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$C^{1,\alpha}$ Theory for the Prescribed Mean Curvature Equation with Dirichlet Data

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Abstract In this work we study solutions of the prescribed mean curvature equation over a general domain that do not necessarily attain the given boundary data. With such a solution we can naturally associate a current with support in the closed cylinder above the domain and with boundary given by the prescribed boundary data and which inherits a natural minimizing property. Our main result is that its support is a $C^{1,\alpha}$ manifold-with-boundary, with boundary equal to the prescribed boundary data, provided that both the initial domain and the prescribed boundary data are of class $C^{1,\alpha}$.

Keywords Prescribed mean curvature equation · Minimal surfaces · Boundary regularity

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1 Introduction

The Dirichlet problem for surfaces of prescribed mean curvature in an open set Ω of $\mathbb{R}^n$ concerns the existence of a solution to the equation

$$\sum_{i=1}^{n} D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = H(x,u) \quad \text{in } \Omega \tag{1.1}$$

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taking prescribed values

$$u = \phi \quad \text{on } \partial\Omega.$$

Here and throughout this paper $\Omega \subset \mathbb{R}^n$ is an open bounded set, $\phi \in L^1(\partial\Omega)$, and $H(x, x_{n+1})$ is a C^1 function defined in $\Omega \times \mathbb{R}$, which is non-decreasing in the x_{n+1}-variable and such that $\|H\|_0 \le n(\omega_n/|\Omega|)^{1/n}$.

It is known [15, 23] that if $\partial\Omega$ is C^2, then a solution exists for any given boundary values $\phi \in C^0(\partial\Omega)$ provided that $H_{\partial\Omega}(x) > |H(x, \phi(x))|$ for each $x \in \partial\Omega$, where $H_{\partial\Omega}$ denotes the mean curvature of the boundary, and furthermore, the regularity of the solution depends on that of $\partial\Omega$ and ϕ. Here and in what follows, we adopt the sign convention according to which the mean curvature of $\partial\Omega$ is non-negative in case Ω is convex. Furthermore, there are examples that indicate that this condition is necessary for the existence of a solution (cf. [13, 14.4]).

Our goal is to study the regularity of such a solution without imposing any curvature conditions for $\partial\Omega$. For this reason we will use a variational approach to the Dirichlet problem (cf. [11, 22]) and look for a minimum of the functional

$$\mathcal{F}(v) = \int_\Omega \sqrt{1 + |Dv|^2}dx + \int_\Omega \int_0^{v(x)} H(x, x_{n+1})dx dx_{n+1} + \int_{\partial\Omega} |v - \phi| dx \quad (1.2)$$

for $v \in \mathrm{BV}(\Omega)$; here $\mathrm{BV}(\Omega)$ denotes the space of all functions in $L^1(\Omega)$ that have bounded variation, i.e., with first distribution derivatives given by signed Radon measures.

Giusti and Miranda [11, 22] have proved that if $\partial\Omega$ is Lipschitz, then there exists a minimizer u of the functional $\mathcal{F}$, which is unique up to translations. Furthermore, this minimizer satisfies (1.1) in Ω (cf. [3, 19, 28]) and attains the prescribed boundary values above any C^2 portion of the boundary where the mean curvature is bigger than $|H(x, \phi(x))|$ [21].

The purpose of this paper is to give a complete and general discussion on the regularity of the hypersurface obtained by taking the union of $\{(x, u(x)) : x \in \overline{\Omega}\}$ and the part of $\partial\Omega \times \mathbb{R}$ which is enclosed by $\{(x, u(x)) : x \in \partial\Omega\}$ and $\{(x, \phi(x)) : x \in \partial\Omega\}$, where u is the minimizer of $\mathcal{F}$. In particular, in our main theorem (Theorem 4.2) we prove that if $\partial\Omega$ is $C^{1,\alpha}$ and $\phi \in C^{1,\alpha}(\partial\Omega)$, then this hypersurface is a $C^{1,\alpha}$ manifold-with-boundary, with boundary equal to graph ϕ. We also show that this regularity result can be extended for boundary data $\phi \in C^{1,\alpha}(\partial\Omega \setminus \{x_0\})$, where at $x_0 \in \partial\Omega$, ϕ has a jump discontinuity.

Furthermore, we will show that this manifold can be obtained as the $C^{1,\alpha}$ limit (as submanifolds of $\mathbb{R}^{n+1}$) of graphs of $C^{1,\alpha}$ functions over $\overline{\Omega}$. The main idea is to approximate the equation of the given Dirichlet problem (1.1) by new equations in which we change the RHS near the boundary by adding a divergence term that will allow us to prove existence of barriers for solutions of the new equations. We then use techniques from the theory of integer multiplicity varifolds, integral currents, and partial differential equations to get uniform $C^{1,\alpha}$ estimates for the graphs of the solutions to the approximating equations.

Concerning the regularity of u (the minimizer of $\mathcal{F}$), it is known [12, 24] that if ϕ is Lipschitz, then above any C^2 portion of the boundary where the mean curvature is bigger than $|H(x, \phi(x))|$, u is Hölder continuous for some positive exponent.

However, above points of the boundary where this condition is not satisfied, we could have $u \neq \phi$, and there are examples that show that the gradient of u does not have to be bounded near these points. In [25] it is proved that if Ω is a C^4 domain and ϕ is a Lipschitz function over $\partial\Omega$ then in the case $H = 0$, u is Hölder continuous at every point $x \in \partial\Omega$ where the mean curvature is negative, and furthermore, the trace of u as a function above $\partial\Omega$ is locally Lipschitz at these points. Note that since $u \in \mathrm{BV}(\Omega)$ it has a well defined trace in $L^1(\partial\Omega)$. In [17] this result was extended for surfaces of prescribed mean curvature $H = H(x)$.

The hypersurface that corresponds to u, as described above, inherits a minimizing property which we now describe: With a function $v \in \mathrm{BV}(\Omega)$ we can associate an integral n-current defined by

$$T_v = [\![\mathrm{graph}\, v]\!] + Q, \tag{1.3}$$

where Q is the multiplicity one n-current with support in $\partial\Omega \times \mathbb{R}$ and boundary $\partial Q = [\![\mathrm{graph}\, \phi]\!] - [\![\mathrm{trace}\, v]\!]$. Here and in what follows, for the orientation of a current $[\![\mathrm{graph}\, v]\!]$ associated with the graph of a function v we use the downward pointing unit normal to the graph. For any multiplicity one n-current S such that $\mathrm{spt}\, S \subset \overline{\Omega} \times \mathbb{R}$ and $\partial S = [\![\mathrm{graph}\, \phi]\!]$ we let $\widetilde{S}$ be the multiplicity one $(n+1)$-current such that $S - [\![(x, z) : x \in \partial\Omega, z \leq \phi(x)]\!] = \partial\widetilde{S}$. Then if u minimizes the functional $\mathcal{F}$, the current $T = T_u$, as defined in (1.3), locally minimizes the functional

$$\underline{\underline{M}}(T) + \int_{\mathrm{spt}\widetilde{T}} H(x', x_{n+1})dx'dx_{n+1} \tag{1.4}$$

among all integral n-currents with support in $\overline{\Omega} \times \mathbb{R}$ and boundary $[\![\mathrm{graph}\, \phi]\!]$ [17], where $\underline{\underline{M}}(T)$ denotes the mass of the current T.

This observation was first made by Lin and Lau [18] for the $H = 0$ case. In particular, they observed that in that case T minimizes area among all integral currents with support in $\overline{\Omega} \times \mathbb{R}$ and boundary equal to $[\![\mathrm{graph}\, \phi]\!]$, thus locally, near points of the trace of u that are away from $\mathrm{graph}\, \phi$, $\mathrm{spt}\, T$ is a solution to a parametric obstacle problem. Hence, using results from [4, 20] in case Ω is a C^2 domain, they showed that $\mathrm{spt}\, T$ is a $C^{1,1}$ manifold near such points.

There are various results concerning the regularity of minimal boundaries respecting a given obstacle [5, 20, 27], however these results (as that of Lin and Lau) do not include any discussion about boundary points and hence, using these results, we cannot conclude anything about the regularity around points in the intersection $\mathrm{trace}\, u \cap \mathrm{graph}\, \phi$.

Finally, we mention that if $\partial\Omega$ is of class C^2 then, following the notation of [8], the current $T = T_u$ is λ-minimizing, i.e.,

$$\underline{\underline{M}}(T) \leq \underline{\underline{M}}(T + \partial Q) + \lambda \underline{\underline{M}}(Q)$$

for all integral $(n+1)$-currents Q, where $\lambda = \max\{\|H\|_0, \|H_{\partial\Omega}\|_0\}$. In [7], Duzaar and Steffen generalized for such currents the boundary regularity results given in [14] for area minimizing currents. In particular, they proved that if ∂T is represented by a multiplicity one, $C^{1,\alpha}$ submanifold, then $\mathrm{spt}\, T$ is a $C^{1,\beta}$ submanifold, for all $\beta \leq \alpha/2$, around each point $a \in \partial T$, where $\Theta_T(a) < 1 + 1/2$.

2 The Dirichlet Problem with Regular Data

2.1 Notation and Definitions

$B_r^m(x)$ will denote the m-dimensional ball of radius r centered at a point $x \in \mathbb{R}^m$, i.e.,

$$B_r^m(x) = \{y \in \mathbb{R}^m : |y - x| < r\},$$

and ω_m will denote the measure of the m-dimensional unit ball.

For any $C^{1,\alpha}$ function $u : V \cap B_r^m(0) \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$ is a $C^{1,\alpha}$ domain and $\alpha \in (0, 1]$, $\|u\|_{1,\alpha,V\cap B_r^m(x)}$ will denote the scaled $C^{1,\alpha}$ norm of u, i.e.,

$$\|u\|_{1,\alpha,V\cap B_r^m(x)} = \frac{1}{r}\|u\|_0 + \|Du\|_0 + r^\alpha[Du]_\alpha.$$

Occasionally, when there is no confusion about the domain of u, we will write $\|u\|_{1,\alpha}$ instead of $\|u\|_{1,\alpha,V\cap B_r^m(x)}$.

For a point $x \in \mathbb{R}^{n+1}$ we will often write $x = (x', x_{n+1})$, where $x' \in \mathbb{R}^n$. Finally, the letter c will denote a constant depending only on the specified parameters, and when different constants appear in the course of a proof we will keep the same letter c unless the constant depends on some different parameters.

Definition 2.1 $M \subset \mathbb{R}^{n+1}$ is an m-dimensional properly embedded $C^{1,\alpha}$ submanifold (where $m \le n$, $\alpha \in (0, 1]$) if for each $x \in M$ there is a $\rho > 0$ such that

$$M \cap B_\rho^{n+1}(x) = \operatorname{graph} u_x \cap B_\rho^{n+1}(x),$$

where $u_x \in C^{1,\alpha}((x + L_x) \cap \overline{B}_\rho^{n+1}(x); L_x^\perp)$ for some m-dimensional subspace L_x of $\mathbb{R}^{n+1}$ and where $\operatorname{graph} u_x = \{\xi + u_x(\xi) : \xi \in (x + L_x) \cap \overline{B}_\rho^{n+1}(x)\}$. We quantify the regularity of $M \cap B_\rho^{n+1}(x)$ by defining

$$\kappa(M, \rho, x) = \inf \|u_x\|_{1,\alpha,(x+L_x)\cap B_\rho^{n+1}(x)},$$

where the infimum is taken over all choices of subspaces L_x and corresponding representing functions u_x.

We say that a sequence M_k of m-dimensional submanifolds converges in the $C^{1,\alpha}$ sense to M in $B_\rho^{n+1}(x)$ (for $\rho > 0$ and $x \in M$) and write

$$M_k \xrightarrow{C^{1,\alpha}} M \text{ in } B_\rho^{n+1}(x)$$

if there exists a subspace L_x and functions $u, u_k \in C^{1,\alpha}((x + L_x) \cap \overline{B}_\rho^{n+1}(x); L_x^\perp)$ with

$$M_k \cap B_\rho^{n+1}(x) = \operatorname{graph} u_k \cap B_\rho^{n+1}(x),\ M \cap B_\rho^{n+1}(x) = \operatorname{graph} u \cap B_\rho^{n+1}(x)$$

and

$$\|u_k - u\|_{1,\alpha,(x+L_x)\cap B_\rho^{n+1}(x)} \to 0.$$

We then say that M_k converges in the $C^{1,\alpha}$ sense to M in $\mathbb{R}^{n+1}$ and write $M_k \xrightarrow{C^{1,\alpha}} M$, if there is a $\rho > 0$ such that $M_k \subset \bigcup_{x\in M} B^{n+1}_\rho(x)$ for all sufficiently large k and if $M_k \xrightarrow{C^{1,\alpha}} M$ in $B^{n+1}_\rho(x)$ for each $x \in M$.

Definition 2.2 (Regular Class) For $\alpha \in (0, 1]$, $r > 0$, we define the (α, r)-regular class, which we denote by $\mathcal{B}^\alpha_r$, to be the set of all pairs (Ω, Φ) satisfying the following:

1. Ω is a domain of $\mathbb{R}^n$ such that $\partial\Omega \cap B^n_r(0)$ is a non-empty, $(n-1)$-dimensional embedded $C^{1,\alpha}$ submanifold of $\mathbb{R}^n$ such that

$$\kappa(\partial\Omega, r, x) < 1, \quad \forall x \in \partial\Omega \cap B^n_r(0).$$

2. There exists a sequence of functions $\{\phi_i\} \subset L^1(\partial\Omega) \cap C^{1,\alpha}(\partial\Omega \cap B^n_r(0))$ such that $\operatorname{graph}\phi_i \xrightarrow{C^{1,\alpha}} \Phi$ in $B^n_r(0) \times \mathbb{R}$ and $\Phi \cap (B^n_r(0) \times \mathbb{R})$ is an $(n-1)$-dimensional embedded $C^{1,\alpha}$ submanifold of $\mathbb{R}^{n+1}$ such that

$$\kappa(\Phi \cap (B^n_r(0) \times \mathbb{R}), r, x) < 1, \quad \forall x \in \Phi \cap (B^n_r(0) \times \mathbb{R}).$$

For $(\Omega, \Phi) \in \mathcal{B}^\alpha_r$ we define

$$\kappa_{(\Omega,\Phi)} = \max\left\{ \sup_{x\in\partial\Omega\cap B^n_r(0)} \kappa(\partial\Omega, r, x), \sup_{x\in\Phi\cap(B^n_r(0)\times\mathbb{R})} \kappa(\Phi \cap (B^n_r(0) \times \mathbb{R}), r, x)\right\}.$$

Remark 2.3 Note that if $(\Omega, \Phi) \in \mathcal{B}^\alpha_r$ then

$$\|\nu_{\partial\Omega}(x) - \nu_{\partial\Omega}(y)\| \le c(n)\kappa_{(\Omega,\Phi)}|x - y|^\alpha$$

$\forall x, y \in \partial\Omega \cap B^n_r(0)$, where for $x \in \partial\Omega$, $\nu_{\partial\Omega}(x)$ denotes the inward pointing unit normal to $\partial\Omega$ at x, and

$$\| \operatorname{proj}_{\operatorname{Nor}_\Phi(x)} - \operatorname{proj}_{\operatorname{Nor}_\Phi(y)} \| \le c(n)\kappa_{(\Omega,\Phi)}|x - y|^\alpha$$

$\forall x, y \in \Phi \cap (B^n_r(0) \times \mathbb{R})$ such that $|x - y| \le r$, where for $x \in \Phi$, $\operatorname{Nor}_\Phi(x)$ denotes the 2-dimensional normal subspace to Φ at x.

The following remark is a direct consequence of the Arzela–Ascoli theorem.

Remark 2.4 Let $(\Omega_i, \Phi_i) \in \mathcal{B}^\alpha_{r_i}$ be a sequence such that $\liminf r_i = \infty$ and for some $r \in (0, \infty)$, $B^n_r(0) \cap \partial\Omega_i \ne \emptyset$ for all i. Then after passing to a subsequence

$$\Omega_i \xrightarrow{C^{1,\alpha'}} \Omega \quad \text{and} \quad \Phi_i \xrightarrow{C^{1,\alpha'}} \Phi$$

for any $\alpha' < \alpha$ in the sense of Definition 2.1 and where $(\Omega, \Phi) \in \mathcal{B}^\alpha_{r'}$ for all $r' > r$.

If in addition $\kappa_{(\Omega_i,\Phi_i)} \to 0$, then for the limit we have that $(\Omega, \Phi) = (H, \Phi)$, where H is an n-dimensional halfspace and $\Phi \subset \partial H \times \mathbb{R}$ is an $(n-1)$-dimensional linear space or $\emptyset$.

2.2 The Dirichlet Problem

Let Ω be a domain of $\mathbb{R}^n$ and $\phi \in L^1(\partial\Omega)$ be such that $(\Omega, \operatorname{graph}\phi) \in \mathcal{B}_r^\alpha$, for some $\alpha \in (0,1]$ and $r > 0$. We consider the following Dirichlet problem:

$$\begin{aligned} \sum_{i=1}^n D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) &= H + \sum_{i=1}^n D_i f^i \quad \text{in } \Omega, \\ u &= \phi \quad \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where $H = H(x,u(x))$, $f^i = f^i(x,u(x)) \in L^1(\overline{\Omega}\times\mathbb{R})$, H is bounded in $(\overline{\Omega} \cap B_r^n(0)) \times \mathbb{R}$, and $f = (f^1,\dots,f^n)$ is a $C^{0,\alpha}$ vector field in $(\overline{\Omega} \cap B_r(0)) \times \mathbb{R}$, so that

$$\|H\|_{0,(\overline{\Omega}\cap B_r^n(0))\times\mathbb{R}} = \sup_{x\in(\overline{\Omega}\cap B_r^n(0))\times\mathbb{R}} |H(x)| < \infty$$

and

$$[f]_{\alpha,(\overline{\Omega}\cap B_r^n(0))\times\mathbb{R}} = \sup_{x,y\in(\overline{\Omega}\cap B_r^n(0))\times\mathbb{R}} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

For notational simplicity and as long as there is no confusion about the domain Ω, we will write $\|H\|_{0,B_r^n(0)\times\mathbb{R}}$ and $[f]_{\alpha,B_r^n(0)\times\mathbb{R}}$ instead of $\|H\|_{0,(\overline{\Omega}\cap B_r^n(0))\times\mathbb{R}}$ and $[f]_{\alpha,(\overline{\Omega}\cap B_r^n(0))\times\mathbb{R}}$, respectively.

The equation in (2.5) above is to be interpreted weakly, i.e.,

$$\int_\Omega \sum_{i=1}^n \frac{D_i u}{\sqrt{1+|Du|^2}} D_i\zeta\, d\mathcal{H}^n = -\int_\Omega H\zeta\, d\mathcal{H}^n + \int_\Omega \sum_{i=1}^n f_i D_i\zeta\, d\mathcal{H}^n \tag{2.6}$$

for any $\zeta \in C_c^1(\Omega)$.

For the rest of Sect. 2 we will let $u \in C^{1,\alpha}(\overline{\Omega})$ be a (weak) solution of the Dirichlet problem (2.5) and $T = [\![\operatorname{graph} u]\!]$ be the multiplicity one n-current associated with the graph of u. Recall that for the orientation of a current associated with the graph of a function we use the downward pointing unit normal to the graph. In our case, for the function u, we extend this vector to be an $\mathbb{R}^{n+1}$-valued function in all of $\overline{\Omega}\times\mathbb{R}$ that is independent of the x_{n+1}-variable and we let ν denote this extension, i.e., for any $(x', x_{n+1}) \in \Omega \times \mathbb{R}$

$$\nu(x', x_{n+1}) = \left(\frac{D_1 u(x')}{\sqrt{1+|Du(x')|^2}}, \dots, \frac{D_n u(x')}{\sqrt{1+|Du(x')|^2}}, \frac{-1}{\sqrt{1+|Du(x')|^2}}\right). \tag{2.7}$$

Furthermore, we associate with the vector field ν an n-form ω defined as follows:

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} e_i \cdot \nu\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}, \tag{2.8}$$

where $e_1, e_2, \dots, e_{n+1}$ denote the standard unit vectors in $\mathbb{R}^{n+1}$.

2.3 Volume Bounds

In this paragraph we will show bounds for the mass of the current T and also prove that it has an "almost minimizing" property (cf. Lemma 2.10). The main ingredient is Lemma 2.9, which allows us to compare T with other currents that have the same boundary and coincide with T outside $(\overline{\Omega} \cap B_r^n(0)) \times \mathbb{R}$. Recall that r is such that for the initial data of the Dirichlet problem (2.5) we have that $(\Omega, \operatorname{graph}\phi) \in \mathcal{B}_r^\alpha$.

Lemma 2.9 *Assume $u \in C^{1,\alpha}(\overline{\Omega})$ is a (weak) solution of the Dirichlet problem (2.5), and let $T = [\![\operatorname{graph} u]\!]$ be the corresponding multiplicity one n-current. Let R be a multiplicity one $(n+1)$-current in $\mathbb{R}^{n+1}$ with $\operatorname{spt} R \subset W \subset\subset (\overline{\Omega} \cap B_r^n(0)) \times \mathbb{R}$. Then for $S = T - \partial R$*

$$(1 - d^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}})\underline{\underline{M}}(T) \le (1 + d^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}})\underline{\underline{M}}(S) + \|H\|_{0, B_r^n(0)\times\mathbb{R}}\underline{\underline{M}}(R),$$

where $d = \operatorname{diam} W$.

Proof Note first that if u is smooth then ω (as defined in (2.8)) is a smooth n-form, and hence

$$\begin{aligned} T(\omega) - S(\omega) = \partial R(\omega) = R(d\omega) &= \int_{\operatorname{spt} R} \Theta(x) \operatorname{div} \nu(x) dx \\ &= \int_{\operatorname{spt} R} \Theta(x) \left(H(x', u(x')) + \operatorname{div} f(x', u(x'))\right) dx' dx_{n+1} \\ &= \int_{\operatorname{spt} R} \Theta(x) \left(H(x', u(x')) + \operatorname{div}(f(x', u(x')) - f(x_0)\right) dx' dx_{n+1}, \end{aligned}$$

where x_0 is any given point in W, $\Theta(x)$ depends on the orientation of R, in particular

$$\Theta(x) = \langle \vec{R}(x), dx_1 \wedge \cdots \wedge dx_{n+1} \rangle \in \{1, -1\}$$

and recall that for any point $x \in \mathbb{R}^{n+1}$ we use the notation $x = (x', x_{n+1})$.

Hence, we get that

$$T(\omega) - S(\omega) = \int_{\operatorname{spt} R} \Theta(x) H(x', x_{n+1}) dx' dx_{n+1} + \partial R(\omega_{f - f_0}), \tag{1}$$

where

$$\omega_{f-f_0} = \sum_{i=1}^{n+1} (-1)^{i+1} e_i \cdot (f - f(x_0))\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1},$$

which implies the lemma, since $T(\omega) = \underline{\underline{M}}(T)$ and $|\omega_{f-f_0}| \le d^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}}$ everywhere in W.

For the general case, when u is $C^{1,\alpha}$ it suffices to show that (1) is still true. For this reason we will approximate ω by smooth n-forms.

