MOTION AND RADIATION IN GENERAL RELATIVITY

Bernard F. Schutz
Department of Applied Mathematics & Astronomy,
University College, P.O. Box 78,
Cardiff CF1 1XL, U.K.

ABSTRACT

These lectures will review recent progress on the connected problems of the motion of bodies in general relativity, the gravitational radiation they generate, and the reaction effects of this radiation on the motion. After a general introduction to the nature of the problem and to some useful theorems, I will treat three separate problems in a unified manner. The first is linearized theory and the post-linear approximations, including a slow-motion expansion. The second is the Newtonian limit, in which self-gravity is weak but not negligible. And the third is the strong-field point-particle limit, in which bodies with strong internal gravity interact by their "Newtonian" gravitational field. I first calculate the lowest-order equations of motion for each case, and then after a review of some aspects of gravitational radiation theory, I calculate the radiation they generate and the reaction effects they experience. In each case, the so-called quadrupole formula is verified. The unified point of view I adopt is based on constructing sequences of space-

times from specified initial data, and calculating their evolution in various limits. In addition to providing more rigor than other methods, it leads naturally to the introduction of certain geometrical structures, the near-zone manifold NM and the far-zone manifold FM, which are the arenas in which, respectively, the dynamics of motion and the gravitational radiation dominate. In particular, FM provides a rigorous treatment of the quadrupole formula in the far zone free from worries about future null infinity $\mathcal{I}^+$. The initial-value method includes a statistical definition of radiation reaction, and I discuss the implications of this for the "arrow of time" for gravitational radiation. I review other approaches to the same problems briefly, trying to show their relation to the initial-value method, and I end by giving some astrophysical applications and suggesting some areas that require further research.

CONTENTS

1. INTRODUCTION

The principal aim of these lectures is to derive from the field equations of general relativity both the quadrupole formula for gravitational radiation and an extra-strong version of the strong equivalence principle, one that applies to the motion of bodies with arbitrary self-gravitation. To do this we shall develop a systematic method of studying the twin problems of motion and radiation in general relativity.

These are two of the theory's oldest problems, and until the 1960s were two of the most actively investigated. But many early results were equivocal, partly because of the complexity of the theory, and partly because of persistent confusion over the "reality" of gravitational radiation, a confusion which was not resolved until the work of Bondi[1], Sachs[2], and Penrose[3]. By that time, new results from astrophysics focussed the attention of relativists on such questions as black holes, neutron stars, collapse, cosmology, tests of gravitational theories, the generation of gravitational radiation and the possibility of detecting gravitational waves. The general problem of motion received little attention[4]; by contrast, the questions of the radiation generated by systems and of the associated reaction effects saw even more attention than before. Landau and Lifshitz[5] had already derived that "far-field quadrupole formula" for the gravitational wave luminosity L_{GW} of nearly-Newtonian systems,

$$L_{GW} = \frac{1}{5} \dddot{\mathcal{I}}_{jk} \dddot{\mathcal{I}}^{jk} \tag{1}$$

in which

$$I_{jk} = \int \rho x_j x_k \, d^3x \tag{2}$$

$$\mathcal{I}_{jk} = I_{jk} - \frac{1}{3} \delta_{jk} I^{\ell}{}_{\ell}, \tag{3}$$

dots denote derivatives, and $c = G = 1$. (Particle physicists, more accustomed to setting $c = \hbar = 1$, please note: in these units mass has the dimension of length.) Following on from this, Mathews and Peters[6] calculated the secular evolution of binary star systems on the assumption that the radiated energy and angular momentum is compensated by changes in the corresponding Newtonian orbital quantities. The first correct direct derivation of the radiation-reaction effects seems to have been by Peres[7]. But the most important work of the 1960s on radiation reaction was that of Chandrasekhar and colleagues[8] and of Burke[9], both of whom derived the following simple form for the quadrupole radiation reaction force,

$$F^i = -\frac{2}{5} \rho x_j \overset{(5)}{\mathcal{I}}{}^{ij}, \tag{4}$$

where the (5) denotes five time derivatives. This may be called the "near-zone quadrupole formula", and the rate of work done by this force,

$$\int v_i F^i \, d^3x \,,$$

when averaged over a periodic motion, gives a mean energy loss rate of

$$\frac{dE}{dt} = -\frac{1}{5}\,\dddot{I\!\!\!{-}}_{jk}\,\dddot{I\!\!\!{-}}^{jk} \quad , \tag{5}$$

exactly accounting for the energy radiated. Chandrasekhar[10] promptly used this force to show a completely unexpected result, that gravitational radiation can induce instabilities in rotating stars that are stable in Newtonian theory.

The discovery of the binary pulsar system by Taylor and colleagues[11] provided the first opportunity for a direct test of Eq.(4). It inspired Ehlers *et al*[12] to look again at the mathematical foundations of Eq.(4), and they found a number of deficiencies. The most serious was that the methods of Chandrasekhar, Burke, and others seemed always to give some divergent integrals which have to be ignored in order to arrive at Eq.(4). This was not satisfactory in a theory as regular as general relativity, and the result has been an explosion of new work on the subject. This work has considerably deepened our understanding of the problem of motion as well as of radiation, and in particular of the approximation methods we must use.

One characteristic of the recent work has been that a number of quite different techniques have been brought to bear on these questions: point-particle calculations using analytic continuation[13], methods of matched asymptotic expansions[14–19], and slow-motion iteration schemes using retarded potentials[20–22] or characteristic initial data[23] or Cauchy initial data[24–28]. Each of these methods has verified the validity of the quadrupole formula under its own particular assumptions, so that we can safely assert that the "quadrupole controversy" has died down. In these lectures I will develop the last-mentioned of the above techniques, calculating motion and radiation from initial data set on a spacelike hypersurface. Not only is this the method I know best; it also offers the opportunity to be more rigorous than any of the others except the characteristic-initial-value problem, to which it is closely related. While concentrating on the Cauchy method, I will not ignore the others. Rather, at appropriate points, I will try to show how the various approaches are related.

I will begin with the problem of motion. First I will try to define it, which is not entirely elementary. Then I will prove a few general theorems about perturbation theory, for later use. In the major part of the first section, to illustrate the versatility of our techniques, I will address three separate kinds of motion: motion in the post-linear approximation, in the Newtonian limit, and of strong-field point particles in nearly-Newtonian fields. In each case we will consider slow motion and stop before the order at which the effects of the emission of gravitational radiation (radiation reaction, radiation damping) manifest themselves.

Then I will go on to review some aspects of the theory of gravitational radiation which are important for our study. Following that I will review various definitions in the literature of an "isolated body", by which is meant a body which is free to emit gravitational radiation without being "driven" by an outside disturbance. These definitions amount to setting asymptotic or initial conditions on the radiation field in order to ensure that the system evolves by some approximation of retarded (rather than advanced) potentials. In particular I will develop the notion of a statistically isolated system, which is particularly appropriate for the Cauchy problem.

In the final section of these lectures I will calculate the emitted radiation and the radiation-reaction effects in isolated systems of the three types whose motion we studied earlier: the post-linear, post-Newtonian, and point-particle limits. The point-particle limit leads in particular to the strongest statement of the strong equivalence principle yet established: that self-gravitational energy radiates gravitational waves in exactly the same manner as any other form of energy executing the same motion. In each type of system we will find that the near-zone and far-zone quadrupole formulas give, respectively, the leading-order reaction effects and radiated energy. I will briefly describe some applications of these results, particularly to the binary pulsar system. Finally I will suggest other limits for which the present techniques might prove useful.

The work I describe here has been done jointly with T. Futamase, who also helped me to plan these lectures.

2. MOTION IN GENERAL RELATIVITY

2.1 Newtonian Motion: How to Calculate the Motion of the Planets

Calculating the motion of bodies is such a commonplace in Newtonian physics that it is easy to overlook the fact that it almost always involves approximation. A look at planetary motion will help guide our expectations when we consider similar problems in general relativity.

Each planet is subject to internal forces (pressure, gravitational forces) and external forces (from other planets and the sun). Adding these up gives the rate of change of its total momentum:

$$\Sigma \underset{\sim}{F}_{\text{internal}} + \Sigma\, \underset{\sim}{F}_{\text{external}} = \frac{d}{dt}\, \underset{\sim}{P}_{\text{total}} \quad .$$

By the equality of action and reaction, the internal forces cancel:

$$\Sigma \underset{\sim}{F}_{\text{internal}} = 0 \ .$$

We can define the center of mass $\underset{\sim}{R}_{CM}$ by the vector integration

$$M_{\text{total}}\, \underset{\sim}{R}_{CM} = \int \rho(\underset{\sim}{y})\underset{\sim}{y}\; d^3y \quad ,$$

with

$$M_{\text{total}} = \int \rho(\underset{\sim}{y})\; d^3y$$

and ρ the mass-density of the body. By conservation of mass, our original force equation reads

$$\Sigma \underset{\sim}{F}_{\text{external}} = M_{\text{total}}\, \frac{d}{dt}\, \underset{\sim}{R}_{CM} \quad . \qquad (6)$$

This is generally called the equation of motion of the body. How good is it in the problem of planetary motion?

As a first approximation, compute the motion of planet A, taking the motion of all the other planets as given and compute $\underset{\sim}{R}^{(A)}_{CM}(t)$ from Eq.(6). The problem with this is that the motion of A affects the motions of the other planets, so it is not possible to take them as given in advance. A better approximation is to write down Eq.(6) for all the planets and the Sun and solve them simultaneously, calculating the forces on the assumption that each body is a point mass. This is in fact good enough for most planetary problems, but it is still an approximation because the planets are extended bodies and their forces on each other depend upon their shapes. A still better approximation might be to treat each planet as a rigid body and to solve simultaneously for the six dynamical degrees of freedom of each planet using Eq.(6) and the analogous torque equation. If we want to study planet-moon systems we must not stop there, for we have to allow for changes of shape and internal dissipation in response to time-dependent torques. And if it is hard to model these with a few parameters (as in the Earth-Moon system), one might have to abandon all approximations and go to the "exact" equations: three-dimensional continuum-mechanical dynamical equations for the internal structure and motion of each planet, solved simultaneously for all the bodies. Naturally, this is only a last resort, not only because of its complexity, but also because the physics of planetary interiors is very uncertain, and we would like a theory of planetary motion which is relatively independent of these uncertainties. The center-of-mass approximation achieves this for us: it is useful precisely because it reduces the number of dynamical degrees of freedom from infinity down to three for each planet.

2.2 Complications Introduced by General Relativity

If we try to repeat the steps leading to Eq.(6) in general relativity, we encounter obstacles:

(i) Nonlinearity. There is no clear separation of $\underset{\sim}{F}_{internal}$ from $\underset{\sim}{F}_{external}$, since gravitational fields do not add linearly.

(ii) Curved geometry. The given definition of $\underset{\sim}{R}_{CM}$ has no invariant meaning in general relativity. Any covariant generalization will be rather complicated[4].

(iii) Rigid bodies, point masses problematical. No body will be rigid against time-dependent forces, although if the forces are weak it may be approximately so. And if we shrink one our planets to a spherical "point" of fixed mass M_{total}, we get a black hole, not a point mass.

(iv) Definition of a "body" obscure. In Newtonian theory, the boundary between the interior and exterior of a body is clear, and in particular the body's inertia resides inside. In general relativity, the gravitational field of the body will have some energy and hence inertia, however hard this may be to define covariantly. The distinction between inside and outside is therefore somewhat fuzzy in dynamical problems.

(v) Dynamical freedom of the gravitational field. The problem of the motion of bodies requires not only initial data for the bodies but also initial data or other conditions to define the gravitational wave freedom in the field. Moreover, retardation of the field also means (here as in electromagnetism) that action and reaction do not necessarily cancel internally: the radiation reaction effects come precisely from this non-cancellation of the body's self-force (which, as we noted above, is imprecisely defined as well).

These difficulties mean that the problem of motion will involve even more approximations in general relativity than in Newtonian gravity. In my view, the problem has only been satisfactorily solved in the three cases which will be the subject of these lectures:

(i) The post-linear approximation, in which fields are weak but velocities may be arbitrary.

(ii) The post-Newtonian approximation, which is a specialization to slow motion of the post-linear approximation, where the field gets weak as the square of the velocity.

(iii) The strong-field point particle limit, in which 'particles' with arbitrarily strong internal fields interact with one another weakly, through their long-range Newtonian fields.

There have been attempts to extend certain concepts, like the center of mass, to more general situations, but I am not optimistic that useful definitions can be found except in some form of weak-field limit. (My criterion for usefulness is, as mentioned before, that one can reduce the problem of motion to one of a finite number of degrees of freedom.) I will mention some likely extensions at the end of these lectures, as well as suggesting ways in which our understanding of the above three problems can be improved.

2.3 Asymptotic Approximations to Sequences of Spacetimes

Our aim is to develop approximations to general relativity which are asymptotically valid as some parameter, say ε, goes to zero. We will see that these approximations are often not analytic[29], in which case there is no question of summing an infinite series even if one could find the general term. Rather, we develop an approximation consisting of a finite number of terms, which gets more accurate as $\varepsilon \to 0$. It is helpful to remind ourselves what this means in the case of a function $f(\varepsilon)$. If $f(\varepsilon) \in C^{n+1}$ (i.e. if f is differentiable n+1 times) then Taylor's theorem

$$f(\varepsilon) = f(o) + \varepsilon f'(o) + \ldots + \frac{\varepsilon^n}{n!} f^{(n)}(o) + R_{n+1}, \qquad (7)$$

with

$$R_{n+1} = \frac{1}{n!} \int_0^1 (1-\ell)^n \left(\frac{\partial}{\partial \ell}\right)^{n+1} f(x\ell)\, d\ell = o(x^n), \qquad (8)$$

is asymptotic to the function $f(\varepsilon)$. This is an approximation to the <u>function</u> f for small ε, not just to one value of the function, which is what one might use an infinite analytic series for. So when we approximate Einstein's equations asymptotically, we should define a <u>sequence</u> of solutions of Einstein's equations[18], parametrized by ε. That is, in some co-ordinate system $\{x^\alpha\}$ we have a sequence[30]

$$\{g_{\mu\nu}(x^\alpha;\varepsilon), T^{\mu\nu}(x^\alpha;\varepsilon),\ 0 \le \varepsilon \le 1\}$$

satisfying

$$G^{\mu\nu}[g_{\alpha\beta}(x^\gamma;\varepsilon)] = 8\pi\, T^{\mu\nu}(x^\gamma;\varepsilon) \tag{9}$$

for each ε, and we develop the approximations

$$g_{\mu\nu}(x^\alpha;\varepsilon) = g_{\mu\nu}(x^\alpha;\, o) + \varepsilon\partial_\varepsilon g_{\mu\nu}(x^\alpha;o) + \ldots\,, \tag{10a}$$

$$T^{\mu\nu}(x^\alpha;\varepsilon) = T^{\mu\nu}(x^\alpha;o) + \varepsilon\partial_\varepsilon T^{\mu\nu}(x^\alpha;o) + \ldots\,. \tag{10b}$$

From these observations, a number of things become evident:

(i) We must have some way of defining each member $\{g_{\mu\nu}(x^\alpha;\varepsilon),\, T^{\mu\nu}(x^\alpha;\varepsilon)\}$ of the sequence and of ensuring that the members exhibit the sort of limit we wish to have as $\varepsilon \to 0$. We shall do this by choosing <u>initial data</u> that have the appropriate limiting behavior. A method which only assumes that a sequence exists with the desired properties cannot <u>prove</u> that the approximation scheme is genuinely asymptotic. Worse, a method which requires that the field equations, Eq.(9), are satisfied only for <u>one</u> member of the sequence, say $\varepsilon = 1$, will certainly not develop an asymptotic approximation through Eqs.(10); indeed, if it were careless it could obtain seriously misleading results for small ε.

(ii) The derivatives with respect to ε in Eq.(10) hold the co-ordinates fixed, and therefore depend upon the co-ordinates adopted. To see the impact of this, it is useful to view the sequence as a five-dimensional manifold $R^4 \times R^1$, indeed as a (trivial) fiber bundle whose base space is R^1 (the parameter ε) and whose fiber is $M(\varepsilon)$, the space-time manifold for each ε, which we shall assume is diffeomorphic to R^4. (We shall actually only use some open region of each $M(\varepsilon)$ and assume it is diffeomorphic to some open set in R^4.) This is illustrated in Fig.(1). There is no natural map from one fiber to another, which means there is no natural relation between the co-ordinates of one manifold and those of another. An ε-dependent co-ordinate trans - formation will change the approximations in Eq.(10), and we will exploit this fact to our advantage when we develop separate approximations to the near zone and the far zone of a single sequence. Put in geomet-

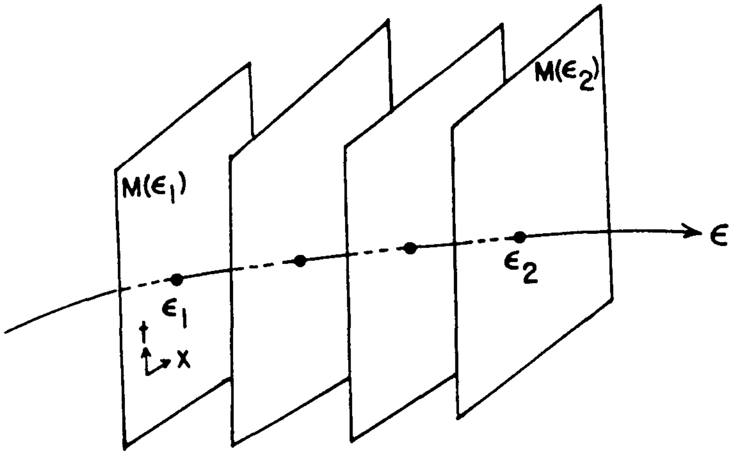


Figure (1). The sequence of spacetimes as a fiber bundle over the base space (ε). Only two of the four dimensions of each manifold are drawn. We imagine that the fiber $\varepsilon = 1$ is a fully nonlinear solution of Einstein's equations, while that for $\varepsilon = 0$ is a simple limit (usually Minkowski spacetime).

rical language, we will define for each approximation a 1-1 sequence of maps from $\varepsilon = 1$ to any other ε, which defines a congruence of curves in the fiber bundle, parametrized by ε. The derivatives with respect to ε in Eq.(10) are really Lie derivatives with respect to the tangent vector to this congruence[31)].

