RELATIVISTIC GRAVITATIONAL INSTABILITIES

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INTRODUCTION

The purpose of these lectures is to review and explain what is known about the stability of relativistic stars and black holes, with particular emphasis on two instabilities which are due entirely to relativistic effects. The first of these is the post-Newtonian pulsational instability discovered independently by Chandrasekhar (1964) and Fowler (1964). This effectively ruled out the then-popular supermassive star model for quasars, and it sets a limit to the central density of white dwarfs. The second instability was also discovered by Chandrasekhar (1970): the gravitational wave induced instability. This sets an upper bound on the rotation rate of neutron stars, which is near that of the millisecond pulsar PSR 1937+214, and which is beginning to constrain the equation of state of neutron matter.

I will follow the notation of Misner, *et al* (1973) and of Schutz (1985): the metric has signature +2; Greek indices run from 0 to 3, Latin from 1 to 3. I set c and G to 1 everywhere. For perfect fluids, my notation is: n is the number density of conserved particles; ρ is the density of total mass-energy; S is the specific entropy; and γ is the adiabatic index, defined by

$$\gamma = \left.\frac{\partial \ln p}{\partial \ln n}\right|_S .$$

All these quantities are defined in the local rest frame of the fluid. In these terms, the first law of thermodynamics (energy conservation) becomes

$$nTdS = d\rho - (\rho+p)dn/n. \tag{1}$$

SPHERICAL PULSATION OF SPHERICAL STARS

Newtonian Stars

Although our subject is relativistic instability, it will help us to get a general feeling for the way instability arises in Newtonian stars before tackling the relativistic case. Not only is the Newtonian case simpler, but also by comparing the Newtonian and relativistic versions of stability criteria we will be able to see exactly which instabilities are attributable to general relativity.

The most convenient way to describe the small-amplitude spherical pulsation of a spherical star is in terms of the displacement ψ of a thin shell of the star from its equilibrium position. In terms of ψ the first-order perturbation in the Euler and continuity equations can be written in the form (Ledoux & Walraven 1958)

$$\rho\psi_{,tt} + C(\psi) = 0, \tag{2}$$

where a subscripted ",t" denotes a partial derivative with respect to time and where C is the operator

$$C(\psi) = -\frac{d}{dr}\left[\frac{\gamma p}{r^2}\frac{d}{dr}(r^2\psi)\right] - \frac{4Gm(r)\rho}{r^3}\psi, \tag{3}$$

in which m(r) is the mass inside radius r. It is easy to see that C is selfadjoint with respect to the L^2 norm weighted by r^2 (*i.e.*, integrating over the volume of the star rather than the 1-dimensional radius), with the boundary condition that ψ vanishes at r=0. It is more relevant to solving Eq.(4) below that $\rho^{-1}C$ is selfadjoint in the density-weighted norm (Eisenfeld 1969).

The stability of the star could be studied directly by showing that all solutions of Eq.(2) are bounded in time if and only if the operator C is positive-definite (Laval, *et al* 1965). But for our later purposes it is useful to introduce here the *normal mode* problem. If we assume that the perturbation ψ has harmonic time dependence, $\psi(x^i) = \chi(x^i)\exp(i\omega t)$, then the dynamical equation (2) becomes the eigenvalue problem:

$$C(\chi) = \rho\omega^2\chi. \tag{4}$$

This problem can essentially be put into Sturm-Liouville form for the eigenvalue ω^2 by a change to the variable χ/r. Several consequences follow immediately. (i) There is an ascending series of eigenvalues $\omega_n{}^2$, (n = 0,1,2,...), which approaches infinity as n does. (ii) The eigenfunction associated with $\omega_n{}^2$ has n nodes in the radial direction. (iii) The eigenfunctions are complete, so that a star is stable if and only if all the ω_n are real, *i.e.* if and only if all $\omega_n{}^2$ are positive. From property (i) it follows that the star is stable if and only if $\omega_0{}^2$ is positive. (iv) This in turn will be true if and only if the integral

$$\int \chi^* C(\chi) r^2 dr > 0 \tag{5}$$

for all χ. Using the explicit form of C gives, after some algebra, the simple stability criterion (Ledoux 1958)

$$\frac{d}{dr}[(\gamma - 4/3)p] < 0 \Rightarrow \text{stability}. \tag{6}$$

Thus, in the linear approximation (small amplitude perturbations), a star with constant γ is stable if $\gamma > 4/3$. If γ is not constant, then the stability is harder to decide. This is a *sufficient* condition for stability.

There is a simple way to understand why 4/3 should be the critical value of the adiabatic index. This is an order-of-magnitude argument based on binding energy (*cf.* Zel'dovich & Novikov 1971). The total binding energy is

$$E = U + W,$$

where U is the internal energy and W the gravitational potential energy. For a star of radius R, mass M, typical pressure p, and typical density ρ, these are

$$U = ap R^3 \quad \text{and} \quad W = -bM^2R^{-1},$$

where a and b are constants of order unity. Given that p is proportional to ρ^γ, we have

$$E = kM^\gamma R^{3-3\gamma} - bM^2R^{-1},$$

where k is another constant. The star will be in equilibrium if its binding energy is an extremum against variations of R with M fixed. This gives

$$\left.\frac{\partial E}{\partial R}\right|_M = 0 \quad \Rightarrow \quad E = -b\,\frac{\gamma - 4/3}{\gamma - 1}\,\frac{M^2}{R},$$

so that a star is bound (hence stable) if γ exceeds 4/3 and unbound (unstable) if γ is less than 4/3. The marginally stable case is $\gamma = 4/3$: if such a star is perturbed, it will experience no net restoring force, so its radius will simply increase or decrease linearly with time, at least until nonlinear effects become important.

Relativistic stars

Spherical motions of a star do not radiate gravitational waves, so we might guess that there is no qualitative difference between the evolution of a perturbation in the relativistic case from that in the Newtonian one. This expectation is basically correct, but it is also easy to see that we should expect the Newtonian criterion of $\gamma = 4/3$ to be different in general relativity. This is because the binding-energy argument presented above must be different for relativistic stars, whose gravitational binding energy is larger than that of their Newtonian counterparts (*cf.* Harrison, *et al* 1965). In the first post-Newtonian approximation (that is, taking into account the first relativistic corrections to Newtonian theory), the appropriate criterion was found independently by Chandrasekhar (1964) and Fowler (1964):

$$\text{stability} \iff \gamma > 4/3 + K, \tag{7}$$

where K is a positive constant that depends on the equation of state and which increases with M/R.

In itself, this represents a small correction to the stability criterion, and it would not be remarkable except for the *coincidence* that γ approaches 4/3 as a fluid becomes more relativistic. (I call this a coincidence because I can see no fundamental relation between the way 4/3 is singled out as special in the binding-energy argument above and the fact that 4/3 is also the relativistic limit of the adiabatic index.) One class of stars in which $\gamma \approx 4/3$ is massive main-sequence stars, where radiation pressure is the dominant support. As Chandrasekhar and Fowler both showed, stars with masses approaching $10^6\ M_\odot$ have γ so close to 4/3 that the small correction K makes them unstable. Thus, stars whose structure is essentially completely Newtonian have their stability decided by effects of general relativity. This happens because the Newtonian forces nearly cancel: the star is almost marginally stable in Newtonian theory, and the issue is decided by tiny relativistic corrections.

Another class of stars where $\gamma \approx 4/3$ is white dwarfs with large central densities. Consider a sequence of white dwarfs with increasing central density. As the central density gets larger, the electrons providing the pressure need to get more and more relativistic, and γ approaches 4/3. This competes with another destabilizing effect: as central density increases, the *neutronization* reaction in which a proton and an electron combine to form a neutron and a neutrino becomes energetically favorable. This reaction removes the pressure-providing electrons, so that the equation of state softens and γ drops below 4/3. For helium and carbon white dwarfs, general relativity limits the central density; for iron white dwarfs, it is neutronization (*cf.* Shapiro & Teukolsky 1983).

The turning point criterion for white dwarfs and neutron stars

There is an easy way to decide the overall stability of stellar models that are members of a one-parameter family of models, such as one obtains by taking "cold" matter (minimum entropy) for various central densities. This method appears to have been devised by Zel'dovich and by Wheeler, and it is described in Harrison, *et al* (1965). Figure (1) contains a plot of the mass versus the radius of such stars, parametrized by their central density. The curve should be regarded as schematic, since quantitative details will depend upon the particular equation of state chosen. Suppose we follow the curve from the low-mass, low-density objects at the left margin (rocks and planets)

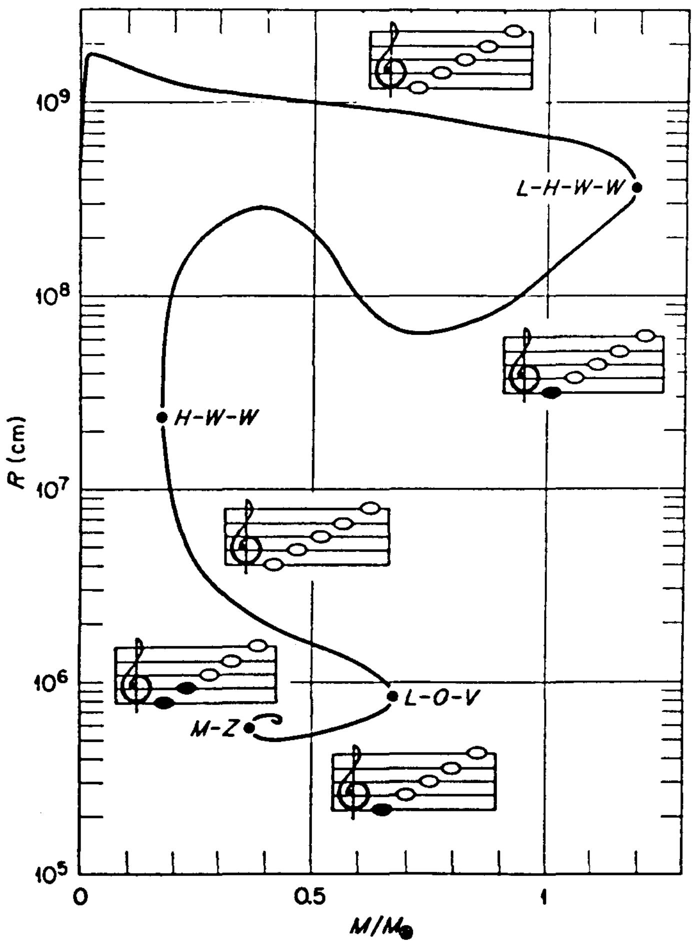

Fig. 1. A typical plot of radius versus mass for stars parametrized by their central densities. The musical notation indicates which modes (overtones of the fundamental) are stable (open circles) or unstable (filled circles). Taken from Harrison, *et al* (1965).

until M reaches its first maximum. This is a place where nearby models have the same masses but different radii. This means that we could perturb the star at the maximum by, say, increasing its radius, and it would remain in equilibrium: there would be no restoring force, and the star would be neutrally stable. Since rocks are stable and this is the first such neutrally stable point on the curve, it is reasonable to expect that the star goes *unstable* at this point. A more careful analysis bears this out and shows, in fact, that if the curve is curling clockwise at an extremum, a mode is going unstable, while if it is curling counterclockwise then an unstable mode is returning to stability. The first maximum of **M** in Figure (1) is the instability point of the white dwarfs. The slide to the minimum represents a sequence of unstable models, and past the minimum we reach the stable neutron stars. They become unstable (and therefore reach their maximum mass) at the next maximum of M, and after that the continued spiralling of the curve indicates that more and more modes are going unstable. This section of the curve is especially sensitive to the assumptions one makes about the largely unknown properties of high-density matter.