Let $\zeta \in C_c^\infty(\mathbb{R}^{n+1})$ be such that $\operatorname{spt}\zeta \subset B_1^{n+1}(0)$, $\zeta \geq 0$, $\int_{\mathbb{R}^{n+1}} \zeta(x)dx = 1$. For $\sigma \in (0,1)$ we let $\zeta_\sigma(x) = \sigma^{-(n+1)}\zeta(x/\sigma)$ and consider the n-form

$$\omega_\sigma = \zeta_\sigma * \omega = \sum_{i=1}^{n+1}(-1)^{i+1}(\zeta_\sigma * \nu_i)dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1},$$

where ν, ω are as defined in (2.7), (2.8).

Then ω_σ is a smooth n-form, and hence

$$\begin{aligned} T(\omega_\sigma) - S(\omega_\sigma) &= \partial R(\omega_\sigma) = R(d\omega_\sigma) \\ &= \int_{\operatorname{spt} R} \Theta(x) \int_{B_\sigma^{n+1}(x)} -D_y\left(\zeta_\sigma(x-y)\right) \cdot \nu(y)dydx. \end{aligned}$$

Using (2.6) (the weak form of the prescribed mean curvature equation) we get

$$\begin{aligned} T(\omega_\sigma) - S(\omega_\sigma) = &- \int_{\operatorname{spt} R} \Theta(x) \int_{B_\sigma^{n+1}(x)} \zeta_\sigma(x-y)H(y', u(y'))dydx \\ &- \int_{\operatorname{spt} R} \Theta(x) \int_{B_\sigma^{n+1}(x)} D_x\zeta_\sigma(x-y) \\ &\times \left(f(y', u(y')) - f(x_0)\right) dydx, \end{aligned}$$

where we have used the fact that

$$\int_{B_\sigma^{n+1}(x)} D_x\zeta_\sigma(x-y) \cdot f(x_0)dy = 0.$$

Hence we have that

$$\begin{aligned} T(\omega_\sigma) - S(\omega_\sigma) = &- \int_{\operatorname{spt} R} \Theta(x) \int_{B_\sigma^{n+1}(x)} \zeta_\sigma(x-y)H(y', u(y'))dydx \\ &- \partial R(\zeta_\sigma * \omega_{f-f_0}), \end{aligned}$$

which by letting $\sigma \to 0$ implies (1). □

Lemma 2.10 *Assume $u \in C^{1,\alpha}(\overline{\Omega})$ is a (weak) solution of the Dirichlet problem (2.5) and let $T = [\![\operatorname{graph} u]\!]$ be the corresponding multiplicity one n-current. Then for any $x_0 \in B_r^n(0) \times \mathbb{R}$ and $\rho > 0$ such that $B_\rho^{n+1}(x_0) \subset B_r^n(0) \times \mathbb{R}$ and $\rho^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}} <$ 1/4:*

$$\underline{\underline{M}}(T \llcorner B_\rho^{n+1}(x_0)) \leq c\left(1 + \rho\|H\|_{0,B_r^n(0)\times\mathbb{R}}\right)\omega_n\rho^n$$

and

$$\underline{\underline{M}}_W(T) \leq \underline{\underline{M}}_W(S) + c\omega_n\rho^n\left(\rho\|H\|_{0,B_r^n(0)\times\mathbb{R}} + \rho^\alpha[f]_{\alpha,B_r^n(0)\times\mathbb{R}}\right)$$

for any $W \Subset B_\rho^{n+1}(x_0)$ and S an integral n-current in $\mathbb{R}^{n+1}$ with $\partial S = \partial T$ and $\operatorname{spt}(T - S)$ a compact subset of $W \cap (\overline{\Omega} \times \mathbb{R})$ and where c is an absolute constant.

Proof Since $\operatorname{spt}(S-T) \subset B_\rho^{n+1}(x_0) \cap ((B_r^n(0) \cap \overline{\Omega}) \times \mathbb{R})$, by Lemma 2.9 and using the assumption $\rho^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}} < 1/4$ we have that

$$\left(1 - 2\rho^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}}\right) \underline{\underline{M}}_W(T) \leq \underline{\underline{M}}_W(S) + 2\omega_n \rho^{n+1} \|H\|_{0, B_r^n(0)\times\mathbb{R}}. \tag{1}$$

Let $U = \{(x', x_{n+1}) \in \Omega \times \mathbb{R} : x_{n+1} \leq u(x')\}$, i.e., U is the region under the graph of u, and also let $U_\sigma = U \cap B_\sigma^{n+1}(x_0)$. By Sard's theorem, for almost all $\sigma > 0$, $\underline{\underline{M}}(T \llcorner \partial B_\sigma^{n+1}(x_0)) = 0$. For such $\sigma \leq \rho$, let $S = T \llcorner B_\sigma^{n+1}(x_0) - \partial U_\sigma$. Then $\operatorname{spt} S \subset \partial B_\sigma^{n+1}(x_0) \cup ((\partial\Omega \times \mathbb{R}) \cap B_\sigma^{n+1}(x_0))$ and so

$$\underline{\underline{M}}(S) \leq 4\omega_n \sigma^n + \omega_n \sigma^n (1+\kappa) \leq 6\omega_n \sigma^n,$$

where $\kappa = \kappa_{(\Omega,\Phi)}$ is as in Definition 2.2. Using this in inequality (1) we have that

$$\underline{\underline{M}}(T \llcorner B_\sigma^{n+1}(x_0)) \leq 12(1 + \sigma \|H\|_{0, B_r^n(0)\times\mathbb{R}})\omega_n \sigma^n,$$

which gives the first assertion of the lemma.

Taking $\sigma \in (0, \rho]$ such that $W \subset B_\sigma^{n+1}(x_0)$, we have that

$$\underline{\underline{M}}_W(T) \leq 12(1 + \rho \|H\|_{0, B_r^n(0)\times\mathbb{R}})\omega_n \rho^n,$$

and using this estimate back in the inequality (1) we get

$$\begin{aligned} \underline{\underline{M}}_W(T) \leq & \underline{\underline{M}}_W(S) + 24\omega_n \rho^n (1 + \rho \|H\|_{0, B_r^n(0)\times\mathbb{R}})\rho^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}} \\ & + 2\omega_n \rho^{n+1} \|H\|_{0, B_r^n(0)\times\mathbb{R}}, \end{aligned}$$

which implies the second assertion of the lemma. □

Definition 2.11 Let M be an n-dimensional manifold in $\mathbb{R}^{n+1}$, $x \in M$ and P an n-dimensional linear space passing through x. We say that M is σ-close to P in $B_\rho^{n+1}(x)$ if

$$M \cap B_\rho^{n+1}(x) \subset q(Q_{\rho,\sigma})$$

for some orthogonal transformation q of $\mathbb{R}^{n+1}$ such that $q(0) = x$, $q(\{0\} \times \mathbb{R}^n) = P$ and where

$$Q_{\rho,\sigma} = [-\sigma\rho, \sigma\rho] \times B_\rho^n(0).$$

Lemma 2.12 *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a (weak) solution of the Dirichlet problem (2.5). Let $x_0 \in \operatorname{graph} u \cap (B_{r/2}^n(0) \times \mathbb{R})$ and $\rho \in (0, r/2]$ be such that $B_\rho^{n+1}(x_0) \subset B_{r/2}^n(0) \times \mathbb{R}$, $B_\rho^{n+1}(x_0) \cap \operatorname{graph} \phi = \emptyset$ and $\rho^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}} < 1/4$. Then*

$$\begin{aligned} \mathcal{H}^n(\operatorname{graph} u \cap q(Q_{\rho,\sigma})) \leq & (1 + 3\rho^\alpha [f]_{\alpha, B_r^n(0)\times\mathbb{R}})\omega_n \rho^n \\ & + c\sigma\omega_n \rho^n (n + \rho \|H\|_{0, B_r^n(0)\times\mathbb{R}}) \end{aligned}$$

for any $\sigma \in (0, \rho)$ and any orthogonal transformation of $\mathbb{R}^{n+1}$, q, such that $q(0) = x_0$ and where c is an absolute constant.

Proof Let q be an orthogonal transformation of $\mathbb{R}^{n+1}$ and let $Q^{\pm}$ be the regions in $q(Q_{\rho,\sigma})$ that lie above and below the graph of u, i.e.,

$$Q^{+} = \{x = (x', x_{n+1}) \in q(Q_{\rho,\sigma}) : x_{n+1} > u(x)\},$$
$$Q^{-} = \{x = (x', x_{n+1}) \in q(Q_{\rho,\sigma}) : x_{n+1} < u(x)\}.$$

Notice that for one of the $\partial Q^{\pm}$, say ∂Q^{+}, we know that

$$\left|\partial Q^{+} \cap \left(q\left(\{\sigma\rho\} \times B^n_\rho(0)\right) \cup q\left(\{-\sigma\rho\} \times B^n_\rho(0)\right)\right)\right| \leq \omega_n \rho^n.$$

Then the lemma is a direct consequence of Lemma 2.9, applied with Q^{+} in place of R. □

Remark 2.13 If in addition to the hypotheses of Lemma 2.12 we have that $\operatorname{graph} u \cap B^{n+1}_\rho(x_0)$ is σ-close to a plane (in the sense of Definition 2.11) then Lemma 2.12 eventually gives

$$\mathcal{H}^n(\operatorname{graph} u \cap B^{n+1}_\rho(x_0)) \leq (1 + 3\rho^\alpha [f]_{\alpha, B^n_r(0)\times\mathbb{R}})\omega_n \rho^n + c\sigma\omega_n\rho^n(n + \rho\|H\|_{0,B^n_r(0)\times\mathbb{R}}).$$

Lemma 2.14 *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a (weak) solution of the Dirichlet problem (2.5). Let $x_0 \in \operatorname{graph} u \cap (B^n_r(0) \times \mathbb{R})$ and $\rho > 0$ be such that $B^{n+1}_\rho(x_0) \subset B^n_r(0) \times \mathbb{R}$, $B^{n+1}_\rho(x_0) \cap \operatorname{graph}\phi = \emptyset$ and $\rho^\alpha [f]_{\alpha, B^n_r(0)\times\mathbb{R}} \leq 1/2$. Then*

$$\mathcal{H}^{n+1}(U^{\pm} \cap B^{n+1}_\rho(x_0)) \geq c\rho^{n+1}, \tag{1}$$

where $U^{\pm}$ are the regions of $\Omega \times \mathbb{R}$ that lie above (U^{+}) and below (U^{-}) the graph of u and

$$\mathcal{H}^n(\operatorname{graph} u \cap B^{n+1}_\rho(x_0)) \geq c\rho^n. \tag{2}$$

The constant c in both inequalities depends on $\|H\|_{0,B^n_r(0)\times\mathbb{R}}$ and n.

Proof We first give the proof of (1) for U^{-}; the argument for U^{+} is similar.

Let

$$U_\rho = U^{-} \cap B^{n+1}_\rho(x_0), \qquad G_\rho = \operatorname{graph} u \cap B^{n+1}_\rho(x_0). \tag{3}$$

By Lemma 2.9 (with U_ρ in place of R) we get that

$$\mathcal{H}^n(G_\rho) \leq 3\frac{d}{d\rho}\mathcal{H}^{n+1}(U_\rho) + 2\|H\|_{0,B^n_r(0)\times\mathbb{R}}\mathcal{H}^{n+1}(U_\rho). \tag{4}$$

Since

$$\mathcal{H}^n(\partial U_\rho) \leq \mathcal{H}^n(G_\rho) + \frac{d}{d\rho}\mathcal{H}^{n+1}(U_\rho),$$

the isoperimetric inequality for U_ρ implies that

$$\mathcal{H}^{n+1}(U_\rho)^{\frac{n}{n+1}} \leq c(n)\mathcal{H}^n(\partial U_\rho) \leq c(n)\left(\mathcal{H}^n(G_\rho) + \frac{d}{d\rho}\mathcal{H}^{n+1}(U_\rho)\right). \tag{5}$$

Using the estimate (4) in (5) we get

$$\mathcal{H}^{n+1}(U_\rho)^{\frac{n}{n+1}} \leq c\frac{d}{d\rho}\mathcal{H}^{n+1}(U_\rho)$$

because we can assume that $2c(n)\|H\|_{0,B_r^n(0)\times\mathbb{R}}\mathcal{H}^{n+1}(U_\rho) \leq \frac{1}{2}\mathcal{H}^{n+1}(U_\rho)^{\frac{n}{n+1}}$, where $c(n)$ is the constant from the isoperimetric inequality, since otherwise the lemma is trivially true.

Hence

$$\frac{d}{d\rho}\left(\mathcal{H}^{n+1}(U_\rho)^{\frac{1}{n+1}}\right) \geq c,$$

and after integrating

$$\mathcal{H}^{n+1}(U_\rho) \geq c\rho^{n+1}.$$

For proving (2) of the lemma we let U_ρ, G_ρ be as defined in (3) above. By inequality (1) we know that

$$\mathcal{H}^{n+1}(U_\rho) \geq c\rho^{n+1}.$$

Let v be a unit vector in $\mathbb{R}^{n+1}$ such that

$$v \cdot (0, \dots, 0, 1) > 0. \tag{6}$$

For such a vector v we define P_v to be the n-dimensional affine subspace of $\mathbb{R}^{n+1}$, passing through x_0 and normal to v, i.e.,

$$P_v = \{x \in \mathbb{R}^{n+1} : (x - x_0) \cdot v = 0\}$$

and $U^+_{\rho,v}$ to be the part of U_ρ that lies above P_v, i.e.,

$$U^+_{\rho,v} = \{x \in U_\rho : (x - x_0) \cdot v > 0\}.$$

We claim that it is enough to prove that for some vector v, satisfying (6), we have that

$$\mathcal{H}^{n+1}(U^+_{\rho,v}) \leq \frac{1}{4}\mathcal{H}^{n+1}(U_\rho). \tag{7}$$

To see this, assume that (7) is true for some v and let $G^\pm_{\rho,v}$ be the parts of G_ρ that lie above ($G^+_{\rho,v}$) and below ($G^-_{\rho,v}$) the affine subspace P_v. Let also

$$\widetilde{G}_\rho = G^-_{\rho,v} \cup \mathrm{ref}_{P_v}(G^+_{\rho,v}) \cup \left(G_\rho \cap P_v\right),$$

where ref_{P_v} denotes the reflection along P_v, and let $\widetilde{U}_\rho$ be the region of $B_\rho^{n+1}(x_0)$ that lies below $\widetilde{G}_\rho$. Then we have that

$$\mathcal{H}^n(\widetilde{G}_\rho) = \mathcal{H}^n(G_\rho)$$

and

$$\mathcal{H}^{n+1}(\widetilde{U}_\rho) \geq \mathcal{H}^{n+1}(U_\rho) - 2\mathcal{H}^{n+1}(U_{\rho,v}^+) \geq \frac{1}{2}\mathcal{H}^{n+1}(U_\rho).$$

Furthermore, since $\widetilde{U}_\rho$ lies below P_v, we have that

$$\mathrm{proj}_{P_v}(\widetilde{G}_\rho) = \mathrm{proj}_{P_v}(\widetilde{U}_\rho),$$

where proj_{P_v} denotes the projection onto the affine subspace P_v. Finally, since

$$\widetilde{U}_\rho \subset \{x - tv : x \in \mathrm{proj}_{P_v}(\widetilde{U}_\rho), 0 \leq t \leq \rho\}$$

we have that

$$\rho\mathcal{H}^n(\mathrm{proj}_{P_v}(\widetilde{U}_\rho)) \geq \mathcal{H}^{n+1}(\widetilde{U}_\rho)$$

and hence

$$\begin{aligned}\mathcal{H}^n(G_\rho) &= \mathcal{H}^n(\widetilde{G}_\rho) \geq \mathcal{H}^n(\mathrm{proj}_{P_v}(\widetilde{G}_\rho)) = \mathcal{H}^n(\mathrm{proj}_{P_v}(\widetilde{U}_\rho)) \\ &\geq \rho^{-1}\mathcal{H}^{n+1}(\widetilde{U}_\rho) \geq \rho^{-1}\frac{1}{2}\mathcal{H}^{n+1}(U_\rho) \geq c\rho^n.\end{aligned}$$

We now need to show that for some vector v that satisfies (6), inequality (7) is true.

For any $t \in (-1, 1)$ let $v_t = (t, 0, \ldots, 0, \sqrt{1-t^2})$,

$$P_t = \{x \in \mathbb{R}^{n+1} : (x - x_0) \cdot v_t = 0\}$$

and

$$U_{\rho,t}^+ = \{x \in U_\rho : (x - x_0) \cdot v_t > 0\}, \qquad U_{\rho,t}^- = \{x \in U_\rho : (x - x_0) \cdot v_t < 0\}.$$

We claim that for some t, $\mathcal{H}^{n+1}(U_{\rho,t}^+) \leq \frac{1}{4}\mathcal{H}^{n+1}(U_\rho)$.

Assume it is not true. Then for all t

$$\mathcal{H}^{n+1}(U_{\rho,t}^-) \leq \frac{1}{4}\mathcal{H}^{n+1}(U_\rho)$$

and hence for any $\varepsilon > 0$

$$\mathcal{H}^{n+1}(U_{\rho,1-\varepsilon}^+ \cap U_{\rho,-1+\varepsilon}^+) \geq \frac{1}{2}\mathcal{H}^{n+1}(U_\rho)$$

which is impossible, since

$$\mathcal{H}^{n+1}(U_{\rho,1-\varepsilon}^+ \cap U_{\rho,-1+\varepsilon}^+) \xrightarrow{\varepsilon\to 0} 0. \qquad \square$$

2.4 The Solutions to (2.5) are Uniformly Close to Planes near the Boundary Cylinder

In this subsection we want to show that given $\varepsilon > 0$, there exists $\rho > 0$ depending only on ε, r, $\|H\|_{0,B_r^n(0)\times\mathbb{R}}$, and $[f]_{\alpha,B_r^n(0)\times\mathbb{R}}$, such that the graph of u is ε-close to some n-dimensional linear space in all balls of radius less than ρ that intersect the boundary cylinder (cf. Theorem 2.16). The main ingredient is the following lemma:

Lemma 2.15 *Let* $(\Omega_k, \operatorname{graph}\phi_k) \in \mathcal{B}^\alpha_{r_k}$ *be a sequence such that* $r_k \to \infty$, $\kappa_{(\Omega_k,\Phi_k)} \to 0$, *and* $\partial\Omega_k \cap B_r^n(0) \neq \emptyset$ *for some* $r \in (0,\infty)$, *where* $\Phi_k = \operatorname{graph}\phi_k$. *Assume that* $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ *is a (weak) solution of the corresponding Dirichlet problem* (2.5) *with* $[f_k]_{\alpha,B^n_{r_k}(0)\times\mathbb{R}} \to 0$ *and* $\|H_k\|_{0,B^n_{r_k}(0)\times\mathbb{R}} \to 0$ *and let* $T_k = [\![\operatorname{graph} u_k]\!]$.

Then, after passing to a subsequence,

$$T_k \llcorner (B^n_{r_k}(0)\times\mathbb{R}) \to T \tag{1}$$

in the weak sense of currents, but also with the corresponding measures converging $\mu_{T_k} \to \mu_T$ *as Radon measures and where for the limit* T *either*

(i) $\partial T = 0$ *and* $\operatorname{spt} T$ *is a vertical hyperplane*

or

(ii) $\partial T \neq 0$ *and* $\operatorname{spt} T$ *is an* n*-dimensional halfspace.*

Furthermore, for the convergence in (1) *we have that for any* $\varepsilon > 0$ *and* W *a compact subset of* $\mathbb{R}^{n+1}$ *such that* $W \cap \operatorname{spt} T \neq \emptyset$, *there exists* k_0 *such that for all* $k \geq k_0$

$$\operatorname{spt} T_k \cap W \subset \varepsilon\text{-neighborhood of } \operatorname{spt} T. \tag{2}$$

Proof We note that given any $\rho > 0$ and for all k large enough we have that $\rho^\alpha[f_k]_{\alpha,B^n_{r_k}(0)\times\mathbb{R}} < 1/4$ and therefore we can apply Lemma 2.10. This implies that the currents $T_k \llcorner (B^n_{r_k}(0)\times\mathbb{R})$ have locally uniformly bounded masses. Hence we can apply the Federer–Fleming compactness theorem [26, Theorem 32.2] which implies that after passing to a subsequence

$$T_k \llcorner (B^n_{r_k}(0)\times\mathbb{R}) \to T$$

in the weak sense of currents in $\mathbb{R}^{n+1}$, where T is an integral n-current. Furthermore, by Remark 2.4, T has support in an $(n+1)$-dimensional closed halfspace. Without loss of generality, we can assume that this halfspace is equal to $\overline{\mathbb{R}}_+ \times \mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_1 \geq 0\}$. According to the Federer–Fleming compactness theorem we also have that $\partial T_k = [\![\Phi^k]\!] \to \partial T$ and thus (using Remark 2.4 again) either $\partial T = 0$ or $\partial T = [\![\Phi]\!]$, where Φ is an $(n-1)$-dimensional affine subspace of $\{0\}\times\mathbb{R}^n$.

We claim that T is area minimizing. In view of Lemma B.4 it suffices to prove that it is area minimizing in the closed halfspace $\overline{\mathbb{R}}_+ \times \mathbb{R}^n$. Note that although the currents T_k are not area minimizing, they do satisfy a minimizing property (cf. Lemma 2.10), which enables us to argue as in the case when T is the limit of area minimizing currents (cf. [26, Theorem 34.5]), as follows:

Since $T_k \llcorner (B^n_{r_k}(0) \times \mathbb{R}) \to T$ in the weak sense of currents, we know that the convergence is also with respect to the flat-metric (cf. [26, Theorem 31.2]), i.e., there exist integral $(n+1)$-currents R_k and integral n-currents P_k such that $T - T_k \llcorner (B^n_{r_k}(0) \times \mathbb{R}) = \partial R_k + P_k$ and for any compact subset W of R^{n+1}

$$\underline{\underline{M}}_W(R_k) + \underline{\underline{M}}_W(P_k) \to 0.$$

Let S be an integral n-current such that $\partial S = 0$ and $\operatorname{spt} S \subset W \cap (\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$, where W is a compact subset of $\mathbb{R}^{n+1}$. Let $W_\varepsilon = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, W) < \varepsilon\}$. We can choose $\varepsilon \in (0,1)$ so that, after passing to a subsequence, we have that for all k:

$$\underline{\underline{M}}(T_k \llcorner \partial W_\varepsilon) = 0, \ \underline{\underline{M}}(T \llcorner \partial W_\varepsilon) = 0 \tag{3}$$

and

$$\partial(R_k \llcorner W_\varepsilon) = (\partial R_k) \llcorner W_\varepsilon + L_k, \tag{4}$$

where L_k is an integral n-current such that $\operatorname{spt} L_k \subset \partial W_\varepsilon$ and $\underline{\underline{M}}(L_k) \to 0$. Then

$$T \llcorner W_\varepsilon - T_k \llcorner W_\varepsilon = \partial \widetilde{R}_k + \widetilde{P}_k,$$

where $\widetilde{R}_k = R_k \llcorner W_\varepsilon$ and $\widetilde{P}_k = P_k \llcorner W_\varepsilon - L_k$, and

$$\underline{\underline{M}}_{W_\varepsilon}(T + S) = \underline{\underline{M}}_{W_\varepsilon}(T_k + S + \partial \widetilde{R}_k + \widetilde{P}_k) \geq \underline{\underline{M}}_{W_\varepsilon}(T_k + S_k) - \underline{\underline{M}}_{W_\varepsilon}(\widetilde{P}_k), \tag{5}$$

where $S_k = S + \partial \widetilde{R}_k$. For S_k we have that $\operatorname{spt} S_k \subset \overline{W}_\varepsilon$ and $\partial S_k = 0$.

