PHYSICAL REVIEW D VOLUME 49, NUMBER 6 15 MARCH 1994

Quasinormal modes of Reissner-Nordström black holes: Phase-integral approach

N. Andersson,* M. E. Araújo,† and B.F. Schutz‡
Department of Physics and Astronomy, University of Wales College of Cardiff, Cardiff CF2 3YB, United Kingdom
(Received 3 February 1993)

This paper presents quasinormal-mode frequencies for Reissner-Nordström black holes calculated using the phase-integral approximation. It is shown that the eigenfrequencies of the first modes (those with small imaginary parts) are accurately determined by the so-called Bohr-Sommerfeld formula. The formula generates frequencies with roughly the same accuracy as obtained in the Schwarzschild case except when the charge is close to its limiting value. For higher modes a generalized formula, recently derived by the present authors, is used. Numerical calculations show that, while it is reasonably accurate for low charges, this formula cannot be trusted for high charges. An intrinsic estimate of the error in the phase-integral calculations is also discussed.

PACS number(s): 97.60.Lf, 02.70.Rw, 04.30.Nk

I. INTRODUCTION

The quasinormal modes of a black hole are solutions to the linearized perturbation equations that correspond to purely outgoing waves at spatial infinity and at the same time agree with the requirement that no waves must escape from within the event horizon of the black hole. The quasinormal modes are expected to dominate the radiation emitted at late times, after, for example, a gravitational collapse, see [1]. The spectrum of quasinormal-mode frequencies is characteristic of the black hole, i.e., depends only on the mass, angular momentum, and charge of the hole. Because of this, a detection of gravitational radiation originating in quasinormal-mode ringing provides one of the best ways to identify a black hole. See [2] for a discussion of the possible extraction of the black hole parameters from a detected gravitational wave signal.

The phase-integral method, recently reviewed by the present authors [1], has been used in the analysis of the quasinormal modes of Schwarzschild black holes. The successful results suggest that we should use the method to study other types of black holes, for which the perturbation equations can be reduced to an ordinary differential equation with a complex effective potential [3]. This is the case for both Reissner-Nordström and Kerr black holes. Despite the unphysical character of the Reissner-Nordström solution, due to the unlikely existence of charged macroscopic bodies in our Universe, we think it provides a useful step towards the much more interesting case presented by the Kerr metric. The main reason is the existence of an internal Cauchy horizon, which is a common characteristic of both metrics.

The phase-integral analysis of the Schwarzschild black hole shows that for the low-lying modes, which are slightly damped, the so-called Bohr-Sommerfeld (BS) formula is a good analytic expression for the determination of the corresponding eigenfrequencies [4, 5]. Indeed, the BS formula has proved to be very accurate for the first few modes, providing results in agreement with different numerical techniques [6–8]. However, in the derivation of the formula only two of the existing transition points are considered. As the overtone index n increases the proximity of a third transition point to the ones already considered leads to a loss of accuracy in the formula. In order to obtain better results for highly damped modes the effect of further transition points *must* be considered. In an attempt to do this we, in a recent paper [9], derived a "generalized" Bohr-Sommerfeld (GBS) formula by considering three well-separated transition points in the complex coordinate plane. Accurate phase-integral formulas for the quasinormal-mode frequencies of the Schwarzschild black hole have also been derived by Andersson and Linnæus using uniform approximations [10].

The purpose of this paper is to use the phase-integral method to analyze the quasinormal modes of a Reissner-Nordström black hole. More specifically, we want to investigate if (and how) the presence of charge, and consequently of the inner horizon, alters the applicability of the BS and GBS formulas.

It should be remembered that the numerical techniques [11, 12] that have been applied to this problem give results that are more accurate than we can hope to achieve with our semianalytic approach. Nevertheless, it is easy to justify the present investigation: Numerical methods may not always be available because they require specific properties of the perturbation equations, e.g., three-term recurrence relations, etc. It is also clear that analytic methods often give a better understanding of the actual physics involved, whereas this kind of information may be hard to extract from a numerical solution. Moreover, as

*Electronic address: andersson@cardiff.ac.uk
†Home address: Departamento de Matemática, Universidade de Brasília, Caixa Postal 04670,70.919-000 Brasília DF, Brazil. Electronic address: marcelo@mat.unb.br
‡Electronic address: schutz@cardiff.ac.uk

we have pointed out in [1], the error analysis in the phase-integral method needs further consideration. In order to obtain a better notion of how errors enter and propagate in the approximation we want to apply the method to various problems, and compare the results with available numerical ones.

