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Perturbations and stability of rotating stars – II. Properties of the eigenvectors and a variational principle

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Summary. General properties of the normal modes of linear pulsation of perfect-fluid rotating stars are considered in some detail. With a view toward numerical applications, the discussion is restricted to finite-dimensional versions of the problem. Because the eigenvalue problem is non-selfadjoint, both left- and right-eigenvectors and their Jordan chains are important. These are defined and their mutual orthogonality properties derived. It is then shown that stars which have the time–azimuth reflection symmetry $(t, \phi) \rightarrow (-t, -\phi)$ (which does not require axisymmetry) in their unperturbed state have the remarkable property that any right-eigenvector can be changed into the adjoint of the left-eigenvector for the same eigenvalue by the simple operation of complex conjugation and ϕ-reversal. This property permits a variational principle for the eigenfrequencies of all the normal modes to be formulated. Two versions are given: one is for the quadratic eigenvalue problem (the natural one, in which the eigenfrequency enters quadratically); the other is in 'phase space', where the problem is first-order in time and gives a linear eigenvalue equation. Possible uses of the variational principle are discussed briefly. Another consequence of the map between left- and right-eigenvectors is a test for Jordan chains, which are associated with the onset of dynamical instability.

1 Introduction

This is the second paper of a series which aims at a systematic treatment, both analytic and numerical, of the normal modes of linear pulsation of rotating, perfect-fluid stellar models. The first paper discussed the relation of normal modes to the initial-value problem for the perturbations (Dyson & Schutz 1979, Paper I) and showed that there is a sense in which the normal modes are complete: any perturbation can be expressed as a linear superposition of normal modes. This paper discusses detailed properties of the normal modes, some of which should considerably simplify the numerical effort involved in finding the modes. Some of the properties were discussed briefly in a recent publication (Schutz 1979), but they will be elaborated upon and extended here.

By contrast with the perturbation problem for non-rotating stars, the present problem is non-selfadjoint, with the result that many new features must be taken into account. From an analytic point of view, the most important new elements are the possibility of Jordan chains (incompleteness of eigenvectors), and the distinction between left- and right-eigenvectors, both types being important in the problem. The first section discusses the eigenvectors and their mutual orthogonality properties. After that, the principal result is derived: for stars having the reflection symmetry $(t, \phi) \rightarrow (-t, -\phi)$, which excludes meridional circulation, there exists a simple transformation which turns any right-eigenvector into a left-eigenvector for the same eigenvalue. One immediate consequence of this relation is a variational principle for the eigenfrequencies, which is presented in two different versions. Another consequence is a test for Jordan chains, and a proof that such chains are associated with the onset of dynamical instability along a sequence of models. A full discussion of the 'onset' of instability, however, requires a study of what happens to the eigenvalues when the operators in the problem are slightly changed, and this will be reserved for the third paper in this series (Schutz 1980).

The notation in this paper follows that of Paper I. The only difference is that here I shall use Dirac bra and ket notation for eigenvectors, because it permits precise labelling of vectors without cumbersome sub- and superscripts. The problem is approached using Lagrangian perturbation theory, which Chandrasekhar (1964) used to formulate his variational principle for non-radial modes of non-rotating stars. The Lagrangian perturbation equations for rotating stars were derived by Lynden-Bell & Ostriker (1967), who were only able to formulate a partially successful variational principle. More recently it was discovered that an ambiguity in the Lagrangian description of perturbations of moving fluids requires one to exercise some care in constructing stability criteria (Schutz & Sorkin 1977; Bardeen *et al.* 1977; Friedman & Schutz 1978a, b). But these considerations are more relevant to secular stability criteria than to dynamical ones. Moreover, as pointed out by Friedman & Schutz (1978b), the normal modes which are the subject of the present paper usually have no ambiguity at all. The Lagrangian description offers such strong advantages over the Eulerian one, particularly the symmetry* of the basic operators it involves, that it seems much more suited to the general treatment intended here.

2 Orthogonality properties of eigenvectors and chains

In the space H_2 introduced in Paper I ('initial-data space') the dynamical operator is the matrix

$$T = \begin{pmatrix} 0 & 1 \\ -C & -B \end{pmatrix}.$$

(The primes on B and C in Paper I will be omitted here.)

A right-eigenvector of T for the eigenvalue λ_n will be called $|\lambda_n; r_0\rangle\rangle$:

$$T|\lambda_n; r_0\rangle\rangle = \lambda_n|\lambda_n; r_0\rangle\rangle.$$

Similarly a left-eigenvector is $\langle\langle\lambda_n; l_0|$:

$$\langle\langle\lambda_n; l_0|T = \lambda_n\langle\langle\lambda_n; l_0|.$$

* For the sake of clarity I shall use the mathematicians' terms 'symmetric' and 'antisymmetric' in place of the physicists' 'Hermitian' and 'anti-Hermitian'. This is because it is important to distinguish between these concepts and full selfadjointness or anti-selfadjointness. (This is discussed below and in Paper I.) But I will depart from the strictness of Paper I and assume that the problem is a finite-dimensional one. This will always be true for numerical calculations.

Any eigenvalue λ_n has both a right- and left-eigenvector, since the condition for the existence of either is the vanishing of the determinant of $T - \lambda$. An eigenvalue will be said to be *non-degenerate* if it has only one right-eigenvector. We will label eigenvalues in such a way that each λ_n has only one eigenvector of each type, so that if there is degeneracy we still maintain separate names for the equal eigenvalues. Thus λ_n is degenerate if and only if $\lambda_n = \lambda_m$ for some $m \neq n$.

A right Jordan chain (a 'right-chain') of length p_n to an eigenvalue λ_n is a sequence of vectors

$$\{|\lambda_n; r_j\rangle\rangle, \quad j = 0, \ldots, p_n\}$$

such that $|\lambda_n; r_0\rangle\rangle$ is a right-eigenvector and

$$(T - \lambda_n)|\lambda_n; r_j\rangle\rangle = |\lambda_n; r_{j-1}\rangle\rangle, \quad j = 1, \ldots, p_n. \tag{2.1}$$

(See Hirsch & Smale (1974) for a discussion of Jordan chains in dynamical problems.) This can be made to apply for $j = 0$ as well as if we adopt the convention that $|\lambda_n; r_j\rangle\rangle = 0$ if $j < 0$. This convention will be convenient below. It follows from equation (2.1) that

$$(T - \lambda_n)^{j+1}|\lambda_n; r_j\rangle\rangle = 0, \quad j = 0, \ldots, p_n. \tag{2.2}$$

Similarly a left-chain of length p_n to λ_n is the sequence

$$\{\langle\langle\lambda_n; l_j|, \quad j = 0, \ldots, p_n\} \tag{2.3}$$

for which $\langle\langle\lambda_n; l_0|$ is a left-eigenvector and

$$\langle\langle\lambda_n; l_j|(T - \lambda_n) = \langle\langle\lambda_n; l_{j-1}|, \quad j = 1, \ldots, p_n.$$

The subspace spanned by the chain is called the *characteristic* subspace for that eigenvalue.

If λ_n has a right-chain of length p_n it has a left-chain of the same length, and vice versa. The set of all right-chains and right-eigenvectors is a basis for H_2:

$$[\{|\lambda_n; r_j\rangle\rangle, \quad j = 0, \ldots, p_n\}, \quad \forall n]$$

is a basis. Similarly, the left-chains and left-eigenvectors form a basis for H_2^*, the dual or adjoint space to H_2 (which can, of course, be identified with H_2). The two bases are in fact dual to one another in the sense that we can choose the eigenvectors in such a way that (even if $\lambda_n = \lambda_m$):

$$n \neq m: \langle\langle\lambda_n; l_j|\lambda_m; r_k\rangle\rangle = 0 \quad \forall j, k; \tag{2.4}$$

$$n = m: \langle\langle\lambda_n; l_j|\lambda_n; r_k\rangle\rangle = 0 \quad \text{if } j + k \neq p_n. \tag{2.5}$$

These two relations imply that, say, $\langle\langle\lambda_n; l_j|$ is orthogonal to all of the elements of the basis of H_2 except for one, $|\lambda_n; r_{p_n - j}\rangle\rangle$. It cannot be orthogonal to this as well, for then it would itself be identically zero. So we can choose a *relative normalization* of the two bases by setting $\langle\langle\lambda_n; l_{p_n}|\lambda_n; r_0\rangle\rangle = 1$, which then leads to a compact statement of equations (2.4) and (2.5):

$$\langle\langle\lambda_n; l_j|\lambda_m; r_k\rangle\rangle = \delta_{nm}\delta_{j+k, p_n}. \tag{2.6}$$

We will not usually demand this normalization. Notice that equation (2.5) means that the only non-zero products are between complementary members of the two chains: the end of one with the beginning of the other. It follows that λ_n has a non-zero chain if and only if its left- and right-eigenvectors are orthogonal:

$$\langle\langle\lambda_n; l_0|\lambda_n; r_0\rangle\rangle = 0 \Leftrightarrow \lambda_n \text{ has a non-zero chain.} \tag{2.7}$$

In cases where a given eigenvalue has only one right-eigenvector, equation (2.7) is in principle a test for whether one should also look for a chain. If there is degeneracy (two or more eigenvectors for a given eigenvalue) and one does not know *a priori* how to identify the appropriate basis for the eigenvectors for which equation (2.6) holds, then the condition for a chain to a particular right-eigenvector to exist is that that eigenvector be orthogonal to *all* the left-eigenvectors for that eigenvalue. If there are no chains then *any* basis for the right-eigenvectors will define a basis for the left-eigenvectors for which equation (2.6) holds.

Note that they are the *only* orthogonality relations satisfied in general: the right-eigenvectors are *not* orthogonal to one another except if T is selfadjoint, because there the right- and left-eigenvectors are simply adjoints of one another.

Until now, T could have been an arbitrary operator. Knowing its form, we can derive the relationship between eigenvectors in H_2 and those in H, plus the appropriate orthogonality relations in H.

