On the existence of ergoregions in rotating stars

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Summary. Very compact rotating stars in general relativity can in principle contain regions, called ergoregions, in which all trajectories must rotate in the direction of the star's rotation. These regions are known to create an instability in the star, and it is therefore of physical interest to know whether realistic stars can contain ergoregions. We develop an approximation scheme which enables us to study ergoregions with a minimum of numerical effort. The method is to calculate the structure of a non-rotating star and then to integrate the equation for the 'dragging of inertial frames' in the slow-rotation approximation. When applied to uniform density stars, the results agree with the more accurate calculations of Butterworth & Ipser to within a few per cent. We apply the method to the more realistic degenerate-neutron equation of state (Harrison–Wakano–Wheeler), for which no other ergoregion calculations have been made, and conclude that no realistic neutron star, even with strong differential rotation, can develop an ergoregion. The ergoregion instability is therefore only likely to have astrophysical importance if stars significantly more compact than ordinary neutron stars exist.

1 Introduction

In general relativity the rotation of a star 'drags' the orbits of particles and photons; for example, the period of a circular, equatorial orbit is shorter in the prograde direction than in the retrograde direction, as measured by an observer far from the star. This effect is known, somewhat loosely, as the dragging of inertial frames. Collapse of a rotating star causes both the star's angular velocity and the dragging to increase. If the collapse proceeds far enough, dragging can, in principle, become so great that in some regions all orbits would be forced to co-rotate with the star. This region is called an ergoregion (ER). Friedman (1975, private communication) has argued that a star without a horizon but having an ER will be unstable to spontaneous emission of electromagnetic and gravitational waves. This would presumably have the effect of carrying away both energy and angular momentum from the star, spinning it down until the ER disappears. If ERs are common in rapidly

rotating compact objects, such as newly formed pulsars, then the instability could have a significant effect on the initial rotation period.

Unfortunately, little is known about what kinds of stars may have ERs: the only accurate results are those of Butterworth & Ipser (1976 — discussed below), which require elaborate and expensive numerical calculations. The purpose of this paper is to present an approximate method of determining the existence and structure of an ER, which is easily to apply and gives remarkably accurate answers. We have examined rigidly rotating stars in some detail, and we find that no such star with a realistic equation of state is likely to develop an ER. We also conjecture that differential rotation is unlikely to be much more effective, but we reserve a detailed examination of differential rotation for the future.

In addition, we have calculated the growth rate of Friedman's ER instability for the case of a massless scalar field. The results are summarized below but the details will appear in a separate paper (Comins & Schutz 1977).

2 Outline of the method

Butterworth & Ipser (1976) have published accurate numerical calculations of the structure of realistic, rigidly rotating, uniform density stars. In sufficiently compact stars they find that ERs form at an angular velocity below that which would cause shedding of matter from the star's equator. These landmark calculations yield a wealth of information about the stars, but there is a quicker way of obtaining information just about the ER, which results from the following simple observation. Most sequences of stars of increasing angular momentum terminate where the angular momentum so dominates the structure of the star that it causes shedding of mass from the equator. If an ER develops well before shedding, then the structure of the star in which it develops is not rotation-dominated: the star is 'slowly' rotating in a dynamical sense. We can therefore restrict our considerations to the slow-rotation approximation developed by Hartle (1967), Hartle & Sharp (1967) and Hartle & Thorne (1968). We shall find that making the additional restriction of considering only very compact stars reduces the problem to one of solving (a) the structure of a non-rotating star, and (b) a single ordinary differential equation for the ER. Our notation is that of Misner, Thorne & Wheeler (1973, hereafter MTW).

