$$

$$\Omega = \text{arbitrary function independent of } t \text{ and } \varphi , \tag{40b}$$

$$\alpha = \mu U_\varphi = \mu \vec{\xi}_{(\varphi)} \cdot \vec{U} , \tag{40c}$$

$$\beta = \varphi - \Omega t , \tag{40d}$$

$$\theta = Tt/U^0 = Tt|\vec{\xi}_{(t)} \cdot \vec{\xi}_{(t)} + 2\Omega \vec{\xi}_{(t)} \cdot \vec{\xi}_{(\varphi)} + \Omega^2 \vec{\xi}_{(\varphi)} \cdot \vec{\xi}_{(\varphi)}|^{1/2} , \tag{40e}$$

$$\psi = (-\mu + TS)t/U^0 . \tag{40f}$$

That these are the correct velocity potentials is most easily demonstrated in the coordinates of equation (38), where the generating equation for U_ν,

$$U_\nu = \mu^{-1}(\psi_{,\nu} + \alpha\beta_{,\nu} - S\theta_{,\nu}) , \tag{41}$$

reduces to an identity for $\nu = t, \varphi$. Demanding that $U_A = 0$ $(A = 1, 2)$ in those same coordinates gives the equation of hydrostatic equilibrium,

$$\frac{1}{\rho_0 \mu} p_{,A} - (\ln U^0)_{,A} + U^0 U_\varphi \Omega_{,A} = 0 . \tag{42}$$

The velocity potentials are scalars, so they keep their same values in the more general coordinates of equation (39). There one ought to regard φ as a scalar field geometrically defined by $\vec{\xi}_{(\varphi)}$.

b) The Conserved Angular Momentum of Pulsation

i) *Noether's Theorem*

The existence of a Killing vector $\vec{\xi}_{(a)}$ in the background spacetime makes it possible to define a conserved quantity if the Lagrangian density L_2 is invariant under translations along $\vec{\xi}_{(a)}$ during which the variables $q_r \equiv \{\mathcal{S}^\mu{}_{\alpha\beta}, h^{\alpha\beta}, \delta\psi, \delta\alpha, \delta\beta, \delta\theta, \delta S\}$ are held fixed.[1] Under such conditions Noether's theorem (cf. Trautman 1962; Taub 1971) implies the following conservation law:

$$P_{(a)}{}^\sigma{}_{;\sigma} = 0 , \tag{43}$$

with

$$P_{(a)}{}^\sigma = \sum_r (\pounds_{\vec{\xi}_{(a)}} q_r) \left(\frac{\partial L_2}{\partial q_{r;\sigma}}\right) - L_2 \xi_{(a)}{}^\sigma . \tag{44}$$

We now show that L_2 is invariant under translations along $\vec{\xi}_{(\varphi)}$ but *not* along $\vec{\xi}_{(t)}$. Though the unperturbed spacetime is invariant under both, the unperturbed velocity

[1] More precisely, they are "Lie-dragged" along $\vec{\xi}_{(a)}$, as opposed to being parallel-transported (cf. Yano 1955).

potentials are not. One must look carefully at the way they enter L_2 in order to determine if L_2 is invariant.

The unperturbed velocity potentials enter L_2 only through the term

$$\delta V_\nu = \delta\psi_{,\nu} + \alpha\,\delta\beta_{,\nu} + \beta_{,\nu}\delta\alpha - S\,\delta\theta_{,\nu} - \theta_{,\nu}\delta S\,, \tag{45}$$

which contributes to L_2 both implicitly (through $\delta\mu$) and explicitly. Consider how it changes in t and φ if the perturbations are held fixed:

$$\left(\frac{\partial}{\partial t}\,\delta V_\nu\right)_{q_r} = -\Omega_{,\nu}\delta\alpha - \left(\frac{T}{U^0}\right)_{,\nu}\delta S \neq 0\,; \tag{46}$$

$$\left(\frac{\partial}{\partial \varphi}\,\delta V_\nu\right)_{q_r} = \left(\frac{\partial\beta}{\partial\varphi}\right)_{,\nu}\delta\alpha = 0\,. \tag{47}$$

So L_2 is φ-invariant but not t-invariant. Note, however, from equation (22) that $\delta\mu$ is t-invariant as well.

This result can be understood as follows: Even if the perturbation eventually dies out completely, $\delta\beta$, $\delta\psi$, and $\delta\theta$ may continue to change linearly in time at rates that vary across the star, just as β, ψ, and θ do in the unperturbed state. Therefore, holding $\delta\psi$, $\delta\beta$, $\delta\theta$ fixed during a translation in time is not the same as holding the physical perturbation fixed. It is not surprising that Noether's theorem fails in our context. Later we will construct the real conserved energy [which must exist because $\xi_{(t)}$ exists] in a different manner. First, however, we use the φ-invariance of L_2 to construct the angular momentum.

ii) *The Angular Momentum Density*

The conservation law (43) can be written in the following form when $\xi_{(a)}$ is $\xi_{(\varphi)}$:

$$\frac{\partial}{\partial t}\left[\sum_r N q_{r,\varphi}\frac{\partial L_2}{\partial q_{r;0}}\right] + \left[\sum_r N q_{r,\varphi}\frac{\partial L_2}{\partial q_{r;i}} - N L_2 \delta^i{}_\varphi\right]_{|i} = 0\,. \tag{48}$$

From now on we use the ADaM notation appropriate to a three-plus-one-dimensional split of spacetime. In particular, we define the lapse function $N \equiv (-g^{00})^{-1/2}$; we denote the determinant of the three-dimensional metric by g and that of the four-dimensional metric by ${}^4g$ [which are related by the identity $(-{}^4g)^{1/2} = Ng^{1/2}$]; and we use a vertical rule or a boldface $\boldsymbol{\nabla}$ to denote differentiation covariant with respect to the three-dimensional metric. Equation (48) implies that if we define

$$\mathcal{J}' \equiv -\frac{1}{32\pi}\sum_r N q_{r,\varphi}\frac{\partial L_2}{\partial q_{r;0}}\,, \tag{49a}$$

then the integral of $\mathcal{J}'$ over the entire hypersurface

$$J \equiv \int \mathcal{J}' g^{1/2} d^3x \tag{49b}$$

is constant in time. Note that any density differing from $\mathcal{J}'$ by a spatial divergence will likewise be conserved, and will give the same value for J provided the perturbed region of space is of finite extent.

From L_2 as given in § II we find

$$\mathcal{J}' = -\frac{1}{16\pi}N\mathcal{S}^0{}_{\sigma\nu,\varphi}\bar{h}^{\sigma\nu} + \frac{1}{16\pi}N\mathcal{S}^\nu{}_{\sigma\nu,\varphi}\bar{h}^{\sigma 0}$$
$$+ g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta V_\varphi - \delta\alpha) + N\rho_0 U^0(\delta\alpha\,\delta\beta_{,\varphi} - \delta S\,\delta\theta_{,\varphi})\,. \tag{50}$$

To cast this in a more familiar form we add the divergence

$$\frac{\partial}{\partial\varphi}\left[\frac{1}{16\pi}N(\mathcal{S}^0{}_{\sigma\nu}\bar{h}^{\sigma\nu}-\mathcal{S}^\nu{}_{\sigma\nu}\bar{h}^{\sigma 0})\right].$$

We define the result as the *angular-momentum density:*

$$\mathcal{J}\equiv\mathcal{J}_G+\mathcal{J}_F\,, \tag{51a}$$

where

$$\mathcal{J}_G\equiv\frac{1}{16\pi}[\tfrac{1}{2}N\bar{h}^{\sigma\nu}{}_{,\varphi}\bar{h}_{\sigma\nu}{}^{;0}-N\bar{h}^{\sigma\nu}{}_{,\varphi}\bar{h}^0{}_{\sigma;\nu}-\tfrac{1}{4}N\bar{h}_{,\varphi}\bar{h}^{,0}] \tag{51b}$$

and

$$\mathcal{J}_F\equiv g^{-1/2}\delta(\rho_0U^0Ng^{1/2})(\delta V_\varphi-\delta\alpha)+N\rho_0U^0(\delta\alpha\,\delta\beta_{,\varphi}-\delta S\,\delta\theta_{,\varphi})\,. \tag{51c}$$

To obtain this form for $\mathcal{J}_G$ we have expressed the $\mathcal{S}$'s in terms of $\bar{h}$'s from equation (33). The split between $\mathcal{J}_G$ and $\mathcal{J}_F$ is arbitrary. Only their sum is conserved.

The flux associated with $\mathcal{J}$ is

$$\mathfrak{N}^k\equiv\mathfrak{N}_G{}^k+\mathfrak{N}_F{}^k\,, \tag{52a}$$

with

$$\mathfrak{N}_G{}^k\equiv-\frac{1}{16\pi}N[\mathcal{S}^k{}_{\sigma\nu,\varphi}\bar{h}^{\sigma\nu}-\mathcal{S}^\nu{}_{\sigma\nu,\varphi}\bar{h}^{\sigma k}]+\frac{1}{16\pi}N\delta^k{}_\varphi\left[\tfrac{1}{2}\mathcal{R}-\frac{\partial}{\partial t}(\mathcal{S}^0{}_{\sigma\nu}\bar{h}^{\sigma\nu}-\mathcal{S}^\nu{}_{\sigma\nu}\bar{h}^{\sigma 0})\right] \tag{52b}$$

and

$$\mathfrak{N}_F{}^k\equiv g^{-1/2}\delta(\rho_0U^kNg^{1/2})(\delta V_\varphi-\delta\alpha)+N\rho_0U^k(\delta\alpha\,\delta\beta_{,\varphi}-\delta S\,\delta\theta_{,\varphi})+\tfrac{1}{2}N\mathcal{P}\delta^k{}_\varphi\,. \tag{52c}$$

Then equation (48) becomes

$$\frac{\partial}{\partial t}(\mathcal{J}_G+\mathcal{J}_F)+(\mathfrak{N}_G{}^k+\mathfrak{N}_F{}^k)_{|k}=0\,. \tag{53}$$

Note that since $\mathcal{J}$ differs from $\mathcal{J}'$ by a divergence, the flux $\mathfrak{N}^k$ differs from $-1/32\pi$ times the flux in equation (48) by the time-derivative

$$-\frac{\partial}{\partial t}\left[\frac{1}{16\pi}N(\mathcal{S}^0{}_{\sigma\nu}\bar{h}^{\sigma\nu}-\mathcal{S}^\nu{}_{\sigma\nu}\bar{h}^{\sigma 0})\delta^k{}_\varphi\right].$$

iii) *Heuristic Interpretation of* $\mathcal{J}$

The terms in $\mathcal{J}$ may be interpreted heuristically (and incompletely) as follows:

1. The terms called $\mathcal{J}_G$ may be *defined* as the angular momentum in the gravitational waves. The reasonableness of this definition becomes apparent in the short-wavelength limit (wavelength small compared to the radius of curvature of the background space-time). There the average of $\mathcal{J}_G$ over a few wavelengths in the hypersurface and over a few cycles of time is just the angular-momentum component of the Isaacson (1968) stress-energy tensor for gravitational radiation, $T^{(GW)0}{}_\varphi$. (More precisely, the average is the "Brill-Hartle" average [cf. Isaacson 1968] of $\mathcal{J}_G/N$.) The short-wavelength limit is most easily compared with the expressions for $T^{(GW)}{}_{\mu\nu}$ given by Misner, Thorne, and Wheeler (1972):

$$\langle\mathcal{J}_G/N\rangle_{\mathrm{BH}}=\frac{1}{16\pi}\langle\tfrac{1}{2}\bar{h}^{\sigma\nu}{}_{;\varphi}\bar{h}_{\sigma\nu}{}^{;0}-\bar{h}^{\sigma\nu}{}_{;\nu}\bar{h}^0{}_{\sigma;\varphi}-\tfrac{1}{4}\bar{h}_{,\varphi}\bar{h}^{,0}\rangle_{\mathrm{BH}}=T^{(GW)0}{}_\varphi\,, \tag{54}$$

independent of any gauge.

We emphasize, however, that the dominant radiation from a pulsating relativistic star may not be of short wavelength near the star. If most of the radiation from a star of mass M has frequency greater than some ω_0, then the short-wavelength approximation is good only in the region

$$r\gg(4\pi M/\omega_0)^{1/2}=(4\pi GM/\omega_0c)^{1/2}\,. \tag{55}$$

For a typical neutron star in quadrupole oscillation as studied by Thorne (1969)

($M \simeq 0.7\ M_\odot$, $\omega_0 \simeq 2 \times 10^4$ s^{-1}, $R \simeq 9$ km) this becomes

$$r \gg 14 \text{ km} ,$$

which puts r well outside the star.

Our expression for $\mathcal{J}$ is only one of many that reduce to the Isaacson tensor in the short-wavelength limit. Only in the radiation zone far from the star can we relate $\mathcal{J}_G$ to the density of angular momentum being lost by the star, because only there is that density truly well defined and measurable.

