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Perturbations and stability of rotating stars – III. Perturbation theory for eigenvalues

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Summary. The normal mode problem for perturbations of a perfect-fluid rotating star is a non-selfadjoint eigenvalue problem. To study the changes in the eigenfrequencies when the operators in the problem change slightly, this paper develops the appropriate lowest-order perturbation theory and discusses two important cases. In the first case the perturbing operator is the inclusion of a 'secular' effect, such as viscosity or radiation reaction. Here the main interest is in unperturbed eigenfrequencies that are dynamically stable, and the perturbed frequencies are analytic functions of the perturbation parameter. Explicit formulae are derived for use in numerical calculations of secular effects in stars. The second case is the study of the onset of dynamical instability itself. Here the perturbing operator is a slight change in the structure of the star. It is found that instability sets in only through eigenvalues which have Jordan chains, and the perturbation theory is non-analytic. It is shown that stable modes do not have Jordan chains, and the marginally stable modes have, in the generic case, chains of length 1. This gives a fairly complete characterization of the manner in which instability arises along a sequence of stars.

1 Introduction

The theory of the secular stability of perfect-fluid rotating stars is in reasonably good shape (Friedman & Schutz 1978), but there have been as yet very few calculations by which we can judge the astrophysical importance of these instabilities. The only published calculations are for the modes of the Maclaurin spheroids (Roberts & Stewartson 1963; Chandrasekhar 1969, 1970; Lindblom & Detweiler 1977; Comins 1978, 1979a, b) or for equally unrealistic systems (Papaloizou & Pringle 1978). This is particularly unfortunate in the case of gravitational-radiation-driven instabilities, which may affect all neutron stars at least in principle, but whose growth times are extremely sensitive to the exact structure of the star and of the unstable eigenfunction (Comins 1978). Because the secular terms are so small, it is likely that there would be substantial numerical errors inherent in any attempt to solve directly the dynamical equations with the secular terms included. It seems, therefore, that

Table 1. Peak line flux densities (S_L) and line/continuum ratios (R).

Galaxy	CH (3.26 GHz)		H_2CO (4.83 GHz)		OH (1.67 GHz)	
	S_L	$\|R\|$	S_L	$\|R\|$	S	$\|R\|$
NGC 253	$\leqslant 0.003$	$\leqslant 0.001$	−0.029	0.020	−0.19	0.079
LMC (N159)	0.013	0.006	−0.022	0.014	−0.10	0.05
NGC 4945	0.017	0.006	−0.16	0.080	−0.82	0.27
NGC 5128	0.079	0.026	−0.14	0.035	−0.19	0.038

continuum values used are generally lower than the observed values, in keeping with the discussions in the references above.

For NGC 4945 and NGC 253 it is likely that the CH emission arises in clouds near the nucleus. This is supported by the following:

(a) $|R|$ for CH is more than a factor of 10 lower than for H_2CO and OH. Such an underabundance of CH is similar to that found in clouds near the centre of our Galaxy (see Whiteoak *et al.* 1978).

(b) The velocities associated with the CH emission for NGC 4945 are similar to those for [N II] 6583 Å emission, which is concentrated in a region (< 5 arcsec) believed to be at the nucleus (Whiteoak & Gardner 1979).

For N159 the similarity of the 3264- and 3335-MHz profiles and the low values of $|R|$, increasing gradually from CH to OH, are typical of clouds not closely associated with H II regions or nuclei of galaxies (see discussion by Whiteoak *et al.* 1978).

The values of R for NGC 5128 in Table 1 are derived on the assumption that only the nuclear source is amplified or absorbed. From observations at adjacent frequencies by Wade *et al.* (1971), Christiansen *et al.* (1977) and Beall *et al.* (1978), it was estimated that the intensity of the nuclear component was 3 Jy (compared with total continuum of 75 Jy observed), 4 and 5 Jy at CH, H_2CO and OH frequencies. The above assumption is supported by the fact that the CH features are narrow and similar in shape and velocity to those of the 14.5-GHz transitions of H_2CO (Gardner & Whiteoak 1979), even though only 10 per cent of the continuum in the telescope beam at 14.5 GHz is from outside the nucleus.

Table 1 indicates that the values of $|R|$ for NGC 5128 are similar for CH, H_2CO and OH – a situation not found in our Galaxy. The high value of R for CH and the predominance of the 3264-MHz emission are more typical of clouds associated with H II regions outside than inside the nuclear region of our Galaxy. We have also argued (Gardner & Whiteoak 1979) from a comparison of line/continuum ratios for H_2CO at 4.8 and 14.5 GHz that the molecular cloud is in an outer dust lane well away from either the nucleus or any H II region. The most plausible explanation for the anomalously high values of R for CH is that there is an overabundance of CH relative to H_2CO in NGC 5128 compared with our Galaxy.