The unstable region between white dwarfs and neutron stars means that neutron stars have a ***minimum*** mass, a fact first appreciated by Oppenheimer & Serber (1938). White dwarfs have no such minimum: they follow smoothly after smaller objects, like planets. The turning point criterion has been used for axisymmetric perturbations of rotating stars by Hartle & Thorne (1969), and extended by Ipser & Horowitz (1979); it has been generalized to many-parameter families of models by Sorkin (1982).

Star clusters

Clusters made of collisionless particles interacting gravitationally have also been studied extensively, mainly as models for globular clusters and elliptical galaxies in Newtonian theory, and for quasars or quasar precursors in general relativity. There is no room to review that work here, but the reader is referred to Ipser (1969). The subject has been considerably enlivened recently by the numerical calculations of Shapiro & Teukolsky (1985a,b,c) showing how an unstable relativistic cluster can quickly form a black hole containing a considerable fraction of its mass. These calculations have been summarized in Shapiro & Teukolsky (1986).

NONSPHERICAL PULSATION OF SPHERICAL STARS

Newtonian stars

Things get a little more complicated when we consider nonspherical perturbations of stars. A good reference for this subject is Unno, *et al* (1979). Since the problem is a linear one and the unperturbed star is spherically symmetric, we can analyze the perturbations into spherical harmonics. Scalar functions, such as ρ, have perturbations expandable in the usual way:

$$\delta\rho(r,\theta,\phi) = \sum_{lm} \delta\rho_{lm}(r)\ P_l^m(\cos\theta)\ e^{im\phi}. \tag{8}$$

Vectors are expanded in vector spherical harmonics, one version of which is as follows:

$$\chi(r,\theta,\phi) = \sum_{lm} [\chi^r_{lm}(r)\ P_l^m(\cos\theta)\ e^{im\phi}\ \mathbf{e}_r + \chi^+_{lm}(r)\ \boldsymbol{\nabla}(P_l^m\ e^{im\phi}) + \chi^-_{lm}(r)\ {}^*\boldsymbol{\nabla}(P_l^m\ e^{im\phi})]. \tag{9}$$

Here the radial component of the displacement vector $\chi(r)$ is expanded as a scalar (because that is how it behaves under rotations), and the tangential components are expanded in terms of the gradient of the spherical harmonic of order (l,m) and of its *dual* gradient in the sphere,

$$^{*}\nabla \equiv \mathbf{e}_r \times \nabla \ . \tag{10}$$

The gradient and dual gradient in the sphere produce linearly independent vectors. Since the sphere is two-dimensional, they form a basis for vectors tangent to the sphere, so Eq.(9) is perfectly general.

This representation of vector perturbations is useful because we will now see that the dual-gradient parts of expressions can essentially be ignored. The argument is one of *parity*. Consider a coordinate change $\phi \to -\phi$. Under such a change, and its associated change of basis $\mathbf{e}_\phi \to -\mathbf{e}_\phi$, true scalars and tensors do not change, but pseudovectors do change sign. Now the spherical harmonics and their gradients are true scalars and vectors, respectively, but the dual gradient is a pseudovector. The unperturbed star is of course *invariant* under this change (not true for rotating stars), so the differential equations for the perturbation will not mix the two classes of perturbations. Since the pressure and gravitational field perturbations are scalars, the pseudovector -- called the *odd-parity* part of the perturbation -- does not elicit any restoring forces, so it is a neutral perturbation, one with zero frequency. An example of an odd-parity perturbation is setting the star into rotation: it simply continues to rotate. The odd-parity normal modes are usually called *toroidal* modes.

The *even-parity* normal modes contain the interesting dynamical information. For a relatively simple star (in a sense to be made clear below), the eigenvalues fall into three classes, called f-, p-, and g-modes. Their typical behavior as a function of l is illustrated in Figure 2, taken from Cox (1980). (Because of the spherical symmetry, the eigenfrequencies do not depend on m.) The p-modes form an ascending sequence of eigenfrequencies, with $\omega \to \infty$. The g-modes form another infinite sequence, but

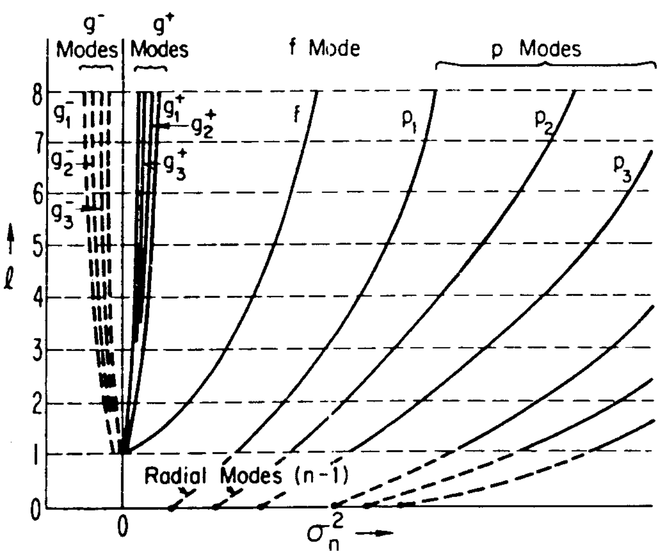

Fig. 2. The qualitative behavior of the even-parity nonradial modes of a typical spherical star. (From Cox 1980.)

with $\omega \to 0$ in the limit. The f-mode is a single mode in between, which shares characteristics of both types. In general, a star may have both stable g-modes (g^+-modes) and unstable (g^-) ones, but the f-mode and p-modes will always be stable. In the limit $\omega \to \infty$, the mode equation approaches a

Sturm-Liouville equation with ω^2 as the eigenvalue, and in the opposite limit of $\omega \to 0$ it approaches a Sturm-Liouville equation with $1/\omega^2$ as the eigenvalue; this explains the asymptotic behavior of the two sequences.

The remarkable feature of the nonradial pulsation problem is the simplicity of its stability criterion (Lebovitz 1966):

$$\text{nonradial stability} \Leftrightarrow dS/dr > 0. \tag{11}$$

This is usually called the *Schwarzschild criterion*, although it was first derived from a local argument based on convection and bouyancy by Lord Kelvin, as related by Chandrasekhar (1939). For an accessible derivation see Cox (1980). If the star is simple enough that dS/dr is of the same sign everywhere, then the g-modes will be either all stable or all unstable. If the star is more complicated (as when it has different zones of convection, ionization, composition, etc.), then it may have both types of g-modes. Physically, the g-modes are associated with convection; their eigenfunctions are dominated by velocity rather than density perturbations. If one has $dS/dr = 0$ throughout the star, then the g-modes will all have zero frequency; this happens for a polytrope, for example, if one takes γ equal to $1 + 1/n$. The p-modes are associated with sound waves; as their order gets larger, they become just local waves travelling at the speed of sound.

An important concept is the *pattern speed* of a mode, which is its phase angular velocity. Since the perturbation is proportional to $\exp(im\phi + i\omega t)$, surfaces of constant phase at some fixed r and θ will satisfy

$$\omega t + m\phi = \text{const.}$$

Differentiating this with respect to t gives

$$d\phi/dt = -\omega/m \equiv \omega_p, \tag{12}$$

where ω_p is called the pattern speed of the mode. It is clear from Figure 2 that the frequency of p-modes typically increases with l less rapidly than linearly, so that the smallest pattern speed associated with any p-mode (obtained by dividing $-\omega$ by l) decreases towards zero as l increases. Therefore, although the p-modes contribute arbitrarily high frequencies, one can find p-modes with arbitrarily small pattern speed. This will be important to us later when we discuss the gravitational-wave-induced instability in rotating stars.

Relativistic stars

When we turn to relativistic stars, we should expect a qualitative difference from the Newtonian theory because nonradial pulsations can emit gravitational radiation. On the other hand, at least for nearly Newtonian systems, we should also expect the quantitative effect of this to be small. If a mode has real frequency in the Newtonian star, then the energy carried away by gravitational radiation should damp the mode slowly, and this should appear as a small positive imaginary part of the eigenfrequency. So we expect stable relativistic stars to have complex eigenfrequencies with $|Re(\omega)| \gg Im(\omega) > 0$. This expectation has been verified by extensive investigations, and it extends essentially unchanged even to highly relativistic stars.

How does the imaginary part of the frequency actually arise in the eigenvalue calculation? It does not come from any qualitative change in the local perturbation equations; rather, it comes form imposing an *outgoing-wave boundary condition* on the eigenfunction. One demands that the energy flux at infinity represent only waves emitted by the star. This is a time-asymmetric condition, and it therefore produces eigenfrequencies which are "biased" -- if ω is an eigenfrequency, then its complex conjugate ω^* is not. (In fact, this ω^* is an eigenfrequency for a mode which satisfies an *incoming* wave boundary condition at infinity.) We shall meet the outgoing-wave boundary condition in the next section and again in Eq.(39) below.

The first calculations of the f- and p-modes of relativistic stars was performed numerically by Thorne (1968, 1969) using the analytic formalism developed by Thorne & Campolattaro (1967). For the l=2 f-mode, he found that the period was typically about a millisecond, and the damping time roughly 10^3 times as long, even for highly relativistic stars. The best calculations of these modes to date have been by Lindblom & Detweiler (1983), using the models that Arnett & Bowers (1977) calculated for a variety of recent equations of state.

The g-modes present special problems for numerical calculations because their very low frequencies and small density perturbations mean they give off extremely small amounts of gravitational radiation. There are two problems because of this: first, a numerical eigenfrequency calculation has to hunt for a tiny imaginary part of the frequency, smaller than the numerical accuracy of typical codes for p-modes; and second, since the wavelength of the waves is very large, the outgoing-wave boundary condition must be imposed very far from the star, where numerical errors can accumulate. Finn (1986) has recently overcome these problems by doing an analytic approximation to the near-zone gravitational wave field of the star and finding the appropriate boundary conditions for numerical computations. He is preparing a further paper with the results of realistic calculations.

Despite the complications of gravitational waves, it seems likely that the Schwarzschild criterion governs the stability of nonradial pulsation in general relativity as well as in Newtonian theory. The argument, basically given by Thorne (1966), is that instability sets in through a zero-frequency mode (established in general by Friedman & Schutz 1975, as we will discuss in detail later in these lectures). But such a mode will not give off gravitational radiation, so the local physics of convection will be essentially the same as in Newtonian theory. Since in Newtonian theory the local criterion for convection is all one needs for the stability of the star, one can conjecture that this will be true as well in general relativity. Chandrasekhar (1965) has extended Lebovitz's Newtonian proofs to post-Newtonian general relativity, and steps toward a rigorous fully relativistic proof have been taken by Thorne (1966) and Islam (1970). But the conjecture has not been fully established.