2. The angular momentum in the fluid per unit coordinate volume, $T^0{}_\varphi\,(-{}^4g)^{1/2}$, can be written as $\rho_0 U^0 V_\varphi N g^{1/2}$. Now V_φ is the angular momentum per particle per unit rest mass:

$$V_\varphi = \mu U_\varphi = \frac{\rho + p}{\rho_0}\, U_\varphi \,.$$

Thus, the angular-momentum density is the product:

(angular-momentum density) = (rest-mass density) $\times$ (angular momentum per particle per unit rest mass),

$$T^0{}_\varphi(-{}^4g)^{1/2} = (\rho_0 U^0 N g^{1/2}) \times (V_\varphi) \,.$$

When the fluid is perturbed, part of the second-order change in this is, from equation (51), $\delta(\rho_0 U^0 N g^{1/2})\,(\delta V_\varphi - \delta\alpha)$. The term $\delta(\rho_0 U^0 N g^{1/2})$ is easy to understand. The term $\delta V_\varphi - \delta\alpha$ can be related to the Lagrangian change in the angular momentum per particle per unit rest mass as follows. If j is the angular momentum per particle per unit rest mass, if Δ denotes a Lagrangian change, and if $\boldsymbol{\xi}$ is defined as the Lagrangian displacement vector of the fluid element (not to be confused with the Killing vectors), then we have

$$\Delta j = \delta j + \boldsymbol{\xi}\cdot\boldsymbol{\nabla} j = \delta V_\varphi + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\alpha \tag{56}$$

because in the unperturbed state $j = V_\varphi = \alpha$. But in Appendix B we show that $\delta\alpha = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\alpha + (\delta\alpha)_0$, where $(\delta\alpha)_0$ is the "initial value" of $\delta\alpha$: its value when $\boldsymbol{\xi}$ is zero. Therefore we have

$$\Delta j - (\delta\alpha)_0 = \delta V_\varphi - \delta\alpha \,. \tag{57}$$

3. The final term in $\mathcal{J}_F$ is $N\rho_0 U^0\,(\delta\alpha\,\delta\beta_{,\varphi} - \delta S\,\delta\theta_{,\varphi})$. This is the same as $\frac{1}{2}N\rho_0 U^0 \delta^2 V_\varphi$, the contribution from the second-order change in V_φ. Because we lack an explicit expression for $\delta\theta_{,\varphi}$ in terms of $\boldsymbol{\xi}$, we have been unable to express this term entirely in terms of $\boldsymbol{\xi}$.

c) *The Conserved Energy of Pulsation*

i) *Calculating the Energy Density*

Although Noether's theorem does not give us a conserved energy, we can construct one from the Hamiltonian. The calculations required to do this appear in Appendix C. The essential steps are summarized here:

First, define the Hamiltonian density,

$$H_2 = \sum_r N q_{r,0} \frac{\partial L_2}{\partial q_{r,0}} - N L_2 \,, \tag{58}$$

where $q_r = \{g^\mu{}_{\alpha\beta},\, h^{\alpha\beta},\, \delta\psi,\, \delta\alpha,\, \delta\beta,\, \delta\theta,\, \delta S\}$. It is degenerate: not all the momenta $\partial L_2/\partial q_{r,0}$ are independent.

Second, find the time-derivative of H_2 by using the method of Dirac (1958*a*) for degenerate theories. Find that

$$\frac{\partial H_2}{\partial t} = \left[\sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|l}}\right]_{|l} - N\left(\frac{\partial L_2}{\partial t}\right)_{\text{holding all } q_r,\, q_{r,0} \text{ fixed}}, \tag{59}$$

$$= -f^l{}_{|l} - 32\pi N \frac{\rho_0}{\mu}\,\delta V^i\left[\Omega_{,i}\delta\alpha + \left(\frac{T}{U^0}\right)_{,i}\delta S\right]. \tag{60}$$

Thus, the Hamiltonian is not conserved. We should expect this from the failure of Noether's theorem.

Third, express the last term in equation (60) in terms of ξ. Define the redshifted temperature,

$$\mathfrak{I} \equiv T/U^0\,, \tag{61a}$$

and a *symmetric* (for proof see Appendix C) tensor

$$M_{ij} = \alpha_{,i}\Omega_{,j} + S_{,i}\mathfrak{I}_{,j}\,. \tag{61b}$$

Find that

$$-32\pi N\frac{\rho_0}{\mu}\,\delta V^i[\Omega_{,i}\delta\alpha + \mathfrak{I}_{,i}\delta S]$$

$$= \frac{\partial}{\partial t}[16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j - 32\pi N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0]$$

$$+ [16\pi N\rho_0 U^l M_{ij}\xi^i\xi^j - 32\pi N\rho_0 U^l\Omega_{,i}\xi^i(\delta\alpha)_0]_{|l}\,, \tag{62}$$

where $(\delta\alpha)_0$ is the "initial value" of the perturbation in α. The time derivative can be brought over to the left-hand side of equation (60) and the divergence absorbed into the divergence of f^i. This defines a conservation law,

$$\frac{\partial \mathcal{E}'}{\partial t} + \mathfrak{F}'^l{}_{|l} = 0\,, \tag{63}$$

for a globally conserved energy density,

$$\mathcal{E}' \equiv \frac{1}{16\pi}H_2 - N\rho_0 U^0 M_{ij}\xi^i\xi^j + 2N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0\,, \tag{64}$$

and its flux,

$$\mathfrak{F}'^l \equiv \frac{1}{16\pi}f^l - N\rho_0 U^l M_{ij}\xi^i\xi^j + 2N\rho_0 U^l\Omega_{,i}\xi^i(\delta\alpha)_0\,. \tag{65}$$

Fourth, the energy density is defined only to within a spatial divergence. Subtract a divergence from $\mathcal{E}'$ and the appropriate time derivative from $\mathfrak{F}'^i$ to arrive at a form of the energy density that is quadratic in derivatives of $h^{\alpha\beta}$. Write the result as

$$\frac{\partial}{\partial t}(\mathcal{E}_G + \mathcal{E}_F) + (\mathfrak{F}_G{}^k + \mathfrak{F}_F{}^k)_{|k} = 0\,, \tag{66}$$

with

$$\mathcal{E}_G = \frac{1}{8\pi}N[g^{\alpha\beta}(\mathcal{S}^\mu{}_{\nu\mu}\mathcal{S}^\nu{}_{\alpha\beta} - \mathcal{S}^\mu{}_{\nu\beta}\mathcal{S}^\nu{}_{\alpha\mu}) - \bar{h}^{\alpha\beta}{}_{,0}\mathcal{S}^0{}_{\alpha\beta} + \bar{h}^{\alpha 0}{}_{,0}\mathcal{S}^\mu{}_{\alpha\mu}]\,, \tag{67}$$

$$\begin{aligned}\mathcal{E}_F = {}& -2g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S) + \frac{\rho_0}{\mu}Ng^{\nu\sigma}\delta V_\nu\delta V_\sigma + 2N\rho_0 U_\sigma h^{\sigma\nu}\delta V_\nu \\ & - N\delta\rho_0\delta\mu + N\frac{\rho_0}{\mu}(\delta\mu)^2 + N\delta(\rho_0 T)\delta S + 2N\rho_0 U^i(\delta\alpha\,\delta\beta_{,i} - \delta S\,\delta\theta_{,i}) \\ & - N\rho_0 U^0(\alpha_{,i}\Omega_{,j} + S_{,i}\mathfrak{I}_{,j})\xi^i\xi^j + 2N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0 \\ & + Nh\delta p + \frac{1}{16\pi}Nhh^{\alpha\beta}R_{\alpha\beta} - N\left(\frac{1}{16\pi}R + p\right)(\tfrac{1}{4}h^2 + \tfrac{1}{2}h^{\alpha\beta}h_{\alpha\beta})\,,\end{aligned} \tag{68}$$

$$\mathfrak{F}_G{}^k = \frac{1}{8\pi} N(-\bar{h}^{\alpha\beta}{}_{,0}\mathcal{S}^k{}_{\alpha\beta} + \bar{h}^{\alpha k}{}_{,0}\mathcal{S}^\mu{}_{\alpha\mu}) , \tag{69}$$

and

$$\mathfrak{F}_F{}^k = -2g^{-1/2}\delta(\rho_0 U^k N g^{1/2})(\delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S) - 2N\rho_0 U^k(\delta\alpha\, \delta\beta_{,0} - \delta S\, \delta\theta_{,0})$$
$$- N\rho_0 U^k(\alpha_{,i}\Omega_{,j} + S_{,i}\mathfrak{I}_{,j})\xi^i\xi^j + 2N\rho_0 U^k\Omega_{,i}\xi^i(\delta\alpha)_0 . \tag{70}$$

The split between $\mathcal{E}_G$ and $\mathcal{E}_F$ (and between $\mathfrak{F}_G{}^k$ and $\mathfrak{F}_F{}^k$) is arbitrary: only their sum, $\mathcal{E} \equiv \mathcal{E}_G + \mathcal{E}_F$, is conserved. As we shall see in the next subsection, $\mathcal{E}$ is really twice what one would normally call the energy density.

ii) *Heuristic Interpretation of* $\mathcal{E}$ *and* $\mathfrak{F}^k$

Because of the great number of terms in $\mathcal{E}$ it is difficult to identify different kinds of energy. We have split off $\mathcal{E}_G$ because it is the only nonvanishing part in vacuum, and because it contains all the terms that have derivatives of $h^{\alpha\beta}$.

1. In the short-wavelength limit in the vacuum region outside the star, the Brill-Hartle average of $\mathcal{E}_G/N$ is proportional to the Isaacson energy density. Outside the star the wave equation is (cf. eq. [34a])

$$\mathcal{S}^\mu{}_{\alpha\beta;\mu} - \mathcal{S}^\mu{}_{\alpha\mu;\beta} = 0 . \tag{71}$$

Then by the identity mentioned in Appendix C (eq. [C15]) we have

$$g^{\alpha\beta}(\mathcal{S}^\mu{}_{\nu\mu}\mathcal{S}^\nu{}_{\alpha\beta} - \mathcal{S}^\mu{}_{\nu\beta}\mathcal{S}^\nu{}_{\alpha\mu}) = \tfrac{1}{2}(\bar{h}^{\mu\nu}\mathcal{S}^\alpha{}_{\mu\nu} - \bar{h}^{\nu\alpha}\mathcal{S}^\mu{}_{\nu\mu})_{;\alpha} .$$

This divergence does not contribute to the Brill-Hartle average of $\mathcal{E}_G/N$, so we obtain in the short-wavelength limit

$$\langle \mathcal{E}_G/N\rangle_{\rm BH} = -2T^{(GW)0}{}_0 . \tag{72}$$

This is in accord with our previous remark that $\mathcal{E}$ is twice the energy density.

2. The interpretation of $\mathcal{E}_F$ is made difficult by the presence of the term

$$2N\rho_0 U^i(\delta\alpha\, \delta\beta_{,i} - \delta S\, \delta\theta_{,i}) = N\rho_0 U^i\delta^2 V_i . \tag{73}$$

As with a similar term in $\mathcal{J}$, we have not been able to express this in terms of $\boldsymbol{\xi}$. Therefore we will not be able to make a comparison of the Newtonian limit of $\mathcal{E}$ with the Newtonian energy density derived in Paper I.[2] However, this term is not present if the unperturbed star is nonrotating, so in that case there is no problem showing that $\mathcal{E}$ reduces to the Newtonian expression derived in Paper I. We will do that later (§ III*e* [ii]). For now we simply note that the similarity between this term and one in $\mathcal{J}_F$ permits us to rewrite $\mathcal{E}_F$ in the form

$$\mathcal{E}_F = -2g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(U^\nu \delta V_\nu + T\delta S)/U^0 + 2\Omega\mathcal{J}_F + \text{remainder} , \tag{74}$$

where "remainder" means all but the term (73) and the first term of $\mathcal{E}_F$ in equation (68). So the kinetic energy associated with the fluid's angular momentum makes an explicit contribution to the total energy density.

3. We can get some feeling for the nature of $\mathcal{E}$ by looking at its flux, which tells us how energy leaves a volume. The flux of gravitational energy, $\mathfrak{F}_G{}^k$, can be averaged over a few wavelengths and cycles of time to give (in the short-wavelength limit)

$$\langle \mathfrak{F}_G{}^k/N\rangle_{\rm BH} = -2T^{(GW)k}{}_0 . \tag{75}$$

[2] This is a Newtonian term and even prevents a direct comparison of the Newtonian energy density derived by analogy with the present procedure with that derived in Paper I. It is difficult to see how they could be different, considering especially that in the nonrotating case one *can* show that they are equal.

Therefore, far from the star this is twice the physically measurable flux of energy in the gravitational waves.

