Statistical formulation of gravitational radiation reaction

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A new formulation of the radiation-reaction problem is proposed, which is simpler than alternatives which have been used before. The new approach is based on the initial-value problem, uses approximations which need be uniformly valid only in compact regions of spacetime, and makes no time-asymmetric assumptions (no *a priori* introduction of retarded potentials or outgoing-wave asymptotic conditions). It defines radiation reaction to be the expected evolution of a source obtained by averaging over a statistical ensemble of initial conditions. The ensemble is chosen to reflect one's complete lack of information (in real systems) about the initial data for the radiation field. The approach is applied to the simple case of a weak-field, slow-motion source in general relativity, where it yields the usual expressions for radiation reaction when the gauge is chosen properly. There is a discussion of gauge freedom, and another of the necessity of taking into account reaction corrections to the particle-conservation equation. The analogy with the second law of thermodynamics is very close, and suggests that the electromagnetic and thermodynamic arrows of time are the same. Because the formulation is based on the usual initial-value problem, it has no spurious "runaway" solutions.

I. WHAT IS RADIATION REACTION?

A. Introduction

The need for an acceptable formulation for radiation reaction in general relativity has been sharpened recently by the identification of the effects of radiation reaction on the orbit of the binary pulsar PSR 1913+16.[1] The observations support the usual quadrupole energy-loss formula[2] [Eq. (4) below], but the theoretical underpinnings of that formula and of a related expression for the reaction force[3] [Eq. (8) below] are not entirely satisfactory.[4] There has even been a report of calculations which disagree with the quadrupole formula for high-speed collisions.[5] I will describe here a statistical approach to the problem, which is mathematically and conceptually simpler than previous derivations, and which also is closer to the nature of the physical problem. In the simplest possible limit—a low-velocity system dominated by nongravitational forces—the standard expressions will be shown to give the statistically most likely evolution of the system. As with the statistical derivation of the second law of thermodynamics, the present approach has the nice feature that it makes no explicitly time-asymmetric assumptions: neither retarded potentials nor time-asymmetric asymptotic wave conditions are put in as *a priori* assumptions. Since the present approach applies as well to electromagnetic radiation reaction, it provides a mathematical basis for the frequently expressed belief[6] that the electromagnetic and thermodynamic arrows of time are in fact the same.

It is useful to begin by defining radiation reaction carefully. The interaction of a body with radiation must produce some effect on it, but the term "radiation reaction" is usually applied only to a formulation which meets the following three requirements.

(i) It separates out the "self-field" of the body (that field of which the body is actually the source) from the total radiation field, and describes the interaction of the body only with its self-field.

(ii) The equations of radiation reaction do not contain the self-field explicitly at all: they involve only the body's own dynamical variables. The field is then regarded only as an intermediary between certain motions of the body and the reactions produced by them.

(iii) The expressions must be much simpler to use than the original coupled equations between the body and the field. This necessarily entails approximation: one hopes to identify the most important characteristics of the body's motion that produce reaction. One would like these approximations to be local, involving only the present condition of the body and not its history in the remote past, but this may not always be possible.

B. The standard expressions

In electromagnetism[7] these requirements are all met by the Abraham-Lorentz equation of motion (for a body of charge Q and small size)

$$m\frac{d\vec{v}}{dt}=\tfrac{2}{3}Q^2\frac{d^2\vec{v}}{dt^2}+\vec{F}_{\rm external}\,, \tag{1}$$

where the radiation-reaction term is the first on the right-hand side (RHS). The energy loss to radiation is the work done on the body by the reaction force

$$\frac{dE}{dt}=\frac{2}{3}Q^2\vec{v}\cdot\frac{d^2\vec{v}}{dt^2} \quad (2)$$

whose average over time, denoted by $\langle\ \rangle$, for a roughly periodic motion is the usual dipole formula[7]

$$\left\langle\frac{dE}{dt}\right\rangle=-\frac{2}{3}Q^2\left\langle\frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}\right\rangle. \quad (3)$$

This is then the *dipole approximation* for electromagnetic radiation reaction. In weak-field general relativity the analog of Eq. (3) is the quadrupole formula

$$\langle dE/dt\rangle=-\frac{1}{5}\langle {}^{(3)}\text{I}^{jk}\,{}^{(3)}\text{I}_{jk}\rangle \quad (4)$$

in which

$$\text{I}_{jk}=I_{jk}-\frac{1}{3}I^l{}_l\delta_{jk}, \quad (5)$$

$$I_{jk}=\int T^{00}x_jx_kd^3x, \quad (6)$$

and ${}^{(3)}I$ means the third time derivative of I.[8] The analog of Eq. (1) was suggested rather recently.[3] It is an addition to the usual Newtonian gravitational potential of the term

$$\Phi_{\text{react}}=\frac{1}{5}x_jx_k\,{}^{(5)}\text{I}^{jk}, \quad (7)$$

which produces a body force

$$F^j_{\text{react}}=-\rho\nabla^j\Phi_{\text{react}}=-\frac{2}{5}x_k\,{}^{(5)}\text{I}^{jk}. \quad (8)$$

This is consistent with (4) via the analog of (2)

$$\langle dE/dt\rangle=\left\langle\int v_jF^j_{\text{react}}d^3x\right\rangle. \quad (9)$$

We will see later that this consistency bears closer examination. It turns out that a very special choice of gauge is required in the relativistic problem in order that the nonrelativistic form of Eq. (9) should give the correct energy loss. We will also see that in some applications it may be necessary to take into account a reactive correction to the particle-conservation law as a supplement to Eq. (8).