For completeness, let us recall the odd-parity toroidal modes of the Newtonian case. The argument that they were zero frequency was purely a symmetry one, so we may apply it in general relativity as well. The difference is that the gravitational field is no longer a scalar As a tensor it can have odd-parity parts as well. But these still do not couple radiation to the star: the pressure and density perturbations are scalars, and velocity perturbations simply cause the star to rotate steadily, without radiating. There are odd-parity gravitational waves that propagate without disturbing the star, and we will consider them in more detail when we study perturbations of spherical black holes.

Strongly damped modes

The above discussion of relativistic modes was motivated by the Newtonian analogy, and all the numerical calculations performed so far have looked only for relativistic modes that may be regarded as small perturbations of Newtonian modes. But gravitational radiation has its own dynamical freedom, and one might ask whether there are any modes associated with this that have no analogy in Newtonian theory. In fact, a family of such modes was discovered by Dyson (1980) in a model problem. An even simpler model problem analyzed recently by Kokkotas & Schutz (1986) will show clearly how they arise. Figure 3 shows the physical system: one finite string of length 2L and one semi-infinite string coupled to it by a massless spring with spring constant k. The strings have the same tension T and wave speed c. The finite string represents the star, while the semi-infinite

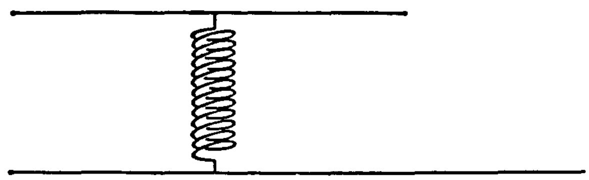

Fig. 3. A simple model system which has both weakly and strongly damped modes.

string is the analogue of the gravitational wave field; the spring couples the two as weakly (the Newtonian case) or strongly (the relativistic one) as we like. The analysis is just elementary mechanics apart from the imposition of an outgoing-wave boundary condition. We want to allow solutions for the amplitude $y(t,x)$ of the semi-infinite string of the form $f(ct-x)$ but to exclude $f(ct+x)$. Therefore we require $y_{,t} + cy_{,x} = 0$, or for a mode with frequency ω:

$$\frac{\partial y}{\partial x} = -i\omega y.$$

The explicit appearance of the frequency in the boundary condition is what makes the eigenvalues complex. Using this, it is not hard to show that the eigenvalue equation is

$$z \cosh z = -K \sinh z \, (2 + e^{-2z}), \tag{13}$$

where z is the dimensionless frequency and K the ratio of the strength of the spring to the tension in the strings:

$$z = i\omega L/c, \qquad K = kL/2T.$$

In the limit of weak coupling ($K \ll 1$), two families of eigenfrequencies emerge. The first has

$$\mathrm{Re}(\omega) \simeq (2n+1)\pi c/2L, \qquad \mathrm{Im}(\omega) = \frac{8K^2 c}{(2n+1)^2\pi^2 L} \tag{14}$$

and the second family has the same $\mathrm{Re}(\omega)$ but $\mathrm{Im}(\omega) = ac/L$, where a solves

$$a = K \exp(2a). \tag{15}$$

The modes of the first sequence are clearly small perturbations of the odd-order modes of the finite string, and are analogous to the modes of the relativistic stars described above. [The even-order modes of the finite string have a node at the attachment point of the spring, so they do not couple to the other string: their eigenfrequencies emerge unchanged from

Eq.(13).] The other modes become more and more strongly damped ($a \to \infty$) as $K \to 0$. The eigenfunctions of these families help us make sense of them. The weakly damped family have their energy primarily in the finite string; it gradually leaks through the spring and is radiated away. The strongly damped modes, on the other hand, have larger amplitude in the semi-infinite string, exciting the finite string only weakly. If we think in terms of the initial-value problem, data that excite the finite string can be represented by the weakly-damped normal modes, but they then have no freedom left to represent any initial excitement of the semi-infinite string. This is the reason for the existence of the strongly damped modes: any initial excitement of the semi-infinite string will be radiated away very quickly, so these modes have strong damping. This physical argument makes it seem plausible to me that strongly damped modes should exist in relativistic stars as well, but so far they have not been seen.

Quadrupole gravitational radiation

In the lectures by Damour in this volume, the reader will find an extensive discussion of the "quadrupole formulas" that describe gravitational radiation in the slow-motion limit. I will simply extract one result from that discussion, namely that a nearly Newtonian system loses energy to gravitational radiation at an average rate given by

$$\frac{dE}{dt} = -\frac{1}{5}\langle \text{Ɨ}^{(3)}_{jk}\text{Ɨ}^{(3)}_{jk}\rangle, \tag{16}$$

where there is an implied sum on j and k, the superscripted "(3)" means three time derivatives, and the reduced quadrupole tensor Ɨ_{jk} is defined by

$$\text{Ɨ}_{jk} = I_{jk} - \frac{1}{3}\delta_{jk}I^{l}_{l}, \qquad I_{jk} = \int T^{00}x_j x_k d^3x. \tag{17}$$

This gives us another method to calculate the normal modes of at least nearly Newtonian stars: calculate the normal modes of a Newtonian star, find its energy radiation from Eq.(16) applied to the eigenfunction, and estimate the damping rate of the relativistic mode from the equation

$$\mathrm{Im}(\omega) = ½\, E_{,t}/E,$$

where E is the energy of the mode. This method gives a good test of both the validity of the quadrupole formula and the accuracy of numerical p-mode eigenfrequencies for weakly relativistic stars. It has been used by Balbinski, *et al* (1985) to show that the quadrupole formula works surprisingly well even for highly relativistic stars, and to improve the numerical methods used for fully relativistic stars.

NONSPHERICAL PERTURBATIONS OF SPHERICAL BLACK HOLES

The other spherical system that has received a lot of attention in general relativity is the Schwarzschild black hole. Because Birkhoff's theorem (*cf.* Misner, *et al* 1973) excludes any nontrivial spherical perturbations of the hole, we need only study its nonradial stability. The problem was first studied by Regge & Wheeler (1957), but at that time the nature of the black hole was not understood (indeed, Wheeler didn't coin the term "black hole" until a decade later), and they used an inappropriate boundary condition at the horizon. A definitive proof of the stability of the Schwarzschild metric was finally given by Vishveshwara (1970). Nevertheless, the Schwarzschild perturbation problem continues to be interesting, partly because normal mode oscillations of a black hole might be seen by gravitational wave antennas, partly as a guide to the much more difficult problem of perturbations of the Kerr metric (the rotating black hole), and partly because the normal mode problem has some peculiar and challenging features (Chandrasekhar 1983).

Formulation as a scattering problem

The interesting features of the problem are most easily illustrated by studying the so-called odd-parity equation for gravitational waves. Unlike the stellar case, the odd-parity perturbations of the Schwarzschild metric are just as interesting as the even-parity ones. At first it was thought that they obeyed a different equation from the even-parity "Zerilli" wave equation (Zerilli 1970), but it has since been shown that the two equations can in fact be transformed into one another (Chandrasekhar 1975, Chandrasekhar & Detweiler 1975; see Chandrasekhar 1983). The equation has a form which is similar to that of scattering problems in quantum mechanics (Regge & Wheeler 1957):

$$\psi'' + (\omega^2 - V)\,\psi = 0, \qquad V = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right], \tag{18}$$

where M is the mass of the hole and primes (') denote, not derivatives with respect to r, but derivatives with respect to r_*, defined by

$$r_* = r + 2M \ln(r/2M-1),$$

which is an affine parameter on the outgoing null geodesics. The amplitude ψ is a metric component, from a knowledge of which all the odd-parity metric components can be calculated. Since r_* is the fundamental variable, we must ask what its limits are and what V looks like in these limits. As $r \to \infty$, so does r_*, and $V \simeq l(l+1)/r_*^2$. As $r \to 2M$, we have $r_* \to -\infty$ and V falling off exponentially as $\exp(r_*/2M)$. Between these two extremes, the potential is smooth, reaching a maximum at some intermediate point, with no complicated wiggles.

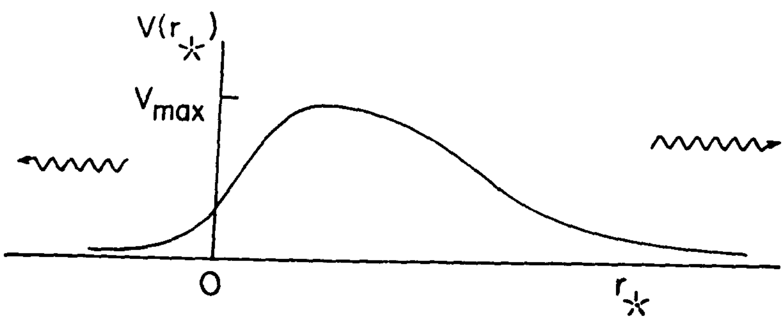

Fig. 4. The qualitative shape of the potential V. The waves show the appropriate boundary conditions for a normal mode.

If this were a standard scattering problem in quantum mechanics, then there would be no difficulty. What makes this problem different are the normal-mode boundary conditions: in order to exclude any outside "forcing" of the hole's oscillations, we demand that the waves in ψ must be outgoing far from the hole and ingoing across the horizon. That is, we are looking for solutions of Eq.(18) which have waves moving *away* from the potential barrier on both sides. Simple flux-conservation arguments of the type one makes for the Schrödinger equation show that it is impossible to satisfy this condition with a purely real frequency ω, and it is not hard to extend them to show that the imaginary part of ω must be positive, so that all the modes are damped away. (We still lack any completeness theorem for the normal modes, however, so that this result cannot be used to infer the stability of the hole.)

In fact, we can go further and use WKB-type arguments to see that the real part of ω^2 must be near the maximum of V. Suppose this were not the case. If ω^2 were larger than the barrier maximum, then a wave outgoing on the right could only match to a wave with a substantial component approaching the barrier from the left, in violation of our boundary condition. If ω^2 is too small, then the left-moving wave on the left of the barrier would "tunnel" through the barrier as a linear combination of two exponentials, one growing and the other dying. Each of these would, in turn, match across the right-hand edge of the barrier to a pair of waves, one ingoing and the other outgoing. For the two ingoing waves to cancel (and thus satisfy the boundary condition), they would have to have the same amplitude; but if the tunneling has been strong, they cannot have the same amplitude. Therefore, the tunnelling has to be essentially negligible: ω^2 has to be near the top of the potential barrier. [A more extended version of this discussion may be found in Schutz (1984).]