(iii) Since Eq.(10) dependsupon x^α as well as ε, the question of the uniformity of the $\varepsilon \to 0$ limit arises, that is, is the error $R_{n+1}(x^\alpha;\varepsilon)$ bounded for fixed ε and all x^α? Physically, the answer is usually obviously no. Consider the Newtonian limit. The lowest order approximation is Newton's theory itself, in which two compact spherical stars would orbit each other in a perfectly periodic orbit for all time. The same orbit in general relativity will precess (the shift of the periastron) and decay (due to the radiation of energy), both of which effects will render the original Newtonian orbit a very poor approximation after a finite time. Moreover, the radiation emitted will radically change the Newtonian field far enough away. Therefore we do not expect Eq.(10) to be uniform in either time or space. For this reason we only develop approximations valid in open regions of each $M(\varepsilon)$, and also for this reason we develop different approximations for different regions of $M(\varepsilon)$.

(iv) We have said before that our sequences will often turn out to be nonanalytic in ε, so that the simple method of developing asymptotic approximations given by Eq.(7) fails after a certain number of terms. There are other ways of extending this expansion, since it appears that the nonanalytic terms are no worse than logarithms of ε[32,33]. These nonanalyticities arise only after the order at which radiative effects manifest themselves, so we will not be concerned with them in these lectures except to note that they do not destroy the asymptotic nature of the lower-order terms.

2.4 General Theorems on Gauge Invariance and Conservation Laws in Perturbation Theory

Another term to describe our asymptotic approximations is perturbation theory. The system for $\varepsilon = 0$ is the unperturbed solution and the coefficients of ε, ε^2, etc., in the asymptotic expansion are called the first-, second-, etc., order perturbations. This language is most often used when the limit $\varepsilon \to 0$ is a regular one, i.e. when the limiting manifold $\varepsilon = 0$ fits the definition of a boundary of the five-dimensional space $M(\varepsilon)$, $0 \leq \varepsilon \leq 1$ given by Geroch[34]. In other cases we are in the regime of singular perturbation theory, and most of the limits we shall treat here are in that category. For example, the Newtonian limit is singular because the field equations go from (quasi-) hyperbolic to elliptic in the limit. Nonsingular perturbations are much easier to treat[35].

Regardless of whether the limit is singular or not, there are a number of useful results which we may derive for general sequences. We shall see special cases of these as we go along. Consider, then a sequence of <u>linear</u> operators $L(\varepsilon)$ and fields $\phi(\varepsilon)$ which solve the sequence of equations

$$L(\varepsilon)[\phi(\varepsilon)] = 0 \tag{11}$$

where we have suppressed the dependence on x^α. An asymptotic expansion of $\phi(\varepsilon)$ may be written

$$\phi(\varepsilon) = {}_0\phi + \varepsilon\,{}_1\phi + \varepsilon^2\,{}_2\phi + \ldots \tag{12}$$

where we have introduced the notation of a prefixed subscript,

$${}_n\phi = (n!)^{-1}(\partial_\varepsilon)^n\phi|_{\varepsilon=0}, \tag{13}$$

that we will use from now on. (We have assumed for simplicity that only powers of ε appear in the approximation Eq.(12) up to the order considered.) Similarly, the explicit ε-dependence of $L(\varepsilon)$ leads to a sequence of operators,

$$L(\varepsilon) = {}_0L + \varepsilon\,{}_1L + \varepsilon^2\,{}_2L + \ldots \quad . \tag{14}$$

Inserting Eqs.(12) and (14) into (11) and collecting terms gives

$${}_0L[{}_0\phi] + \varepsilon\{{}_0L[{}_1\phi] + {}_1L[{}_0\phi]\} + \ldots = 0 \;.$$

For this to vanish for all ε sufficiently near zero, each order in ε must vanish separately. This gives the hierarchy of equations

$$\begin{aligned} {}_0L[{}_0\phi] &= 0 \\ {}_0L[{}_1\phi] &= -{}_1L[{}_0\phi], \\ {}_0L[{}_2\phi] &= -{}_1L[{}_1\phi] - {}_2L[{}_0\phi], \ \ldots \quad . \end{aligned} \tag{15}$$

These may be solved in succession; at each step the lower-order solutions generate inhomogeneous terms in the equation for the next correction to ϕ. Notice that at each level one solves the same differential equations, ${}_0L[\phi] = f$; whatever initial data or boundary conditions may be required to determine a solution uniquely must be given at each order. It may of course happen that at some order the inhomogeneous terms are so badly behaved that no solution is possible. This may be a signal that one should introduce nonanalytic functions of ε into the approximations given by Eqs. (12) and (14).

Now let us suppose that there is some sort of invariance group G associated with the physical problem leading to Eq.(11). If the solutions ϕ are elements of a function space S, then G has a representation that acts in S, associating with any element $g \in G$ the operator g_S acting in S, so that ϕ and $g_S\phi$ are physically equivalent. We define a <u>gauge transformation</u> to be a sequence $g(\varepsilon) \in G$ for which $g(o) = e$, the identity element of G. (We shall assume that $g(\varepsilon)$ is as smooth in ε as we require. The case where $g(o) \neq e$ may be viewed as the gauge transformation $g(o)^{-1} g(\varepsilon)$ followed by the constant symmetry transformation $g(o)$.) The gauge transformation produces a new but equivalent sequence

$$\begin{aligned} g_S(\varepsilon)\phi(\varepsilon) &= {}_0\phi + \varepsilon({}_1\phi + {}_1g_S\, {}_0\phi) \\ &+ \varepsilon^2({}_2\phi + {}_1g_S\, {}_1\phi + {}_1\phi + {}_2g_S\, {}_0\phi) + \dots \end{aligned} \tag{16}$$

From this we see that if, for example, ${}_0\phi = 0$, then ${}_1\phi$ is <u>gauge-invariant</u>; similarly, if ${}_0\phi$, ${}_1\phi$, ..., ${}_{n-1}\phi$ all vanish, then ${}_n\phi$ will be gauge-invariant. This result was first pointed out in relativity by Sachs[36)]. It applies not just to solutions ϕ of the fundamental equations, but also to any functional of them, such as the energy or its time-derivative.

The generalization to a nonlinear system,

$$F[\varepsilon;\ \phi(\varepsilon)] = 0,$$

is straightforward. The gauge results are unchanged, while the approximation hierarchy can be written in terms of the linear first-variation of the operator F

$${}_1F[f] \equiv \frac{\delta F}{\delta \phi}[0;\ {}_0\phi][f] = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} F[0; {}_0\phi + \varepsilon f].$$

We have then

$$F[0;{}_0\phi] = 0,$$

$${}_1F[{}_1\phi] = H_1\ ({}_0\phi),$$

$${}_1F[{}_2\phi] = H_2\ ({}_0\phi,\ {}_1\phi),\ \ldots\ , \tag{17}$$

where the H_ns are nonlinear functionals. So we still find ourselves solving linear equations for the hierarchy.

Our most important result in this section is on broken conservation laws. Suppose that the hierarchy of approximations through order n admits a conservation law,

$$Q(\varepsilon;{}_0\phi + \varepsilon\ {}_1\phi + \ldots + \varepsilon^n\ {}_n\phi) = \text{const}, \tag{18}$$

but that at order n+1 no extension of Q can be found: the conservation law is broken. (Of course, Q may be any constant of the motion, not necessarily a conservation law like energy or momentum. For example, a Newtonian binary system has a fixed periastron, which moves in post-Newtonian gravity. Therefore post-Newtonian effects break the periastron "conservation law".) How do we compute dQ/dt to its lowest non-vanishing order? We proceed by distinguishing between Q's explicit dependence on ε and its implicit dependence through the function $\phi(\varepsilon)$. Let ${}_0Q$ represent the lowest-order conservation law,

$${}_0Q(\psi) \equiv Q(0;\psi), \tag{19}$$

and ΔQ represent the rest

$$Q(\varepsilon;\psi) = {}_0Q(\psi) + \Delta Q(\varepsilon;\psi)\ . \tag{20}$$

Then it is clear that for any ψ

$$\Delta Q(\varepsilon;\psi) = O(\varepsilon\,{}_0Q(\psi)). \tag{21}$$

From the conservation law, Eq.(18), we learn

$$\frac{d}{dt}Q(\varepsilon;\ {}_0\phi + \ldots + \varepsilon^n\ {}_n\phi) = \frac{d}{dt}\ {}_0Q({}_0\phi + \ldots)$$

$$+ \frac{d}{dt}\Delta Q(\varepsilon;\ {}_0\phi + \ldots),$$

$$= \frac{\delta_0 Q}{\delta\psi}\ [{}_0\dot\phi + \ldots + \varepsilon^n\ {}_n\dot\phi] + \frac{\delta\Delta Q}{\delta\psi}\ [{}_0\dot\phi + \ldots + \varepsilon^n\ {}_n\dot\phi],$$

$$= 0, \tag{22}$$

where $\delta_0 Q/\delta\psi$ and $\delta\Delta Q/\delta\psi$ are functional derivatives defined in the same way $\delta F/\delta\phi$ was. Now suppose that $\phi(\varepsilon)$ is a solution of the full hierarchy of approximations. Then we have

$$\frac{d}{dt}Q(\varepsilon;\phi(\varepsilon)) = \frac{\delta Q}{\delta\psi}[\dot\phi(\varepsilon)] + \frac{\delta\Delta Q}{\delta\psi}[\dot\phi(\varepsilon)]$$

$$= \frac{\delta Q}{\delta\psi}\ [\varepsilon^{n+1}\ {}_{n+1}\dot\phi + O(\varepsilon^{n+2})] + \frac{\delta\Delta Q}{\delta\psi}\ [O(\varepsilon^{n+1})], \tag{23}$$

the lower-order terms having cancelled by virtue of Eq. (22). Equation (21) tells us that the final term in Eq.(23) is $O(\varepsilon^{n+2})$, so we have

$$\frac{dQ}{dt} = \varepsilon^{n+1}\ \frac{\delta_0 Q}{\delta\psi}\ [{}_{n+1}\dot\phi] + O(\varepsilon^{n+2}). \tag{24}$$

This means that, no matter how high the order n+1 of conservation breaking is, we only need the lowest-order conservation law ${}_0Q$ to calculate it. The rule given by Eq.(24) is that one computes the change of ${}_0Q$ using only the terms ${}_{n+1}\dot\phi$ in the "equation of motion" at the conservation-breaking order n+1, not any lower-order terms in $\dot\phi$. Even though lower-order terms are larger, they must cancel out to leave Eq.(24). This result has been independently "discovered" in the context of the energy conservation law in the Newtonian limit by Thorne[20)], Breuer & Rudolph[37)], Lapiedra *et al.*,[38)], and Futamase[26)].

The formulation here is my own[39].

Combining this result with our earlier one on gauge-invariance, we see that, if Q has a gauge group, then since dQ/dt vanishes up to order n, its lowest non-vanishing part will be gauge invariant. For example, the perihelion of a planet is constant at Newtonian order, so its rate of change due to post-Newtonian effects is invariant under post-Newtonian gauge (co-ordinate) transformations. The actual position of the perihelion is not invariant, since it does not vanish at Newtonian order.

2.5 Defining Sequences of Spacetimes by Initial Data

The most straightforward way to define a solution of Einstein's equations is to give initial data for it on some Cauchy hypersurface, to which we will assign the co-ordinate time $t = 0$. This will be our method, but it has the disadvantage that it is not immediately clear what data we should adopt for the free,dynamical part of the gravitational field. It will turn out that the effects of gravitational radiation occur at high order in the problems we shall treat in detail, so for the purposes of this section (§2) we shall simply set the wave data to zero, in a way made precise below[40]. The initial-value approach takes advantage of the fact that the Cauchy problem in relativity is well-studied and reasonably well understood[41,42].

An alternative would be to try to define the wave degrees of freedom asymptotically in each spacetime, by putting a zero-incoming-radiation condition on I^-. (The symbols I^+ and I^- refer, respectively, to future and past null infinity, which we will discuss in §3.3) This boundary condition has been favored, at least in principle, by a number of relativists, notably Ehlers[43]. This approach is conceptually appealing from the point of view of causality, but it has a number of practical disadvantages. One is that we understand very little of how to"tie" such a condition to sources, i.e. how to make an asymptotic approximation covering all of spacetime and including such a wave condition. An essential first step has been made by Friedrichs[44], who showed that the conventional definition of $I^{\pm}$ is general enough to include a large class of radiative spacetimes, but much more is necessary. Another problem is that we do not know what sort of data for the matter is compatible with asymptotic data for the radiation. Eder[45], studying a simplified version of electromagnetism, has provided a partial answer, but an unsatisfying one, because it involves the sources' behavior in the infinite past rather than at a particular time. We have already pointed out that we do not expect our approximations to be uniform in the remote past. These problems make it difficult to define spacetimes, let alone to calculate their metrics, with asymptotic wave conditions. We will return to this point in §4.

Returning to our initial-value formulation, we write Einstein's field equations in the Fock-Anderson-DeCanio[21,46] form. We define

$$\bar{h}^{\mu\nu} = \eta^{\mu\nu} - (-g)^{\frac{1}{2}}\, g^{\mu\nu} \tag{25}$$

and adopt the de Donder (harmonic, Lorentz, Lanczos) gauge,

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0. \tag{26}$$

Then we may write Einstein's equations as

$$\Box \bar{h}^{\mu\nu} = -\,16\pi\, \Lambda^{\mu\nu}, \tag{27}$$

$$\Lambda^{\mu\nu} = \Theta^{\mu\nu} - (16\pi)^{-1}\left(\bar{h}^{\mu\nu}\bar{h}^{\alpha\beta} - \bar{h}^{\alpha\mu}\bar{h}^{\beta\nu}\right)_{,\alpha\beta} \tag{28}$$

$$\Theta^{\mu\nu} = (-g)\left(T^{\mu\nu} + t^{\mu\nu}_{LL}\right) \tag{29}$$

where $t^{\mu\nu}_{LL}$ is the Landau-Lifshitz stress-energy pseudotensor[5], expressed in our gauge as

$$\begin{aligned} t^{\mu\nu}_{LL} = (-16\pi g)^{-1}\{ & g_{\lambda\sigma} g^{\alpha\beta} \bar{h}^{\mu\lambda}{}_{,\alpha} \bar{h}^{\nu\sigma}{}_{,\beta} + \tfrac{1}{2} g^{\mu\nu} g_{\lambda\sigma} \bar{h}^{\lambda\alpha}{}_{,\beta} \bar{h}^{\beta\sigma}{}_{,\alpha} \\ & - 2 g_{\alpha\beta} g^{\lambda(\mu} \bar{h}^{\nu)\alpha}{}_{,\sigma} \bar{h}^{\beta\sigma}{}_{,\lambda} + \tfrac{1}{2} T^{\mu\alpha\nu\beta} T_{\lambda\sigma\rho\tau} \bar{h}^{\lambda\rho}{}_{,\alpha} \bar{h}^{\sigma\tau}{}_{,\beta}\}, \end{aligned} \tag{30}$$

with

$$T^{\mu\alpha\nu\beta} = g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta} ,$$

and where $\Box$ represents the flat-space wave operator (d'Alembertian) in our co-ordinates. The initial data for $\bar{h}^{\mu\nu}$ are not all free, because the gauge condition imposes the constraints

$$\bar{h}^{\mu 0}{}_{,0} = - \bar{h}^{\mu i}{}_{,i} \tag{31}$$

and the time-derivative of this condition can be used to eliminate second time derivatives in Eq.(27) for $\nu = 0$, giving the remaining constraints

$$\nabla^2 \bar{h}^{\mu 0} = - 16\pi\, \Lambda^{\mu 0} - \bar{h}^{\mu i}{}_{,i0}. \tag{32}$$

We shall therefore take $\bar{h}^{ij}$ and $\bar{h}^{ij}{}_{,0}$ as our free wave initial data. (This freedom contains a considerable amount of gauge freedom, of course.) The data for the matter, ρ, p, and U^i, are also free.

We shall use the field equation, Eq.(27), in its formally solved form, using the Kirchhoff formula[47):

$$\begin{aligned} \bar{h}^{\mu\nu}(t,x^i) &= 4 \int_{C(t,x^j)} \Lambda^{\mu\nu}(t-r,y^i)\, r^{-1} d^3y \\ &+ \frac{t}{4\pi} \oint_{S(t,x^j)} \bar{h}^{\mu\nu}{}_{,0}(t=0,y^j)\, d\Omega_y \\ &+ \frac{1}{4\pi}\frac{\partial}{\partial t}\left\{ t \oint_{S(t,x^j)} \bar{h}^{\mu\nu}(t=0,y^j)\, d\Omega_y \right\} , \end{aligned} \tag{33}$$

with

$$r \equiv |\underset{\sim}{x} - \underset{\sim}{y}| ,$$

where the truncated retarded flat-space light cone $C(t,x^j)$ of the event

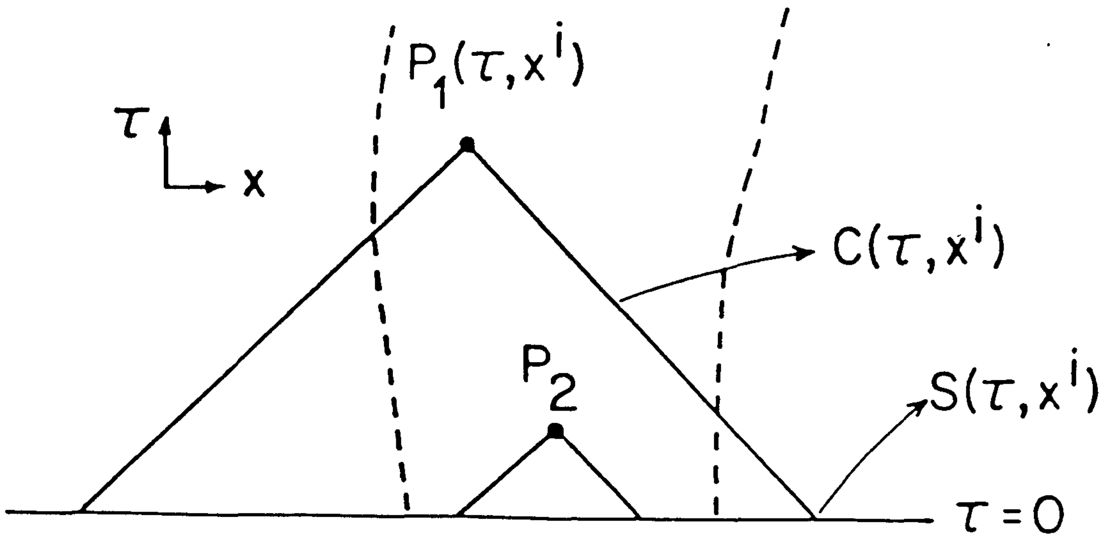


Figure 2. The initial-data problem for the operator $\Box$ is solved by (i) an inhomogeneous solution integrated over the flat space light cone of the event (t, x^j), truncated at $t=0$, and (ii) a solution of the homogeneous equation determined by initial data integrals [last two terms in Eq.(33)] over the intersection S of C with $t=0$.