II. THE STATISTICAL DEFINITION OF RADIATION REACTION

A. How to define the self-field

The first requirement of a radiation-reaction formalism is the separation of the self-field from the total field. The differential equations do not define such a separation. The self-field must satisfy an inhomogeneous equation [e.g., Eq. (12) below], but such a solution is not unique: any solution of the associated homogeneous equation (a "free field") can be added to it. In order to define one particular solution as *the* self-field one must add some other condition. The choice preferred by Ehlers *et al.*[4] is that the solution have no incoming radiation from $\mathscr{I}^-$, past null infinity.[9] This is a mathematical way of requiring the source to be "causally isolated." Another choice, more commonly used[3] because it is easier to impose, is that the self-field solution have only outgoing radiation far from the source. This amounts to a definition of a "retarded" solution to the nonlinear equations. It is not known whether it is equivalent to the zero-incoming-radiation condition. An alternative, initial-value approach will be adopted here. The self-field is defined only in a statistical sense, and the radiation-reaction equations are those which give the expected evolution of the source under random initial conditions for the radiation field.

Because the universe appears to have a finite age, all radiating systems we want to study have an initial moment of formation. For an antenna broadcasting radio waves this is the moment the power is switched on; for the binary pulsar this might be the time the pulsar itself was formed in a supernova explosion, or it might be the original formation of the binary system from the collapse and fragmentation of a gas cloud. The system's subsequent evolution is determined by the initial data for both the field and the "matter" dynamical variables. From theory or observations, we may have a good idea about the matter variables at the initial moment, but it is generally true that we know nothing about the radiation field then. What we are confronted with is an incompletely posed initial-value problem. Rather than to assume a particular kind of initial field, it seems to me more reasonable to *average* over all possible free data for the field.

Consider again the same system when it is first "discovered," as when the binary pulsar was first detected (by a method having nothing to do with gravitational radiation).[10] We again have information about the system's variables—the initial positions and velocities of the stars—but not about the initial gravitational wave field. Again, if we want to predict its evolution we are forced to average over the initial data for the field.

In each case the averaging expresses the assumption that the initial data for the radiation is uncorrelated with the subsequent motion of the system. This assumption may be questioned in the second case, on the grounds that the initial data should contain outgoing waves emitted earlier. But these have no subsequent effect on the system,[11] so their omission causes no practical problem. In the first case the initial-value problem for the moment of formation of the system does not have these outgoing waves.

It must be stressed that this definition of the self-field as the average field will generally be inequivalent to the definition which holds that it is the zero-incoming radiation field, although one would hope that the observable predictions of the two approaches might be very close. The no-incoming-radiation approach can justify its relevance to real problems, in which there certainly is incoming radiation from all sorts of sources, only by assuming that this radiation is uncorrelated with the motion of the system of interest and so will have no effect on its motion, on average. It seems preferable to make this assumption explicit by *defining* radiation reaction to be the average motion. This is particularly useful in view of the difficulty of defining incoming radiation in our universe at all. The usual definition is that the initial data on $\mathscr{I}^-$ should vanish. But our universe does not have a $\mathscr{I}^-$: the big-bang singularity intercepts every null geodesic at a finite affine-parameter distance in the past.

The alternative definition of the self-field, which demands only outgoing radiation, is equally open to criticism. Apart from the unjustified assumption that nature "prefers" outgoing to incoming radiation, it also seems to assume that the source somehow knows what part of the local field will eventually show up as radiation far away. Now, the nature of the radiation far away depends to some extent on the asymptotic structure of spacetime, but the source must react to the radiation it emits (more properly, to its self-field) essentially immediately. It cannot wait to discover whether the self-field actually reaches a distant observer as radiation or has some other fate, such as being swallowed by a nearby black hole.

This highlights a semantic problem that can be the source of some difficulty when one first studies the subject: "radiation reaction" ought to be called "self-field reaction" and has nothing directly to do with the radiation a distant observer might detect. Global conservation laws ensure that the energy lost by the source will turn up elsewhere, but not necessarily as radiation. The local radiation-reaction force exists independently of this asymptotic conservation property, and it seems preferable to derive it in a manner independent of any asymptotic properties of spacetime.

B. The ensemble of initial data

The statistical approach makes no explicitly time-asymmetric assumptions and imposes few global conditions, but it does require the prescription of a rule for averaging the initial data, i.e., the assignment of a probability distribution to the initial data. In turn, this depends on what information is taken as given. We have already addressed one aspect of this: we will not assume the initial data to contain waves emitted earlier by the system. This still leaves the choice of the distribution function open. For linear problems (including weak gravitational waves on a curved background), one plausible restriction suffices: any freely specifiable initial datum for the radiation field and its negative should have equal probability. This ensures that the mean free initial data are zero and that the mean evolution of this system solves the initial-value problem for these zero data. This assumption will not suffice for nonlinear problems, in which the subsequent evolution of the two sets of data will not necessarily cancel. In such a case one will have to make some added constraint on the data, such as a requirement that its averaged effect on the background curvature is less than some fixed limit. Note also that in the nonlinear problem, the mean motion resulting from an ensemble of initial data may not itself be a solution of the equations for any particular data. In such a case, the present approach must give rather different results from those which impose specific asymptotic conditions and identify radiation reaction as a particular solution.