4. The flow of fluid energy across some surface is

$$\left\{\begin{matrix}\text{transport of fluid energy in the hypersurface across} \\ \text{a two-surface } \Sigma \text{ with unit normal } n_k\end{matrix}\right\} = \int_\Sigma \mathfrak{F}_F{}^k n_k d\sigma \, .$$

If the surface Σ is parallel to the unperturbed streamlines ($U^k n_k = 0$), this becomes

$$\left\{\begin{matrix}\text{transport of fluid energy across} \\ \text{the unperturbed streamlines}\end{matrix}\right\} = -2\int_\Sigma N(\delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S)\rho_0 \delta v^k n_k d\sigma \, , \qquad (76)$$

where by v^k we mean the coordinate velocity U^k/U^0 (not to be confused with $V^k \equiv \mu U^k$). It can be shown that

$$-(\delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S) = \frac{1}{\rho_0 U^0}\delta p + \frac{\mu}{2U^0} U_\alpha U_\beta h^{\alpha\beta} + \Omega(\delta V_\varphi - \delta\alpha) \, . \qquad (77)$$

Thus the energy carried by the perturbations across the unperturbed streamlines is heuristically of three types: (*a*) work done (or gained) because of local changes in pressure; (*b*) "gravitational potential energy" (note that in the Newtonian limit, $\frac{1}{2}U_\alpha U_\beta h^{\alpha\beta} \to \frac{1}{2}h^{00} \to \delta\Phi$, the change in the Newtonian potential); and (*c*) rotational kinetic energy (recall that $\delta V_\varphi - \delta\alpha$ is related to the Lagrangian change in j by eq. [57]).

iii) *The Outgoing-Energy Boundary Condition*

Far from the star, where the short-wavelength approximation is valid for all but a negligible part of the gravitational energy, it is possible to formulate a physically meaningful condition that the net flux of energy be away from the star. On a closed surface Σ in the short-wavelength region, the net flux of energy will not be inward if

$$\int_\Sigma T^{(GW)k0} n_k d\sigma \geq 0 \, . \qquad (78)$$

By equation (75) this is equivalent to

$$\int_\Sigma \langle \mathfrak{F}_G{}^k/N\rangle_{\rm BH} n_k d\sigma \geq 0 \, . \qquad (79)$$

From this and equation (66) follows the important result: *The total energy of pulsation* ($\int \mathcal{E} g^{1/2} d^3x$) *inside* Σ *never increases if the radiation satisfies the outgoing-energy boundary condition on* Σ.

Note that this is a very weak condition compared to the usual outgoing-wave boundary condition, which requires that the flux be outward at every point of Σ. For our purposes we will need only the weak condition, equation (79).

d) The Total Energy and Angular Momentum

Three conclusions help us understand the physical meaning of the total energy, $E \equiv \int \mathcal{E} g^{1/2} d^3x$, and the total angular momentum, $J \equiv \int \mathcal{J} g^{1/2} d^3x$.

1. *E and J are gauge-independent.* This follows from reasoning similar to that used to prove the coordinate-independence of pseudo-tensor energies (cf. Landau and Lifshitz 1962). Briefly, assume that E or J is different in two different gauges. Choose a third gauge that matches the first on one hypersurface and goes smoothly into the second on a later hypersurface. Then conservation of E and J in every gauge contradicts the assumption. This does not imply that the densities $\mathcal{E}$ and $\mathcal{J}$ are gauge-invariant. Conservation of E and J is fundamental to the argument, and the conservation law is valid only if the perturbations satisfy the initial-value equations on every hypersurface. Therefore

the argument implies only that under a gauge transformation $\mathcal{E}$ and $\mathcal{J}$ change by terms that become spatial divergences after the initial-value equations are applied.

2. Suppose that a distant observer (outside the furthest wave front) measures the active gravitational mass M^* and total angular momentum L^* of the pulsating star. Suppose also that the star is stable, so that the pulsations eventually die out and leave behind a star of mass M and angular momentum L. *For a stable star, the differences $M^* - M$ and $L^* - L$ are at most second order in the perturbations.*

The difference $M^* - M$ is conserved at all orders. If there were a first-order piece in $M^* - M$, it would have to be radiated away as the stable star's pulsations damp out. It could not remain localized inside or near the star because by assumption M is the mass left behind. On the other hand, the work of Isaacson (1968) shows that there can be no first-order radiation of physically measurable energy on the stationary background far from the star. Therefore the first-order contribution to $M^* - M$ must vanish. The same argument applies to $L^* - L$.

This result is similar to the theorem of Bardeen (1970) that the equilibrium configuration of a rotating star extremizes the active gravitational mass of all nearby momentarily stationary configurations with the same total baryon number, angular momentum, and entropy that satisfy the initial-value equations. (This was proved for nonrotating stars by Cocke 1965 and Harrison *et al.* 1965.) Where Bardeen compares momentarily stationary configurations with different masses but identical angular momenta, we compare momentarily stationary configurations whose masses and angular momenta are related by the requirement that one configuration can be obtained from another by the emission or absorption of gravitational radiation. (The configuration with mass M^* can be considered to be momentarily stationary at the moment the perturbation is applied, just before it begins to emit gravitational waves.)

3. *In the notation of conclusion* 2, *the following equations are correct to second order in the perturbations:*

$$M^* = M + \tfrac{1}{2}E \tag{80a}$$

$$L^* = L + J\,, \tag{80b}$$

where the background star is the star of mass M and angular momentum L that is left behind. This result follows from three properties of E and J: (a) They are unique apart from additive and multiplicative constants because they depend only on the Killing vectors $\xi_{(t)}$ and $\xi_{(\varphi)}$. (b) They vanish when the perturbation vanishes. (c) The change in $\int \mathcal{E} g^{1/2} d^3x$ and $\int \mathcal{J} g^{1/2} d^3x$ inside any fixed surface surrounding the star and far from it is determined solely by the physically measurable fluxes $T^{(GW)k}{}_0$ and $T^{(GW)k}{}_\varphi$.

If there were any other second-order contribution to M^* or L^*, it would have to be globally conserved. By (c) it would also have to be confined forever within a closed surface at some large but finite distance from the star. The use of (b) and of arguments similar to those of conclusion 2 above then implies equations (80).

e) The Spherically Symmetric, Nonrotating Star

i) *Expressions for the Energy and Flux*

We turn now to a special case in which our expressions simplify considerably: the nonrotating star. In curvature coordinates the background metric is

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\,. \tag{81}$$

Then we have in the background

$$N = -U_0 = 1/U^0 = e^{\nu/2}\,, \quad g^{1/2} = r^2 \sin^2 \vartheta e^{\lambda/2}\,, \quad U^i = \Omega = 0\,. \tag{82}$$

In Appendix D we simplify $\mathcal{E}_F$ for this case as much as possible by substituting for the perturbed fluid quantities their expressions in terms of ξ (cf. Appendix B). The result is

$$\begin{aligned} e^{-\nu/2}\mathcal{E}_F = {} & \rho_0\mu e^{-\nu}\xi_{,0}\cdot\xi_{,0} + \gamma p(\nabla\cdot\xi)^2 + \rho_0^{-1}(\xi\cdot\nabla\rho_0)(\xi\cdot\nabla p) \\ & + 2(\nabla\cdot\xi)(\xi\cdot\nabla p) - \tfrac{1}{2}\rho_0 T(\xi\cdot\nabla S)(\xi\cdot\nabla\nu) - \mu\ell\nabla\cdot(\rho_0\xi) \\ & + \rho_0\ell\delta\mu + \delta p(h^j{}_j - \ell) + \tfrac{1}{8}(\rho + 3p)\ell^2 + \tfrac{1}{4}(\rho + 3p)\ell h^j{}_j \\ & - \tfrac{1}{2}(\rho + 3p)k_jk^j - \tfrac{1}{4}(\rho - p)h^{jk}h_{jk} + \tfrac{1}{8}(\rho - p - 2\gamma p)(h^j{}_j)^2 . \end{aligned} \tag{83}$$

In this expression we have defined

$$\ell \equiv e^{-\nu}h_{00} \tag{84a}$$

and

$$k_j \equiv e^{-\nu/2}h_{0j} ; \tag{84b}$$

and we mean by $\delta\mu$ and δp

$$\delta\mu = -\frac{\gamma p}{\rho_0}(\nabla\cdot\xi - \tfrac{1}{2}h^j{}_j) - \xi\cdot\nabla\mu , \tag{85a}$$

$$\delta p = -\gamma p(\nabla\cdot\xi - \tfrac{1}{2}h^j{}_j) - \xi\cdot\nabla p . \tag{85b}$$

The flux $\mathfrak{F}_F{}^k$ is especially simple in this case. Equation (76) applies because *all* surfaces are orthogonal to the unperturbed streamlines:

$$e^{-\nu/2}\mathfrak{F}_F{}^k = e^{\nu/2}(\delta p + \tfrac{1}{2}\rho_0\mu\ell)\xi^k{}_{,0} . \tag{86}$$

The energy density and flux of gravitational waves do not simplify very much from their full form (eqs. [67] and [69]) so we will not reproduce them here.

Our previous remark that $\mathcal{E}$ is really twice the energy density is again verified by the "kinetic energy" term in equation (83), which has the form mv^2.

ii) *The Newtonian Limit*

The Newtonian limit of $\mathcal{E}$ for the nonrotating star is obtained by neglecting p and $\rho\Phi$ compared to ρ (Φ is the Newtonian gravitational potential). In equation (83), the fifth, seventh, and subsequent terms are all of post-Newtonian order or higher. In the Newtonian limit we have $\ell = 2\delta\Phi$, so that $\mathcal{E}_F$ becomes

$$\begin{aligned} (\mathcal{E}_F)_{\text{NEWT}} = {} & \rho\xi_{,0}\cdot\xi_0 + \gamma p(\nabla\cdot\xi)^2 + \rho^{-1}(\xi\cdot\nabla\rho)(\xi\cdot\nabla p) \\ & + 2(\nabla\cdot\xi)(\xi\cdot\nabla p) - 2\delta\Phi\nabla\cdot(\rho\xi) . \end{aligned} \tag{87}$$

The perturbed source equation for $\delta\Phi$ (analog of relativistic initial-value equation) is (cf. Paper I)

$$\nabla^2\delta\Phi = 4\pi\delta\rho = -4\pi\nabla\cdot(\rho\xi) . \tag{88}$$

Therefore the last term in $(\mathcal{E}_F)_{\text{NEWT}}$ becomes

$$-2\delta\Phi\nabla\cdot(\rho\xi) = -\frac{2}{4\pi}\nabla\delta\Phi\cdot\nabla\delta\Phi + (\text{divergence}) . \tag{89}$$

We will discard the divergence. By comparison with equation (27) of Paper I, we see that $\mathcal{E}_F$ differs from the Newtonian energy density only in that the term in equation (89) is twice as large as it should be. We therefore expect the Newtonian limit of $\mathcal{E}_G$ to be $(4\pi)^{-1}\nabla\delta\Phi\cdot\nabla\delta\Phi$.

Rather than find the Newtonian limit of $\mathcal{E}_G$ for arbitrary nonradial pulsations, we will restrict ourselves at first to the case of radial pulsations, for which we have explicitly calculated the relativistic expressions (Appendix D). We will then argue that the nonradial Newtonian limit differs from the radial limit in no important respects.

For relativistic radial pulsations we can choose a gauge such that the only two nonzero metric perturbations are

$$\delta\nu = -h^0{}_0 , \tag{90a}$$

$$\delta\lambda = -h^r{}_r . \tag{90b}$$

In terms of the fluid perturbations these are

$$\delta\lambda = -8\pi r e^{\lambda} \rho_0 \mu \xi , \tag{91a}$$

$$\delta\nu' = 8\pi r e^{\lambda}\left[\delta p - \rho_0\mu\left(\nu' + \frac{1}{r}\right)\xi\right] , \tag{91b}$$

where primes denote differentiation with respect to r. The Newtonian limits of these expressions are

$$\delta\lambda = -8\pi r \rho \xi , \tag{92a}$$

$$\delta\nu' = -8\pi\rho\xi . \tag{92b}$$

From equation (88) applied to the radial case we see that indeed $\delta\nu = 2\delta\Phi$. Moreover, it is clear that $\delta\lambda$ is of the same order as $\delta\nu$.

The energy $\mathcal{E}_G$ for radial pulsations is

$$(\mathcal{E}_G)_{\mathrm{RADIAL,\,RELATIVISTIC}} = \frac{e^{\nu/2-\lambda}}{8\pi}\left[\tfrac{1}{4}\nu'(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{r}\delta\lambda(\delta\lambda' + \delta\nu')\right] . \tag{93}$$

The first term is post-Newtonian compared to the second ($\nu' \ll 1/r$). From equations (92) we find the Newtonian limit to be

$$(\mathcal{E}_G)_{\mathrm{RADIAL,\,NEWT}} = \frac{1}{8\pi}\{2(\delta\nu')^2 + \tfrac{1}{2}r[(\delta\nu')^2]'\} . \tag{94}$$

If we add the divergence

$$-\frac{1}{16\pi}g^{-1/2}[g^{1/2}r(\delta\nu')^2]' = -\frac{1}{16\pi r^2}[r^3(\delta\nu')^2]' , \tag{95}$$

we obtain

$$(\mathcal{E}_G)_{\mathrm{RADIAL,\,NEWT}} = \frac{1}{16\pi}(\delta\nu')^2 = \frac{1}{4\pi}\delta\Phi'\delta\Phi' . \tag{96}$$

This is exactly what we require to make $\mathcal{E} = \mathcal{E}_F + \mathcal{E}_G$ reduce to the Newtonian energy density for radial pulsations.