To conclude: the 3264-MHz transition of CH has been detected in three galaxies and possibly in a fourth. As in our Galaxy, the line profiles are similar to those for H_2CO. The results for N159 in the LMC are similar to those for CH clouds in our Galaxy; for NGC 4945 and NGC 253 they suggest relative underabundance of CH in their nuclear regions, as in our Galaxy; for NGC 5128, however, they indicate an overabundance of CH relative to our Galaxy.

Because the 3335-MHz transition of CH observed towards continuum regions is usually much fainter than the 3264-MHz transition, it is not surprising that it was observed in only one galaxy (LMC), and possibly in another (NGC 4945). The results confirm that, as in our Galaxy, the molecular clouds are not in local thermodynamic equilibrium, a situation also suggested by the observations of other molecules in these clouds.

the best procedure is to solve the pulsation problem without the secular terms and to use the resulting eigenfunctions to calculate the effect of the secular terms in a perturbation approximation. This is made possible by the completeness of the unperturbed eigenfunctions (Dyson & Schutz 1979 – Paper I). This paper develops such a method and derives explicit equations for the change in an eigenfrequency in terms of its zero-order eigenfunction when the secular terms arise from viscosity or radiation reaction.

The onset of dynamical instability is in some sense a simpler problem to study than that of secular instability, because one does not need to introduce extra terms into the equations: one simply monitors the behaviour of the eigenfrequencies along the sequence. Nevertheless, the actual onset of instability is poorly understood. Given that we have as yet no general criterion for dynamical instability (which is in contrast with the secular case), a better understanding of how dynamical instabilities make their appearance may be helpful. This can also be treated by perturbation theory, since one member of a sequence of stars may be thought of as a perturbation of an earlier member. We will find that the instability point is always marked by a Jordan chain, whose length is usually 1.

At this point it may be helpful to explain the principal differences between secular and dynamical instabilities in this context. The word 'dynamical' refers to our basic system of equations, those which govern linear perturbations of a perfect-fluid star. These equations are conservative, so the eigenfrequencies either are real or occur in complex-conjugate pairs (of which one leads to exponential growth of the perturbation). Suppose a sequence of models is constructed, parameterized by the real variable μ, along which the eigenfrequencies are continuous functions of μ. Then dynamical instability in a particular mode sets in at the value $\mu = \mu_d$ if its eigenfrequency is real below μ_d and complex above it. At μ_d the mode is said to be marginally unstable. (We will see below that the property of marginal instability, although defined here for a particular sequence, is in fact a sequence-independent property of the star itself.) Although the eigenfrequency is continuous at μ_d it is clearly not analytic in μ there (in the complex-variable sense).

By contrast, 'secular' instabilities are caused by changing, not the unperturbed model, but the dynamical equations governing the perturbation. If the added terms are dissipative then a zero-order real eigenfrequency will acquire an imaginary part, but there is no need for these to occur in complex-conjugate pairs. Again we say that secular instability in a particular mode sets in at the value $\mu = \mu_s$ if its secularly perturbed eigenfrequency corresponds to exponential decay below μ_s and to exponential growth above it. This is only meaningful if the mode is not dynamically unstable. In addition to μ there is a parameter ϵ giving the 'strength' of the secular terms, and the eigenfrequencies can be – and usually are – analytic functions of ϵ at $\epsilon = 0$.

The important difference between the eigenvalue perturbation theory that must be used here and the usual perturbation theory of selfadjoint operators that physicists are familiar with from quantum mechanics is that our problem, the oscillation of a rotating star, is essentially non-selfadjoint. This means that the eigenvectors are not necessarily orthogonal to one another, so that explicit use has to be made of left-eigenvectors as well as right-eigenvectors in this problem. A previous paper (Schutz 1980 – Paper II) discussed right-eigenfunctions and left-eigenfunctions in some detail, and showed that for stars which possess the reflection symmetry $(t, \phi) \rightarrow (-t, -\phi)$ there was a direct relation between the right-eigenfunctions and left-eigenfunctions for any particular eigenvalue. We shall use this relation in some of our perturbation formulae.

This paper is unfortunately not self-contained. It relies on an understanding of the results of Paper II and it follows the notation of Paper II and of Paper I. Nevertheless, it will be helpful to write down the basic formulae. The dynamical equation can be written (Lynden-