Calculations of the normal modes

Numerical calculations bear out this simple argument. The first extensive calculations were by Chandrasekhar & Detweiler (1975). They were able to get the first few modes for each l, but soon lost accuracy. They found that for each l the modes formed a sequence in which the real part of the frequency did not change very much, but the imaginary part increased rapidly (reminiscent of the strongly damped modes of our model string problem, above).

An approximate solution can be sought by replacing the potential V with the simpler *Price* potential V_P,

$$V_P = \begin{cases} l(l+1)/r_*^2 & r_* > a \\ 0 & r_* < a \end{cases}$$

for some a. The solutions for this potential are obtainable in terms of Bessel functions. Remarkably, there are only a finite number of modes for each l. This raises the question of whether the real potential V only has a finite number of modes, as well. It also forces one to wonder about the completeness problem: since only a finite number of eigenfunctions exist, they cannot be a basis for arbitrary initial data. What is the evolution of the data that cannot be expressed in terms of normal modes? The completeness problem for radiating systems has not received the attention it deserves. The first extensive investigation of which I am aware was in the Ph.D. thesis of Dyson (1980), and it led to the discovery of the strongly damped modes. Leaver (1986b) has recently studied the problem for the Schwarzschild metric, showing in particular that there are non-normal-mode contributions that give rise to a radiating, power-of-time falloff in the "tails" of gravitational wave perturbations, eventually becoming the nonradiative decay tails discovered by Price (1972). It seems to me that this is one of the most important and potentially fruitful areas for research in the black-hole normal mode problem.

Another approximation to the Schwarzschild potential has been explored by Blome & Mashhoon (1984) and Ferrari & Mashhoon (1984). Here another analytic form is substituted for the full potential V, permitting an analytic solution. This one has a couple of parameters that can be adjusted so that it is a good fit at the maximum, rather than having the correct behavior at infinity, as does the Price potential. This method yields good approximations for the lowest modes, but it is hard to improve it to do better for higher-order modes.

In another attempt at approximation, Schutz & Will (1985) introduced WKB methods to obtain the eigenfrequencies. This also gave good results only for the lowest modes, but it has the advantage that it can be extended to higher orders. Iyer & Will (1986) and Iyer (1986) have gone up to fifth order, giving vastly improved accuracy. The WKB approach also gives an analytic formula for the behavior of the eigenfrequencies as l gets large:

$$M\omega \simeq [(l+\tfrac{1}{2}) + i(n+\tfrac{1}{2})]/3^{3/2} + O(1/l),$$

where n is the order of the mode. The real part of the frequency gives, for modes with l=m, a pattern speed $M\omega_p \simeq 3^{-3/2}$, which is the orbital frequency of a photon in the (unstable) circular photon orbit around Schwarzschild (Goebel 1972). The fact that, for n=0, the imaginary part of the frequency approaches the limit .0962, had been noticed by Detweiler (1979), but not shown analytically until this WKB work.

Great progress on the development of suitable numerical techniques for this problem has recently been made by Leaver (1985, 1986a). He has given a very detailed discussion of the spheroidal wave equation, of which Eq. (18) is a special case, and he has adapted approximation methods developed in atomic physics to this problem. Future work on this problem will surely take this work as its starting point. Leaver's methods strongly suggest that there will be an infinite number of normal modes for any l, but they don't quite prove it: this is still one of the most important unsolved problems in this field.

STABILITY OF ROTATING STARS: GENERAL REMARKS

In the preceding sections I have reviewed a subject that is reasonably well understood, and I was able only to highlight some important results. When we turn to the study of the perturbations and stability of rotating stars, we find a very different story: despite considerable interest in the problem, there are few general theoretical results; there have been no extensive calculations of the modes of relativistic, rapidly rotating stars; and even the Newtonian theory is poorly understood. Nevertheless, some remarkable features of the problem have been discovered, including the fact that the emission of gravitational radiation can actually *destabilize* a rotating star; even more, *all* perfect-fluid rotating stars are unstable in this way! It appears that these instabilities may play an important role in the formation and subsequent evolution of neutron stars, and so the main aim of the remainder of these lectures will be to try to understand them.

Before going on, it is well to ask why adding rotation to the star makes the problem so much more difficult. Except in the limit of slow rotation, where the problem is not much harder than the spherical one (Newtonian theory reviewed by Tassoul 1978; relativistic theory treated by a number of authors, *e. g.*: Hartle 1967; Hartle & Thorne 1968, 1969; Hartle, *et al*, 1972; Hartle & Munn 1975; Chandrasekhar & Friedman 1972; Abramowicz & Wagoner 1978), there are two factors that make the normal mode problem harder in the rotating star:

(i) *It is harder to compute results.* When we lose spherical symmetry, the mode problem becomes an elliptic boundary value problem in two dimensions, rather than an ordinary differential equation. This is not an insuperable problem with today's computers, but it has inhibited progress in the past. I suspect that the motivation to tackle the problem has been lacking until recently, but the possibility of gravitational wave observations in the near future may change that.

(ii) *Even in the Newtonian case, the eigenfrequencies are not eigenvalues of a selfadjoint operator.* In fact, this is true of the spherical problem as well, but there we are able to solve the problem in terms of a selfadjoint operator whose eigenvalues are ω^2. But ω itself is the eigenvalue of the square root of this operator, which is not necessarily selfadjoint. It is easy to see why: in order to allow for the possibility of the star being unstable, it must be possible to have complex eigenfrequencies ω; but a selfadjoint operator has real eigenvalues, so we cannot expect ω to be the eigenvalue of such an operator. This argument is as true for the rotating star as for the nonrotating one. But there is no lucky way around it in the rotating case. For example, in terms of the Lagrangian displacement vector field χ, the dynamical equation for the perturbations of a Newtonian perfect fluid star has the general form (Lynden-Bell & Ostriker 1967)

$$\rho\chi_{,tt} + \mathbf{B}(\chi_{,t}) + \mathbf{C}(\chi) = 0, \tag{19}$$

where **B** and **C** are operators. The operator **B** involves Coriolis-type terms, and is present for nonaxisymmetric modes only when the background star is rotating. In the absence of rotation, when $\mathbf{B} = 0$, then there is an associated eigenvalue problem for ω^2 which is selfadjoint, as we have seen. The axisymmetric normal mode problem is also selfadjoint. But in the rotating case, the eigenvalue problem for nonaxisymmetric modes is genuinely quadratic, and there is no associated simple selfadjoint problem. Because the nonaxisymmetric problem is so different from the axisymmetric one, and because it has unique astrophysical implications, I will concentrate on it in what follows. The axisymmetric problem resembles in many ways the spherical one. In the spherical problem, we are lucky. For the nonaxisymmetric modes of rotating stars, our luck runs out.

The formal structure of the Newtonian problem is, nevertheless, very regular and interesting. The operator of which ω is the eigenvalue is, of course, essentially the time-evolution operator. This turns out (Schutz 1980a) to be symmetric with respect to a non-positive-definite inner product [in fact the symplectic inner product of Eq.(44) below]. Such operators have been studied by Bognar (1974), and extensively in a series of papers by Barston (1967a,b; 1968; 1971a,b; 1972; 1974; 1977). The structure and some aspects of the stability of rotating stars in Newtonian theory have been developed in the monograph by Tassoul (1978). See also Schutz (1983, 1984) for recent reviews of this subject in more mathematical depth than we shall go into here.

Despite a considerable amount of work on the problem, the outstanding problem for Newtonian stars is still unsolved: to give a necessary and sufficient criterion for the absence of complex eigenfrequencies in the spectrum of the operator, one which can be used without actually solving for the eigenvalues. In the spherical case, this criterion is simply the positive-definiteness of **C**. Here, positive-definiteness of **C** is sufficient but not necessary for stability, and as we shall see it is never satisfied. Therefore, we are still in the position of having to compute all the normal modes of a star to see if it is stable. For this reason, although we know the way some instabilities behave in stars, it is still possible that there are others we do not suspect. Indeed, new classes of instabilities have recently been found (Balbinski 1985, Papaloizou & Pringle 1984).

In some respects, the relativistic problem is easier: the destabilizing influence of gravitational radiation makes it possible to show that positive-definiteness of the relativistic counterpart of C is in fact necessary and sufficient for stability. Here, the fact that C is not positive means that all stars are formally unstable. But because the growth rate of the instability depends on the efficiency with which gravitational waves can carry energy away, the importance of this instability depends strongly on how relativistic the star is. Moreover, the instabilities of Newtonian stars should still be present as well in the relativistic case. Again, therefore, to answer the physical questions we have to compute normal modes.

Many of these difficulties have been addressed outside the context of astrophysics, especially in meteorology and oceanography. Useful references are Drazin & Reid (1981), Greenspan (1968), and Holm, *et al* (1985).

THE MACLAURIN SPHEROIDS

I shall begin our study of rotating stability by describing the modes and instabilities of the simplest self-gravitating rotating system in Newtonian theory, the Maclaurin spheroid. This is not just because it is well to begin simply; it is also because these models are still the only sequence of models that has received extensive study. What intuition we have about rotational instabilities in astrophysics, we have to a large extent developed in the Maclaurin spheroids.

The Maclaurin spheroids are axisymmetric models of uniform density ρ and uniform angular velocity Ω. When the equilibrium equations are solved in Newtonian gravity, one finds that the surface is a perfect ellipsoid. These models are obviously a crude approximation to realistic stars, but the instabilities we see in them also seem to be present in compressible models. For reviews of their structure and some aspects of their stability, see Lyttleton (1953) and Chandrasekhar (1969). Their perturbations were first studied by Bryan (1889), who obtained a complete analytic solution to the problem: the mode equation separates in ellipsoidal harmonics, and the eigenfrequency equation is a polynomial. Unfortunately, the problem was by and large too difficult to compute by hand, and so many features of the solutions of the eigenfrequency equation remained undiscovered until the work of Comins (1979a,b).

The nonaxisymmetric modes

I shall restrict myself to the discussion of the nonaxisymmetric modes; axisymmetric modes and stablility are easier to treat but (apparently) less interesting. The modes which seem easiest to destabilize for a given spheroidal index l are those with l = m. Figure 5 (next page) shows the behavior of the pattern speed ω_p of the m = 2 and m = 5 modes, with various significant points indicated.

First note the general shapes of the curves. The vertical axis is normalized to Ω, which goes to zero as e approaches zero. (Therefore, the curves near the left margin go to infinity; this way of displaying frequencies is not the best for slow rotation.) For very slow rotation, the curve for any fixed m is fairly symmetrical about the horizontal axis. This is because the nonrotating star has a single eigenvalue ω^2, which gives equal values $\pm|\omega|$ for the forward- and backward-going modes. These two branches respond differently to increasing rotation. The backward-going mode is dragged forward by the rotation of the star, and eventually it is pulled so far that it goes forward in the inertial frame, although still backward relative to the star ($\omega_p/\Omega < 1$). Its forward-going counterpart also gets dragged faster forward, but not as much as the rotation of the star, so ω_p/Ω decreases.

At some point the two modes join. This is the onset of instability: I have not shown it in the diagram, but after this the modes have complex-conjugate eigenfrequencies. This signature, that instability in Newtonian theory sets in by the merging of two related eigenfrequencies, turns out to be very general (Schutz 1980b). This instability is called the *dynamical instability*, and it typically grows on the dynamical time scale, one rotation period.