We should expect the same result for nonradial pulsations. The nonradial case is made difficult because the appropriate limiting values of $h^{\alpha\beta}$ depend upon the gauge. Even in the radial case we saw that $\delta\lambda$ was comparable in size to $\delta\nu$. Nevertheless, the Newtonian limit of $\mathcal{E}_G$ cannot depend upon the gauge. It should be possible to construct a gauge in which the only two metric perturbations that have nonzero Newtonian limits will be $h^0{}_0$ and $h^r{}_r$. Dragging of inertial frames (given by $h^0{}_i$) and the nonexistence of post-intrinsically spherical two-surfaces (due to $h^\vartheta{}_\varphi$ and $h^\vartheta{}_\vartheta - h^\varphi{}_\varphi$) are physically of post-Newtonian order. Moreover, gauge freedom can be used to made $h^\vartheta{}_\vartheta$, $h^\vartheta{}_r$, and $h^\varphi{}_r$ of post-Newtonian order, leaving only $h^0{}_0$ and $h^r{}_r$ at the Newtonian level. In such a gauge $\mathcal{E}_G$ will have a Newtonian limit substantially like equation (94), only with three-dimensional gradients replacing r-derivatives. Then $\mathcal{E}$ will limit to the correct Newtonian energy density.

IV. STABILITY

a) *The Sufficient Condition*

The energy density $\mathcal{E}$ has three properties that qualify it as a Liapunov function (see, e.g., La Salle and Lefschetz 1961): (i) it is homogeneous and quadratic in the perturbation variables; (ii) it is globally conserved; and (iii) its integral over the interior of a large but finite sphere surrounding the star must decrease if the radiation satisfies a physically meaningful outgoing wave boundary condition on the sphere. Therefore *a sufficient condition for stability is that $\mathcal{E}$ be positive-definite*, i.e., that the integral of $\mathcal{E}$ over the interior of the large sphere be positive for all nontrivial physically acceptable perturbations.

By "physically acceptable" we mean that *the perturbation and its time-derivative must be consistent with the perturbed initial-value equations.* If one specifies $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{,0}$ on the hypersurface, one is not free to specify all ten $h^{\alpha\beta}$ and their derivatives. The initial-value equations (perturbed versions of $G^{\mu}{}_{0} - 8\pi T^{\mu}{}_{0} = 0$) set four restrictions on the 20 functions $h^{\alpha\beta}$ and $h^{\alpha\beta}{}_{,0}$. In addition, the choice of a gauge sets twelve more restrictions: The gauge completely determines four of the $h^{\alpha\beta}$ throughout spacetime (four conditions on $h^{\alpha\beta}$ *and* four conditions on $h^{\alpha\beta}{}_{,0}$ on the hypersurface), plus it permits solving for the four perturbed lapse and shift functions in terms of the remaining variables (cf. ADaM 1962 or Wheeler 1964). Another way to do this counting is to realize that the perturbed *geometry* is completely specified by giving the 12 functions h_{ij} and $h_{ij,0}$ on the hypersurface, though coordinate (gauge) arbitrariness off the hypersurface leaves some indeterminacy in $h^{\alpha\beta}$ off the hypersurface. Then imposing a gauge in the hypersurface (four conditions) and solving the four initial-value equations in the hypersurface reduce the number of free functions to four. *Thus, $\mathcal{E}$ must be positive-definite for arbitrary values of the six functions ξ^i and $\xi^i{}_{,0}$ plus the four independent functions among $h^{\alpha\beta}$ and $h^{\alpha\beta}{}_{,0}$.* (Unfortunately one is not likely to be able to prove $\mathcal{E}$ positive-definite without imposing the initial-value equations, as we show in the next paragraph.)

b) *Obstacles to the Application of This Condition*

Both the solution of the initial-value equations and the imposition of a gauge appear to be crucial before the sufficient condition can be used. In Newtonian theory the analog of the initial-value equations is the source equation for the gravitational potential, $\nabla^2\Phi = 4\pi\rho$. The contribution of the perturbed potential, $\delta\Phi$, to the energy of pulsation is negative-definite (cf. Paper I). Only by solving for $\delta\Phi$ as a Green's function integral over $\boldsymbol{\xi}$, or in terms of the longitudinal part of $\rho\boldsymbol{\xi}$ (as was done in Paper I), can the entire pulsation energy by shown to be positive-definite.

The imposition of a gauge is important because $\mathcal{E}$ is not gauge-invariant (though its integral over the hypersurface is). It may happen that even after solving the initial-value equations one may be able to prove the positive-definiteness of the energy density easily only in some gauges. Thus part of the problem is to find a gauge in which $\mathcal{E}_G$ (or $\mathcal{E}_G$ plus some of the terms in $\mathcal{E}_F$ that are quadratic in $h^{\alpha\beta}$) is *manifestly* positive-definite in the four free gravitational variables that remain. If such a gauge can be found, then the contribution to $\mathcal{E}$ from $\mathcal{E}_G$ can be discarded, and the sufficient condition reduced to an integral just over the interior of the star (plus possible surface integrals, as in Paper I). In that form, with the remaining energy a function only of $\boldsymbol{\xi}$, the condition will be tractable and ready for application to realistic stellar models.

We should remark that the gauge problem can probably be solved without going to a specific stellar model. The purpose of the gauge is to prove that the "free" gravitational waves—those that can be specified on the hypersurface independently of the star's perturbation $\boldsymbol{\xi}$—have positive energy. We should also remember that the gauge that solves the radiation problem may not be the same gauge that makes the dynamical equations simple (e.g., the Regge-Wheeler gauge used by Thorne and Campolattaro 1967 for the nonradial pulsations of spherical stars). Generally, one might expect the dynamical fluid equations to be simplest in the "near zone" or "Coulomb"-type gauge, which might be poorly behaved at spatial infinity. The gauge that proves the gravitational wave energy to be positive-definite, on the other hand, is likely to be a "radiation" or "Lorentz"-type gauge. This conflict may pose no problem since one need never solve the dynamical fluid equations to use the criterion: one need only prove that a certain functional of ξ is positive-definite.

c) *An Example: Radial Pulsation*

To illustrate the procedure outlined above on a problem whose solution is known, we evaluate $\mathcal{E}$ for the radial pulsations of a spherical star. We will find that $\mathcal{E}$ reduces to the same functional whose positive-definiteness Chandrasekhar (1964) proved was necessary and sufficient for stability.[3] The details of the calculations are contained in Appendix D.

i) *Choice of a Gauge*

The unperturbed metric is given by equation (81). For radial pulsations it is possible to choose a gauge in which the only nonzero metric perturbations are $\delta\nu = -h^0{}_0$ and $\delta\lambda = -h^r{}_r$ (see, for example, Landau and Lifshitz 1962). Both can be made to vanish outside the star.

ii) *Eliminating Non-dynamical Gravitational Variables*

Since there are no gravitational waves, both $\delta\lambda$ and $\delta\nu$ are determined completely by the fluid perturbations. The two "initial-value equations" that are relevant are

$$r^{-2}(re^{-\lambda})' - \frac{1}{r^2} = 8\pi T^0{}_0 \tag{97a}$$

and

$$e^{-\lambda}\left(\frac{1}{r}\,\nu' + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi T^r{}_r \tag{97b}$$

(where primes denote $\partial/\partial r$). Following Chandrasekhar (1964), the perturbed versions of these equations can be solved to give

$$\delta\lambda = -8\pi r e^{\lambda}\rho_0\mu\xi \tag{98a}$$

and

$$\delta\nu' = 8\pi r e^{\lambda}\left[\delta p - \rho_0\mu\left(\nu' + \frac{1}{r}\right)\xi\right]. \tag{98b}$$

We will not need the last equation for $\delta\nu$ because $\mathcal{E}$ will contain only $\delta\lambda$.

iii) *Calculating the Energy Density*

In Appendix D we show that the two parts of the gravitational energy density, equation (67), are

$$g^{\alpha\beta}(\mathcal{S}^\mu{}_{\nu\mu}\mathcal{S}^\nu{}_{\alpha\beta} - \mathcal{S}^\mu{}_{\nu\beta}\mathcal{S}^\nu{}_{\alpha\mu}) = \tfrac{1}{4}\nu' e^{-\lambda}(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{r}\,e^{-\lambda}\delta\lambda(\delta\lambda' + \delta\nu') \tag{99}$$

and

$$-\bar{h}^{\alpha\beta}{}_{,0}\mathcal{S}^0{}_{\alpha\beta} + \bar{h}^{\alpha 0}{}_{,0}\mathcal{S}^\mu{}_{\alpha\mu} = 0\,.$$

Adding to the energy density the divergence

$$-\frac{1}{64\pi}\,g^{-1/2}\left[g^{1/2}N\nu' e^{-\lambda}(\delta\nu - \delta\lambda)^2 + 4g^{1/2}N\,\frac{1}{r}\,e^{-\lambda}\delta\lambda^2\right]',$$

[3] Taub (1969) derived Chandrasekhar's criterion from the second variation of a variational principle of his own. This appears to be the first application of the second variation to stability problems in relativistic astrophysics.

we obtain

$$e^{-\nu/2}(\mathcal{E}_G)_{\text{RADIAL}} = \frac{1}{8\pi r}e^{-\lambda}\delta\lambda\delta\nu' - \frac{1}{64\pi}e^{-\lambda}\delta\nu^2\left(\nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \frac{2}{r}\nu'\right)$$
$$+ \frac{1}{32\pi}e^{-\lambda}\delta\nu\delta\lambda\left(\nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \frac{2}{r}\nu'\right)$$
$$- \frac{1}{64\pi}e^{-\lambda}\delta\lambda^2\left(\nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \frac{4}{r}\nu' - \frac{2}{r}\lambda' + \frac{4}{r^2}\right). \quad (100)$$

In Appendix D we also show that $\mathcal{E}_F$ becomes

$$e^{-\nu/2}(\mathcal{E}_F)_{\text{RADIAL}} = -\tfrac{1}{8}(\rho - p + 2\gamma p)\delta\lambda^2 - \tfrac{1}{4}(\rho + 3p)\delta\lambda\delta\nu + \tfrac{1}{8}(\rho + 3p)\delta\nu^2$$
$$- \delta p\delta\lambda + \rho_0 T\delta S\delta\nu - \mu\delta\nu\nabla\cdot(\rho_0\xi) + \rho_0\mu e^{\lambda-\nu}(\xi_{,0})^2$$
$$+ \gamma p(\nabla\cdot\xi)^2 + 2(\nabla\cdot\xi)p'\xi + \rho_0^{-1}p'\rho_0'\xi^2 - \tfrac{1}{2}\rho_0 TS'\nu'\xi^2. \quad (101)$$

When $\mathcal{E}_G$ is added to $\mathcal{E}_F$, and a convenient divergence added as well, the coefficients of all terms containing $\delta\nu$ vanish by virtue of equation (98a) and the unperturbed field equations. When $\delta\lambda$ is expressed in terms of ξ from equation (98a) and another divergence added, the resultant expression can be simplified to

$$(\mathcal{E})_{\text{RADIAL}} = \rho_0\mu e^{\lambda-\nu/2}(\xi_{,0})^2 + p\gamma e^{\nu/2}\chi^2 - \frac{e^{\nu/2}}{\rho_0\mu}(p')^2\xi^2$$
$$+ \frac{4}{r}e^{\nu/2}p'\xi^2 + 8\pi e^{\lambda+\nu/2}\rho_0\mu p\xi^2, \quad (102)$$

where χ stands for

$$\chi \equiv r^{-2}e^{\nu/2}(r^2 e^{-\nu/2}\xi)'. \quad (103)$$

Then positive-definiteness of the total energy,

$$E_{\text{RADIAL}} = \int_0^\infty \mathcal{E}4\pi r^2 e^{\lambda/2}dr, \quad (104)$$

for all possible ξ and $\xi_{,0}$ guarantees stability.