(t, x^j) and its sphere of intersection $S(t, x^j)$ with the hypersurface $t=0$ are illustrated in Fig.2. Equation (33) gives an implicit solution, since the unknown $\bar{h}^{\mu\nu}$ appears on both sides. But it is an ideal form for iterating in a weak-field approximation, since the "source" terms on the right-hand-side are quadratic in $\bar{h}^{\mu\nu}$. It is important to understand, also, that Eq.(33) involves no approximation even though it involves an integration over flat-space light cones, which are not the physical characteristics of the system. The solution of Eq.(27) is unique for the given data, and Eq.(33) represents this solution. Any "non-causal" behavior from the light-cone integral must be cancelled by the solution of the homogeneous equation (integrals over S). It would be an approximation (frequently adopted, in fact[21,22]) to use a non-truncated flat-space cone for the retarded integral and not to put in the homogeneous solution. But our expression is exact.

2.6 Linearized Theory and the Post-Linear (pl) Expansion

We are now in a position to construct a particular sequence for a particular problem. We will begin with the simplest one, the weak-field (post-linear) approximation. We want a sequence in which $\bar{h}^{\mu\nu}$ approaches zero but the matter system is otherwise arbitrary. We define an asymptotically linear sequence with parameter λ by the sequence of data

$$\left.\begin{aligned} \rho(t=0,x^i;\lambda) &= \lambda a(x^i) \\ p(t=0,x^i;\lambda) &= \lambda b(x^i) \\ v^j(t=0,x^i;\lambda) &= c^j(x^i) \end{aligned}\right\} \quad (34)$$

$$\bar{h}^{ij}(t=0,x^k,\lambda) = \bar{h}^{ij}{}_{,0}(t=0,x^k,\lambda) = 0,$$

where

$$v^j = U^j/U^0. \quad (35)$$

Note that at $t = 0$ the "speed of sound" $(p/\rho)^{\frac{1}{2}}$ is independent of λ, as is the velocity v^i. The decrease in density ρ ensures that the field gets weak. The dynamics of this system therefore will be dominated by material stresses as $\lambda \to 0$. The lowest order of this approximation is derived in all serious textbooks, but it will be instructive to test our more rigorous approach on it first.

The constraint equations (31) and (32) for the case where $\bar{h}^{ij}$ and $\bar{h}^{ij}{}_{,0}$ vanish at $t = 0$ imply

$$\left.\begin{aligned}
&\bar{h}^{i0}{}_{,0}(t=0,x^j) = 0, \\
&\bar{h}^{0\mu}(t=0,x^j;\lambda) = 4\int \Lambda^{0\mu}(t=0,x^j;\lambda)r^{-1}\,d^3y, \\
&\bar{h}^{00}{}_{,0}(t=0,x^j;\lambda) = -\bar{h}^{0i}{}_{,i}(t=0,x^j;\lambda).
\end{aligned}\right\} \quad (36)$$

Because $\Lambda^{0\mu}$ is $O(\lambda)$ at $t=0$, we have that $\bar{h}^{0\mu}$ will also be $O(\lambda)$, so the contributions of $\bar{h}^{0\mu}$ to $\Lambda^{0\mu}$ will be $O(\lambda^2)$. To lowest order, therefore, we can differentiate Eq.(36) with respect to λ at $\lambda = 0$ to obtain

$$\left.\begin{aligned}
{}_1\bar{h}^{0\mu}(t=0,x^j) &= 4\int {}_1T^{0\mu}(t=0,x^j)r^{-1}d^3y, \\
{}_1\bar{h}^{00}(t=0,x^j) &= 4\int \{[a(y^j)+b(y^j)]/[1-c_i(y^j)c_i(y^j)] - b(y^j)\}r^{-1}d^3y, \\
{}_1\bar{h}^{0i}(t=0,x^j) &= 4\int [a(y^j)+b(y^j)]c^i(y^j)/[1-c_k(y^j)c_k(y^j)]r^{-1}d^3y.
\end{aligned}\right\} \quad (37)$$

Both integrands are of compact support, which is significant in what follows. We have here our first concrete example of how to use Eqs.(31) - (33) iteratively: ${}_1\bar{h}^{\mu\nu}$ is determined by ${}_1T^{\mu\nu}$ and in turn determines ${}_2t^{\mu\nu}_{LL}$, which with ${}_2T^{\mu\nu}$ determines ${}_2\bar{h}^{\mu\nu}$, and so on.

We may now compute the evolution of the homogeneous solution, ${}^{H}\bar{h}^{\mu\nu}$, the last two terms of Eq.(33). For the case where the initial data are of the form ∇^{-2} (function of compact support), the evolution is particularly simple[24]. We divide spacetime into three regions as in Fig. (3). In region I, we find

$$\left.\begin{aligned}
{}^{H}_{1}\bar{h}^{00}(t,x^j\epsilon I) &= 4\int {}_1T^{00}(t=0,y^j)r^{-1}d^3y - 4t\int {}_1T^{0i}{}_{,i}(t=0,y^j)r^{-1}d^3y, \\
{}^{H}_{1}\bar{h}^{0i}(t,x^j\epsilon I) &= 4\int {}_1T^{0i}(t=0,y^j)r^{-1}d^3y.
\end{aligned}\right\} \quad (38)$$

Thus, in region I, where the inhomogeneous field vanishes to lowest order because the cone $C(t,x^j)$ is truncated before it intersects the support of ${}_1T^{\mu\nu}$, the homogeneous solution carries the Newtonian

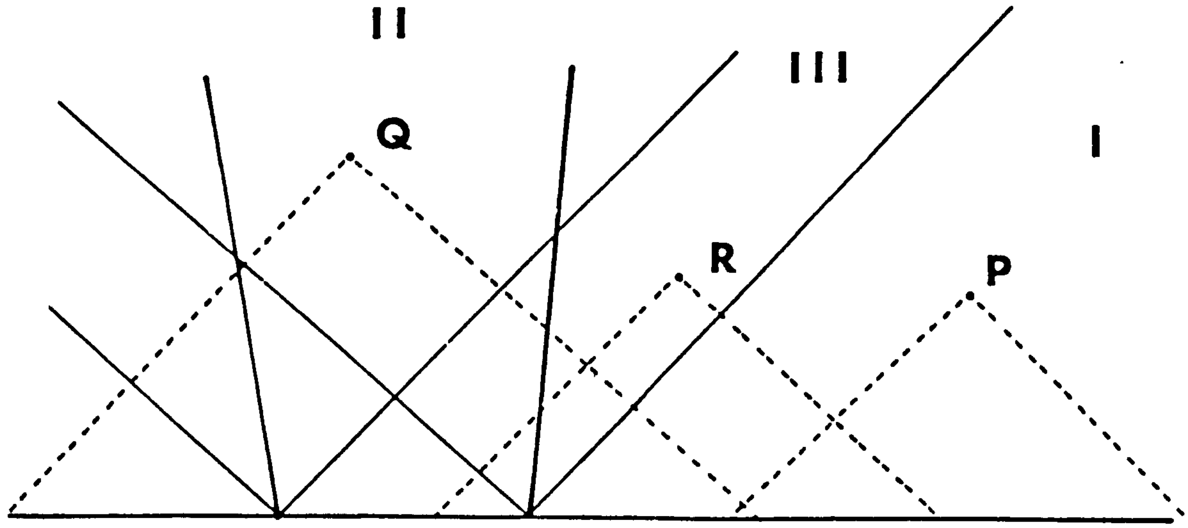


Figure (3). Spacetime is divided into three regions by the null cones of the points of the boundary of the support of the matter density in the initial hypersurface. In regions I and II the lowest-order homogeneous solution is particularly simple.

("Coulomb") part of the field. Note that ${}_1\overset{H\leftarrow}{h}{}^{00}$ contains a term which adjusts the field with time according to the motion (momentum) of the source. In region II, by contrast, the homogeneous field ${}_1\overset{H\leftarrow}{h}(t,x^j)$ vanishes identically. In this region the inhomogeneous field is found by integrating over a cone which intersects the whole of the support of ${}_1T^{\mu\nu}$, so it picks up all the Newtonian part. The field ${}_1\overset{H\leftarrow}{h}$ is superfluous, and it obligingly vanishes. Region III is a transition region, but it is small and we will not need to worry about it.

The inhomogeneous field ${}_1\overset{H\leftarrow}{h}{}^{\mu\nu}$ vanishes in I, as noted above, and has the usual retarded form in II:

$$ {}_1\bar{h}^{\mu\nu}(t,x^j\varepsilon \mathrm{II}) = \int_{G(t,x^j)} {}_1T^{\mu\nu}(t-r,y^j)r^{-1}d^3y, \tag{39}$$

where we have omitted the preceding superscript designation I for inhomogeneous because this is also the <u>total</u> field in region II.

The gauge condition implies

$$ {}_1T^{\mu\nu}{}_{,\nu} = 0, \tag{40}$$

which is the lowest-order equation of motion. It says that to lowest order the system behaves as it would in special relativity. This is consistent with our choice of initial data, which we arranged to be dominated by material stresses. The gauge condition at the next order implies

$$ {}_2\Theta^{\mu\nu}{}_{,\nu} = 0 $$

or

$$ {}_2T^{\mu\nu}{}_{,\nu} = -[(-{}_1g)\,{}_1T^{\mu\nu}]_{,\nu} - {}_2t^{\mu\nu}_{LL,\nu} \quad . $$

This equation is equivalent to the usual covariant conservation law at this order

$$ {}_2T^{\mu\nu}{}_{,\nu} = -{}_1\Gamma^{\mu}{}_{\alpha\nu}\,{}_1T^{\alpha\nu} - {}_1\Gamma^{\nu}{}_{\alpha\nu}\,{}_1T^{\mu\alpha}. \tag{41}$$

This is the first order at which self-gravitational effects appear. These are called the first post-linear equations of motion. We could calculate the next order in the metric, ${}_2\bar{h}^{\mu\nu}$, but this would not affect the equations of motion until second post-linear order.

One sometimes hearsit said that because the lowest-order equations of motion do not involve $\bar{h}^{\mu\nu}$, linearized theory (the zero-order equations of motion and the linearized field equation) is inconsistent: the energy that goes into the waves is not lost from the system's motion. This is a misunderstanding of the nature of the approximation. The energy in the waves, however it is measured, is quadratic in $\bar{h}^{\mu\nu}$ and therefore of order λ^2, so it is perfectly consistent to neglect it in the first-order equations of motion. It is accounted for in the correct way in Eq.(41).

Linearized theory presents us with a well-known example of our theorem on gauge invariance in perturbation theory. Since the zero-th order is flat spacetime, we have ${}_0R_{\alpha\beta\mu\nu} = 0$. Therefore ${}_1R_{\alpha\beta\mu\nu}$ will be gauge-invariant[36).

2.7 The Slow-Motion Post-Linear Expansion

Many problems involve not only weak fields but also slow motion, so an important modification of the pl approximation is obtained by introducing an independent slowness parameter ε into the initial data:

$$\left.\begin{aligned} \rho(t=0,x^i;\lambda,\varepsilon) &= \lambda a(x^i), \\ p(t=0,x^i;\lambda,\varepsilon) &= \lambda\varepsilon^2 b(x^i), \\ v^j(t=0,x^i;\lambda,\varepsilon) &= \varepsilon c^j(x^i). \end{aligned}\right\} \qquad (42)$$

The scaling of p ensures that bulk stresses (p) scale the same way as kinetic ones ($\rho v^i v^j$), so that the dynamics will preserve the smallness of the initial velocities. Equations(42) define a two-parameter family of solutions, which may be approximated to any desired order in both parameters. However, the slow-motion expansion in ε must be handled with care. Imagine we are dealing with a simple system like a mechanical spring. As $\varepsilon \to 0$, velocities slow down and the spring oscillates with a longer period, proportional in the limit to ε^{-1}. We are not interested in the limit $\varepsilon \to 0$ at fixed time t, since less and less happens then. Rather, we want a limit at fixed dynamical state, *e.g.* a limit to $\varepsilon \to 0$ for the system after one cycle of oscillation. To accomplish this we define the dynamical time τ

$$\tau = \varepsilon t \qquad (43)$$

and take the slow motion limit holding τ fixed. This has the effect that the time derivative

$$\frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial \tau} \tag{44}$$

scales with ε, which is usually assumed as part of a slow-motion expansion. As a matter of notation we use a tilde above a letter to denote the new function of τ generated by replacing t by τ/ε:

$$T^{\mu\nu}(\tau/\varepsilon, x^i; \lambda, \varepsilon) \equiv \tilde{T}^{\mu\nu}(\tau, x^i; \lambda, \varepsilon) \ . \tag{45}$$

Geometrically, the limit at fixed τ is illustrated in Fig.(4). In effect, we are adopting co-ordinates (τ, x^i) to fix the map from one manifold to another. We must therefore also convert indices t to τ. For instance, we have

$$\eta^{\tau\tau} = - \varepsilon^2. \tag{46}$$

Let us see what the lowest-order terms in ${}_1\bar{h}^{\mu\nu}$ look like. For notation we write

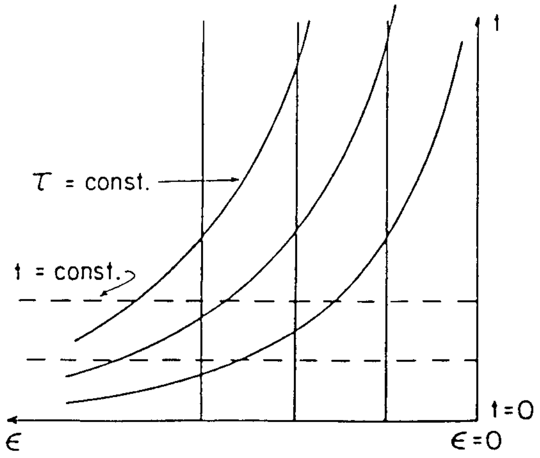
$$_{(n,m)}f(\tau, x^i) = \frac{1}{n!}\frac{1}{m!} \lim_{\varepsilon \to 0} \lim_{\lambda \to 0} (\partial_\lambda)^n (\partial_\varepsilon)^m f(\tau, x^i; \lambda, \varepsilon). \tag{47}$$

Then Eq.(39) must be written

$${}_1\bar{h}^{\mu\nu} = 4 \int_{C(\tau/\varepsilon, x^i)} {}_1\tilde{T}^{\mu\nu}(\tau - \varepsilon r, y^j; \varepsilon) r^{-1} \, d^3y \ , \tag{48}$$

where here the indices μ and ν refer to τ and x^i. Of course, the cone C depends on ε, getting bigger as $\varepsilon \to 0$ since the time t at its apex increases, but this does not affect Eq.(48) for sufficiently small ε because ${}_1\tilde{T}^{\mu\nu}$ is of compact support. To lowest order we have

$$_{(1,2)}\bar{h}^{\tau\tau}(\tau, x^i) = 4 \int {}_{(1,0)}\rho(\tau, y^j) r^{-1} \, d^3y = - 4\phi \tag{49}$$



Figure(4). Taking the limit $\varepsilon \to 0$ at fixed τ means following the hyperbolae εt = const. The limit to $\varepsilon = 0$ is <u>not</u> the fiber defined by the initial data for $\varepsilon = 0$, which is Minkowski space-time. The nature of the slow-motion limiting manifold depends upon how λ behaves in the limit.

where ϕ is the Newtonian potential. Similarly we find

$$ {}_{(1,2)}\bar{h}^{\tau i}(\tau, x^i) = 4 \int {}_{(1,0)}\rho(\tau, y^j) \; {}_{(0,1)}v^i(\tau, y^j) \; d^3y \tag{50} $$

$$ {}_{(1,2)}\bar{h}^{ij}(\tau, x^k) = 4 \int [\, {}_{(1,0)}\rho(\tau, y^k) \; {}_{(0,1)}v^i(\tau, y^k) \; {}_{(0,1)}v^j(\tau, y^k) + \delta^{ij} \, {}_{(1,2)}p(\tau, y^k)\,] r^{-1} \, d^3y \tag{51} $$

Recalling Eq.(46), we find that the lowest-order (in slowness) force in the 2pl equation of motion is

$$ {}_{(2,0)}T^{i\nu}{}_{,\nu} = - {}_{(1,0)}\rho\phi^{,i} \, , \tag{52} $$

which is Newton's gravitational force. We shall go beyond this order in ε when we consider radiation reaction later.