These considerations have analogs in statistical mechanics, where the derivation of the second law of thermodynamics has obvious analogies with the statistical approach to radiation reaction. The second problem, of constraining the initial data, is solved by requiring a fixed total energy (and angular momentum, for a rotating thermodynamic system). The first problem, of deciding what information is known and what is random, is usually answered by pleading complete ignorance of the microscopic motions. This is perhaps a little naive, since it could be argued that for a system which has been isolated for some time we also know (from experience with other systems) that its entropy was lower in the past than it is now. This implies a certain correlation among the particles' positions and velocities, a microscopic correlation not evident in the macroscopic state. I will return to this point in Sec. V below, pointing out that ignoring these correlations is the analog of ignoring the outgoing radiation generated earlier by our system in the present problem. In statistical mechanics, these correlations are probably truly ignorable, since they are presumably wiped out quickly by collisions. In the radiation problem the information is always present in the outgoing radiation. This difference is one reason for being somewhat cautious in drawing the analogy between the two problems.

There is another important difference between the two problems: Our one constraint on the initial data for the radiation field is not enough to determine the probability of any initial data in the ensemble. This means that we have no knowledge that the "average" behavior is in fact highly probable. It might turn out to be the average of two much more probable kinds of behavior. This is almost certainly not the case, and is one of the questions one would expect to resolve in a study of the nonlinear problem. The situation in statistical mechanics is more complete, since the imposed constraints determine the probability distribution of microstates more completely.

These differences do not obscure the striking conceptual similarity between the two problems. If the Universe has a thermodynamic "arrow of time" evident in the increase of entropy, then it also has an electromagnetic arrow evident in radiation reaction and the presence of outgoing radiation correlated with earlier motions of the source; these arrows can be derived by analogous statistical arguments.

III. SOLUTION OF THE RANDOM INITIAL-VALUE PROBLEM FOR WEAK FIELDS

A. The weak-field approximation

The definition of radiation reaction described in the previous section can be applied to any situation, but there is one which is the most straightforward. This is one in which nongravitational forces dominate the motion; gravitational effects, both Newtonian and post-Newtonian, are small. (There is reason to believe that our results will apply even when the Newtonian gravitational field dominates the dynamics,[12] and we plan to consider such situations in a later paper.) For now, the weak-field case suffices to illustrate the approach.

The lowest approximation is the linearized theory,[12] with the Lorentz or harmonic gauge condition:

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\,,\quad \bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha}\,, \tag{10}$$

$$\bar{h}^{\mu\nu}{}_{,\nu}=0\,, \tag{11}$$

$$\Box\bar{h}^{\mu\nu}=-16\pi T^{\mu\nu}\quad (\Box=-\partial_t^{\,2}+\nabla^2)\,, \tag{12}$$

$$T^{\mu\nu}{}_{,\nu}=0 \tag{13}$$

with indices raised and lowered by $\eta_{\mu\nu}=\mathrm{diag}(-1, 1, 1, 1)$, the metric of flat spacetime. Commas denote partial derivatives. These equations describe a system whose motion is nongravitational [Eq. (13)]—such as a mass on a spring—and they give the first-order gravitational field this system generates [Eq. (12)]. Effects of this field on the motion of the system appear in the equation of motion at the next order, using the metric generated in (12):

$$T^{\mu\nu}{}_{,\nu}=-\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu}-\Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha}\,, \tag{14}$$

with

$$\Gamma^{\mu}{}_{\alpha\nu}=\tfrac{1}{2}\eta^{\mu\beta}(h_{\beta\alpha,\nu}+h_{\beta\nu,\alpha}-h_{\alpha\nu,\beta})\,. \tag{15}$$

In Eq. (15) one computes $\Gamma^{\mu}{}_{\alpha\nu}$, using the zero-order solution from Eq. (12), and this plus the zero-order $T^{\mu\nu}$ determines the RHS of Eq. (14). From the point of view of a Minkowskian observer, the RHS of Eq. (14) appears to be a new force acting on the matter, the self-interaction force

$$F^{\mu}_{\text{self}}=-\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu}-\Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha}\,. \tag{16}$$

These equations are merely the first of a recursive sequence of approximations to Einstein's nonlinear equations, the convergence of which is poorly understood. But we are not interested in convergence here. All we require at this point is that our calculations be an asymptotic approximation in the limit of small $|h_{\mu\nu}|$. Of this there seems to be little doubt, particularly as we require this approximation to be valid only in a compact region of spacetime.[13]

B. The initial-value problem

The initial data will be set on the initial hypersurface H of coordinate time $t=0$. The data for Eq. (13) or (14) are data for the matter variables, and are assumed arbitrary but fixed for the rest of this calculation. The initial data for Eq. (12) are $\bar{h}_{\mu\nu}(0)$ and $\bar{h}_{\mu\nu,0}(0)$, but these are constrained by the gauge condition, Eq. (11), and by its time derivative, simplified by the use of Eq. (12):

$$0=\bar{h}^{\mu\nu}{}_{,\nu0}=\bar{h}^{\mu0}{}_{,00}+\bar{h}^{\mu i}{}_{,i0}=-\Box\bar{h}^{\mu0}+\nabla^2\bar{h}^{\mu0}+\bar{h}^{\mu i}{}_{,i0}$$

or

$$0=16\pi T^{\mu0}+\nabla^2\bar{h}^{\mu0}+\bar{h}^{\mu i}{}_{,i0}\,. \tag{17}$$

This is, of course, the usual initial-value constraint. Its form makes it natural to take $(\bar{h}^{ij}, \bar{h}^{ij}{}_{,0})$ as the free data on H and to solve Eqs. (11) and (17) for $(\bar{h}^{\mu0}, \bar{h}^{\mu0}{}_{,0})$.