Let $R > 0$ be such that $\overline{W}_\varepsilon \subset B^{n+1}_R(0)$. For k big enough, so that $r_k > R$, let ℓ_k be a Lipschitz retraction of $(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \cap B^{n+1}_R(0)$ to $(\overline{\Omega}_k \times \mathbb{R}) \cap B^{n+1}_R(0)$ such that

$$\frac{|\ell_k(x) - \ell_k(y)|}{|x - y|} \leq \kappa_{(\Omega_k, \Phi_k)}, \quad \forall x, y \in \{0\} \times \mathbb{R}^n.$$

Then the current $\ell_{k\#} S_k$ has support in $\overline{\Omega}_k \times \mathbb{R}$ and no boundary, hence using the minimizing property of T_k (Lemma 2.10) we have that

$$\begin{aligned} \underline{\underline{M}}_{W_\varepsilon}(T_k) &\leq \underline{\underline{M}}_{W_\varepsilon}(T_k + \ell_{k\#} S_k) + c\left(R\|H\|_{0, B^n_R(0) \times \mathbb{R}} + R^\alpha [f]_{\alpha, B^n_R(0) \times \mathbb{R}}\right) \omega_n R^n \\ &\leq |J\ell_k| \underline{\underline{M}}_{W_{\varepsilon'}}(T_k + S_k) + c\left(R\|H\|_{0, B^n_R(0) \times \mathbb{R}} + R^\alpha [f]_{\alpha, B^n_R(0) \times \mathbb{R}}\right) \omega_n R^n \end{aligned}$$

for any $\varepsilon' > \varepsilon$ such that $W_{\varepsilon'} \subset B^{n+1}_R(0)$ and where $J\ell_k$ denotes the Jacobian of ℓ_k and thus $|J\ell_k| < 1 + c\kappa_{(\Omega_k, \Phi_k)}$. Letting $\varepsilon' \downarrow \varepsilon$ we get

$$\begin{aligned} \underline{\underline{M}}_{W_\varepsilon}(T_k) \leq &|J\ell_k| \left(\underline{\underline{M}}_{W_\varepsilon}(T_k + S_k) + \underline{\underline{M}}(L_k)\right) \\ &+ c\left(R\|H\|_{0, B^n_R(0) \times \mathbb{R}} + R^\alpha [f]_{\alpha, B^n_R(0) \times \mathbb{R}}\right) \omega_n R^n, \end{aligned} \tag{6}$$

where we have used (3) and (4). Hence, using (6) to estimate $\underline{\underline{M}}_{W_\varepsilon}(T_k+S_k)$ in (5), we get

$$\underline{\underline{M}}_{W_\varepsilon}(T+S) \geq (1-c\kappa_{(\Omega_k,\Phi_k)})\underline{\underline{M}}_{W_\varepsilon}(T_k) - \underline{\underline{M}}_{W_\varepsilon}(\widetilde{P}_k) - \underline{\underline{M}}(L_k)$$
$$- c\left(R\|H\|_{0,B_R^n(0)\times\mathbb{R}} + R^\alpha[f]_{\alpha,B_R^n(0)\times\mathbb{R}}\right)\omega_n R^n,$$

where c depends only on n. Letting $k\to\infty$ and using the lower semicontinuity of the mass and the fact that $\underline{\underline{M}}_{W_\varepsilon}(\widetilde{P}_k)\to 0$, $\underline{\underline{M}}(L_k)\to 0$ we get

$$\underline{\underline{M}}_{W_\varepsilon}(T) \leq \underline{\underline{M}}_{W_\varepsilon}(T+S)$$

which implies that

$$\underline{\underline{M}}_{W}(T) \leq \underline{\underline{M}}_{W}(T+S)$$

since $S=0$ outside W, and hence T is area minimizing.

We claim now that $\mu_{T_k}\to\mu_T$ as Radon measures.

Using $S=0$ in the above argument we have that

$$\underline{\underline{M}}_{W_\varepsilon}(T) \geq (1-c\kappa_{(\Omega_k,\Phi_k)})\underline{\underline{M}}_{W_\varepsilon}(T_k) - \underline{\underline{M}}_{W_\varepsilon}(\widetilde{P}_k)$$
$$- c\left(R\|H\|_{0,B_R^n(0)\times\mathbb{R}} + R^\alpha[f]_{\alpha,B_R^n(0)\times\mathbb{R}}\right)\omega_n R^n$$

and letting $k\to 0$

$$\underline{\underline{M}}_{W_\varepsilon}(T) \geq \limsup_k \underline{\underline{M}}_{W_\varepsilon}(T_k).$$

Since $W\subset W_\varepsilon$

$$\limsup \mu_{T_k}(W) \leq \underline{\underline{M}}_{W_\varepsilon}(T) = \mu_T(W_\varepsilon)$$

and because we can repeat the argument for $\varepsilon\downarrow 0$ we have that

$$\mu_T(W) \geq \limsup_k \mu_{T_k}(W)$$

which along with the lower semicontinuity of Radon measures implies the measure convergence.

Next we will show that T is either an n-dimensional halfspace or a vertical hyperplane.

Assume first that we are in case (i) $\partial T=0$. This is the case when for any $W\Subset\mathbb{R}^{n+1}$, $W\cap\Phi_k=\emptyset$ and hence $\partial T_k \llcorner W=0$, for all k large enough (cf. Remark 2.4).

Using the uniform area ratio bounds, Lemma 2.10 and the interior monotonicity formula [1] we have

$$1\leq \omega_n^{-1}r^{-n}\mu_T(B_r^{n+1}(x)) = \omega_n^{-1}r^{-n}\lim_k \mu_{T_k}(B_r^{n+1}(x)) \leq c$$

for all $x\in\operatorname{spt}T$ and any $r>0$, where c is an absolute constant.

Hence for a sequence $\{\Lambda_i\} \uparrow \infty$ we can apply the Federer–Fleming compactness theorem to the sequence $T_{x,\Lambda_i} = \eta_{x,\Lambda_i\#}T$, where for $x \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$, $\eta_{x,\lambda} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$. So, after passing to a subsequence,

$$T_{x,\Lambda_i} \to C$$

in the weak sense of currents, where C is an integral n-current. Since T_{x,Λ_i} are area minimizing, C is an area minimizing cone and $\mu_{T_{x,\Lambda_i}} \to \mu_C$ as Radon measures. Furthermore, since $\operatorname{spt} T \subset \overline{\mathbb{R}}_+ \times \mathbb{R}^n$ we have that $\operatorname{spt} T_{x,\Lambda_i} \subset \{y \in \mathbb{R}^{n+1} : y_1 \geq -\Lambda_i^{-1} x_1\}$, where x_1, y_1 denote the first coordinates of x and y, respectively, and hence $\operatorname{spt} C \subset \overline{R}_+ \times \mathbb{R}^n$. This implies [26, Theorems 36.5, 26.27] that

$$C = m[\![\{0\} \times \mathbb{R}^n]\!]$$

for some integer $m \geq 1$. We claim that in fact $m = 1$.

For $\sigma \in (0, 1)$, let $Q_{1,\sigma} = [-\sigma, \sigma] \times B_1^n(0)$. Then $\mu_C(Q_{1,\sigma}) = m\omega_n$. By the measure convergence $\mu_{T_{x,\Lambda_i}} \to \mu_C$ and $\mu_{T_k} \to \mu_T$, we have that for any $\delta > 0$ there exists some $\Lambda > 0$ and k_0 such that for all $k \geq k_0$

$$m - \delta \leq \frac{1}{\Lambda^n \omega_n} \mu_{T_k}(x + \Lambda Q_{1,\sigma}).$$

Using Lemma 2.12, the RHS of the above inequality is less than $1 + \Lambda^\alpha [f_k]_{\alpha, B_{r_k}^n(0)\times\mathbb{R}}$ $+ c\sigma(1 + \Lambda \|H_k\|_0)$ and hence taking σ small enough we conclude that m has to be 1.

Hence we get that

$$\omega_n^{-1} r^{-n} \mu_T(B_r^{n+1}(x)) = 1, \quad \forall x \in \operatorname{spt} T \text{ and } r > 0$$

which implies that T itself is a hyperplane of multiplicity one, and since $\operatorname{spt} T \subset \overline{R}_+ \times \mathbb{R}^n$ it has to be a vertical hyperplane.

Assume now that we are in case (ii) $\partial T = [\![\Phi]\!]$. In this case $\Phi_k \xrightarrow{C^{1,\alpha'}} \Phi$ for all $\alpha' < \alpha$ and hence Φ is an $(n-1)$-dimensional linear subspace of $\{0\} \times \mathbb{R}^n$ (cf. Remark 2.4). Without loss of generality we can assume that $\Phi = \{0\} \times \mathbb{R}^{n-1} \times \{0\}$.

Using the uniform area ratio bounds, Lemma 2.10 and the boundary monotonicity formula [2] we get

$$\frac{1}{2} \leq \omega_n^{-1} r^{-n} \mu_T(B_r^{n+1}(0)) = \omega_n^{-1} r^{-n} \lim_k \mu_{T_k}(B_r^{n+1}(0)) \leq c$$

for any $r > 0$ and where c is an absolute constant.

Hence for a sequence $\{\Lambda_i\} \uparrow \infty$ we can apply the Federer–Fleming compactness theorem to the sequence $T_{\Lambda_i} = \eta_{0,\Lambda_k\#}T$ to conclude (as in case (i)) that after passing to a subsequence

$$T_{\Lambda_i} \to C,$$

where C is an area minimizing cone with $\operatorname{spt} C \subset \overline{\mathbb{R}}_+ \times \mathbb{R}^n$, $\partial C = \Phi = \{0\} \times \mathbb{R}^{n-1} \times \{0\}$ and also $\mu_{T_{\Lambda_i}} \to \mu_C$ as Radon measures.

Hence we can apply Lemma B.1 and in particular Corollaries B.2, B.3 to C to conclude that C is either an n-dimensional halfspace or

$$C = mP_1 + (m-1)P_2$$

for some integer $m \geq 1$, where P_1, P_2 denote the n-dimensional halfspaces $\{0\} \times \mathbb{R}^{n-1} \times \mathbb{R}_{\pm}$.

We claim that in the latter case $m = 1$ and hence C is a halfspace in either case. To see this, take $x \in \operatorname{spt} C$ such that for $Q_{1,\sigma}(x) = x + [-\sigma, \sigma] \times B_1^n(0)$ we have that $Q_{1,\sigma}(x) \cap \Phi = \emptyset$ and $\mu_C(Q_{1,\sigma}(x)) = m\omega_n$. We can argue now as in case (i) and using the measure convergence $\mu_{T_{\Lambda_i}} \to \mu_C$, $\mu_{T_k} \to \mu_T$ and Lemma 2.12 we have that for any $\delta > 0$ there exists some $\Lambda > 0$ and k_0 such that for all $k \geq k_0$

$$m - \delta \leq \frac{1}{\Lambda^n \omega_n} \mu_{T_k}(x + \Lambda Q_{1,\sigma}(x)) \leq 1 + 3\Lambda^{\alpha}[f_k]_{\alpha, B^n_{r_k}(0)\times\mathbb{R}} + c\sigma(1 + \Lambda \|H_k\|_0)$$

and hence taking σ small enough we conclude that m has to be 1.

Hence C is a halfspace and therefore for any $r > 0$

$$\omega_n^{-1} r^{-n} \mu_T(B_r^{n+1}(0)) = \frac{1}{2}$$

so that T is an area minimizing cone with vertex 0. Hence we can apply Lemma B.1 and Corollaries B.2, B.3 to T, which along with the fact that the density at 0 is $1/2$ imply that T is an n-dimensional halfspace.

We finally have to prove statement (2) of the theorem.

Assume that for some $W \Subset \mathbb{R}^{n+1}$ such that $W \cap \operatorname{spt} T \neq \emptyset$ and $\varepsilon > 0$, statement (2) of the theorem is not true. Hence, after passing to a subsequence, we have that for every k there exists $x_k \in \operatorname{spt} T_k \cap W$ such that

$$B_\varepsilon^{n+1}(x_k) \cap \operatorname{spt} T = \emptyset.$$

Since either $\Phi_k \cap W = \emptyset$ for k big enough or $\Phi_k \cap (B^n_{r_k}(0) \times \mathbb{R}) \xrightarrow{C^{1,\alpha'}} \Phi$ for all $\alpha' < \alpha$, we have (after passing to a further subsequence if necessary) that $B^{n+1}_{\varepsilon/2}(x_k) \cap \Phi_k = \emptyset$. Hence, for k big enough so that $B^{n+1}_{\varepsilon/2}(x_k) \subset B^n_{r_k}(0) \times \mathbb{R}$ and $(\varepsilon/2)^{\alpha}[f_k]_{\alpha, B^n_{r_k}\times\mathbb{R}} \leq 1/2$, we can apply (2) of Lemma 2.14 with $x_0 = x_k$ to conclude that for any $\rho \leq \varepsilon/2$

$$\mu_{T_k}(B_\rho^{n+1}(x_k)) \geq c\rho^n$$

where c depends only on n.

Since $x_k \in W \Subset \mathbb{R}^{n+1}$ and $B_\varepsilon^{n+1}(x_k) \cap \operatorname{spt} T = \emptyset$ for all k, we have that, after passing to a subsequence, $x_k \to x_0$ for some $x_0 \in W$, such that $B^{n+1}_{\varepsilon/2}(x_0) \cap \operatorname{spt} T = \emptyset$. Then for k large enough we have that

$$B^{n+1}_{\varepsilon/4}(x_k) \subset B^{n+1}_{\varepsilon/2}(x_0) \quad \Rightarrow \quad \mu_{T_k}(B^{n+1}_{\varepsilon/2}(x_0)) \geq \mu_{T_k}(B^{n+1}_{\varepsilon/4}(x_k)) \geq c(\varepsilon/4)^n.$$

By the mass convergence $\mu_{T_k} \to \mu_T$, this implies that

$$\mu_T(B^{n+1}_{\varepsilon/2}(x_0)) > 0$$

which contradicts the fact that $B^{n+1}_{\varepsilon/2}(x_0) \cap \operatorname{spt} T = \emptyset$. $\square$

Theorem 2.16 *Let* $(\Omega, \Phi) \in \mathcal{B}^\alpha_r$, *with* Φ *given by the graph of a function*; $\Phi = \operatorname{graph}\phi$ *and let also* $u \in C^{1,\alpha}(\overline{\Omega})$ *be a* (*weak*) *solution of the Dirichlet problem* (2.5) *with* H, f *satisfying*

$$\|H\|_{0,B^n_r(0)\times\mathbb{R}} \leq K, \qquad [f]_{\alpha,B^n_r(0)\times\mathbb{R}} \leq K \tag{1}$$

for some $K > 0$.

Then $\forall\, \varepsilon > 0$, *there exists* $\rho = \rho(r, \varepsilon, K) < r$ *such that the following holds*:

For any $x \in (B^n_{r/2}(0) \times \mathbb{R}) \cap \operatorname{graph} u$ *and* $\lambda \in (0, \rho]$ *such that* $\operatorname{dist}(x, \partial\Omega \times \mathbb{R}) < \lambda$,

$$\lambda^{-1} \sup_{\operatorname{graph} u \cap B^{n+1}_\lambda(x)} \operatorname{dist}(y - x, P) < \varepsilon, \tag{2}$$

for some n*-dimensional linear subspace* $P = P(x, \lambda)$

$$\omega_n^{-1}\lambda^{-n}|\operatorname{graph} u \cap B^{n+1}_\lambda(x)| \leq 1 + \varepsilon. \tag{3}$$

In particular, if $x \in \Phi$ *then inequality* (2) *holds with an* n*-dimensional halfspace* $P_+ = P_+(x, \lambda)$ *in place of* P, *such that* $0 \in \partial P_+$ *and*

$$\lambda^{-1} \sup_{\Phi \cap B^{n+1}_\lambda(x)} \operatorname{dist}(y - x, \partial P_+) < \varepsilon$$

and inequality (3) *holds with the* RHS *replaced by* $1/2 + \varepsilon$.

Proof Assume that the theorem is not true. Then for some $\varepsilon > 0$, there exist a sequence of boundary data $(\Omega_i, \Phi_i) \in \mathcal{B}^\alpha_r$ and corresponding Dirichlet problems (as in (2.5)) with H_i, f_i satisfying (1) such that the following holds: there exists a sequence $\lambda_i \downarrow 0$ and $x_i \in (B^n_{r/2}(0) \times \mathbb{R}) \cap \operatorname{graph} u_i$, where $u_i \in C^{1,\alpha}(\Omega_i)$ are weak solutions of the corresponding problems, with

$$\operatorname{dist}(x_i, \partial\Omega_i \times \mathbb{R}) < \lambda_i$$

but such that at least one of the assertions (2), (3) with $x = x_i$ and $\lambda = \lambda_i$ fails.

Let $\widetilde{\Omega}_i = \eta_{x_i,\lambda_i}(\Omega_i)$ and $\widetilde{\Phi}_i = \eta_{x_i,\lambda_i}(\Phi_i)$, where $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$. Then (after a vertical translation so that $\widetilde{\Omega}_i \subset \mathbb{R}^n \times \{0\}$)

$$(\widetilde{\Omega}_i, \widetilde{\Phi}_i) \in \mathcal{B}^\alpha_{r/(2\lambda_i)}$$

and $\kappa_{(\widetilde{\Omega}_i,\widetilde{\Phi}_i)} \leq \lambda_i^\alpha \kappa_{(\Omega_i,\Phi_i)}$. Also $\widetilde{\Phi}_i = \operatorname{graph}\widetilde{\phi}_i$, where $\widetilde{\phi}_i \in C^{1,\alpha}(\partial\widetilde{\Omega}_i)$ is defined by $\widetilde{\phi}_i(x') = \eta_{x_i,\lambda_i}(\phi_i(\lambda_i x' + x_i))$. Furthermore, for $\widetilde{T}_i = \eta_{x_i,\lambda_i\#}(T_i)$, where $T_i = [\![\operatorname{graph} u_i]\!]$, we have that $\widetilde{T}_i = [\![\operatorname{graph}\widetilde{u}_i]\!]$, where $\widetilde{u}_i \in C^{1,\alpha}(\widetilde{\Omega}_i)$ is defined by $\widetilde{u}_i(x') = \eta_{x_i,\lambda_i}(u_i(\lambda_i x' + x_i))$ and is therefore a solution to the Dirichlet problem

$$\sum_{j=1}^n D_j\left(\frac{D_j\widetilde{u}_i}{\sqrt{1 + |D\widetilde{u}_i|^2}}\right) = \widetilde{H}_i + \sum_{j=1}^n D_j \widetilde{f}_i^j \quad \text{in } \widetilde{\Omega}_i,$$

$$\widetilde{u}_i = \widetilde{\phi}_i \quad \text{on } \partial\widetilde{\Omega}_i,$$

with

$$\widetilde{H}_i(x) = \lambda_i H_i(x_i + \lambda_i x) \quad \Rightarrow \quad \|\widetilde{H}_i\|_{0, B^n_{r/(2\lambda_i)} \times \mathbb{R}} \le \lambda_i \|H_i\|_{0, B^n_r(0)\times\mathbb{R}} \overset{i\to\infty}{\longrightarrow} 0,$$

$$\widetilde{f}_i(x) = f_i(x_i + \lambda_i x) \quad \Rightarrow \quad [\widetilde{f}_i]_{\alpha, B^n_{r/(2\lambda_i)}(0)\times\mathbb{R}} \le \lambda_i^\alpha [f_i]_{\alpha, B^n_r(0)\times\mathbb{R}} \overset{i\to\infty}{\longrightarrow} 0.$$

Hence we can apply Lemma 2.15 to the sequence $\widetilde{T}_i$, which implies that

$$\widetilde{T}_i \llcorner (B^n_{r/(2\lambda_i)}(0) \times \mathbb{R}) \to T$$

in the weak sense of currents, but also $\mu_{\widetilde{T}_i} \to \mu_T$ as Radon measures and where for the limit T the following holds:

(i) If $\liminf \lambda_i^{-1} \operatorname{dist}(x_i, \Phi_i) = \infty$ then $\partial T = 0$ and $\operatorname{spt} T$ is a vertical hyperplane.
(ii) If $\liminf \lambda_i^{-1} \operatorname{dist}(x_i, \Phi_i) < \infty$ then $\partial T \neq 0$, $\operatorname{spt} T$ is an n-dimensional halfspace and $\partial T = [\![\Phi]\!]$ with Φ being determined by $\widetilde{\Phi}_i$ as follows:

$$\widetilde{\Phi}_i \cap (B^n_{r/(2\lambda_i)}(0) \times \mathbb{R}) \overset{C^{1,\alpha'}}{\longrightarrow} \Phi, \quad \text{for all } \alpha' < \alpha.$$

By the measure convergence $\mu_{\widetilde{T}_i} \to \mu_T$, $\forall\, \varepsilon > 0$ there exists i_0 such that $\forall\, i \ge i_0$:

$$\lambda_i^{-n} \mu_{T_i}(B^{n+1}_{\lambda_i}(x_i)) = \mu_{\widetilde{T}_i}(B^{n+1}_1(0)) \le |\operatorname{spt} T \cap B^{n+1}_1(0)| + \varepsilon.$$

Furthermore, because of (2) of Lemma 2.15 for any $\varepsilon > 0$, there exists i_0 such that for all $i \ge i_0$

$$\frac{1}{\lambda_i} \sup_{y \in B^{n+1}_{\lambda_i}(x_i) \cap \operatorname{spt} T_i} \operatorname{dist}(y - x_i, \operatorname{spt} T) \le \sup_{y \in B^{n+1}_1(0) \cap \operatorname{spt} \widetilde{T}_i} \operatorname{dist}(y, \operatorname{spt} T) < \varepsilon$$

and if $\partial T \neq 0$, we also have that

$$\frac{1}{\lambda_i} \sup_{y \in B^{n+1}_{\lambda_i}(x_i) \cap \Phi_i} \operatorname{dist}(y - x_i, \operatorname{spt} \partial T) \le \sup_{y \in B^{n+1}_1(0) \cap \widetilde{\Phi}_i} \operatorname{dist}(y, \operatorname{spt} \Phi) < \varepsilon$$

since $\widetilde{\Phi}_i \cap (B^n_{r/(2\lambda_i)}(0) \times \mathbb{R}) \overset{C^{1,\alpha'}}{\longrightarrow} \Phi$.

Hence taking P to be the n-dimensional linear subspace that contains the support of T we get a contradiction.

In the special case when $x_i \in \Phi_i$ we argue in the same way. In this situation, for the limit T we are in case (ii) $\partial T = [\![\Phi]\!] \neq 0$ and furthermore $0 \in \Phi$. Hence we get a contradiction by taking $P_+ = \operatorname{spt} T$. $\square$

3 Approximating the MCE

Throughout this section we let Ω be a $C^{1,\alpha}$ bounded domain in $\mathbb{R}^n$ and Φ a compact, embedded $C^{1,\alpha}$ submanifold of $\partial\Omega \times \mathbb{R}$, such that for a sequence $\phi_i \in C^{1,\alpha}(\partial\Omega)$, graph $\phi_i \xrightarrow{C^{1,\alpha}} \Phi$, where the convergence is as in Definition 2.1. By translating Ω we can assume that $0 \in \partial\Omega$ and hence for some $r > 0$, $(\Omega, \Phi) \in \mathcal{B}_r^\alpha$, with $\mathcal{B}_r^\alpha$ as in Definition 2.2. We let also $H = H(x', x_{n+1})$ be a C^1 function in $\overline{\Omega} \times \mathbb{R}$, which is non-decreasing in the x_{n+1}-variable and such that $\|H\|_0 \leq n\omega_n^{1/n}|\Omega|^{-1/n}$.