We write the two components of any chain vector in H_2 as

$$|\lambda_n; r_k\rangle\rangle = \begin{pmatrix} |\lambda_n; r_k^1\rangle \\ |\lambda_n; r_k^2\rangle \end{pmatrix} \tag{2.8}$$

(2.9)

and similarly in H_2^*:

$$\langle\langle\lambda_n; l_k| = (\langle\lambda_n; l_k^1|, \quad \langle\lambda_n; l_k^2|).$$

It is easy to show that equation (2.1) implies

$$L(\lambda_n)|\lambda_n; r_0^1\rangle = 0, \tag{2.10a}$$

$$|\lambda_n; r_0^2\rangle = \lambda_n|\lambda_n; r_0^1\rangle \tag{2.10b}$$

for eigenvectors and, for chains (with $k \geqslant 1$ and $|\lambda_n; r_{-1}^1\rangle \equiv 0$),

$$L(\lambda_n)|\lambda_n; r_k^1\rangle + (2\lambda_n + B)|\lambda_n; r_{k-1}^1\rangle + |\lambda_n; r_{k-2}^1\rangle = 0, \tag{2.11a}$$

$$|\lambda_n; r_k^2\rangle = \lambda_n|\lambda_n; r_k^1\rangle + |\lambda_n; r_{k-1}^1\rangle. \tag{2.11b}$$

Two things are notable: first, as one should expect, one can solve the problem in H by dealing with the $|\lambda_n; r_k^1\rangle$ vectors alone, afterwards deriving the second components; and second, the chain relation is far more complicated here than in H_2. Equation (2.11) enables us to replace equation (2.8) by

$$|\lambda_n; r_k\rangle\rangle = \begin{pmatrix} |\lambda_n; r_k^1\rangle \\ \lambda_n|\lambda_n; r_k^1\rangle + |\lambda_n; r_{k-1}^1\rangle \end{pmatrix} \tag{2.12}$$

with the convention that $|\lambda_n; r_{-1}\rangle = 0$. Notice that the chain $\{|\lambda_n; r_k^1\rangle\}$ of $L(\lambda_n)$ has the same length p_n as the corresponding chain of T.

Similar results obtain for left-chains. They can be summarized by

$$\langle\langle\lambda_n; l_k| = (\langle\lambda_n; l_k^2|(\lambda_n + B) + \langle\lambda_n; l_{k-1}^2|, \quad \langle\lambda_n; l_k^2|) \tag{2.13}$$

and

$$\langle\lambda_n; l_k^2|L(\lambda_n) + \langle\lambda_n; l_{k-1}^2|(2\lambda_n + B) + \langle\lambda_n; l_{k-2}^2| = 0, \tag{2.14}$$

again with the convention that $\langle\lambda_n; l_k|$ vanishes for $k < 0$. Notice that here it is the *second* component of $\langle\langle\lambda_n; l_k|$ which forms the chain in H and generates the correspondence with H_2.

Since the set $\{|\lambda_n; r_k^1\rangle, \forall n, k\}$ generates the set $\{|\lambda_n; r_k\rangle\rangle\}$ and the set $\{\langle\lambda_n; l_k^2|\}$ generates $\{\langle\langle\lambda_n; l_k|\}$ it is possible to solve the problem in H without reference to H_2. From now on we shall denote by $|\lambda_n; r_k\rangle$ the right-chains of $L(\lambda)$ in H and by $\langle\lambda_n; l_k|$ the left-chains; that is

$$|\lambda_n; r_k\rangle \equiv |\lambda_n; r_k^1\rangle, \qquad \langle\lambda_n; l_k| \equiv \langle\lambda_n; l_k^2|.$$

These vectors have a more complicated set of orthogonality relations in H. If $m \neq n$ we find

$$\langle\lambda_n; l_j|(\lambda_n + \lambda_m + B)|\lambda_m; r_k\rangle + \langle\lambda_n; l_{j-1}|\lambda_m; r_k\rangle + \langle\lambda_n; l_j|\lambda_m; r_{k-1}\rangle = 0, \tag{2.15}$$

again with the convention that $|\lambda_n; r_k\rangle$ and $\langle\lambda_n; l_k|$ vanish for negative k. The more tractable case is $n = m$:

$$\langle\lambda_n; l_j|(2\lambda_n + B)|\lambda_n; r_k\rangle + \langle\lambda_n; l_{j-1}|\lambda_n; r_k\rangle + \langle\lambda_n; l_j|\lambda_n; r_{k-1}\rangle = 0, \qquad j + k \neq p_n. \tag{2.16}$$

From this one deduces (setting $j = k = 0$)

$$\langle\lambda_n; l_0|(2\lambda_n + B)|\lambda_n; r_0\rangle = 0 \Leftrightarrow \lambda_n \quad \text{has a Jordan chain,} \tag{2.17}$$

which is the analogue in H of equation (2.7). (Again the same remarks about degeneracy apply here.)

3 Consequences of symmetry relations

All the formulae of Section 2 are valid for any perturbation problem which is second-order in time-derivatives, since no properties of the operators B and C were used. From now on we shall restrict attention to perturbations of a perfect fluid star*. In Section 3.1 we will look at the results that follow merely because C is selfadjoint and B anti-selfadjoint. In Section 3.2 we will make the additional, very fruitful assumption that the star is symmetric under the reflection $(t, \phi) \to (-t, -\phi)$.

3.1 SELFADJOINT OPERATORS AND THE EXISTENCE OF JORDAN CHAINS

If the star is axisymmetric without meridional circulation, Paper I showed that C is self-adjoint and B anti-selfadjoint. It is likely that this is also true of non-axisymmetric stars with arbitrary (time-dependent) velocity fields, but this has not been proved. In any case it should always be possible to construct finite-dimensional approximations for numerical work that do have these symmetries. The symmetries allow us to show that if λ_n is an eigenvalue of $L(\lambda)$ with right-eigenvector $|\lambda_n; r_0\rangle$ then $-\bar{\lambda}_n$ is also an eigenvalue whose left-eigenvector is $\langle\lambda_n; r_0|$, the adjoint of $|\lambda_n; r_0\rangle$. (A bar denotes complex conjugation, and an * denotes the adjoint transformation.)

By assumption we have

$$B^* = -B, \qquad C^* = C, \tag{3.1}$$

from which follows

$$[L(\lambda)]^* = (\lambda^2 + \lambda B + C)^* = L(-\bar{\lambda}). \tag{3.2}$$

Then the adjoint of

$$L(\lambda_n)|\lambda_n; r_0\rangle = 0 \tag{3.3a}$$

* Our assumption of a perfect fluid can be relaxed to, say, a perfectly elastic body. What matters is that the equations are derivable from a real Lagrangian, which means the equations conserve energy. See Friedman & Schutz (1975) for a discussion of this point.

is

$$\langle \lambda_n; r_0 | L(-\bar{\lambda}_n) = 0 \tag{3.3b}$$

which proves the proposition. If λ_n is imaginary (purely oscillatory mode), then it follows that $|\lambda_n; r_0\rangle$ is simultaneously a right- and (in adjoint form) a left-eigenvector of λ_n. More generally one can show from equation (2.14) that right-chains for λ_n generate left ones for $-\bar{\lambda}_n$:

$$\langle -\bar{\lambda}_n; l_k | = (-1)^k \langle \lambda_n; r_k |. \tag{3.4}$$

By taking the inner product of equation (3.3a) with $\langle \lambda_n; r_0 |$ we get a quadratic equation for λ_n, which I shall call the *Hermitian eigenfrequency equation*:

$$\begin{aligned} 0 &= \langle \lambda_n; r_0 | L(\lambda_n) | \lambda_n; r_0 \rangle \\ &= \lambda_n^2 \langle \lambda_n; r_0 | \lambda_n; r_0 \rangle + \lambda_n \langle \lambda_n; r_0 | B | \lambda_n; r_0 \rangle + \langle \lambda_n; r_0 | C | \lambda_n; r_0 \rangle. \end{aligned} \tag{3.5}$$

Its two roots can be written

$$\langle \lambda_n; r_0 | (2\lambda_n + B) | \lambda_n; r_0 \rangle = \pm \left[\langle \lambda_n; r_0 | B | \lambda_n; r_0 \rangle^2 - 4 \langle \lambda_n; r_0 | \lambda_n; r_0 \rangle \langle \lambda_n; r_0 | C | \lambda_n; r_0 \rangle \right]^{1/2}. \tag{3.6}$$

By comparing this with equation (2.17) we see that Jordan chains occur on the imaginary axis (where $\langle \lambda_n; r_0 | = \langle \lambda_n; l_0 |$) only if the Hermitian eigenfrequency equation has a double root, and can only occur for $|\lambda_n| \leqslant ||B||/2$, where $||B||$ is the norm of B. This is the same restriction found in Paper I on the imaginary part of a complex eigenfrequency. This and the double root suggest a connection between Jordan chains and marginal stability. Let us look more closely at this.

Of the two roots for λ_n implicit in equation (3.6), one must be the true eigenfrequency. If this is complex then the other root is the eigenfrequency $-\bar{\lambda}_n$. But if λ_n is pure imaginary then there is apparently no need for the other root to be an eigenfrequency at all, and it certainly is not one with eigenvector $|\lambda_n; r_0\rangle$ unless that is also an eigenvector simultaneously of B and C (a very unlikely circumstance). What significance is there, then, in the coincidence of the two roots in equation (3.6)? Suppose we have a *sequence* of unperturbed stars which is smooth enough for the eigenfrequencies to be continuous functions of the sequence's parameter. If we move along the sequence in a direction which makes a particular mode change from unstable to stable, then before the change both roots of equation (3.6) represent eigenfrequencies, and these roots coincide (by hypothesis) at the changeover point. In the absence of degeneracy, equation (2.17) guarantees a Jordan chain here. Therefore, *whenever instability sets in along such a sequence, the Hermitian eigenfrequency equation will have a double root and the marginal mode will have a Jordan chain.*

In the next paper in this series (Schutz 1979b) the onset of instability will be studied in more detail. The association between instability points and chains will be established by a different method, and it will be demonstrated that chains do not exist along the stable part of a sequence (except at its end) except under very special conditions. Moreover, the marginally stable chain will usually be of length 1 ($p_n = 1$), as can easily be understood from the fact that generally only *two* modes will merge to become unstable. Regarding the possible existence of chains to unstable eigenvalues, we have no information.

It should also be noted that the discussion preceding the previous paragraph is qualitatively the same for non-rotating stars, where $B = 0$. Although we are not accustomed to considering Jordan chains in the selfadjoint problem of a non-rotating star, they do exist in H_2 for a zero-frequency (marginally stable) mode, where they provide a good illustration of the merging of eigenvectors to form a chain. But in H the chain condition (2.11a) is trivial for chains of length 1 and $\lambda_n = B = 0$.