2.8 The Newtonian Limit

All serious introductory textbooks on general relativity derive Newton's equations from Einstein's by specializing to weak fields and slow motions in such a way that the typical gravitational potential M/R is of the same order as the typical squared velocity v^2. This is evidently a special case of the slow-motion pl expansion, in which we set

$$\lambda = \varepsilon^2 . \tag{53}$$

Why does this produce a Newtonian set of equations? The fundamental answer is that Newton's equations themselves have the following scale-invariance property[48]. If the fields $\rho(x^i,t)$, $p(x^i,t)$, $v^j(x^i,t)$, and $\phi(x^i,t)$ satisfy the equations

$$\left.\begin{aligned} \nabla^2\phi &= 4\pi\rho \\ \rho_{,t} + \nabla_i(\rho v^i) &= 0 \\ \rho v^i{}_{,t} + \rho v^j\nabla_y v^i + \nabla^i p + \rho\nabla^i\phi &= 0 \end{aligned}\right\} \tag{54}$$

then so do the fields $\varepsilon^2\rho(x^i,\varepsilon t)$, $\varepsilon^4 p(x^i,\varepsilon t)$, $\varepsilon v^j(x^i,\varepsilon t)$, and $\varepsilon^2\phi(x^i,\varepsilon t)$ for <u>any</u> ε.

This scaling is exactly what we would have for our initial data, Eq.(42), and our time co-ordinate τ, Eq.(43), if we set $\lambda = \varepsilon^2$. So if we used Newton's equations (54) to evolve the initial data, the ε-dependence of the data would be preserved exactly. Using Einstein's equations instead produces only higher-order corrections in ε. We call the sequence of solutions definedby the data

$$\left.\begin{aligned} \rho(t=0,x^i;\varepsilon) &= \varepsilon^2 a(x^i) \\ p(t=0,x^i;\varepsilon) &= \varepsilon^4 b(x^i) \\ v^j(t=0,x^i;\varepsilon) &= \varepsilon c^j(x^i) \end{aligned}\right\} \qquad (55)$$

a regular, asymptotically Newtonian sequence.

We can obtain the asymptotic approximations to this sequence from those of the slow-motion pl sequence by the following simple rule. If $g(\tau,x^i;\lambda,\varepsilon)$ is a function in the slow- motion pl sequence, define $f(\tau,x^i;\varepsilon) = g(\tau,x^i;\varepsilon^2,\varepsilon)$. Then we have

$$\frac{1}{q!}\lim_{\varepsilon\to 0}(\partial_\varepsilon)^q f(\tau,x^i;\varepsilon) \equiv {}_q f(\tau,x^i) = \sum_{p=0}^{[q/2]} {}_{(p,q-2p)}g(\tau,x^i), \qquad (56)$$

where $[q/2]$ is the largest integer contained in $q/2$. This means that any order q in the post-Newtonian expansion involves both slow-motion terms and terms from the nonlinearity of Einstein's equations.

The lowest order of the expansion is[25)]

$${}_4T^{\tau\tau} = \rho,\ {}_4T^{\tau i} = {}_2\rho\,{}_1v^i,\ {}_4T^{ij} = {}_2\rho\,{}_1v^i\,{}_1v^j + {}_4p\delta^{ij}, \qquad (57)$$

$${}_4\bar{h}^{\tau\tau} = 4\phi,\ {}_0\Gamma^i{}_{\tau\tau} = -\nabla^i\phi,\ {}_4T^{i\nu}{}_{,\nu} = -\,{}_2\rho\nabla^i\phi \qquad (58)$$

where

$$\nabla^2\phi = 4\pi\,{}_2\rho \qquad (59)$$

and ∇^i is the three-dimensional Euclidean covariant derivative. These are obviously the Newtonian equations of motion.

Notice that in the limit $\varepsilon \to 0$ we have (recall Eq.(46))

$${}_0g^{\tau\tau} = 0,\quad {}_0g^{ij} = \delta^{ij},\quad {}_0g^{\tau i} = 0,\quad {}_0\Gamma^i{}_{\tau\tau} = -\nabla^i\phi\ . \qquad (60)$$

This is precisely the geometry of the Cartan formulation of Newtonian

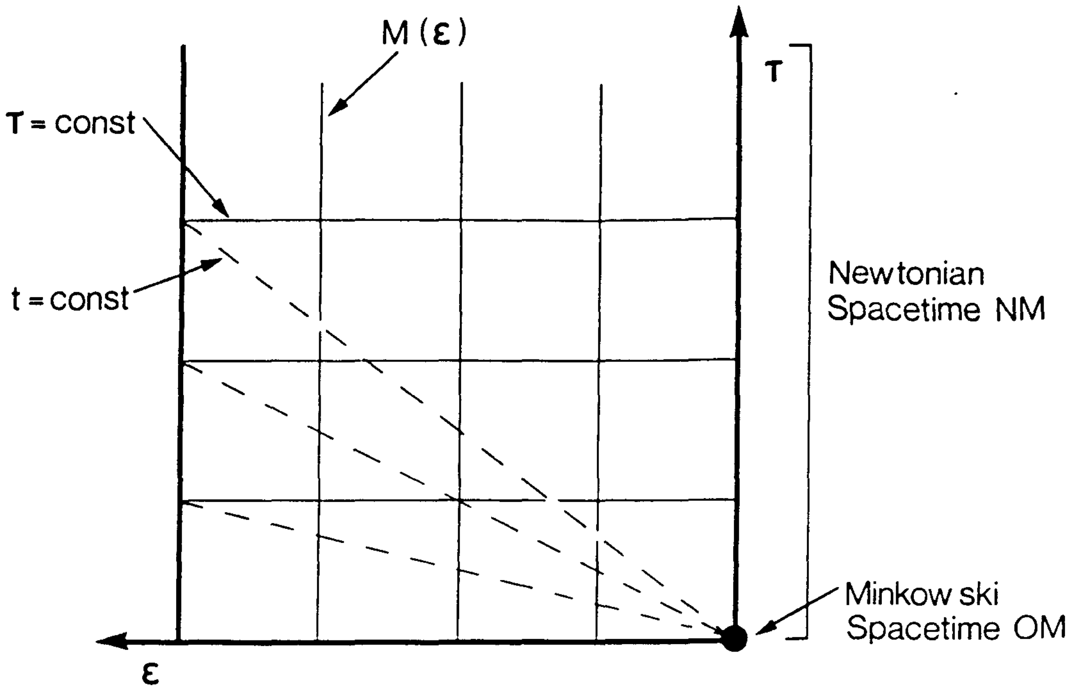


Figure (5). As Fig.(4), but with the vertical co-ordinate rescaled to τ. NM is the limit of the horizontal lines, while the $\varepsilon = 0$ spacetimes is the limit OM of the diagonal lines converging on the corner.

mechanics[49]. Since this limit at constant τ takes us up the hyperbolae of Fig.(4), the limiting manifold is not the fiber $\varepsilon = 0$. We call it the near-zone limiting manifold, NM. It is better displayed in Fig.(5), which is Fig.(4) with the time co-ordinate rescaled to τ.

We may continue to differentiate the implicit solution Eq.(33) for $\bar{h}^{\mu\nu}$ and the equations of motion $\theta^{\mu\nu}{}_{,\nu} = 0$ with respect to ε. The resulting equations of motion are ordered as follows:

$O(\varepsilon^4)$: Newtonian, requiring only ${}_4\bar{h}^{\tau\tau}$.
$O(\varepsilon^6)$: post-Newtonian (pN), requiring ${}_4\bar{h}^{\mu\nu}$, ${}_6\bar{h}^{\tau\tau}$.
$O(\varepsilon^8)$: post-post-Newtonian (2pN), requiring ${}_4\bar{h}^{\mu\nu}$, ${}_6\bar{h}^{\mu\nu}$, ${}_8\bar{h}^{\tau\tau}$.
$O(\varepsilon^9)$: $2\frac{1}{2}$-post-Newtonian (2.5pN), radiation reaction order.

Because ${}_4\bar{h}^{ij}$ does not enter the Newtonian equations of motion, it is usually called a post-Newtonian term[26] in spite of its being of order ε^4. The two lowest odd-order equations of motion have zero solutions, *e.g.* ${}_3\rho = 0$, ${}_2v^i = 0$, ${}_5p^i = 0$ all follow from ${}_5\theta^{\mu\nu}{}_{,\nu} = 0$. This is not surprising, since changing the sign of ε is equivalent to solving the original sequence for $t < 0$. Only time-odd effects ought to appear at odd orders, and these are the radiation-reaction terms. We will discuss them below.

2.9 The Strong-Field Point-Particle Limit

Although the techniques of the last section enable one to approximate nearly-Newtonian systems within general relativity, they do not extend to all situations in which we expect Newtonian motion. One important system of this type is the binary pulsar[11]. This seems to consist of two neutron stars orbiting one another at a sufficient distance that their orbits are well described by Newtonian theory with post-Newtonian corrections[50,90] But internally the stars may be poorly-described by Newtonian theory. Is there a way of showing, therefore, that bodies with strong internal gravity nevertheless interact in a Newtonian manner if they are sufficiently far apart?

Naively, one might think of doing this by shrinking the size of a body of fixed mass M, increasing its internal gravity while not disturbing its external field at large distances. Unfortunately, before the body shrinks to zero size, it forms a black hole, whose size is then fixed by its mass. For this reason it is sometimes said that there is no true point-particle limit in general relativity. This conclusion, however, would be incorrect.

The key is to realize that one wants to keep the "strength" of the internal field fixed as the radius shrinks. Since a measure of this strength is M/R, one ought to allow M to go to zero in proportion to R. This can be fitted into our previous approximation schemes very conveniently, because we already have the masses of our bodies going to zero as ε^2 (or λ), since their densities behave this way. If we modify these schemes so that the densities do not go to zero, but rather the radii also scale as ε^2 or λ (so that the densities go as ε^{-4} or λ^{-2})

then the internal gravity of the bodies will remain large but their masses will behave as before, so their interaction will be as weak as before. This strong-field point-particle limit was to my knowledge first devised by Kates[16)] and later incorporated by Futamase[28)] into the present framework, both authors being interested only in the Newtonian limit of the motion of two interacting bodies. The more general problem of incorporating point particles into the pl expansion (i.e. without assuming small velocities) has not yet been studied in this manner[13)]. In our treatment we follow Futamase. At the end of this section we will discuss the relation of this work to other work.

We begin by defining <u>body-zone</u> <u>co-ordinates</u> for each body (bodies are labelled I and II),

$$x^{\underline{i}}_{I,II} = \epsilon^{-2}(x^i - \xi^i_{I,II}) \, , \tag{61}$$

where x^i is the usual co-ordinate and (ξ^i_I, ξ^i_{II}) are the co-ordinates of the origin of the two body-zone co-ordinate systems, which generally may change with time. The scaling by ϵ^{-2} means that as the body shrinks with respect to x^i it remains of fixed size in $x^{\underline{i}}$. Body-zone indices are always denoted by an underscore. We define the <u>body zones</u> to be the regions

$$|x^i - \xi^i_{I,II}| < \epsilon R \tag{62}$$

for some fixed R. Since this boundary shrinks as ϵ while the body shrinks as ϵ^2, this boundary actually expands in body-zone co-ordinates. This leads to a clean separation of the body from the exterior geometry in the limit $\epsilon \to 0$. We refer to x^i from now on as the <u>near-zone</u> <u>co-ordinates</u>.

We shall develop separate asymptotic expansions in the different zones, keeping $(x^{\underline{i}}_{I,II}, \tau)$ fixed in body zone I and II and keeping (x^i, τ) fixed in the near zone, i.e. all x^i except the body zones. Therefore we use τ as the time co-ordinate everywhere. Is this permissible in the body zones? After all, if M/R remains fixed, then the typical internal velocity $v \sim (M/R)^{1/2}$ will be fixed as well, and so the dynamical timescale R/v will scale as ε^2 relative to proper time t in the near zone. By contrast, fixed τ scales as ε^{-1} times proper time. So in general it would be necessary to define a body-zone time $\alpha^0 = \varepsilon^{-2}t = \varepsilon^{-3}\tau$ and develop asymptotic approximations for fixed α^0. This would be appropriate if we were trying to study the internal dynamics of each body, say the radiation emitted by its pulsations. But we are only interested in the orbital dynamics here, so we shall choose initial data for <u>stationary</u> bodies in the body-zone co-ordinates, i.e. solutions that would be independent of time if they were isolated. Then any internal motions are due to tidal effects, which we will consider below.

Therefore, suppose $\{T^{\mu\nu}_{S_{I,II}}(y^i), g_{S_{I,II}}(y^i)\}$ are two stationary solutions (denoted by the subscripts I or II) of Einstein's equations for a perfect fluid, given in terms of some coordinate system $\{y^i\}$. We shall identify y^i with α^i, so that as $\varepsilon \to 0$ the solutions will automatically shrink relative to near-zone coordinates. We cannot ensure that the body-zone solutions are strictly stationary, but we can come as close to that as possible by using these stationary solutions as initial data.

To convert these to near-zone coordinates we need

$$\Lambda^{\mu}_{\underline{\alpha}} = \partial x^{\mu}/\partial x^{\underline{\alpha}} .$$

From Eq. (61) we find

$$\Lambda^{\tau}_{\underline{\tau}} = 1, \quad \Lambda^{\tau}_{\underline{i}} = 0, \quad \Lambda^{i}_{\underline{\tau}} = d\xi^i_{I,II}/d\tau \equiv v^i_{I,II}, \quad \Lambda^{i}_{\underline{j}} = \varepsilon^2\delta^i_j . \tag{63}$$

The velocities of the origins, $v^i_{I,II}$, defined here, have another interpretation. As $\varepsilon \to 0$, the bodies shrink (in the near-zone picture) to point particles whose positions are given by $\xi^i_{I,II}(\tau)$, so $v^i_{I,II}(\tau)$ are point-particle velocities. The initial data include $\{\xi^i_{I,II}, v^i_{I,II}\}(0)$ and the following free data

$$T^{\mu\nu}(t=0,x^i;\varepsilon) = \begin{cases} \Lambda^{\mu}_{\underline{\alpha}}\,\Lambda^{\nu}_{\underline{\beta}}\,T^{\underline{\alpha\beta}}_{S_{I,II}}\,(\alpha^{\underline{k}}_{I,II}) & \text{in each body zone} \\ 0 & \text{elsewhere} \end{cases} \tag{64}$$

$$\bar{h}^{ij}(t=0,x^i;\varepsilon) = \begin{cases} \Lambda^{i}_{\underline{\alpha}}\,\Lambda^{j}_{\underline{\beta}}\,\bar{h}^{\underline{\alpha\beta}}_{S_{I,II}}\,(\alpha^{\underline{k}}_{I,II}) & \text{body zones} \\ 0 & \text{elsewhere} \end{cases} \tag{65}$$

$$\bar{h}^{ij}_{\ ,\tau}(t=0,x^i;\varepsilon) = \begin{cases} -\varepsilon^{-2}\Lambda^{i}_{\alpha}\,\Lambda^{j}_{\beta}\,v^{\ell}_{I,II}\,\dfrac{\partial}{\partial\alpha^{\underline{\ell}}_{I,II}}\,[\bar{h}^{\underline{\alpha\beta}}_{S_{I,II}}\,(\alpha^{\underline{k}}_{I,II})] & \text{body zones} \\ 0 & \text{elsewhere} \end{cases} \tag{66}$$

where $\Lambda^{\underline{\mu}}_{\alpha}$ and $v^{\ell}_{I,II}$ are evaluated at $t = 0$ in their appropriate body zones, and where the components $T^{\underline{\alpha\beta}}_{S_{I,II}}$ are given in terms of the initial stationary solutions $T^{\mu\nu}_{S_{I,II}}$ byidentifying y^i with α^i and scaling with the following factors of ε:

$$T^{\underline{\tau\tau}}_{S_{I,II}} = \varepsilon^{-2}\,T^{00}_{S_{I,II}}\,,\quad T^{\underline{\tau i}}_{S_{I,II}} = \varepsilon^{-5}T^{0i}_{S_{I,II}}\,,\quad T^{\underline{ij}}_{S_{I,II}} = \varepsilon^{-8}T^{ij}_{S_{I,II}} \tag{67}$$

and similarly for $\bar{h}^{\underline{\mu\nu}}$. The scaling of $T^{\underline{\tau\tau}}$ ensures that the physical density in the body zone scales as ε^{-4}, since $T^{\underline{\tau\tau}}$ is obtained from the physical T^{tt} by multiplying by two factors of $\partial\tau/\partial t = \varepsilon$. the other scalings in Eq. (67) are most easily found from the realization that τ is not the 'natural' time coordinate in the body zone, as we discussed above, but its rather a factor of ε^3 slower than the body-zone dynamical time. Thus, whenever an index changes from τ to a spatial one in Eq. (67) we pick up a factor ε^{-3}.

Notice that our initial data include a non-zero $\bar{h}^{\underline{ij}}$ in the body zone, by contrast with our earlier Newtonian calculations. This is to ensure that the initial body-zone solution is as near to stationarity as possible. Because the body zone shrinks to a point, this will not substantially affect the subsequent $\bar{h}^{ij}$ field in the near zone.

The remaining initial data are determined by the constraints. Of course, if there were only one body zone, the body zone solution would remain strictly stationary, moving with uniform velocity $v^{\ell}(0)$ in the near-zone coordinates. The following argument shows the extent to which we may expect non-stationary behaviour in the case of two bodies.

As $\varepsilon \to 0$, the metric in each body zone remains a highly nonlinear solution, approaching the stationary solution assumed in the initial data. As $\varepsilon \to 0$, the mass of the other body scales as ε^2, and since the body shrinks as ε^2, the tidal force on one body due to the other scales as ε^6 relative to the internal forces. This is then the order at which deviations from stationarity occur in the body-zone limit. We will not need them in our calculation of the motion of the bodies at lowest order.

The motion is determined from the near-zone limit, i.e. the limit at fixed (τ, x^i). The metric in the near zone is given as an integral over the light cone, as before, but in this case the integral is dominated by the contribution of the body zones, as shown in Fig.(6).