In this section we will show that the Dirichlet problem of prescribed mean curvature equal to H (cf. (1.1)) and with boundary data (Ω, Φ), can be approximated by a sequence of new Dirichlet problems for the prescribed mean curvature equation which have the form of the one defined in (2.5). We will construct the new equations in such a way that

(a) We have uniform $C^{1,\alpha}$ bounds for the graphs of the solutions of the approximating problems.
(b) We can construct barriers for the solutions and prove gradient bounds and hence existence of the solutions.

3.1 Constructing the Approximating Sequence

For $(\Omega, \Phi) \in \mathcal{B}_r^\alpha$, let $\{\Omega_k\}$ be a sequence of bounded, C^∞ domains with $\Omega_k \subset \Omega$ for all k, $\phi_k \in C^\infty(\partial\Omega_k)$ and $\Phi_k = \operatorname{graph}\phi_k$ be such that $(\Omega_k, \Phi_k) \in \mathcal{B}_r^\alpha$ and

$$\Omega_k \xrightarrow{C^{1,\alpha}} \Omega, \qquad \Phi_k \xrightarrow{C^{1,\alpha}} \Phi$$

with the convergence being as in Definition 2.1.

For each k we consider the following Dirichlet problem

$$\begin{aligned} \sum_{i=1}^n D_i\left(\frac{D_i u_k}{\sqrt{1+|Du_k|^2}}\right) &= \sum_{i=1}^n D_i f_k^i + H_k \quad \text{in } \Omega_k, \\ u_k &= \phi_k \quad \text{on } \partial\Omega_k, \end{aligned} \tag{3.1}$$

where the equation above is to be interpreted weakly (as is (2.6)) and $H_k : \overline{\Omega}_k \times \mathbb{R} \to \mathbb{R}$, $f_k = (f_k^1, \ldots, f_k^n) : \overline{\Omega}_k \times \mathbb{R} \to \mathbb{R}$ satisfy the following properties for a sequence $\delta_k \downarrow 0$:

(i) $H_k = H_k(x', x_{n+1})$ is a C^1 function in $\overline{\Omega}_k \times \mathbb{R}$, which is non-decreasing in the x_{n+1}-variable and such that $\|H_k\|_0 \leq \|H\|_0$ and

$$H_k(x', x_{n+1}) = \begin{cases} H(x', x_{n+1}) & \text{for } x' \in \Omega_k : \operatorname{dist}(x', \partial\Omega_k) > 2\delta_k, \\ 0 & \text{for } x' \in \Omega_k : \operatorname{dist}(x', \partial\Omega_k) < \delta_k. \end{cases}$$

(ii) There exists a neighborhood V_k of $\partial\Omega_k$ in Ω_k such that

$$\{x' \in \Omega_k : \operatorname{dist}(x', \partial\Omega_k) < \delta_k\} \subset V_k, \qquad B_{r/4}^n(x') \subset V_k \quad \forall x' \in \partial\Omega_k \cap B_r^n(0)$$

and such that f_k is $C^{0,\alpha}$ when restricted in $V_k \times \mathbb{R}$, and in particular it satisfies the estimate

$$\begin{aligned} &\|f_k\|_{0, B_{r/4}^n(x') \times \mathbb{R}} + r^\alpha [f_k]_{\alpha, B_{r/4}^n(x') \times \mathbb{R}} \\ &\quad \leq C\left(\|\eta_k\|_{0, (\partial\Omega_k \cap B_r^n(x')) \times \mathbb{R}} + r^\alpha [\eta_k]_{\alpha, (\partial\Omega_k \cap B_r^n(x')) \times \mathbb{R}}\right) \end{aligned}$$

for all $x' \in \partial\Omega_k \cap B_r^n(0)$. Here η_k is the inward pointing unit normal to the cylinder $\Omega_k \times \mathbb{R}$, and C is a constant that depends only on n, in particular it is independent of k (recall that r is such that $(\Omega, \Phi) \in \mathcal{B}_r^\alpha$). Also

$$\sum_{i=1}^n D_i f_k(x', x_{n+1}) = \begin{cases} \operatorname{div} \eta_k(x', x_{n+1}) & \text{for } x' \in \partial\Omega_k,\ x_{n+1} > \phi_k(x'), \\ -\operatorname{div} \eta_k(x', x_{n+1}) & \text{for } x' \in \partial\Omega_k,\ x_{n+1} < \phi_k(x'), \\ 0 & \text{for } x' \in \Omega_k : \operatorname{dist}(x', \partial\Omega_k) > \delta_k \text{ (weakly)}. \end{cases}$$

We now show how to construct H_k, f_k satisfying the above properties.
For any $\delta > 0$ we define Ω_k^δ to be the δ-neighborhood of $\partial\Omega_k$ in Ω_k, i.e.,

$$\Omega_k^\delta = \{x' \in \Omega_k : \operatorname{dist}(x', \partial\Omega_k) < \delta\}.$$

Let $\{\delta_k\}$ be a sequence such that

$$\delta_k \to 0 \quad \text{and} \quad \delta_k^{1/2} \|H_{\partial\Omega_k}\|_0 \to 0, \tag{3.2}$$

where $H_{\partial\Omega_k}$ denotes the mean curvature of $\partial\Omega_k$ with respect to the inward pointing unit normal. We also take δ_k small enough so that the nearest point projection, which we will denote by $\operatorname{proj}_{\partial\Omega_k}(x')$, is well defined for all $x' \in \Omega_k^{2\delta_k}$. Notice that we can do this since $\partial\Omega_k$ is C^∞, and using (3.2) we have that

$$|\operatorname{proj}_{\partial\Omega_k \times \mathbb{R}}(x) - \operatorname{proj}_{\partial\Omega \times \mathbb{R}}(y)| \le C|x - y| \quad \forall x, y \in \Omega_k^{2\delta_k} \times \mathbb{R}, \tag{3.3}$$

where C is a constant that is independent of k. This enables us to extend η_k in $\Omega_k^{2\delta_k} \times \mathbb{R}$ by letting $\eta_k(x', x_{n+1}) = \eta_k(\operatorname{proj}_{\partial\Omega_k}(x'), x_{n+1}) = \eta_k(\operatorname{proj}_{\partial\Omega_k}(x'), 0)$ and for this extension, using (3.3) we have that

$$[\eta_k]_{\alpha, (\Omega_k^{\delta_k} \cap B_r^n(x')) \times \mathbb{R}} \le C[\eta_k]_{\alpha, (\partial\Omega_k \cap B_r^n(x'))} \quad \forall x' \in \partial\Omega_k, \tag{3.4}$$

where C is a constant independent of k. Similarly, we can extend ϕ_k in $\Omega_k^{2\delta_k}$ by $\phi_k(x') = \phi_k(\operatorname{proj}_{\partial\Omega_k}(x'))$. Furthermore, we pick the sequence $\{\delta_k\}$ so that

$$\delta_k^{\alpha/2} \|D\phi_k\|_0 \to 0. \tag{3.5}$$

We remark that this is a technical assumption that will be used later for proving global gradient estimates for a solution of (3.1) (cf. Lemma 3.16).

With δ_k as above, we let

$$H_k(x', x_{n+1}) = \begin{cases} H(x', x_{n+1}) & \text{in } (\Omega_k \setminus \Omega_k^{2\delta_k}) \times \mathbb{R}, \\ 0 & \text{in } \Omega_k^{\delta_k} \times \mathbb{R}, \end{cases}$$

and extend it in the rest of the domain $\Omega_k \times \mathbb{R}$ so that it is C^1, non-decreasing in the x_{n+1}-variable and so that $\|H_k\|_{0, \overline{\Omega}_k \times \mathbb{R}} \le \|H\|_{0, \overline{\Omega} \times \mathbb{R}}$. Hence we have constructed H_k, satisfying the properties described in (i) above.

To construct f_k, we define $U^+, U^- \subset \Omega_k^{\delta_k/2} \times \mathbb{R}$ by

$$\begin{aligned} U^+ &= \{(x', x_{n+1}) : x' \in \Omega_k^{\delta_k/2}, x_{n+1} \geq \phi(x') + \operatorname{dist}(x', \partial\Omega_k \times \mathbb{R})\}, \\ U^- &= \{(x', x_{n+1}) : x' \in \Omega_k^{\delta_k/2}, x_{n+1} < \phi(x') - \operatorname{dist}(x', \partial\Omega_k \times \mathbb{R})\}. \end{aligned} \tag{3.6}$$

By Lemma A.1, Remark A.2, there exists a smooth vector field $X = (X^1, \ldots, X^n, X^{n+1})$ in U^-, independent of the x_{n+1}-variable, such that

$$\operatorname{div} X = \sum_{i=1}^{n} D_i X^i = 0, \qquad X(x', x_{n+1}) = 2\eta_k(x', x_{n+1}) \quad \text{for } x' \in \partial\Omega_k$$

and

$$\begin{aligned} &\|X\|_{0,(B_r^n(x')\times\mathbb{R})\cap U^-} + r^\alpha [X]_{\alpha,(B_r^n(x')\times\mathbb{R})\cap U^-} \\ &\quad \leq C\left(\|\eta_k\|_{0,(B_r^n(x')\cap\partial\Omega_k)\times\mathbb{R}} + r^\alpha[\eta_k]_{\alpha,(B_r^n(x')\cap\partial\Omega_k)\times\mathbb{R}}\right) \end{aligned}$$

for any $x' \in \partial\Omega_k$, and where C is independent of k (for sufficiently large k).

Using again Lemma A.1, Remark A.2, there exists a neighborhood V_k of $\partial\Omega_k$ in Ω_k such that

$$\Omega_k^{\delta_k} \subset V_k, \qquad B_{r/4}^n(x) \subset V_k \quad \forall x \in \partial\Omega_k \cap B_r^n(0)$$

and a smooth vector field $Y = (Y^1, \ldots, Y^n, Y^{n+1})$ in $(V_k \setminus \Omega_k^{\delta_k}) \times \mathbb{R}$, independent of the x_{n+1} variable, such that

$$\operatorname{div} Y = \sum_{i=1}^{n} D_i Y^i = 0, \qquad Y(x', x_{n+1}) = \eta_k(x', x_{n+1}) \quad \text{for } x' \in \partial V_k \setminus \partial\Omega_k$$

and

$$\begin{aligned} &\|Y\|_{0,(B_r^n(x')\cap(V_k\setminus\Omega_k^{\delta_k}))\times\mathbb{R}} + r^\alpha [Y]_{\alpha,(B_r^n(x')\cap(V_k\setminus\Omega_k^{\delta_k}))\times\mathbb{R}} \\ &\quad \leq C\left(\|\eta_k\|_{0,(B_r^n(x')\cap\partial\Omega_k)\times\mathbb{R}} + r^\alpha[\eta_k]_{\alpha,(B_r^n(x')\cap\partial\Omega_k)\times\mathbb{R}}\right) \end{aligned}$$

for any $x' \in \partial\Omega_k$, and where C is independent of k (for sufficiently large k).

We then define $f_k : U = U^+ \cup U^- \cup ((V_k \setminus \Omega_k^{\delta_k}) \times \mathbb{R}) \to \mathbb{R}^n$ as follows: For each $i \in \{1, \ldots, n\}$ we let

$$f_k^i(x) = \begin{cases} \eta_k^i(x) & \text{if } x \in U^+, \\ X^i - \eta_k^i(x) & \text{if } x \in U^-, \\ Y^i(x) & \text{if } x \in (V_k \setminus \Omega_k^{\delta_k}) \times \mathbb{R}, \\ 0 & \text{if } x \in (\Omega_k \setminus V_k) \times \mathbb{R}. \end{cases} \tag{3.7}$$

One can easily check now, using the estimates for the norms of X and Y, the definition of $U^\pm$ as well as (3.3), (3.4), that $f_k = (f_k^1, \ldots, f_k^n)$ is a $C^{0,\alpha}$ vector field in the domain $U = U^+ \cup U^- \cup ((V_k \setminus \Omega_k^{\delta_k}) \times \mathbb{R})$ and

$$\begin{aligned}
&\|f_k\|_{0,(B^n_{r/4}(x')\times\mathbb{R})\cap U} + r^\alpha [f_k]_{\alpha,(B^n_{r/4}(x')\times\mathbb{R})\cap U} \\
&\quad \le C \left(\|\eta_k\|_{0,(B^n_r(x')\cap\partial\Omega_k)\times\mathbb{R}} + r^\alpha [\eta_k]_{\alpha,(B^n_r(x')\cap\partial\Omega_k)\times\mathbb{R}}\right)
\end{aligned} \tag{3.8}$$

for any $x' \in \partial\Omega_k \cap B^n_r(0)$, and where C is independent of k. Hence we can extend f_k in $\Omega_k \times \mathbb{R}$ so that the estimate (3.8) still holds (with U replaced by $\Omega_k \times \mathbb{R}$).

Remark 3.9 By the construction of f_k and using Lemma A.1, Remark A.2 we note that $\sum_{i=1}^n D_i f_k^i$ is well defined and smooth in $(\overline{\Omega}_k \times \mathbb{R}) \setminus \operatorname{graph}\phi_k|_{\partial\Omega_k}$ and in this domain

$$\left\|\sum_{i=1}^n D_i f_k^i\right\|_0 \le c\|\operatorname{div}\eta_k\|_0 \quad \Rightarrow \quad \delta_k^{1/2}\left\|\sum_{i=1}^n D_i f_k^i\right\|_0 \stackrel{k\to\infty}{\longrightarrow} 0$$

since $\delta_k^{1/2}\|H_{\partial\Omega_k}\|_0 \to 0$ (cf. (3.2)).

Furthermore, since f_k, as defined in (3.7), is independent of the x_{n+1}-variable in each of the domains U^+, U^- and $(V_k \setminus \Omega_k^{\delta_k})$, we can extend it in $\Omega_k \times \mathbb{R}$ so that it is still independent of the x_{n+1} variable in each of the domains

$$\{(x', x_{n+1}) : x' \in \Omega_k^{\delta_k}, x_{n+1} \ge \phi(x') + \operatorname{dist}(x', \partial\Omega_k \times \mathbb{R})\}$$

and

$$\{(x', x_{n+1}) : x' \in \Omega_k^{\delta_k}, x_{n+1} < \phi(x') - \operatorname{dist}(x', \partial\Omega_k \times \mathbb{R})\}.$$

In these domains we also have that $\sum_{i=1}^n D_i f_k^i(x)$ is equal to $\operatorname{div}\eta_{\partial\Omega_k}(x)$ and $-\operatorname{div}\eta_{\partial\Omega_k}(x)$, respectively. This extra property of f_k will be used later for proving global gradient estimates for a solution of (3.1) (cf. Lemma 3.16).

Remark 3.10 The solutions u_k of the approximating problems satisfy a uniform sup estimate, i.e., if $u_k \in C^{1,\alpha}(\Omega_k)$ are solutions of the problems (3.1), then

$$\|u_k\|_0 \le M$$

for some constant M independent of k.

To see this, note that by the assumption on $\|H\|_0$ and by Remark 3.9, for δ_k small enough

$$\left|\int_\Omega \left(H_k + \sum_{i=1}^n D_i f_k^i\right)\zeta\, d\mathcal{H}^n\right| \le (1-\varepsilon_0)\int_\Omega |D\zeta|\, d\mathcal{H}^n$$

for all $\zeta \in C_0^1(\Omega)$ and where $\varepsilon_0 < 1$ is a constant independent of k. Hence we get a uniform sup estimate [13, p. 408].

3.2 $C^{1,\alpha}$ Regularity of the Approximating Graphs

In the following theorem, which is essentially an application of Theorem 2.16 and Allard's regularity theorem [1], we prove that the graphs of the solutions are close to planes in uniform sized balls.

Theorem 3.11 *For each k let $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ be a (weak) solution of (3.1). For any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon)$ such that the following holds:*

For any k, $x_k \in \operatorname{graph} u_k \cap (B^n_{r/8}(0) \times \mathbb{R})$ and $\lambda \leq \lambda_0$

$$\lambda^{-1} \sup_{\operatorname{graph} u_k \cap B^{n+1}_\lambda(x_k)} \operatorname{dist}(y - x_k, P) < \varepsilon \tag{1}$$

for some n-dimensional linear subspace $P = P(x_k, \varepsilon, \lambda)$.

In particular, if $x_k \in \operatorname{graph} \phi_k \cap (B^n_{r/8}(0) \times \mathbb{R})$ then (1) holds with an n-dimensional halfspace $P_+ = P_+(x_k, \varepsilon, \lambda)$ in place of P, such that $0 \in \partial P_+$ and

$$\lambda^{-1} \sup_{\Phi_k \cap B^{n+1}_\lambda(x_k)} \operatorname{dist}(y - x_k, \partial P_+) < \varepsilon. \tag{2}$$

Proof We assume that the conclusion is not true. Then for some $\varepsilon_0 > 0$ and for any $\lambda_0 > 0$ there exist k_j, $x_j \in \operatorname{graph} u_{k_j} \cap (B^n_{r/8}(0) \times \mathbb{R})$ and $\lambda_j < \lambda_0$ such that the conclusion of the lemma for $k = k_j$, $x_k = x_j$, and $\lambda = \lambda_j$ fails. Hence there exist sequences $\{k_j\}$, $\{x_j\}$ such that $x_j \in \operatorname{graph} u_{k_j}$ and a sequence $\{\lambda_j\} \downarrow 0$ such that the conclusion (1) of the lemma with this ε_0 and with $k = k_j$, $x_k = x_j$, $\lambda = \lambda_j$ fails for all j.

Since for all k, $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$, we can assume that $k_j \to \infty$. Hence without loss of generality we can take $k_j = j$.

Let $d_j = \operatorname{dist}(x_j, \partial\Omega_j \times \mathbb{R})$. Standard PDE theory implies uniform interior $C^{1,\alpha}$ estimates for the solutions of the of the problems (3.1), therefore we can assume that $d_j \to 0$ (cf. Remark 3.12).

In the special case when $x_j \in \Phi_j$, we can apply Theorem 2.16 with $x = x_j$, $\lambda = \lambda_j$. Hence for any $\varepsilon > 0$ there exists j_0 such that for $j \geq j_0$

$$\lambda_j^{-1} \sup_{\operatorname{graph} u_j \cap B^{n+1}_{\lambda_j}(x_j)} \operatorname{dist}(y - x_j, P_+) \leq \varepsilon$$

for some n-dimensional linear halfspace P_+ with $0 \in \partial P_+$, such that

$$\lambda_j^{-1} \sup_{\Phi_j \cap B^{n+1}_{\lambda_j}(x_j)} \operatorname{dist}(y - x_j, \partial P_+) \leq \varepsilon$$

and so by taking $\varepsilon = \varepsilon_0$ we get a contradiction, which proves the special case (2) of the theorem.

We assume now that $x_j \notin \Phi_j$. Applying Theorem 2.16 with $x = x_j$ and $\lambda = d_j + \lambda_j$ we get that for any $\varepsilon > 0$ there exists j_0 such that for $j \geq j_0$

$$(d_j + \lambda_j)^{-1} \sup_{\operatorname{graph} u_j \cap B^{n+1}_{d_j + \lambda_j}(x_j)} \operatorname{dist}(y - x_j, P) \leq \varepsilon \tag{3}$$

for some n-dimensional linear subspace P and

$$\omega_n^{-1}(d_j+\lambda_j)^{-n}|\operatorname{graph} u_j \cap B^{n+1}_{d_j+\lambda_j}(x_j)| \le 1+\varepsilon. \tag{4}$$

We will consider two different cases, namely $\liminf d_j^{-1}\lambda_j > 0$ or $d_j = 0$ and $\liminf d_j^{-1}\lambda_j = 0$ and show that in both cases we are led to a contradiction.

Case 1: $\liminf d_j^{-1}\lambda_j > 0$ or $d_j = 0$.

In this case (3) implies that

$$\lambda_j^{-1} \sup_{\operatorname{graph} u_j \cap B^{n+1}_{\lambda_j}(x_j)} \operatorname{dist}(y-x_j, P) \le \frac{\lambda_j+d_j}{\lambda_j}\varepsilon \le c\varepsilon.$$

Hence by taking ε small enough, so that $c\varepsilon < \varepsilon_0$, where c is as in the above inequality, we get a contradiction.

Case 2: $\liminf d_j^{-1}\lambda_j = 0$.

In this case for any $p \ge 2n$

$$d_j^{1-n/p}\left(\int_{B^{n+1}_{d_j}(x_j)} \left| H_j + \sum_{i=1}^n D_i f_j^i \right|^p d\mu_{T_j}\right)^{\frac{1}{p}} \overset{j\to\infty}{\longrightarrow} 0 \tag{5}$$

since either $d_j > 2\delta_j$, which implies that $B^{n+1}_{d_j/2}(x_j) \subset \{x \in \Omega_j \times \mathbb{R} : \operatorname{dist}(x, \partial\Omega_j \times \mathbb{R}) > \delta_j\}$ where $\sum_{i=1}^n D_i f_j^i = 0$, or $d_j \le 2\delta_j$, in which case (5) is true because of Remark 3.9.

Furthermore, (4) implies that

$$\omega_n^{-1} d_j^{-n}|\operatorname{graph} u_j \cap B^{n+1}_{d_j}(x_j)| \le \left(1+\frac{\lambda_j}{d_j}\right)^n (1+\varepsilon).$$

Hence for any $\varepsilon' > 0$ there exists j_0 such that for all $j \ge j_0$, $\operatorname{graph} u_j \cap B^{n+1}_{d_j}(x_j)$ satisfies the hypothesis of Allard's interior regularity theorem and thus there exists $\theta \in (0,1)$ such that $\operatorname{graph} u_j \cap B^{n+1}_{\theta d_j}(x_j)$ is the graph of a $C^{1,\alpha}$ function v_j above an n-dimensional linear space P, with the $C^{1,\alpha}$ norm of v_j is less than ε'. Hence for all $j \le j_0$ such that $\lambda_j < \theta d_j$

$$\lambda_j^{-1} \sup_{\operatorname{graph} u_j \cap B^{n+1}_{\lambda_j}(x_j)} \operatorname{dist}(y-x_j, P) \le \varepsilon'$$

which for $\varepsilon' = \varepsilon_0$ gives a contradiction. □

We will show next that the graphs of the solutions u_k are not only ε-close to planes, as we proved in Theorem 3.11, but in fact they are ε-close in the $C^{1,\alpha}$ sense, i.e., we will prove that around each point there exists a uniform sized ball in which graph u_k is a $C^{1,\alpha}$ manifold with uniformly bounded $C^{1,\alpha}$ norm.