3.2 REFLECTION-SYMMETRIC STARS AND THE SYMMETRY OPERATORS

If the star is invariant under the reflection $(t, \phi) \to (-t, -\phi)$ then many more results follow. Because the equations of motion are real, any solution to equation (1.1)

$$\xi(t, r, \theta, \phi) = \{\xi^r(t, r, \theta, \phi), \quad \xi^\theta(t, r, \theta, \phi), \quad \xi^\phi(t, r, \theta, \phi)\}$$

can be mapped into another solution by combining complex conjugation with (t, ϕ)-reflection:

$$S\xi = \{\bar{\xi}^r(-t, r, \theta, -\phi), \quad \bar{\xi}^\theta(-t, r, \theta, -\phi), \quad -\bar{\xi}^\phi(-t, r, \theta, -\phi)\}. \tag{3.7}$$

This defines the symmetry operator S. Notice that a star may have this symmetry property without being axisymmetric: the Dedekind ellipsoids are an example (*cf.* Chandrasekhar 1969), provided the plane $\phi = 0$ contains the axis of rotation and the semi-major or semi-minor axis of the equatorial section.

If ξ is a mode with time-dependence $\exp(\lambda t)$ then $S\xi$ has time-dependence $\exp(-\bar{\lambda}t)$. We already observed that the symmetries of B and C imply that if λ is an eigenfrequency so is $-\bar{\lambda}$. Now we see that the operator S enables us to find one of the eigenfunctions from the other. If the star is axisymmetric as well, then the solutions may be Fourier-analysed in ϕ, each component of $\exp(im\phi)$ being independent of those for other m's. If ξ belongs to a particular m, so does $S\xi$, so that S has a natural restriction to fixed m.

In any case, the eigenvectors will be elements of some Hilbert space H, and from now on we shall use the symbol S to denote the restriction of equation (3.7) to H. There it consists of ϕ-reflection and complex conjugation.

The operator S has two important properties: it is antilinear, which means that

$$S(\alpha\xi) = \bar{\alpha}S\xi \tag{3.8}$$

for any complex function α, and it is its own inverse,

$$S^2 = 1. \tag{3.9}$$

Because it is antilinear an adjoint operator cannot be defined, so we shall be careful to apply S to vectors *before* taking their adjoints. The adjoint of $S|\lambda_n; r_j\rangle$ will be denoted by $\langle S; \lambda_n; r_j|$.

How does S affect the operators B and C? In H, consider the equation for any solution $\xi(t, r, \theta, \phi)$ at, say, $t = 0$:

$$\xi_{tt} + B\xi_t + C\xi = 0.$$

Because $S^2 = 1$ we can apply S to this and get

$$S(\xi_{tt}) + SBS\,S(\xi_t) + SCS\,S\xi = 0.$$

Since S contains a time-reversal this becomes

$$(S\xi)_{tt} - SBS(S\xi)_t + SCS(S\xi) = 0. \tag{3.10}$$

But $S\xi$ is also a solution:

$$(S\xi)_{tt} + B(S\xi)_t + C(S\xi) = 0. \tag{3.11}$$

Since at $t = 0$ the functions ξ_t and ξ are arbitrary and independent, the only way equations (3.10) and (3.11) can both hold is if

$$SBS = -B, \quad SCS = C. \tag{3.12}$$

An explicit demonstration of this symmetry for B was given in Schutz (1979) (but beware

that the notation is slightly different, in that the present B is $-i$ times the B of the previous paper).

From equations (3.8) and (3.12) it follows immediately that if

$$L(\lambda_n)|\lambda_n, r_0\rangle = 0$$

then

$$L(-\bar{\lambda}_n)S|\lambda_n; r_0\rangle = 0,$$

i.e. that

$$|-\bar{\lambda}_n; r_0\rangle = S|\lambda_n; r_0\rangle. \tag{3.13}$$

So as previously remarked S maps eigenvectors of λ_n into those of $-\bar{\lambda}_n$. But when combined with equation (3.3) we get a very useful result:

$$\langle\lambda_n; l_0| = \langle S; \lambda_n; r_0|, \tag{3.14}$$

that is, the adjoint of S times a right-eigenvector of $L(\lambda)$ is the left-eigenvector for the same eigenvalue. This is a remarkable result, that we can find the left-eigenvector from the right one without knowing any others. For a general matrix the left-eigenvectors are simply the basis dual to the right ones, and as such a change in *any* right-eigenvector generally changes *all* the left-eigenvectors. But our problem has sufficient symmetry to reduce the problem to a 1–1 pairing of the eigenvectors, independently of the others.

For members of chains in H the corresponding relations are

$$|-\bar{\lambda}_n, r_k\rangle = (-1)^k S|\lambda_n; r_k\rangle \tag{3.15}$$

and

$$\langle\lambda_n, l_k| = \langle S; \lambda_n; r_k|. \tag{3.16}$$

This immediately enables us to write down a useful criterion for the existence or not of a Jordan chain to a particular eigenvalue λ_n. From equations (2.17) and (3.14) we get, for the non-degenerate case,

$$\langle S; \lambda_n; r_0|(2\lambda_n I + B)|\lambda_n, r_0\rangle = 0 \Leftrightarrow \text{Jordan chain}. \tag{3.17}$$

In concrete terms for the case of an axisymmetric star and perturbations with azimuthal eigenvalue m, where $\boldsymbol{\xi} = \boldsymbol{\xi}_m(r, \theta)\exp(im\phi)$ and

$$B\boldsymbol{\xi} = 2\Omega(im\boldsymbol{\xi} + \mathbf{e}_z \times \boldsymbol{\xi}) \tag{3.18}$$

($\mathbf{e}_z$ being the unit vector along the axis of rotation), this criterion becomes

$$0 = \int\int \rho\varpi d\varpi dz\,\{(\lambda_n + im\Omega)[(\xi_m^{\varpi})^2 + (\xi_m^z)^2 - (\xi_m^{\phi})^2] + 2\Omega\xi_m^{\varpi}\xi_m^{\phi}\} \tag{3.19}$$

where the components are on the orthonormal basis of a cylindrical polar coordinate system. Notice that, although $\boldsymbol{\xi}_m$ is complex, there are no complex conjugations in equation (3.19) because the operator S and the adjoint operation in equation (3.14) both involve conjugation. So equation (3.19) represents a complex equation that a non-degenerate eigenfunction satisfies if and only if it is associated with a chain. This should be a convenient test to use numerically. In the degenerate case, $|\lambda_n; r_0\rangle$ must be orthogonal, in the sense of equation (3.17), to S applied to *each* right-eigenvector in order for it to have a chain.

3.3 S AS A CONJUGATION

Because S is so useful, it is worthwhile looking at it in more detail. The first useful property it has is, for any $|\xi\rangle$ and $|\eta\rangle$ in H,

$$\overline{\langle\xi|\eta\rangle} = \langle S;\xi|S|\eta\rangle, \tag{3.20}$$

where $\langle S;\xi|$ is the adjoint of $S|\xi\rangle$. This was proved in Schutz (1979) for the axisymmetric star. In the general case, we refer back to equation (3.7). The elements of H are vector fields, $\boldsymbol{\xi}(r,\theta,\phi)$. The inner product $\langle\xi|\eta\rangle$ involves

$$\bar{\boldsymbol{\xi}}\cdot\boldsymbol{\eta}(r,\theta,\phi) = \bar{\xi}^r(r,\theta,\phi)\eta^r(r,\theta,\phi) + \bar{\xi}^\theta(r,\theta,\phi)\eta^\theta(r,\theta,\phi) + \bar{\xi}^\phi(r,\theta,\phi)\eta^\phi(r,\theta,\phi) \tag{3.21}$$

again with components referred to an orthonormal basis. The right-hand side of equation (3.20) involves

$$\begin{aligned}\overline{(S\boldsymbol{\xi})}\cdot(S\boldsymbol{\eta}) &= \xi^r(r,\theta,-\phi)\bar{\eta}^r(r,\theta,-\phi) + \xi^\theta(r,\theta,-\phi)\bar{\eta}^\theta(r,\theta,-\phi) + [-\xi^\phi(r,\theta,-\phi)]\\ &\quad\times[-\bar{\eta}^\phi(r,\theta,-\phi)],\\ &= \boldsymbol{\xi}\cdot\bar{\boldsymbol{\eta}}(r,\theta,-\phi).\end{aligned} \tag{3.22}$$

So the integrand of the left-hand side of equation (3.20) equals that of the right-hand side evaluated at the reflected point ($\phi\to-\phi$). Since the weight-factor in the integral (1.4) is just ρ and is invariant under reflection, it follows that the integrals in equation (3.20) are equal.

Since $S^2 = 1$, it is possible to divide any vector into its even and odd parts under S:

$$|\xi\rangle = |\xi_+\rangle + |\xi_-\rangle, \tag{3.23a}$$

$$|\xi_\pm\rangle = \tfrac{1}{2}(|\xi\rangle \pm S|\xi\rangle), \tag{3.23b}$$

$$S|\xi_\pm\rangle = \pm|\xi_\pm\rangle. \tag{3.23c}$$

Similarly any operator D is decomposable into

$$D = D_+ + D_-, \tag{3.24a}$$

$$D_\pm = \tfrac{1}{2}(D \pm SDS), \tag{3.24b}$$

$$SD_\pm S = \pm D_\pm. \tag{3.24c}$$

For the operators in this problem, we conclude

$$B = B_- \tag{3.25a}$$

$$C = C_+ \tag{3.25b}$$

$$L_+(\lambda_r + i\lambda_i) = \lambda_r^2 - \lambda_i^2 + i\lambda_i B + C \tag{3.25c}$$

$$L_-(\lambda_r + i\lambda_i) = 2i\lambda_r\lambda_i + \lambda_r B. \tag{3.25d}$$

Clearly L_+ is selfadjoint and L_- anti-selfadjoint. The evenness or oddness of an expression is determined by its constituents. Thus $D_+|\xi_+\rangle$ and $D_-|\xi_-\rangle$ are even vectors while $D_+|\xi_-\rangle$ and $D_-|\xi_+\rangle$ are odd.