Under our assumptions for linear problems, the free data average to zero, which implies that the average solution to (12) is the solution to (12) which evolves from the data on H:

$$\bar{h}^{\mu0}=-16\pi\nabla^{-2}(T^{\mu0})\,, \tag{18}$$

$$\bar{h}^{i0}{}_{,0}=0\,, \tag{19}$$

$$\bar{h}^{00}{}_{,0}=16\pi\nabla^{-2}(T^{i0}{}_{,i})\,, \tag{20}$$

where ∇^{-2} is the inverse Laplacian for solutions well behaved at infinity. It is not hard to show that these initial data satisfy the criteria of York[19] for the absence of gravitational radiation: The intrinsic geometry of the hypersurface $t=0$ is conformally flat and its extrinsic curvature has zero transverse-tracefree part.

The general solution of Eq. (12) can be written

$$\bar{h}_{\mu\nu}=\bar{h}^{\text{ret}}_{\mu\nu}+\bar{h}^{H}_{\mu\nu}\,. \tag{21}$$

Here $\bar{h}^H_{\mu\nu}$ is a solution of the homogeneous equation

$$\Box\bar{h}^H_{\mu\nu}=0\,, \tag{22}$$

and $\bar{h}^{\rm ret}_{\mu\nu}$ is a retarded solution

$$\bar{h}^{\rm ret}_{\mu\nu}(x^i,t)=4\int_{c(x,t)}\frac{T_{\mu\nu}(y^i,t_{\rm ret})}{R}d^3y\,, \tag{23}$$

where

$$R^i=x^i-y^i,\quad R=(R^iR_i)^{1/2},\quad t_{\rm ret}=t-R$$

and the integral in (23) is over the past light cone $c(x,t)$ of the event (x^i,t). Writing $\bar{h}_{\mu\nu}$ this way with no explicit mention of the advanced solution involves no loss of generality, because the retarded and advanced solutions differ from one another by a homogeneous solution, which can be absorbed by $\bar{h}^H_{\mu\nu}$. We have chosen the form (21) because it enables us to solve the initial-value problem explicitly.

In order to connect with the initial-value problem, we now take a crucial step. We are only interested in $t>0$ in Eq. (23). We can, therefore, again without loss of generality, cut off the integral in (23) at $t=0$. (See Fig. 1.) The resulting $\bar{h}^{\rm ret}_{\mu\nu}$ is still a solution of Eq. (12) for all $t>0$, since we can accomplish the cutoff by multiplying the retarded Green's function $G^{\rm ret}(x^i,t_x;y^i,t_y)$ implicit in Eq. (23) by a Heaviside step function $H(t_y)$. This does not alter the fact that $G^{\rm ret}$ is a Green's function for $t_y>0$, the test for which involves differentiating $G^{\rm ret}$ with respect to x^i and t_x, not y^i or t_y. The advantage that placing such a cutoff on (23) gives us is that now

$$\lim_{t\to 0}\bar{h}^{\rm ret}_{\mu\nu}(x^i,t)=0\,. \tag{24}$$

It follows that the initial data (18)–(20) are the initial data for the homogeneous solution $\bar{h}^H_{\mu\nu}$ and this defines $\bar{h}_{\mu\nu}$ completely. [Had we not cut off the integral, the initial data for $\bar{h}^H_{\mu\nu}$ would have included terms compensating $\bar{h}^{\rm ret}_{\mu\nu}(x^i,0)$, in order to make $\bar{h}_{\mu\nu}$ have the correct values at $t=0$. But $\bar{h}^{\rm ret}_{\mu\nu}(x^i,0)$ cannot be computed directly from the initial data.]

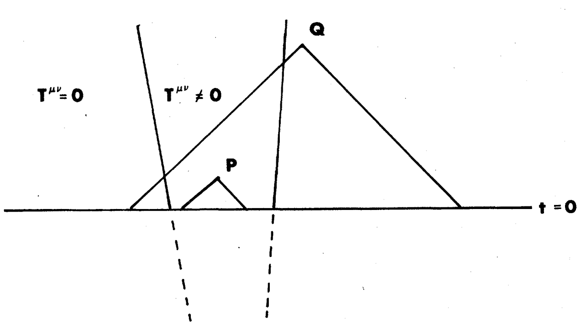

FIG. 1. The integral over the past light cone of event P is cut off at $t=0$, so it is not the usual full retarded integral. The cutoff for the light cone of Q occurs after it passes through the source; in this case it is the usual retarded integral.

It must be stressed that despite the appearance of a retarded Green's function no time-asymmetric assumptions have been made. The solution to the *past* of H would most conveniently be expressed using an advanced Green's function.