Bell & Ostriker 1967; Dyson & Schutz 1979 – Paper I)

$$\frac{\partial^2}{\partial t^2}\xi + B\left(\frac{\partial}{\partial t}\xi\right) + C(\xi) = 0, \tag{1.1}$$

where B and C are, respectively, antisymmetric and symmetric operators in a Hilbert space H whose inner product is

$$\langle \xi | \eta \rangle = \int \rho \bar{\xi} \cdot \eta \, dv, \tag{1.2}$$

where ρ is the density of the star and a bar denotes complex-conjugation. In equation (1.2) Dirac bra and ket notation has been introduced, and it will be used from now on. The eigenvalue problem for equation (1.1) comes from the assumption

$$|\xi(t)\rangle = |\xi\rangle \exp(\lambda t): \tag{1.3}$$

$$(\lambda^2 + \lambda B + C)|\xi\rangle = 0. \tag{1.4}$$

(Note the convention adopted here, that λ is the eigenvalue. The 'frequency' is $i\lambda$ or $-i\lambda$, depending on one's convention.) In discussing perturbation theory it is more convenient to deal with the first-order eigenvalue problem in the space $H_2 = H \oplus H$, which is 'initial-data' space, the space of pairs $(|\xi(t)\rangle, d/dt\,|\xi(t)\rangle) = |\eta\rangle\rangle$. This double-bracket notation will be used to distinguish elements of H_2 from their counterparts in H. The associated eigenvalue problem is

$$T|\eta\rangle\rangle = \lambda|\eta\rangle\rangle, \qquad T = \begin{pmatrix} 0 & 1 \\ -C & -B \end{pmatrix}. \tag{1.5}$$

The matrix T is not symmetric with respect to the inner product of H_2 induced by H:

$$\langle\langle \xi | \eta \rangle\rangle = \langle \xi_1 | \eta_1 \rangle + \langle \xi_2 | \eta_2 \rangle$$

where $|\eta_1\rangle$ and $|\eta_2\rangle$ are the 'components' of $|\eta\rangle\rangle$. Since the eigenvalues λ are eigenvalues of T, it is T whose perturbation theory we must investigate. As in Paper II we shall assume that T is a finite-dimensional matrix, as it always is in numerical approximations to the continuous problem.

2 General remarks on perturbation theory

We shall suppose that a matrix T has eigenvalues $\{\lambda_n^0\}$ and right- (left-) eigenvectors and Jordan chain vectors $\{|\lambda_n^0;\, r_k\rangle\rangle\}$ $(\{\langle\langle\lambda_n^0;\, l_k|\})$, with $k = 0$ denoting the eigenvector and $k = 1, \ldots, p_n$ forming the chain. Since T is a function of the sequence's parameter μ, the first question is how the eigenvalues depend on μ. We quote three results from Kato (1966, Chapter II). First, if T is continuous in μ then its eigenvalues are continuous in μ. Moreover, the dimension of the characteristic subspace of an eigenvalue (that is, the total number of eigenvectors and Jordan chain vectors associated with that eigenvalue) is 'conserved', in the sense that if an eigenvalue splits into distinct eigenvalues the sum of the dimensions of the characteristic subspaces of the distinct eigenvalues equals the total dimension of the original subspace. In fact, the total projection operator on to these subspaces is continuous in μ. The second result is that if T is differentiable in μ at some μ_1 then this same total projection operator is likewise differentiable, but this is not necessarily true of the eigenvalues. If an eigenvalue has no Jordan chain at μ_1 (a 'semisimple' eigenvalue) then it *is* differentiable at μ_1 even if it splits into several eigenvalues for $\mu > \mu_1$, and in fact

$$\lambda(\mu) = \lambda(\mu_1) + (\mu - \mu_1)\lambda^{(1)}(\mu_1) + o(\mu - \mu_1) \tag{2.1}$$

where $\lambda^1(\mu_1)$ is an eigenvalue of $(dT/d\mu)_{\mu_1}$ restricted to the total characteristic subspace, and where $o(\mu-\mu_1)$ is the usual order symbol for a function which goes to zero faster than $\mu-\mu_1$. So if a sequence of models is differentiable, first-order perturbation theory gives at least an asymptotic expression for the change in the eigenvalue. We can immediately conclude that if a semisimple eigenvalue at μ_1 splits into several eigenvalues for $\mu>\mu_1$ in such a way that the various $\lambda^{(1)}(\mu_1)$'s are not zero, then it must split into the same number of eigenvalues for $\mu<\mu_1$, or in other words that μ_1 is merely a crossing point for the eigenvalues. Now, a dynamical instability point is a value of μ where eigenvalues split, since on the unstable side of it there is a complex-conjugate pair which become equal at the instability point. But it is not a crossing point, since the eigenvalues are real on the other side of it. So along a differentiable sequence dynamical instability must occur through a mode which is not semisimple: it must have a Jordan chain. We concluded this for continuous sequences by a different argument in Paper II.

Kato's third result concerns sequences which are in fact (complex-) analytic functions of their parameter μ in some simply connected region D of the complex plane. Here the number of distinct eigenvalues is constant except at a finite number of exceptional points, where degeneracy may occur without Jordan chains (semisimple case) or with them. The total projection operator discussed above is holomorphic even at the exceptional points. In the semisimple case the eigenvalues are analytic functions of μ even at exceptional points, while those with Jordan chains at an exceptional point are branches of analytic functions with at most algebraic singularities at the exceptional points. This means that the lowest-order change in an eigenvalue away from μ_1 is proportional at worst to some fractional power of $\mu-\mu_1$. If the Jordan chain has length p the branch point has order no larger than p. In treatments of secular stability the perturbed operator is usually taken to have the form $T+\epsilon P$, which is of course analytic in ϵ, so that the onset of secular instability can be studied by analytic perturbation theory. This is consistent with the perturbation analysis of the Maclaurin spheroids by Roberts & Stewartson (1963), who did not assume that the problem was finite dimensional.