II. THE PERTURBATION EQUATIONS

The equations governing a small perturbation of a Reissner-Nordström black hole can be reduced to a single ordinary differential equation by separating out the dependence on time and angles. Assuming that the time dependence is $\exp(-i\omega t)$, and fixing the spherical-harmonic index ℓ, we obtain [3]

$$\frac{d^2}{dr_*^2} Z_j^{(\pm)} + \left[\omega^2 - V_j^{(\pm)}\right] Z_j^{(\pm)} = 0 , \tag{1}$$

where $j = 1, 2$ and the tortoise coordinate r_* is defined by

$$\frac{dr}{dr_*} = \frac{\Delta}{r^2} = \tilde{\Delta} \tag{2}$$

and

$$\Delta = r^2 - 2r + e^2 \tag{3}$$

(the simplifying notation $\tilde{\Delta}$ will be used below). The outer and inner horizons of the Reissner-Nordström black hole, r_+ and r_-, are then solutions to $\Delta = 0$.

The superscripts + and − label the independent axial and polar perturbations, respectively. The terminology is due to the fact that the equations describing axial perturbations contain only quantities, or products of quantities, which change sign under the transformation of the angular coordinate φ to $-\varphi$, whereas in the equations for polar perturbations all quantities are invariant under this transformation. It is worth pointing out that the expressions *odd* and *even parity* have often been used in the literature instead of *axial* and *polar*, respectively.

The effective potentials $V_j^{\pm}$ are given by

$$V_j^{(-)} = \frac{\Delta}{r^5}\left[\ell(\ell+1)r - \beta_j + \frac{4e^2}{r}\right] , \tag{4}$$

$$V_j^{(+)} = V_j^{(-)} + 2\beta_j \frac{d}{dr_*}\left[\frac{\Delta}{r^3\left[(\ell-1)(\ell+2)r + \beta_j\right]}\right] , \tag{5}$$

where

$$\beta_{1,2} = 3 \mp \left[9 + 4(\ell-1)(\ell+2)e^2\right]^{1/2} , \tag{6}$$

and e is the charge of the black hole. We use units $c = G = M = 1$, where M is the mass of the black hole. For further details on the theory of perturbations of Reissner-Nordström black holes we refer the reader to the book of Chandrasekhar [3].

In the uncharged case, i.e., when $e = 0$, the two potentials $V_2^{(+)}$ and $V_2^{(-)}$ reduce to the Zerilli [13] and the Regge-Wheeler [14] potentials (describing gravitational perturbations of a Schwarzschild black hole), respectively. The two potentials $V_1^{(+)}$ and $V_1^{(-)}$ both reduce to the effective potential that describes the evolution of a perturbation of the electromagnetic field in the Schwarzschild space-time [3]. An important characteristic of a charged environment is that a variation in the electromagnetic field generates a small variation in the gravitational field and vice versa. This means that when the charge of the black hole is nonzero, an emitted electromagnetic wave will give rise to a gravitational wave with the same frequency. Hence, a quasinormal mode of a Reissner-Nordström black hole will in general correspond to the emission of *both* electromagnetic and gravitational radiation.

Chandrasekhar [15] has shown that the solutions $Z_j^{(+)}$ and $Z_j^{(-)}$, corresponding to the different potentials (4) and (5), can be obtained from each other. Hence, the two equations contain the same information and give rise to identical sets of quasinormal-mode frequencies. One may, therefore, restrict an investigation to one of the corresponding differential equations (1). However, a comparison of approximate results, which should in theory be equivalent, obtained from the formally different equations for axial and polar perturbations provides an intrinsic error estimate that may be useful. Moreover, the investigation of Fröman *et al.* [4] showed that the BS formula was actually more accurate when applied to the equation governing polar perturbations than when used in the axial case. It seems as if these points have not been fully appreciated in the previous investigations of the problem. Therefore, we shall in this paper perform calculations for both potentials.