We can find $\bar{h}^H_{\mu\nu}(x^i,t)$ from the Kirchoff formula[14] for the wave equation. This gives, for initial data (18)–(20),

$$\bar{h}^H_{ij}(x^k,t)=0, \tag{25}$$

$$\bar{h}^H_{i0}(x^k,t)=-4\frac{\partial}{\partial t}\left[t\oint_{s(t,x)}(\nabla^{-2}T_{i0})d\Omega\right], \tag{26}$$

$$\bar{h}^H_{00}(x^k,t)=-4t\oint_{s(t,x^k)}(\nabla^{-2}T^i{}_{0,i})d\Omega - 4\frac{\partial}{\partial t}\left[t\oint_{s(t,x^k)}(\nabla^{-2}T_{00})d\Omega\right], \tag{27}$$

where the integrals are over the sphere $s(t,x^k)$ in the initial hypersurface H of radius t centered at x^k. These spheres are illustrated in Fig. 2. The integrals are most easily evaluated[15] by using the spherical-harmonic representation of ∇^{-2}:

$$(\nabla^{-2}f)(x^k)=-\frac{1}{4\pi}\int d^3y\frac{f(y^k)}{[(x^j-y^j)(x_j-y_j)]^{1/2}}$$
$$=-\sum_{l,m}\frac{1}{2l+1}Y_{m}(\theta,\phi)\int d^3y\frac{r_<^{\,l}}{r_>^{\,l+1}}Y^*_{lm}(\psi,\chi)f(y^k)\,,$$

where ψ and χ are the spherical coordinates associated with y^k and where $(r_>,r_<)$ are the (larger, smaller) of $|x^jx_j|^{1/2}$ and $|y^jy_j|^{1/2}$. When this expression is integrated over a sphere of radius t centered at $x=0$, only the $l=m=0$ piece survives, and one finds

$$\oint_{s(t,0)}(\nabla^{-2}f)d\Omega=-\int d^3y\frac{f(y^k)}{r_>}. \tag{28}$$

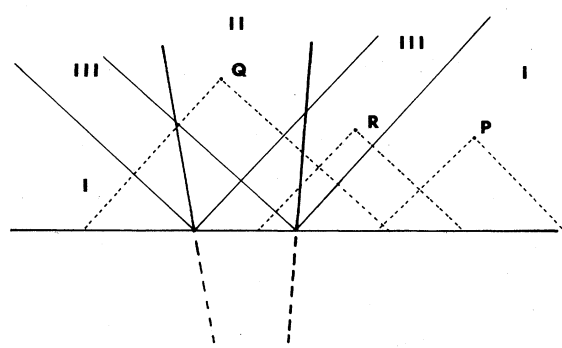

Fig. 2. Spacetime forward of $t=0$ is divided into three regions: (I) the outer region consists of events spacelike separated from all events at $t=0$ inside the source; (III) the inner region is all events timelike separated from all events at $t=0$ inside the source; and (II) the middle region is everything between I and III. The homogeneous field is simple at events like P and Q but complicated at R.

If $f(y^k)$ is of compact support, there are two easy cases. The first is when the support of f is entirely outside the sphere. Then $r_>$ is just $|y^k y_k|^{1/2}$ and Eq. (28) is independent of t. For this case, Eqs. (26) and (27) become

$$\bar{h}^H_{i0}=4\int d^3y\frac{T_{i0}(y^k)}{|y|}\quad \text{with}\ |y|\equiv|y^k y_k|^{1/2},$$

$$\bar{h}^H_{00}=4\int d^3y\frac{T_{00}(y^k)}{|y|}+4t\int d^3y\frac{T^i{}_{0,i}(y^k)}{|y|}.$$

Since the origin of x^k was arbitrary, we can generalize these to arbitrary position as

$$\bar{h}^H_{i0}(x^k,t)=4\int d^3y\frac{T_{i0}(y^k)}{|x-y|}$$

$$\text{with}\ |x-y|\equiv[(x^k-y^k)(x_k-y_k)]^{1/2}, \quad (29)$$

$$\bar{h}^H_{00}(x^k,t)=4\int d^3y\frac{T_{00}(y^k)}{|x-y|}+4t\int d^3y\frac{T^i{}_{0,i}(y^k)}{|x-y|}. \quad (30)$$

These integrals are evaluated in the initial hypersurface and are consequently independent of t. This gives the metric in the outer region of Fig. 2, at events which have not been able to communicate with the source region since $t=0$.

The second easy case is the metric in the interior region, at events which are future timelike separated from *all* points in the source at $t=0$. In this case, the support of f in Eq. (28) is entirely inside the sphere, so that $r_>$ is just $|x^k x_k|^{1/2}=t$. Then Eqs. (26) and (27) become

$$\bar{h}^H_{i0}=0\,, \quad (31)$$

$$\bar{h}^H_{00}=4\int T^i{}_{0,i}(y^k)d^3y=0. \quad (32)$$

The last equality results simply from integrating a divergence. So in this region, which is the principal one of interest for the motion of the source, the homogeneous solution vanishes identically.