In order to put some flesh on the bare bones of these theorems, let us consider the operator family $T+\epsilon P$. Here ϵP is either a change in the character of the equations (secular stability) or a change in T itself brought on by increasing the sequence's parameter μ by ϵ (dynamical stability). In the latter case, we will therefore be able to apply our first-order results only to sequences which are at least differentiable, as in equation (2.1).

Suppose that $T+\epsilon P$ has an eigenvalue $\lambda_n(\epsilon)$ and an associated right-eigenvector $|\lambda_n(\epsilon); r_0\rangle\rangle$. We expand this in terms of the basis $\{|\lambda_n^0; r_k\rangle\rangle\}$:

$$|\lambda_n(\epsilon); r_0\rangle\rangle = \sum_m \sum_{k=0}^{p_m} a_{mk}(\epsilon)|\lambda_m^0; r_k\rangle\rangle, \tag{2.2}$$

where p_m is the length of the Jordan chain to λ_m^0. Applying $T+\epsilon P$ to equation (2.2) gives

$$(T+\epsilon P)|\lambda_n(\epsilon); r_0\rangle\rangle = \lambda_n(\epsilon)|\lambda_n(\epsilon); r_0\rangle\rangle$$

$$\sum_{m,k} a_{mk}(\epsilon)\,(\lambda_m^0|\lambda_m^0; r_k\rangle\rangle + |\lambda_m^0; r_{k-1}\rangle\rangle + \epsilon P|\lambda_m^0; r_k\rangle\rangle) = \sum_{m,k} a_{mk}(\epsilon)\lambda_n(\epsilon)|\lambda_m^0; r_k\rangle\rangle, \tag{2.3}$$

provided we adopt the convention that $|\lambda_n^0; r_k\rangle\rangle = 0$ if $k<0$. Now, if we contract equation (2.2) with $\langle\langle\lambda_s^0; l_j|$ for arbitrary s, employing the notation

$$P_{j,k}^{s,m} = \langle\langle\lambda_s^0; l_j|P|\lambda_m^0; r_k\rangle\rangle, \tag{2.4a}$$

$$I_j^s = \langle\langle\lambda_s^0; l_j|\lambda_s^0; r_{p_s-j}\rangle\rangle\ (\neq 0), \tag{2.4b}$$

then the orthogonality relations given in Paper II give

$$[a_{s,p_s-j}(\epsilon)\lambda_s^0 + a_{s,p_s-j+1}(\epsilon)]I_j^s + \epsilon \sum_{m,k} a_{mk}(\epsilon)P_{j,k}^{s,m} = \lambda_n(\epsilon)a_{s,p_s-j}(\epsilon)I_j^s, \tag{2.5}$$

where $a_{s,k} = 0$ if $k > p_s$. By continuity, the limit $\epsilon \to 0$ gives

$$(\lambda_s^0 - \lambda_n^0)a_{s,p_s-j}(0) + a_{s,p_s-j+1}(0) = 0,$$

This is a system of equations in j for fixed s and n, whose solution is

$$\begin{aligned} &a_{s,j}(0) = 0 \qquad \forall s, \forall\, j > 0, \\ &a_{s,0}(0) = 0 \qquad \text{if } \lambda_s^0 \neq \lambda_n^0. \end{aligned} \tag{2.6}$$

This means that the $\epsilon \to 0$ limit of $|\lambda_n(\epsilon); r_0\rangle\rangle$ is a linear combination of those eigenvectors which have eigenvalues equal to λ_n^0. For simplicity, we shall assume there is no degeneracy at λ_n^0: no other eigenvalues equal it. Degeneracy can be handled exactly as in selfadjoint perturbation theory.

If we assume that λ_n and the a's are *analytic* in ϵ at $\epsilon = 0$ then equation (2.6) says that $a_{n,0} = 1 + 0(\epsilon)$ and $a_{n,j} = 0(\epsilon), j \geqslant 1$. Expanding λ_n in its Taylor series,

$$\lambda_n(\epsilon) = \lambda_n^0 + \delta\lambda_n + 0(\epsilon^2), \tag{2.7}$$

we find that equation (2.5) gives for $s = n$

$$(a_{n,p_n-j+1} - \delta\lambda_n\delta_{j,p_n})I_j^n + \epsilon P_{j,0}^{n,n} + 0(\epsilon^2) = 0, \qquad j = 0, \ldots, p_n, \tag{2.8}$$

where δ_{j,p_n} is the Kronecker delta.