III. PHASE-INTEGRAL FORMULAS

We have recently written a comprehensive review of the phase-integral method as applied to a general investigation of black hole normal modes [1]. Hence, we will restrict ourselves in this section to the presentation of specific formulas that are relevant to an analysis of the Reissner-Nordström problem.

Introducing a new dependent variable Ψ according to

$$Z_j^{(\pm)} = \tilde{\Delta}^{-1/2}\Psi_j^{(\pm)} , \tag{7}$$

the differential equation (1) can be written as a Schrödinger-like differential equation

$$\frac{d^2}{dr^2}\Psi_j^{(\pm)} + R_j^{(\pm)}\Psi_j^{(\pm)} = 0 . \tag{8}$$

The analytic function $R_j^{(\pm)}$ is explicitly given by

$$R_j^{(\pm)} = \frac{1}{\tilde{\Delta}^2}\left[\omega^2 - V_j^{(\pm)} + \frac{1}{4}\left(\frac{d\tilde{\Delta}}{dr}\right)^2 - \frac{\tilde{\Delta}}{2}\frac{d^2\tilde{\Delta}}{dr^2}\right] . \tag{9}$$

In the phase-integral method the general solution to (8) is given by a linear combination of the two functions

$$f_{1,2}(r) = q^{-1/2}(r)\exp\left[\pm i\int q(r)dr\right] , \tag{10}$$

with the function $q(r)$ given by an asymptotic series. The first term in this expansion is a somewhat arbitrary function $Q(r)$, and the contributions of the higher order approximations are expressed in terms of this function [1]. It is convenient to choose

$$Q^2(r) = R_j^{(\pm)} - \frac{1}{4(r-r_+)^2} , \tag{11}$$

in order to ensure that the approximation remains accurate in the vicinity of the outer horizon r_+ where a boundary condition has to be imposed.

In a phase-integral analysis of the quasinormal-mode problem the pattern of Stokes and anti-Stokes lines play an important role. These are curves along which Qdr is purely imaginary or real, respectively. From each transition point, a first order zero of $Q^2(r)$, emanates three curves of each type.

It is natural to assume that an analysis of the quasinormal modes for Schwarzschild black holes will remain valid for small values of the charge e. We have studied how the pattern of Stokes and anti-Stokes lines changes [for each of the potentials (4) and (5)] as the charge is increased. This investigation suggests that the derivations of the BS and the GBS formulas for the Schwarzschild case will remain valid in the Reissner-Nordström case (at least for reasonably low charges). An illustration of this behavior is provided by a comparison of Figs. 1 and 2 with Fig. 2 in [1] and Fig. 1 in [9], respectively. It can be clearly seen that the general pattern of Stokes and anti-Stokes lines are very similar. This is a characteristic feature for all values of the charge e up to a certain limit.

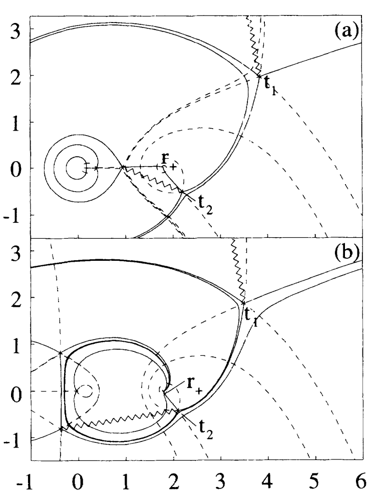

FIG. 1. Complex r plane showing the pattern of Stokes (dashed) and anti-Stokes (continuous) lines emerging from the transition points t_i under consideration for $\ell = 2$, $n = 0$, $e = 0.6$, and (a) axial and (b) polar perturbations, respectively. Note the threefold symmetry around each transition point, e.g., from each of them emerges three Stokes and three anti-Stokes lines. The cuts are represented by the zigzag lines and r_+ denotes the event horizon.