C. Interpretation of the solution

The two limiting forms of $\bar{h}^H_{\mu\nu}$ given by Eqs. (29)–(32) have a sensible physical interpretation. Since we have eliminated any dynamical freedom in $\bar{h}^H_{\mu\nu}$ by our choice of initial conditions, the only information it can be expected to carry is the Coulomb field of the source. In the outer region where (29) and (30) are valid, the retarded field is identically zero (since the retarded integral is cut off at $t=0$), so the information about the mass and momentum of the source must be contained in the homogeneous solution. This is assured by the initial-value constraints, whose solution (29) and (30) may be expanded in powers of $r^{-1}\equiv|x|^{-1}$ to give

$$\bar{h}^H_{0i}(x^k,t)=-4P_i/r+O(r^{-2}),\quad P_i=\int T^0{}_i d^3y \quad (33)$$

$$\bar{h}^H_{00}(x^k,t)=4M/r+4tx^j/r^3\int T^i{}_{0,i}(y^k)y_j d^3y+O(r^{-3}),$$

$$M=\int T^{00}d^3y\,. \quad (34)$$

This is in a coordinate system whose origin is at the center of mass of the initial data [i.e., $\int T^{00}(y^k)y^i d^3y=0$]. The integral in the $O(r^{-2})$ term in (34) can be evaluated by parts to give

$$\bar{h}^H_{00}(x^k,t)=4M/r+4tP^j x_j/r^3+O(r^{-3})\,. \quad (35)$$

The r^{-1} terms in (33) and (34) give the mass and momentum of the source correctly, and the linear term in t in (35) is exactly the right correction at this order for the motion of the source. That is, if $P^j\neq 0$ then at fixed x the Coulomb field far away should change with time to reflect the changing distance to the source. This is automatically contained in (29) and (30).

But if we wait at fixed x for a long enough time, then the retarded integral eventually begins to include some of the source. After a sufficiently long time, we pass into the "interior" region of Fig. 2 in which (31) and (32) apply. In this region the retarded integral "sees" the whole of the source, so it can contain the information about the mass and momentum of the source. The homogeneous field is not needed, and obligingly vanishes.

It follows that, after one light-crossing time, the field inside the source is purely the retarded field, and from this will result all the radiation-reaction effects. We have therefore recovered, from unbiased initial data, the usual retarded solution which other approaches put in by hand. There has been no loss of time symmetry, however. At times earlier than $t=0$, the solution inside the source would be governed by the *advanced* solution. This result is not as disturbing as it might at first seem. We will consider it further in the discussion in Sec. V.

IV. RADIATION REACTION IN THE SLOW-MOTION APPROXIMATION

A. The slow-motion approximation

For the first light-crossing time after $t=0$, the solution inside the source is very complicated. If we now specialize to a slowly changing source, for which one light-crossing time is a very short time, we can ignore it for now and concentrate on the situation after the retarded solution is fully developed. The problem is then to evaluate the right-hand side of Eq. (14)

$$T^{\mu\nu}{}_{,\nu} = -\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu} - \Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha} \tag{14}$$

when the metric is given by Eq. (23) integrated over the past light cone c:

$$\bar{h}_{\mu\nu}(x^i, t) = 4\int_c \frac{T_{\mu\nu}(y^i, t_{\rm ret})}{|R^k R_k|^{1/2}} d^3y, \qquad R^i = x^i - y^i. \tag{23}$$

The slow-motion approximation consists of approximating the integrand in (23) by a Taylor expansion back to the light cone from time t:

$$T_{\mu\nu}(y^i, t_{\rm ret}) = T_{\mu\nu}(y^i, t-R) = \sum_{n=0}^{\infty} \frac{(-R)^n}{n!}\left[\frac{\partial^n}{\partial t^n} T_{\mu\nu}(y^i, t)\right]_t . \tag{36}$$

This is the same method as one uses in electromagnetism,[15] and it is very close to that used to derive the radiation-reaction formulas of linearized gravity by Misner, Thorne, and Wheeler.[16] Accordingly, there is no need here to go through the tedious algebra. What we aim to do is to discuss features that have received little attention previously. It should be noted, however, that the integrals in (23) are considerably simplified by the use of the zero-order equation satisfied by $T_{\mu\nu}$,

$$T^{\mu\nu}{}_{,\nu} = 0 . \tag{13}$$

The results may be expressed in terms of the quadrupole tensor defined earlier,

$$I_{jk} = \int T^{00}(y^l) y_j y_k d^3y . \tag{6}$$

Among the useful identities Eq. (13) leads to are

$$\frac{\partial^2}{\partial t^2} I_{jk} = 2\int T_{jk}\, d^3y$$

and

$$\frac{\partial}{\partial t}\int T^{00} d^3y = 0 .$$

The expansion (36) must be taken out as far as $n = 5$ to get radiation-reaction terms. It is helpful to order the terms in each expression, using a typical velocity v as an order parameter. For a low-velocity system we have $T_{00} = O(1)$, $T_{0i} = O(v)$, $T_{ij} = O(v^2)$, and $\partial/\partial t = O(v)$. The expansions for the metric elements, written with orders from 1 to v^5 separated by parentheses, are

$$\begin{aligned} h_{00} = {} & 2\,{}_{(-1)}J + 0 + ({}^{(2)}_{(1)}J + 2\,{}_{(-1)}L^l_l) \\ & -\tfrac{4}{3}\,{}^{(3)}I^l_l + (\tfrac{1}{12}\,{}^{(4)}_{(3)}J + {}^{(2)}_{(1)}L^l_l) \\ & + (-\tfrac{1}{60}\,{}^{(5)}_{(4)}J - \tfrac{1}{3}\,{}^{(3)}_{(2)}L^l_l) + O(v^6) , \end{aligned} \tag{37a}$$

$$\begin{aligned} h_{0j} = {} & 0 + 4\,{}_{(-1)}K_j + 0 + 2\,{}^{(2)}_{(1)}K_j \\ & -\tfrac{2}{3}\,{}^{(3)}_{(2)}K_j + \tfrac{1}{6}\,{}^{(4)}_{(3)}K_j + O(v^6) , \end{aligned} \tag{37b}$$