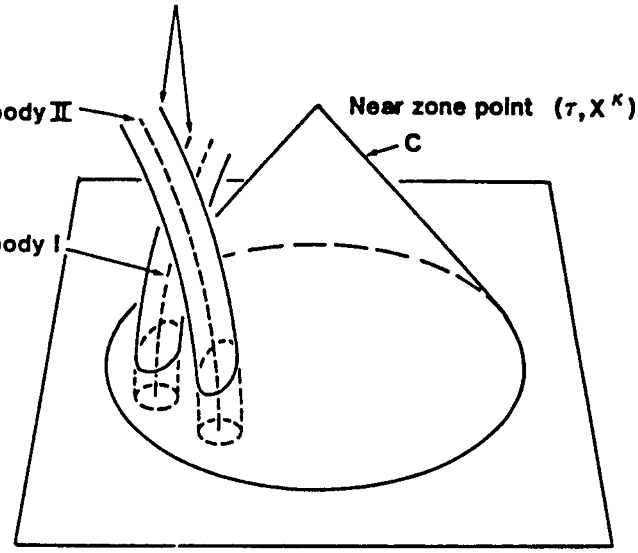


Figure (6). The integral over the truncated past light cone of a near-zone point (τ, x^i) always includes body-zone contributions for small enough ε, and these dominate the integrand at lowest order.

The lowest order contribution in the near zone is

$$ {}_4\bar{h}^{\tau\tau}(\tau, x^i) = 4 \left(\frac{{}_2M_I}{r_I} + \frac{{}_2M_{II}}{r_{II}} \right) \tag{68} $$

where

$$ r_{I,II}(\tau) = \left| x^i - \xi^i_{I,II}(\tau) \right| \tag{69} $$

is the distance of the point x^i from body I or II, and where the 'mass' constants ${}_2M_{I,II}$ are defined by

$$ {}_2M_I = \lim_{\varepsilon \to 0} \varepsilon^6 \int_{|\alpha^i_I| < R/\varepsilon} \underline{\Theta}^{\tau\tau}(\tau, \alpha^j_I; \varepsilon)\, d^3\alpha_I $$

and similarly for ${}_2M_{II}$. Here we use $\underline{\Theta}^{\mu\nu}$ to denote the function derived from $\Theta^{\mu\nu}(t,x^j;\varepsilon)$ by

$$\underline{\Theta}^{\mu\nu}(\tau,\alpha^j_I;\varepsilon) = \Theta^{\mu\nu}(t/\varepsilon,\xi^j_I(\tau) + \varepsilon^2\alpha^j_I;\varepsilon) \ . \tag{71}$$

I put a prefix 2 on ${}_2M_I$ because the mass is physically of order ε^2: if Θ^{tt} is integrated over d^3x^i the result is of order ε^2. But the constant ${}_2M_I$ has another meaning: from Eq.(67) it is clearly the integral of $\Theta^{00}_{S_I}$: the ADM mass of the initial stationary solution adopted for zone I. We therefore see that the near-zero field ${}_4\bar{h}^{\tau\tau}$ is identical to that which would be produced by two spherical Newtonian bodies with masses equal to the total gravitational (ADM) mass of the point particles.

One can calculate ${}_4\bar{h}^{\tau i}$ and ${}_4\bar{h}^{ij}$ in the near zone as well. The first is simple[28):

$${}_4\bar{h}^{\tau i}(\tau,x^j) = 4\left(\frac{{}_2M_I v^i_I}{r_I} + \frac{{}_2M_{II}v^i_{II}}{r_{II}}\right) \ . \tag{72}$$

The integral for the second has contributions of the same order from the integrals over the body zones and the near zones. Doing the body-zone integrals gives[28)

$$\begin{aligned}{}_4\bar{h}^{ij}(\tau,x^k) &= 4\left(\frac{{}_2M_I v^i_I v^j_I}{r_I} + \frac{{}_2M_{II}v^i_{II}v^j_{II}}{r_{II}}\right) \\ &\quad + \int_N |x^k-y^k|^{-1}\ {}_4t^{ij}_{LL}(\tau,y^k)d^3y \ ,\end{aligned} \tag{73}$$

where N is the near zone and

$${}_4t^{ij}_{LL} = (64\pi)^{-1}(\nabla^i\ {}_4\bar{h}^{\tau\tau}\ \nabla^j\ {}_4\bar{h}^{\tau\tau} - \tfrac{1}{2}\delta^{ij}\nabla^k\ {}_4\bar{h}^{\tau\tau}\ \nabla_k\ {}_4\bar{h}^{\tau\tau}) . \tag{74}$$

Equations (68) and (72) automatically satisfy the gauge condition ${}_4\bar{h}^{\tau\alpha}{}_{,\alpha} = 0$, since ${}_2M_{I,II}$ are constants. The Newtonian motion of the bodies follows from the gauge condition ${}_4\bar{h}^{i\alpha}{}_{,\alpha} = 0$. The divergence of the integral in Eq.(73) is easily calculated from the identity

$$\frac{\partial}{\partial x^j} \int |x^k - y^k|^{-1} f^j(y^k) d^3y =$$

$$\int |x^k - y^k|^{-1} \frac{\partial}{\partial y^j} f^j(y^k) d^3y - \oint |x^k - y^k|^{-1} f^j(y^k) n_j d^2S_y$$

where the second integral is over the boundary of the integration region. Since the divergence ${}_4t^{ij}_{LL,j}$ vanishes in N, the divergence of the integral in Eq.(73) reduced to surface integrals over the boundaries of the two body zones. (The surface integral at infinity vanishes.) The result is that the gauge condition implies[28)]

$$\frac{dv^i_I}{d\tau} = - \frac{{}_2M_{II}}{|\xi^k_I - \xi^k_{II}|^3} (\xi^i_I - \xi^i_{II}), \qquad (75)$$

and the companion equation with the indices I and II reversed. This is the equation of motion for Newtonian gravity. It confirms the strong equivalence principle as applied to bodies with strong internal gravity: when they interact with external bodies the single integral mass ${}_2M$ defined by Eq.(70) serves as the active gravitational mass and the inertial mass. Gravitational potential energy contributes to ${}_2M$ and therefore creates as much gravitational field as any other energy. We will extend this to the emission of radiation in a later section.

These results were first obtained by Kates[16)] by the technique of matched asymptotic expansions. Futamase's treatment[28)] shows that the results are genuinely asymptotic to well-defined sequences of solutions. Many of the ideas involved, however, go back at least to Einstein, Infeld, and Hoffmann[51)], who computed the motion of a 'point' mass by assuming that its exterior field was characterized by a few simple parameters (such as the mass). They obtained the equation of motion from the vacuum Einstein equations outside the body, just as we do here. A much more elegant modern point-mass calculation has been carried out by Damour and collaborators[13)], in which the singularity of the point mass is regularized by a method of analytic continuation[52)], leaving an external field depending on parameters. Again the equations of motion follow from Einstein's equations. The Damour approach is computationally efficient, but it does not attempt to prove that the results are asymptotic to any sequence of regular solutions (i.e. without point masses). Indeed, it is not at all clear how to prove that the analytic continuation regularisation method turns a point mass into a sensible continuum, except by comparing its results with those derived for continuous bodies *ab initio*[53]. The method of Kates[16] has recently been adapted and extended beyond Newtonian order by Thorne and Hartle[89], again using asymptotic matching.

3. GRAVITATIONAL RADIATION

3.1 Free Waves in Linearized Theory

We begin our review of the theory of gravitational-wave propagation by reminding ourselves of some properties of gravitational waves in linearized theory. Most standard textbooks contain a treatment along these lines[30]. In linearized theory we have the vacuum field equation

$$\Box\, {}_1\bar{h}^{\mu\nu} = 0. \tag{76}$$

We can remain in Lorentz gauge

$${}_1\bar{h}^{\mu\nu}{}_{,\nu} = 0 \tag{77}$$

with a change of gauge generated by a vector ξ^μ satisfying

$$\Box\, \xi^\mu = 0. \tag{78}$$

For any Fourier component of ${}_1\bar{h}^{\mu\nu}$ with frequency $\omega \neq 0$ one can choose a corresponding Fourier component of ξ^μ to achieve the transverse-traceless (TT) gauge, defined by Eq. (77) and

$${}_1\bar{h}^{0\mu} = 0, \qquad {}_1\bar{h}^{\mu}{}_{\mu} = 0. \tag{79}$$

The time index in Eq. (79) refers to a particular background Lorentz frame, which may be chosen arbitrarily, but whose choice is part of the definition of the TT gauge. The exclusion of $\omega = 0$ means that one cannot put a stationary field into TT gauge. Since Eq.(79) implies that the 0-component of Eq.(77) is satisfied identically, there are eight independent restrictions on the ten components ${}_1\bar{h}^{\mu\nu}$, leaving two degrees of freedom, which carry the polarization information[30].

3.2 Fluxes of Energy and Angular Momentum

There is of course no covariant local prescription for calculating the energy flux of a wave, but there are several useful measures which are physically sensible in certain circumstances.

The Isaacson[54] stress-energy "tensor" deals with the situation in which the waves are of small amplitude and short wavelength (short by comparison with the radius of curvature of the background metric, which need not be flat). In the case where the background is flat, so we are in linearized theory, then in TT gauge the Isaacson measure is

$$T_I^{\mu\nu} = \frac{1}{32\pi} < {}_1\bar{h}_{ij}^{TT,\mu} \quad {}_1\bar{h}_{ij}^{TT,\nu} >_{BH} \tag{80}$$

where $< >_{BH}$ denotes the Brill-Hartle average, which is an average over several wavelengths and periods of the wave. This measure of energy and momentum is invariant under gauge transformations whose wavelength is comparable to that of the gravitational wave's.

There are a number of pseudotensor[55] measures of energy and momentum, the most popular of which is the Landau-Lifshitz pseudotensor, given by Eq.(30). For short wavelengths, this is consistent with the Isaacson measure:

$$< t_{LL}^{\mu\nu} >_{BH} = T_I^{\mu\nu} \tag{81}$$

Angular momentum fluxes may be defined formally from either of the two above measures by constructing some version of $M^{\alpha\beta} = x^{\alpha}T^{0\beta} - x^{\beta}T^{0\alpha}$, but the result turns out to be very sensitive to which $T^{\mu\nu}$ one uses and how (if at all) it is averaged. I know of only one published measure for short-wavelength linearized radiation, the DeWitt angular momentum flux[20)]

$$dS_j/dtd\Omega = r^2 < \varepsilon_{jkl}\, x^k\, t_{LL}^{lm}\, n_m >_{BH} \tag{82}$$

where

$$n_m = x_m/r, \qquad r = |x^j| . \tag{83}$$

In TT gauge this becomes

$$dS_j/dtd\Omega = (r^2/16\pi)\varepsilon_{jkl}\, x^k < ({}_1\bar{h}^{TTlm}\; {}_1\bar{h}^{TT}_{mn,o})^{,n} -$$

$$\frac{1}{2}\, {}_1\bar{h}^{TT,l}_{mn}\; {}_1\bar{h}^{TTmn}{}_{,0} >_{BH} . \tag{84}$$

I do not know whether this is gauge-invariant, but I suspect it is not, even to short-wavelength gauge transformations.

3.3 Radiation at I^+ (or I^-)

Historically, the question of the "reality" of gravitational radiation was finally settled by the elegant work of Bondi[1)], Sachs[2)], and Penrose[3)] on radiation at null infinity. In contrast to the local measures of energy and flux we have just seen, it is possible to define [56)] invariant measures on future (past) null infinity, I^+ (I^-). In particular, the <u>Bondi news function</u> N is a complex function giving the energy flux of radiation. The relation of N to fields in spacetime is[20,75)]

$$\frac{\partial}{\partial t}\bar{h}^{TT}_{jk} = \text{Real}\left\{ \frac{2N}{r}\, \bar{m}_j\, \bar{m}_k \right\} + O(r^{-2}) \tag{85}$$

where

$$\bar{\underset{\sim}{m}} = \frac{1}{\sqrt{2}}\left(\vec{e}_{\hat\theta} - i\vec{e}_{\hat\phi} \right) \tag{86}$$

where $\vec{e}_{\hat\theta}$ and $\vec{e}_{\hat\phi}$ are <u>unit</u> polar basis vectors. The limit to large r in Eq.(85) is, of course, toward I^+. Equation (85) allows us to convert between asymptotic measures (N) and 'local' measures far away.

In particular, the Isaacson and Landau-Lifshitz energy fluxes asymptotically approach the Bondi flux.

3.4 The Irrelevance of Energy Measures

Why do we spend so much effort calculating the energy carried by waves leaving a source, and debating whether the far-zone quadrupole formula gives this energy correctly? A number of possibilities suggest themselves.

(i) In order to calculate the response of a wave detector to the waves? The answer is no: it is simpler to use $\bar{h}^{TT}_{\mu\nu}$ far away for this, without worrying about fluxes.

(ii) In order to calculate the behavior of the source as it loses energy or angular momentum equal to that measured far away? No, again: although our calculations show that there is global energy balance in specific limits, the radiation-reaction forces are purely local (as we shall see) and do not depend on properties of the radiation far away. Knowledge of $\bar{h}^{TT}_{\mu\nu}$ far away gives us all the information we need to infer these forces.

(iii) In order to feel good? Apparently, yes: we want to know that general relativity fits into the rest of theoretical physics in having global conservation laws. From this point of view, any of the above energy measures will suffice. We will therefore spend most of the section on the radiation produced by systems in approximating $h^{TT}_{\mu\nu}$. After that, energy calculations are simple (and secondary).

4. ISOLATED SYSTEMS

4.1 Definitions Using $\mathcal{I}^{\pm}$ (Exactly or Approximately)

In studying the radiation emitted by systems, one wants to take care to exclude agencies that might force oscillations of the system. This means that any derivation of radiation from a system begins with a definition of an isolated system. We will discuss three such definitions, and then digress a bit to consider the "arrow of time" for radiation. The first of the definitions is that a system is isolated if there is no radiation incident on it from $\mathcal{I}^{-}$, i.e. if the news function vanishes there.

This definition serves as an "initial condition" for the radiative degrees of freedom of the gravitational field. As such it plays the same role as the adoption of retarded potentials does in a linear field theory: to"freeze out" the wave freedom in such a way as to allow no incoming radiation. This condition has been advocated most strongly by Ehlers[43)] and his associates. A somewhat easier asymptotic condition to apply in some situations is that near $\mathcal{I}^+$ the radiation should be purely outgoing. This is not quite equivalent to a no-incoming-radiation condition because some outgoing radiation will backscatter off the spacetime curvature and produce incoming radiation; in order to cancel this, some radiation from $\mathcal{I}^-$ is required. This is a small effect, however, and may be neglected in most cases. This condition is used in the matched-asymptotic-expansion approaches[15-18)].

Both of these criteria are sometimes implemented in an approximate form by using the retarded Green's function of flat spacetime in deDonder co-ordinates to solve Eq.(27). Mathematically, this consists of ignoring the homogeneous field in Eq.(33) and extending the light cone $C(t,x^j)$ infinitely far to the past. But this light cone does not in fact reach $\mathcal{I}^-$ when so extended, because the lines $t \pm r = \text{const}$ are not null. Such approximations, therefore, are suspect[57)].

Even if one adopts the "best" of these asymptotic wave conditions, namely that the news function on $\mathcal{I}^-$ should vanish, there are, as we have remarked before, a number of drawbacks[24)]. First, it is very hard to implement this approach in a practical calculation. Second, the real spacetime in which an astrophysical system finds itself may not have $\mathcal{I}^-$, which might be replaced by the big-bang singularity. Third, real systems are not perfectly isolated, so even if the problem could be solved for an isolated system, one would have to show that the result was stable against the random amounts of incoming radiation a real system is subject to. And fourth, the mixed-data problem - initial data at some finite time for the matter, asymptotic wave conditions on $\mathcal{I}^-$ -- may not be solveable or even well-posed. The alternative is to set the matter data in the infinite past[45)], and as we remarked before this data may be very contrived indeed. For example, carried to the infinite past, a nearly-Newtonian binary system with quadrupole radiation reaction started out as two separate stars in a marginally hyperbolic orbit, which capture each other by giving off gravitational radiation on their first encounter[58].

4.2 Thorne's Definition of an Isolated System

The second, third, and fourth difficulties posed above come about from insisting that the asymptotic definitions of isolation be applied rigorously. Thorne[20], however, makes the point that most real systems will be "almost" isolated even though they do not have a rigorous I^-: they will be separated from other matter by large but <u>not</u> <u>infinite</u> distances. Suppose there is a zone around a system of size $r \gg \lambda$ (where λ is the typical wavelength of the gravitational waves), and in which the waves are weak ($|\bar{h}^{\mu\nu}| \ll 1$) and the background curvature is negligible. Thorne calls this the <u>local</u> <u>wave</u> <u>zone</u>, and defines a source to be isolated if it has such a zone. The local wave zone has an outer boundary set by other matter: nearby stars that can deflect the radiation, etc. Such effects have nothing to do with generating waves, so they are ignored.

Thorne's definition is of course approximate. No realistic wave emission can be expected to be exactly zero for all sufficiently long wavelengths, so the local wave zone's inner boundary has to be chosen according to the longest wavelength at which there is significant radiation. But this approximate character of the definition is a practical blessing: it allows one to make other approximations in the same spirit. For example, it is clear that in the local wave zone the waves obey linearized theory, and so their wave fluxes may be calculated by the Isaacson or Landau-Lifshitz expressions, and their propagation is on flat-space light cones.

In this connection, let us compare the "true" light cones with the false flat-space cones in the local wave zone. If the source has mass M, the false cone obeys

$$u = t - r = \text{const}$$

and the true cone

$$u_* = t - r_* = \text{const}, \qquad r_* = r - 2M\ln(r/2M-1). \tag{87}$$

If we adjust the constants so that $u = u_*$ at the inner edge r_1 of the wave zone, then at the outer edge r_2 they will differ by a time interval

$$\Delta t \sim 2M\ln(r_2/r_1), \tag{88}$$

since both r_2 and r_1 are much greater than M. Taking waves of wavelength λ and $r_1 = 10\lambda$, then the time Δt represents a phase $\pi/2$ of the wave when r_2 is given by

$$\lambda/4 = 2M \ln(r_2/10\lambda), \qquad r_2 = 10\lambda \exp(\lambda/8M)$$

Taking $\lambda = 8\times10^9$ km and $M = 3M_\odot \simeq 4.5$ km (appropriate to the binary pulsar) gives $r_2 \sim 10^{(10^8)}$ km, certainly bigger than any distance we might wish to know about! On the other hand, if we consider waves from a supernova, then we should take $\lambda = 300$ km and $M = 10M_\odot = 15$ km, giving $r_2 = 3.6\times10^4$ km, very close to the system. So the wisdom of using flat-space cones depends very much on how relativistic the source of waves is.