Notice that if $\mathrm{Re}(\lambda)$ is zero then $L(\lambda)$ is an even operator. This means that if $|\xi\rangle$ is an eigenvector for imaginary λ_n then so is $S|\xi\rangle$. If there is only a one-dimensional eigensubspace for λ_n then $|\xi\rangle$ and $S|\xi\rangle$ are linearly dependent. This in turn means that one can choose the phase of $|\xi\rangle$ (see equation (3.32) below) in such a way as to make $|\xi\rangle$ even (or odd). (If there is more than one eigenvector then linear combinations can be taken to make them all

even or odd.) So without loss of generality one can take the eigenvector of any imaginary eigenvalue to be even under S.

If S were simply the operation of complex conjugation then equation (3.23) would correspond to splitting a vector into its real and imaginary parts. A vector which is 'real' (even under S) has components (in coordinates r, θ, ϕ),

$$\xi_+ = (a, b, ic) \tag{3.26}$$

where a, b and c are real. An 'imaginary' vector is just i times this. This can be formalized by speaking of a *complex structure* in a six-real-dimensional space R, elements of which have real components $(a_1, a_2, b_1, b_2, c_1, c_2)$. This structure is an operator J which acts on this vector to give

$$J(a_1, a_2, \ldots, c_2) = (-a_2, a_1, -b_2, b_1, c_2, -c_1) \tag{3.27}$$

This operator has the property

$$J^2 = -1$$

required of a complex structure. A 'real' vector in R is $(a, 0, b, 0, 0, c)$, and any vector in R can be written as a real one plus J times a real one – an imaginary one:

$$(a_1, a_2, b_1, b_2, c_1, c_2) = (a_1, 0, b_1, 0, 0, c_2) + J(a_2, 0, b_2, 0, 0, c_1). \tag{3.28}$$

We identify this six-dimensional space R with the three-complex dimensional space H by

$$\begin{aligned} &\mathrm{Re}(\xi^r) = a_1, \quad \mathrm{Im}(\xi^r) = a_2, \quad \mathrm{Re}(\xi^\theta) = b_1, \\ &\mathrm{Im}(\xi^\theta) = b_2, \quad \mathrm{Re}(\xi^\phi) = c_1, \quad \mathrm{Im}(\xi^\phi) = c_2. \end{aligned} \tag{3.29}$$

Then S is the conjugation operator with respect to J: it changes the vector (3.28) into its complex conjugate, $J \to -J$:

$$\begin{aligned} S(a_1, a_2, b_1, b_2, c_1, c_2) &= (a_1, -a_2, b_1, -b_2, -c_1, c_2) = (a_1, 0, b_1, 0, 0, c_2) \\ &\quad - J(a_2, 0, b_2, 0, 0, c_1). \end{aligned} \tag{3.30}$$

The natural Hilbert-space norm in R determined by its complex structure J is the sum of the squares of the real and imaginary parts of a vector:

$$\begin{aligned} |(a_1, a_2, b_1, b_2, c_1, c_2)|^2 &= |(a_1, 0, b_1, 0, 0, c_2)|^2 + |(a_2, 0, b_2, 0, 0, c_1)|^2 \\ &= a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2. \end{aligned}$$

Apart from the weighting factor ρ, this is the same as the usual norm of H if equation (3.29) is used. This is just another way of understanding equation (3.20).

One should note that the identification of an element of H, $|\xi\rangle$, with a physical perturbation is unchanged if the overall phase of $|\xi\rangle$ is changed: both $|\xi\rangle$ and

$$|\xi'\rangle = \exp(i\chi)|\xi\rangle, \quad \chi = \text{real constant}, \tag{3.31}$$

give the same physics. Therefore the even and odd parts of $|\xi\rangle$ can be transformed into one another:

$$\begin{aligned} |\xi'_+\rangle &= \cos\chi|\xi_+\rangle + i\sin\chi|\xi_-\rangle, \\ |\xi'_-\rangle &= i\sin\chi|\xi_+\rangle + \cos\chi|\xi_-\rangle. \end{aligned} \tag{3.32}$$

3.4 SYMMETRIES IN H_2

So far we have discussed the symmetries in H, but they must have an expression in H_2. In this section I shall briefly list them. They will be used to develop the variational principle in H_2.

Because S involves time-reversal, and because the second component of a vector in H_2 is, physically, the time-derivative of the first, it should not be surprising that the analogue in H_2 of S is

$$S_2 = \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}, \quad (S_2)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.33}$$

Recalling that T is

$$T = \begin{pmatrix} 0 & 1 \\ -C & -B \end{pmatrix},$$

one can easily show that

$$S_2 T S_2 = -T, \tag{3.34}$$

again a natural result in view of the time-reversal in S_2. Thus T is 'pure imaginary' with respect to the natural complex structure associated with S_2.

In order to find how to turn a right-eigenvector $|\lambda_n;\, r_0\rangle\rangle$ into the left-eigenvector $\langle\langle\lambda_n;\, r_0|$ for the same eigenvalue, we need only consult equation (3.16) and the correspondences (2.12) and (2.13). The result is that if we define

$$M = \begin{pmatrix} SB & S \\ S & 0 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 0 & S \\ S & SB \end{pmatrix} \tag{3.35}$$

then the map is

$$\langle\langle\lambda_n;\, l_k| = \langle\langle M;\, \lambda_n;\, r_k|, \tag{3.36}$$

where

$$|M;\, \lambda_n;\, r_k\rangle\rangle \equiv M|\lambda_n;\, r_k\rangle\rangle. \tag{3.37}$$

(Again we have to employ this notation because M has no adjoint.)

It is instructive to factor S_2 out of M:

$$M = WS_2, \quad W = \begin{pmatrix} -B & -1 \\ 1 & 0 \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -B \end{pmatrix}. \tag{3.38}$$

The operator W is just the symplectic structure on H_2, the anti-selfadjoint symplectic form that converts H_2 into the phase space of Hamiltonian mechanics. To see this, observe that if $(|\xi\rangle, |\xi_t\rangle)$ and $(|\eta\rangle, |\eta_t\rangle)$ are any two vectors in H_2 then we have

$$\begin{aligned} (\langle\xi|, \langle\xi_t|) \begin{pmatrix} -B & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\eta\rangle \\ |\eta_t\rangle \end{pmatrix} &= -\langle\xi|\eta_t\rangle + \langle\xi_t|\eta\rangle - \langle\xi|B|\eta\rangle \\ &= -\langle\xi|(|\eta_t\rangle + \tfrac{1}{2}B|\eta\rangle) + \overline{\langle\eta|(|\xi_t\rangle + \tfrac{1}{2}B|\xi\rangle)} \\ &= \langle p_\xi|\eta\rangle - \langle\xi|p_\eta\rangle, \end{aligned} \tag{3.39}$$

where $|p_\eta\rangle$ is the canonical momentum derivable from the Lagrangian which leads to the dynamical equations:

$$|p_\eta\rangle = |\eta_t\rangle + \tfrac{1}{2}B|\eta\rangle. \tag{3.40}$$

There is a full discussion of this symplectic structure in Friedman & Schutz (1978a), where it is used extensively. For our purposes later it suffices to note here that

$$E = WT = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \tag{3.41}$$

is the energy operator. That is, the canonical energy of any perturbation $|\xi\rangle\rangle$ is $\frac{1}{2}\langle\langle\xi|E|\xi\rangle\rangle$. The selfadjointness of E in H_2 will be important in the variational principle set out in the next section. Since T is the time-evolution operator, equation (3.41) is an example of the fact that time and energy are canonically conjugate quantities in Hamiltonian theory.

4 Variational principles for the eigenfrequencies

The variational principle for the quadratic eigenvalue problem in H was given in Schutz (1979). As is well known (*cf.* Goertzel & Tralli 1960; Chandrasekhar 1961), whenever one has a method of transforming a right-eigenvector into a left-eigenvector, independently of the eigenvalue, then one has at least a candidate for a variational expression. In our case, the candidates in H and H_2 are, respectively,

$$\langle S;\xi|(\lambda^2 + \lambda B + C)|\xi\rangle = 0, \tag{4.1}$$

and

$$\langle\langle M;\eta|(T-\lambda)|\eta\rangle\rangle = -\langle\langle S_2;\eta|(E + W\lambda|\eta\rangle\rangle = 0. \tag{4.2}$$

If one does not have a way of mapping any right-eigenvector into its associated left-eigenvector, then variational expressions of this sort probably cannot be found. To prove that equations (4.1) and (4.2) are variational expressions, one must show that if λ is a solution of equations (4.1) (or (4.2)) for a given $|\xi\rangle$ ($|\eta\rangle\rangle$) and λ is unchanged to first order by arbitrary changes in $|\xi\rangle$ ($|\eta\rangle\rangle$), then λ is an eigenvalue and $|\xi\rangle$($|\eta\rangle\rangle$) is its associated eigenvector.

Expression (4.1) will be called the *symmetric eigenfrequency equation.* If $|\xi\rangle$ is an eigenvector for a complex eigenfrequency (i.e. not an even vector under S) then one of the roots of equation (4.1) will be an eigenfrequency, but there is no reason to suppose that the other one is. If $|\xi\rangle$ is even under S (e.g. an eigenvector for an imaginary eigenfrequency) then the symmetric eigenfrequency equation and the Hermitian one (3.5) are identical.