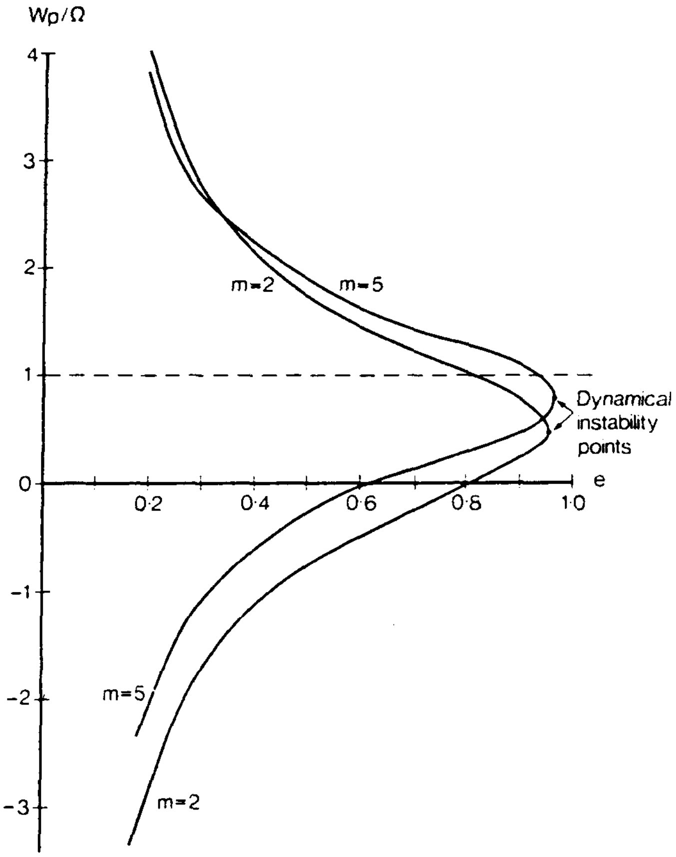

Fig. 5. The pattern speeds of the m = 2 and m = 5 modes of the Maclaurin spheroids as a function of the eccentricity e of the surface of the unperturbed star; e increases monotonically with angular momentum. Various important stability points are explained in the text. [From Schutz (1983).]

The secular instabilities

The addition of a small amount of viscosity or gravitational radiation reaction to the problem can induce other instabilities, which are called *secular instabilities* because their (long) time scale depends on the size of the added effect (the coefficient of viscosity or the ratio GM/Rc^2). The viscous instability was first studied by Roberts & Stewartson (1963), who showed that the instability sets in when a mode has zero frequency in the frame rotating with the star. This means that its pattern speed will be the same as Ω in the inertial frame, so the signal for instability is that the forward-going mode should drop below the horizontal line at $\omega_p/\Omega = 1$. Clearly, the m = 2 instability occurs before the m = 5, and this is part of a general pattern: the viscous secular instability sets in first for m = 2, and after that for each successive m in turn. Note that stars that are dynamically stable still can be secularly unstable to viscosity.

The gravitational-radiation-induced secular instability was discovered by Chandrasekhar (1970), and had been completely unexpected. In retrospect, we will see that it has an easy explanation. It sets in when the backward-going mode is dragged forward in the inertial frame, so the instability point is where the bottom branch of a curve in Fig.5 goes through $\omega_p = 0$. Chandrasekhar calculated where the $m = 2$ instability point was, and made the reasonable assumption that it represented the onset of this secular instability along the sequence. But Fig.5 shows a different story: the $m = 5$ mode goes unstable earlier. This is again part of a general pattern. The $m = 2$ mode is actually the *last* to go unstable, while in any Maclaurin spheroid there is some value of m such that all m larger than this are unstable. Therefore every Maclaurin spheroid is unstable. This is a special case of the generic gravitational radiation instability of all rotating stars discovered by Friedman & Schutz (1978b) and proved rigorously in general relativity by Friedman (1978a).

The Maclaurin spheroids also provide us with the "escape route" from this instability, that is, they tell us why it is not catastrophic for rotation in ordinary stars. Comins (1979b) shows that the growth time for the instability increases exponentially with m, so that even for very relativistic stars the instability in the modes for $m > 10$ or so grows too slowly to matter. Moreover, he shows that the growth rate is very sensitive to the compactness of the star, so that it is unimportant even for $m = 2$ in main-sequence stars. And finally, Detweiler & Lindblom (1977) and Lindblom & Detweiler (1977) showed that gravitational radiation and viscosity actually *compete*, the one acting to stabilize where the other destabilizes. Since viscosity is more effective on short length scales, it becomes relatively stronger with increasing m. It is no surprise, therefore, that Comins was also able to show that even a tiny viscosity would stabilize all m larger than some minimum, for a star of given compactness. Thus, since real stars have some viscosity, slowly rotating stars are stable even in principle.

The T/W criterion for instability

The general picture painted by the Maclaurin spheroids seems to be essentially unchanged for sequences of compressible stars. It is true that there are likely to be other instabilities as well, but at least the ones we see in the Maclaurin spheroids are also seen in calculations of compressible stars and disks, and even collisionless systems like model galaxies. A rule of thumb for calculating where some of these instabilities should set in emerged from one of the earliest extensive studies of differentially rotating, compressible stars, by Ostriker and colleagues (Tassoul & Ostriker 1968, Ostriker & Tassoul 1969, Ostriker & Bodenheimer 1973). If we let T stand for the rotational kinetic energy of the star and W for the absolute value of its gravitational potential energy, then it is found that the dynamical instability for $m = 2$ sets in when the ratio T/W is roughly 0.26. Since this may be the earliest significant dynamical instability on a sequence, this gives a rough idea of how far a sequence of rotating stars can reasonably be pushed. The $m = 2$ secular instability to gravitational radiation sets in when T/W is roughly 0.14. These criteria seem to be fairly robust, giving predictions to perhaps 10% for a wide variety of systems. But the T/W criterion does not extend as a good predictor to the more interesting gravitational radiation instabilities for $m = 4$ or 5. For these we either have to compute the normal modes or use the variational principle for zero-frequency modes of Newtonian stars recently discovered by Ipser & Managan (1985).

When, then, does the gravitational radiation instability matter? Only a mode calculation seems to be able to tell us. I will return to this question below, after we have understood more about how this instability operates.

A RELATIVISTIC APPROACH TO STABILITY

Having learned from the Maclaurin spheroids a number of things which we would like to understand better, we now turn to an analysis of stability in general relativity. In many ways the formalism of perturbation theory is more natural in a 4-dimensional context, and it is certainly true that it is easier to find general stability criteria in general relativity than in Newtonian gravity. We will return to the Newtonian case later. The approach I will take follows Friedman & Schutz (1975, 1978c), and has been reviewed by me in somewhat greater detail elsewhere (Schutz 1984).

Perfect fluids in general relativity

We will consider in detail only perfect fluid systems, although the main results can be extended to any dissipationless physical theory. The fluid has an equation of state of the form $p = p(\rho,S)$, a stress-energy tensor

$$T^{\alpha\beta} = (\rho+p)U^{\alpha}U^{\beta} + pg^{\alpha\beta}, \tag{20}$$

and dynamical equations

$$T^{\alpha\beta}{}_{;\beta} = 0, \tag{21}$$

$$(nU^{\alpha})_{;\alpha} = 0, \tag{22}$$

$$U^{\alpha}S_{,\alpha} = 0. \tag{23}$$

In fact, Eq.(21) and either of (22) or (23) imply the other. It is useful to define the *specific momentum* associated with a fluid element,

$$V_{\alpha} \equiv \frac{\rho+p}{n}U_{\alpha}. \tag{24}$$

In terms of the specific momentum, the vorticity conservation law has a very simple expression. If the fluid is *isentropic* (uniform entropy), then we have

$$\pounds_U(\nabla_{\alpha}V_{\beta} - \nabla_{\beta}V_{\alpha}) = 0. \tag{25a}$$

Here $\pounds_U$ is the Lie derivative with respect to the vector field U^{α}. For an introduction to the Lie derivative, see Carter's lectures in this volume or Schutz (1980c). It can be taken to be the ordinary partial derivative of a tensor's components if the coordinate system includes U^{α} as on of its coordinate basis vectors. If there is an entropy gradient in the fluid, then this law is replaced by the slightly weaker version,

$$\pounds_U(\nabla_{[\alpha}S\,\nabla_{\beta}V_{\gamma]}) = 0, \tag{25b}$$

where square brackets denote antisymmetrization. This is called *Ertel's theorem* (Ertel 1942a,b), and its relativistic form was found by Friedman (1978a).

Definition of a perturbation in terms of a sequence of solutions

Consider a smooth sequence of manifolds $M(\epsilon)$, each a sloution of Einstein's equations, the family parametrized by ϵ. Let $\epsilon = 0$ denote a *stationary* solution, which we call the unperturbed manifold. The other members of the sequence deviate more and more from the stationary state, but the limit $\epsilon \to 0$ is sufficiently differentiable in ϵ, in a sense which we won't need to define more precisely. Technically, this sequence is a trivial fiber bundle (Schutz 1980c), with base space R^1 (coordinate ϵ) and fiber R^4 (coordinates t,x,y,z). It is clear that in general there is no preferred or natural map from one spacetime in the sequence to another, no natural way to associate a point of a perturbed manifold with one of the unperturbed spacetime. Such associations are useful, however, so we imagine introducing a family of maps such that $f(\epsilon)$ maps points of the $\epsilon = 0$ manifold in a 1-1 fashion to points of $M(\epsilon)$.

We can use these maps to define a perturbation (Schutz & Sorkin 1977). Suppose there is a tensor field $Q(\epsilon)$ on each manifold, also smooth in the limit $\epsilon \to 0$ (for example, the metric tensor or the four-velocity). Then we can drag $Q(\epsilon)$ from any manifold $M(\epsilon)$ back to $\epsilon = 0$, thereby defining a tensor field $Q_f(\epsilon)^*$ on the unperturbed manifold. As ϵ varies, we therefore have a family of tensor fields on $M(0)$ which are the images under f of the family on the sequence. Given a sufficiently smooth family, then it will be approximated for small ϵ by the expansion

$$Q_f(\epsilon)^* = Q(0) + \epsilon\, \delta_f Q + O(\epsilon^2), \tag{26}$$

where we define

$$\delta_f Q \equiv \left.\frac{dQ_f(\epsilon)^*}{d\epsilon}\right|_{\epsilon=0} . \tag{27}$$

This is *defined* as the first-order perturbation in Q following f. It is a tensor field on $M(0)$. Clearly we could define second-order and higher perturbations in terms of higher derivatives with respect to ϵ along f. But without introducing f, it is impossible to define a perturbation as a tensor field on $M(0)$.