Remark 3.12 For all k, $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ and so graph u_k is a $C^{1,\alpha}$ manifold-with-boundary equal to $\Phi_k = \operatorname{graph} \phi_k$. Therefore, given $\varepsilon_0 \in (0, 1/2)$, for any k and $x \in \operatorname{graph} u_k$ there exists some $\rho = \rho(k, x, \varepsilon_0)$ such that

$$\rho^\alpha \frac{|\nu_k(y) - \nu_k(z)|}{|y - z|^\alpha} \le \varepsilon_0/4 \quad \forall y, z \in B_\rho^{n+1}(x) \cap \operatorname{graph} u_k, \tag{1}$$

where for any point $x = (x', u_k(x')) \in \operatorname{graph} u_k$, $\nu_k(x)$ is the downward pointing unit normal of graph u_k at x. Note that provided $\operatorname{dist}(x, \partial\Omega \times \mathbb{R}) \ge d > 0$, the radius ρ satisfying (1) is independent of k and x, i.e., there exists $\rho_0 = \rho_0(d, \varepsilon_0) < d$ such that the inequality in (1) holds with any k and $x \in \operatorname{graph} u_k$ such that $B_\rho^{n+1}(x) \subset (\Omega_k \setminus \Omega_k^d) \times \mathbb{R}$; recall that $\Omega_k^d = \{x \in \Omega_k : \operatorname{dist}(x, \partial\Omega_k) < d\}$. That is because standard PDE estimates [13, 16, Chap. 13] imply that for any $d > 0$ we have that

$$\sup_{\{k : \delta_k < d/2\}} \|u_k\|_{1,\alpha,\Omega_k \setminus \Omega_k^d} < C(d),$$

where $C(d)$ is a constant independent of k.

For any k, $x \in \operatorname{graph} u_k$ and $\rho = \rho(k, x, \varepsilon_0)$ satisfying (1), we have that

$$\operatorname{graph} u_k \cap B_\rho^{n+1}(x) = \operatorname{graph} v \cap B_\rho^{n+1}(x) \tag{2}$$

with $v \in C^{1,\alpha}((x + (L_x \cap U)) \cap B_\rho^{n+1}(x); L_x^\perp)$ and $L_x = T_x \operatorname{graph} u_k$, the tangent space of graph u_k at x. Since $v(0) = 0$, $|Dv(0)| = 0$ and for all $x = (x', v(x')) \in \operatorname{graph} u_k$ we have that

$$\nu_k(x) = \left(\frac{D_1 v(x')}{\sqrt{1 + |Dv(x')|^2}}, \ldots, \frac{D_n v(x')}{\sqrt{1 + |Dv(x')|^2}}, -\frac{1}{\sqrt{1 + |Dv(x')|^2}} \right).$$

It is easy to check that (1) implies that

$$\|v\|_{1,\alpha} \le \varepsilon_0.$$

Also, U is a $C^{1,\alpha}$ domain of L_x, since either

(i) $\Phi_k \cap B_\rho^{n+1}(x) = \emptyset$, in which case $U = L_x$ or
(ii) $x + (\partial U \cap B_\rho^{n+1}(0)) = \operatorname{proj}_{T_x \operatorname{graph} u_k}(\Phi_k \cap B_\rho^{n+1}(x))$.

Furthermore, the function v satisfies the equation

$$\sum_{i=1}^n D_i \left(\frac{D_i v}{\sqrt{1 + |Dv|^2}} \right) = \operatorname{div} f_k + H_k \quad \text{in } U \cap B_\rho^n(0),$$

where for $x' \in \Omega_k$, $y' \in x + (L_x \cap U)$ we have identified $(x', u_k(x'))$ with $(y', v(y'))$ using (2).

Given ε_0, k, and $x \in \operatorname{graph} u_k$, let ρ be such that (1) holds and assume furthermore that for some $\varepsilon < \varepsilon_0$, $B^{n+1}_\rho(x) \cap \operatorname{graph} u_k$ is ε-close to some n-dimensional linear space P, i.e.,

$$\rho^{-1} \sup_{y \in B^{n+1}_\rho(x) \cap \operatorname{graph} u_k} \operatorname{dist}(y - x, P) < \varepsilon.$$

We then have that

$$\rho^{-1} \operatorname{dist}(P \cap B_\rho(0), T_x \operatorname{graph} u_k \cap B_\rho(0)) \leq \varepsilon + \varepsilon_0$$

and it is then easy to check (by writing P as the graph of a linear function above $T_x \operatorname{graph} u_k$) that this last inequality implies that

$$\|N - \nu_k(x)\| < 5\varepsilon_0/2,$$

where N is the normal to P. Using this and (1) we have that

$$\begin{aligned} &\|\nu_k(y) - N\| \leq \|\nu_k(y) - \nu_k(x)\| + \|\nu_k(x) - N\| < 3\varepsilon_0 \\ &\forall y \in \operatorname{graph} u_k \cap B^{n+1}_\rho(x). \end{aligned} \tag{3}$$

This implies that (2) holds with $v \in C^{1,\alpha}((x + (P \cap \widetilde{U})) \cap B^{n+1}_\rho(x); P^\perp)$, such that $\|v\|_{1,\alpha} \leq 6\varepsilon_0$ and where $\widetilde{U}$ is a $C^{1,\alpha}$ domain of P.

Theorem 3.13 *Let $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ be a solution of (3.1). For $0 < \varepsilon_0 < 1/4$ there exists a constant $\rho_0 = \rho_0(\varepsilon_0)$, independent of k, such that*

$$\rho_x = \sup_r \{\kappa(\operatorname{graph} u_k, x, r) < \varepsilon_0\} < \rho_0$$

for all $x \in \operatorname{graph} u_k \cap B^n_{r/16}(0)$, where κ is as in Definition 2.1.

Proof Let $\sigma = r/8$. By Theorem 3.11 we have that for any $\varepsilon > 0$ there exists $\lambda_0 = \lambda_0(\varepsilon)$ such that for any k, $x_k \in \operatorname{graph} u_k \cap (B^n_\sigma(0) \times \mathbb{R})$ and $\lambda \leq \lambda_0$

$$\lambda^{-1} \sup_{\operatorname{graph} u_k \cap B^{n+1}_\lambda(x_k)} \operatorname{dist}(y - x_k, P) < \varepsilon \tag{1}$$

for some n-dimensional linear subspace $P = P(x_k, \varepsilon, \lambda)$.

We fix a k, and define the following:

$$\begin{aligned} d(x) &= \operatorname{dist}(x, \partial B^n_\sigma(0) \times \mathbb{R}), \\ \theta_1 &= \min\left\{\frac{\rho_x}{d(x)} : x \in \operatorname{graph} u_k \cap \Phi_k \cap (B^n_\sigma(0) \times \mathbb{R})\right\}, \\ \theta_2 &= \min\left\{\frac{\rho_x}{d(x)} : x \in (\operatorname{graph} u_k \setminus \Phi_k) \cap (B^n_\sigma(0) \times \mathbb{R})\right\}. \end{aligned}$$

Note that both these minima are attained. Given ε small enough (that will be determined later), let $\lambda_0 = \lambda_0(\varepsilon)$ be such that (1) holds. We can assume that

$$\min\{\theta_1, \theta_2\} \leq \frac{1}{8} \min\{1/2, \lambda_0/\sigma\} \tag{2}$$

since otherwise for all $x \in \operatorname{graph} u_k \cap (B^n_{\sigma/2}(0) \times \mathbb{R})$

$$\rho_x \geq \min\{\theta_1, \theta_2\} d(x) \geq \frac{1}{8} \min\{\sigma/4, \lambda_0/2\}$$

and hence the lemma is trivially true.

Let $x \in \operatorname{graph} u_k \cap (\overline{B}^n_\sigma(0) \times \mathbb{R})$ be such that the following hold:

(i) if $\theta_1 \leq 4\theta_2$ then $x \in \Phi_k$ and $\rho_x = \theta_1 d(x)$,
(ii) if $\theta_1 > 4\theta_2$ then $x \notin \Phi_k$ and $\rho_x = \theta_2 d(x)$.

In both cases, using (2), we have that

$$\rho_x \leq 4 \min\{\theta_1, \theta_2\} d(x) \leq \lambda_0/2.$$

Therefore, there exists an n-dimensional linear subspace P_0 such that

$$(2\rho_x)^{-1} \sup_{\operatorname{graph} u_k \cap B^{n+1}_{2\rho_x}(x)} \operatorname{dist}(y - x, P_0) < \varepsilon. \tag{3}$$

Remark 3.12, the definition of ρ_x, and (3) imply that

$$\operatorname{graph} u_k \cap B^{n+1}_{\rho_x}(x) = \operatorname{graph} v \cap B^{n+1}_{\rho_x}(x), \tag{4}$$

where $v \in C^{1,\alpha}((x + (P_0 \cap U)) \cap B^{n+1}_{\rho_x}(x); P_0^\perp)$ is such that $\|v\|_{1,\alpha} < 6\varepsilon_0$ and where U is a $C^{1,\alpha}$ domain of P_0, provided that $2\varepsilon < \varepsilon_0/4$.

We will show that we can extend v in $B^{n+1}_{(1+\gamma)\rho_x}(x)$ for some $\gamma \in (0, 1)$ such that (4) still holds with $(1 + \gamma)\rho_x$ in place of ρ_x and $\|v\|_{1,\alpha} < c\varepsilon_0$ for some constant c.

Let $z \in B^{n+1}_{2\rho_x}(x) \cap \operatorname{graph} u_k$. Then because of the choice of x and using (2) we have that

$$\rho_z \geq d(z) \min\{\theta_1, \theta_2\} \geq \frac{1}{4} \rho_x \frac{d(z)}{d(x)} \geq \frac{1}{4} \rho_x \left(1 - \frac{2\rho_x}{d(x)}\right) \geq \rho_x/8.$$

Furthermore, by (3)

$$(\rho_x/8)^{-1} \sup_{\operatorname{graph} u_k \cap B^{n+1}_{\rho_x/8}(z)} \operatorname{dist}(y - z, P_0) < 16\varepsilon.$$

Hence, by Remark 3.12, $B^{n+1}_{\rho_x/8}(z) \cap \operatorname{graph} u_k$ can be written as the graph of a $C^{1,\alpha}$ function above $z + P_0$ with $C^{1,\alpha}$ norm less than $6\varepsilon_0$, provided that $16\varepsilon < \varepsilon_0/4$. Since $\operatorname{dist}(z - x, P_0) < 2\varepsilon\rho_x$, by a translation we have that

$$\operatorname{graph} u_k \cap B^{n+1}_{\rho_x/8}(z) = \operatorname{graph} v_z \cap B^{n+1}_{\rho_x/8}(z), \tag{5}$$

where $v_z \in C^{1,\alpha}(x + (P_0 \cap U_z); P_0^\perp)$ is such that $\|v_z\|_{1,\alpha} < 7\varepsilon_0$ and where U_z is a $C^{1,\alpha}$ domain of P_0.

Note that for $z \in \operatorname{graph} u_k \cap \partial B^{n+1}_{\rho_x}$, because of (3) we have that

$$\operatorname{graph} v_z \cap B^{n+1}_{\rho_x/8}(z) \cap B^{n+1}_{\rho_x}(x) \neq \emptyset$$

and so

$$\operatorname{graph} v_z \cap \operatorname{graph} v \neq \emptyset,$$

where v is as defined in (4). Therefore, using the graphical representations in (5) for any $z \in \operatorname{graph} u_k \cap \partial B^{n+1}_{\rho_x}$, we can extend the function v so that $v \in C^{1,\alpha}((x + (P_0 \cap U)) \cap B^{n+1}_{(1+1/8)\rho_x}(x); P_0^\perp)$ and with

$$\|v\|_{1,\alpha} < c\varepsilon_0, \tag{6}$$

where c is an absolute constant. For $\gamma \in (0, 1/16)$, we can check (again using (3)) that for any $y \in \operatorname{graph} u_k \cap B^{n+1}_{(1+\gamma)\rho_x}(x)$ there exists $z \in \operatorname{graph} u_k \cap \partial B^{n+1}_{r_x}(x)$ such that $|z - y| < \rho_x/8$. Hence for the extended function v we have that

$$\operatorname{graph} u_k \cap B^{n+1}_{(1+\gamma)\rho_x}(x) = \operatorname{graph} v \cap B^{n+1}_{(1+\gamma)\rho_x}(x). \tag{7}$$

Furthermore, v satisfies the following equation:

$$D_i\left(\frac{D_i v}{\sqrt{1+|Dv|^2}}\right) = \operatorname{div} f_k + H_k \quad \text{in } U \cap B^n_{(1+\gamma/2)\rho_x}(0),$$

where as in Remark 3.12, for $x' \in \Omega_k$, $y' \in x + P_0$ we identify $(x', u_k(x'))$ with $(y', v(y'))$ using (7).

For this function v, if either $U = \mathbb{R}^n$ or $0 \in \partial U$ we can apply the interior or boundary $C^{1,\alpha}$ Schauder estimates, respectively, in $B^n_{\rho_x}(0) \subset B^n_{(1+\gamma/2)\rho_x}(0)$, which imply that

$$\|v\|_{1,\alpha,B^n_{\rho_x}(0)} \leq C\left(\varepsilon + \rho_x^\alpha [f_k]_{\alpha,B^n_{(1+\gamma/2)\rho_x}(0)} + \rho_x \|H_k\|_{0,B^n_{(1+\gamma/2)\rho_x}(0)}\right), \tag{8}$$

where C is a constant depending only on α, n, γ. In this case, (8) implies a lower bound for ρ_x. To see this, note that the LHS is bounded below by $c^{-1}\varepsilon_0$, where c is the absolute constant in (6), since if it wasn't true then (by (6)) we would have that $\|v_k\|_{1,\alpha,B^n_{(1+\gamma/2)\rho_x}(0)} \leq \varepsilon_0$, which would contradict the definition of ρ_x. Hence, taking ε small enough, the inequality gives a lower bound on ρ_x that is independent of k.

To finish the proof we need to show that we can indeed apply the Schauder estimates, i.e., we need to show that either $0 \in \partial U$ or $U = \mathbb{R}^n$.

If $x \in \Phi_k$ then $0 \in \partial U$. So we can assume that $x \notin \Phi_k$ which implies that $\theta_1 > 4\theta_2$ (i.e., we are in case (ii), as described at the beginning of the proof). We will show that $\Phi_k \cap B^{n+1}_{(1+\gamma)\rho_x}(x) = \emptyset$ and hence $U = \mathbb{R}^n$. Assume that for some $\overline{x} \in \Phi_k$, $|x - \overline{x}| < 2\rho_x$. Then

$$\frac{\rho_{\overline{x}}}{d(\overline{x})} \geq \theta_1 > 4\theta_2 = 4\frac{\rho_x}{d(x)}$$
$$\Rightarrow \quad \rho_{\overline{x}} > 4\left(1 - \frac{|x-\overline{x}|}{d(x)}\right)\rho_x \geq 4(1-2\theta_2) > \frac{7}{2}\rho_x,$$

where we have used (2). Hence $B^{n+1}_{3\rho_x/2}(x) \subset B^{n+1}_{\rho_{\overline{x}}}(\overline{x})$. But this would contradict the definition of ρ_x, and so $\mathrm{dist}(x, \Phi_k) > 2\rho_x$. □

Remark 3.14 Theorem 3.13 implies that for any $\varepsilon > 0$ there exists $\rho_0 = \rho_0(\varepsilon)$ such that for all k

$$\|\nu_k(x) - \nu_k(y)\| \leq \varepsilon|x-y|^\alpha,$$
$$\forall x, y \in \mathrm{graph}\, u_k \cap (B^n_{r/16}(0) \times \mathbb{R}) : |x-y| < \rho_0,$$

where ν_k denotes the downward unit normal of graph u_k.

Recall that $\partial\Omega$, Φ are compact, $C^{1,\alpha}$ embedded submanifolds, which along with Remark 3.12 imply that for any $\varepsilon > 0$ there exists $\widetilde{\rho}_0 = \widetilde{\rho}_0(\varepsilon)$ such that for all k

$$\|\nu_k(x) - \nu_k(y)\| \leq \varepsilon|x-y|^\alpha, \quad \forall x, y \in \mathrm{graph}\, u_k : |x-y| < \widetilde{\rho}_0,$$

where $\widetilde{\rho}_0$ now also depends on

$$\sup_r\{r : \kappa(\partial\Omega, x, r) < 1, \forall x \in \partial\Omega \text{ and } \kappa(\Phi, x, r) < 1, \forall x \in \Phi\}.$$

Let $M = \sup_{\Omega_k} |u_k|$ and recall that M is independent of k (cf. Remark 3.9). Covering $\Omega_k \times [-M, M]$ by balls of radius $\widetilde{\rho}_0$ we conclude that

$$\|\nu_k(x) - \nu_k(y)\| \leq C|x-y|^\alpha, \quad \forall x, y \in \mathrm{graph}\, u_k,$$

where C does not depend on k.

3.3 Gradient Estimates for the Solutions to the Approximating Problems (3.1)

Our goal here is to show a priori $C^{1,\alpha}$ estimates for the solutions $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ of (3.1). That will allow us (cf. Theorem 3.18) to apply the Leray–Schauder theory to prove the existence of such solutions.

We first show that for each k we have boundary gradient estimates for a solution u_k, by using local barriers at each boundary point. In particular we have the following:

Lemma 3.15 *Let $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ be a solution of* (3.1). *Then*

$$\|Du_k\|_{0,\partial\Omega_k} \leq C,$$

where C depends on δ_k, $\|u_k\|_0$, $\|H_{\partial\Omega_k}\|_0$, and $\|\phi_k\|_2$.

Proof For any $x_0' \in \partial\Omega_k$, let $N = B_r^n(x_0') \cap \Omega_k^{\delta_k/2}$ and let $\phi_k^1, \phi_k^2 : N \to \mathbb{R}$ be C^2 functions that satisfy the following:

$$\begin{aligned}
&\phi_k^1(x_0') = \phi_k^2(x_0') = \phi_k(x_0'),\\
&\phi_k^1(x') \geq \phi_k(x') \geq \phi_k^2(x') \quad \forall x' \in \partial N \cap \partial\Omega_k,\\
&\phi_k^1(x') \geq \|u_k\|_0, \qquad \phi_k^2(x') \leq -\|u_k\|_0 \quad \forall x' \in \partial N \setminus \partial\Omega_k
\end{aligned}$$

and

$$\phi_k^1(x') > \phi_k(x') + \operatorname{dist}(x', \partial\Omega_k), \qquad \phi_k^2(x') < \phi_k(x') - \operatorname{dist}(x', \partial\Omega_k) \quad \forall x' \in N$$

so that

$$(x', \phi_k^1(x')) \in U^+, \qquad (x', \phi_k^2(x')) \in U^- \quad \forall x' \in N,$$

where $U^\pm$ are as defined in (3.6), so that

$$\sum_{i=1}^n D_i f_k^i = \begin{cases} \operatorname{div} \eta_k & \text{in } U^+,\\ -\operatorname{div} \eta_k & \text{in } U^-, \end{cases}$$

where η_k is the inward pointing unit normal to $\partial\Omega_k \times \mathbb{R}$, extended in $\Omega_k^{\delta_k} \times \mathbb{R}$, so that $\eta_k(x', x_{n+1}) = \eta_k(\operatorname{proj}_{\partial\Omega_k}(x'), x_{n+1})$.

We look at the following Dirichlet problems:

$$\begin{aligned}
\sum_{i=1}^n D_i \left(\frac{D_i u_k^1}{\sqrt{1 + |D u_k^1|^2}} \right) &= \operatorname{div} \eta_k \quad \text{in } N,\\
u_k^1 &= \phi_k^1 \quad \text{on } \partial N,
\end{aligned} \tag{1}$$

$$\begin{aligned}
\sum_{i=1}^n D_i \left(\frac{D_i u_k^2}{\sqrt{1 + |D u_k^2|^2}} \right) &= -\operatorname{div} \eta_k \quad \text{in } N,\\
u_k^2 &= \phi_k^2 \quad \text{on } \partial N.
\end{aligned} \tag{2}$$

By standard PDE theory [13, Theorem 14.6] we know that there exists a positive function $\psi : \mathbb{R} \to \mathbb{R}_+$, such that for the functions defined by

$$\psi^+(x') = \phi_k^1(x') + \psi(d(x')), \qquad \psi^-(x') = \phi_k^2(x') - \psi(d(x')),$$

where $d(x') = \operatorname{dist}(x', \partial\Omega_k)$, we have that

$$\sum_{i=1}^n D_i \left(\frac{D_i \psi^+}{\sqrt{1 + |D\psi^+|^2}} \right) \leq \operatorname{div} \eta_k \quad \text{on } N$$

and

$$\sum_{i=1}^{n} D_i \left(\frac{D_i \psi^-}{\sqrt{1+|D\psi^-|^2}} \right) \geq -\operatorname{div} \eta_k \quad \text{on } N.$$

So ψ^+ is an upper barrier for a solution $u_k^1 \in C^{1,\alpha}(\overline{\Omega})$ of the problem (1) at the point x_0', ψ^- is a lower barrier for a solution $u_k^2 \in C^{1,\alpha}(\overline{\Omega})$ of the problem (2) at the point x_0', and their gradients satisfy an estimate

$$|D\psi^+(x_0')|, |D\psi^-(x_0')| \leq C(\|H_{\partial\Omega_k}\|_0, \|\phi_k\|_2, \delta_k^{-1}\|u_k\|_0), \tag{3}$$

a constant depending on $\|H_{\partial\Omega_k}\|_0$, $\|\phi_k\|_2$ and $\delta_k^{-1}\|u_k\|_0$.

We claim that ψ^+ and ψ^- are also an upper and, respectively, a lower barrier for u_k at x_0'.

Let

$$N^+ = \{x' \in N : u_k(x') \geq \psi^+(x')\}, \qquad N^- = \{x' \in N : u_k(x') \leq \psi^-(x')\}.$$

Since $\psi^+(x') \geq u_k(x') \geq \psi^-(x')$ for all $x' \in \partial N$, we have that

$$u_k(x') = \psi^+(x') \quad \text{on } \partial N^+ \quad \text{and} \quad u_k(x') = \psi^-(x') \quad \text{on } \partial N^-.$$

Furthermore, since $\psi^+(x') \geq \phi_k^1(x')$, $\psi^-(x') \leq \phi_k^2(x')$, we have that

$$(x', u_k(x')) \in U^+, \quad \forall x' \in N^+ \quad \text{and} \quad (x', u_k(x')) \in U^-, \quad \forall x' \in N^-.$$

Hence

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u_k}{\sqrt{1+|Du_k|^2}} \right) = \sum_{i=1}^{n} D_i f_k^i = \operatorname{div} \eta_k \quad \text{in } N^+$$

and

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u_k}{\sqrt{1+|Du_k|^2}} \right) = \sum_{i=1}^{n} D_i f_k^i = -\operatorname{div} \eta_k \quad \text{in } N^-.$$

Thus, by the comparison principle we have that $u_k(x') = \psi^+(x')$ for all $x' \in N^+$, $u_k(x') = \psi^-(x')$ for all $x' \in N^-$, and so ψ^+ and ψ^- are upper and, respectively, lower barriers for u_k. □

We claim now that the boundary gradient bounds (Lemma 3.15) along with the $C^{1,\alpha}$ estimates that we have shown for the graph of u_k as a submanifold (Theorem 3.13), imply global gradient bounds for the function u_k.

Lemma 3.16 *Let $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ be a solution of (3.1). Then*

$$\sup_{\overline{\Omega}_k} |Du_k| \leq C,$$

where C depends on $\|Du_k\|_{0,\partial\Omega_k}$, $\sup_{\Omega_k} |H_k| + |DH_k|$, $\|\phi_k\|_2$ and $\sup_{\partial\Omega_k} |H_{\partial\Omega_k}| + |DH_{\partial\Omega_k}|$.