The proof that equation (4.1) is the basis of a variational principle is simple. It relies on the following property: for any vectors $|1\rangle$, $|2\rangle$ and any selfadjoint operator D_+ even under S,

$$\langle S;1|D_+|2\rangle = \langle S;2|D_+|1\rangle. \tag{4.3}$$

(For an odd anti-selfadjoint operator D_- the relation is exactly the same.) By equation (3.20) the left-hand side has complex conjugate $\langle 1|SD_+|2\rangle$, which by the evenness of D_+ is $\langle 1|D_+S|2\rangle$, whose complex conjugate is the right-hand side of equation (4.3) because D_+ is selfadjoint. This helps us simplify the variation of equation (4.1),

$$\delta\lambda\langle S;\xi|(2\lambda + B)|\xi\rangle + \langle S;\delta\xi|(\lambda^2 + \lambda B + C)|\xi\rangle + \langle S;\xi|(\lambda^2 + \lambda B + C)|\delta\xi\rangle = 0, \tag{4.4}$$

by applying equation (4.3) to each coefficient of the last polynomial in λ in equation (4.4):

$$\delta\lambda\langle S;\xi|(2\lambda + B)|\xi\rangle + 2\langle S;\delta\xi|(\lambda^2 + \lambda B + C)|\xi\rangle = 0. \tag{4.5}$$

If $\delta\lambda$ vanishes for arbitrary $|\delta\xi\rangle$ (hence arbitrary $S|\delta\xi\rangle$) then equation (4.5) implies

$$(\lambda^2 + \lambda B + C)|\xi\rangle = 0.$$

This proves the variational principle. The proof that equation (4.2) provides a variational principle in H_2 is analogous. It is easier to use the second form, noting that, with respect to S_2,

$$E = E_+, \quad W = W_-.$$

This formulation is instructive because it practically forces us to write the linear eigenvalue problem in the form

$$E|\xi\rangle\rangle = -\lambda W|\xi\rangle\rangle. \tag{4.6}$$

As this is a linear equation in λ involving only selfadjoint and anti-selfadjoint operators it may be easier to solve numerically than the conventional formulation. As a formal expression it means that the problem is a selfadjoint eigenvalue problem in phase space, that is in H_2 using W as an inner product. (The usual selfadjointness theorems do not apply here because neither E nor iW is of definite sign.) See Paper I for a further discussion of this point.

4.1 USING THE VARIATIONAL PRINCIPLE

As the two principles are clearly equivalent, I shall confine my remarks to the one in H. If equation (4.1) or (4.5) is actually to be used in a numerical search for modes, the search would be made easier if some of the eigenvalues were not merely stationary but actually extremal values, as happens in the case of compact selfadjoint operators. But even in the non-rotating case the eigenvalues cannot be found this way very easily, and in the present case there is no reason to expect anything better.

One can still attempt a variant on this approach, namely to use a certain chosen set of functions $\{|i\rangle, i = 1, \ldots, N\}$ as a basis for a subspace of H, and to solve the problem restricted to this subspace. This restriction has two versions, according to whether one adopts the *Hermitian* representation of, say, C,

$$_{(H)}C_{ij} = \langle i|C|j\rangle, \qquad _{(H)}C_{ij} = \overline{_{(H)}C_{ji}}$$

or the *symmetric* representation,

$$_{(S)}C_{ij} = \langle S; i|C|j\rangle, \qquad _{(S)}C_{ij} = {}_{(S)}C_{ij} = \overline{_{(S)}C_{ji}}$$

These are equivalent if every vector $S|i\rangle$ is a linear combination of the basis. When they are inequivalent, the symmetric representation should give better approximations to the eigenvalues, for reasons discussed in Schutz (1979a).

One other use to which variational principles have been put is the derivation of stability criteria. We still have no necessary and sufficient condition for the existence of complex eigenvalues in this problem. Can the variational principle be used to formulate one? The Hermitian eigenvalue equation (3.5) has proved useful in setting bounds on the complex eigenvalues (Paper I). It may be hoped that the symmetric eigenvalue equation (4.1), which contains different information if λ_n is complex, will similarly prove useful.

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Perturbations and stability of rotating stars – III. Perturbation theory for eigenvalues

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Summary. The normal mode problem for perturbations of a perfect-fluid rotating star is a non-selfadjoint eigenvalue problem. To study the changes in the eigenfrequencies when the operators in the problem change slightly, this paper develops the appropriate lowest-order perturbation theory and discusses two important cases. In the first case the perturbing operator is the inclusion of a 'secular' effect, such as viscosity or radiation reaction. Here the main interest is in unperturbed eigenfrequencies that are dynamically stable, and the perturbed frequencies are analytic functions of the perturbation parameter. Explicit formulae are derived for use in numerical calculations of secular effects in stars. The second case is the study of the onset of dynamical instability itself. Here the perturbing operator is a slight change in the structure of the star. It is found that instability sets in only through eigenvalues which have Jordan chains, and the perturbation theory is non-analytic. It is shown that stable modes do not have Jordan chains, and the marginally stable modes have, in the generic case, chains of length 1. This gives a fairly complete characterization of the manner in which instability arises along a sequence of stars.

1 Introduction

The theory of the secular stability of perfect-fluid rotating stars is in reasonably good shape (Friedman & Schutz 1978), but there have been as yet very few calculations by which we can judge the astrophysical importance of these instabilities. The only published calculations are for the modes of the Maclaurin spheroids (Roberts & Stewartson 1963; Chandrasekhar 1969, 1970; Lindblom & Detweiler 1977; Comins 1978, 1979a, b) or for equally unrealistic systems (Papaloizou & Pringle 1978). This is particularly unfortunate in the case of gravitational-radiation-driven instabilities, which may affect all neutron stars at least in principle, but whose growth times are extremely sensitive to the exact structure of the star and of the unstable eigenfunction (Comins 1978). Because the secular terms are so small, it is likely that there would be substantial numerical errors inherent in any attempt to solve directly the dynamical equations with the secular terms included. It seems, therefore, that

Table 1. Peak line flux densities (S_L) and line/continuum ratios (R).

Galaxy	CH (3.26 GHz)		H_2CO (4.83 GHz)		OH (1.67 GHz)	
	S_L	$\|R\|$	S_L	$\|R\|$	S	$\|R\|$
NGC 253	$\leqslant 0.003$	$\leqslant 0.001$	−0.029	0.020	−0.19	0.079
LMC (N159)	0.013	0.006	−0.022	0.014	−0.10	0.05
NGC 4945	0.017	0.006	−0.16	0.080	−0.82	0.27
NGC 5128	0.079	0.026	−0.14	0.035	−0.19	0.038

continuum values used are generally lower than the observed values, in keeping with the discussions in the references above.

For NGC 4945 and NGC 253 it is likely that the CH emission arises in clouds near the nucleus. This is supported by the following:

(a) $|R|$ for CH is more than a factor of 10 lower than for H_2CO and OH. Such an underabundance of CH is similar to that found in clouds near the centre of our Galaxy (see Whiteoak *et al.* 1978).

(b) The velocities associated with the CH emission for NGC 4945 are similar to those for [N II] 6583 Å emission, which is concentrated in a region (< 5 arcsec) believed to be at the nucleus (Whiteoak & Gardner 1979).

For N159 the similarity of the 3264- and 3335-MHz profiles and the low values of $|R|$, increasing gradually from CH to OH, are typical of clouds not closely associated with H II regions or nuclei of galaxies (see discussion by Whiteoak *et al.* 1978).

The values of R for NGC 5128 in Table 1 are derived on the assumption that only the nuclear source is amplified or absorbed. From observations at adjacent frequencies by Wade *et al.* (1971), Christiansen *et al.* (1977) and Beall *et al.* (1978), it was estimated that the intensity of the nuclear component was 3 Jy (compared with total continuum of 75 Jy observed), 4 and 5 Jy at CH, H_2CO and OH frequencies. The above assumption is supported by the fact that the CH features are narrow and similar in shape and velocity to those of the 14.5-GHz transitions of H_2CO (Gardner & Whiteoak 1979), even though only 10 per cent of the continuum in the telescope beam at 14.5 GHz is from outside the nucleus.

Table 1 indicates that the values of $|R|$ for NGC 5128 are similar for CH, H_2CO and OH – a situation not found in our Galaxy. The high value of R for CH and the predominance of the 3264-MHz emission are more typical of clouds associated with H II regions outside than inside the nuclear region of our Galaxy. We have also argued (Gardner & Whiteoak 1979) from a comparison of line/continuum ratios for H_2CO at 4.8 and 14.5 GHz that the molecular cloud is in an outer dust lane well away from either the nucleus or any H II region. The most plausible explanation for the anomalously high values of R for CH is that there is an overabundance of CH relative to H_2CO in NGC 5128 compared with our Galaxy.

To conclude: the 3264-MHz transition of CH has been detected in three galaxies and possibly in a fourth. As in our Galaxy, the line profiles are similar to those for H_2CO. The results for N159 in the LMC are similar to those for CH clouds in our Galaxy; for NGC 4945 and NGC 253 they suggest relative underabundance of CH in their nuclear regions, as in our Galaxy; for NGC 5128, however, they indicate an overabundance of CH relative to our Galaxy.

Because the 3335-MHz transition of CH observed towards continuum regions is usually much fainter than the 3264-MHz transition, it is not surprising that it was observed in only one galaxy (LMC), and possibly in another (NGC 4945). The results confirm that, as in our Galaxy, the molecular clouds are not in local thermodynamic equilibrium, a situation also suggested by the observations of other molecules in these clouds.

the best procedure is to solve the pulsation problem without the secular terms and to use the resulting eigenfunctions to calculate the effect of the secular terms in a perturbation approximation. This is made possible by the completeness of the unperturbed eigenfunctions (Dyson & Schutz 1979 – Paper I). This paper develops such a method and derives explicit equations for the change in an eigenfrequency in terms of its zero-order eigenfunction when the secular terms arise from viscosity or radiation reaction.

The onset of dynamical instability is in some sense a simpler problem to study than that of secular instability, because one does not need to introduce extra terms into the equations: one simply monitors the behaviour of the eigenfrequencies along the sequence. Nevertheless, the actual onset of instability is poorly understood. Given that we have as yet no general criterion for dynamical instability (which is in contrast with the secular case), a better understanding of how dynamical instabilities make their appearance may be helpful. This can also be treated by perturbation theory, since one member of a sequence of stars may be thought of as a perturbation of an earlier member. We will find that the instability point is always marked by a Jordan chain, whose length is usually 1.

At this point it may be helpful to explain the principal differences between secular and dynamical instabilities in this context. The word 'dynamical' refers to our basic system of equations, those which govern linear perturbations of a perfect-fluid star. These equations are conservative, so the eigenfrequencies either are real or occur in complex-conjugate pairs (of which one leads to exponential growth of the perturbation). Suppose a sequence of models is constructed, parameterized by the real variable μ, along which the eigenfrequencies are continuous functions of μ. Then dynamical instability in a particular mode sets in at the value $\mu = \mu_d$ if its eigenfrequency is real below μ_d and complex above it. At μ_d the mode is said to be marginally unstable. (We will see below that the property of marginal instability, although defined here for a particular sequence, is in fact a sequence-independent property of the star itself.) Although the eigenfrequency is continuous at μ_d it is clearly not analytic in μ there (in the complex-variable sense).