If $p_n = 0$ (no Jordan chains in the unperturbed mode) then equation (2.8) immediately gives the first-order change in the eigenvalue:

$$\delta\lambda_n = \epsilon P_{0,0}^{n,n}/I_0^n = \epsilon \frac{\langle\langle \lambda_n^0; l_0|P|\lambda_n^0; r_0\rangle\rangle}{\langle\langle \lambda_n^0; l_0|\lambda_n^0; r_0\rangle\rangle}. \tag{2.9}$$

This is clearly analogous to the familiar formula for the perturbation theory of selfadjoint operators, but here one must be careful to use $\langle\langle\lambda_n^0; l_0|$, the left-eigenvector of T, which is usually not the adjoint of $|\lambda_n^0; r_0\rangle\rangle$.

We can see how the assumption of analyticity breaks down if there is a Jordan chain to λ_n^0. With $p_n > 0$, equation (2.8) for $j = 0$ gives

$$\epsilon P_{0,0}^{n,n} = 0. \tag{2.10}$$

The element of the matrix P cannot be expected to vanish in general, leading to a contradiction. We shall look further at this case in our study of dynamical instability in Section 4. The analytic formula, equation (2.9), suffices for the study of secular instability, to which we now turn.

3 Analytic perturbation theory and secular stability

Secular instability in a mode is really only of interest when that mode has a purely real unperturbed frequency (purely imaginary λ_n^0). As we shall see in Section 4, such modes cannot possess Jordan chains if they are not marginally dynamically unstable, so we shall assume in this section that $p_n = 0$.

3.1 VISCOSITY

In a stationary, isolated star, the only stationary axisymmetric state in the presence of viscosity is rigid rotation with angular velocity Ω. So we shall restrict ourselves to such stars in this section. They clearly have the reflection symmetry $(t, \phi) \to (-t, -\phi)$.

Viscosity changes the dynamical equation to (Friedman & Schutz 1978)

$$(\lambda^2 + \lambda B + C)\boldsymbol{\xi} = -(\lambda + im\Omega)F\boldsymbol{\xi} \tag{3.1}$$

where

$$-(F\boldsymbol{\xi})^i = \frac{2}{\rho}\nabla_j(\eta\Sigma^{ij}) + \frac{1}{\rho}\nabla^i(\zeta\theta) \tag{3.2}$$

$$\Sigma^{ij} = \tfrac{1}{2}(\nabla^i\xi^j + \nabla^j\xi^i) - \tfrac{1}{3}g^{ij}\theta \tag{3.3}$$

$$\theta = \nabla_i\xi^i \tag{3.4}$$

(summation on repeated indices). Here η is the coefficient of shear viscosity and ζ the coefficient of bulk viscosity. We wish to find a representation of equation (3.1) in H_2. The following operator gives the correct result:

$$P = \begin{pmatrix} 0 & 0 \\ -im\Omega F & -F \end{pmatrix}. \tag{3.5}$$

Then if $|\lambda_n^0; r_0\rangle$ is the right-eigenvector of $L(\lambda_n^0)$ we have from Paper II

$$|\lambda_n^0; r_0\rangle\rangle = \begin{pmatrix} |\lambda_n^0; r_0\rangle \\ \lambda_n^0|\lambda_n^0; r_0\rangle \end{pmatrix} \tag{3.6}$$

and

$$\langle\langle\lambda_n^0; l_0| = (\langle S; \lambda_n^0; r_0|(\lambda_n^0 + B), \quad \langle S; \lambda_n^0; r_0|). \tag{3.7}$$

This gives, from equation (2.9),

$$\delta\lambda_n = -(\lambda_n^0 + im\Omega)\frac{\langle S; \lambda_n^0; r_0|F|\lambda_n^0; r_0\rangle}{\langle S; \lambda_n^0; r_0|(2\lambda_n^0 + B)|\lambda_n^0; r_0\rangle}. \tag{3.8}$$

In more concrete terms, suppose that the zero-order eigenfunction is

$$|\lambda_n^0; r_0\rangle = \begin{pmatrix} \xi^{\varpi}(\varpi, z) \\ \xi^{\phi}(\varpi, z) \\ \xi^{z}(\varpi, z) \end{pmatrix} \exp(im\phi), \tag{3.9}$$

referred to an *orthonormal* basis in cylindrical polar coordinates. Then

$$\langle S; \lambda_n^0; r_0| = (\xi^{\varpi}, -\xi^{\phi}, \xi^{z}) \exp(-im\phi) \tag{3.10}$$

is the left-eigenfunction. A rather long but straightforward calculation gives that the numera-