3 Structure of the ergoregion

The ER may be defined as the region in which no time-like world line can remain at rest relative to an observer at infinity. This notion of 'at rest' is definable in general relativity only in stationary spacetimes, that is, spacetimes in which a (Killing) vector field ξ_t exists which moves each hypersurface forward in time invariantly. To remain at rest is to follow a world line tangent to ξ_t. This vector field is time-like at infinity; inside the ER it must be space-like, so that nothing can remain at rest. The boundary of the ER is therefore the surface on which ξ_t is null:

$$\xi_t \cdot \xi_t = g_{00} = 0.$$

A non-rotating star has the metric (MTW)

$$ds^2 = - \exp\left[2\Phi(r)\right] dt^2 + \exp\left[2\Lambda(r)\right] dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2).$$

Slow rotation, with angular velocity Ω, causes corrections (Hartle & Sharp 1967) to $g_{00}, g_{tt}, g_{\theta\theta}$ and $g_{\phi\phi}$ that are of order Ω^2 and adds an entirely new metric term of order Ω

$$g_{0\phi} = -\omega g_{\phi\phi}.$$

This equation defines ω, the angular velocity of frame dragging. Since ω is the only term linear in Ω, it obeys a differential equation decoupled from all other metric corrections. If Ω is a constant (rigid rotation) this becomes (Hartle 1967) the ordinary differential equation

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j\frac{d\varpi}{dr}\right) + \frac{4}{r}\frac{dj}{dr}\varpi = 0, \tag{1a}$$

where

$$\varpi \equiv \Omega - \omega, \tag{1b}$$

$$j(r) = \exp\left[-\Phi(r) - \Lambda(r)\right]. \tag{1c}$$

We are interested in g_{00}. Including correction terms, it is

$$g_{00} = -\exp\left[2\Phi(r)\right]\left[1 + h(r,\theta)\right] + \omega^2 r^2 \sin^2\theta.$$

Inspection of the equations governing h (Hartle 1967) shows that h and $\omega^2 r^2$ are of the same order. This permits our second approximation: we neglect $h(r,\theta)$ in g_{00}. This is not strictly a consistent approximation, but it can be expected to hold in very compact stars, where $\exp(2\Phi) \ll 1$. In such stars, a small ω^2 can change the sign of g_{00}, while h would have to be of order 1 to have any effect. (This justification was suggested to us by K. S. Thorne.) Since ERs only form in compact stars, we can therefore take the boundary of the ER to be given by the solution to

$$0 = -\exp\left[2\Phi(r)\right] + \omega^2(r)\,r^2\sin^2\theta.$$

If a solution exists it is topologically a toroid. Its boundaries in the equatorial plane satisfy

$$r\omega(r) = \exp\left[\Phi(r)\right]. \tag{2}$$

To find these boundaries we need only solve for $\Phi(r)$ of the non-rotating star and for ω via equation (1). Integration of equation (1) begins at the origin with $d\varpi/dr = 0$ and ϖ finite but arbitrary. The exterior solution for ω is $2J/r^3$, where J is the angular momentum of the star (MTW). Therefore the exterior solution for ϖ is $\varpi = \Omega(1 - 2I/r^3)$, where I is interpreted as the moment of inertia of the star. Both ϖ and $d\varpi/dr$ must be continuous across the surface of the star, and these two conditions uniquely determine ϖ/Ω at the centre of the star and I. Since equation (1) is homogeneous, the angular velocity Ω is an arbitrary scale factor, and so we automatically obtain the entire sequence of slowly rotating stars. This is the same procedure as that used by Chandrasekhar & Miller (1974).

