NON-VACUUM ADaM FIELD EQUATIONS*

Bernard F. Schutz, Jr. †

California Institute of Technology, Pasadena, California 91109

The canonical version of the vacuum Einstein field equations formulated ten years ago by Arnowitt, Deser, and Misner (ADaM) [1] has stimulated several attempts to quantize certain cosmological models, most notably Misner's so-called Mixmaster Universe [2]. Some researchers have begun recently to extend these methods to non-vacuum spacetimes; for example, Nutku earlier at this conference described the canonical theory of a scalar field in Schwarzschild spacetime. The purpose of this talk is to generalize the ADaM field equations to include an arbitrary stress-energy tensor. This is not a "first step" toward a canonical formulation of the full non-vacuum field equations; rather, it is simply a possible starting point.

Essentially, the ADaM field equations are a linear combination of Einstein's $G_{\mu\nu} = 0$ equations that is particularly well-suited to a "three-plus-one split" of spacetime, i.e., a division of spacetime into three-dimensional spacelike sections labelled by the parameter time. The metric of each section is the spacelike part of the metric for all of spacetime:

$$g_{ij} \equiv {}^4g_{ij} \quad . \tag{1a}$$

(Superscript "4" denotes quantities referred to the full four-dimensional spacetime, while no superscript implies three-dimensional quantities. Latin indices run from

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1 to 3, Greek from 0 to 3. Signature is - 2.) ADaM replace the remaining four metric components - which give information on how one hypersurface fits into the next 3 - with: a three-scalar

$$N \equiv (-\,{}^4g^{oo})^{-\frac{1}{2}} \tag{1b}$$

and a covariant three-vector

$$N_i \equiv {}^4g_{oi} \; . \tag{1c}$$

The ADaM field equations are derived from the usual variational principle,

$$\delta I = \delta \int {}^4R(-\,{}^4g)^{\frac{1}{2}}\, d\,{}^4x = 0 \; . \tag{2}$$

Were one to use $\{{}^4g^{\mu\nu}\}$ as the set of independent variables, one would obtain $G_{\mu\nu} = 0$ from Eq. (2) [4] . Using the ADaM variables $\{N, N_i, g_{ij}\}$, on the other hand, gives the ADaM equations.

To obtain the non-vacuum equations, let L be the Lagrangian for the non-gravitational fields. Then Eq. (2) generalizes to

$$\delta I = \delta \int ({}^4R + 2\kappa L)(-\,{}^4g)^{\frac{1}{2}}\, d\,{}^4x = 0 \; . \tag{3}$$

Using $\{{}^4g^{\mu\nu}\}$ as the variables gives [5]

$$G_{\mu\nu} = \kappa T_{\mu\nu} \; , \tag{4}$$

where

$$T_{\mu\nu} = L\,{}^4g_{\mu\nu} - 2\frac{\partial L}{\partial\,{}^4g^{\mu\nu}} + \frac{2}{(-\,{}^4g)^{\frac{1}{2}}}\left[(-\,{}^4g)^{\frac{1}{2}}\frac{\partial L}{\partial\,{}^4g^{\mu\nu}{}_{,\beta}}\right]_{,\beta} \; . \tag{5}$$

The non-vacuum ADaM equations follow from Eq. (3) if one uses the set $\{a_{\alpha\beta}\}$ of ADaM variables, defined by

$$a_{oo} \equiv (-\,{}^4g^{oo})^{-\frac{1}{2}};\; a_{oi} \equiv {}^4g_{oi};\; a_{io} \equiv {}^4g_{io}\; ;\; a_{ij} \equiv {}^4g_{ij} \; . \tag{6}$$

It is convenient in what follows to ignore the symmetry of $a_{\alpha\beta}$ and ${}^4g_{\mu\nu}$. For instance, variations of a_{oi} will be taken while holding a_{io} fixed. The final results will, of course, be symmetrized.

Because the transformation from $\{{}^4g^{\mu\nu}\}$ to $\{a_{\alpha\beta}\}$ is nonsingular and does not involve derivatives of ${}^4g^{\mu\nu}$ or explicit dependence upon the spacetime coordinates, the equations obtained from varying $a_{\alpha\beta}$ will be the linear combination

$$0 = \frac{\delta I}{\delta a_{\alpha\beta}} = \frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{\alpha\beta}} \frac{\delta I}{\delta\, {}^4g^{\mu\nu}} \tag{7}$$

of the equations obtained from varying ${}^4g^{\mu\nu}$. We therefore need only find $\partial\, {}^4g^{\mu\nu}/\partial a_{\alpha\beta}$, in which it is understood that the derivative is taken holding all other $a_{\gamma\delta}$ fixed. This is the key to the difference between Einstein and ADaM: it means, for example, that $\partial\, {}^4g^{01}/\partial a_{01}$ is not the same as $\partial\, {}^4g^{01}/\partial\, {}^4g_{01} = -{}^4g^{oo}\, {}^4g^{11}$, because in the first case one holds $\{{}^4g^{oo}, {}^4g_{02}, {}^4g_{03}, {}^4g_{ij}\}$ fixed while in the second case one holds $\{{}^4g_{oo}, {}^4g_{02}, {}^4g_{03}, {}^4g_{ij}\}$ fixed. Bearing this in mind, we write down the equations of transformation:

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{ij}} = -\, {}^4g^{\mu i}\, {}^4g^{\nu j} + {}^4g^{o\mu}\, {}^4g^{o\nu} N^i N^j \quad ; \tag{8a}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{oi}} = -\, {}^4g^{o\mu}\, {}^4g^{\nu i} - {}^4g^{o\mu}\, {}^4g^{o\nu} N^i \quad ; \tag{8b}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{io}} = -\, {}^4g^{\mu i}\, {}^4g^{\nu o} - {}^4g^{o\mu}\, {}^4g^{o\nu} N^i \quad ; \tag{8c}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{oo}} = 2\, {}^4g^{o\mu}\, {}^4g^{o\nu} N \quad . \tag{8d}$$

It is straightforward to use Eqs. (7) and (8) to find the non-vacuum ADaM field equations. (Here π^{ij} is the momentum canonical to g_{ij}, defined by Eq. (9c) below. Indices on it and N_i are raised and lowered by the three-dimensional metric, covariant differentiation with respect to which is denoted by a slash, "|".)

$$-\, g^{\frac{1}{2}}\left[{}^3R + g^{-1}(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}) \right] = -\, 2\kappa N^2 g^{\frac{1}{2}} T^{oo} \quad ; \tag{9a}$$

$$-\pi^{ij}{}_{|j} = \kappa N g^{\frac{1}{2}} (T^{oi} + N^i T^{oo}) \quad ; \tag{9b}$$

$$\partial_t g_{ij} = 2Ng^{-\frac{1}{2}}(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi) + N_{i|j} + N_{j|i} \quad ; \tag{9c}$$

$$\partial_t \pi^{ij} = - Ng^{\frac{1}{2}}({}^3R^{ij} - \tfrac{1}{2}g^{ij}\,{}^3R) + \tfrac{1}{2}Ng^{-\frac{1}{2}}g^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2)$$

$$-2Ng^{-\frac{1}{2}}(\pi^{im}\pi_m{}^j - \tfrac{1}{2}\pi\pi^{ij}) + g^{\frac{1}{2}}(N^{|ij} - g^{ij}N^{|m}{}_{|m})$$

$$+ (\pi^{ij}N^{|m})_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi}$$

$$+ \kappa Ng^{\frac{1}{2}}(T^{ij} - T^{oo}N^iN^j) \quad . \tag{9d}$$

I wish to remark on a few features of these equations. First, as we would expect, they do not contain L, since they are simply a linear combination of Eqs. (4). This means they can be used even if a Lagrangian is not available. Second, Eqs. (9) are instructive in understanding even the ADaM vacuum equations, since the particular linear combination used by ADaM is manifest. And third, the equations contain $T^{\mu\nu}$, the contravariant components of the four-dimensional stress-energy tensor. In many situations (e.g., scalar field) one might feel that the covariant components, $T_{\mu\nu}$, are physically more meaningful in a 3 + 1 split, in which case one can rewrite the equations as follows. Using the unit normal to the three-hypersurface, $n^\alpha = - N\,{}^4g^{o\alpha}$, one can define a "preferred" energy and momentum density for the matter:

$$\mathcal{E} \equiv n^\alpha n^\beta\,{}^4T_{\alpha\beta} \ , \tag{10a}$$

$$\mathcal{P}_i \equiv n^\alpha\,{}^4T_{\alpha i} \quad . \tag{10b}$$

Then the stress tensor in the hypersurface is

$$\mathcal{T}_{ik} = {}^4T_{ik} \quad . \tag{10c}$$

In terms of these quantities, the relevant parts of Eqs. (9) become

$$- 2\kappa N^2 g^{\frac{1}{2}} T^{oo} = - 2\kappa g^{\frac{1}{2}} \mathcal{E} \quad ; \qquad (11a)$$

$$\kappa N g^{\frac{1}{2}} (T^{oi} + N^i T^{oo}) = - \kappa g^{\frac{1}{2}} \mathcal{P}^i \quad ; \qquad (11b)$$

$$\kappa N g^{\frac{1}{2}} (T^{ij} - N^i N^j T^{oo}) = \kappa g^{\frac{1}{2}} (N \mathcal{T}^{ij} + N^i \mathcal{P}^j + N^j \mathcal{P}^i) \quad , \qquad (11c)$$

where all indices on $\mathcal{P}$ and $\mathcal{T}$ are raised by the three-dimensional metric.

Steps toward a full canonical theory could well begin here. One method would be to specify in advance the motion of the matter in terms of the metric tensor (e.g., homogeneous cosmology), and then to solve the constraint Eqs. (9a,b) by analogy with vacuum ADaM. A more general approach must include a canonical formulation for the fields present in spacetime. In any case, the basic gravitational constraints and dynamical equations will be Eqs. (9).

REFERENCES

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5. ibid., § 94.