By contrast, 'secular' instabilities are caused by changing, not the unperturbed model, but the dynamical equations governing the perturbation. If the added terms are dissipative then a zero-order real eigenfrequency will acquire an imaginary part, but there is no need for these to occur in complex-conjugate pairs. Again we say that secular instability in a particular mode sets in at the value $\mu = \mu_s$ if its secularly perturbed eigenfrequency corresponds to exponential decay below μ_s and to exponential growth above it. This is only meaningful if the mode is not dynamically unstable. In addition to μ there is a parameter ϵ giving the 'strength' of the secular terms, and the eigenfrequencies can be – and usually are – analytic functions of ϵ at $\epsilon = 0$.

The important difference between the eigenvalue perturbation theory that must be used here and the usual perturbation theory of selfadjoint operators that physicists are familiar with from quantum mechanics is that our problem, the oscillation of a rotating star, is essentially non-selfadjoint. This means that the eigenvectors are not necessarily orthogonal to one another, so that explicit use has to be made of left-eigenvectors as well as right-eigenvectors in this problem. A previous paper (Schutz 1980 – Paper II) discussed right-eigenfunctions and left-eigenfunctions in some detail, and showed that for stars which possess the reflection symmetry $(t, \phi) \rightarrow (-t, -\phi)$ there was a direct relation between the right-eigenfunctions and left-eigenfunctions for any particular eigenvalue. We shall use this relation in some of our perturbation formulae.

This paper is unfortunately not self-contained. It relies on an understanding of the results of Paper II and it follows the notation of Paper II and of Paper I. Nevertheless, it will be helpful to write down the basic formulae. The dynamical equation can be written (Lynden-

Bell & Ostriker 1967; Dyson & Schutz 1979 – Paper I)

$$\frac{\partial^2}{\partial t^2}\xi + B\left(\frac{\partial}{\partial t}\xi\right) + C(\xi) = 0, \tag{1.1}$$

where B and C are, respectively, antisymmetric and symmetric operators in a Hilbert space H whose inner product is

$$\langle \xi | \eta \rangle = \int \rho \bar{\xi} \cdot \eta \, dv, \tag{1.2}$$

where ρ is the density of the star and a bar denotes complex-conjugation. In equation (1.2) Dirac bra and ket notation has been introduced, and it will be used from now on. The eigenvalue problem for equation (1.1) comes from the assumption

$$|\xi(t)\rangle = |\xi\rangle \exp(\lambda t): \tag{1.3}$$

$$(\lambda^2 + \lambda B + C)|\xi\rangle = 0. \tag{1.4}$$

(Note the convention adopted here, that λ is the eigenvalue. The 'frequency' is $i\lambda$ or $-i\lambda$, depending on one's convention.) In discussing perturbation theory it is more convenient to deal with the first-order eigenvalue problem in the space $H_2 = H \oplus H$, which is 'initial-data' space, the space of pairs $(|\xi(t)\rangle, d/dt\,|\xi(t)\rangle) = |\eta\rangle\rangle$. This double-bracket notation will be used to distinguish elements of H_2 from their counterparts in H. The associated eigenvalue problem is

$$T|\eta\rangle\rangle = \lambda|\eta\rangle\rangle, \qquad T = \begin{pmatrix} 0 & 1 \\ -C & -B \end{pmatrix}. \tag{1.5}$$

The matrix T is not symmetric with respect to the inner product of H_2 induced by H:

$$\langle\langle \xi | \eta \rangle\rangle = \langle \xi_1 | \eta_1 \rangle + \langle \xi_2 | \eta_2 \rangle$$

where $|\eta_1\rangle$ and $|\eta_2\rangle$ are the 'components' of $|\eta\rangle\rangle$. Since the eigenvalues λ are eigenvalues of T, it is T whose perturbation theory we must investigate. As in Paper II we shall assume that T is a finite-dimensional matrix, as it always is in numerical approximations to the continuous problem.

2 General remarks on perturbation theory

We shall suppose that a matrix T has eigenvalues $\{\lambda_n^0\}$ and right- (left-) eigenvectors and Jordan chain vectors $\{|\lambda_n^0;\ r_k\rangle\rangle\}$ $(\{\langle\langle\lambda_n^0;\ l_k|\})$, with $k = 0$ denoting the eigenvector and $k = 1, \ldots, p_n$ forming the chain. Since T is a function of the sequence's parameter μ, the first question is how the eigenvalues depend on μ. We quote three results from Kato (1966, Chapter II). First, if T is continuous in μ then its eigenvalues are continuous in μ. Moreover, the dimension of the characteristic subspace of an eigenvalue (that is, the total number of eigenvectors and Jordan chain vectors associated with that eigenvalue) is 'conserved', in the sense that if an eigenvalue splits into distinct eigenvalues the sum of the dimensions of the characteristic subspaces of the distinct eigenvalues equals the total dimension of the original subspace. In fact, the total projection operator on to these subspaces is continuous in μ. The second result is that if T is differentiable in μ at some μ_1 then this same total projection operator is likewise differentiable, but this is not necessarily true of the eigenvalues. If an eigenvalue has no Jordan chain at μ_1 (a 'semisimple' eigenvalue) then it *is* differentiable at μ_1 even if it splits into several eigenvalues for $\mu > \mu_1$, and in fact

$$\lambda(\mu) = \lambda(\mu_1) + (\mu - \mu_1)\lambda^{(1)}(\mu_1) + o(\mu - \mu_1) \tag{2.1}$$

where $\lambda^1(\mu_1)$ is an eigenvalue of $(dT/d\mu)_{\mu_1}$ restricted to the total characteristic subspace, and where $o(\mu-\mu_1)$ is the usual order symbol for a function which goes to zero faster than $\mu-\mu_1$. So if a sequence of models is differentiable, first-order perturbation theory gives at least an asymptotic expression for the change in the eigenvalue. We can immediately conclude that if a semisimple eigenvalue at μ_1 splits into several eigenvalues for $\mu>\mu_1$ in such a way that the various $\lambda^{(1)}(\mu_1)$'s are not zero, then it must split into the same number of eigenvalues for $\mu<\mu_1$, or in other words that μ_1 is merely a crossing point for the eigenvalues. Now, a dynamical instability point is a value of μ where eigenvalues split, since on the unstable side of it there is a complex-conjugate pair which become equal at the instability point. But it is not a crossing point, since the eigenvalues are real on the other side of it. So along a differentiable sequence dynamical instability must occur through a mode which is not semisimple: it must have a Jordan chain. We concluded this for continuous sequences by a different argument in Paper II.

Kato's third result concerns sequences which are in fact (complex-) analytic functions of their parameter μ in some simply connected region D of the complex plane. Here the number of distinct eigenvalues is constant except at a finite number of exceptional points, where degeneracy may occur without Jordan chains (semisimple case) or with them. The total projection operator discussed above is holomorphic even at the exceptional points. In the semisimple case the eigenvalues are analytic functions of μ even at exceptional points, while those with Jordan chains at an exceptional point are branches of analytic functions with at most algebraic singularities at the exceptional points. This means that the lowest-order change in an eigenvalue away from μ_1 is proportional at worst to some fractional power of $\mu-\mu_1$. If the Jordan chain has length p the branch point has order no larger than p. In treatments of secular stability the perturbed operator is usually taken to have the form $T+\epsilon P$, which is of course analytic in ϵ, so that the onset of secular instability can be studied by analytic perturbation theory. This is consistent with the perturbation analysis of the Maclaurin spheroids by Roberts & Stewartson (1963), who did not assume that the problem was finite dimensional.

In order to put some flesh on the bare bones of these theorems, let us consider the operator family $T+\epsilon P$. Here ϵP is either a change in the character of the equations (secular stability) or a change in T itself brought on by increasing the sequence's parameter μ by ϵ (dynamical stability). In the latter case, we will therefore be able to apply our first-order results only to sequences which are at least differentiable, as in equation (2.1).

Suppose that $T+\epsilon P$ has an eigenvalue $\lambda_n(\epsilon)$ and an associated right-eigenvector $|\lambda_n(\epsilon); r_0\rangle\rangle$. We expand this in terms of the basis $\{|\lambda_n^0; r_k\rangle\rangle\}$:

$$|\lambda_n(\epsilon); r_0\rangle\rangle = \sum_m \sum_{k=0}^{p_m} a_{mk}(\epsilon)|\lambda_m^0; r_k\rangle\rangle, \tag{2.2}$$

where p_m is the length of the Jordan chain to λ_m^0. Applying $T+\epsilon P$ to equation (2.2) gives

$$(T+\epsilon P)|\lambda_n(\epsilon); r_0\rangle\rangle = \lambda_n(\epsilon)|\lambda_n(\epsilon); r_0\rangle\rangle$$

$$\sum_{m,k} a_{mk}(\epsilon)\,(\lambda_m^0|\lambda_m^0; r_k\rangle\rangle + |\lambda_m^0; r_{k-1}\rangle\rangle + \epsilon P|\lambda_m^0; r_k\rangle\rangle) = \sum_{m,k} a_{mk}(\epsilon)\lambda_n(\epsilon)|\lambda_m^0; r_k\rangle\rangle, \tag{2.3}$$

provided we adopt the convention that $|\lambda_n^0; r_k\rangle\rangle = 0$ if $k<0$. Now, if we contract equation (2.2) with $\langle\langle\lambda_s^0; l_j|$ for arbitrary s, employing the notation

$$P_{j,k}^{s,m} = \langle\langle\lambda_s^0; l_j|P|\lambda_m^0; r_k\rangle\rangle, \tag{2.4a}$$

$$I_j^s = \langle\langle\lambda_s^0; l_j|\lambda_s^0; r_{p_s-j}\rangle\rangle\ (\neq 0), \tag{2.4b}$$

then the orthogonality relations given in Paper II give

$$[a_{s,p_s-j}(\epsilon)\lambda_s^0 + a_{s,p_s-j+1}(\epsilon)]I_j^s + \epsilon \sum_{m,k} a_{mk}(\epsilon)P_{j,k}^{s,m} = \lambda_n(\epsilon)a_{s,p_s-j}(\epsilon)I_j^s, \tag{2.5}$$

where $a_{s,k} = 0$ if $k > p_s$. By continuity, the limit $\epsilon \to 0$ gives

$$(\lambda_s^0 - \lambda_n^0)a_{s,p_s-j}(0) + a_{s,p_s-j+1}(0) = 0,$$

This is a system of equations in j for fixed s and n, whose solution is

$$\begin{aligned} &a_{s,j}(0) = 0 \qquad \forall s, \forall\, j > 0, \\ &a_{s,0}(0) = 0 \qquad \text{if } \lambda_s^0 \neq \lambda_n^0. \end{aligned} \tag{2.6}$$

This means that the $\epsilon \to 0$ limit of $|\lambda_n(\epsilon); r_0\rangle\rangle$ is a linear combination of those eigenvectors which have eigenvalues equal to λ_n^0. For simplicity, we shall assume there is no degeneracy at λ_n^0: no other eigenvalues equal it. Degeneracy can be handled exactly as in selfadjoint perturbation theory.