4 Results for uniform density stars

A non-rotating uniform density model is determined by two numbers, which we take to be the mass M and radius R_* (*cf.* MTW). A rigidly rotating star is further characterized by its angular momentum J, where $J = I\Omega$. For three sequences parameterized by J, the boundaries of the ER in the equatorial plane are plotted in Fig. 1. Each curve refers to a different rotating model, labelled by $2M/R_*$. The overall scale given by M has, of course, been factored out. The two points on each curve at a given value of J/M^2 are the radii of the ER boundary. The minimum J for each curve gives the point at which the ER first forms. The curves terminate at some maximum J, where mass-shedding occurs for each sequence. In our approximation, mass-shedding occurs when the centrifugal force on the equator exceeds the gravitational force of the non-rotating star there. Observe that the sizes
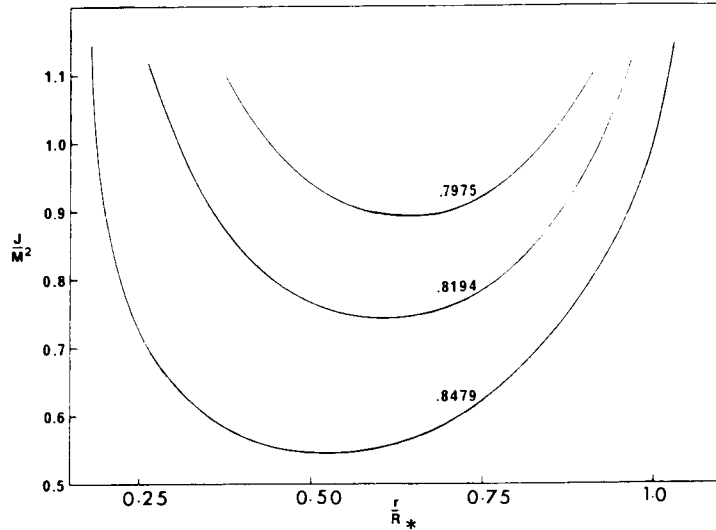
Figure 1. Size of the ergoregion (ER) for uniform density, rigidly rotating stars. The vertical axis is the angular momentum of the star, in units of the square of its mass. (This ratio is dimensionless in units where $c = G = 1$.) The horizontal axis gives the radii of the boundaries of the ER in the equatorial plane in units of the star's radius. Each curve refers to a different sequence of stars, labelled by the value of $2M/R_*$ for the non-rotating member. Curves terminate at the value of J/M^2 at which shedding occurs at the equator of the star. Each curve has a minimum, below which there is no ER. To find the size of an ER for a certain value of J/M^2, draw a horizontal line at that value; its intersections with the curve relating to the sequence of interest give the radii of the boundaries.

of the ER in stars of low $2M/R_*$ are more strongly limited by mass shedding than those for compact stars. Shedding sets a lower limit on $2M/R_*$ of 0.77. Stars less compact than this do not form ER before shedding. For a star of one solar mass this corresponds to a density of $1.4 \times 10^{16} \, \text{g/cm}^3$, which is larger than one expects to find in neutron stars.

In Table 1 we list the characteristics of stars having incipient ERs and compare them with the more accurate results of Butterworth & Ipser (1976). Our results are within a few per cent of theirs. The moments of inertia of these relativistic stars are displayed in Fig. 2(a), where the dimensionless $I/(2/5 \, MR_*^2)$ is plotted against $2M/R_*$. These agree well with the results of Chandrasekhar & Miller (1974). The angular velocity of dragging, ω, for a typical compact star is depicted in Fig. 2(b). The good agreement of our approximate results with the calculations of Butterworth & Ipser encourage us to apply them to the more realistic equation of state discussed below, for which more accurate calculations do not exist to date.

Table 1. Comparison of present approximate results with those of Butterworth & Ipser (1976). Comparison of results for the sequences of uniform-density, rigidly rotating stars described in Fig. 1. Columns (1)–(3) describe the present calculations: (1) the value of $2M/R_*$ for the sequence; (2) the radius at which an ER first forms along the sequence, in units of the star's radius R_*; and (3) the angular momentum of the star in which the ER first forms, in units of the square of the star's mass. Column (4) gives the Butterworth–Ipser value of this angular momentum and (5) the percentage difference between (3) and (4).