tor in equation (3.8) is, after an integration by parts,

$$\langle S;\lambda_n^0;r_0|F|\lambda_n^0;r_0\rangle = \int\int \varpi d\varpi\, dz \left\{\eta\left[2(\partial_\varpi \xi^\varpi)^2 + (\partial_\varpi \xi^z)^2 + 2\partial_z \xi^\varpi \partial_\varpi \xi^z \right.\right.$$
$$+ (\partial_z \xi^\varpi)^2 + 2(\partial_z \xi^z)^2 - \frac{m^2}{\varpi^2}(\xi^z)^2 + \frac{1-m^2}{\varpi^2}(\xi^\varpi)^2 - (\partial_z \xi^\phi)^2 - (\partial_\varpi \xi^\phi)^2$$
$$\left. - \frac{2}{\varpi}\xi^\phi \partial_\varpi \xi^\phi - \frac{1-m^2}{\varpi^2}(\xi^\phi)^2\right] + (\zeta - \tfrac{2}{3}\eta)\left[\left(\partial_z \xi^z + \partial_\varpi \xi^\varpi + \frac{1}{\varpi}\xi^\varpi\right)^2\right.$$
$$\left.\left. + \frac{m^2}{\varpi^2}(\xi^\phi)^2\right]\right\}. \tag{3.11}$$

Similarly, the denominator is

$$\langle S, \lambda_n^0; r_0|(2\lambda_n + B)|\lambda_n^0, r_0\rangle = -2\iint \rho\varpi\, d\varpi\, dz\, \{(\lambda_n^0 + im\Omega)[(\xi^\varpi)^2 + (\xi^z)^2$$
$$- (\xi^\phi)^2] - \Omega\xi^\phi\xi^\varpi\}. \tag{3.12}$$

Although complicated, these expressions should be easy enough to evaluate numerically.

3.2 GRAVITATIONAL RADIATION

The gravitational radiation emitted by a pulsating object is generally a very complicated function of the pulsation, but in the case of a nearly-Newtonian star emitting wavelengths large compared to the stellar radius, considerable simplification is possible. We will follow the treatment of Thorne (1969). The important point is that when the wavefield is analysed in spherical harmonics Y_{lm} at infinity, the dominant radiation usually comes from the lowest value of $l \geqslant 2$ contributing to the radiation ($l = 0$ and 1 do not contribute any radiation in general relativity). The strict statement is that in an asymptotic expansion in powers of c^{-1}, the radiation from each l is proportional to $c^{-(2l+2)}$. While it is possible to have a star whose quadrupole moment ($l = 2$) is changing much less rapidly than its octopole moment ($l = 3$) and so contributes less radiation, it should be safe to assume that this is a rather contrived circumstance for any star whose unperturbed shape is reasonably non-spherical. Since for spherical harmonics one always has $|m| \leqslant l$, it follows that an axisymmetric star with a perturbation of azimuthal eigenvalue m gives off its dominant gravitational radiation with $l = |m|$. We shall restrict our attention to this case, and assume the star has the reflection-symmetry $(t, \phi) \to (-t, -\phi)$.

In Thorne's treatment, the effect of radiation on the star, which is what we are interested in, is accounted for by a change in the Newtonian gravitational potential, the addition of a new radiation-reaction potential. For $l = m$ this is

$$\delta\Phi_{RR} = 4\pi G \epsilon_m F_m Y_{mm}(\theta, \phi) r^m, \tag{3.13}$$

$$\epsilon_m = (-1)^{m+1} \frac{(m+1)(m+2)}{m(m-1)(2m+1)^2\,[(2m-1)!!]^2}, \tag{3.14a}$$

$$F_m(t) = \frac{1}{c^{(2m+1)}}\left(\frac{d}{dt}\right)^{2m+1} \int \rho(t) r^m (Y_{mm})^*\, dV. \tag{3.14b}$$

Because F_m contains an odd number of time-derivatives, it is an energy-dissipating term. If

we define an element of H

$$|RR;m\rangle = \nabla[r^m Y_{mm}(\theta,\phi)] \tag{3.15}$$

then it is not hard to show that the integral in equation (3.14b) is just

$$\langle RR;m|\xi\rangle, \tag{3.16}$$

since

$$\rho(t) = \delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}).$$

(Recall that the inner product of equation (3.16) is weighted by ρ.)

The equation of motion involves $\nabla\delta\Phi_{RR}$ and so can be written

$$(\lambda^2 + \lambda B + C)|\xi\rangle = -4\pi G\epsilon_m\left[\left(\frac{d}{dct}\right)^{2m+1}\langle RR;m|\xi\rangle\right]|RR;m\rangle. \tag{3.17}$$

It is clear that this does not fit into the scheme we have employed up till now because it contains higher than second time-derivatives. On the other hand, the right-hand side of equation (3.17) must be small for perturbation theory to work, so we do not make any error at this order if we use the unperturbed time-dependence of $|\xi\rangle$ on the right-hand side. For a mode $|\lambda_n^0; r_0\rangle$, this amounts to replacing d/dt by λ_n^0. We find, then, that each mode has its own perturbation operator G_n:

$$(\lambda^2 + \lambda B + C)|\lambda_n; r_0\rangle = G_n|\lambda_n; r_0\rangle \tag{3.18}$$

$$G_n = -4\pi G\epsilon_m(\lambda_n^0/c)^{2m+1}|RR;m\rangle\langle RR;m|. \tag{3.19}$$