If we assume that λ_n and the a's are *analytic* in ϵ at $\epsilon = 0$ then equation (2.6) says that $a_{n,0} = 1 + 0(\epsilon)$ and $a_{n,j} = 0(\epsilon), j \geqslant 1$. Expanding λ_n in its Taylor series,

$$\lambda_n(\epsilon) = \lambda_n^0 + \delta\lambda_n + 0(\epsilon^2), \tag{2.7}$$

we find that equation (2.5) gives for $s = n$

$$(a_{n,p_n-j+1} - \delta\lambda_n\delta_{j,p_n})I_j^n + \epsilon P_{j,0}^{n,n} + 0(\epsilon^2) = 0, \qquad j = 0, \ldots, p_n, \tag{2.8}$$

where δ_{j,p_n} is the Kronecker delta.

If $p_n = 0$ (no Jordan chains in the unperturbed mode) then equation (2.8) immediately gives the first-order change in the eigenvalue:

$$\delta\lambda_n = \epsilon P_{0,0}^{n,n}/I_0^n = \epsilon \frac{\langle\langle \lambda_n^0; l_0|P|\lambda_n^0; r_0\rangle\rangle}{\langle\langle \lambda_n^0; l_0|\lambda_n^0; r_0\rangle\rangle}. \tag{2.9}$$

This is clearly analogous to the familiar formula for the perturbation theory of selfadjoint operators, but here one must be careful to use $\langle\langle \lambda_n^0; l_0|$, the left-eigenvector of T, which is usually not the adjoint of $|\lambda_n^0; r_0\rangle\rangle$.

We can see how the assumption of analyticity breaks down if there is a Jordan chain to λ_n^0. With $p_n > 0$, equation (2.8) for $j = 0$ gives

$$\epsilon P_{0,0}^{n,n} = 0. \tag{2.10}$$

The element of the matrix P cannot be expected to vanish in general, leading to a contradiction. We shall look further at this case in our study of dynamical instability in Section 4. The analytic formula, equation (2.9), suffices for the study of secular instability, to which we now turn.

3 Analytic perturbation theory and secular stability

Secular instability in a mode is really only of interest when that mode has a purely real unperturbed frequency (purely imaginary λ_n^0). As we shall see in Section 4, such modes cannot possess Jordan chains if they are not marginally dynamically unstable, so we shall assume in this section that $p_n = 0$.

3.1 VISCOSITY

In a stationary, isolated star, the only stationary axisymmetric state in the presence of viscosity is rigid rotation with angular velocity Ω. So we shall restrict ourselves to such stars in this section. They clearly have the reflection symmetry $(t, \phi) \to (-t, -\phi)$.

Viscosity changes the dynamical equation to (Friedman & Schutz 1978)

$$(\lambda^2 + \lambda B + C)\boldsymbol{\xi} = -(\lambda + im\Omega)F\boldsymbol{\xi} \tag{3.1}$$

where

$$-(F\boldsymbol{\xi})^i = \frac{2}{\rho}\nabla_j(\eta\Sigma^{ij}) + \frac{1}{\rho}\nabla^i(\zeta\theta) \tag{3.2}$$

$$\Sigma^{ij} = \tfrac{1}{2}(\nabla^i\xi^j + \nabla^j\xi^i) - \tfrac{1}{3}g^{ij}\theta \tag{3.3}$$

$$\theta = \nabla_i\xi^i \tag{3.4}$$

(summation on repeated indices). Here η is the coefficient of shear viscosity and ζ the coefficient of bulk viscosity. We wish to find a representation of equation (3.1) in H_2. The following operator gives the correct result:

$$P = \begin{pmatrix} 0 & 0 \\ -im\Omega F & -F \end{pmatrix}. \tag{3.5}$$

Then if $|\lambda_n^0; r_0\rangle$ is the right-eigenvector of $L(\lambda_n^0)$ we have from Paper II

$$|\lambda_n^0; r_0\rangle\rangle = \begin{pmatrix} |\lambda_n^0; r_0\rangle \\ \lambda_n^0|\lambda_n^0; r_0\rangle \end{pmatrix} \tag{3.6}$$

and

$$\langle\langle\lambda_n^0; l_0| = (\langle S; \lambda_n^0; r_0|(\lambda_n^0 + B), \quad \langle S; \lambda_n^0; r_0|). \tag{3.7}$$

This gives, from equation (2.9),

$$\delta\lambda_n = -(\lambda_n^0 + im\Omega)\frac{\langle S; \lambda_n^0; r_0|F|\lambda_n^0; r_0\rangle}{\langle S; \lambda_n^0; r_0|(2\lambda_n^0 + B)|\lambda_n^0; r_0\rangle}. \tag{3.8}$$

In more concrete terms, suppose that the zero-order eigenfunction is

$$|\lambda_n^0; r_0\rangle = \begin{pmatrix} \xi^{\varpi}(\varpi, z) \\ \xi^{\phi}(\varpi, z) \\ \xi^{z}(\varpi, z) \end{pmatrix} \exp(im\phi), \tag{3.9}$$

referred to an *orthonormal* basis in cylindrical polar coordinates. Then

$$\langle S; \lambda_n^0; r_0| = (\xi^{\varpi}, -\xi^{\phi}, \xi^{z}) \exp(-im\phi) \tag{3.10}$$

is the left-eigenfunction. A rather long but straightforward calculation gives that the numera-

tor in equation (3.8) is, after an integration by parts,

$$\langle S;\lambda_n^0;r_0|F|\lambda_n^0;r_0\rangle = \int\int \varpi d\varpi\, dz \left\{\eta\left[2(\partial_\varpi \xi^\varpi)^2 + (\partial_\varpi \xi^z)^2 + 2\partial_z \xi^\varpi \partial_\varpi \xi^z \right.\right.$$
$$+ (\partial_z \xi^\varpi)^2 + 2(\partial_z \xi^z)^2 - \frac{m^2}{\varpi^2}(\xi^z)^2 + \frac{1-m^2}{\varpi^2}(\xi^\varpi)^2 - (\partial_z \xi^\phi)^2 - (\partial_\varpi \xi^\phi)^2$$
$$\left. - \frac{2}{\varpi}\xi^\phi \partial_\varpi \xi^\phi - \frac{1-m^2}{\varpi^2}(\xi^\phi)^2\right] + (\zeta - \tfrac{2}{3}\eta)\left[\left(\partial_z \xi^z + \partial_\varpi \xi^\varpi + \frac{1}{\varpi}\xi^\varpi\right)^2\right.$$
$$\left.\left. + \frac{m^2}{\varpi^2}(\xi^\phi)^2\right]\right\}. \tag{3.11}$$

Similarly, the denominator is

$$\langle S, \lambda_n^0; r_0|(2\lambda_n + B)|\lambda_n^0, r_0\rangle = -2\iint \rho\varpi\, d\varpi\, dz\, \{(\lambda_n^0 + im\Omega)[(\xi^\varpi)^2 + (\xi^z)^2$$
$$- (\xi^\phi)^2] - \Omega\xi^\phi\xi^\varpi\}. \tag{3.12}$$

Although complicated, these expressions should be easy enough to evaluate numerically.

3.2 GRAVITATIONAL RADIATION

The gravitational radiation emitted by a pulsating object is generally a very complicated function of the pulsation, but in the case of a nearly-Newtonian star emitting wavelengths large compared to the stellar radius, considerable simplification is possible. We will follow the treatment of Thorne (1969). The important point is that when the wavefield is analysed in spherical harmonics Y_{lm} at infinity, the dominant radiation usually comes from the lowest value of $l \geqslant 2$ contributing to the radiation ($l = 0$ and 1 do not contribute any radiation in general relativity). The strict statement is that in an asymptotic expansion in powers of c^{-1}, the radiation from each l is proportional to $c^{-(2l+2)}$. While it is possible to have a star whose quadrupole moment ($l = 2$) is changing much less rapidly than its octopole moment ($l = 3$) and so contributes less radiation, it should be safe to assume that this is a rather contrived circumstance for any star whose unperturbed shape is reasonably non-spherical. Since for spherical harmonics one always has $|m| \leqslant l$, it follows that an axisymmetric star with a perturbation of azimuthal eigenvalue m gives off its dominant gravitational radiation with $l = |m|$. We shall restrict our attention to this case, and assume the star has the reflection-symmetry $(t, \phi) \to (-t, -\phi)$.

In Thorne's treatment, the effect of radiation on the star, which is what we are interested in, is accounted for by a change in the Newtonian gravitational potential, the addition of a new radiation-reaction potential. For $l = m$ this is

$$\delta\Phi_{RR} = 4\pi G \epsilon_m F_m Y_{mm}(\theta, \phi) r^m, \tag{3.13}$$

$$\epsilon_m = (-1)^{m+1} \frac{(m+1)(m+2)}{m(m-1)(2m+1)^2\,[(2m-1)!!]^2}, \tag{3.14a}$$

$$F_m(t) = \frac{1}{c^{(2m+1)}}\left(\frac{d}{dt}\right)^{2m+1} \int \rho(t) r^m (Y_{mm})^*\, dV. \tag{3.14b}$$

Because F_m contains an odd number of time-derivatives, it is an energy-dissipating term. If

we define an element of H

$$|RR;m\rangle = \nabla[r^m Y_{mm}(\theta,\phi)] \tag{3.15}$$

then it is not hard to show that the integral in equation (3.14b) is just

$$\langle RR;m|\xi\rangle, \tag{3.16}$$

since

$$\rho(t) = \delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}).$$

(Recall that the inner product of equation (3.16) is weighted by ρ.)