$2M/R_*$	r/R_*	J/M^2_{crit}	$J/M^2_{\text{crit B\&I}}$	Percentage difference
0.7975	0.642	0.89	0.83	6.7
0.8194	0.600	0.74	0.73	1.8
0.8479	0.520	0.54	0.57	5.6
0.8556	0.486	0.48	0.49	2.1
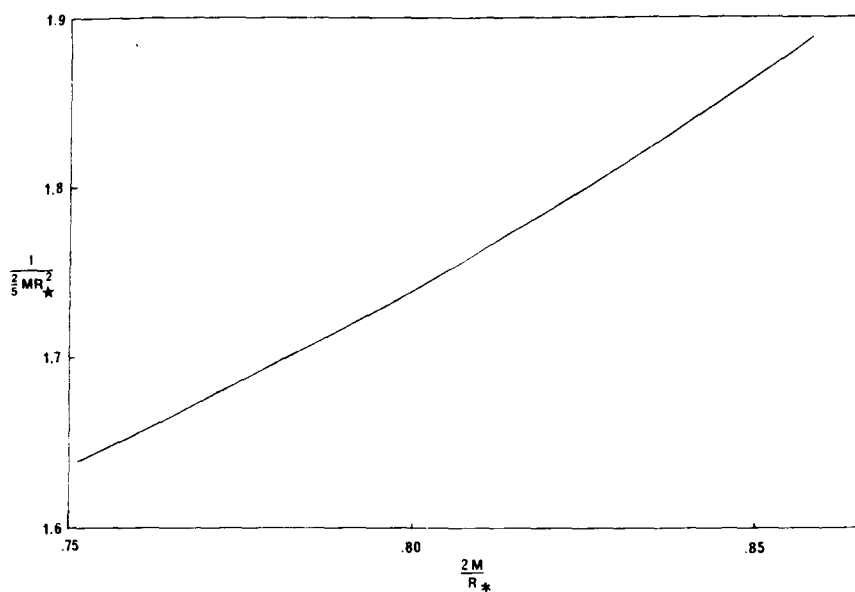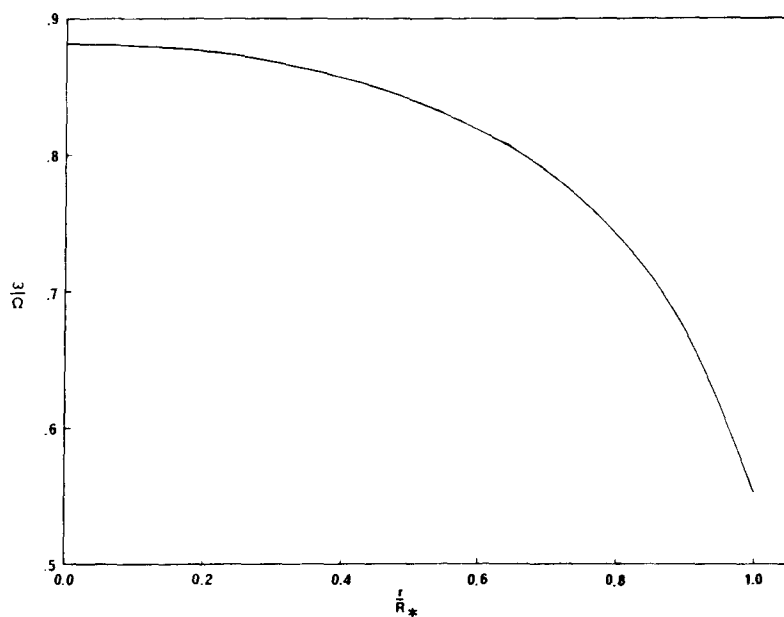
Figure 2. (a) The moment of inertia I versus $2M/R_*$ for uniform-density, rigidly rotating stars. I is normalized by the (non-relativistic) moment of inertia of a sphere $\frac{2}{5}MR_*^2$. (b) The angular velocity of the dragging of inertial frames (in units of the star's angular velocity), ω/Ω, plotted as a function of radius (in units of the star's radius, for a uniform density, rigidly rotating star). The star has $2M/R_* = 0.7975$.