Chandrasekhar (1964) proved that the positive-definiteness of this $\mathcal{E}$ integrated from $r = 0$ to $r = R$ (surface of the star) is necessary and sufficient for stability. Since $\mathcal{E}$ is zero for $r > R$ and contains no delta-functions at $r = R$, we see that our results demonstrate the sufficiency of Chandrasekhar's criterion. In the next section we use our methods to show that his criterion is also necessary.

iv) *Lagrangian for Radial Pulsation*

The radial pulsations of a relativistic star are very similar to Newtonian pulsations: there is no gravitational radiation, and the perturbed gravitational field ($\delta\lambda$ and $\delta\nu$) can be expressed entirely in terms of ξ on a given hypersurface, without reference to the dynamics on previous hypersurfaces (cf. eqs. [98]). It is therefore possible to follow the procedure of Paper I here: one can substitute ξ directly into the Lagrangian density, equation (31), and use the resultant expression as the reduced Lagrangian density for the radial pulsations. The calculations are very similar to those required to reduce $\mathcal{E}$. The result is

$$(L_2)_{\text{RADIAL}} = -\rho_0\mu e^{\lambda-\nu}(\xi_{,0})^2 + p\gamma\chi^2 - \frac{1}{\rho_0\mu}(p')^2\xi^2 + \frac{4}{r}p'\xi^2 + 8\pi e^{\lambda}\rho_0\mu p\xi^2, \quad (105)$$

where χ was defined by equation (103). Clearly the energy density $\mathcal{E}$ is the Hamiltonian density associated with this Lagrangian density.

The theorem of Laval, Mercier, and Pellat (1965) applies to this case and implies that the positive-definiteness of E_{RADIAL} (eq. [104]) is necessary and sufficient for stability. This demonstrates how Chandrasekhar's theorem can be obtained with our approach. Needless to say, Chandrasekhar's own methods are much better for such a simple case. We used ours only to illustrate the more general procedure.

V. OUTLOOK

The stability criterion derived in this paper is only the first step in what promises to be a difficult but rewarding search for a useful stability criterion for relativistic stars. I have already discussed what steps may be needed before the goal is achieved. The most promising approach seems to me to be the analogue of the ADaM approach to the full field equations: choose a transverse-traceless gauge and solve the initial-value equations. There may be other workable approaches, however. In Appendix C the rate of transfer of energy from $\mathcal{E}_F$ to $\mathcal{E}_G$ is derived; it may happen that with the "outgoing energy" boundary condition and a careful choice of gauge, the initial-value equations imply that this rate is positive. Then $\mathcal{E}_G$ itself must decrease in time and so *its* positive-definiteness alone would guarantee stability. Both these approaches are under investigation.

Moreover, the Lagrangian, equation (31), has applications beyond the derivation of the sufficient criterion of this paper. It should be possible to derive from it the results of Chandrasekhar and Friedman (1971) in the zero-frequency approximation. It should also be possible to derive from it general criteria for the stability of standing-wave modes. Such criteria might well be less complicated than the one presented in this paper, and might serve as reasonably good indicators of the stability of realistic, outgoing-wave pulsations. The Lagrangian may prove to be an even more useful tool than the sufficient criterion for stability.

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APPENDIX A

THE SECOND VARIATION OF THE VELOCITY-POTENTIAL LAGRANGIAN

The full velocity potential Lagrangian is (Schutz 1970)

$$\mathfrak{L} = (R + 16\pi p)(-g)^{1/2} . \tag{A1}$$

Its second variation is the part that is quadratic in the perturbations when the full perturbed values of the independent variables (Palatini style: $g^{\sigma\nu}$, $\Gamma^{\gamma}{}_{\sigma\nu}$, ψ, α, β, θ, S) are substituted into equation (A1). By definition, the second variation of any of the independent variables themselves is zero. We treat the two parts of $\mathfrak{L}$ separately.

a) Second Variation of the Fluid Lagrangian

The fluid Lagrangian is $p(-g)^{1/2}$. Its second variation is

$$\delta^2[p(-g)^{1/2}] = \delta^2 p(-g)^{1/2} + 2\delta p\delta[(-g)^{1/2}] + p\delta^2[(-g)^{1/2}] . \tag{A2}$$

Now, the middle term is easy:

$$\delta[(-g)^{1/2}] = -\tfrac{1}{2}(-g)^{1/2}g_{\sigma\nu}h^{\sigma\nu} = -\tfrac{1}{2}(-g)^{1/2}h , \tag{A3}$$

and

$$\delta p = \rho_0\delta\mu - \rho_0 T\delta S , \tag{A4}$$

with

$$\delta\mu = \delta[(-g^{\nu\sigma}V_\nu V_\sigma)^{1/2}] = -\frac{1}{2\mu}h^{\nu\sigma}V_\nu V_\sigma - \frac{1}{\mu}g^{\nu\sigma}V_\sigma\delta V_\nu , \tag{A5}$$

where δV_ν stands for

$$\delta V_\nu = \delta\psi_{,\nu} + \alpha\delta\beta_{,\nu} + \delta\alpha\beta_{,\nu} - S\delta\theta_{,\nu} - \delta S\theta_{,\nu} . \tag{A6}$$

The second variation of $(-g)^{1/2}$ is also not hard to find:

$$\delta^2[(-g)^{1/2}] = \delta[\delta(-g)^{1/2}] = (-g)^{1/2}(\tfrac{1}{4}h^2 + \tfrac{1}{2}h^{\mu\sigma}h_{\mu\sigma}) . \tag{A7}$$

The second variation of p comes from equation (A4):

$$\delta^2 p = \delta(\rho_0\delta\mu - \rho_0 T\delta S) = \delta\rho_0\delta\mu + \rho_0\delta^2\mu - \delta(\rho_0 T)\delta S \tag{A8}$$

(recall that $\delta^2 S \equiv 0$). From equation (A5) we can compute $\delta^2\mu$:

$$\delta^2\mu = -\frac{1}{\mu}(\delta\mu)^2 - \frac{2}{\mu}h^{\sigma\nu}V_\sigma\delta V_\nu - \frac{1}{\mu}g^{\nu\sigma}\delta V_\sigma\delta V_\nu - \frac{1}{\mu}g^{\nu\sigma}V_\sigma\delta^2 V_\nu . \tag{A9}$$

Finally, we can find $\delta^2 V_\nu$ from equation (A6):

$$\delta^2 V_\nu = 2\delta\alpha\delta\beta_{,\nu} - 2\delta S\,\delta\theta_{,\nu} . \tag{A10}$$

Equations (A8)–(A10) combine to give

$$\begin{aligned}\delta^2 p = {} & \delta\rho_0\delta\mu - \delta(\rho_0 T)\delta S - \frac{\rho_0}{\mu}(\delta\mu)^2 - 2\rho_0 h^{\sigma\nu}U_\sigma\delta V_\nu \\ & - \frac{\rho_0}{\mu}g^{\nu\sigma}\delta V_\sigma\delta V_\nu - 2\rho_0 U^\nu(\delta\alpha\delta\beta_{,\nu} - \delta S\,\delta\theta_{,\nu}) .\end{aligned} \tag{A11}$$

This equation plus equations (A3), (A4), and (A7), when substituted into equation (A2), give equation (25) in the body of this paper.

b) Second Variation of the Curvature Lagrangian

In the Palatini method, the curvature Lagrangian is $g^{\alpha\beta}R_{\alpha\beta}(\Gamma)(-g)^{1/2}$. Its second variation is

$$\begin{aligned}\delta^2[R(-g)^{1/2}] = {} & 2h^{\alpha\beta}\delta R_{\alpha\beta}(-g)^{1/2} + 2h^{\alpha\beta}R_{\alpha\beta}\delta[(-g)^{1/2}] + g^{\alpha\beta}\delta^2(R_{\alpha\beta})(-g)^{1/2} \\ & + 2g^{\alpha\beta}\delta R_{\alpha\beta}\delta[(-g)^{1/2}] + g^{\alpha\beta}R_{\alpha\beta}\delta^2[(-g)^{1/2}] .\end{aligned} \tag{A12}$$

The only terms here that we have not yet computed are

$$\delta R_{\alpha\beta} = \delta[\Gamma^\mu{}_{\alpha\beta,\mu} - \Gamma^\mu{}_{\alpha\mu,\beta} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\Gamma^\nu{}_{\alpha\mu}] = \mathcal{S}^\mu{}_{\alpha\beta;\mu} - \mathcal{S}^\mu{}_{\alpha\mu;\beta} , \tag{A13}$$

and

$$\delta^2 R_{\alpha\beta} = 2\mathcal{S}^\mu{}_{\nu\mu}\mathcal{S}^\nu{}_{\alpha\beta} - 2\mathcal{S}^\mu{}_{\nu\beta}\mathcal{S}^\nu{}_{\alpha\mu} . \tag{A14}$$

It is straightforward to plug equations (A3), (A7), (A13), and (A14) into (A12) to obtain equation (29) in the body of this paper.

APPENDIX B

EULERIAN PERTURBATIONS

In this paper we often have occasion to convert from $\delta\psi$, $\delta\alpha$, . . . to the fluid displacement, $\boldsymbol{\xi}$. We shall write down the necessary expressions. More details can be found in Lebovitz (1961) or Lynden-Bell and Ostriker (1967). We use the language of the $3 + 1$ split of the background spacetime: $\boldsymbol{\xi}$ is the displacement of the fluid in the hypersurface of constant time, whose metric is g_{ij}. The determinant of g_{ij} is g. Covariant derivatives in the hypersurface are denoted by ∇ or by a subscripted vertical rule, $|$.

Because baryons are conserved, the change in rest mass inside a coordinate volume equals the transport of rest mass across its surface:

$$\delta(\rho_0 U^0 N g^{1/2}) = -g^{1/2}\nabla\cdot(\rho_0 U^0 N \xi) \tag{B1}$$

Because entropy per baryon is conserved, $\rho_0 S$ obeys the same equation as ρ_0. Together with equation (B1) this implies

$$\delta S = -\xi\cdot\nabla S\,. \tag{B2}$$

The velocity potentials α and β obey the same equation as S, so their perturbations are

$$\delta\alpha = -\xi\cdot\nabla\alpha + (\delta\alpha)_0\,, \tag{B3a}$$

$$\delta\beta = -\xi\cdot\nabla\beta + (\delta\beta)_0\,, \tag{B3b}$$

where $(\delta\alpha)_0$ and $(\delta\beta)_0$ are the values of $\delta\alpha$ and $\delta\beta$ when $\xi = 0$. They represent an initial velocity perturbation. They are "constants" of integration in the following sense:

$$U^\nu[(\delta\alpha)_0]_{,\nu} = U^\nu[(\delta\beta)_0]_{,\nu} = 0\,. \tag{B4}$$

Note that for δS the constant of integration is zero (cf. Paper I). The potentials $\delta\psi$ and $\delta\theta$ do not have equations as nice as equations (B3) because they are not "conserved" in the way α, β, and S are.

The changes in p, μ, ρ, T, ... can be computed from equations (B1) and (B2) and the equation of state. We obtain

$$\delta p = -\gamma p(\nabla\cdot\xi + g^{-1/2}\delta g^{1/2}) - \xi\cdot\nabla p - \frac{\gamma p}{U^0 N}[\delta(U^0 N) + \xi\cdot\nabla(U^0 N)]\,, \tag{B5}$$

$$\delta\mu = -\frac{\gamma p}{\rho_0}(\nabla\cdot\xi + g^{-1/2}\delta g^{1/2}) - \xi\cdot\nabla\mu - \frac{\gamma p}{\rho_0 U^0 N}[\delta(U^0 N) + \xi\cdot\nabla(U^0 N)]\,, \tag{B6}$$

$$\delta T = \left(\frac{\partial T}{\partial p}\right)_S \delta p + \left(\frac{\partial T}{\partial S}\right)_p \delta S\,, \tag{B7}$$

with the Maxwell identity

$$\left(\frac{\partial T}{\partial p}\right)_S = \frac{1}{\rho_0 p\gamma}\left(\frac{\partial p}{\partial S}\right)_{\rho_0} = \frac{1}{\rho_0^2}\left(\frac{\partial \rho_0}{\partial S}\right)_p\,. \tag{B8}$$

If we define the three-dimensional coordinate velocity, v, by the equation

$$v^i = U^i/U^0\,, \tag{B9}$$

then we have

$$\delta v^i = \xi^i{}_{,0} + \pounds_v \xi^i\,, \tag{B10a}$$

$$= \xi^i{}_{,0} + \xi^i{}_{|j}v^j - v^i{}_{|j}\xi^j\,. \tag{B10b}$$

This equation and equation (B2) render the perturbed entropy equation,

$$\delta\left(\frac{1}{U^0}U^\nu S_{,\nu}\right) = \delta S_{,0} + \delta v^i S_{,i} + v^i \delta S_{,i} = 0\,, \tag{B11}$$

an identity, and similarly for the α, β, and ρ_0 equations.

APPENDIX C

THE ENERGY OF PULSATION

a) The Hamiltonian

The generalized momenta of the problem, $\partial L_2/\partial \xi_{r,0}$, are not all independent, so one cannot solve for the velocities in terms of the momenta. Dirac (1958*a*, *b*) has developed an algorithm for expressing the equations of motion in Hamiltonian form in such situations, and Schutz (1971) has applied the method to the relativistic perfect fluid, starting from the full velocity-potential Lagrangian, equation (9). The only result we will need here is a result demonstrated in the appendix to Schutz (1971) for the time derivative of the Hamiltonian.