Since f is arbitrary, we could do the same with another 1-1 family of maps $h(\epsilon)$. Then there will be a different definition of a first-order perturbation, $\delta_h Q$. To see how this differs, let us consider a smooth family of *maps* $m(\epsilon;\lambda)$, such that $m(\epsilon;0) = f(\epsilon)$ and $m(\epsilon;1) = h(\epsilon)$. This family gives us a smooth transition from one map to the other. Now, if we hold ϵ fixed, say at ϵ_1, then in the manifold $M(\epsilon_1)$ the map $m(\epsilon_1;\lambda)$ traces out a one-parameter family of points as λ varies; *i.e.*, it defines a curve in $M(\epsilon_1)$. Let us denote the tangent vector to this curve as $\chi^\alpha(\epsilon_1;\lambda)$. Since all the maps reduce to the identity as $\epsilon \to 0$, χ^α goes to zero in that limit: it is a first-order quantity in ϵ. It therefore has an approximation in $M(0)$ in the spirit of Eq.(26) given by

$$\chi^\alpha(\epsilon;\lambda) = \epsilon\, \chi^\alpha{}_1(\lambda) + O(\epsilon^2). \tag{28}$$

Moreover, if the maps f and h are not too far apart for some ϵ_1, then $\chi^\alpha(\epsilon_1;\lambda)$ will be well approximated by $\epsilon_1 \chi^\alpha{}_1(0)$. This will always be true for sufficiently small ϵ_1, since the maps f and h approach each other in this limit. Therefore, keeping only the lowest order terms, it is not hard to show that the first-order perturbations are related by

$$\delta_h Q = \delta_f Q + \pounds_{\chi_1(0)} Q(0). \tag{29}$$

This change of the perturbation under a change of the map is called a *gauge transformation*. The gauge transformations of linearized theory (Misner, *et al* 1973, Schutz 1985) can be viewed in this framework.

Two preferred perturbations: Eulerian and Lagrangian

Although perturbation theory should be covariant under changes of gauge, some choices are singled out by either computational or physical considerations. Normally, tensor fields on $M(\epsilon)$ will be described in terms of some family of coordinate systems $\{x^\alpha(\epsilon)\}$ that is also smooth as $\epsilon \to 0$. I shall define the *Eulerian* map as the one which connects points that are at the same coordinate positions in the different manifolds. It is conventional to represent the perturbation with respect to this map simply by δ, with no subscript. Thus we have the component equations

$$\epsilon\ \delta g_{\alpha\beta}(x^\mu) \simeq g_{\alpha\beta}(x^\mu,\epsilon) - g_{\alpha\beta}(x^\mu,0),$$

$$\epsilon\ \delta\rho(x^\mu) \simeq \rho(x^\mu,\epsilon) - \rho(x^\mu,0),$$

etc. In the Newtonian theory, "Eulerian" has a somewhat different meaning, although in practice the difference is not usually important. Since Newtonian theory has a fixed Euclidean metric and a universal time, there exist isometries between the different manifolds. Choosing an isometry defines the Eulerian map. In practice one always chooses coordinates on the family $M(\epsilon)$ such that the isometry preserves the coordinates, in which case the relativistic and Newtonian definitions of Eulerian coincide. But in general relativity there is generally no isometry, so our definition is the only one possible.

The existence of a fluid in the manifolds allows us to define another map, called the *Lagrangian* map, which connects the "same" fluid elements in different manifolds. This is physically reasonable only in certain circumstances, namely where the sequence can be thought of as a deformation of a single system. More precisely, since the motion of the fluid preserves entropy, particle numbers, and vorticity (or Ertel's constant), then we shall require that the Lagrangian map also preserve these quantities. (That this usually defines the map almost uniquely was shown by Friedman & Schutz 1978a.) It is customary to denote the Lagrangian perturbation by Δ. The vector field χ relating δ to Δ by Eq.(29) is called the Lagrangian displacement vector field, because it can be interpreted as representing the first-order change in the position of a fluid element relative to the Eulerian map:

$$\Delta = \delta + \pounds_\chi. \tag{30}$$

The defining conditions for Lagrangian perturbations are therefore

$$\Delta S = 0, \qquad \Delta(nU^\alpha g^{1/2}) = 0, \qquad \text{and} \quad \Delta(\nabla_{[\alpha}S\ \nabla_\beta V_{\gamma]}) = 0, \tag{31}$$

where g is the absolute value of the determinant of the metric components.

Perturbations of Einstein's Equations

Using these definitions, Friedman & Schutz (1975, 1978c) were able to show that the first-order perturbed Einstein equations have an important symmetry property. The Eulerian perturbation of the field equations can be written

$$g^{-1/2}\delta(G^{\alpha\beta}g^{1/2}) = -\tfrac{1}{2}\,\epsilon^{\alpha\gamma\lambda\tau}\epsilon^{\beta\mu\sigma}{}_{\tau}\,\nabla_{(\gamma}\nabla_{\mu)}\delta g_{\lambda\sigma} + G^{\alpha\beta\lambda\sigma}\delta g_{\lambda\sigma}, \tag{32}$$

$$G^{\alpha\beta\lambda\sigma} \equiv \tfrac{1}{2}R^{\alpha(\lambda\sigma)\beta} - (\tfrac{1}{4}R^{\alpha(\lambda}g^{\sigma)\beta} + \tfrac{1}{4}R^{\beta(\lambda}g^{\sigma)\alpha} - \tfrac{1}{2}R^{\alpha\beta}g^{\lambda\sigma} - \tfrac{1}{2}R^{\lambda\sigma}g^{\alpha\beta})$$
$$+ \tfrac{1}{4}R(g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\lambda} - g^{\alpha\beta}g^{\lambda\sigma}). \tag{33}$$

Here round brackets denote symmetrization on a pair of indices. It is easy to see that the tensor $G^{\alpha\beta\lambda\sigma}$ has the symmetries

$$G^{\alpha\beta\lambda\sigma} = G^{(\alpha\beta)(\lambda\sigma)} = G^{\lambda\sigma\alpha\beta}, \qquad G^{\alpha\beta\lambda}{}_{\lambda} = 0. \tag{34}$$

Similarly, the Lagrangian perturbation of the stress-energy tensor density is

$$g^{-\frac{1}{2}}\Delta(T^{\alpha\beta}g^{\frac{1}{2}}) = W^{\alpha\beta\lambda\sigma}\Delta g_{\lambda\sigma}, \tag{35}$$

$$\begin{aligned} \Delta g_{\lambda\sigma} &= \delta g_{\lambda\sigma} + \nabla_{\lambda}\chi_{\sigma} + \nabla_{\sigma}\chi_{\lambda}, \\ W^{\alpha\beta\lambda\sigma} &= \tfrac{1}{2}(\rho+p)U^{\alpha}U^{\beta}U^{\lambda}U^{\sigma} + \tfrac{1}{2}p(g^{\alpha\beta}g^{\lambda\sigma} - g^{\alpha\lambda}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\lambda}) \\ &\quad - \tfrac{1}{2}\gamma p(g^{\alpha\beta} + U^{\alpha}U^{\beta})(g^{\lambda\sigma} + U^{\lambda}U^{\sigma}). \end{aligned} \tag{36}$$

This tensor has the same symmetries as $G^{\alpha\beta\lambda\sigma}$, Eq.(34), but not the traceless property.

The result of these symmetries (which reflect the fundamental fact that the combined hydrodynamical and Einstein equations can be derived from a variational principle, provided the Lagrangian conditions in Eq.(31) are respected by the variations -- see Schutz & Sorkin 1977) is that we have the following identity for any $\delta g_{\alpha\beta}$, χ^{α}, $\delta g_{\alpha\beta}$, $\tilde{\chi}^{\alpha}$, regardless of whether they satisfy the field equations or not:

$$\begin{aligned} &16\pi\tilde{\chi}_{\alpha}\Delta(\nabla_{\beta}T^{\alpha\beta}) + \delta g_{\alpha\beta}\delta(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) \\ &\quad = 16\pi\chi_{\alpha}\tilde{\Delta}(\nabla_{\beta}T^{\alpha\beta}) + \delta g_{\alpha\beta}\delta(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) + \nabla_{\alpha}R^{\alpha}, \end{aligned} \tag{37}$$

where R^{α} is bilinear in $\delta g_{\alpha\beta}$ and $\delta g_{\alpha\beta}$.

<u>A stability criterion</u>

Now we shall see how to use Eq.(37) to derive a stability criterion. We suppose that $h_{\alpha\beta}$ and y^{α} are the eigenfunctions of a normal mode solution of the first-order perturbed field equations, with complex frequency ω, so that the time-dependent solutions are

$$\chi^{\alpha}(t,x^{k}) = y^{\alpha}(x^{k})e^{i\omega t}, \qquad \delta g_{\alpha\beta}(t,x^{k}) = h_{\alpha\beta}(x^{k})e^{i\omega t}.$$

We define the tilde-perturbation in terms of this eigenfunction:

$$\tilde{\chi}^{\alpha}(t,x^{k}) = [y^{\alpha}(x^{k})]^{*}\, e^{-i\omega t}, \qquad \delta g_{\alpha\beta}(t,x^{k}) = [h_{\alpha\beta}(x^{k})]^{*}\, e^{-i\omega t}.$$

Here * denotes the complex conjugate. Notice that χ^{α} and $\tilde{\chi}^{\alpha}$ are not complex conjugates of each other, since ω is not real. When substituted into Eq.(37), the time-dependences of these functions cancel and the complex-conjugate relationships of the eigenfunctions combine with the symmetry of the equation to give the following:

$$\omega^{2}A - \omega(iB) - C + \nabla_{k}R^{k} = 0, \tag{38}$$

where A, iB, and C are *real* (*i.e.*, by virtue of the symmetry in Eq.(37), they are Hermitian forms).

Now suppose we integrate Eq.(38) over a spacelike hypersurface which is asymptotically null outgoing, so that it intersects future null infinity. Then

Friedman & Schutz (1975, 1978c) have shown that the integral over the sphere at infinity becomes, with an outgoing-wave boundary condition,

$$\oint R^k n_k dS = -4i\omega |\text{Bondi news function}|^2. \tag{39}$$

Because the r.h.s. of this equation is pure imaginary if ω is real, we immediately have from Eqs.(38) and (39) that *ω can be real only if it is zero.* But now imagine a sequence of unperturbed stellar models, such as a relativistic version of the Maclaurin spheroids. If a particular mode is stable somewhere along the sequence, then it can go unstable only by going through a real value of ω. By our result here, this can only be zero. *Instability in modes sets in only through zero frequency.* Moreover, if somewhere along a sequence the form C is positive-definite for all possible eigenfunctions ($h_{\alpha\beta}$, χ^α), then instability occurs at the first point along the sequence where C becomes semi-definite. (It is easy to see that this is necessary by substituting $\omega = 0$ into Eq.(38). To show that it is sufficient requires more work.)