Proof Recall that by the construction of the approximating problems, in the domain

$$((\Omega_k \setminus \Omega_k^{\delta_k}) \times \mathbb{R}) \cup \{(x', x_{n+1}) : x' \in \Omega_k^{\delta_k}, |x_{n+1} - \phi_k(x')| > \operatorname{dist}(x', \partial\Omega_k)\}$$

the mean curvature of graph u_k, $H_k + \sum D^i f_k^i$ is smooth and its derivative with respect to the x_{n+1}-variable is non-negative (cf. Remark 3.9). Hence, by standard gradient estimates for a solution to the prescribed mean curvature equation [16], it suffices to show that for any $x_0' \in \partial\Omega_k$

$$|Du_k(x')| \le C, \quad \forall x' \in B^n_{2\delta_k}(x_0') : |u_k(x') - \phi_k(x')| < 2\operatorname{dist}(x', \partial\Omega_k) \tag{1}$$

for some constant C as in the statement of the lemma.

Let $x_0 = (x_0', u_k(x_0')) \in \Phi_k$. Recall that we picked δ_k so that we can extend ϕ_k in $\Omega^{2\delta_k}$ by letting $\phi_k(x') = \phi_k(\operatorname{proj}_{\partial\Omega_k}(x'))$, and by (3.3) we have that for any $x', y' \in B^n_{2\delta_k}(x_0')$

$$|\phi_k(x') - \phi_k(y')| \le C\delta_k \|D\phi_k\|_0,$$

where C is a constant independent of k. We have also picked δ_k so that $\delta_k^{\alpha/2}\|D\phi_k\|_0 \to 0$ (cf. (3.5)), hence for δ_k small enough: If $x' \in B^n_{2\delta_k}(x_0')$ and $|u_k(x') - \phi_k(x')| < 2\operatorname{dist}(x', \partial\Omega_k)$, then $(x', u_k(x')) \in B^{n+1}_{\delta_k^{1/2}}(x_0)$.

By the uniform $C^{1,\alpha}$ estimates for graph u_k (cf. Theorem 3.13, Remark 3.14), there exists ρ_0 such that for all k

$$\|\nu_k(x) - \nu_k(y)\| \le |x - y|^\alpha, \quad \forall x, y \in \operatorname{graph} u_k : |x - y| \le \rho_0, \tag{2}$$

where ν_k is the downward pointing unit normal to graph u_k.

Let $K = 1 + \sup_k \|D\phi_k\|_0$. We will show that (1) holds for all δ_k small enough so that $\delta_k^{1/2} < \rho_0$ and

$$\delta_k^{\alpha/2} < \frac{1}{2\sqrt{1 + 4K^2}} \quad \Leftrightarrow \quad 4\delta_k^\alpha + 16(\delta_k^{\alpha/2}K)^2 < 1. \tag{3}$$

We will consider two different cases:

Case 1: $|Du_k(x_0')| \le 2K$. Then for any $x = (x', u(x')) \in B^{n+1}_{\delta_k^{1/2}}(x_0)$ we have, by (2), (3):

$$\frac{1}{\sqrt{1 + |Du_k(x')|^2}} > \frac{1}{\sqrt{1 + |Du_k(x_0')|^2}} - \delta_k^{\alpha/2} \quad \Rightarrow \quad |Du(x')| \le 4(1 + K)$$

in which case (1) holds.

Case 2: $|Du(x_0')| > 2K$. Then, by (2), (3), for any $x = (x', u(x')) \in B^{n+1}_{\delta_k^{1/2}}(x_0)$ we have that

$$\frac{1}{\sqrt{1 + |Du_k(x')|^2}} < \frac{1}{\sqrt{1 + |Du_k(x_0')|^2}} + \delta_k^{1/2} \quad \Rightarrow \quad |Du(x')| > 1 + K.$$

Hence in this case

$$(B^{n+1}_{\delta_k^{1/2}}(x_0) \cap \operatorname{graph} u_k \cap (B^n_{2\delta_k}(x_0') \times \mathbb{R})) \setminus \Phi_k \subset \{(x', x_{n+1}) : |x_{n+1} - \phi_k(x')| \geq 2d(x)\}$$

and so (1) is trivially true. □

3.4 Existence of a Solution to the Approximating Problems (3.1)

Lemma 3.16 and standard applications of the de Giorgi, Nash, Moser theory give the following $C^{1,\alpha}$ estimates [13, Theorem 13.2].

Corollary 3.17 *Let $u_k \in C^{1,\alpha}(\overline{\Omega}_k)$ be a solution of* (3.1). *Then*

$$\|u_k\|_{1,\alpha,\overline{\Omega}_k} \leq C,$$

where C depends on δ_k, $\sup_{\Omega_k} |H_k| + |DH_k|$, $\|\phi_k\|_2$ and $\sup_{\partial\Omega_k} |H_{\partial\Omega_k}| + |DH_{\partial\Omega_k}|$.

We can now prove the existence of a solution to the problem (3.1).

Theorem 3.18 *The Dirichlet problem* (3.1) *has a solution in $C^{1,\alpha}(\overline{\Omega}_k)$.*

Proof Let $p \geq n/(1-\alpha)$. We define the family of operators

$$T_\sigma : C^{1,\alpha}(\overline{\Omega}_k) \to W^{2,p}(\Omega_k), \quad \sigma \in [0,1]$$

such that for any $v \in C^{1,\alpha}(\overline{\Omega}_k)$, $u = T_\sigma v$ is defined to be the solution of the linear Dirichlet problem

$$\begin{gathered} \sum_{i,j=1}^{n} a_{ij}(x', Dv) D_{ij} u = g_\sigma(x', v) \quad \text{in } \Omega_k, \\ u = \sigma\phi_k \quad \text{on } \partial\Omega_k, \end{gathered} \tag{1}$$

where

$$a_{ij}(x', p) = \frac{\delta_{ij}}{\sqrt{1+|p|^2}} - \frac{p_i p_j}{(\sqrt{1+|p|^2})^3}$$

and

$$g_\sigma(x', x_{n+1}) = \sigma H_k(x', x_{n+1}) + \sum_{i=1}^{n} D_i f^i_{k,\sigma}(x', x_{n+1}),$$

where the vector field $f_{k,\sigma}$ is constructed in the same way as f_k was constructed in the beginning of this section (cf. construction under (3.1)), but with boundary data $\sigma\phi_k$ instead of ϕ_k and δ_k replaced by $s(\sigma)\delta_k$, where $s : [0,1] \to [0,1]$ is a continuous increasing function such that $s(1) = 1$ and $s(0) = 0$. Note that then $[f_{k,\sigma}]_\alpha \leq c[\eta_k]_\alpha$, where η_k is the inward pointing unit normal to $\partial\Omega_k \times \mathbb{R}$.

Claim *For all $\sigma \in [0, 1]$, T_σ is well defined, compact, and continuous.*

$a_{ij} \in C^0(\overline{\Omega}_k)$ and $g_\sigma(x', v(x'))$ is bounded and thus in $L^p(\Omega_k)$. Hence, there exists a solution $T_\sigma v = u \in W^{2,p}(\Omega_k)$ of (1) [13, Theorem 9.18] and by the Calderón–Zygmund inequality, for this solution we have that

$$\|u\|_{W^{2,p}} \leq C(\|\phi_k\|_{W^{2,p}} + \|g_\sigma\|_p). \tag{2}$$

Therefore, T_σ is well defined.

Assume now that $\{v_i\}$ is a sequence of functions in $C^{1,\alpha}(\overline{\Omega}_k)$, such that for some $K > 0$

$$\|v_i\|_{1,\alpha,\overline{\Omega}_k} \leq K, \quad \forall i.$$

Then, by the Arzela–Ascoli theorem, after passing to a subsequence, $v_i \to v \in C^{1,\alpha}(\overline{\Omega}_k)$, where the convergence is with respect to the $C^{1,\alpha'}$ norm, for all $\alpha' < \alpha$.

Let $u_i = T_\sigma v_i$. Then, because of (2) and by the Sobolev embedding theorem [13, Theorem 7.26], after passing to a subsequence, $u_i \to u$ with respect to the $C^{1,\alpha'}$ norm for all $\alpha' < \alpha$ and where $u \in W^{2,p}(\Omega_k)$. Furthermore, u is a solution to the equation

$$a_{ij}(x', Dv)D_{ij}u = g_\sigma(x', v) \quad \text{in } \Omega_k$$

and since $u_i = \sigma\phi_k$ on $\partial\Omega_k$ for all i, we have that $u = \sigma\phi_k$ on $\partial\Omega_k$. Hence

$$T_\sigma v = u$$

and thus T_σ is compact and continuous.

If u_σ is a fixed point of the operator T_σ, then u_σ is a solution of the following Dirichlet problem:

$$\begin{aligned} \sum_{i=1}^n D_i\left(\frac{D_i u_\sigma}{\sqrt{1+|Du_\sigma|^2}}\right) &= \sum_{i=1}^n D_i f^i_{k,\sigma}(x', u_\sigma) + \sigma H_k(x', u_\sigma) \quad \text{in } \Omega_k, \\ u &= \sigma\phi_k \quad \text{on } \partial\Omega_k. \end{aligned} \tag{3}$$

Since 0 is the unique solution for $T_0(0)$, and a fixed point of T_1 corresponds to a solution of the problem (3.1), the Leray–Schauder theory implies that (3.1) is solvable in $C^{1,\alpha}(\overline{\Omega}_k)$ if there exists a constant C such that

$$\|u_\sigma\|_{1,\alpha} \leq C, \quad \forall\sigma \in [0, 1] \tag{4}$$

where for each σ, $u_\sigma \in C^{1,\alpha}(\overline{\Omega}_k)$ is a solution to (3).

Since $\|u_\sigma\|_0$ are uniformly bounded (cf. Remark 3.10), and because of standard PDE estimates [13, Theorem 13.2], for proving (4) it suffices to show that

$$\|Du_\sigma\|_0 \leq C, \quad \forall\sigma \in [0, 1]. \tag{5}$$

Note first that the results of this section concerning the regularity of the approximating graphs of the solutions to (3.1) (Theorems 3.11 and 3.13) are applicable for the family of problems given in (3) for $\sigma \in [0,1]$. Hence Theorem 3.13 implies uniform (independent of σ) $C^{1,\alpha}$ estimates for the graphs of the solutions u_σ as manifolds. Furthermore, for each u_σ, Lemma 3.15 implies a boundary gradient estimate $\|Du_\sigma\|_{0,\partial\Omega_k} \leq C_\sigma$, possibly depending on σ, and consequently we can apply Lemma 3.16 to get a global gradient estimate for each u_σ (depending on C_σ).

We will show that by choosing the function $s(\sigma)$ appropriately, C_σ is in fact independent of σ, which would imply (5).

By the construction of the barriers in Lemma 3.15 (cf. estimate (3) in proof of Lemma 3.15, [13, Chap. 14]) we note that it suffices to show that $s(\sigma)^{-1}\|u_\sigma\|_0$ are uniformly bounded. The following sup estimate shows that this can be achieved as long as we take $s = s(\sigma) = \sigma^{1/2}$.

Let $v = (u_\sigma - l)_+$, where $l = \sup_{\partial\Omega_k}\|\sigma\phi_k\|_0$. For some $\varepsilon_1, \varepsilon_2 \in (0,1)$, also let

$$\Omega_1 = (\Omega_k \setminus \Omega_k^{s\delta_k}) \cap \{x : |Du_\sigma(x)| > \varepsilon_1\}, \qquad \Omega_2 = (\Omega_k \setminus \Omega_k^{s\delta_k}) \cap \{x : |Du_\sigma(x)| \leq \varepsilon_1\}$$

and

$$\Omega_1^s = \Omega_k^{s\delta_k} \cap \{x : |Du_\sigma(x)| > \varepsilon_2\}, \qquad \Omega_2^s = \Omega_k^{s\delta_k} \cap \{x : |Du_\sigma(x)| \leq \varepsilon_2\}.$$

By using v as a test function in the weak form of (3) we have:

$$\begin{aligned}
&\frac{\varepsilon_1}{\sqrt{1+\varepsilon_1^2}}\int_{\Omega_2}|Dv|d\mathcal{H}^n + \frac{1}{\sqrt{1+\varepsilon_1^2}}\int_{\Omega_2}|Dv|^2 d\mathcal{H}^n \\
&\quad + \frac{\varepsilon_2}{\sqrt{1+\varepsilon_2^2}}\int_{\Omega_2^s}|Dv|d\mathcal{H}^n + \frac{1}{\sqrt{1+\varepsilon_2^2}}\int_{\Omega_1^s}|Dv|^2 d\mathcal{H}^n \\
&\leq C\sigma\left(\int_{\Omega_2}|Dv|d\mathcal{H}^n + \int_{\Omega_1}|Dv|^2 d\mathcal{H}^n + 1\right) \\
&\quad + Cs^{1/2}\left(\int_{\Omega_2^s}|Dv|d\mathcal{H}^n + \int_{\Omega_1^s}|Dv|^2 d\mathcal{H}^n + s\right),
\end{aligned}$$

where C is a constant depending only on $\|H_k\|_0, |\Omega_k|$ (cf. Remark 3.10). So for $\varepsilon_1 = \sigma^{1/2}$, $\varepsilon_2 = 4Cs^{1/2}$ (where C is as in the above estimate), and for s, σ small enough, we get the following:

$$\begin{aligned}
&\sigma^{1/2}\int_{\Omega_2}|Dv|d\mathcal{H}^n + s^{1/2}\int_{\Omega_2^s}|Dv|d\mathcal{H}^n \leq C(\sigma + s^{1+1/2}) \\
&\Rightarrow \quad \sigma^{1/2}\int_{\Omega_2\cup\Omega_1}|Dv|d\mathcal{H}^n + s^{1/2}\int_{\Omega_2^s\cup\Omega_1^s}|Dv|d\mathcal{H}^n \\
&\qquad \leq C(\sigma + s^{1+1/2}) + C(\sigma + s^2) \\
&\Rightarrow \quad \int_{\Omega_k}|Dv|d\mathcal{H}^n \leq C(\sigma^{1/2} + \sigma^{-1/2}s^{1+1/2})
\end{aligned}$$

and by the Sobolev inequality

$$\left(\int_{\Omega_k} |v|^{\frac{n}{n-1}} d\mathcal{H}^n\right)^{\frac{n-1}{n}} \leq C\sigma^{-1/2}(\sigma + s^{1+1/2}).$$

This implies a sup estimate (cf. [13, Sect. 10.5]):

$$\|v\|_0 \leq Cs,$$

where C is independent of s, σ, provided that $\sigma^{1/2} \leq s$. □

4 Main Theorem and Applications

As in Sect. 3, we let Ω be a $C^{1,\alpha}$ bounded domain in $\mathbb{R}^n$ and Φ a compact, embedded $C^{1,\alpha}$ submanifold of $\partial\Omega \times \mathbb{R}$. We will use the following notation:

The set $(\partial\Omega \times \mathbb{R}) \setminus \Phi$ is the union of two disjoint open connected components U_Φ, V_Φ, where $U_\Phi \supset \{(x', x_{n+1}) : x' \in \partial\Omega,\ x_{n+1} > R\}$ and $V_\Phi \supset \{(x', x_{n+1}) : x' \in \partial\Omega,\ x_{n+1} < -R\}$ for sufficiently large R. We can think of these components as the parts of the cylinder $\partial\Omega \times \mathbb{R}$ that lie "above" and "below" Φ, respectively. Then for any multiplicity one n-current S with $\operatorname{spt} S \subset \overline{\Omega} \times \mathbb{R}$ and $\partial S = [\![\Phi]\!]$, there exists a multiplicity one $(n+1)$-current, which we will denote by $\widetilde{S}$, such that

$$S = [\![V_\Phi]\!] + \partial\widetilde{S}, \qquad \operatorname{spt}\widetilde{S} \subset \Omega \times \mathbb{R}. \tag{4.1}$$

We now state our Main Theorem:

Theorem 4.2 (Main Theorem) *Given Ω a $C^{1,\alpha}$ bounded domain of $\mathbb{R}^n$, Φ a compact, embedded $C^{1,\alpha}$ submanifold of $\partial\Omega \times \mathbb{R}$, such that for a sequence $\phi_i \in C^{1,\alpha}(\partial\Omega)$, $\operatorname{graph}\phi_i \xrightarrow{C^{1,\alpha}} \Phi$ and $H = H(x', x_{n+1})$ a C^1 function in $\overline{\Omega} \times \mathbb{R}$, which is non-decreasing in the x_{n+1}-variable and such that $\|H\|_0 \leq n\omega_n^{1/n}|\Omega|^{-1/n}$, we let u be a function in* $\mathrm{BV}(\Omega)$ *that minimizes the functional*

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2} dx' + \int_\Omega \int_0^{u(x)} H(x', x_{n+1}) dx' dx_{n+1}$$
$$+ \lim_i \int_{\partial\Omega} |u - \phi_i| dx'.$$

Then for the current $T = [\![\operatorname{graph} u]\!] + Q$, where Q is the multiplicity one n-current such that $\operatorname{spt} Q \subset \partial\Omega \times \mathbb{R}$ and $\partial Q = [\![\Phi]\!] - [\![\operatorname{trace} u]\!]$, $\operatorname{spt} T$ is a $C^{1,\alpha}$ manifold-with-boundary, with boundary given by Φ.

Moreover, this current $T = T(\Omega, \Phi, H)$ locally minimizes the functional

$$\underline{\underline{M}}(T) + \int_{\operatorname{spt}\widetilde{T}} H(x', x_{n+1}) dx' dx_{n+1} \tag{1}$$

among all n-currents with boundary $[\![\Phi]\!]$ and support in $\overline{\Omega} \times \mathbb{R}$.

Proof Without loss of generality we can assume that $0 \in \partial\Omega$. We also let $r > 0$ be such that $(\Omega, \Phi) \in \mathcal{B}_r^\alpha$ (cf. Definition 2.2).

We use the approximating method described in Sect. 3 with the given boundary data (Ω, Φ). Let $u_k \in C^1(\overline{\Omega}_k)$ be the solutions to the approximating problems defined in (3.1):

$$\begin{aligned} \sum_{i=1}^n D_i \left(\frac{D_i u_k}{\sqrt{1+|Du_k|^2}} \right) &= \sum_{i=1}^n D_i f_k^i + H_k \quad \text{in } \Omega_k, \\ u_k &= \phi_k \quad \text{on } \partial\Omega_k, \end{aligned} \tag{2}$$

where $\Omega_k \xrightarrow{C^{1,\alpha}} \Omega$ and $\Phi_k|_{\partial\Omega_k} \xrightarrow{C^{1,\alpha}} \Phi$ with $\Phi_k = \operatorname{graph} \phi_k$. Note that the solutions u_k exist by Theorem 3.18.

Then, by Theorem 3.13, we have uniform $C^{1,\alpha}$ estimates for the graphs of u_k (independent of k). In particular, given ε_0, there exists ρ such that

$$\kappa(\operatorname{graph} u_k, \rho, x_k) < \varepsilon_0, \quad \forall x_k \in \operatorname{graph} u_k \text{ and } \forall k,$$

where κ is as in Definition 2.1.

Assume now that $B_{\rho/2}^{n+1}(x) \subset \mathbb{R}^{n+1}$ is such that $B_{\rho/2}^{n+1}(x) \cap \operatorname{graph} u_k \neq \emptyset$ for infinitely many k. Then for these k, $B_{\rho/2}^{n+1}(x) \cap \operatorname{graph} u_k$ is the graph of a $C^{1,\alpha}$ function, of norm less than ε_0, above some n-dimensional affine space P_k. After passing to a subsequence $P_k \to P$, where P is an n-dimensional affine space and hence

$$\operatorname{graph} u_k \cap B_{\rho/2}^{n+1}(x) = \operatorname{graph} v_k \cap B_{\rho/2}^{n+1}(x), \tag{3}$$

where $v_k \in C^{1,\alpha}(P \cap U_k; P^\perp)$ is such that $\|v_k\|_{1,\alpha} < 2\varepsilon_0$ and where U_k is a $C^{1,\alpha}$ domain of P. Notice that either $U_k = P$ or else $\Phi_k \cap B_\rho(x) \neq \emptyset$, in which case $\partial U_k \cap B_{\rho/2}^{n+1}(x) = \operatorname{proj}_P(\Phi_k \cap B_{\rho/2}^{n+1}(x))$. In the latter case, since $\Phi_k \xrightarrow{C^{1,\alpha}} \Phi$, we have that $U_k \xrightarrow{C^{1,\alpha}} U$, where U is a $C^{1,\alpha}$ domain of P such that $\partial U \cap B_{\rho/2}^{n+1}(x) = \operatorname{proj}_P(\Phi \cap B_{\rho/2}^{n+1}(x))$.

Hence we can apply the Arzela–Ascoli theorem to the sequence $\{v_k\}$, to conclude that, after passing to a subsequence,

$$v_k \to v \tag{4}$$

with respect to the $C^{1,\alpha'}$ norm, for any $\alpha' < \alpha$, where $v \in C^{1,\alpha}(P \cap U; P^\perp)$ is such that $\|v\|_{1,\alpha} \leq 2\varepsilon_0$, and furthermore

$$\operatorname{graph} u_k \cap B_\rho^{n+1}(x) \xrightarrow{C^{1,\alpha'}} \operatorname{graph} v \cap B_\rho^{n+1}(x).$$

Since $\|u_k\|_0$ are uniformly bounded, there exists a compact subset $D \subset \mathbb{R}^{n+1}$ such that $\operatorname{graph} u_k \subset D$ for all k. Covering D with finitely many balls of radius ρ and applying the above discussion in each of them we get that after passing to a subsequence

$$\operatorname{graph} u_k \xrightarrow{C^{1,\alpha'}} M \tag{5}$$

for all $\alpha' < \alpha$ and where M is an embedded $C^{1,\alpha}$ manifold-with-boundary Φ and such that for any $x \in M$, $\kappa(M, \rho, x) < 2\varepsilon_0$.

For the sequence $\{u_k\}$, by standard PDE estimates (as discussed in Remark 3.12) we have uniform $C^{1,\alpha}$ estimates in compact sets of Ω (independent of k) and thus, after passing to a subsequence, $\{u_k\}$ converges with respect to the $C^{1,\alpha'}$ norm on compact sets of Ω, for any $\alpha' < \alpha$, to a function $u \in C^{1,\alpha}(\Omega)$. Hence we have that $M \cap (\Omega \times \mathbb{R}) = \operatorname{graph} u$. Furthermore, since u_k satisfy the equation in (2) and $\Omega_k \xrightarrow{C^{1,\alpha}} \Omega$, u satisfies

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = H(x', u(x')), \quad \text{in } \Omega. \tag{6}$$

Let T be the multiplicity one n-current such that $\operatorname{spt} T = M$. Then $\partial T = [\![\Phi]\!]$ and $T = [\![\operatorname{graph} u]\!] + Q$, where Q is the multiplicity one n-current such that $\operatorname{spt} Q \subset \partial\Omega \times \mathbb{R}$ and $\partial Q = [\![\Phi]\!] - [\![\operatorname{trace} u]\!]$.

u satisfies (6) and therefore T minimizes the functional in (1) of the theorem. To see this, let $W \Subset \overline{\Omega} \times \mathbb{R}$ and let S be a multiplicity one n-current with boundary $[\![\Phi]\!]$ and such that $\operatorname{spt} T = \operatorname{spt} S$ outside W.