5 Results for the Harrison, Wakano & Wheeler (HWW) equation of state

This equation of state (Harrison *et al.* 1965, hereafter HTWW), based on treating neutrons as a cold ideal Fermi gas, is more realistic than the uniform density case, but is still of course an idealization. It has the advantages of being analytic and easy to program; our results do not justify the use of a more realistic equation for rigidly rotating stars. The non-rotating star for this case is determined completely by the central number density. No extra scale factor such as the total mass exists since the equation of state already contains the mass of the neutron. Fig. 3 gives the angular momentum of the sequence of stars (parameterized by central mass
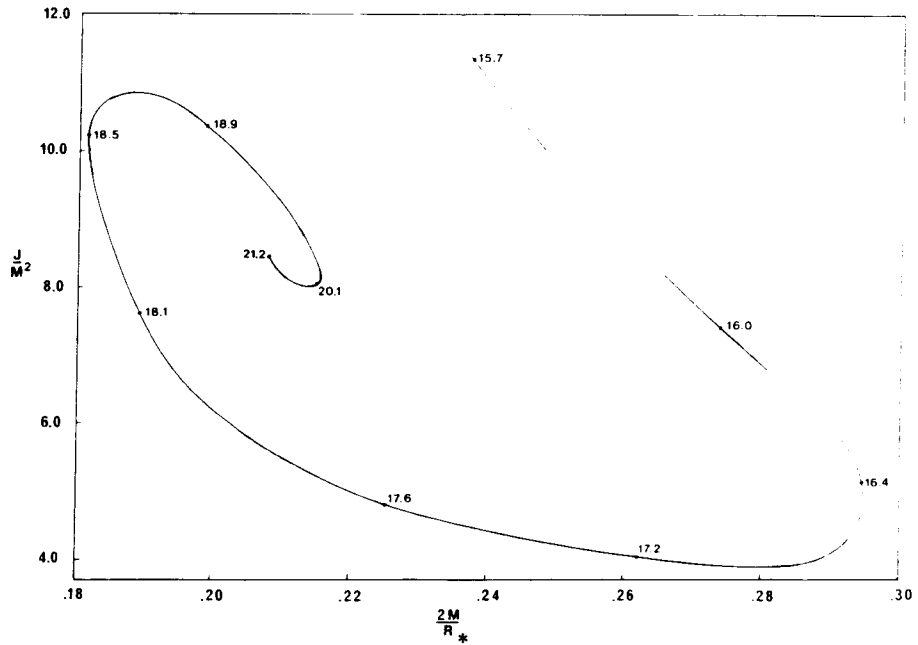
Figure 3. Angular momentum versus compactness (as measured by $2M/R_*$) for rigidly rotating HWW neutron stars which have an *incipient* ER. The curve is parameterized by $\log_{10} \rho_c$, where ρ_c is the central density of the star in $g\,cm^{-3}$. To compare these stars with the uniform-density stars, note that every point here corresponds to the *minimum* of a curve of the type drawn in Fig. 1. HWW stars form ERs at much higher values of J/M^2 than do uniform-density stars.

density) which have an *incipient* ER. All of these models, however, are *above* the shedding limit. We will discuss this in detail in a moment. A feature not present in the constant density case which appears for the HWW stars is a minimum angular momentum below which no star can have an ER. Notice how much larger J/M^2 must be in order for an ER to develop in these stars than in the uniform density stars. Although perhaps not unexpected this has the effect that all rigidly rotating HWW stars with ERs also shed matter at the equator. Fig. 4 shows ω versus r for a typical HWW star.

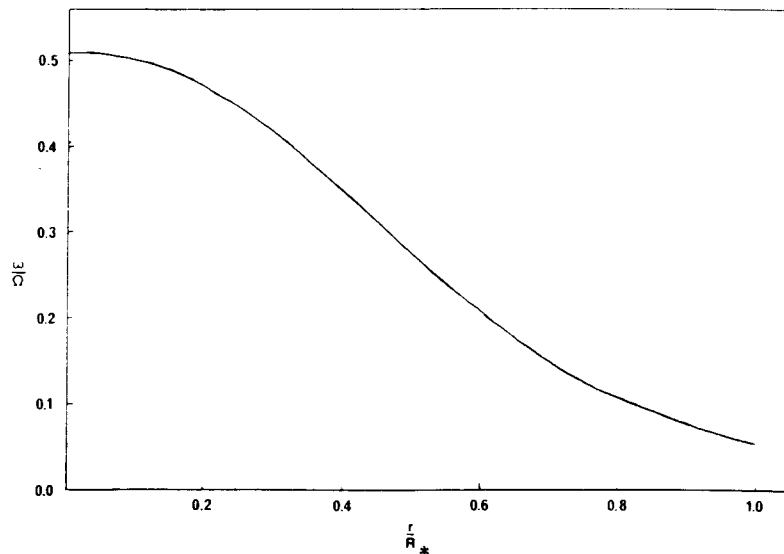**Figure 4.** Same as Fig. 2(b) for an HWW star of central density $\rho_c = 1.1 \times 10^{16}\,g\,cm^3$. This star has $2M/R_* = 0.274$.