Thorne's approach, therefore, has much to recommend itself in the weak-field limit, which is where he uses it.[20,89] Unfortunately, when carried to high enough order in the post-Newtonian radiation problem, it will share the same drawback as many calculations using $\mathcal{I}^\pm$ have encountered: divergent integrals in the approximation[12,21,22]. We will show how they may be avoided below.

4.3 Statistically Isolated Systems

Any criterion based on excluding incoming radiation is clearly an idealization, since real systems sit in a "bath" of gravitational radiation generated elsewhere in the universe. All that we really want to do is to exclude radiation that is correlated with our system's motions, radiation that can build up a cumulative effect over a period of time that will rival radiation-reaction effects. Only if the incoming radiation is matched in frequency and phase with the system's own motions can this happen. If such radiation is absent, or present in negligible amounts, then the incoming radiation is random. We define a system to be statistically isolated if the incoming radiation is random. This definition has the great advantage that it may be imposed at any finite time, not just in the remote past. One can therefore study statistically isolated systems in terms of the initial-value problem, which is much more regular than methods involving I^-, and which incidentally allows us to apply the methods of §2 to the radiation problem.

For definiteness, we will define our initial free-wave data $\bar{h}^{ij}$ and $\bar{h}^{ij}{}_{,0}$ to be randomly drawn from a suitable function space equipped with a probability measure P having the following property.

$$P(\bar{h}^{ij}, \bar{h}^{ij}{}_{,0}) = P(-\bar{h}^{ij}, -\bar{h}^{ij}{}_{,0}) \ . \tag{89}$$

This is essentially a random phase approximation. If we denote by angle brackets $\langle f \rangle$ the expectation value of a function f over the probability measure P, then Eq.(89) implies that

$$\langle \bar{h}^{ij} \rangle = \langle \bar{h}^{ij}{}_{,0} \rangle = 0 \quad \text{at } t = 0 \ . \tag{90}$$

In a linear theory, the expectation value of the field would evolve from the expectation of the initial data; in particular in linearized theory the homogeneous field ${}_1\bar{h}^{ij}_H$ would have zero expectation value at all times. This is then the solution that we have already studied. But in full general relativity this will not be true: the expected evolution of a system is the ensemble average $\langle \ \rangle$ of its evolution from random initial data, not the evolution from the expected initial data. That is, we must average after evolving, not before. The expected evolution is generally not even a solution of Einstein's equations, but it nevertheless has a clear and acceptable physical interpretation.

As a consequence of these considerations, we must also specify the amplitude of the random initial data, since the quadratic terms will not average to zero. We shall only need to do this on the Newtonian and point-particle limits, where we shall take

$$\left.\begin{aligned} \bar{h}^{ij}(t=0,x^k) &= \varepsilon^4\, d^{ij}(x^k) \\ \bar{h}^{ij}{}_{,\tau}(t=0,x^k) &= \varepsilon^4\, f^{ij}(x^k) \end{aligned}\right\} \qquad (91)$$

where the functions d^{ij} and f^{ij} obey the following asymptotic regularity conditions on the hypersurface $t = 0$[26]:

$$d^{ij} = O(r^{-1}), \quad d^{ij}{}_{,k} = O(r^{-2}), \quad f^{ij} = O(r^{-2}), \quad f^{ij}{}_{,k} = O(r^{-3}). \qquad (92)$$

These suffice to guarantee the unique existence of solutions of the constraint equations (31) and (32).

In the pN hierarchy, initial data for $\bar{h}^{ij}$ of order ε^4 is as large as the first post-Newtonian field, much larger than the radiation-reaction terms, and much larger as well than realistic estimates of the random radiation in the binary pulsar system. Nevertheless, we will see that these data do not affect the radiation reaction behavior when averaged, in the sense that the expected evolution has radiation-reaction effects equal to those we would obtain if we simply set the initial data for $\bar{h}^{ij}$ and $\bar{h}^{ij}{}_{,0}$ to zero. Thus, a nearly-Newtonian system may be bathed in very strong gravitational waves, but will still evolve by the standard radiation-reaction terms if the incoming waves have random phases.

The statistical definition of isolation has two principal advantages. First, it is "local" and therefore easy to calculate. And second, we will see that it eliminates all the divergent integrals at higher orders: the asymptotic approximation it produces is finite at any order.

4.4 The Arrow of Time for Radiation

Radiation phenomena of all kinds share a common property: radiation spreads out away from its source in the forward direction of our "psychological" sense of time[59]. For water waves on a pond, generated by dropping a rock in it, this can be explained from the point of view of molecular statistics. The time-reversed phenomenon, in which waves travel inwards to meet the rock and the pond is still after the rock hits it, would require very unlikely correlations among molecular velocities on the edge of the pond. But the coincidence between the water-wave arrow of time and the similar arrow for electromagnetism has not tempted most physicists to look for a statistical explanation for the electromagnetic one as well. Instead, the standard view seems to be that put forward by Ritz[60], that one must solve the electromagnetic field equations using retarded potentials, essentially on grounds of causality. Modern quantum field theory has continued this point of view. Interestingly, Einstein himself objected to this, at least in his early years:

> ... one can no more conclude from the fact that we have not observed [advanced potentials] that the elementary processes of electromagnetism are irreversible than one can infer the irreversiblity of the elementary atomic processes from the second law of thermodynamics[61].

Einstein objected most strongly to the removal of the dynamical freedom of electromagnetism in Ritz's view, and pointed out that conservation laws for energy and momentum are awkward and nonlocal if the field is removed in favor of particle variables at earlier times. As the above quotation would suggest, Einstein tried to deduce the electromagnetic arrow from the statistics of the molecules which are the source of the radiation, but he was unsuccessful.

Progress on these lines was made by Wheeler and Feynman[62], who showed that if the universe contained perfect absorbers of radiation but the electromagnetic interaction uses half-advanced and half-retarded potentials, then particles could behave as if they were interacting with fully retarded potentials. Here the time asymmetry is introduced by the postulation of absorbers (rather than emitters), which presumably can be justified in terms of molecular statistics. Many physicists object to the Wheeler-Feynman picture on aesthetic grounds, but it also has two more concrete drawbacks: (i) in an open expanding Universe the absorbers cannot absorb everything; and (ii) perfect absorption of gravitational radiation is impossible in any spacetime. Therefore, one would have to come up with another explanation of why the gravitational arrow of time (which has now been observed in the binary pulsar) matches the electromagnetic one, and why they both match the one for water waves.

The statistical picture[24)] I have described above is the only one I know which works for any radiation field in any spacetime. If we simply take the point of view that at some "initial" moment a physical system is uncorrelated with the ambient radiation field, then we find that its expected evolution shows outoing radiation and the associated reation effects. It has in addition further advantageous features:

(i) Because the retarded integral is truncated at $t = 0$, there is no radiation reaction at $t = 0$, so the initial-value problem is well-posed. In particular, there are no runaway solutions in this formulation.

(ii) Radiation reaction is calculated locally, so one need not worry about whether the energy lost really becomes radiation or is swallowed by a black hole or something. Radiation reaction occurs instantly, and this derivation of it frees it from distant radiation conditions.

(iii) The structure of spacetime in the distant past does not affect radiation reaction.

By averaging over the initial radiation field, which is not something we have much information about or control over in most radiation problems, this approach is closer to the modern view that physics should deal only with what it can measure, and should treat uncontrollables statistically. Interestingly in this context, Einstein objected to Ritz's view that a body "cannot receive energy from infinity unless another body loses a corresponding quantum of energy"[60)] on the grounds that this is not in principle a testable assertion: "we cannot speak of infinity but only of regions that lie outside of observed regions"[61)].

Moreover, it is hard to use "causality" as an argument for the Ritz view; the initial-value problem is nothing if not causal.

The statistical argument links all the radiative arrows of time with that of thermodynamics and the second law, because they are both derivable statistically. But neither derivation says where either of them really comes from: Why does the universe show arrows in the first place? Why is it not in perfect equilibrium? The answers to these questions presumably come from cosmology, but they would take us further afield than our interest here in gravitational radiation permits[59).

5. RADIATION FROM ISOLATED SYSTEMS

5.1 Radiation Reaction in Slow-Motion Linearized Theory

The slow-motion post-linear approximation may be carried out recursively to higher orders in λ and ε than we considered in §2. Eventually one reaches terms which are generally called radiation-reaction effects because they introduce nonconservative terms into the equations of motion, which cause the energy, momentum, and angular momentum of the system to change in a way that balances (at least in simple theories like electromagnetism) the amounts of these quantities that appear in the radiation field. It is important, however, to understand that these terms may be derived locally (as we shall show), without reference to the radiation produced[63). In a conservative theory one expects the local decrease in energy to be balanced by that in the radiation, but if one had a nonconservative field theory this would not happen, and one would not describe these terms as radiation-reaction terms. These terms originate[63) in the self-interaction of the system: the retardation delay in the interaction between different parts of the system means that "action" and "reaction" do not necessarily balance each other. The net residual sum of these "internal" or "self-" forces is called the radiation-reaction force. (It is absent in Newtonian mechanics with its instantaneous interactions.) So a better term for this force would probably be simply to call it the net self-interaction.

From this discussion it is clear that in order to calculate radiation reaction one has to separate the total field into a self-field and an "external" field. A system with no external field is an isolated system, and so we can take the self-field to be the total field of an isolated system. In linear theories this is done by using retarded potentials to compute the self-field. (Choosing advanced potentials instead would change the sign of the force, leading to radiation anti-damping.) As we discussed in the previous section, a possible but difficult choice of the self-field is to define it by the asymptotic condition that the radiation vanishes in $\mathcal{I}^{-}$. Our approach is to define the self-field statistically.

We have defined radiation reaction to be the part of the force which does not give zero when integrated over the whole system, but of course the local force density will contain many terms that do give zero when integrated and are therefore not of interest to us. The simplest way to separate out the radiation-reaction terms is to exploit their link with retardation: they change sign if advanced interactions replace retarded ones. Now, in our initial-value picture we get retarded potentials naturally for $t > 0$. But if we solve for the behavior for $t < 0$ from the same initial data, we would find that <u>advanced</u> potentials arise just as naturally. (Turn Fig.(2) upside-down.) Moreover, examination of the slow-motion initial data in Eqs.(42) and (55) shows that solving our problem for $t < 0$ is the same as solving the problem posed with ε changed to $-\varepsilon$ for $t > 0$. Therefore the radiation-reaction terms in the equations of motion are those which are of odd power in ε. (In the ε-reversed problem we would define $\tau = |\varepsilon| t$, so that τ does <u>not</u> depend on the sign of ε.)

We begin by examining radiation reaction in linearized theory. As ε approaches zero for fixed τ, any point at fixed x^j eventually enters region II of Fig.(3), in which ${}_1^{H}\bar{h}^{\mu\nu} = 0$, so we may expand Eq. (48) in ε at fixed x^i and τ:

$$ {}_1\bar{h}^{\mu\nu} = 4 \int_{C(\tau/\varepsilon, x^i)} {}_1\tilde{T}^{\mu\nu} (\tau - \varepsilon r, y^j; \varepsilon) r^{-1} \, d^3y \quad . \tag{48} $$

There are three places where ε enters the expression:

(i) The intersection of the cone C with the hypersurface $t = 0$ gets larger as $\varepsilon \to 0$, but since ${}_1\tilde{T}^{\mu\nu}$ is of compact support, we may set the upper limit of integration on $|y^i|$ to ∞ if ε is sufficiently small. So there is no ε-dependence here.

(ii) How does ${}_1\tilde{T}^{\mu\nu}(\tau, y^j; \varepsilon)$ depend on ε (ignoring the explicit retardation terms for the moment)? In the initial data for ${}_1\tilde{T}^{\tau\tau}$, ${}_1\tilde{T}^{\tau i}$, ${}_1\tilde{T}^{ij}$ only even powers of ε appear. The equation of evolution,

$${}_1\tilde{T}^{\mu\nu}{}_{,\nu} = 0 ,$$

contains no explicit factors of ε, so, $\tilde{T}^{\mu\nu}(\tau, y^j; \varepsilon)$ contains only even powers of ε.

(iii) As we anticipated earlier, odd powers of ε arise only from the explicit retardation terms. Let us examine the successive odd derivatives. For ${}_1\tilde{T}^{\tau\tau}$, which begins at order ε^2, the first odd term is

$${}_{(1,3)}\bar{h}^{\tau\tau} = -4 \int {}_{(1,2)}\tilde{T}^{\tau\tau}{}_{,\tau}(\tau, y^j)\, d^3y = -4 \frac{d}{d\tau} \int {}_1\tilde{T}^{\tau\tau} d^3y = 0, \tag{93}$$

which vanishes by conservation of energy. The next term is

$$\begin{aligned} {}_{(1,5)}\bar{h}^{\tau\tau} &= -4 \int {}_{(1,2)}\tilde{T}^{\tau\tau}{}_{,\tau\tau\tau}(\tau, y^j) r^2\, d^3y - 4 \int {}_{(1,4)}\tilde{T}^{\tau\tau}{}_{,\tau}(\tau, y^j)\, d^3y \\ &= -4 \frac{d^3}{d\tau^3} \int {}_{(1,2)}\tilde{T}^{\tau\tau}\, y_j y^j\, d^3y . \end{aligned} \tag{94}$$

The second term vanishes by conservation of energy (i.e. ${}_1\tilde{T}^{\tau\tau}$ is conserved at all orders in ε). This iterative evaluation of integrals has to be carried to order (1,7) in each component of $\bar{h}^{\mu\nu}$. The terms are catalogued, in somewhat different notation, by Schutz[24].

These terms affect the 2pl equation of motion, so that we have the energy loss at second order in λ

$$\frac{d}{d\tau}\,{}_2E = \int {}_2\tilde{T}^{\tau\tau}{}_{,\tau}\; d^3x$$

$$= -\frac{1}{2}\int {}_1h_{\alpha\beta,\tau}\;{}_1\tilde{T}^{\alpha\beta}\; d^3x - \frac{d}{d\tau}\int \left({}_1h^{\tau}{}_{\alpha}\tilde{T}^{\tau\alpha} + \frac{1}{2}\,{}_1h^{\alpha}{}_{\alpha}\tilde{T}^{\tau\tau}\right) d^3x. \qquad (95)$$

If, $\tilde{T}^{\mu\nu}$ is periodic in time then so will be ${}_1\bar{h}^{\mu\nu}$, so that if we average Eq.(95) over one period of motion the second term will go away. The first term turns out to contribute first at order ε^6:

$$\left(\frac{d}{d\tau}\,{}_{(2,6)}E\right)_{avg} = -\frac{1}{5}\left(\overset{(3)}{\text{Ɨ}}_{jk}\;\overset{(3)}{\text{Ɨ}}{}^{jk}\right)_{avg} , \qquad (96)$$

where

$$I_{jk} = \int {}_{(1,2)}\tilde{T}^{\tau\tau}\; y_j y_k\; d^3y \quad , \qquad (97)$$

$$\text{Ɨ}_{jk} = I_{jk} - \frac{1}{3}\,\delta_{jk}\, I^{l}{}_{l}, \qquad (98)$$

and where (3) above Ɨ means three τ-derivatives.

Equation (96) is the <u>near-zone quadrupole formula</u> for linearized theory, and seems first to have been obtained by Eddington[64]. It is gauge-invariant[24]. This fits with our earlier discussion: this is the lowest order at which dE/dt fails to vanish.

The radiation-reaction force density may be defined by

$${}_{(2,5)}F^{i}{}_{react} = \frac{1}{5!}\left(\frac{d}{d\varepsilon}\right)^5 \left({}_2\tilde{T}^{\tau i}{}_{,\tau}\right)_{\varepsilon=0} \quad . \qquad (99)$$

This is not gauge-invariant, because the non-reactive force density is of course non-zero at lower orders. The expression in our gauge is rather complicated[24], but Chandrasekhar[8] and Burke[9] found that there exists a gauge in which it is particularly simple:

$${}_{(2,5)}F^{i}{}_{react} = -\frac{2}{5}\,x_k\,\overset{(5)}{\text{Ɨ}}{}^{ik}\;{}_{(1,2)}\tilde{T}^{\tau\tau} , \qquad (100)$$

There might in principle be radiation-reaction terms in F^i at order $\lambda^3\varepsilon^3$ (in fact, there <u>are</u> such terms), so that if we are to use Eq.(95) and (100) we have to assume that, roughly, $\lambda^3\varepsilon^3 \ll \lambda^2\varepsilon^5$, or $\lambda \ll \varepsilon^2$, which means

$$M/R \ll v^2 \ . \tag{101}$$

This certainly applies in the laboratory, but it of course does not apply to Newtonian systems, which we will have to treat separately.

5.2 The Radiation Generated by Sources in Slow-Motion Linearized Theory

Here we look for the first time at what happens far from the system, where the radiation is. In discussing the equations of motion one naturally solves for the metric inside the system. To study the radiation, we want to be in the <u>far zone</u>, which means several wavelengths from the system. Now, as $\varepsilon \to 0$ the typical wavelength of radiation goes to ∞, so any fixed point x^i eventually enters the near zone. That is why we called the Newtonian manifold NM the near-zone manifold. In order to stay a fixed number of wavelengths from the source we define

$$\eta^{i'} = \varepsilon x^i, \tag{102}$$

and we approximate the metric at fixed $\eta^{i'}$ and τ as $\varepsilon \to 0$. We call $(\tau,\eta^{i'})$ the <u>far-zone</u> or <u>characteristic</u> <u>co-ordinates</u>. We put primes on indices that refer to the far-zone co-ordinates.