$$\begin{aligned} h_{jk} = {} & 2\,{}_{(-1)}J\delta_{jk} + 0 + (4\,{}_{(-1)}L_{jk} - 2\,{}_{(-1)}L^l_l\delta_{jk} + {}^{(2)}_{(1)}J\delta_{jk}) \\ & - 2({}^{(3)}I_{jk} - \tfrac{1}{3}\,{}^{(3)}I^l_l\delta_{jk}) \\ & + (2\,{}^{(2)}_{(1)}L_{jk} + \tfrac{1}{12}\,{}^{(4)}_{(3)}J\delta_{jk} - {}^{(2)}_{(1)}L^l_l\delta_{jk}) \\ & + (-\tfrac{2}{3}\,{}^{(3)}_{(2)}L_{jk} - \tfrac{1}{60}\,{}^{(5)}_{(4)}J\delta_{jk} + \tfrac{1}{3}\,{}^{(3)}_{(2)}L^l_l\delta_{jk}) + O(v^6) , \end{aligned} \tag{37c}$$

with the definitions

$$I_{jk} = \int y_j y_k T_{00}(y^i) d^3y ,$$

$${}_{(n)}J = \int |x - \mathbf{y}|^n T_{00}(\mathbf{y}^i) d^3y , \qquad |x - y| \equiv [(x^i - y^i)(x_i - y_i)]^{1/2}$$

$${}_{(n)}K_j = \int |x - y|^n T_{0j}(y^i) d^3y ,$$

$${}_{(n)}L_{jk} = \int |x - y|^n T_{jk}(y^i) d^3y .$$

(Recall that bracketed numbers in superscripts indicate time derivatives.)

B. Radiation reaction from the self-field

The effect of the self-field on the energy of the source may be deduced from Eq. (14) with $\mu = 0$:

$$T^{0\nu}{}_{,\nu} = -\Gamma^{0}{}_{\alpha\nu}T^{\alpha\nu} - \Gamma^{\nu}{}_{\alpha\nu}T^{0\alpha} . \tag{38}$$

This equation includes *all* the effects on the energy of the source that are first order in the weak-field approximation. Some merely transfer energy from one part of the source to another. These can be eliminated by integrating (38) over the whole source at a fixed time t. Using Eq. (15) for the Christoffel symbols, integrating freely by parts, and using (13) gives us

$$\begin{aligned} \frac{dE}{dt} &= \int T^{00}{}_{,0}\, d^3x \\ &= -\tfrac{1}{2}\int h_{\alpha\nu,0} T^{\alpha\nu} d^3x \\ &\quad -\frac{d}{dt}\int (h^0{}_\alpha T^{0\alpha} + \tfrac{1}{2} h^\alpha{}_\alpha T^{00}) d^3x . \end{aligned} \tag{39}$$

The effects that remain are transfers of energy between the source and the external gravitational field. Not all these transfers show up as radiation, i.e., as permanent losses from the source. Some may only be stored inductively in the gravitational field, ready to be transferred back to the source later. The easiest way to distinguish between these is to assume that the zero-order $T_{\mu\nu}$, and hence $h_{\mu\nu}$, are periodic in time. Then any inductive storage of energy in the field will also be periodic, so that the *average* of dE/dt over one period will equal the irrevocable energy loss of the system. Under the assumption of periodicity the average of the second integral on the rhs of Eq. (39) vanishes, and we have

$$\langle dE/dt\rangle = -\tfrac{1}{2}\left\langle \int h_{\alpha\nu,0}T^{\alpha\nu}d^3x\right\rangle , \tag{40}$$

where angular brackets denote a time average over one period of a periodic zero-order motion. This expression is what is usually called the radiation-reaction energy loss, but no discussion of the actual radiation is needed to evaluate it. It is a simple matter to evaluate (40), and one finds that all terms in the expansions (37) below $O(v^5)$ cancel, with the result (4):

$$\langle dE/dt\rangle = -\tfrac{1}{5}\langle {}^{(3)}\mathcal{I}_{jk}\,{}^{(3)}\mathcal{I}^{jk}\rangle + O(v^7) . \tag{41}$$

One might object to the above discussion of "inductive" versus "irrevocable" energy transfers on the grounds that there is no unique separation of energy between the field and the source, that any such split is gauge dependent. Indeed, the local expressions (38) and (39) *are* gauge dependent, but not surprisingly the *average* result (40) is independent of gauge. That is, a change to

$$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{42}$$

for *any* ξ_α makes no change in (41). This is easy to deduce from Eq. (40), using the zero-order conservation law (13).

While (41) gives the average energy-loss rate of the source, it does not give enough information to enable one to calculate the detailed motion of the source. This information comes from the spatial components of Eq. (14):

$$T^{i\nu}{}_{,\nu} = -\Gamma^i{}_{\alpha\nu}T^{\alpha\nu} - \Gamma^\nu{}_{\alpha\nu}T^{i\alpha} . \tag{43}$$

As with Eq. (38), this includes all the self-interaction effects, not simply those associated with the long-term energy loss. But it does no good to integrate over space and time here, since we want the detailed, local corrections to the equation of motion. These will inevitably be gauge independent, so we shall pay some attention below to the choice of a convenient gauge. For the moment, we remain in the Lorentz gauge.