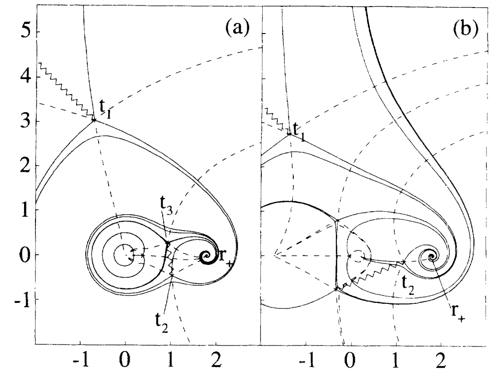

FIG. 2. Complex r plane showing the pattern of Stokes (dashed) and anti-Stokes (continuous) lines emerging from the transition points t_i under consideration for $\ell = 2$, $n = 4$, $e = 0.6$, and (a) axial and (b) polar perturbations, respectively. Note the threefold symmetry around each transition point, e.g., from each of them emerges three Stokes and three anti-Stokes lines. The cuts are represented by the zigzag lines and r_+ denotes the event horizon.

It should be pointed out that the Stokes and anti-Stokes lines in our figures were determined using frequencies obtained numerically by Andersson [12]. (The reader interested in a larger number of tables than those printed in the latter is referred to Ref. [16]. Copies of these tables can be obtained from the author.) For cases where such numerical values are not already available one may calculate the frequencies using the phase-integral formula that is believed to be appropriate (by comparing with the Schwarzschild case). The pattern of Stokes and anti-Stokes lines for the obtained values of ω should remain similar to that for which the formula was originally derived if the result is to be trusted. It turns out that the frequency need not be very accurate to make this kind of analysis effective; the pattern of Stokes and anti-Stokes lines hardly changes if the frequency is changed as much as one part in, say, 10^{-2}.

The Bohr-Sommerfeld formula is given by [1]

$$\gamma_{21} = \frac{1}{2}\oint_C q(r)dr = \left(n+\frac{1}{2}\right)\pi . \tag{12}$$

The contour C encircles the two transition points t_1 and t_2 (in such a way that Re γ_{21} is positive), see Fig. 1. From a comparison with the Schwarzschild case one would expect this formula to give reliable results for the first few modes. On the other hand, the GBS formula is valid for highly damped modes of axial perturbations of the Schwarzschild black hole. It was derived by considering three transition points [9], and can be written

$$\gamma_{21} = \left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln\left[1+\exp\left(2i\gamma_{32}\right)\right] , \tag{13}$$

with obvious notation. One would expect that this formula can be used with confidence also for axial perturbations of the Reissner-Nordström black hole.

IV. NUMERICAL RESULTS

Quasinormal-mode frequencies for the Reissner-Nordström black hole have been presented in other papers. Using the numerical integration method that was developed by Chandrasekhar and Detweiler [17] for the Schwarzschild black hole, Gunter did calculations for the fundamental mode and several different values of ℓ [18, 19]. The WKB method, originally devised by Schutz and Will [20], was used by Kokkotas and Schutz [21]. They also did further calculations using the numerical

TABLE I. Phase-integral results for Reissner-Nordström black holes: A sample of quasinormal-mode frequencies for $\psi_1^{\pm}$ and $\ell = 2$ calculated using the Bohr-Sommerfeld formula (12). The optimal order of approximation and the estimated error are determined by the simple rule of thumb. As an estimate of the actual error the absolute difference between the phase-integral and the numerical results of Andersson [12] has been used. The first entry for each value of e corresponds to axial perturbations, while the second is for polar perturbations.