The Hamiltonian is

$$H_2 = \sum_r N q_{r,0} \frac{\partial L_2}{\partial q_{r,0}} - N L_2 , \tag{C1}$$

where $q_r \equiv \{\mathfrak{S}^\mu{}_{\alpha\beta},\ h^{\alpha\beta},\ \delta\psi,\ \delta\alpha,\ \delta\beta,\ \delta\theta,\ \delta S\}$. The overall factor of $N \equiv (-g^{00})^{-1/2}$ in H_2 arises from our abandoning general covariance: The action is to be expressed in the form

$$I_2 = \int\Big(\sum_r p^r q_{r,0} - H_2\Big) g^{1/2} d^3x dt . \tag{C2}$$

In order that this should be the same as

$$I_2 = \int L_2(-{}^4g)^{1/2} d^4x = \int L_2 N g^{1/2} d^3x dt , \tag{C3}$$

we need to include the factor of N in H_2 and in the generalized momenta, p^r.

By the theorem from Schutz (1971), the time derivative of H_2 is

$$\frac{\partial H_2}{\partial t} = \Big[\sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|i}}\Big]_{|i} - \frac{\partial}{\partial t}(N L_2)_{\text{holding } q_r,\ q_{r,0} \text{ fixed}} . \tag{C4}$$

This is the same as for a nondegenerate Hamiltonian.

If L_2 did not depend explicitly on time, then H_2 would be globally conserved. However, L_2 does depend upon time. From the remarks in § III*b* (i) we find that

$$\frac{\partial}{\partial t} N L_2 = -32\pi N \frac{\rho_0}{\mu}(g^{\sigma\nu}\delta V_\sigma + \mu U_\sigma h^{\sigma\nu})(\Omega_{,\nu}\delta\alpha + \mathfrak{T}_{,\nu}\delta S) . \tag{C5}$$

Here we have defined the "redshifted" temperature,

$$\mathfrak{T} \equiv T/U^0 . \tag{C6}$$

The first expression in parentheses in equation (C5) is just δV^ν. In terms of the coordinate velocity, $v^i \equiv U^i/U^0$, equation (C5) becomes

$$\frac{\partial}{\partial t} N L_2 = -32\pi N \rho_0 U^0 \delta v^i(\Omega_{,i}\delta\alpha + \mathfrak{T}_{,i}\delta S) . \tag{C7}$$

In obtaining this we used the fact that Ω and $\mathfrak{T}$ are independent of t and φ.

We can express δv^i, $\delta\alpha$, and δS in terms of ξ by using equations (B10), (B2), and (B3a). Then manipulations similar to those of Appendix A of Paper I can simplify equation (C7) considerably. The crucial idea in the manipulations is that the quantity

$$M_{ij} \equiv \alpha_{,i}\Omega_{,j} + S_{,i}\mathfrak{T}_{,j} \tag{C8}$$

is symmetric; its antisymmetric part is

$$\frac{\partial}{\partial t} V_{[i;j]} = \frac{\partial}{\partial t}(\alpha_{[,j}\beta_{,i]} - S_{[,j}\theta_{,i]}) = M_{[ij]} ,$$

which must vanish because the unperturbed flow is stationary. The final result of the manipulations is

$$\begin{aligned} &-32\pi N\rho_0 U^0 \delta v^i(\Omega_{,i}\delta\alpha + \mho_{,i}\delta S) \\ &= \frac{\partial}{\partial t}[16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j - 32\pi N\rho_0 U^0 \Omega_{,i}\xi^i(\delta\alpha)_0] \\ &+ [16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j v^l - 32\pi N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0 v^l]_{|l} . \end{aligned} \tag{C9}$$

Notice that the initial perturbation in α appears explicitly.

From this equation we see that the term that prevents H_2 from being conserved is itself a time-derivative plus a divergence! We can therefore rewrite equation (C4) in the form

$$\begin{aligned} &\frac{\partial}{\partial t}[H_2 - 16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j + 32\pi N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0] \\ &= \left[\sum_r q_{r,0}\frac{\partial H_2}{\partial q_{r|l}} + 16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j v^l - 32\pi N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0 v^l\right]_{|l} . \end{aligned} \tag{C10}$$

b) The Energy and Its Flux

We may tentatively identify the energy density of the pulsations as

$$\mathcal{E}' \equiv H_2 - 16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j + 32\pi N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0 . \tag{C11}$$

Its uniqueness and gauge properties are discussed in § III*d*. Here we are interested in evaluating $\mathcal{E}'$ and its flux.

From the Lagrangian $L_2 = \mathcal{R} + 16\pi\mathcal{P}$ given in equations (25) and (29) we find

$$N\frac{\partial L_2}{\partial \mathcal{S}^\mu{}_{\alpha\beta,0}} = 2N\bar{h}^{\alpha\beta}\delta^0{}_\mu - 2N\bar{h}^{\alpha\beta}\delta^\beta{}_\mu , \tag{C12a}$$

$$N\frac{\partial L_2}{\partial h^{\alpha\beta}{}_{,0}} = 0 , \tag{C12b}$$

$$N\frac{\partial L_2}{\partial \delta\psi_{,0}} = -32\pi g^{-1/2}\delta(\rho_0 U^0 N g^{1/2}) , \tag{C12c}$$

$$N\frac{\partial L_2}{\partial \delta\alpha_{,0}} = 0 , \tag{C12d}$$

$$N\frac{\partial L_2}{\partial \delta\beta_{,0}} = -32\pi g^{-1/2}\delta(\rho_0 U^0\alpha N g^{1/2}) , \tag{C12e}$$

$$N\frac{\partial L_2}{\partial \delta S_{,0}} = 0 , \tag{C12f}$$

$$N\frac{\partial L_2}{\partial \delta\theta_{,0}} = +32\pi g^{-1/2}\delta(\rho_0 U^0 S N g^{1/2}) . \tag{C12g}$$

These imply that H_2 is

$$\begin{aligned} H_2 = {} & 2N\bar{h}^{\alpha\beta}\mathcal{S}^0{}_{\alpha\beta,0} - 2N\bar{h}^{\alpha 0}\mathcal{S}^\mu{}_{\alpha\mu,0} - 32\pi g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta\psi_{,0} + \alpha\delta\beta_{,0} - S\delta\theta_{,0}) \\ & - 32\pi N\rho_0 U^0(\delta\alpha\delta\beta_{,0} - \delta S\delta\theta_{,0}) - N\mathcal{R} - 16\pi N\mathcal{P} . \end{aligned} \tag{C13}$$

Consider the gravitational part first:

$$H_{2(G)} \equiv 2N\bar{h}^{\alpha\beta}\mathcal{S}^{0}{}_{\alpha\beta,0} - 2N\bar{h}^{\alpha 0}\mathcal{S}^{\mu}{}_{\alpha\mu,0} - N\mathcal{R}\,. \tag{C14}$$

This would appear to contain second time-derivatives of $h^{\alpha\beta}$. Actually it does not, as we can see with the help of an identity that follows from the definition of $\mathcal{S}^{\mu}{}_{\alpha\beta}$ in terms of $h^{\alpha\beta}$ (eq. [33]):

$$\begin{aligned}\bar{h}^{\alpha\beta}(\mathcal{S}^{\mu}{}_{\alpha\beta;\mu} - \mathcal{S}^{\mu}{}_{\alpha\mu;\beta}) \equiv {}& (-{}^4g)^{-1/2}[(-{}^4g)^{1/2}(\bar{h}^{\mu\nu}\mathcal{S}^{\alpha}{}_{\mu\nu} - \bar{h}^{\nu\alpha}\mathcal{S}^{\mu}{}_{\nu\mu})]_{,\alpha} \\ & - 2g^{\alpha\beta}(\mathcal{S}^{\mu}{}_{\nu\mu}\mathcal{S}^{\nu}{}_{\alpha\beta} - \mathcal{S}^{\mu}{}_{\nu\beta}\mathcal{S}^{\nu}{}_{\alpha\mu})\,.\end{aligned} \tag{C15}$$

This identity converts $\mathcal{R}$ (eq. [29]) to

$$\begin{aligned}\mathcal{R} = {}& -2g^{\alpha\beta}(\mathcal{S}^{\mu}{}_{\nu\mu}\mathcal{S}^{\nu}{}_{\alpha\beta} - \mathcal{S}^{\mu}{}_{\nu\beta}\mathcal{S}^{\nu}{}_{\alpha\mu}) + 2(\bar{h}^{\alpha\beta}\mathcal{S}^{0}{}_{\alpha\beta} - \bar{h}^{\alpha 0}\mathcal{S}^{\mu}{}_{\alpha\mu})_{,0} \\ & + \frac{2}{N}[N\bar{h}^{\mu\nu}\mathcal{S}^{i}{}_{\mu\nu} - N\bar{h}^{\nu i}\mathcal{S}^{\mu}{}_{\nu\mu}]_{|i} - hh^{\alpha\beta}R_{\alpha\beta} + R(\tfrac{1}{4}h^2 + \tfrac{1}{2}h_{\alpha\beta}h^{\alpha\beta})\,.\end{aligned} \tag{C16}$$

With this, $H_{2(G)}$ becomes

$$\begin{aligned}H_{2(G)} = {}& -2N\bar{h}^{\alpha\beta}{}_{,0}\mathcal{S}^{0}{}_{\alpha\beta} + 2N\bar{h}^{\alpha 0}{}_{,0}\mathcal{S}^{\mu}{}_{\alpha\mu} + 2N(\mathcal{S}^{\mu}{}_{\nu\mu}\mathcal{S}^{\nu}{}_{\alpha\beta} - \mathcal{S}^{\mu}{}_{\nu\beta}\mathcal{S}^{\nu}{}_{\alpha\mu}) \\ & + Nhh^{\alpha\beta}R_{\alpha\beta} - NR(\tfrac{1}{4}h^2 + \tfrac{1}{2}h_{\alpha\beta}h^{\alpha\beta}) + 2[N\bar{h}^{\mu\nu}\mathcal{S}^{i}{}_{\mu\nu} - N\bar{h}^{\nu i}\mathcal{S}^{\mu}{}_{\nu\mu}]_{|i}\,.\end{aligned} \tag{C17}$$

This is quadratic in derivatives of $h^{\alpha\beta}$ after we throw away the divergence (we must remember to discard the appropriate time derivative from the flux to compensate this divergence).

We make no modification of the rest of H_2 except to note that

$$\delta\psi_{,0} + \alpha\delta\beta_{,0} - S\delta\theta_{,0} = \delta V_0 - \beta_{,0}\delta\alpha + \theta_{,0}\delta S = \delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S\,. \tag{C18}$$

When all terms are assembled and divided by 16π, the result is equations (67) and (68).

The energy flux (Poynting vector) is, from equation (C10),

$$\mathfrak{F}'^{l} = -\sum_r q_{r,0}\frac{\partial H_2}{\partial q_{r|l}} - 16\pi N\rho_0 U^0 M_{ij}\xi^i\xi^j v^l + 32\pi N\rho_0 U^0\Omega_{,i}\xi^i(\delta\alpha)_0 v^l\,. \tag{C19}$$

From expressions similar to equations (C12) we find that

$$\begin{aligned}-\sum_r q_{r,0}\frac{\partial H_2}{\partial q_{r|l}} &= \sum_r Nq_{r,0}\frac{\partial L_2}{\partial q_{r|l}} \\ &= 2N\bar{h}^{\alpha\beta}\mathcal{S}^{l}{}_{\alpha\beta,0} - 2N\bar{h}^{\alpha l}\mathcal{S}^{\mu}{}_{\alpha\mu,0} - 32\pi g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta\psi_{,0} + \alpha\delta\beta_{,0} - S\delta\theta_{,0}) \\ &\quad - 32\pi N\rho_0 U^0(\delta\alpha\delta\beta_{,0} - \delta S\delta\theta_{,0}) \\ &= -2N\bar{h}^{\alpha\beta}{}_{,0}\mathcal{S}^{l}{}_{\alpha\beta} + 2N\bar{h}^{\alpha l}{}_{,0}\mathcal{S}^{\mu}{}_{\alpha\mu} - 32\pi N g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S) \\ &\quad - 32\pi N\rho_0 U^0(\delta\alpha\delta\beta_{,0} - \delta S\delta\theta_{,0}) + (2N\bar{h}^{\alpha\beta}\mathcal{S}^{l}{}_{\alpha\beta} - 2N\bar{h}^{\alpha l}\mathcal{S}^{\mu}{}_{\alpha\mu})_{,0}\,.\end{aligned} \tag{C20}$$

The last term in this equation is exactly the one required to cancel the divergence in equation (C12)! So when we discard it and divide by 16π we get equations (69) and (70) for the flux.