The linear metric follows from Eq.(48) again, which we write as

$${}^{F}_{1}\bar{h}^{\mu\nu}(\tau,\eta^{i'};\varepsilon) \equiv {}_{1}\bar{h}^{\mu\nu}(\tau,x^i = \eta^{i'}/\varepsilon;\varepsilon) \tag{103}$$

$$= 4 \int_{C(\tau/\varepsilon,\eta^{i'}/\varepsilon)} {}_{1}\tilde{T}^{\mu\nu}(\tau-\varepsilon r,y^j;\varepsilon)\, r^{-1}\, d^3y \ . \tag{104}$$

We do not scale the integration co-ordinate y^j here because the source remains in the near zone. Now, however, we have $r = |\eta^{i'}/\varepsilon - y^i|$, so that r^{-1} contributes a factor of ε to ${}^{F}_{1}\bar{h}^{\mu\nu}$ as $\varepsilon \to 0$. The resulting metric is derived in a number of standard sources[65]. In our notation we have, with $\eta \equiv |\eta^{i'}|$:

$$ {}_{(1,3)}^{\;\;F}\bar{h}^{\tau\tau} = 4M/\eta \quad , \qquad M = \int {}_{(1,0)}T^{00}\, d^3y \; ; \tag{105}$$

$$ \left.\begin{aligned} {}_{(1,4)}^{\;\;F}\bar{h}^{\tau\tau} &= 4P_i n^i/\eta + 4D_i(u)n^i/\eta^2, \quad n^i = \eta^{i'}/\eta \; , \quad u = \tau - \eta, \\ P_i &= \int {}_{(1,0)}T^{00}\; {}_{(0,1)}v_i\, d^3y, \quad D_i = \int {}_{(1,0)}T^{00}\, y_i\, d^3y; \end{aligned}\right\} \tag{106}$$

$$ {}_{(1,5)}^{\;\;F}\bar{h}^{\tau\tau} = 2\,\overset{(2)}{I}_{ij}(u)n^i n^j/\eta + 6\,\overset{(1)}{\not{I}}_{ij} n^i n^j/\eta^2 + 6\not{I}_{ij}(u)n^i n^j/\eta^3 ; \tag{107}$$

$$ {}_{(1,4)}^{\;\;F}\bar{h}^{\tau i'} = 4P^i/\eta ; \tag{108}$$

$$ \left.\begin{aligned} {}_{(1,5)}^{\;\;F}\bar{h}^{\tau i'} &= 2\,\overset{(2)}{I}{}^{ij}(u)n_j/\eta + (M^{ij} + \overset{(1)}{I}{}^{ij})n_j/\eta^2 \; , \\ M^{ij} &= \int {}_{(1,0)}T^{00}\,({}_{(0,1)}v^i y^j - {}_{(0,1)}v^j y^i)\,d^3y; \end{aligned}\right\} \tag{109}$$

$$ {}_{(1,5)}^{\;\;F}\bar{h}^{i'j'} = 2\,\overset{(2)}{I}{}^{ij}(u)/\eta \; . \tag{110}$$

Note that the conservation laws in the near zone ensure that M, P^i, and M^{ij} are constants, while D^i changes at most linearly in time. A choice of the zero-order Lorentz frame can eliminate P^i and D^i. The quantity u, the retarded time, is the only variable upon which I^{ij} and D^i depend. These expressions for ${}_{(1,n)}^{\;\;F}\bar{h}^{\mu'\nu'}$ are exact: the fact that the $1/r$ expansion is linked to the ε-expansion means that there are no higher-order terms in η^{-1} in Eq. (105) - (110).

It is clear that the radiative terms contribute to ${}_{1}^{F}\bar{h}^{\mu'\nu'}$ first at order ε^5. This must be compared with the limiting flat metric itself,

$$ \eta^{\mu'\nu'} = \varepsilon^2 \eta^{\mu\nu} \tag{111}$$

because of the co-ordinate scaling. We therefore remove a conformal factor of ε^2 from the metric $g^{\mu'\nu'}$ and define[27] the <u>far-zone manifold</u> FM to be the $\varepsilon \to 0$ limit of the manifold of the congruence $x^{\mu'}$ = const equipped with the metric $\underline{g}^{\mu'\nu'} = \varepsilon^{-2} g^{\mu'\nu'}$. Therefore FM is a flat Minkowski manifold with co-ordinates $(\tau, \eta^{i'})$. We shall denote tensors on FM by an underline, e.g. $\underline{g}^{\mu'\nu'}$. From the point of view of FM, the near-zone manifold NM with co-ordinates t and x^i has shrunk to the line $\eta^{i'} = 0$: the near zone for any τ is a singular point in the far-zone manifold. This is illustrated in Fig. (7).

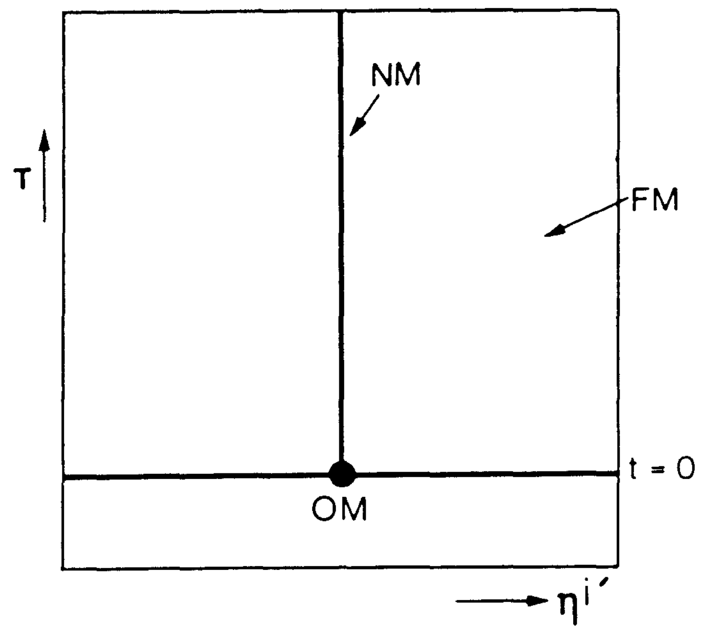


Figure (7). In the far-zone manifold FM, the near-zone manifold NM is the singular line $\eta^{i'} = 0$.

The terms in ${}_1\overset{F}{\underline{h}}{}^{\mu'\nu'}$ in Eq.(105)-(110) can be viewed as perturbations of the metric of FM, provided we likewise remove the conformal factor of ε^2. This means that the lowest-order terms in the far zone having non-vanishing second time derivatives are of order ε^3. These may be put into TT-gauge[30] and the various measures of energy flux calculated. The result, after integrating over a sphere of radius η for large η, is

$$\frac{dE}{d\tau} = \frac{1}{5} \varepsilon^6 \left(\overset{(3)}{I}_{jk} \overset{(3)}{I}{}^{jk} \right)_{avg} + O(\varepsilon^7), \tag{112}$$

exactly compensating Eq.(96). This is called the far-zone quadrupole formula. It was first derived in this context by Einstein[66], and Einstein's factor-of-two error was corrected by Eddington[64]. Note that, in principle, there could be time-dependent far-zone terms of order $\lambda^2\varepsilon^3$ in ${}^{F}h^{TT}$, which might then contribute to the TT part of the metric. Until we examine these terms, therefore, we can only use Eq.(112) if $\lambda \ll \varepsilon^2$, the same restriction as we had in Eq.(101).

5.3 Radiation Reaction in the Newtonian Limit

Let us recall the order-counting we have just mentioned for the near-zone quadrupole formula. In linearized theory F^i_{react} was of order $\lambda^2\varepsilon^5$. In the post-Newtonian approximation, where $\lambda = \varepsilon^2$, this is of order ε^9. Terms of the same order will come from those slow-motion post-linear terms of order $\lambda^3\varepsilon^3$ and $\lambda^4\varepsilon$. (Since we have no inverse powers of ε, we need not go higher in λ than this.) This means that to calculate radiation-reaction in the Newtonian limit we have to calculate the metric two orders of λ beyond linearized theory: this is called the third iteration of Einstein's equations. This fact was first pointed out[13] by Eddington[64], but it was often ignored by his successors, with the result that the history of the calculation of radiation reaction is peppered with incomplete calculations which served to confuse relativists and to make them further doubt the reality of gravitational radiation[13,67], and which also led to distrust[12] of the initial Burke[9] and Chandrasekhar[10] calculations until their mathematical shortcomings had been rectified[68]. (In principle, radiation reaction terms could arise at order $\lambda^3\varepsilon$, making them stronger than those in linearized theory, but these terms do not in fact contribute[26].)

Another complication over the linearized theory calculation is the treatment of the random radiation field. In the linear theory it averages to zero, but here it will not vanish at order λ^2, so it can in principle contribute to the subsequent motion. If we put in random data of order ε^4, as in Eq.(91), then these will affect the 1pN equations of motion linearly. After averaging they will not contribute to these equations but they might affect the 2pN equations of motion.

However, Futamase[26)] has shown that, in fact, they do not influence the equations of motion until 3pN order (ε^{10}), safely beyond radiation raction order (ε^{9}).

The first complete calculation of the metric through radiation reaction order was by Chandrasekhar and colleagues[8)]. The calculation was repeated and simplified by Anderson and DeCanio[21)], whose choice of gauge and manner of iterating the field equations have been the standard method ever since. Kerlick[22)] managed to show that some of the apparently divergent light-cone integrals in Anderson & DeCanio (and Chandrasekhar) were well-defined, and Breuer and Rudolph[37,69)] and Futamase[26)] removed all the remaining apparent divergences up to and including radiation-reaction order. We shall return to the subject of diverent integrals in detail below.

As in linearized theory, the radiation-reaction force is gauge dependent, and various expressions for it may be found in Kerlick[22)], Breuer and Rudolph[37)], and Futamase[26)]. The simplest form[30)] is identical to the one we gave in linearized theory, Eq.(100), with the same definition of $\mathcal{I}^{ij}$, Eqs.(97) and (98), and with the understanding that ${}_{(1,2)}\tilde{T}^{\tau\tau}$ is the near-zone ${}_{4}\tilde{T}^{\tau\tau}$ here, i.e. the lowest-order Newtonian mass-density ${}_{2}\rho$:

$$
{}_{9}F^{i}_{\text{react}} = -\frac{2}{5}x_k \overset{(5)}{\mathcal{I}}{}^{ik}\,{}_{2}\rho \;. \tag{113}
$$

At first sight it is surprising that the extra $\lambda^{3}\varepsilon^{3}$ and $\lambda^{4}\varepsilon$ terms in the pN approximation seem to make no contribution to the radiation-reaction force. Closer examination shows that they make a contribution in other gauges, but in the simple gauge leading to Eq.(113), there is no room for them: higher terms make no contribution to ${}_{2}\rho$ or I^{ij}, which depend on Newtonian-order quantities. There is, however, a hidden contribution in Eq.(113), because the time derivatives assume the Newtonian equation of motion rather than the linearized one, so the field enters here.

The force, of course, is not directly observable in systems like the binary pulsar, so its calculation is only an intermediate step in the calculation of observable effects. The quantity usually calculated is the rate of loss of energy, but this is also not directly observable in the near zone. Many authors have shown that the linearized theory near-zone quadrupole formula applies here also, in that the rate of change of the Newtonian energy E_N is

$$\left(\frac{d}{d\tau} E_N\right)_{avg} = -\frac{1}{5}\left(\overset{(3)}{I}_{jk}\ \overset{(3)}{I}{}^{jk}\right)_{avg} , \qquad (114)$$

but there has been some confusion over the interpretation of this. Since energy can only be defined globally in general relativity, it is hard to decide what relation the "local" energy E_N has to the total. Evidently its rate of change is the same as that of the total (we will show this in the next section), but does this really mean that energy is conserved? What about the rates of change of the pN and 2pN energies, which may also contribute?

I think it is helpful to break this problem into two parts. First, how do we understand conservation of energy? And second, what is it that is observable about the loss of energy? The first problem is easy to settle in the light of our results on breaking conservation laws in §2.4. Energy is conserved exactly through 2pN order, but the near-zone equations of motion incorporate energy-changing terms at order ε^9 (2.5pN). The lowest-order rate of energy change is then calculated by using these nonconservative terms in the lowest-order energy functional E_N. Thus, although Eq.(114) uses only E_N, it should be interpreted as the rate of change of the total energy through 2pN order. This still does not explain why this energy turns up in the radiation far away. For this we would have to invoke a global conservation law, an argument which to my knowledge has never been satisfactorily made. More important from the observer's point of view is not what energy the waves have far away, but what effect the change in the 2pN energy will have on the dynamics of the system. This is the second part of our problem.

In the binary pulsar system, the direct observable is the period of the orbit of the two stars[26]. Now, the period is another example of a broken conservation law. Through 2pN order the binary orbit will be strictly periodic, so the period may be considered to be a constant function of the dynamical variables. Only the lowest-order period functional is required to calculate the lowest-order period change, so we need only the Newtonian period P_N. As is well known, this is a function only of the energy E_N, so that the near-zone quadrupole formula immediately predicts the rate of change of the period:

$$P_N^{-1}\ dP_N/d\tau = -\frac{3}{2}\,E_N^{-1}\ dE_N/d\tau\ . \tag{115}$$

But how is the period defined? After all, the real system is fully relativistic and strictly speaking has no period at all. To answer this we have to look at the way the observers calculate the period[50]. They take observations over a limited amount of time, say a month, and fit them within observational errors to a 2pN orbit, which gives them a value for the period of that orbit. (In fact, observational errors allow them to determine only some of the 1pN orbital characteristics and none -- yet -- at 2pN order. So in practice they use only the 1pN orbit equations. This does not change our argument). This fit to the period changes from month to month, and this rate of change is what the observers report. Observers with ideally accurate information could refine this procedure further, taking shorter and shorter fitting times, until they reached a continuously varying period; this $P_{2pN}(\tau)$ would be interpretable as the period the system would have if it evolved from its configuration at time τ according to the 2pN equations of motion rather than the fully relativistic ones. Mathematically, we say that the period is the period of the osculating 2pN approximation to the orbit. The idea of using osculating orbital approximations is standard in celestial mechanics[70], but to my knowledge it was first introduced into our subject by Walker and Will[58]. It removes any remaining ambiguity in the interpretation of the near-zone quadrupole formula.

Let us return now to the question of divergent integrals. The nonlinearities of the field contribute to the inhomogeneous solution

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as one integrates over the light cone C. In "global" approaches to the problem, the light cone extends an infinite distance to the past, and it can happen that when the gravitational waves themselves enter the integrand, the integrand will not fall off rapidly enough to give a finite integral at some order. This is bound to happen at some order in a slow-motion expansion, because as we have seen in Eq.(93) and (94), this expansion introduces higher and higher powers of r into the integrands, which arise from expanding the $\tau-\varepsilon r$ argument of Eq.(33) reformulated to look like the argument in Eq.(48):

$$\Lambda^{\mu\nu}(\tau-\varepsilon r,y^j;\varepsilon) = \Lambda^{\mu\nu}(\tau,y^j;\varepsilon) + \Lambda^{\mu\nu}{}_{,\tau}(\tau,y^j;\varepsilon)r + \Lambda^{\mu\nu}{}_{,\tau\tau}(\tau,y^j;\varepsilon)r^2 + \ldots \quad . \tag{116}$$

A number of such divergent expressions were identified by Kerlick[22]. All of those below radiation-reaction order are in fact not diverent: angular integration removes the apparently diverent terms[26,69]. But some divergences at 3pN order and beyond do seem un-removable in this picture.

The divergences can be removed, however, in other approaches, and this is one of the principal advantages of both the matched asymptotic expansions method and the initial-value method. It is easy to see that the divergences have a clear origin in an initial-value framework. As $\varepsilon \to 0$, our light-cone $C(\tau/\varepsilon, x^k)$ in Fig.(2) grows infinitely large, so the divergences will still be present. Here, however, the upper limit of integration is ε-dependent, so what we will find is that for sufficiently small ε, the n^{th} derivative of some field quantity f behaves like

$$\partial^n f/\partial\varepsilon^n = a\ln\varepsilon + O(1) \tag{117}$$

where we have chosen to illustrate the weakest type of divergence, a logarithmic one. This means that f is not C^n in ε at $\varepsilon = 0$, but it does not prevent one from developing the asymptotic approximation beyond this order[33,71]. We conclude from Eq.(117) that

$$f(\varepsilon) = p(\varepsilon) + \frac{a}{n!}\,\varepsilon^n \ln\varepsilon + O(\varepsilon^n), \tag{118}$$

where $p(\varepsilon)$ is a polynomial of order n-1 in ε. There is in principle no difficulty extending this approximation to order n and beyond[33]. We see, therefore, that the divergences of the "global" approaches arise because these approaches assume an analytic, power-of-ε behavior to all orders, so that the $\varepsilon^n \ln\varepsilon$ terms with perfectly finite coefficients $a/n!$ are forced to appear at the higher order of ε^n, where they must make an infinite contribution. If the "global" methods give worse divergences, say those that lead to ε^{-1} terms rather than $\ln\varepsilon$ in Eq.(117) then these will have to be shifted down by a full order. No such terms are known, and they would be unlikely to arise in a properly-carried-out initial-value formulation, where all terms at each order are calculated when that order is reached. They might arise at high order in the global approach, however, say as a disguise for $(\ln\varepsilon)^2$ terms.

The method of matched asymptotic expansions can also remove the divergent terms by introducing explicit logarithmic terms in the asymptotic expansion of the near zone and showing that these match properly to far-zone terms. This is how, in fact, the removal was first accomplished[17], with terms introduced by analogy with terms in simpler model calculations by Anderson and others[71,72,73]. It is hard to see, however, how this can be made as systematic as the initial-value formulation makes it; in the latter one has no choice, no freedom to introduce new terms: one takes only what the formalism gives.

Our discussion of Newtonian radiation-reaction has been long on words and short on equations. This is not because calculations are lacking or the equations are short, as a glance through the literature will show. Rather, the calculations here are so close in spirit to those of the linearized theory, that I have decided to give more space to the many conceptual problems that one encounters here, whose resolution is essential to our confidence in the predictions of radiation-reaction theory. One can say with confidence now that the near-zone quadrupole formula does provide an asymptotic approximation to a well-defined sequence of solutions in general relativity.