Then $T - S = \partial(\widetilde{T} - \widetilde{S})$, where $\widetilde{T}, \widetilde{S}$ are as defined at the beginning of this section (cf. (4.1)) and note that $\widetilde{T} - \widetilde{S}$ has support in $W \cap (\Omega \times \mathbb{R})$. Let

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} e_i \cdot \nu dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1},$$

where ν is the downward pointing unit normal to $\operatorname{graph} u$ extended to be an $\mathbb{R}^{n+1}$-valued function in $\Omega \times \mathbb{R}$ so that it is independent of the x_{n+1}-variable. Then, because of the convergence in (5), $\vec{T}_k \to \vec{T}$ (where $T_k = [\![\operatorname{graph} u_k]\!]$) with respect to the $C^{0,\alpha'}$ norm, for any $\alpha' < \alpha$. Hence $T(\omega) = \underline{\underline{M}}(T)$ and arguing as in Lemma 2.9 we have

$$\begin{aligned} & T(\omega) - S(\omega) = (\widetilde{T} - \widetilde{S})(d\omega) \\ \Rightarrow \quad & \underline{\underline{M}}(T) - \underline{\underline{M}}(S) \le -\int_{\operatorname{spt} \widetilde{T} \setminus \operatorname{spt} \widetilde{S}} H(x', x_{n+1}) dx dx_{n+1} \\ & \qquad + \int_{\operatorname{spt} \widetilde{S} \setminus \operatorname{spt} \widetilde{T}} H(x', x_{n+1}) dx dx_{n+1}. \end{aligned} \tag{7}$$

Hence T minimizes the functional defined in (1) and so u minimizes $\mathcal{F}$. $\square$

Remark 4.3 If S is another n-current with boundary $[\![\Phi]\!]$ and support in $\overline{\Omega} \times \mathbb{R}$ that minimizes the functional defined in (1) of Theorem 4.2, then $\vec{T} = \vec{S}$ almost everywhere. To see this, note that by the argument (7) in the proof of Theorem 4.2 we have that

$$\underline{\underline{M}}(T) = \underline{\underline{M}}(S) = \int_{\operatorname{spt} S} \langle \vec{S}, \vec{T} \rangle d\mu_S.$$

Therefore, $\operatorname{spt} S \cap (\Omega \times \mathbb{R}) = \operatorname{graph} u + c$, for some constant c, and hence $u + c$ (as well as u) minimizes the functional $\mathcal{F}$. Hence if $\operatorname{trace} u \cap \Phi \neq \emptyset$ then $S = T$ and T is the unique current with this minimizing property. In particular, we know that $u(x) = \phi(x)$ for any $x \in \partial\Omega$ such that $H_{\partial\Omega}(x) > |H(x, \phi(x))|$ [21].

Remark 4.4 The regularity of $\operatorname{spt} T$, where $T = (\Omega, \Phi, H)$ (as defined in Theorem 4.2) depends on that of $\partial\Omega$, Φ, and H in a continuous way and the boundary regularity of $\operatorname{spt} T$ is a local result:

By the proof of Theorem 4.2 we see that $\operatorname{spt} T$ can be approximated in the $C^{1,\alpha}$ sense (cf. Definition 2.1) by a sequence of graphs of solutions u_k to the approximating problems (3.1) (as described in Sect. 3). Furthermore, for this sequence we also have that $\|u_k - u\|_{1,\alpha',W} \to 0$ for all $W \Subset \Omega$, $\alpha' < \alpha$ and $\|u_k\|_{1,\alpha,W} \le C$, with the constant C depending only on $\|H\|_{C^1}$, the C^1 norm of H. Hence $\|u\|_{1,\alpha,W}$ and therefore $\sup\{r : \kappa(\operatorname{spt} T, x, r) < \infty\}$ for any $x \in \operatorname{spt} T \cap (\Omega \times \mathbb{R})$ depends only on $\|H\|_{C^1}$.

For points $x = (x', x_{n+1}) \in \operatorname{spt} T \cap (\partial\Omega \times \mathbb{R})$ we have (by Theorem 3.13 and the proof of Theorem 4.2) that $\sup\{r : \kappa(\operatorname{spt} T, x, r) < \infty\}$ depends on $\|H\|_{C^1}$ but also on $\sup\{r : \kappa(\partial\Omega, x', r) < \infty\}$ and $\sup\{r : \kappa(\Phi, x, r) < \infty\ ,\ \forall x = (x', t) \in \Phi\}$.

In the following corollary, which is an immediate consequence of Theorem 4.2, we give some properties for the trace of the function u that minimizes $\mathcal{F}$, as defined in Theorem 4.2.

Corollary 4.5 *Let* (Ω, Φ), H, u, *and* $T = T(\Omega, \Phi, H)$ *be as in the statement of* Theorem 4.2.

(i) *If* $\operatorname{trace} u \cap \Phi \neq 0$, *then* $\forall x = (x', x_{n+1}) \in \operatorname{trace} u \cap \Phi$ *there exists* $\rho > 0$ *such that either* $B^{n+1}_\rho(x) \cap V_\Phi = \emptyset$ *or* $B^{n+1}_\rho(x) \cap U_\Phi = \emptyset$. *The radius* ρ *depends only on* $\sup\{r : \kappa(\partial\Omega, x', r) < \infty\}$, $\sup\{r : \kappa(\Phi, x, r) < \infty,\ \forall x = (x', t) \in \Phi\}$ *and* $\|H\|_{C^1}$ *(because of Remark* 4.4*).*

(ii) *If for some* $x' \in \partial\Omega$, $x = (x', x_{n+1}) \in \operatorname{trace} u \cap U_\Phi$ *then* $\vec{T}(x)$ *coincides with the inward pointing unit normal of* $\partial\Omega$ *at* x', *and if* $x = (x', x_{n+1}) \in \operatorname{trace} u \cap V_\Phi$ *then* $\vec{T}(x)$ *coincides with the outward pointing unit normal of* $\partial\Omega$ *at* x'.

U_Φ, V_Φ *are as defined at the beginning of Sect.* 4.

4.1 Higher Regularity

In this paragraph we show higher regularity for $\operatorname{spt} T$, where T is as in Theorem 4.2, provided that we impose some additional regularity conditions on Φ, $\partial\Omega$.

Lemma 4.6 *If in addition to the hypotheses of Theorem* 4.2 *we assume that* $\partial\Omega$ *and* Φ *are* $W^{2,p}$ *submanifolds, for some* $p > n$, *then for the current* $T = T(\Omega, \Phi, H)$, *as defined in Theorem* 4.2, *we have that* $\operatorname{spt} T$ *is a* $W^{2,p}$ *manifold-with-boundary, with boundary given by* Φ.

Proof Note first that since $\partial\Omega$, Φ are $W^{2,p}$, they are also $C^{1,\alpha}$ for $\alpha = 1 - n/p$ and hence (by Theorem 4.2) $\operatorname{spt} T$ is $C^{1,\alpha}$.

Following the proof of Theorem 4.2 we can construct the vector fields f_k^i in the approximating problems, as defined in (2) in the proof of Theorem 4.2, so that

$$\sup_k \left\| \sum_{i=1}^n D_i f_k^i \right\|_p < \infty \quad \text{and} \quad \sup_k \|\phi_k\|_{W^{2,p}} < \infty.$$

This is possible because $\partial\Omega$, Φ are in $W^{2,p}$ (see also Lemma A.1, Remark A.2).

Let v_k be the local graphical representations of $\operatorname{graph} u_k$, as defined in (3) in the proof of Theorem 4.2: $v_k \in C^{1,\alpha}(P \cap U_k; P^\perp)$, for some n-dimensional affine space P, are such that

$$\operatorname{graph} u_k \cap B^{n+1}_{\rho/2}(x) = \operatorname{graph} v_k \cap B^{n+1}_{\rho/2}(x).$$

Recall (Remark 3.12) that v_k satisfy the following equations

$$\sum_{i=1}^n D_i \left(\frac{D_i v_k}{\sqrt{1 + |Dv_k|^2}} \right) = \sum_{i=1}^n f_k^i + H_k^i \quad \text{in } U_k$$

and if $\partial\Phi_k \cap B^{n+1}_{\rho/2}(x) \neq \emptyset$ we also have that

$$v_k = \phi_k \quad \text{on } \partial U_k \cap B^{n+1}_{\rho/2}(x).$$

Applying the Calderón–Zygmund inequality to the solutions v_k and noticing that

$$\sup_k \left\| \sum_{i=1}^n D_i f_k^i + H_k \right\|_p < \infty, \qquad \sup_k \|\phi_k\|_{W^{2,p}} < \infty$$

we conclude that $\|v_k\|_{W^{2,p}}$ are uniformly bounded. This implies that v is in $W^{2,p}$, where v is as in (4) in the proof of Theorem 4.2, and in particular

$$\|v\|_{W^{2,p}} \leq \sup_k \|v_k\|_{W^{2,p}}.$$

Hence $M = \operatorname{spt} T$ is a $W^{2,p}$ submanifold (see (5) in proof of Theorem 4.2). □

Let $\Omega, \Phi, H, u, T = T(\Omega, \Phi, H)$ be as in Theorem 4.2. Then by standard PDE theory, $\operatorname{spt} T \cap (\Omega \times \mathbb{R})$ is a C^2 manifold. However, near points $x \in \operatorname{trace} u$ the best we can expect is that $\operatorname{spt} T$ is $C^{1,1}$. We will show that this is the case, provided that we impose higher regularity conditions on Ω, Φ. In particular, we will show that around those points $\operatorname{spt} T$ can be expressed as the graph of a function that satisfies a variational inequality, an observation that for the case $H = 0$ and for points $x \in \operatorname{trace} u \setminus \Phi$ was first made in [18]. Thus, using regularity results for such functions [4, 10], we will show that $\operatorname{spt} T$ is a $C^{1,1}$ manifold-with-boundary provided that Ω is a C^2 domain and Φ is a C^3 manifold.

Theorem 4.7 *If in addition to the hypotheses of Theorem 4.2, Ω is a C^2 domain and Φ is a C^3 submanifold of $\partial\Omega \times \mathbb{R}$, then for the current $T = T(\Omega, \Phi, H)$, as defined* *in Theorem 4.2, we have that* $\operatorname{spt} T$ *is a* $C^{1,1}$ *manifold-with-boundary, with boundary given by* Φ.

Proof Let $x = (x', x_{n+1}) \in \operatorname{trace} u$. By Theorem 4.2 there exists $\rho > 0$ such that $B_\rho^{n+1}(x) \cap \operatorname{spt} T$ can be represented as the graph of a $C^{1,\alpha}$ function above $P = T_x(\operatorname{spt} T)$, the tangent space of $\operatorname{spt} T$ at x, i.e.,

$$B_\rho^{n+1}(x) \cap \operatorname{spt} T = \operatorname{graph} v \cap B_\rho^{n+1}(x), \tag{1}$$

where $v \in C^{1,\alpha}((x + (P \cap U)) \cap B_\rho^{n+1}(x); P^\perp)$ is a $C^{1,\alpha}$ function and U a $C^{1,\alpha}$ domain of P.

Assume that $P \neq T_x(\partial\Omega \times \mathbb{R})$, the tangent space of $\partial\Omega \times \mathbb{R}$ at x. Then, because of Corollary 4.5 (ii), we have that $x \in \Phi$ and after replacing ρ with a smaller radius if necessary we have that

$$(B_\rho^{n+1}(x) \cap \operatorname{spt} T) \setminus \Phi \subset \Omega \times \mathbb{R}$$

since $\operatorname{spt} T$ is $C^{1,\alpha}$. Hence v satisfies

$$\sum_{i=1}^{n} D_i \left(\frac{D_i v}{\sqrt{1 + |Dv|^2}} \right) = H \quad \text{in } U$$

(cf. Remark 3.12). Also for ∂U we have that $\operatorname{graph} v|_{x+\partial U} \cap B_\rho^{n+1}(x) = \Phi \cap B_\rho^{n+1}(x)$. Hence standard PDE estimates imply that $v \in C^2((x + (P \cap \overline{U})) \cap B_\rho^{n+1}(x); P^\perp)$, provided that Φ is $C^{2,\alpha}$.

Hence we will assume that $P = T_x(\partial\Omega \times \mathbb{R})$. In this case v doesn't satisfy a prescribed mean curvature equation. However, we will show that due to the minimizing property of T (Theorem 4.2), it satisfies a variational inequality.

Note that we can also represent $(\partial\Omega \times \mathbb{R}) \cap B_\rho^{n+1}(x)$ as the graph of a $C^{1,\alpha}$ function above $P = T_x(\partial\Omega \times \mathbb{R})$, i.e.,

$$(\partial\Omega \times \mathbb{R}) \cap B_\rho^{n+1}(x) = \operatorname{graph} \psi \cap B_\rho^{n+1}(x)$$

for some $\psi \in C^{1,\alpha}((x + P) \cap B_\rho^{n+1}(x); P^\perp)$.

In case $\Phi \cap B_\rho^{n+1}(x) \neq \emptyset$, we define $f : (x + \partial U) \cap B_\rho^{n+1}(x) \to P^\perp$ to be the restriction of v on ∂U, so that

$$\Phi \cap B_\rho^{n+1}(x) = \operatorname{graph} f \cap B_\rho^{n+1}(x).$$

We also define the following set

$$\mathbf{K} = \Big\{ w \in C^{0,1}((x + (P \cap U)) \cap B_\rho^{n+1}(x); P^\perp) : w \geq \psi,$$
$$w = f \text{ on } (x + \partial U) \cap B_\rho^{n+1}(x), w = v \text{ on } \partial B_\rho^{n+1}(x) \cap (x + U) \Big\}.$$

Then because of the minimizing property of T, v minimizes the functional

$$B(v) = \int_{(x+U)\cap B_\rho^{n+1}(x)} \sqrt{1+|Dv|^2}\, d\mathcal{H}^n + \int_{(x+U)\cap B_\rho^{n+1}(x)} \int_{\psi(x')}^{v(x')} H(x', x_{n+1})\, dx_{n+1}\, dx'$$

among all functions in $\mathbf{K}$, where as usual for $x' \in \Omega$, $y' \in U$ we identify $(x', u(x'))$ with $(y', v(y'))$ using (1); here u is such that $\operatorname{spt} T \cap (\Omega \times \mathbb{R}) = \operatorname{graph} u$.

Since $\mathbf{K}$ is a convex set, for any $w \in \mathbf{K}$, the function $\lambda : [0,1] \to B(v + \lambda(w - v))$, attains its minimum when $\lambda = 0$, therefore

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} B(v + \lambda(w - v)) \geq 0$$
$$\Rightarrow \quad \int_{U\cap B_\rho^{n+1}(x)} \sum_{i=1}^{n} \frac{D_i v}{\sqrt{1+|Dv|^2}} D_i(w - v) + H(x', v(x'))(w - v)\, dx' \geq 0$$

and hence v satisfies the variational inequality

$$\langle Av + Hv, w - v\rangle \geq 0, \quad \forall w \in \mathbf{K}, \tag{2}$$

where

$$Av = -\sum_{i=1}^{n} D_i \left(\frac{D_i v}{\sqrt{1+|Dv|^2}} \right) \quad \text{and} \quad Hv = H(x', v(x')).$$

Therefore, by a theorem of Gerhardt [10], if ψ is of class C^2 and f is of class C^3 then v is a $C^{1,1}$ function.

We remark here that in case $(x + U) \cap B_\rho^{n+1}(x) = (x + P) \cap B_\rho^{n+1}(x)$ (i.e., if $x \in \operatorname{trace} u \setminus \Phi$) then v satisfies (2), but with the set $\mathbf{K}$ defined by

$$\mathbf{K} = \{w \in C^{0,1}((x + P) \cap B_\rho^{n+1}(x); P^\perp) : w \geq \psi,\ w = v \text{ on } \partial B_\rho^{n+1}(x)\}.$$

In this case, as was first shown in [18], we can also derive that v is a $C^{1,1}$ function provided that ψ is of class C^2 by a result in [4]. $\square$

Finally, we state a result about the regularity of the trace. It is known [17, 25] that above a C^4 portion of $\partial\Omega$ where $H_{\partial\Omega}(x) < |H(x, \phi(x))|$, the trace of u is a Lipschitz manifold. In the following theorem we show that because of Theorem 4.7, we can apply a result of Caffarelli [6] to show that it is actually C^1.

Theorem 4.8 *In addition to the hypotheses of Theorem 4.2, assume that Ω is a C^4 domain and Φ is a C^3 submanifold of $\partial\Omega \times \mathbb{R}$. Let $S = \{x' \in \partial\Omega : H_{\partial\Omega}(x') < |H(x', x_{n+1})|,\ \forall (x', x_{n+1}) \in \Phi\}$. Then the trace of u above S is C^1, where $u \in$* $\mathrm{BV}(\Omega)$ *is the minimizer of $\mathcal{F}$, as in Theorem 4.2.*

Proof Following the notation in the proof of Theorem 4.7, we introduce the function $\overline{v} = v - \psi$ and define the set

$$F(\overline{v}) = \partial\{y \in U \cap B^{n+1}_{\rho}(x) : \overline{v}(y) > 0\} \cap \partial\{x \in U \cap B^{n+1}_{\rho}(x) : \overline{v}(y) = 0\}.$$

Then $F(\overline{v}) = \operatorname{trace} u \cap B^{n+1}_{\rho}(x)$. By [17, 25], $F(\overline{v})$ is a Lipschitz manifold. If we furthermore know that $\overline{v}$ is a $C^{1,1}$ function, then we can apply the results in [6] to conclude that $F(\overline{v})$ is C^1. This completes the proof, since if $\partial\Omega, \Phi \in C^3$, then by Theorem 4.7 $\overline{v} \in C^{1,1}$. □

4.2 Corollaries, Applications

We show that for $u \in BV(\Omega)$ minimizing the functional $\mathcal{F}$ (as in Theorem 4.2), the trace of u changes monotonely if we change the boundary data monotonely:

Lemma 4.9 *Let* $H, \Omega, \Phi_j, T_j = T(\Omega, \Phi_j, H)$, *for* $j = 1, 2$, *be as in Theorem* 4.2 *and such that* $V_{\Phi_1} \subset V_{\Phi_2}$. *Then*

$$\operatorname{spt} \widetilde{T}_1 \subset \operatorname{spt} \widetilde{T}_2$$

where $V_{\Phi_j}, \widetilde{T}_j$, $j = 1, 2$ *are as defined at the beginning of Sect.* 4, (*cf.* 4.1).

Proof For $j = 1, 2$, we approximate $\operatorname{spt} T_j$ by graphs of solutions u^j_k to the approximating problems defined in (3.1) (as in (2) in the proof of Theorem 4.2). Note that we can take u^j_k, for $j = 1, 2$, to be solutions to the same equation

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u^j_k}{\sqrt{1 + |Du^j_k|^2}} \right) = \sum_{i=1}^{n} D_i f^i_k + H_k \quad \text{in } \Omega_k$$

and their boundary values to satisfy

$$\phi^1_k(x') \leq \phi^2_k(x'), \quad \forall x' \in \partial\Omega_k.$$

The above equation satisfies the comparison principle and hence $u^1_k(x') \leq u^2_k(x')$ for all $x' \in \Omega_k$. The lemma follows by letting $k \to \infty$. □

Finally, we want to show (Theorem 4.11) that in case $n = 2$, for a large class of boundary data Ω, Φ the trace of the minimizer of the functional $\mathcal{F}$ with H equal to a non-negative constant has a jump discontinuity at a point where the mean curvature of $\partial\Omega$ is less than $-H$, and along this discontinuity it attaches to the prescribed boundary in a subset with non-empty interior (relative to the boundary manifold). For this we will need the following lemma.

Lemma 4.10 *Let* Ω, Φ, H *be as in Theorem* 4.2. *Suppose that* $\{\Phi_t\}_{t\in[0,1]}$ *is a continuous* (*as a map from* $[0, 1]$ *into the space of* $C^{1,\alpha}$ *manifolds*) 1*-parameter family*, *where for each* $t \in [0, 1]$, Φ_t *is the limit of* $C^{1,\alpha}$ *graphs above* $\partial\Omega$ *and such that it satisfies the following*:

There exists $x_0' \in \partial\Omega$ *and* $\sigma > 0$ *such that*:

(i) $\Phi_t \cap (B^n_\sigma(x_0') \times \mathbb{R}) = \Phi \cap (B^n_\sigma(x_0') \times \mathbb{R})$ *for all* $t \in [0, 1]$.
(ii) $\{(x', \operatorname{trace} u_0(x')) : x' \in \partial\Omega \cap B^n_\sigma(x_0')\} \subset V_\Phi$.
(iii) $\{(x', \operatorname{trace} u_1(x')) : x' \in \partial\Omega \cap B^n_\sigma(x_0')\} \subset U_\Phi$.

Here for each $t \in [0, 1]$, $u_t \in BV(\Omega)$ *is a minimizing function of the functional* $\mathcal{F}$ *with given data* (Ω, Φ_t, H).

Then for each $x_1 \in \Phi \cap (B^n_\sigma(x_0') \times \mathbb{R})$ *there exists* $t = t(x_1) \in (0, 1)$ *and* $\varepsilon > 0$ *with* $\{(x', \operatorname{trace} u_t(x')) : x' \in \partial\Omega\} \cap B^{n+1}_\varepsilon(x_1) = \Phi \cap B^{n+1}_\varepsilon(x_1)$.

Proof For each t, $(\partial\Omega \times \mathbb{R}) \setminus \operatorname{trace} u_t$ is the union of two disjoint connected components U_t, V_t, where $U_t \supset \{(x', x_{n+1}) : x' \in \partial\Omega, x_{n+1} > R\}$ and $V_t \supset \{(x', x_{n+1}) : x' \in \partial\Omega, x_{n+1} < -R\}$ for sufficiently large R. Given $x_1 \in \Phi \cap (B^n_\sigma(x_0') \times \mathbb{R})$, let

$$t_1 = \sup\{t \in [0, 1] : x_1 \in U_t\}$$

and

$$t_2 = \inf\{t \in [0, 1] : x_1 \in V_t\}.$$

Note that by the assumptions (ii), (iii), and because of Remark 4.3, $t_1, t_2 \in (0, 1)$. Take any sequence $t_i \downarrow t_2$. Then by ii of Corollary 4.5, $\vec{T}(\Omega, \Phi_{t_i})(x_1)$ coincides with the outward pointing normal of $\partial\Omega \times \mathbb{R}$ at x_1 for all t_i in the sequence and therefore, by Remark 4.3, it is also true for t_2. On the other hand, if we take a sequence $t_i \uparrow t_1$, then similarly we get that $\vec{T}(\Omega, \Phi_{t_1})(x_1)$ coincides with the inward pointing normal of $\partial\Omega \times \mathbb{R}$ at x_1.