For completeness we write down what the first three terms of $H_{2(G)}$ (eq. [C17]) become if we substitute for the $\mathcal{S}^{\mu}{}_{\alpha\beta}$'s their expressions in terms of $h^{\alpha\beta}$. This is what in the body of the paper we call $16\pi\,\mathcal{E}_G$:

$$\begin{aligned}16\pi\mathcal{E}_G = {}& -\tfrac{1}{2}N\bar{h}^{\alpha\beta}{}_{,0}\bar{h}_{\alpha\beta}{}^{;0} + \tfrac{1}{2}N\bar{h}^{\alpha\beta}{}_{;i}\bar{h}_{\alpha\beta}{}^{;i} - N\bar{h}^{\alpha\beta}{}_{;i}\bar{h}^{i}{}_{\alpha;\beta} + N\bar{h}^{\alpha\beta}{}_{,0}\bar{h}^{0}{}_{\alpha;\beta} \\ & + N\bar{h}_{,0}\bar{h}^{,0} + \tfrac{1}{2}N\bar{h}^{,i}\bar{h}_{,i} + \tfrac{1}{2}N(\Gamma^{\alpha}{}_{\sigma 0}\bar{h}^{\sigma\beta} + \Gamma^{\beta}{}_{\sigma 0}\bar{h}^{\alpha\sigma})(\bar{h}_{\alpha\beta}{}^{;0} - 2\bar{h}^{0}{}_{\alpha;\beta})\,.\end{aligned} \tag{C21}$$

Similarly, the gravitational part of the flux (first two terms of eq. [C20]) becomes

$$16\pi\mathfrak{F}_G{}^l = -N\bar{h}^{\alpha\beta}{}_{,0}\bar{h}_{\alpha\beta}{}^{;l} + 2N\bar{h}^{\alpha\beta}{}_{,0}\bar{h}^l{}_{\alpha;\beta} + \tfrac{1}{2}N\bar{h}_{,0}\bar{h}^{,l} . \tag{C22}$$

c) Transfer of Energy between Fluid and Radiation

The Hamiltonian formalism permits us to calculate not only the rate of change of the total energy density $\mathcal{E}$, but also the rate at which different parts of $\mathcal{E}$ change. In the body of this paper we define

$$\begin{aligned}\mathcal{E}_F &= \mathcal{E} - \mathcal{E}_G \\ &= \frac{1}{16\pi} H_{2(F)} - N\rho_0 U^0 M_{ij}\xi^i\xi^j + 2N\rho_0 U^0 \Omega_{,i}\xi^i(\delta\alpha)_0 + \frac{1}{16\pi} Nhh^{\alpha\beta}R_{\alpha\beta} \\ &\quad - \frac{1}{16\pi} NR(\tfrac{1}{4}h^2 + \tfrac{1}{2}h^{\alpha\beta}h_{\alpha\beta}) ,\end{aligned} \tag{C23}$$

where $H_{2(F)}$ is the Hamiltonian obtained just from the Lagrangian $\mathcal{P}$:

$$\begin{aligned}\frac{1}{16\pi} H_{2(F)} &= -2g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta V_0 + \Omega\delta\alpha + \mathfrak{I}\delta S) \\ &\quad - 2\rho_0 U^0 N(\delta\alpha\delta\beta_{,0} - \delta S\delta\theta_{,0}) - \mathcal{P} .\end{aligned} \tag{C24}$$

The time-derivative of $\mathcal{E}_F$ can be found in this manner:

The time-derivative of $H_{2(F)}$ is of three parts: a part due to the time-derivatives of the fluid variables, a part due to the time-derivatives of the gravitational variables, and a part due to its explicit time dependence. The last part is canceled by the time-derivative of the second and third terms in equation (C23) (by the construction of the previous section!). The first part is just a divergence because $H_{2(F)}$ is the Hamiltonian that governs the time-derivatives of the fluid variables. Thus we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{1}{16\pi} H_{2(F)} - N\rho_0 U^0 M_{ij}\xi^i\xi^j + 2N\rho_0 U^0 \Omega_{,i}\xi^i(\delta\alpha)_0\right] \\ &\quad = -\mathfrak{F}_F{}^k{}_{|k} + \frac{1}{16\pi} h^{\alpha\beta}{}_{,0}\frac{\partial H_{2(F)}}{\partial h^{\alpha\beta}} + \frac{1}{16\pi} \mathcal{S}^\mu{}_{\alpha\beta,0}\frac{\partial H_{2(F)}}{\partial \mathcal{S}^\mu{}_{\alpha\beta}} ,\end{aligned} \tag{C25}$$

where $\mathfrak{F}_F{}^k$, which is defined in the body of this paper (eq. [70]), represents the energy carried out of some volume by the fluid itself. Now $H_{2(F)}$ does not depend upon $\mathcal{S}^\mu{}_{\alpha\beta}$; from equation (35) we find

$$\frac{1}{16\pi}\frac{\partial H_{2(F)}}{\partial h^{\alpha\beta}} = -N\frac{\partial\mathcal{P}}{\partial h^{\alpha\beta}} = g^{-1/2}\delta[T_{\alpha\beta}(-{}^4g)^{1/2}] , \tag{C26a}$$

$$= N[\mu U_\alpha U_\beta\delta\rho_0 + \rho_0 U_\alpha U_\beta\delta\mu + 2\rho_0\mu U_{(\alpha}\delta U_{\beta)} + \delta p g_{\alpha\beta} - p h_{\alpha\beta} - \tfrac{1}{2}T_{\alpha\beta}h] . \tag{C26b}$$

[From eq. (C24) one might conclude that $H_{2(F)}$ depends on $h^{\alpha\beta}$ not only through $\mathcal{P}$ but through the first term, which includes $\delta(\rho_0 U^0 N g^{1/2})$. This is not true: $-2g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})$ is the momentum conjugate to $\delta\psi$, $\partial_2 L/\partial\delta\psi_{,0}$. It is a fluid variable, and its time rate of change is included in $\mathfrak{F}_F{}^k$.]

Since the last two terms in $\mathcal{E}_F$ also depend only on $h^{\alpha\beta}$, we can write down $\partial\mathcal{E}_F/\partial t$ immediately:

$$\begin{aligned}\frac{\partial\mathcal{E}_F}{\partial t} + \mathfrak{F}_F{}^k{}_{|k} &= Nh^{\alpha\beta}{}_{,0}[\mu U_\alpha U_\beta\delta\rho_0 + \rho_0 U_\alpha U_\beta\delta\mu + 2\rho_0\mu U_\alpha\delta U_\beta + \delta p g_{\alpha\beta} \\ &\quad - \tfrac{1}{2}(\rho - p)\bar{h}_{\alpha\beta} + \tfrac{1}{2}\rho_0\mu g_{\alpha\beta}U_\mu U_\nu h^{\mu\nu}] .\end{aligned} \tag{C27}$$

Since the divergence of $\mathfrak{F}_F{}^k$ represents transport of energy by the fluid, the total rate of transfer of energy from $\mathcal{E}_F$ to $\mathcal{E}_G$ is negative of the integral of the right-hand side of equation (C27) over the entire star.

APPENDIX D

THE NONROTATING STAR

a) Arbitrary Pulsations

The nonrotating star has the background metric

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \tag{D1}$$

From the equation $U^\alpha U_\alpha = -1$ we find

$$\delta U^0 = -\tfrac{1}{2}e^{-3\nu/2}h_{00} \tag{D2a}$$

and

$$\delta U_0 = -\tfrac{1}{2}e^{-\nu/2}h_{00} . \tag{D2b}$$

From Appendix B we learn

$$\delta S = -\xi \cdot \nabla S = -\xi^r S_{,r} \tag{D3a}$$

$$g^{-1/2}\delta(\rho_0 U^0 N g^{1/2}) = g^{-1/2}\delta(\rho_0 g^{1/2}) = -\nabla \cdot (\rho_0 \xi) \tag{D3b}$$

$$\delta v^i = \xi^i{}_{,0} . \tag{D3c}$$

In order to put $\mathcal{E}_F$ in terms only of ξ and $h^{\alpha\beta}$ it is convenient to treat separately the following pieces of $\mathcal{E}_F$:

$$A \equiv -2g^{-1/2}\delta(\rho_0 U^0 N g^{1/2})(\delta V_0 + \mathfrak{I}\delta S) , \tag{D4a}$$

$$B \equiv \frac{\rho_0}{\mu} N g^{\nu\sigma}\delta V_\nu \delta V_\sigma + 2N\rho_0 U_\sigma h^{\sigma\nu}\delta V_\nu + N\frac{\rho_0}{\mu}(\delta\mu)^2 , \tag{D4b}$$

$$C \equiv -N\delta\rho_0\delta\mu + N\delta(\rho_0 T)\delta S - N\rho_0 U^0 S_{,i}\mathfrak{I}_{,j}\xi^i\xi^j , \tag{D4c}$$

$$D \equiv Nh\delta p + \frac{1}{16\pi}Nhh^{\alpha\beta}R_{\alpha\beta} - N\left(\frac{1}{16\pi}R + p\right)(\tfrac{1}{4}h^2 + \tfrac{1}{2}h^{\alpha\beta}h_{\alpha\beta}) . \tag{D4d}$$

i) *A*. From the above equations and Appendix B, we find

$$\delta(\rho_0 U^0 N g^{1/2}) = -g^{1/2}\nabla \cdot (\rho_0 \xi) ,$$

$$\delta V_0 + \mathfrak{I}\delta S = -e^{\nu/2}\delta\mu - \tfrac{1}{2}\mu e^{-\nu/2}h_{00} + e^{\nu/2}T\delta S ,$$

and

$$A = -2e^{\nu/2}\nabla \cdot (\rho_0\xi)[\rho_0{}^{-1}\delta p + \tfrac{1}{2}\mu e^{-\nu}h_{00}] . \tag{D5}$$

ii) *B*. This term contains the kinetic energy of the fluid:

$$\begin{aligned}\delta V_\sigma &= U_\sigma\delta\mu + \mu\delta U_\sigma = U_\sigma\delta\mu + \mu\delta(g_{\sigma\nu}U^\nu) \\ &= U_\sigma\delta\mu + \mu g_{\sigma 0}\delta U^0 + \mu U^0 g_{\sigma i}\delta v^i - \mu U^0 h_{0\sigma} \\ &= -\delta^0{}_\sigma e^{\nu/2}(\delta\mu + \tfrac{1}{2}\mu e^{-\nu}h_{00}) + \delta^j{}_\sigma \mu e^{-\nu/2}(g_{ij}\delta v^i - h_{j0}) .\end{aligned} \tag{D6}$$

From this we find

$$g^{\sigma\nu}\delta V_\sigma \delta V_\nu = -(\delta\mu + \tfrac{1}{2}\mu e^{-\nu}h_{00})^2 + \mu^2 e^{-\nu}[g_{ij}\xi^i{}_{,0}\xi^j{}_{,0} - 2h_{0i}\xi^i{}_{,0} + h_0{}^j h_{0j}]$$

and

$$U_\sigma h^{\sigma\nu}\delta V_\nu = e^{-\nu}(\delta\mu + \tfrac{1}{2}\mu e^{-\nu}h_{00})h_{00} + \mu e^{-\nu}h_{0i}\xi^i{}_{,0} - \mu e^{-\nu}h_{0j}h_0{}^j .$$

These combine to give

$$B = \rho_0\mu e^{-\nu/2}\xi_{,0}\cdot\xi_{,0} + \rho_0 e^{-\nu/2}h_{00}\delta\mu - \rho_0\mu e^{-\nu/2}h_0{}^j h_{0j} + \tfrac{3}{4}\rho_0\mu e^{-3\nu/2}h_{00}{}^2 \,. \tag{D7}$$

iii) C. If we add to C the first term of A from equation (D5) and call the result E, we get

$$E = N\rho_0{}^{-1}g^{-1/2}\delta(\rho_0 g^{1/2})\delta p + N\rho_0\delta T\delta S - N\rho_0 S_{,i}T_{,j}\xi^i\xi^j$$
$$+ N\delta p g^{-1/2}\delta(g^{1/2}) - N\rho_0 U^0\left(\frac{1}{U^0}\right)_{,j} TS_{,i}\xi^i\xi^j \,. \tag{D8}$$

The first three terms of this can be written as

$$Ng^{-1/2}\delta(g^{1/2})\delta p + N\rho_0{}^{-1}\delta\rho_0\delta p + N\rho_0(\Delta T)\delta S \,, \tag{D9}$$

where ΔT is the *Lagrangian* change in T,

$$\Delta T = \left(\frac{\partial T}{\partial p}\right)_S \Delta p = -\frac{1}{\rho_0{}^2}\left(\frac{\partial\rho_0}{\partial S}\right)_p \Delta p \,. \tag{D10}$$