n	e	Re ω	Im ω	Optimal order	Estimated error bound	Actual error
0	0.30	0.46986525	-0.09582592	7	1.4×10^{-6}	5.4×10^{-7}
		0.46986477	-0.09582610	5	9.1×10^{-7}	3.6×10^{-7}
	0.60	0.5120117	-0.0980201	7	6.2×10^{-6}	3.6×10^{-6}
		0.51201107	-0.09801677	7	2.7×10^{-7}	2.1×10^{-7}
	0.90	0.6193978	-0.0975919	5	1.7×10^{-5}	9.0×10^{-6}
		0.61939782	-0.09758338	7	9.3×10^{-7}	5.5×10^{-7}
	0.99	0.692745	-0.088634	7	1.9×10^{-5}	1.1×10^{-5}
		0.69275213	-0.08864263	7	6.9×10^{-7}	4.4×10^{-7}
1	0.30	0.4494287	-0.2929916	5	5.0×10^{-6}	3.2×10^{-6}
		0.44943008	-0.29299418	5	2.4×10^{-6}	6.5×10^{-7}
	0.60	0.493748	-0.298907	7	1.8×10^{-5}	1.0×10^{-5}
		0.49375709	-0.29891065	7	4.8×10^{-7}	3.6×10^{-7}
	0.90	0.606589	-0.295596	5	8.6×10^{-5}	3.9×10^{-5}
		0.6066245	-0.2956092	7	2.3×10^{-6}	1.4×10^{-6}
	0.99	0.678611	-0.267543	5	1.2×10^{-4}	6.2×10^{-5}
		0.6786583	-0.2675039	7	1.4×10^{-6}	1.0×10^{-6}
2	0.30	0.4150791	-0.5048038	5	1.3×10^{-5}	5.8×10^{-6}
		0.4150797	-0.5048101	5	6.7×10^{-6}	1.2×10^{-6}
	0.60	0.462970	-0.512629	5	5.7×10^{-5}	3.8×10^{-5}
		0.46293407	-0.51261456	7	9.9×10^{-7}	6.7×10^{-7}
	0.90	0.58434	-0.50128	3	3.2×10^{-4}	1.8×10^{-4}
		0.5842145	-0.5011578	7	5.0×10^{-6}	3.0×10^{-6}
	0.99	0.65120	-0.45118	3	4.5×10^{-4}	2.5×10^{-4}
		0.6509526	-0.4511785	7	2.2×10^{-6}	1.9×10^{-6}
3	0.30	0.377420	-0.733780	5	3.1×10^{-5}	1.0×10^{-5}
		0.3774175	-0.7337936	5	1.9×10^{-5}	3.6×10^{-6}
	0.60	0.42863	-0.74159	5	1.6×10^{-4}	1.0×10^{-4}
		0.4285909	-0.7414895	7	2.3×10^{-6}	1.3×10^{-6}
	0.90	0.55712	-0.71654	3	1.0×10^{-3}	4.9×10^{-4}
		0.5570577	-0.7160636	5	1.3×10^{-5}	7.3×10^{-6}
	0.99	0.61121	-0.64364	3	1.4×10^{-3}	5.9×10^{-4}
		0.6108070	-0.6432101	7	2.2×10^{-6}	3.6×10^{-6}
4	0.30	0.344185	-0.975302	5	6.4×10^{-5}	1.8×10^{-5}
		0.344175	-0.975329	5	5.1×10^{-5}	1.1×10^{-5}
	0.60	0.39728	-0.98197	3	3.7×10^{-4}	3.4×10^{-4}
		0.3976162	-0.9819481	7	5.5×10^{-6}	2.5×10^{-6}
	0.90	0.5301	-0.9377	1	2.8×10^{-3}	1.6×10^{-3}
		0.529337	-0.939085	7	1.8×10^{-5}	1.1×10^{-5}
	0.99	0.56081	-0.8489	3	4.5×10^{-3}	1.5×10^{-3}
		0.5608431	-0.8474286	7	8.0×10^{-6}	9.6×10^{-6}

technique applied by Gunter. Approximate analytic formulas for the frequencies were obtained by Ferrari and Mashhoon [22], by employing an exact relation between the quasinormal modes and bound states of the inverted black-hole potential. Finally, Leaver used recurrence relations together with his continued fraction method to determine very accurate results [11]. The accuracy of Leaver's results has been discussed elsewhere by one of the present authors [12].

We have done calculations using the two formulas (12) and (13) for $\ell = 2$. Calculations were performed in the first five orders of approximation (labeled by odd numbers 1 to 9) for each mode. A sample of results for the BS formula are given in Tables I and II. The GBS results are in Table III.