5.4 The Radiation Generated by Newtonian Sources

As in linearized theory, here also we examine the field produced at a point a fixed number of wavelengths away from the source as $\varepsilon \to 0$. But here, as in the previous section, we need to take account of the nonlinearities of the field equations. The outgoing radiation in linearized theory in FM was of order $\lambda\varepsilon^3$ (after removing the ε^2 conformal factor), but the dominant far-field term was of order $\lambda\varepsilon$. When we set $\lambda = \varepsilon^2$ we see that we might expect far-zone terms of order $\lambda^2\varepsilon$ to be of the same order as the radiation. This means we have to consider the second iteration of the field equations far from the source, as has been pointed out by Walker and Will[67]. In our gauge it will turn out that these extra terms are non-radiative, so that the far field will look much like that of linearized theory, and in particular the far-zone quadrupole formula will be unchanged.

Because our integrands are not necessarily of compact support, we divide space into two regions: the near zone, defined by $\eta < R$ (or $|x^i| < R/\varepsilon$) and the far zone outside it. In the near zone we will approximate our integrands at fixed x^i and τ, while outside we will approximate at fixed $\eta^{i'}$ and τ. We can expect these approximations to be uniform in their respective domains. The situation is illustrated in Fig.(8). Note that the near-zone integral includes part of the vacuum outside the source, where the 'Newtonian' energy density dominates. In the far zone the wave energy dominates. It is possible to show[27] that the integral over the far zone does not begin to contribute to $\overset{F}{h}{}^{\bar{\mu}'\nu'}$ until order ε^8, one order beyond that which we shall need to calculate radiation. The calculation reduces, therefore, to an integration over the near zone, and this makes it very similar to that in linearized theory. There are only two differences. One is that the integrand Λ^{ij} is not of compact support; but the Newtonian near-zone equations of motion nevertheless allow the appropriate integral to be expressed in terms of I^{ij} in the same way as in Eq.(110). The other

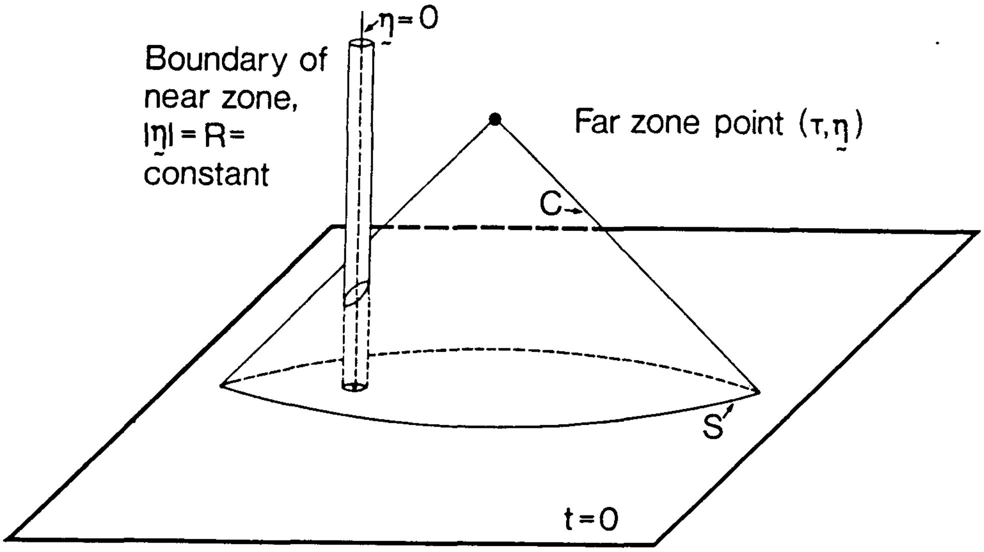

Figure (8) When evaluating the field ${}^{F}h^{\mu'\nu'}$ at a far-zone point $(\tau, \eta^{i'})$, the integral over the light cone C includes contributions from the near zone ($\eta < R$) and the far zone ($\eta > R$), where we make separate approximations.

difference is that the post-Newtonian mass,

$$M_2 = \int {}_6\Lambda^{\tau\tau} \, d^3y \, , \tag{119}$$

a time-independent near-zone integral, contributes a 'Coulomb' term to ${}_7^{F}h^{\tau\tau}$; this is the only explicitly nonlinear term in this gauge, and it is non-radiative. Apart from this, the far-zone metric is identical to that in Eqs.(105) - (110), with orders in λ combined with those in ε according to the rule $\lambda = \varepsilon^2$.

It follows that the far-zone manifold FM may be constructed just as before, and the energy flux in it will give the far-zone quadrupole formula, Eq.(112), unchanged. It is here in the Newtonian limit that the utility of FM is most striking. As we remarked before, the discussion of radiation in a manifold with mass is much more complicated and delicate than in linearized theory. Since Newtonian systems have mass, it seems to have been universally assumed until recently that a proper treatment of $\mathcal{I}^+$ was necessary to obtain a rigorous description of the energy radiated, even in the Newtonian limit. Our formulation makes it clear that this is not true: in FM we have a flat manifold with linear radiation. The linking of the limits $\lambda \to 0$ and $\varepsilon \to 0$, which causes the Newtonian mass to vanish in the limit, and the linking of the limits $\varepsilon \to 0$ and $r \to \infty$ by introducing the co-ordinate $\eta^{i'}$, which reduces the radiation to that of linearized theory, literally makes these anticipated complications disappear.

Here it is appropriate to mention three calculations which in fact do take care to treat I^+ in the Newtonian limit. The calculation of Anderson, *et al.*,[17] in the frame work of matched asymptotic expansions, uses the method of " strained co-ordinates" to make sure that the far-zone solution is matched on the correct light cones (rather than on the characteristics of the flat-space wave operator in our co-ordinates). The quadrupole formula emerges unscathed. The second calculation is that of Winicour and collaborators[23], which begins by setting initial data not on a spacelike hypersurface but on a true <u>null</u> hypersurface. In many respects this is a very similar approach to ours, and we have shown[27] that if we were to apply the conformal transformation by ε^2 to our whole sequence of manifolds and not just to FM, the resulting sequence (still solutions of Einstein's equations since ε = const) would be the one to which Winicour, *et al.*, construct asymptotic approximations. The map from I^+ in the Winicour picture to I^+ of our FM is regular. However, where we have completely free initial data for radiation, which we remove by averaging, Winicour has no freedom in the radiative initial data[23]. This difference is as yet unexplained. Not surprisingly, but gratifyingly, the far-field quadrupole formula emerges from the Winicour work as well.

The third calculation is a semi-heuristic argument devised by Walker and Will[74] to show that the use of the "wrong", flat-space light cones to compute the outgoing energy flux makes a negligible error from using the correct light cones. Their calculation may be viewed as showing that if one uses the flat-space cones in Thorne's intermediate wave zone[20] (which I described in §4.2) and then uses the correct null cones from there to infinity, then the energy flux calculated in the intermediate wave zone and at null infinity are essentially equal.

5.5 Radiation Reaction for Strong-Field Point Particles

This is a further development of the near-zone expansion of the metric between the bodies. The calculation[16,76] strongly resembles the post-linear one, in which certain constants in the near-zone metric are given as integrals over the body zones. At each order, the gauge condition $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ constrains the motion of the centers of the body zones. The radiation-reaction order expressions are identical to the post-linear form[76], in which the particle masses are again given by the ADM masses of the body zones. Therefore, we find the same reaction force: the near-zone quadrupole formula is still valid here.

This is a particularly important result. In the first place, it gives us complete assurance that Einstein's equations do predict the observed change of period of the binary pulsar, despite the fact that the neutron stars are strong-field sources. And in the second place, it extends the strong equivalence principle to the radiative effects in bodies with significant self-gravitation.

5.6 The Radiation Generated by Strong-Field Point Particles

The results of §2.9 imply that the near-zone metric of the point-mass problem is the same as that of the ordinary Newtonian limit sufficiently far from the bodies, so it follows by continuity of solutions of Einstein's equations that the dominant far-zone metric in the point particle case will be the same as in the Newtonian case. One has to check that the only post-Newtonian term, ${}_7\bar{h}^{\tau\tau}$, also comes out in the same way. It does[28], so we may then conclude that the far-zone metric is the same as in Newtonian (or linearized) theory, provided the mass, etc., is defined appropriately as an integral over the body zone. In particular, the mass felt far away is the sum of the ADM masses minus a small contribution giving the relative Newtonian gravitational potential energy of the two bodies, calculated as if they were Newtonian point masses. This is a further extension of the strong equivalence principle.

5.7 Implications for Numerical Calculations

Thorne[20], Schumaker and Thorne[77], and Futamase[28] have suggested that there may be some strong-field situations (besides the point-particle limit) in which the far-zone metric may reasonably be approximated by the quadrupole terms that we have calculated. In particular, if there are strong-field regions in which for some reason the velocities are abnormally small, then there will be a near zone outside the bodies and inside a sphere of one wavelength's radius. The metric there will be dominated by Newtonian terms, just as it is for the point-particle calculation, and by continuity these will lead to a far-zone metric with the standard quadrupole terms. In any such situation it may be hard to calculate the near-zone metric in terms of the source, which is fully nonlinear, but if this can be done then the radiation calculation is easy.

Schumaker and Thorne[77] applied this idea to the calculation of the torsional oscillations of a neutron star, which are very slow compared to, say, the orbital time at the star's surface. One can also imagine using this to simplify certain numerical collapse problems. If an axisymmetric collapse with angular momentum is "delayed" by "centrifugal" effects,then its external field may change relatively slowly. A nummerical calculation would only have to use a grid large enough to extend into the near zone, after which the radiation could be calculated analytically. This might represent a significant savings of storage and computing time over present methods, in which the grids extend into the wave zone, computing the radiation numerically[78]. As Thorne[20] has suggested, further savings might be made by eliminating the (now useless) dynamical degrees of freedom of the field inside the collapsing body. This amounts to using a "Coulomb-plus-magnetostatic" approximation inside. It would be interesting to see comparisons of such approximate calculations with more complete ones. Note also that the problem in which collapse is delayed by rotation is not simply an artificial test problem. Slowly-rotating collapse will be nearly spherical and may be poor emitters of gravitational radiation despite their high velocities. The most energetic emissions of gravitational waves may indeed come from systems to which these approximations apply.

5.8 Astrophysical Applications

Once one accepts that general relativity predicts radiation reaction effects, then the universality of gravitational fields in astrophysics means there will almost inevitably be important applications. We have no room to discuss these in detail, but we will mention three: the binary pulsar, cataclysmic variables, and the unexpected gravitational-wave induced instability of rotating stars.

We have mentioned the binary pulsar system before, so here we will describe why it has become such an important test of general relativity. A more complete discussion is given by Will[79]. The system appears to consist of two neutron stars; the pulsar (PSR 1913+16) is certainly a neutron star, while its companion must be small enough to avoid being stripped of its outer envelope by the pulsar's gravitational field and dim enough not to be seen optically, which leads to its being a neutron star as well. (Whether it is a pulsar beaming in some other direction or a radio-quiet star is not known.) The mean pulse period is intrinsically so stable (varying by less than one part in 10^{17} per pulse) that very precise measurements of the orbit may be made from observations of the Doppler shift of the pulse period. Fitting a Newtonian orbit to such a "spectroscopic" binary does not yield enough data to fix the masses of the stars uniquely; rather it provides the value of the "mass function", a relation between the masses and the inclination of the orbit. But two post-Newtonian effects are observable in this system: the periastron shift (4° per year, rather larger than for Mercury's orbit!), and a combined transverse-Doppler/gravitational redshift measurement. When combined with the mass function, these two extra numbers allow all the physical quantities to be determined. The results are[50]: pulsar mass 1.42 $M_\odot$, companion mass 1.41 $M_\odot$, orbital period 27907 s, orbital eccentricity 0.61714, relative semimajor axis 1.95×10^{11} cm. With these values one can calculate the expected change in the orbital period due to radiation reaction[80], and this is

$dP/dt = -2.40\times10^{-12}$. The observed period change agrees with this to within observational errors of about five percent[81]. The fact that the system is so clean, so that non-gravitational interactions between the stars (such as mass flow) may be ruled out, and the confidence we now have in the quadrupole formula in the near zone, means that this agreement may be regarded as another successful test of general relativity.

Although the binary pulsar's stars are relatively close (never separated by more than three times our Sun's diameter), there are many other binary systems of comparable or smaller size. In some of them, one star is relatively normal and the other is a white dwarf, whose tidal effects on its companion lead to mass transfer onto the dwarf. These systems are irregular variables called cataclysmic variables, and it may be that gravitational radiation plays a major role in controlling the variability of at least some of them[82]. When mass transfer occurs, the change in the relative masses may push the stars further apart, reducing the tidal effects and turning off mass transfer. But after a while the effects of radiation reaction will bring the stars closer together again, and a new phase of mass transfer (and observed activity) will begin.

The third application is the discovery by Chandrasekhar[10] that rotating stars that are stable in Newtonian gravity may become unstable when gravitational radiation reaction effects are included. He studied the simplest rotating models: rigidly rotating, uniform-density axisymmetric models called Maclaurin spheroids[83]. They radiate no waves in their equilibrium state, but a small *nonaxisymmetric* perturbation will do so. One might expect that since the waves carry energy away, the perturbation would be damped out. This is indeed what happens if the unperturbed star is rotating sufficiently slowly. But if it rotates faster than a certain amount (at which it would still be stable in Newtonian gravity), the reverse happens: the perturbation begins to <u>grow</u> by converting the rotational energy of the star into gravitational wave energy. The effect is presumably to spin the star down to the point of marginal instability.

Chandrasekhar investigated only the modes with symmetry $\exp(2i\phi)$, where ϕ is the angle about the axis of symmetry of the unperturbed star. Friedman and Schutz[84], investigating the same instability for general $\exp(im\phi)$ modes in more realistic, compressible, differentially-rotating stars, made the remarkable discovery that it was easier to destabilize modes for high values of m, and that in fact every rotating star is unstable to this instability for sufficiently large m. Fortunately for the persistence of such stars, Comins[85] showed that such instabilities have growth times that increase exponentially with m, so that only the lowest few modes ($m \lesssim 5$) in very relativistic stars have any practical importance to astronomy. It now seems that this may indeed determine the rotational velocity of some pulsars[86]. Moreover, a neutron star that accretes angular momentum from an accretion disc will be spun up to an instability point, at which it may then remain, radiating away further accreted angular momentum[87]. Such a star can be a strong source of gravitational radiation.

6. LOOKING AHEAD

We have studied those problems which have had the most attention, but there are a number of unanswered questions and new problems which might yield to these techniques. I will list somehere, beginning with questions that would increase our understanding of the present work:

(i) Because the Newtonian approximation can be uniformly valid only for a finite time, each separate time-interval so approximated generates its own aysmptotic manifolds NM and FM. Can these be pieced together so that there is a single, complete NM and FM for the sequence? If we adopt the attractive notion that the Newtonian limit is an osculating one, what happens to NM and FM then?

(ii) Is there a geometrical, co-ordinate-free definition of, say NM? Such a definition would have to include a characterization of what kinds of sequences have Newtonian limits. Similar considerations of interest for FM.

(iii) Can one resolve the "paradox" that we have complete freedom to specify initial gravitational waves while in the characteristic-initial-value problem there seems to be none?[23)]

(iv) Look into the near-zone logarithmic term in ε. There seems to be disagreement between different methods about the order at which these first appear[17,28)] Which is correct? Futamase[28)] noticed that if a system is giving off no quadrupole radiation then the appearance of logarithmic terms is delayed until after octupole-radiation-reaction order. Does this reflect some fundamental (and simple) property of all wave equations?

(v) What are the next terms beyond the lowest radiation order in the far zone and beyond the lowest radiation-reaction order in the near zone? Can they be used to bound errors in the approximation?

(vi) What are the peeling properties of radiation near I^{+}_{FM}?

The next group of problems are "new" research problems on which the same techniques might be brought to bear.

(vii) What gravitational radiation reaction does a body falling freely in an external gravitational field experience? The electromagnetic problem has been solved for slow motion of a charge in an everywhere-weak gravitational field by DeWitt and DeWitt[88)], and the formula has recently been extended using initial value methods to slow motion in arbitrarily strong stationary fields by Nuala O'Donnell, Clifford Will and myself (paper in preparation). The gravitational case has not, to my knowledge, been staitsfactorily solved, and we plan to look at that next.

(viii) One can study spinning bodies in the point-particle limit, keeping the rotational velocity constant as $\varepsilon \rightarrow$. Does the Papapetrou equation[30)] result? What other effects are there?

(ix) Is there a satisfactory method of using global (pesudotensorial) conservation laws to derive the energy balance implied by the equality of the near- and far-zone quadrupole formulas for energy? The heart of such a demonstration would be to give a clear reason why the decrease in the integral of the pseudotensorial energy density should be given by the rate of change of the Newtonian energy. This might be a variant on our theorem about exact conservation laws of approximation equations.

(x) We have developed the general weak-field, slow-motion approximation for initial data in which $\rho \sim \lambda$ and $v \sim \varepsilon$. Then we saw that the case $\lambda = \varepsilon^2$ was "well-conditioned" in the sense that if $\rho \sim \varepsilon^2$, $v \sim \varepsilon$, etc., at $t = 0$, then the dynamical equations preserve this ordering: no terms of order ε in ρ, say, are created for $t > 0$. This would not be the case, if, say, we took $\rho \sim \varepsilon^2$ and $v \sim \varepsilon^2$, for then the gravitational fields created by ρ would induce accelerations that would make v of order ε for $t > 0$. Is the Newtonian limit the only one-dimensional curve through (λ,ε) space which is well-conditioned in this sense? If other such limits exist they might be instructive. If they do not, what property of Einstein's equations precludes them?

(xi) In the strong-field point-particle limit <u>without</u> small velocities, do the point particles still interact as in linearized theory, or are there extra effects from their internal structure?

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