This would be a very powerful stability criterion for perfect fluid configurations, except for one unfortunate fact: it turns out that C is not positive definite for any star, because the term $-W^{\alpha\beta\lambda\sigma}[\Delta g_{\alpha\beta}]^*\Delta g_{\lambda\sigma}$ in C contains derivatives with respect to ϕ quadratically, and these are negative-definite. One can always choose the trial function and the azimuthal eigenvalue m in such a way that this term dominates all others. Therefore the simple idea of a sequence going unstable at some point does not hold. One can still show, however, that the indefiniteness of C does imply the existence of unstable modes (Friedman 1978a). These are the gravitational-wave-excited modes we encountered in the Maclaurin spheroids. The generality of our treatment shows that they are a feature of all compressible stars as well.

There are two remarks on the general problem which I wish to make. One is that the proof of instability offerred here is mode-based. It would be more satisfying and perhaps illuminating if we had one which did not rely so heavily on modes. The second is that the only thing we needed to make the proof work was the symmetry property of $W^{\alpha\beta\lambda\sigma}$, and that followed from the fact that the theory has an action principle. Many other systems therefore have the same basic stability theorem, that instablility sets in through a zero frequency mode: see for example Carter & Quintana (1972) for a discussion of elastic media in general relativity.

A SIMPLE APPROACH TO THE RADIATION INSTABILITY

Now that we have seen that the radiation-excited instability of the Maclaurin spheroids is in fact a general feature of relativistic systems, we should expect to be able to find some very general and fundamental way of understanding it even in a Newtonian context. The conserved quantities of the Newtonian problem prove to be the key to such an understanding.

The equations for a perturbation of a rotating Newtonian perfect fluid, when written in their Lagrangian form (Lynden-Bell & Ostriker 1967), follow from a variational principle. From the action we can derive, via Noether's theorem, two interesting conserved quantities: the canonical energy E_c (the value of the Hamiltonian) and the canonical angular momentum J_c. Both are conserved if the unperturbed star is stationary and axisymmetric. Both are quadratic in the Lagrangian displacement vector χ^i. (The fact that they are quadratic might suggest that they are the second-order changes in the energy and angular momentum of the system. In fact, however, they are only *part* of these changes. We will examine this in more detail in the next section.)

Conserved quantities for wave fields

These conserved quantities have such useful properties that it is worthwhile studying them in some detail. Let us consider *any* dynamical system for a field χ^i whose equations follow from the unconstrained variations of a Lagrangian of the form

$$L = \tfrac{1}{2}\,\chi^i{}_{,t}\,A_{ij}(\chi^j{}_{,t}) + \tfrac{1}{2}\,\chi^i{}_{,t}\,B_{ij}(\chi^j) - \tfrac{1}{2}\,\chi^i C_{ij}(\chi^j), \tag{40}$$

where A_{ij} and C_{ij} are selfadjoint operators, B_{ij} is antiselfadjoint, and all are independent of time t and azimuthal angle ϕ. These properties will hold for the perturbations of essentially any conservative, stationary, and axisymmetric system. If the field is a tensor rather than a vector, similar properties will still obtain if we interpret i and j as multi-indices.

The field equations that follow are

$$0 = A_{ij}(\chi^j{}_{,tt}) + B_{ij}(\chi^j{}_{,t}) + C_{ij}(\chi^j). \tag{41}$$

The conserved energy is (for possibly complex solutions, such as normal modes)

$$E_c = \tfrac{1}{2}\int[\chi^*_{i,t}A^i{}_j(\chi^j{}_{,t}) + \chi^*_i C^i{}_j(\chi^j)]\,d^3x, \tag{42}$$

while the conserved angular momentum is

$$J_c = -\mathrm{Re}\int\chi^*_{i,\phi}[A^i{}_j(\chi^j) + \tfrac{1}{2}\,B^i{}_j(\chi^j)]\,d^3x. \tag{43}$$

Both of these are closely related to and derivable from a simpler conserved quadratic form, which I shall call the *symplectic form*: given any two independent fields χ^i and ψ^i, we define

$$W(\chi,\psi) = \int\{\psi^*_i[A^i{}_j(\chi^j_t) + \tfrac{1}{2}\,B^i{}_j(\chi^j)] - \chi^*_i[A^i{}_j(\psi^j_t) + \tfrac{1}{2}\,B^i{}_j(\psi^j)]\}\,d^3x. \tag{44}$$

Notice that the terms in square brackets in this equation are the canonical momenta conjugate to χ^i and ψ^i, respectively, so that this is just the antisymmetric product of the canonical coordinates and momenta. It obviously has a close relation to the Poisson bracket, and so we should not be surprised if it is also related to the conservation laws. (This relationship is discussed in more detail in Schutz 1980c.) In fact, if both χ^i and ψ^i satisfy the dynamical equation, then their symplectic product is conserved. Moreover, if χ^i satisfies the dynamical equation, then so do its derivatives $\chi^i{}_{,t}$ and $\chi^i{}_{,\phi}$, and the associated conserved quantities are none other than E_c and J_c:

$$E_c = \tfrac{1}{2}\,W(\chi_{,t},\chi), \qquad J_c = -\tfrac{1}{2}\,W(\chi_{,\phi},\chi). \tag{45}$$

Now we come to the fundamental result we have been building up to. Since a normal mode solution is proportional to $\exp(im\phi+i\omega t)$, the derivatives in Eq.(45) are trivial, and we immediately find that for any normal modes,

$$mE_c = -\omega J_c. \tag{46}$$

This has a number of consequences: (i) If ω is complex, then the fact that m, E_c and J_c are real means that $E_c = J_c = 0$. (ii) If ω is real, then consequence (i) and a continuity assumption imply that a mode is marginally stable if and only if $W(\chi^*,\chi) = 0$. This is an *intrinsic* characterization of marginal stability, and is as close as we have come to a stability criterion for rotating Newtonian stars. But since it is mode-based, it is not very useful. This criterion for marginal stability is arrived at by very different methods in Schutz (1980b). (iii) For nonaxisymmetric modes, where $m \neq 0$, we have a relation that illustrates the importance of the pattern speed:

$$E_c = \omega_p J_c. \tag{47}$$

This is a general property of linear waves, be they stellar perturbations or gravitational waves.

Mechanism for the gravitational wave instability

This equation will be the basis of our understanding of the way the gravitational wave instability develops along a sequence of stellar models. If we remind ourselves that J_c is at least part of the second-order change in the angular momentum of the star, then it will perhaps not be hard to believe that the sign of J_c will be determined by the relative rotation rates of the star and the wave's pattern: if the mode goes faster than the star then it has positive angular momentum, and conversely if it is slower then it has negative angular momentum. (This is not exactly true, but becomes more and more accurate as m gets larger. Rigorous bounds on the range of pattern speeds in which J_c can change sign are given in Friedman & Schutz 1978b.) On the other hand, the relative sign of E_c and J_c is determined by the sign of ω_p, the mode's pattern speed in the inertial frame. If we denote the star's angular velocity by $\Omega > 0$ (in differentially rotating stars, this should be taken to be the *mean* of the angular velocity over the mode's eigenfunction), we have the following table of signs:

	J_c	E_c
$\omega_p > \Omega > 0$	+	+
$\Omega > \omega_p > 0$	-	-
$\Omega > 0 > \omega_p$	-	+

The crucial entry is the one where E_c is negative: *if a mode rotates forwards in the inertial frame and backwards relative to the star, then its canonical energy will be negative.* This is exactly the region of instability of the gravitational wave excited modes of the Maclaurin spheroids, as in Fig.5. Why does this signal an instability of a mode when it is coupled to radiation, but not necessarily when it obeys simply the Newtonian equations? The answer is that since the Newtonian equations preserve E_c, its sign is not automatically an indication of stability: it is necessary that the symplectic form change sign, not the energy. But if the Newtonian system is coupled to another system in such a way that E_c must decrease with time, then if E_c is already negative its absolute value will increase, and the coupling will make the mode unstable. This is excactly what happens when the mode is coupled to gravitational radiation. Gravitational waves also obey equations like (41) and (47), and far from the star their physical energy *is* their E_c. So conservation of total E_c and an outgoing-wave boundary condition ensure that the Newtonian E_c will decrease.

Gravitational wave instability as a two-stream instability

Readers familiar with hydrodynamical instabilities may recognize this instability mechanism, for this is just a version of the *two-stream instability*. Energy arguments like these were first used by Sturrock (1962) to explain the two-stream instability; the only difference was that there the linear momentum of the mode replaces the angular momentum in our argument. One finds that the unstable modes have pattern speeds intermediate between the speeds of the two fluids. In our case, the two "fluids" are the star and the nonrotating inertial vacuum outside it; when modes exist which rotate at an angular velocity intermediate between the two, the instability is present. Another related instability is the Kelvin-Helmholtz instability, which is the mechanism by which the wind raises the waves on the ocean. This is often discussed in a nonlinear context, but its initial linear development is that of a two-stream instability, with the unstable modes again being intermediate in speed between the two fluids. The ocean waves travel at a fixed speed. As long as the wind moves faster than this, the ocean waves will be unstable and grow.

This analogy can help us to understand at least one reason why all rotating stars are unstable. The waves in a rotating star will, in the short-wavelength limit, move at the speed of sound relative to the fluid of the star. If the star rotates "supersonically," so that in some region even the backward-going sound waves will go forward in the inertial frame, then it will be unstable. But all compressible perfect fluid stellar models have a speed of sound that goes to zero at the surface, so any star will be unstable if the surface has any finite angular velocity. Another way of seeing that we should expect this instability in all rotating stars is the observation we made in our discussion of the p-modes of nonrotating stars, that there are p-modes of arbitrarily small pattern speeds for sufficiently large m. If a star rotates slowly, then the p-modes will have the same pattern speeds relative to the star, and so these very slowly moving ones will go unstable.

Other ways of exciting the instability

Our explanation for this instability used only one property of gravitational radiation, that it causes E_c to decrease. Therefore it can be excited by any mechanism that does the same, and in particular by any coherent form of radiation. *Magnetic fields* in stars will emit electromagnetic radiation when the star is perturbed, but one can show that the energy radiated in electromagnetic waves is always smaller then that in gravitational waves, essentially because the magnetic field energy is always less than (usually much less than) the gravitational potential energy. Therefore this does not seem to be an important mechanism except where gravitational radiation is absent, as in the $l=1$ (dipole) modes of the star. This has not received any attention, to my knowledge. But a mechanism that might indeed be important is *acoustic radiation*. Imagine a collapsing stellar core surrounded by an envelope that is collapsing on a much longer time-scale. Then nonaxisymmetric perturbations of the core will "stir" the envelope and this will extract energy from the modes of the core. As long as the core rotates rapidly compared to the envelope (a condition likely to be easy to satisfy in a rotating collapse), then the energy argument will parallel the one for gravitational radiation. I have made a crude estimate of the size of this effect, and it may dominate the one due to gravitational radiation near the onset of instability (Schutz 1983).

The instability due to viscosity

Finally, we may ask about viscosity. In the Maclaurin spheroids, it does not have at all the same effect as gravitational radiation. Do our energy arguments help us here? Friedman & Schutz (1978b) have discussed this at some length, but the essence of the result is that viscosity dissipates energy measured in the rest frame of the star rather than in the inertial frame. It has no systematic effect on E_c, but rather on its analogue in the rotating frame. We shall see why in the next section. This fact can be used to devise as systematic a description of the viscosity secular instability as we have given of the one due to gravitational radiation.