In H_2 the appropriate operator is

$$P_n = \begin{pmatrix} 0 & 0 \\ G_n & 0 \end{pmatrix}.$$

Following equations (3.6), (3.7) and (2.9), we find

$$\delta\lambda_n = \frac{\langle S;\lambda_n^0;r_0|G_n|\lambda_n^0;r_0\rangle}{\langle S;\lambda_n^0;r_0|(2\lambda_n^0 + B)|\lambda_n^0;r_0\rangle}. \tag{3.21}$$

If we use the same form (3.9) for $|\lambda_n^0; r_0\rangle$, the denominator of equation (3.21) is the same as equation (3.12). To evaluate the numerator it is more convenient to use an orthonormal basis in spherical polar coordinates, in which $\boldsymbol{\xi}$ has components $(\xi^r, \xi^\theta, \xi^\phi)$. Then the numerator is

$$\langle S;\lambda_n^0;r_0|G_n|\lambda_n^0;r_0\rangle = -4\pi G\epsilon_m(\lambda_n^0/c)^{2m+1}(N_{mm})^2 \times\left[\int\int \rho r^{2m}\,dr\sin\theta\,d\theta\left(m\xi^r P_m^m + \xi^\theta\partial_\theta P_m^m - \frac{im}{\sin\theta}\xi^\phi P_m^m\right)\right]^2, \tag{3.22}$$

where we have rewritten Y_{mm} as

$$Y_{mm} = N_{mm}P_m^m\exp(im\phi), \tag{3.23}$$

with P_m^m the usual associated Legendre polynomial and N_{mm} is the usual normalization factor

$$N_{mm} = \left(\frac{2m+1}{4\pi(2m)!}\right)^{1/2}. \tag{3.24}$$

Equation (3.22) should be particularly easy to evaluate because it does not involve derivatives of ξ.

4 Non-analytic perturbation theory and the onset of dynamical instability

Here we assume that we have a differentiable sequence of stars with parameter μ, so that for any μ_0 $T(\mu)$ can be approximated by $T(\mu_0) + (\mu - \mu_0)P$ for sufficiently small $(\mu - \mu_0)$, with $P = dT/d\mu|_{\mu_0}$. We are interested in what happens if μ_0 has a Jordan chain to one of its eigenvalues, λ_n^0. Our starting point is equation (2.5), where ϵ stands for $\mu - \mu_0$:

$$\{a_{s,p_s-j}(\epsilon)\lambda_s^0 + a_{s,p_s-j+1}(\epsilon)\}\, I_j^s + \epsilon \sum_{m,k} a_{mk}(\epsilon) P_{j,k}^{s,m} = \lambda_n(\epsilon) a_{s,p_s-j}(\epsilon) I_j^s. \tag{2.5}$$

It is easy to verify that the assumption that $\{a_{s,k},\ k = 1, \ldots, p_s\}$ are analytic in ϵ when $\lambda_s \neq \lambda_n$ is consistent. Moreover in this case equation (2.5) gives no information about $\lambda_n(\epsilon)$. Since we continue to assume that $\lambda_s \neq \lambda_n$ whenever $s \neq n$ (no 'irrelevant' degeneracy), we shall concentrate on the case $s = n$. Then equation (2.5) is solved by allowing $\lambda_n(\epsilon)$ and $a_{n,k}(\epsilon)$ to depend on fractional powers of ϵ. We try the forms:

$$\lambda_n(\epsilon) = \lambda_n^0 + \epsilon^{f_\lambda}\delta\lambda + \ldots \tag{4.1}$$

$$a_{n,0}(\epsilon) = 1 + \epsilon^{f_0}\alpha_0 + \ldots \tag{4.2}$$

$$a_{n,k}(\epsilon) = \epsilon^{f_k}\alpha_k + \ldots, \qquad k = 1, \ldots, p_n, \tag{4.3}$$

where $\{\delta\lambda,\ \alpha_k\}$ are constants and all higher-order terms are dropped. The powers $\{f_\lambda, f_k\}$ must all be non-negative, since the $\epsilon \to 0$ limit exists. Let us examine equation (2.5) for various values of j, keeping only the lowest powers of ϵ in each term. We shall assume that $P_{0,0}^{n,n} \neq 0$. This must be regarded as the typical situation. The analysis becomes very much more complicated if $P_{0,0}^{n,n}$ vanishes, and we hope to return to it in a later publication. Equation (2.5) for $j = 0$ becomes

$$\epsilon P_{0,0}^{n,n} = \epsilon^{f_\lambda + f_{p_n}}\delta\lambda\alpha_{p_n} I_0^n, \tag{4.4}$$

since $a_{n,k} = 0$ for $k > p_s$. From this we conclude

$$f_\lambda + f_{p_n} = 1, \tag{4.5a}$$

$$\alpha_{p_n}\delta\lambda = P_{0,0}^{n,n}/I_0^n. \tag{4.5b}$$

Next consider arbitrary j in the range $0 < j < p_n$:

$$\epsilon^{f_{p_n-j+1}}\alpha_{p_n-j+1} I_j^n = \epsilon^{f_\lambda + f_{p_n-j}}\alpha_{p_n-j}\delta\lambda I_j^n.$$