Hence, again by Remark 4.3, for some t between t_1 and t_2, $\vec{T}(\Omega, \Phi_t)(x_1)$ is not parallel to the normal vector to $\partial\Omega \times \mathbb{R}$ at x_1 and so for this t, Corollary 4.5 implies that $\{(x', \operatorname{trace} u_t(x')) : x' \in \partial\Omega\} \cap B^{n+1}_\varepsilon(x_1) = \Phi \cap B^{n+1}_\varepsilon(x_1)$ for some $\varepsilon > 0$. □

Theorem 4.11 *Let* Ω *be a bounded* C^2 *domain of* $\mathbb{R}^2$, $H \geq 0$ *a given constant, and* $x_0' \in \partial\Omega$ *is such that*

$$H_{\partial\Omega}(x_0') < -H, \tag{1}$$

where $H_{\partial\Omega}(x_0')$ *denotes the mean curvature of* $\partial\Omega$ *with respect to the inward pointing unit normal.*

Then there exists a large class of $C^{1,\alpha}$ *boundary data* Φ, *for which the function* u *that minimizes the functional* $\mathcal{F}$ *with given data* $\{\Omega, \Phi, H\}$ *has trace with a jump discontinuity at* x_0' *along which it attaches to* Φ *in a subset with non-empty interior.*

Proof Let Φ be an embedded $C^{1,\alpha}$ submanifold of $\partial\Omega \times \mathbb{R}$ such that for a sequence $\phi_i \in C^{1,\alpha}(\partial\Omega)$, $\operatorname{graph} \phi_i \xrightarrow{C^{1,\alpha}} \Phi$ and assume that

$$\{x_0'\} \times I \subset \Phi$$

for some interval I.

We will show that for any such Φ there exist $C^{1,\alpha}$ boundary data $\widetilde{\Phi}$ such that $\widetilde{\Phi} \cap (B_\sigma(x_0') \times \mathbb{R}) = \Phi \cap (B_\sigma(x_0') \times \mathbb{R})$ for some $\sigma > 0$ and for which the conclusion of the theorem holds, i.e., the function u that minimizes the functional $\mathcal{F}$ with given data $\{\Omega, \widetilde{\Phi}, H\}$ has trace with a jump discontinuity at x_0' along which it attaches to $\widetilde{\Phi}$ in a subset with non-empty interior. Here and in the rest of the proof $B_r(x')$ will denote the 2-dimensional ball of radius r centered at x'. By Lemma 4.10, it suffices to show that for such Φ we can construct a continuous 1-parameter family of boundary data $\{\Phi_t\}$ satisfying properties (i)–(iii) of Lemma 4.10.

For Φ as above and $\sigma > 0$, which will be determined later, let $\{\Phi_t\}_{-\infty<t<\infty}$ be any monotone, continuous, 1-parameter family of boundary data satisfying the following:

$$\Phi_t \cap (B_\sigma(x_0') \times \mathbb{R}) = \Phi \cap (B_\sigma(x_0') \times \mathbb{R})$$

and outside $B_\sigma(x_0') \times \mathbb{R}$, Φ_t is given by the graph of a $C^{1,\alpha}$ function ϕ_t with

$$\lim_{t\to-\infty} \phi_t(x') = -\infty, \qquad \lim_{t\to+\infty} \phi_t(x') = \infty \quad \forall x' \in \partial\Omega \setminus B_\sigma(x_0').$$

We will show that for $t_0 > 0$ big enough, $\{\Phi_t\}_{\{-t_0 \le t \le t_0\}}$ (after a reparameterization) satisfies properties ii and iii of Lemma 4.10. In particular, we will show that there exists $t_0 > 0$ and $\sigma > 0$, such that for all $t \ge t_0$

$$\{(x', \operatorname{trace} u_t(x')) : x' \in \partial\Omega \cap B_\sigma(x_0')\} \subset U_{\Phi_t} \tag{2}$$

and

$$\{(x', \operatorname{trace} u_{-t}(x')) : x' \in \partial\Omega \cap B_\sigma(x_0')\} \subset V_{\Phi_{-t}}, \tag{3}$$

where u_t is the minimizer of $\mathcal{F}$ with data (Ω, Φ_t, H).

For any $x' \in \partial\Omega$ such that $H_{\partial\Omega}(x') < 0$, there exists a circumference $\mathcal{C}_{x'}$ passing through x', such that a neighborhood of x' in $\partial\Omega$ lies inside $\mathcal{C}_{x'}$. Since $H_{\partial\Omega}(x_0') < 0$ we can choose $\sigma > 0$ such that the following holds: For all $x' \in B_\sigma(x_0')$ there exists a circumference $\mathcal{C}_{x'}$ passing through x' and such that $B_\sigma(x_0') \cap \partial\Omega$ lies inside $\mathcal{C}_{x'}$.

Let $x = (x', x_3) \in \Phi \cap (B_\sigma^n(x_0') \times \mathbb{R})$, $\mathcal{C}_{x'}$ be as described above and let Δ be the region defined as follows: If $H = 0$ then Δ is the region of $\mathbb{R}^2$ outside $\mathcal{C}_{x'}$, and if $H > 0$ then Δ is an annulus with inner boundary $\mathcal{C}_{x'}$ and width H^{-1}. Then (cf. [9]) there exist functions $v^\pm$ defined in $\Delta \cap \Omega$ such that $x \in \operatorname{graph} v^\pm$,

$$\sum_{i=1,2} D_i \left(\frac{D_i v^\pm}{\sqrt{1 + |Dv^\pm|^2}} \right) = \pm H$$

and $Dv^\pm = \pm\infty$ on $\partial(\Delta \cap \Omega) \setminus \partial\Omega$. Taking t small enough we have that

$$v^+ \ge u_t \quad \text{on } \partial\Omega \cap \Delta$$

and taking t big enough we have that

$$v^- \le u_t \quad \text{on } \partial\Omega \cap \Delta.$$

Hence we can apply the comparison principle in [9] to conclude that for t small enough, $v^+ \geq u_t$ on $\Delta \cap \Omega$ and so x lies above the trace of u_t. Similarly for t big enough, $v^- \leq u_t$ on $\Delta \cap \Omega$ and so x lies below the trace of u_t. □

Remark 4.12 By Remark 4.3 and Corollary 4.5, Theorem 4.11 still holds if Ω is a $C^{1,\alpha}$ domain. In this case condition (1) of the theorem should be replaced by the following: There exists some $r > 0$ for which

$$-\int_{\partial\Omega} \nu_{\partial\Omega} \cdot D\zeta < \int_{\partial\Omega} H\zeta$$

for all positive $\zeta \in C^\infty$ with support in $B_r(x_0') \cap \partial\Omega$.

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Appendix A: A Technical Lemma

Lemma A.1 *Let Ω be a $C^{1,\alpha}$ domain of $\mathbb{R}^n$ and $r > 0$ be such that*

$$\kappa(\partial\Omega, r, x) < 1, \quad \forall x \in \partial\Omega$$

with κ as defined in Definition 2.1.

Given $\eta = (\eta_1, \ldots, \eta_n) \in C^{0,\alpha}(\partial\Omega; \mathbb{R}^n)$, there exists a $C^{0,\alpha}$ vector field X on $\Omega^{r/4} = \{x \in \Omega : 0 \leq \operatorname{dist}(x, \partial\Omega) \leq r/4\}$ such that $\operatorname{div} X = 0$ (*weakly*), $X|_{\partial\Omega} = \eta$ *and for any $x \in \partial\Omega$*

$$\|X\|_{0, B_r^n(x)\cap\Omega^{r/4}} + r^\alpha [X]_{\alpha, B_r^n(x)\cap\Omega^{r/4}} \leq C\left(\|\eta\|_{0, B_r^n(x)\cap\partial\Omega} + r^\alpha[\eta]_{\alpha, B_r^n(x)\cap\partial\Omega}\right),$$

where C depends on $\partial\Omega$ and n.

Proof We will construct X around each point using local transformations that flatten the boundary. Therefore, we will first show that the lemma holds in the case of flat boundary:

Claim *Given a $C^{0,\alpha}$ vector field $g = (g_1, \ldots, g_n) : \mathbb{R}^{n-1} \to \mathbb{R}^n$ with compact support, there exists a $C^{0,\alpha}$ vector field $X = (X_1, \ldots, X_n)$ on $\mathbb{R}^{n-1} \times [0, 1]$ such that $X(x', 0) = g(x')$,* $\operatorname{div} X = 0$ (*weakly*) *on $\mathbb{R}^{n-1} \times [0, 1]$ and $\|X\|_{0,\alpha} \leq C\|g\|_{0,\alpha}$, where C is an absolute constant.*

Let ϕ be a non-negative, smooth function with compact support in $B_1^{n-1}(0)$ and such that $\int_{\mathbb{R}^{n-1}} \phi(\xi)d\xi = 1$. For $x = (x', x_n)$ define X by the following formula:

$$X(x', x_n) = \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} g_j(x' - x_n\xi)\,(\phi(\xi) - \operatorname{div}(\phi(\xi)\xi))\,d\xi\, e_j + \left(g_n(x') - \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} g_j(x' - x_n\xi) D_j\phi(\xi)d\xi\right) e_n. \qquad (1)$$

For X it is easy to check that $X(x', 0) = g(x')$ and $\|X\|_{0,\alpha} \leq C\|g\|_{0,\alpha}$. Furthermore, we have that $\operatorname{div} X = 0$. To see this, we only need to check it for smooth g, for which we can integrate by parts to get that

$$X(x', x_n) = \sum_{j=1}^{n-1} D_n f_j(x) e_j + \left(g_n(x') - \sum_{j=1}^{n-1} D_j f_j(x) \right) e_n,$$

where

$$f_j(x', x_n) = x_n \int_{\mathbb{R}^{n-1}} g_j(x' - \xi x_n)\phi(\xi) d\xi.$$

Then

$$\operatorname{div} X = \sum_{j=1}^{n-1} D_j D_n f_j + D_n g_n - \sum_{j=1}^{n-1} D_n D_j f_j = 0$$

since g_n is independent of the x_n-variable. Hence the claim is true.

For the general case, consider a finite cover $\{B^n_{r/4}(x_i)\}$ of $\partial\Omega$, where $x_i \in \partial\Omega$ and such that $B^n_{r/8}(x_i) \cap B^n_{r/8}(x_j) = \emptyset$ for $x_i \neq x_j$. Let ϕ_i be a partition of unity subordinate to this cover. Let $\eta_i(x) = \phi_i(x)\eta(x)$, for all $x \in \partial\Omega$.

For each i, let ψ_i be a diffeomorphism that flattens $\partial\Omega$ in $B^n_r(x_i)$, i.e.,

$$\psi_i(\Omega \cap B^n_r(x_i)) = B^n_r(0) \cap \mathbb{R}^n_+.$$

We can also take ψ_i so that $\psi_i(\Omega \cap B^n_{r/4}(x_i)) = B^n_{r/4}(0) \cap \mathbb{R}^n_+$ and

$$\psi_i(\Omega^{r/4} \cap B_r(x_i)) \subset B^n_r(0) \cap (\mathbb{R}^{n-1} \times [0, r/4]).$$

Let

$$g_i(x) = (D_{\psi_i^{-1}(x)}\psi_i)\eta_i(\psi_i^{-1}(x)), \quad \text{for } x \in B^n_r(0) \cap (\mathbb{R}^{n-1} \times \{0\}), \tag{2}$$

where $(D_{\psi_i^{-1}(x)}\psi_i)$ denotes the matrix of the Jacobian of ψ_i at the point $\psi_i^{-1}(x)$.

Then, since $\psi_i(\Omega \cap B^n_{r/4}(x_i)) = B^n_{r/4}(0) \cap \mathbb{R}^n_+$ and $\eta_i = 0$ outside $B^n_{r/4}(x_i)$, we have that

$$g_i(x) = 0 \quad \text{for } x \in (B^n_r(0) \cap (\mathbb{R}^{n-1} \times \{0\})) \setminus B^n_{r/4}(0). \tag{3}$$

For each g_i let X_i be the vector field given by (1). Then, by (3)

$$X_i = 0 \quad \text{on } \partial B^n_r(0) \cap (\mathbb{R}^{n-1} \times [0, r/4]). \tag{4}$$

Hence

$$X(x) = \sum_i (D_{\psi_i(x)}\psi_i{}^{-1}) X_i(\psi_i(x))$$

is a $C^{0,\alpha}$ vector field on $\Omega^{r/4}$ and is such that for any $x \in \partial\Omega$

$$\begin{aligned}\|X\|_{0,\Omega^{r/4}\cap B_r^n(x)} &\le \sum_i \|D\psi_i^{-1}\|_{0,B_r^n(0)}\|X_i\|_{0,\psi_i(B_r^n(x))}\\ &\le C\sum_i \|g_i\|_{0,\psi_i(B_r^n(x))\cap(\mathbb{R}^{n-1}\times\{0\})} \le C\|\eta\|_{0,\partial\Omega\cap B_r^n(x)}\end{aligned}$$

and

$$\begin{aligned}r^\alpha[X]_{\alpha,\Omega^{r/4}\cap B_r^n(x)} \le &\sum_i r^\alpha[D\psi_i^{-1}]_{\alpha,B_r^n(0)}\|X_i\|_{0,\psi_i(B_r^n(x))}\\ &+\|D\psi_i^{-1}\|_{0,B_r^n(0)}r^\alpha[X_i]_{\alpha,\psi_i(B_r^n(x))}\\ \le &C\sum_i \|g_i\|_{0,\psi_i(B_r^n(x))\cap(\mathbb{R}^{n-1}\times\{0\})} + r^\alpha[g_i]_{\alpha,\psi_i(B_r^n(x))\cap(\mathbb{R}^{n-1}\times\{0\})}\\ \le &C\left(\|\eta\|_{0,B_r^n(x)\cap\partial\Omega} + r^\alpha[\eta]_{\alpha,B_r^n(x)\cap\partial\Omega}\right),\end{aligned}$$

where C depends on n.

Finally, we have that $X|_{\partial\Omega} = \eta$ and $\operatorname{div} X = 0$. To see this, let $x \in \partial\Omega$. Then:

$$\begin{aligned}X(x) &= \sum_i (D_{\psi_i(x)}\psi_i{}^{-1})g_i(\psi_i(x)) = \sum_i (D_{\psi_i(x)}\psi_i{}^{-1})(D_x\psi_i)\eta_i(x)\\ &= \sum_i \phi_i(x)\eta(x) = \eta(x).\end{aligned}$$

Let $\zeta \in C^\infty(\Omega^{r/4})$ and having compact support. We will show that

$$\int_{\Omega^{r/4}} X(x)\cdot D_x\zeta\, d\mathcal{H}^n(x) = 0,$$

$$\begin{aligned}&\int_{\Omega^{r/4}} X(x)\cdot D_x\zeta\, d\mathcal{H}^n(x)\\ &= \sum_i \int_{B_r^n(x_i)} (D_{\psi_i(x)}\psi_i{}^{-1})X_i(\psi_i(x))\cdot D_x\zeta\, d\mathcal{H}^n(x)\\ &= \sum_i \int_{B_r^n(x_i)} (D_{\psi_i(x)}\psi_i{}^{-1})X_i(\psi_i(x))\cdot D_{\psi(x)}(\zeta\circ\psi^{-1})D_x\psi\, d\mathcal{H}^n(x)\\ &= \sum_i \int_{B_r^n(x_i)} X_i(\psi_i(x))\cdot D_{\psi(x)}(\zeta\circ\psi^{-1})d\mathcal{H}^n(x)\\ &= \sum_i \int_{B_r^n(0)} X_i(y)\cdot D_y(\zeta\circ\psi^{-1})d\mathcal{H}^n(y)\end{aligned}$$

which is equal to zero because by construction $\operatorname{div} X_i = 0$ (weakly). $\square$

Remark A.2 We remark that Lemma A.1 is true with higher regularity of the boundary and of the vector field η. In particular, we have the following:

Let Ω be a $C^{k,\alpha}$ domain of $\mathbb{R}^n$ and $\eta \in C^{l,\beta}(\partial\Omega, \mathbb{R}^n)$, where $k, l \geq 0$, $\alpha, \beta \in [0, 1]$, and $l + \beta \leq k + \alpha + 1$. Then there exist a neighborhood V of $\partial\Omega$ in Ω and a $C^{l,\beta}$ vector field X on V such that $\operatorname{div} X = 0$, $X|_{\partial\Omega} = \eta$, and

$$\|X\|_{l,\beta,V} \leq C\|\eta\|_{l,\beta,\partial\Omega},$$

where the neighborhood V and the constant C depend on $\partial\Omega$.

Appendix B: Varifolds, Currents

Let

$$P : \mathbb{R}^{n+1} \to \mathbb{R}^2 \times \{0\}^{n-1}$$

denote the projection onto the (x_1, x_2)-plane in $\mathbb{R}^{n+1}$.
Let

$$\zeta : \mathbb{R}^{n+1} \setminus (\{0\}^2 \times \mathbb{R}^{n-1}) \to \mathbb{R}^2 \times \{0\}^{n-1}$$

be the function defined by

$$\zeta(x) = \zeta(x_1, x_2, \ldots, x_{n+1}) = (-x_2, x_1, 0, \ldots, 0)$$

so that ζ is the projection P followed by a counterclockwise $\pi/2$-rotation.

Lemma B.1 (Allard) *Let $C = (\operatorname{spt} C, \vec{C}, \theta)$ be an n-dimensional cone in R^{n+1}, such that $0 \in \partial C$, $\theta(x) \geq 1$ for all $x \in \operatorname{spt} C \setminus \partial C$, $\partial C = \{0\}^2 \times \mathbb{R}^{n-1}$, and $\|\delta C\|(\mathbb{R}^{n+1} \setminus \partial C) = 0$.*

For each $\phi \in C^\infty((R^2 \times \{0\}^{n-1}) \cap S^n)$ define

$$T(\phi) = \int_{B_1^{n+1}(0)\setminus\partial C} \phi\left(\frac{P(x)}{|P(x)|}\right) \frac{|p_x(\zeta(x))|^2}{|P(x)|^2} \theta(x) d\mathcal{H}^n(x),$$

where ζ is as defined above and p_x denotes the projection onto the tangent space of C at x.

Then

(1) *T is a multiple of $\mathcal{H}^1((R^2 \times \{0\}^{n-1}) \cap S^n)$, i.e.,*

$$T(\phi) = c \int_{(\mathbb{R}^2\times\{0\}^{n-1})\cap S^n} \phi(x) d\mathcal{H}^1(x)$$

for any $\phi \in C^\infty((R^2 \times \{0\}^{n-1}) \cap S^n)$.

(2) *If $T = 0$ then $P(\operatorname{spt} C) \cap S^n$ is finite.*

For the proof of this lemma we refer to [2].

Part (2) of Lemma B.1 directly implies the following:

Corollary B.2 *Let C be an n-dimensional cone in $\mathbb{R}^{n+1}$ such that $0 \in \partial C$, ∂C is an $(n-1)$-dimensional subspace, $\theta(x) \geq 1$ for all $x \in \operatorname{spt} C \setminus \partial C$, and $\operatorname{spt} C \subset H$, where H is a halfspace with $\partial C \subset \partial H$.*

Then

$$C = \sum_{i=1}^{k} P_i,$$

where P_i are n-dimensional halfspaces, with $\partial P_i = \pm \partial C$ and $P_i \subset H$.

Corollary B.3 *If in addition to the hypotheses of Corollary B.2, we assume that C is area minimizing, then we have that either C is an n-dimensional halfspace or*

$$C = mH_1 + lH_2,$$

where H_1, H_2 are the two halfspaces in ∂H defined by ∂C.

Furthermore, $|m - l|$ gives the multiplicity of ∂C.

Proof Without loss of generality we can assume that $H = \overline{R}_+ \times \mathbb{R}^n$. By Corollary B.2 we can write $C = \sum_{i=1}^{k} P_i$, where P_i are now of multiplicity one (so that we could have that $P_i = P_j$),

$$P_i = \pm [\![\{y + tu_i, y \in \partial C, t > 0\}]\!]$$

for some unit vector u_i, normal to ∂C and such that $u_i \cdot e_1 \geq 0$.

Take $j \in \{1, \dots, k\}$ such that $\partial P_j = \partial C$. Then

$$\partial (C - P_j) = 0$$

and for any compact set $W \subset \mathbb{R}^{n+1}$

$$\underline{\underline{M}}_W(C - P_j) = \underline{\underline{M}}_W(C) - \underline{\underline{M}}_W(P_j).$$

We claim that $C - H_j$ is also area minimizing. Assume that it is not true. Then for a compact set $W \subset \mathbb{R}^{n+1}$ there exists a current S with $\operatorname{spt} S \subset W$, $\partial S = 0$, and such that

$$\underline{\underline{M}}_W(C - P_j + S) < \underline{\underline{M}}_W(C - P_j) = \underline{\underline{M}}_W(C) - \underline{\underline{M}}_W(P_j).$$

Then

$$\underline{\underline{M}}_W(C + S) \leq \underline{\underline{M}}_W(C + S - P_j) + \underline{\underline{M}}_W(P_j) < \underline{\underline{M}}_W(C)$$

which contradicts the fact that C is area minimizing.

So $C - P_j$ is area minimizing and hence the associated varifold is stationary. Computing $\|\delta(C - P_j)\|(B_R(0))$ we get that

$$0 = \sum_{\substack{i=1 \\ i \neq j}}^{k} u_i.$$

This is true for any j such that $\partial P_j = \partial C$, hence there can only be one such different u_j.

Similarly, picking a j such that $\partial P_j = -\partial C$ we get that $C + H_j$ will be area minimizing, and computing the first variation of the corresponding varifold we get that

$$0 = u_j + \sum_{i=1}^{k} u_i.$$

Hence, as before, there can only be one such different u_j.

So either $k = 1$, in which case we get that C is an n-dimensional halfspace, or if $k > 1$ we showed that C must be of the form

$$C = kH_1 + lH_2,$$

where H_1, H_2 are the two halfspaces in ∂H defined by ∂C. $\square$

Lemma B.4 *Let C be an n-dimensional integral current such that* $\operatorname{spt} C$ *lies in a closed halfspace H, $\partial C \subset \partial H$ and C minimizes area in H. Then C is area minimizing.*

Proof Suppose not. Then there exists an integer multiplicity current S, with $\partial S = \partial C$, $W = \operatorname{spt}(S - C)$ compact in $\mathbb{R}^{n+1}$, and such that

$$\underline{\underline{M}}_W(C) > \underline{\underline{M}}_W(S). \tag{1}$$

Let f be the reflection along $L = \partial H$:

$$f(x) = L(x) - L^{\perp}(x), \quad x \in \mathbb{R}^{n+1},$$

where, for a subspace P, $P(x)$ denotes the projection of x on P.

Define the function $g : \mathbb{R}^{n+1} \to H$, by:

$$g(x) = \begin{cases} x, & x \in H, \\ f(x), & x \in \mathbb{R}^{n+1} \setminus H. \end{cases}$$

Then for the current $g_\# S$ we have that it has support in H, $\partial(g_\# S) = g_\# \partial S = \partial C$, and it satisfies the estimate

$$\underline{\underline{M}}_V(g_\# S) \le \sup_{g^{-1}(V)} |Dg|^n \underline{\underline{M}}_{g^{-1}(V)}(S), \quad \forall V \Subset H, \tag{2}$$

where, if $S = (\operatorname{spt} S, \theta, \vec{S})$, then:

$$(g_\# S)(\omega) = \int_{\operatorname{spt} S} \langle \omega(g(x)), dg_{x\#}\vec{S}(x)\rangle \theta(x) d\mathcal{H}^n(x).$$

Using now the assumption on C and (2) we have that for any compact subset of $\mathbb{R}^{n+1}$, W:

$$\underline{\underline{M}}_W(C) = \underline{\underline{M}}_{W\cap H}(C) \leq \underline{\underline{M}}_{W\cap H}(g_\# S) \leq \underline{\underline{M}}_W(S)$$

which contradicts (1). □

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