By writing the second term in expression (D9) as

$$N\rho_0{}^{-1}\delta\rho_0\Delta p - N\rho_0{}^{-1}(\xi\cdot\nabla p)\delta\rho_0 \tag{D11}$$

and using equation (D10), we find that E becomes

$$E = N\rho_0{}^{-1}\left(\frac{\partial\rho_0}{\partial p}\right)_S \Delta p\delta p - N\rho_0{}^{-1}\delta\rho_0(\xi\cdot\nabla p) + 2Ng^{-1/2}\delta(g^{1/2})\delta p$$
$$- \tfrac{1}{2}N\rho_0 T(\xi\cdot\nabla S)(\xi\cdot\nabla\nu) \,. \tag{D12}$$

But Appendix B tells us that

$$\Delta p = -\gamma p[\nabla\cdot\xi + g^{-1/2}\delta(g^{1/2})] \,. \tag{D13a}$$

Moreover, the definition of γ is

$$\gamma = \frac{\rho_0}{p}\left(\frac{\partial p}{\partial\rho_0}\right)_S \,. \tag{D13b}$$

Therefore, E becomes

$$E = e^{\nu/2}\gamma p(\nabla\cdot\xi)^2 + 2e^{\nu/2}(\nabla\cdot\xi)(\xi\cdot\nabla p) + e^{\nu/2}\rho_0{}^{-1}(\xi\cdot\nabla p)(\xi\cdot\nabla\rho_0)$$
$$- \tfrac{1}{2}e^{\nu/2}\rho_0 T(\xi\cdot\nabla S)(\xi\cdot\nabla\nu) - \tfrac{1}{4}e^{\nu/2}\gamma p(h^j{}_j)^2 \,. \tag{D14}$$

iv) D. Using the unperturbed Einstein equations, we obtain

$$D = e^{\nu/2}h\delta p + \tfrac{1}{2}e^{-\nu/2}\rho_0\mu h_{00}h + \tfrac{1}{8}e^{\nu/2}(\rho - p)h^2 - \tfrac{1}{4}e^{\nu/2}(\rho - p)h^{\alpha\beta}h_{\alpha\beta} \,. \tag{D15}$$

If we assemble all these terms we obtain equation (83).

b) Radial Pulsations

If Nature is reasonable, the stability criterion proved in this paper ought to reduce to Chandrasekhar's (1964) necessary and sufficient condition for stability against radial pulsations. In this section we show that $\mathcal{E}$ does indeed reduce to Chandrasekhar's variational function.

We can choose a gauge such that the only two nonzero metric perturbations are

(see, e.g., Landau and Lifshitz 1962)

$$\delta\nu = e^{-\nu}h_{00} = -h^0{}_0 \tag{D16a}$$

and

$$\delta\lambda = -e^{-\lambda}h_{rr} = -h^r{}_r\,. \tag{D16b}$$

In this gauge we have (ξ has only an r-component)

$$\delta\mu = -\frac{\gamma p}{\rho_0}(\nabla\cdot\xi + \tfrac{1}{2}\delta\lambda) - \xi\cdot\nabla\mu \tag{D17a}$$

$$\delta p = -\gamma p(\nabla\cdot\xi + \tfrac{1}{2}\delta\lambda) - \xi\cdot\nabla p\,. \tag{D17b}$$

Since there is no dynamical freedom in the gravitational field (no spherical gravitational waves), we ought to be able to express $\delta\nu$ and $\delta\lambda$ in terms of ξ. We use the $\binom{0}{0}$ and $\binom{r}{r}$ Einstein equations:

$$\frac{1}{r^2}(re^{-\lambda})' - \frac{1}{r^2} = 8\pi T^0{}_0 \tag{D18a}$$

and

$$e^{-\lambda}\left(\frac{1}{r}\nu' + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi T^r{}_r \tag{D18b}$$

(where primes denote $\partial/\partial r$). Their perturbed versions can be solved to give (cf. Chandrasekhar 1964)

$$\delta\lambda = -8\pi re^{\lambda}\rho_0\mu\xi\,, \tag{D19a}$$

$$\delta\nu' = 8\pi re^{\lambda}\left[\delta p - \rho_0\mu\left(\nu' + \frac{1}{r}\right)\xi\right]. \tag{D19b}$$

We will never need $\delta\nu$ itself; will only need to substitute for $\delta\lambda$.

To calculate $\mathcal{E}_G$ we need the following $\mathcal{S}$'s (which can be read off the table of Christoffel symbols in Landau and Lifshitz 1962, § 97)

$$\mathcal{S}^0{}_{00} = \tfrac{1}{2}\delta\nu_{,0}\,,\quad \mathcal{S}^r{}_{00} = \tfrac{1}{2}e^{\nu-\lambda}[\delta\nu' + \nu'(\delta\nu - \delta\lambda)]\,,$$
$$\mathcal{S}^0{}_{0r} = \tfrac{1}{2}\delta\nu'\,,\quad \mathcal{S}^r{}_{0r} = \tfrac{1}{2}\delta\lambda_{,0}\,,\quad \mathcal{S}^r{}_{\vartheta\vartheta} = re^{-\lambda}\delta\lambda\,,$$
$$\mathcal{S}^0{}_{rr} = \tfrac{1}{2}e^{\lambda-\nu}\delta\lambda_{,0}\,,\quad \mathcal{S}^r{}_{rr} = \tfrac{1}{2}\delta\lambda'\,,\quad \mathcal{S}^r{}_{\varphi\varphi} = r\sin^2\vartheta e^{-\lambda}\delta\lambda\,. \tag{D20}$$

All others that cannot be obtained from these by the symmetry $\mathcal{S}^\mu{}_{\alpha\beta} = \mathcal{S}^\mu{}_{\beta\alpha}$ are zero.

With these we find

$$g^{\alpha\beta}(\mathcal{S}^\mu{}_{\nu\mu}\mathcal{S}^\nu{}_{\alpha\beta} - \mathcal{S}^\mu{}_{\nu\beta}\mathcal{S}^\nu{}_{\alpha\mu}) = \tfrac{1}{4}\nu' e^{-\lambda}(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + r^{-1}e^{-\lambda}\delta\lambda(\delta\lambda' + \delta\nu')\,, \tag{D21}$$

and

$$-\bar{h}^{\alpha\beta}{}_{,0}\mathcal{S}^0{}_{\alpha\beta} + \bar{h}^{\alpha 0}{}_{,0}\mathcal{S}^\beta{}_{\alpha\beta} = 0\,. \tag{D22}$$

Then from equation (67) $\mathcal{E}_G$ is

$$\mathcal{E}_G = \frac{\nu'}{32\pi}e^{\nu/2-\lambda}(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{8\pi r}e^{\nu/2-\lambda}\delta\lambda(\delta\lambda' + \delta\nu')\,. \tag{D23}$$

By adding the divergence

$$-\frac{1}{64\pi}g^{-1/2}\left\{g^{1/2}e^{\nu/2-\lambda}\left[\nu'(\delta\nu - \delta\lambda)^2 + \frac{4}{r}\delta\lambda^2\right]\right\}'\,, \tag{D24}$$

we can eliminate almost all terms that have derivatives of $\delta\lambda$ and $\delta\nu$. (Note that the factors of $g^{1/2}$ in eq. [D24] ensure that the expression will be a divergence when integrated over proper volume in the hypersurface, $g^{1/2}d^3x$.) The result is equation (100).

To calculate $\mathcal{E}_F$ we begin with equation (83). We shall need the following field equations:

$$\frac{1}{16\pi} R_{00} = \frac{1}{32\pi} e^{\nu-\lambda}\left[\nu'' - \tfrac{1}{2}\nu'\lambda' + \tfrac{1}{2}(\nu')^2 + \frac{2}{r}\nu'\right] = \tfrac{1}{4}e^{\nu}(\rho + 3p)\,, \tag{D25}$$

$$\frac{1}{16\pi} R_{rr} = \frac{1}{32\pi}\left[-\nu'' + \tfrac{1}{2}\nu'\lambda' - \tfrac{1}{2}(\nu')^2 + \frac{2}{r}\lambda'\right] = \tfrac{1}{4}e^{\lambda}(\rho - p)\,, \tag{D26}$$

$$\frac{1}{16\pi} R = -\frac{1}{16\pi}\left[\nu'' - \tfrac{1}{2}\nu'\lambda' + \tfrac{1}{2}(\nu')^2 + \frac{2}{r}(\nu' - \lambda') + \frac{2}{r^2}(1 - e^{\lambda})\right] = \tfrac{1}{2}(\rho - 3p)\,. \tag{D27}$$

Equation (83) becomes

$$\begin{aligned} e^{-\nu/2}\mathcal{E}_F = {}& \rho_0\mu e^{\lambda-\nu}(\xi_{\,0})^2 + \gamma p(\nabla\cdot\xi)^2 + 2(\nabla\cdot\xi)p'\xi + \rho_0^{-1}p'\rho_0'\xi^2 \\ &- \tfrac{1}{2}\rho_0 TS'\nu'\xi^2 - \mu\delta\nu\nabla\cdot(\rho_0\xi) + \rho_0 T\delta S\delta\nu - \delta p\delta\lambda \\ &+ \tfrac{1}{8}(\rho + 3p)\delta\nu^2 - \tfrac{1}{4}(\rho + 3p)\delta\nu\delta\lambda - \tfrac{1}{8}(\rho - p + 2\gamma p)\delta\lambda^2\,. \end{aligned} \tag{D28}$$

By adding to $\mathcal{E}_F$ the divergence

$$g^{-1/2}(\mu e^{\nu/2}g^{1/2}\rho_0\xi\delta\nu)'\,, \tag{D29}$$

and by adding $\mathcal{E}_F$ to $\mathcal{E}_G$, we obtain for $\mathcal{E}$

$$\begin{aligned} e^{-\nu/2}\mathcal{E} = {}& \frac{-1}{16\pi r^2} e^{-\lambda}(1 + r\nu' + 4\pi r^2\gamma p)\delta\lambda^2 - \delta\lambda\delta p + \rho_0\mu e^{\lambda-\nu}(\xi_{.0})^2 \\ &+ \gamma p(\nabla\cdot\xi)^2 + 2(\nabla\cdot\xi)p'\xi + \rho_0^{-1}p'\rho_0'\xi^2 - \tfrac{1}{2}\rho_0 TS'\nu'\xi^2\,. \end{aligned} \tag{D30}$$

All terms containing $\delta\nu$ have canceled out by virtue of equations (D19a), (D25)–(D27), and the equation of hydrostatic equilibrium,

$$p' = -\tfrac{1}{2}\rho_0\mu\nu'\,. \tag{D31}$$

Now we define

$$\chi \equiv r^{-2}e^{\nu/2}(r^2 e^{-\nu/2}\xi)' = \nabla\cdot\xi + \tfrac{1}{2}\delta\lambda\,. \tag{D32}$$

The last step follows from equation (D19a) and the equation

$$\nu' + \lambda' = 8\pi r\rho_0\mu e^{\lambda}\,. \tag{D33}$$

This equation and the useful identity

$$\nu' + \frac{1}{r} = \frac{1}{r}e^{\lambda}(1 + 8\pi r^2 p) \tag{D34}$$

both follow from the unperturbed Einstein equations. From the definition of χ and equations (D31) and (D34) we obtain for $\mathcal{E}$

$$\begin{aligned} e^{-\nu/2}\mathcal{E} = {}& \rho_0\mu e^{\lambda-\nu}(\xi_{\,0})^2 + \gamma p\chi^2 + 2p'\xi\chi - \frac{1}{16\pi}\left(\frac{1}{r^2} + 8\pi p\right)\delta\lambda^2 \\ &+ \frac{1}{\rho_0}p'\rho_0'\xi^2 - \tfrac{1}{2}\rho_0 TS'\nu'\xi^2\,. \end{aligned} \tag{D35}$$

If we now substitute equation (D19a) for $\delta\lambda$, add to $\mathcal{E}$ the divergence

$$-\,g^{-1/2}(g^{1/2}p'e^{\nu/2}\xi^2)'\,, \tag{D36}$$

and use the unperturbed Tolman-Oppenheimer-Volkoff equation, $p' = \rho_0 \mu e^{\lambda}\ (m + 4\pi r^3 p)/r^2$, we find that $\mathcal{E}$ simplifies to

$$e^{-\nu/2}\mathcal{E} = \rho_0 \mu e^{\lambda-\nu}(\xi_{,0})^2 + p\gamma\chi^2 - \frac{(p')^2}{\rho_0\mu}\,\xi^2 + \frac{4p'}{r}\,\xi^2 + 8\pi e^{\lambda}\rho_0\mu p\xi^2\,. \tag{D37}$$

This is exactly the function whose positive-definiteness Chandrasekhar (1964) proved was necessary and sufficient for stability. Our "energy density" $\mathcal{E}$ differs from Chandrasekhar's function by the "redshift" factor $e^{\nu/2}$, which arises from our 3 + 1 split of spacetime. Our "total energy" is the same as his: his is the integral of equation (D37) over $(-{}^4g)^{1/2}d^3x = 4\pi e^{(\nu+\lambda)/2}r^2 dr$, while ours is the integral of $\mathcal{E}$ over $g^{1/2}d^3x = 4\pi e^{\lambda/2}r^2 dr$.

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