The equation of motion involves $\nabla\delta\Phi_{RR}$ and so can be written

$$(\lambda^2 + \lambda B + C)|\xi\rangle = -4\pi G\epsilon_m\left[\left(\frac{d}{dct}\right)^{2m+1}\langle RR;m|\xi\rangle\right]|RR;m\rangle. \tag{3.17}$$

It is clear that this does not fit into the scheme we have employed up till now because it contains higher than second time-derivatives. On the other hand, the right-hand side of equation (3.17) must be small for perturbation theory to work, so we do not make any error at this order if we use the unperturbed time-dependence of $|\xi\rangle$ on the right-hand side. For a mode $|\lambda_n^0; r_0\rangle$, this amounts to replacing d/dt by λ_n^0. We find, then, that each mode has its own perturbation operator G_n:

$$(\lambda^2 + \lambda B + C)|\lambda_n; r_0\rangle = G_n|\lambda_n; r_0\rangle \tag{3.18}$$

$$G_n = -4\pi G\epsilon_m(\lambda_n^0/c)^{2m+1}|RR;m\rangle\langle RR;m|. \tag{3.19}$$

In H_2 the appropriate operator is

$$P_n = \begin{pmatrix} 0 & 0 \\ G_n & 0 \end{pmatrix}.$$

Following equations (3.6), (3.7) and (2.9), we find

$$\delta\lambda_n = \frac{\langle S;\lambda_n^0;r_0|G_n|\lambda_n^0;r_0\rangle}{\langle S;\lambda_n^0;r_0|(2\lambda_n^0 + B)|\lambda_n^0;r_0\rangle}. \tag{3.21}$$

If we use the same form (3.9) for $|\lambda_n^0; r_0\rangle$, the denominator of equation (3.21) is the same as equation (3.12). To evaluate the numerator it is more convenient to use an orthonormal basis in spherical polar coordinates, in which $\boldsymbol{\xi}$ has components $(\xi^r, \xi^\theta, \xi^\phi)$. Then the numerator is

$$\langle S;\lambda_n^0;r_0|G_n|\lambda_n^0;r_0\rangle = -4\pi G\epsilon_m(\lambda_n^0/c)^{2m+1}(N_{mm})^2 \times\left[\int\int \rho r^{2m}\,dr\sin\theta\,d\theta\left(m\xi^r P_m^m + \xi^\theta\partial_\theta P_m^m - \frac{im}{\sin\theta}\xi^\phi P_m^m\right)\right]^2, \tag{3.22}$$

where we have rewritten Y_{mm} as

$$Y_{mm} = N_{mm}P_m^m\exp(im\phi), \tag{3.23}$$

with P_m^m the usual associated Legendre polynomial and N_{mm} is the usual normalization factor

$$N_{mm} = \left(\frac{2m+1}{4\pi(2m)!}\right)^{1/2}. \tag{3.24}$$

Equation (3.22) should be particularly easy to evaluate because it does not involve derivatives of ξ.

4 Non-analytic perturbation theory and the onset of dynamical instability

Here we assume that we have a differentiable sequence of stars with parameter μ, so that for any μ_0 $T(\mu)$ can be approximated by $T(\mu_0) + (\mu - \mu_0)P$ for sufficiently small $(\mu - \mu_0)$, with $P = dT/d\mu|_{\mu_0}$. We are interested in what happens if μ_0 has a Jordan chain to one of its eigenvalues, λ_n^0. Our starting point is equation (2.5), where ϵ stands for $\mu - \mu_0$:

$$\{a_{s,p_s-j}(\epsilon)\lambda_s^0 + a_{s,p_s-j+1}(\epsilon)\}\, I_j^s + \epsilon \sum_{m,k} a_{mk}(\epsilon) P_{j,k}^{s,m} = \lambda_n(\epsilon) a_{s,p_s-j}(\epsilon) I_j^s. \tag{2.5}$$

It is easy to verify that the assumption that $\{a_{s,k},\ k = 1, \ldots, p_s\}$ are analytic in ϵ when $\lambda_s \neq \lambda_n$ is consistent. Moreover in this case equation (2.5) gives no information about $\lambda_n(\epsilon)$. Since we continue to assume that $\lambda_s \neq \lambda_n$ whenever $s \neq n$ (no 'irrelevant' degeneracy), we shall concentrate on the case $s = n$. Then equation (2.5) is solved by allowing $\lambda_n(\epsilon)$ and $a_{n,k}(\epsilon)$ to depend on fractional powers of ϵ. We try the forms:

$$\lambda_n(\epsilon) = \lambda_n^0 + \epsilon^{f_\lambda}\delta\lambda + \ldots \tag{4.1}$$

$$a_{n,0}(\epsilon) = 1 + \epsilon^{f_0}\alpha_0 + \ldots \tag{4.2}$$

$$a_{n,k}(\epsilon) = \epsilon^{f_k}\alpha_k + \ldots, \qquad k = 1, \ldots, p_n, \tag{4.3}$$

where $\{\delta\lambda,\ \alpha_k\}$ are constants and all higher-order terms are dropped. The powers $\{f_\lambda, f_k\}$ must all be non-negative, since the $\epsilon \to 0$ limit exists. Let us examine equation (2.5) for various values of j, keeping only the lowest powers of ϵ in each term. We shall assume that $P_{0,0}^{n,n} \neq 0$. This must be regarded as the typical situation. The analysis becomes very much more complicated if $P_{0,0}^{n,n}$ vanishes, and we hope to return to it in a later publication. Equation (2.5) for $j = 0$ becomes

$$\epsilon P_{0,0}^{n,n} = \epsilon^{f_\lambda + f_{p_n}}\delta\lambda\alpha_{p_n} I_0^n, \tag{4.4}$$

since $a_{n,k} = 0$ for $k > p_s$. From this we conclude

$$f_\lambda + f_{p_n} = 1, \tag{4.5a}$$

$$\alpha_{p_n}\delta\lambda = P_{0,0}^{n,n}/I_0^n. \tag{4.5b}$$

Next consider arbitrary j in the range $0 < j < p_n$:

$$\epsilon^{f_{p_n-j+1}}\alpha_{p_n-j+1} I_j^n = \epsilon^{f_\lambda + f_{p_n-j}}\alpha_{p_n-j}\delta\lambda I_j^n.$$

The terms depending on P do not appear here because ϵ^1 is of higher-order than $\epsilon^{f_{p_n-j+1}}$, as we shall show. So we conclude

$$f_\lambda + f_{p_n-j} = f_{p_n-j+1}, \tag{4.6a}$$

$$\alpha_{p_n-j}\delta\lambda = \alpha_{p_n-j+1}. \tag{4.6b}$$

Finally for $j = p_n$ we find

$$\epsilon^{f_1}\alpha_1 I_{p_n}^n = \epsilon^{f_\lambda}\delta\lambda I_{p_n}^n,$$

where we have only used the lowest-order part of equation (4.2) in the right-hand side. This leads to

$$f_\lambda = f_1 \tag{4.7a}$$

$$\delta\lambda = \alpha_1 \tag{4.7b}$$

Equations (4.5)–(4.7) can be solved easily. We add equation (4.7a) and all equations (4.6a) (for all allowed j) together to get

$$p_n f_\lambda = f_{p_n},$$

which when combined with equation (4.5a) gives

$$f_\lambda = \frac{1}{1+p_n}, \qquad f_j = \frac{j}{1+p_n}. \tag{4.8a}$$

Thus, all fractional powers are terms of *lower* order than ϵ. Similarly if we multiply equation (4.7b) and all of equations (4.6b) together we get

$$(\delta\lambda)^{p_n} = \alpha_{p_n}$$

which together with equation (4.5b) gives

$$(\delta\lambda)^{p_n+1} = P^{n,n}_{0,0}/I^n_0, \qquad \alpha_j = (\delta\lambda)^j. \tag{4.8b}$$

We therefore find that the new eigenvalue is

$$\lambda_n(\epsilon) = \lambda^0_n + (\epsilon P^{n,n}_{0,0}/I^n_0)^{1/(1+p_n)} + \ldots. \tag{4.9}$$

There are $1+p_n$ distinct roots one can use in equation (4.9), and for each there is an eigenvector linearly independent of the others. So the perturbation completely destroys the chain if $P^{n,n}_{0,0} \neq 0$.

Nothing so far has restricted the sign of ϵ, and it should be true for reasonable sequences that $T + \epsilon P$ describes the sequence well for small enough ϵ of either sign*. But we have assumed that for $\epsilon < 0$ the eigenvalues are all purely imaginary: no unstable modes. From equation (4.9) this is possible in only one case: $p_n = 1$ and $P^{n,n}_{0,0}/I^n_0$ real and positive. Any larger value of p_n will necessarily give roots elsewhere in the complex plane. Moreover, in this one case, the other side of the critical point, $\epsilon > 0$, will have an unstable mode. We can therefore draw the remarkable conclusion that, as long as $P^{n,n}_{0,0}$ is never zero, *the dynamically stable part of the sequence can have no Jordan chains at all and dynamical instability sets in through a Jordan chain of length 1.* So dynamical instability sets in in the simplest possible way. As μ increases, our picture shows us that pure-imaginary eigenvalues with no Jordan chains converge on one another in pairs; where they meet they form a Jordan chain of length 1, and for larger μ they diverge again, this time away from the imaginary axis.

This picture of instability is expressed in terms of one-dimensional sequences of models, because that is closest to the way models are constructed numerically. But it is worthwhile pointing out that every model is just one point in an infinite-dimensional space of all stellar models. All nearby points are probably accessible along some sequence of models for which

* An interesting exception to this is the beginning of the sequence with a non-rotating, spherical star. Such a star has an infinite number of zero-frequency modes, which occur as Jordan chains of length 1, and correspond to setting the star into rotation. They are split – in a stable manner, presumably – as soon as one moves along the sequence with some rotation, but there is no meaning to going in the 'other' direction along the sequence (where they would have complex frequencies).

the present analysis holds. We are now able to characterize the stability of a model by its *intrinsic* properties, without reference to a particular sequence. If a model has only imaginary $\{\lambda_n\}$ and no chains, then it is not only dynamically stable but also 'structurally' stable: all nearby models are also dynamically stable. If all its λ_n's are imaginary, but it has at least one chain, then it is *marginally stable*: most small changes in the star will produce a dynamically unstable model. Moreover, it is a point of marginal stability for nearly any sequence of stars one may happen to place it on.

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