In the tables are listed the result for each mode in the *optimal order of approximation.* This, in a sense vague, concept is here defined by the rule of thumb that we have used elsewhere [9]. That is, we consider the numerical results obtained for each mode and increasing order of approximation as an asymptotic series. Theoretically, the terms in this series, i.e., the contributions of consec-

TABLE II. Phase-integral results for Reissner-Nordström black holes: A sample of quasinormal-mode frequencies for $\psi_2^{\pm}$ and $\ell = 2$ calculated using the Bohr-Sommerfeld formula (12). The optimal order of approximation and the estimated error are determined by the simple rule of thumb. As an estimate of the actual error the absolute difference between the phase-integral and the numerical results of Andersson [12] has been used. The first entry for each value of e corresponds to axial perturbations, while the second is for polar perturbations. For axial perturbations and high charges the BS formula cannot be used. These values are denoted by an asterisk.

n	e	Re ω	Im ω	Optimal order	Estimated error bound	Actual error
0	0.30	0.37594	-0.08936	3	3.5×10^{-4}	2.9×10^{-4}
		0.376180	-0.089182	5	3.2×10^{-5}	3.5×10^{-5}
	0.60	0.38600	-0.08992	3	2.8×10^{-4}	2.4×10^{-4}
		0.3862153	-0.0898217	7	3.7×10^{-5}	8.3×10^{-6}
	0.90	0.41339	-0.08803	3	5.4×10^{-4}	3.6×10^{-4}
		0.41353	-0.08824	5	1.6×10^{-4}	1.0×10^{-4}
	0.99	0.4288	-0.0834	3	1.7×10^{-3}	1.0×10^{-3}
		0.429271	-0.084176	5	1.9×10^{-4}	9.4×10^{-5}
1	0.30	0.3479	-0.2749	3	2.2×10^{-3}	1.5×10^{-3}
		0.349377	-0.274637	5	9.9×10^{-5}	3.2×10^{-5}
	0.60	0.3590	-0.2763	3	1.8×10^{-3}	1.2×10^{-3}
		0.360247	-0.276200	5	1.1×10^{-4}	9.2×10^{-5}
	0.90	0.39034	-0.27022	1	2.7×10^{-3}	2.4×10^{-4}
		0.39067	-0.27010	5	4.0×10^{-4}	2.3×10^{-4}
	0.99	0.3971	-0.2563	1	1.1×10^{-2}	6.5×10^{-3}
		0.40341	-0.25742	3	4.7×10^{-4}	4.2×10^{-4}
2	0.30	0.3039	-0.4865	1	8.4×10^{-3}	7.2×10^{-3}
		0.30335	-0.47913	3	2.6×10^{-4}	5.2×10^{-4}
	0.60	0.3162	-0.4872	1	7.7×10^{-3}	6.3×10^{-3}
		0.31511	-0.48071	3	4.4×10^{-4}	7.9×10^{-4}
	0.90*	0.3484	-0.4653	3	7.8×10^{-4}	1.2×10^{-3}
	0.99*	0.35312	-0.44350	3	9.1×10^{-4}	7.9×10^{-4}
3	0.30	0.2542	-0.7214	1	1.1×10^{-2}	1.5×10^{-2}
		0.2537	-0.7050	3	3.3×10^{-4}	1.4×10^{-3}
	0.60	0.2672	-0.7206	1	1.1×10^{-2}	1.4×10^{-2}
		0.2661	-0.7052	3	6.2×10^{-4}	2.0×10^{-3}
	0.90*	0.2979	-0.6746	3	1.0×10^{-3}	2.7×10^{-3}
	0.99*	0.2880	-0.6515	3	2.3×10^{-3}	1.7×10^{-2}
4	0.30	0.206	-0.975	3	5.8×10^{-3}	2.7×10^{-2}
		0.2107	-0.9446	3	1.3×10^{-3}	3.6×10^{-3}
	0.60	0.218	-0.973	3	5.1×10^{-3}	2.5×10^{-2}
		0.2233	-0.9426	3	1.8×10^{-3}	5.2×10^{-3}
	0.90*	0.2459	-0.8946	3	2.9×10^{-3}	5.9×10^{-3}
	0.99*	0.225	-0.882	3	3.0×10^{-3}	3.5×10^{-2}