THE PERTURBED ENERGY OF A ROTATING SYSTEM

In the previous section we used the conserved canonical energy to build stability criteria for Newtonian stars, while cautioning that it is not the whole of the second-order change in the energy of a rotating star. Indeed, our remark at the end about the effect of viscosity would not make sense if the canonical energy in the inertial frame were the total energy, since surely viscosity dissipates energy. But if E_c is not the total energy, what is the missing piece? And if E_c and the total energy are both conserved, then so must be their difference, so is there yet another conserved quantity in the problem that we have overlooked until now? In this section I shall address these questions directly. The best way to start is with the simplest system that exhibits these features: the energy of a particle in a perturbed, nearly circular orbit in a central force.

Orbiting particle: an elementary example

Consider a particle in a central potential $V(r)$ in an orbit with angular momentum J. Its energy is

$$E = \tfrac{1}{2}\, m\dot{r}^2 + J^2/2mr^2 + V(r), \tag{48}$$

which is a nonlinear function of r. The solution which is stationary in r is the circular orbit, which we will take to be the unperturbed state. For this we have

$$\dot{r} = 0 \quad \text{and} \quad \left.\frac{\partial E}{\partial r}\right|_{J=\text{const}} = 0. \tag{49}$$

In other words, a stationary solution is an extremum of the energy provided the angular momentum is held constant. Now suppose that the orbit is perturbed, but remains in the same plane for simplicity. The first-order change in the energy is

$$\delta E = \frac{\partial E}{\partial \dot{r}}\,\delta\dot{r} + \frac{\partial E}{\partial r}\,\delta r + \frac{\partial E}{\partial J}\,\delta J. \tag{50}$$

But the first two terms vanish because of Eq.(49), and it is easy to see that the third term is just

$$\delta E = \Omega\, \delta J, \tag{51}$$

where Ω is the angular velocity of the unperturbed orbit. Physically, it is clear why there *has* to be a term of this type: part of the unperturbed energy is kinetic, and it is possible to change the kinetic energy to *first* order by changing the component of the velocity *along* the unperturbed velocity. Notice that this change is conserved: we expect energy to be conserved at all orders, and at this order this follows from angular-momentum conservation.

The second-order change in the energy is simplest if we set the first-order change δJ to zero:

$$\delta^2 E = \Omega\, \delta^2 J + \tfrac{1}{2}\, m(\delta\dot{r})^2 + \tfrac{1}{2}\,[V''(r) + 3J^2/mr^4](\delta r)^2. \tag{52}$$

The first term is the second-order analogue of Eq.(51), allowing for a second-order change in J. The remaining terms are the Hamiltonian of the radial oscillation that the particle's first-order perturbed motion undergoes. When $V(r)$ is the Kepler potential, then evaluating the coefficient of $(\delta r)^2$ gives an oscillation frequency exactly equal to Ω: the perturbed orbit is (of course) closed. In general this frequency is called the epicyclic frequency.

So here we see that even in the simplest example, the total energy at second order is not the Hamiltonian of the second-order motion. There is another piece, related to the second-order change in the angular momentum. Both pieces are separately conserved if the motion obeys the original dynamical equations.

The second-order energy of a rotating fluid

The energy of a fluid, perturbed about a stationary, differentially rotating state, has a strong analogy with Eqs.(51) and (52), but there are important differences that arise because the system is a continuum. To first order the change is (Schutz & Sorkin 1977)

$$\delta E = \int [nT\Delta S + \rho\Omega\Delta j + g^{-1/2}\mu\Delta(ng^{1/2})]d^3x, \tag{53}$$

where j is the specific angular momentum of the fluid, μ is the injection energy per particle ($\mu = \frac{1}{2}v^2 + h + V$, where h is the enthalpy), and Δ is the usual Lagrangian change. Each of these terms has a ready interpretation: the first is the energy change if we add heat; the second is the kinetic energy change, as in the particle analogy above; and the third is the energy added if we add particles. It is significant that the kinetic energy term involves the specific angular momentum, because it is possible to show that this term is *conserved* by virtue of the vorticity-conservation law of the perfect fluid (Schutz 1984). It is obvious that the evolution equations for a perfect fluid also preserve each of the other two terms separately as well, so the analogy with Eq.(51) is very good.

If we set the first-order changes ΔS, Δj, and of the vorticity to zero, then we have what we have earlier defined to be a Lagrangian perturbation of the fluid, and δE vanishes. Then the second-order change in E turns out to be (Friedman & Schutz 1978a, Schutz 1984)

$$\delta^2 E = \int [nT\Delta^2 S + \rho\Omega\Delta^2 j + g^{-1/2}\mu\Delta^2(ng^{1/2})]d^3x + E_C. \tag{54}$$

The first group of terms is just the analogue at second order of Eq.(53), and the remaining term is the canonical energy, the Hamiltonian of the first-order perturbation equations. This is very closely analogous to Eq.(52). The difference between E_c and $\delta^2 E$ is indeed a separately conserved quantity, provided the perfect-fluid dynamical equations are satisfied.

None of this is absolutely necessary for computing the gravitational wave instability, but it does illuminate the viscosity instability. Viscosity does not conserve vorticity, so the second term in Eq.(54) is not constant. The total $\delta^2 E$ must decrease, but it can (and does) sometimes happen that the vorticity term decreases faster, allowing E_c to increase. A further investigation of the second-order changes in J reveals that the canonical energy in the rotating frame of the star monotonically decreases under the action of viscosity, and that this happens when a forward-going mode starts going backwards in the rotating frame. (It is possible to speak of a single rotating frame here, because we must exclude the possibility that the unperturbed star is differentially rotating. In the presence of viscosity, only a rigidly rotating star can be stationary.)

MAXIMUM ROTATION RATE OF NEUTRON STARS

The most immediate consequence of the gravitational radiation instability is that no star can rotate faster than whatever rate would render it unstable on a sufficiently short timescale. This limit is set by the largest value of m for which a mode grows on an interesting timescale (such as the age of the universe), since the larger values of m are unstable in the more slowly rotating stars but have the longer growth times. Unfortunately, this is a difficult calculation to make, since it involves knowing the normal modes of the star, their growth rates when coupled to gravitational radiation, and the size of the damping effect of viscosity. For main sequence stars and white dwarfs, the gravitational radiation timescales are too long to be of interest. But for neutron stars it seems that even modes as high as $m = 4$ or 5 may be important. Friedman (1983) estimated the likely effect of viscosity in neutron stars and concluded that the $m = 4$ mode would set the limit on rotation, since the $m = 5$ mode would be damped by viscosity. He used the Maclaurin spheroids to estimate the various timescales. Based on these estimates, Friedman concluded that the existence of the millisecond pulsar PSR 1937+214 with a rotation rate of some 642 Hz might already rule out the stiffer equations of state if the star has a baryon mass of 1.4 $M_\odot$.

Realistic models of rotating neutron stars, using the same equations of state as Arnett & Bowers (1977) used, have recently been calculated by Friedman, *et al* (1984). They concluded again that the stiffest equations of state were on the verge of being ruled out if the millisecond pulsar is a 1.4 $M_\odot$ star. Conversely, if the millisecond pulsar's rotation rate is limited by this instability, then the equation of state must be fairly stiff.

In a very recent paper, Lindblom (1986) has taken a new look at the relative importance of gravitational radiation and viscosity in realistic stars rather than in the Maclaurin spheroids, and concluded that Friedman (1983) may have overestimated the effect of viscosity, and that consequently the rotation limit may in fact be set by the $m = 5$ mode. This significantly lowers the critical rotation rate and rules out the two stiffest equations of state used by Arnett & Bowers (1977), again provided that the millisecond pulsar has a baryon mass of 1.4 $M_\odot$. Figure 6 shows the result of his calculations for various assumed kinematic viscosities ν, in units of cm^2s^{-1}. But these results must also be treated with some caution, because the stellar models which he used were nonrotating, and rotation may make some changes in one's estimates of the critical quantities. We need full calculations of the normal modes of realistic rotating relativistic stars to answer these questions, and we don't have any yet. This is probably the most important untouched problem in this subject today. Not only would it help us to constrain equations of state, but such calculations would be a useful testbed for comparison with the results of full nonlinear three-dimensional hydrodynamics codes in general relativity, which will certainly need such comparison problems to ensure that their results are reliable.

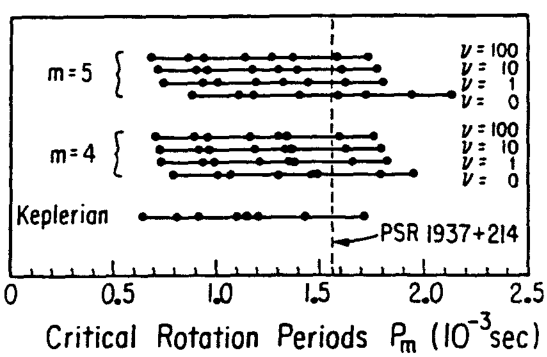

Fig. 6. Critical rotation periods of neutron stars with various kinematic viscosities (ν) and various equations of state (dots) for the $m = 4$ and 5 modes. The millisecond pulsar (dashed line) appears to rule out the stiffest equations of state. (From Lindblom 1986.)

STABILITY OF THE KERR BLACK HOLE

Just as the stability of rotating stars is much harder to analyze than that of nonrotating ones, so is the stability of Kerr more difficult than that of Schwarzschild. There is an unpublished calculation by Whiting (1985) that is reliably said to establish that all the normal modes of Kerr are stable, but I have not seen it. However, because there is some uncertainty even for Schwarzschild about whether the modes are complete (or even finite in number for any l), the stability of the modes of Kerr does not establish the stability of the metric itself. This reinforces the importance of studying the completeness problem for radiating systems.

The fullest discussion of this problem in the literature is by Chandrasekhar (1983). The analysis of Leaver (1986a,b) has made a substantial advance, and gives hope of further important progress soon. I have elsewhere given a brief introduction to the mode problem from a point of view analogous to that which I took earlier in this article for Schwarzschild (Schutz 1984). There is no space here for a full discussion of this interesting and complex problem, but I should not leave it without drawing attention to its relation to the rotating star problem. The possibility of instability in Kerr comes from the negative-energy nonaxisymmetric modes of wave fields that must exist because of the ergosphere. (The axisymmetric modes of Kerr are known to be stable: Friedman & Schutz 1973.) When a star has an ergosphere (Schutz & Comins 1978), these modes *do* result in an instability, for the same reason as in the star: as they lose energy to infinity, their already negative energy must get more negative, hence larger in absolute value (Friedman 1978b, Comins & Schutz 1978). But the boundary conditions for the Kerr problem are different: the ingoing waves at the horizon have, for these negative-energy modes, an *outward* energy flux, which in all calculated modes seems to more than compensate for the energy radiated to infinity, and allows the mode amplitude to decrease. The question is, does this happen for every wave disturbance of Kerr? Nobody yet knows, but all the evidence is that it does.

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