The terms depending on P do not appear here because ϵ^1 is of higher-order than $\epsilon^{f_{p_n-j+1}}$, as we shall show. So we conclude

$$f_\lambda + f_{p_n-j} = f_{p_n-j+1}, \tag{4.6a}$$

$$\alpha_{p_n-j}\delta\lambda = \alpha_{p_n-j+1}. \tag{4.6b}$$

Finally for $j = p_n$ we find

$$\epsilon^{f_1}\alpha_1 I_{p_n}^n = \epsilon^{f_\lambda}\delta\lambda I_{p_n}^n,$$

where we have only used the lowest-order part of equation (4.2) in the right-hand side. This leads to

$$f_\lambda = f_1 \tag{4.7a}$$

$$\delta\lambda = \alpha_1 \tag{4.7b}$$

Equations (4.5)–(4.7) can be solved easily. We add equation (4.7a) and all equations (4.6a) (for all allowed j) together to get

$$p_n f_\lambda = f_{p_n},$$

which when combined with equation (4.5a) gives

$$f_\lambda = \frac{1}{1+p_n}, \qquad f_j = \frac{j}{1+p_n}. \tag{4.8a}$$

Thus, all fractional powers are terms of *lower* order than ϵ. Similarly if we multiply equation (4.7b) and all of equations (4.6b) together we get

$$(\delta\lambda)^{p_n} = \alpha_{p_n}$$

which together with equation (4.5b) gives

$$(\delta\lambda)^{p_n+1} = P^{n,n}_{0,0}/I^n_0, \qquad \alpha_j = (\delta\lambda)^j. \tag{4.8b}$$

We therefore find that the new eigenvalue is

$$\lambda_n(\epsilon) = \lambda^0_n + (\epsilon P^{n,n}_{0,0}/I^n_0)^{1/(1+p_n)} + \ldots. \tag{4.9}$$

There are $1+p_n$ distinct roots one can use in equation (4.9), and for each there is an eigenvector linearly independent of the others. So the perturbation completely destroys the chain if $P^{n,n}_{0,0} \neq 0$.

Nothing so far has restricted the sign of ϵ, and it should be true for reasonable sequences that $T + \epsilon P$ describes the sequence well for small enough ϵ of either sign★. But we have assumed that for $\epsilon < 0$ the eigenvalues are all purely imaginary: no unstable modes. From equation (4.9) this is possible in only one case: $p_n = 1$ and $P^{n,n}_{0,0}/I^n_0$ real and positive. Any larger value of p_n will necessarily give roots elsewhere in the complex plane. Moreover, in this one case, the other side of the critical point, $\epsilon > 0$, will have an unstable mode. We can therefore draw the remarkable conclusion that, as long as $P^{n,n}_{0,0}$ is never zero, *the dynamically stable part of the sequence can have no Jordan chains at all and dynamical instability sets in through a Jordan chain of length 1.* So dynamical instability sets in in the simplest possible way. As μ increases, our picture shows us that pure-imaginary eigenvalues with no Jordan chains converge on one another in pairs; where they meet they form a Jordan chain of length 1, and for larger μ they diverge again, this time away from the imaginary axis.

This picture of instability is expressed in terms of one-dimensional sequences of models, because that is closest to the way models are constructed numerically. But it is worthwhile pointing out that every model is just one point in an infinite-dimensional space of all stellar models. All nearby points are probably accessible along some sequence of models for which

★ An interesting exception to this is the beginning of the sequence with a non-rotating, spherical star. Such a star has an infinite number of zero-frequency modes, which occur as Jordan chains of length 1, and correspond to setting the star into rotation. They are split – in a stable manner, presumably – as soon as one moves along the sequence with some rotation, but there is no meaning to going in the 'other' direction along the sequence (where they would have complex frequencies).

the present analysis holds. We are now able to characterize the stability of a model by its *intrinsic* properties, without reference to a particular sequence. If a model has only imaginary $\{\lambda_n\}$ and no chains, then it is not only dynamically stable but also 'structurally' stable: all nearby models are also dynamically stable. If all its λ_n's are imaginary, but it has at least one chain, then it is *marginally stable*: most small changes in the star will produce a dynamically unstable model. Moreover, it is a point of marginal stability for nearly any sequence of stars one may happen to place it on.

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