TABLE III. Phase-integral results for Reissner-Nordström black holes: A sample of quasinormal-mode frequencies for ψ_2^- and $\ell = 2$ calculated using the generalized Bohr-Sommerfeld formula (13). The optimal order of approximation and the estimated error are determined by the simple rule of thumb. As an estimate of the actual error the absolute difference between the phase-integral and the numerical results of Andersson [12] has been used.

n	e	Re ω	Im ω	Optimal order	Estimated error bound	Actual error
4	0.30	0.2074	-0.9468	3	6.4×10^{-3}	2.9×10^{-3}
	0.50	0.2139	-0.9476	3	6.1×10^{-3}	2.8×10^{-3}
	0.60	0.2195	-0.9468	3	6.2×10^{-3}	3.0×10^{-3}
	0.65	0.2220	-0.9462	3	9.4×10^{-3}	
	0.70	0.241	-0.947	1	2.3×10^{-2}	1.1×10^{-2}
5	0.30	0.1684	-1.1961	3	4.1×10^{-3}	2.8×10^{-3}
	0.50	0.1736	-1.1970	3	3.9×10^{-3}	2.8×10^{-3}
	0.60	0.1776	-1.1961	3	3.0×10^{-3}	3.9×10^{-3}
	0.65	0.178	-1.196	3	1.3×10^{-2}	
	0.70	0.168	-1.197	3	3.1×10^{-2}	2.2×10^{-2}
6	0.30	0.1302	-1.4483	3	3.5×10^{-3}	
	0.50	0.1318	-1.4501	3	3.3×10^{-3}	
	0.60	0.1309	-1.4508	3	4.5×10^{-3}	
	0.65	0.162	-1.446	1	4.0×10^{-2}	
	0.70	0.156	-1.445	1	7.3×10^{-2}	
7	0.30	0.0868	-1.7031	5	3.1×10^{-3}	
	0.50	0.0651	-1.7106	3	5.9×10^{-3}	
	0.60	0.073	-1.777	3	1.9×10^{-2}	
	0.65	0.108	-1.782	3	1.9×10^{-2}	
	0.70	0.1364	-1.7814	3	9.7×10^{-4}	

utively higher orders, should become smaller and smaller until, at a certain order, they start to increase. The estimated error bound in the phase-integral calculation is given by the smallest contribution to the series, and the optimal order precedes that giving rise to the smallest contribution. As an example this means that, although we performed calculations in the ninth order of approximation, this order can *never* be optimal. If the ninth order contribution is the smallest in the series the seventh order will be considered as optimal, and the error should supposedly be smaller than the ninth order contribution. It turns out, cf. the tables, that this rule of thumb is extremely useful. Whenever the phase-integral formula used is appropriate the rule of thumb gives a good estimate of the actual error. In fact, in the few cases where the rule of thumb does not pick out the most accurate result (as compared to reliable numerical results) the chosen order of approximation has an error of the same order of magnitude as the most accurate one. Of course, the smallness of this error estimate is not directly related to the accuracy of the phase-integral result, see the discussion in [9]. Nevertheless, an increase in the estimated error together with a decreasing optimal order is a useful indication that the formula used may be losing its validity.

In Tables I and II the phase-integral results are compared with the numerical results of Andersson [12]. These numerical results are considered as reliable, especially since they agree very well with those of Leaver [11]. Hence, we quote as the "actual error" the absolute difference between the phase-integral results and these numerical results.

From our tables it is clear that the BS formula gives very accurate results for the two potentials $V_1^{\pm}$. As already mentioned above, these potentials describe the evolution of an electromagnetic test field in the Schwarzschild background in the uncharged case. The estimated error provided by the rule of thumb is in good agreement with (often slightly larger than) the actual error for all the results in Table I. Unfortunately, the situation is not as favourable for the potentials $V_2^{\pm}$, i.e., those corresponding to gravitational perturbations in the uncharged case. As expected from the investigation of the Schwarzschild black hole, see [4, 9], the BS formula is reasonably accurate for the first three modes, but cannot be trusted for higher overtones.