One might hope to get around the shedding restriction by considering differentially rotating stars. The idea is that formation of the ER may be dominated by the angular momentum in the compact core; the shedding limit, on the other hand, is set by the diffuse outer layers, whose angular momentum may not contribute substantially to formation of the ER. For normal HWW stars, however, our calculation makes this seem like a false hope. We find that by the time the ER has formed in these stars (at least for $\rho_c < 2 \times 10^{17} \mathrm{g\,cm^{-3}}$), the angular velocity is so large that centrifugal force dominates gravity not only at the surface but practically all the way through the star, to well inside the core. Differential rotation, then, probably could not help in forming an ER in these stars. The inner region, which is already rotating too fast, would have to rotate even faster in order to maintain an ER while the outer region rotates slowly enough not to shed. In stars with central densities larger than $2 \times 10^{17} \mathrm{g/cm^3}$ there is a central region which has not reached its 'shedding limit' when the ER forms; differential rotation may be effective in forming ERs in these stars. Unfortunately these stars are almost certainly unstable: the non-rotating ones of these central densities have one unstable radial mode of pulsation (HTWW).

Thus it appears that realistic, rotating neutron stars will not have ERs. We must note that the HWW stars are less relativistic than the constant density stars and so the approximation used in g_{00} is not as good here as in the previous case. This suggests that the exact structure of the ER (Fig. 4) may not be given as accurately here as before, but it is unlikely to affect the qualitative conclusions that ERs are unlikely to form.

6 Growth rates of the ER instability for scalar fields

The timescale of the ER instability is examined in another paper (Comins & Schutz 1977). It has considerable interest as a problem in its own right, because this instability has a close similarity to other instabilities in general relativity. We quote here the major results. Attention is restricted to radiation of scalar waves on the background of a rotating star.

The scalar field is assumed to have an angular dependence of exp ($im\phi$). Friedman (1975, private communication) suggests that the instability mentioned in the Introduction sets in through the high m modes. In this limit the scalar wave equation is amenable to WKBJ analysis. We find that instability is indeed caused by an ER. In a given ER, all modes with m greater than some m_0 are unstable, and m_0 decreases as the size of the ER increases. The *e*-folding time τ for the unstable modes of the scalar field has the asymptotic form

$$\tau = \tau_0 \exp(\beta m), \quad \beta = O(1),$$

for large m, where τ_0 is a constant. The m dependence means that the instability is a weak one: we find that all unstable modes will have growth times much longer than the dynamic timescales of the star.

7 Astrophysical implications

It is clear from the calculation using the HWW equation of state there are no realistic rigidly rotating stars with ERs. There are two situations in which ERs could conceivably arise: ultra-stiff equations of state and ultra-dense equations of state. Equations of state for which $p > \rho$ ('ultra-stiff') have sometimes been suggested in order to permit spherical neutron stars of $6 M_\odot$ or more. While these are unrealistic to begin with, it is interesting to note that rotation should not raise the mass limit significantly, as it can for less-dense stars. The reason is that these stars ought to be so compact that they would easily form an ER at low angular velocities, creating the instability.

An 'ultra-dense' equation of state need not have $p > \rho$: it need only provide a stiff regime at densities higher than nuclear, in order that dense stars of moderate mass may be stable. These are sometimes called hyperon stars. If one can get central densities of 10^{17} or $10^{18}\,\mathrm{g\,cm^{-3}}$, our calculations indicate that ERs may form quite easily. (A stable 'ultradense' star based on quark confinement and a repulsive nuclear core is given, along with consequent limitations on neutron star masses, in Ipser, Kislinger & Morley (1975, unpublished).)

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