Interestingly, the BS formula is more accurate when applied to the potential for polar perturbations than when it is used together with the potential describing the axial perturbations, see also [4]. The explanation for this is probably that a third transition point (the one considered in the derivation of the GBS formula) becomes important for high modes of the axial perturbations. This is not the case for the polar perturbations, see Fig. 2(b).

Although the GBS formula generates results accurate to a few parts in 10^{-3} for low charges, it is not reliable for $e > 0.6$, see Table III. The reason for this is that the pattern of Stokes and anti-Stokes lines for higher charges does not remain similar to that used in the derivation of the GBS formula.

V. CONCLUSIONS

We have used two different phase-integral formulas to determine the quasinormal-mode frequencies of a Reissner-Nordström black hole. The first of the formulas, the so-called Bohr-Sommerfeld formula, provides results in perfect agreement with other investigations of the lowest-lying modes and all but extreme values of the charge. An error estimate obtained from a simple rule of thumb is proved to be surprisingly reliable, providing a useful measure of the error in each phase-integral result. On the other hand, a generalized formula that gave good results for higher overtones in the Schwarzschild case can only be used with confidence for black holes with moderate charges.

Finally, it should be remembered that an astrophysical black hole is not really expected to have a significant charge. Nevertheless, it seems probable (or even likely) that an analysis that gives accurate results for the Reissner-Nordström black hole may also prove useful in the Kerr case, the reason being that a common characteristic of the two black-hole solutions is the existence of an inner horizon. From this analogy it seems probable that the Bohr-Sommerfeld formula will, in some sense, be useful also for Kerr black holes.

ACKNOWLEDGMENTS

M.E.A. wishes to thank the Royal Society for financial support in visiting the Cardiff Relativity group, the Conselho Nacional de Desenvolvimento Científico e Tecnológico—CNPq for a research grant, and the Department of Physics of The University of Wales College of Cardiff for its hospitality. N.A. acknowledges an SERC research grant.

---

[1] N. Andersson, M. E. Araújo, and B. F. Schutz, Class. Quantum Grav. **10**, 735 (1993).
[2] F. Echeverria, Phys. Rev. D **40**, 3194 (1989).
[3] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[4] N. Fröman, P. O. Fröman, N. Andersson, and A. Hökback, Phys. Rev. D **45**, 2609 (1992).
[5] M. E. Araújo, D. Nicholson, and B. F. Schutz, Class. Quantum Grav. **10**, 1127 (1993).
[6] N. Andersson, Proc. R. Soc. London **A439** 47 (1992).
[7] E. W. Leaver, Proc. R. Soc. London **A402**, 285 (1985).
[8] H.-P. Nollert and B. G. Schmidt, Phys. Rev. D **45**, 2617 (1992).
[9] N. Andersson, M. E. Araújo, and B. F. Schutz, Class. Quantum Grav. **10**, 757 (1993).
[10] N. Andersson and S. Linnæus, Phys. Rev. D **46**, 4178 (1992).
[11] E. W. Leaver, Phys. Rev. D **41**, 2986 (1991).
[12] N. Andersson, Proc. R. Soc. London **A442**, 427 (1993).
[13] F. J. Zerilli, Phys. Rev. D **10**, 2141 (1970).
[14] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[15] S. Chandrasekhar, Proc. R. Soc. London **A369**, 425 (1980).
[16] N. Andersson, Doctoral thesis, Uppsala University, 1991.
[17] S. Chandrasekhar and S. Detweiler, Proc. R. Soc. London **A344** 441 (1975).
[18] G. L. Gunter, Philos. Trans. R. Soc. London **296**, 497 (1980).
[19] G. L. Gunter, Philos. Trans. R. Soc. London **301**, 705 (1981).
[20] B. F. Schutz and C. M. Will, Astrophys. J. **291**, L33 (1985).
[21] K. D. Kokkotas and B. F. Schutz, Phys. Rev. D **37**, 3378 (1988).
[22] V. Ferrari and B. Mashhoon, Phys. Rev. D **30**, 295 (1984).