We may argue, with Misner *et al.*,[2] that the radiation-reaction terms in (43) are those whose sign depends upon whether retarded or advanced potentials are used in the calculation of expansions such as (37). It is not hard to see that these are the terms of odd order in v, in h_{00} and h_{jk}, and of even order in v in h_{0j}. These give a reaction force which is of order v^5:

$$\begin{aligned} F^i_{\text{react}} = T^{00}\Big(&\tfrac{3}{5}x_k\,{}^{(5)}\mathcal{I}^{ik} - \tfrac{6}{5}\int y_k\,{}^{(3)}T^{ki}d^3y \\ &+ \tfrac{2}{5}\int y^i\,{}^{(3)}T^k{}_k d^3y\Big) \\ &+ 2T^{0k}\big({}^{(4)}\mathcal{I}^i{}_k - \tfrac{1}{3}\delta^i{}_k\,{}^{(4)}I\big) + O(v^7) . \end{aligned} \tag{44}$$

Though these terms deserve the name "radiation-reaction force," they are by no means the whole of the self-interaction terms, and lower-order terms can certainly have observable effects on the system. Any observational interpretations must be careful to consider *all* the terms in (43). But the reaction terms are special because they tell us in detail how the irrevocable energy loss actually shows up in the source. For this reason they are studied separately and even, sometimes naively, used without including the rest of (43).

C. The standard gauge

Our reaction force, Eq. (44), is rather cumbersome, so we shall now use our gauge freedom to leave the Lorentz gauge for a quasi-Newtonian gauge in which the expressions simplify. Perhaps the simplest criterion for choosing a gauge is to try to eliminate the lowest-order radiation-reaction terms in the metric expansion (37). Since these appear at $O(v^3)$ in h_{00} and h_{ij} and at $O(v^4)$ in h_{0j}, while the reaction force itself is $O(v^5)$, this seems a reasonable hope. A little experimentation reveals that the following is the most general vector field that generates a gauge transformation (42) which eliminates the $O(v^3)$ terms in h_{00} and h_{ij}:

$$\xi_0 = -\tfrac{2}{3}\,{}^{(2)}I^l{}_l + f(x) , \tag{45a}$$

$$\xi_i = -x_j\,{}^{(3)}\mathcal{I}^j{}_i + g_i(t) , \tag{45b}$$

where $f(x)$ is an arbitrary function independent of time and $g_i(t)$ an arbitrary vector independent of position. This gauge transformation does not eliminate the $O(v^4)$ terms in h_{0j}. In fact, it complicates them. In this new gauge the reaction force simplifies to

$$F^i_{\text{react}} = T^{00}\left[-\tfrac{2}{5}x_k\,{}^{(5)}\mathcal{I}^{ik} - \tfrac{6}{5}\int y_k\,{}^{(3)}T^{ki}(y^l)d^3y + \tfrac{2}{5}\int y^i\,{}^{(3)}T^l{}_l(y^k)d^3y + {}^{(2)}g_i(t)\right] . \tag{46}$$

In this gauge, reaction terms couple only to T^{00}. By choosing

$$g_i(t) = \tfrac{6}{5}\int y_k\dot{T}^{ki}(y^l)d^3y - \tfrac{2}{5}\int y^i\dot{T}^l{}_l(y^k)d^3y \tag{47}$$

we can eliminate these cumbersome terms and arrive at the standard expression

$$F^i_{\text{react}} = (-\tfrac{2}{5}x_k\,{}^{(5)}\mathcal{I}^{ik})T^{00} . \tag{48}$$

The gauge choice (45), (47) will be called the *standard reaction gauge.*

D. A Newtonian problem?

The standard gauge has one other advantage besides the simplicity of (48). It enables one to pretend that the problem is a Newtonian one, that the only relativistic effect is to produce the force (48). Not only is the force derivable from the potential, Eq. (7), but it is consistent with the energy-loss formula (4) via the nonrelativistic expression for the work done by a force, Eq. (9). This is in fact a remarkable feature, not shared by most gauges. For example, an expression nearly as simple as (48) can be obtained from the standard-gauge expression (48) by a gauge change generated by the vector field

$$\xi_0 = -\tfrac{2}{3}\,{}^{(2)}I^l{}_l, \quad \xi_i = -\tfrac{2}{15}x^i\,{}^{(3)}I, \tag{49}$$

which gives

$$F^i_{\text{react}} = -\tfrac{2}{5}\,(x_k\,{}^{(5)}\not{I}^{ik} - \tfrac{1}{3}x^i\,{}^{(5)}I^l{}_l)T^{00}. \tag{50}$$

This again couples only to T^{00} and is derivable from a potential. But for this expression, the mean Newtonian work done, $\langle \int F^i_{\text{react}} v_i d^3x\rangle$, does not equal the energy-loss rate. This gauge restores the $O(v^3)$ terms in h_{00} and h_{ij} and may therefore be suspect, but this does not explain how the Newtonian intuition in this problem fails.

To understand this we must look at the fully relativistic analog of the $v_i F^i$ expression for the energy balance. Consider a perfect fluid whose stress-energy tensor is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}.$$

Then the following is an identity:

$$T^{0\nu}{}_{,\nu} = \left[U_j T^{j\nu}{}_{,\nu} + \frac{\rho + p}{n}(nU^\nu)_{,\nu} + nTS_{,\nu}U^\nu + \tfrac{1}{2}(\rho + p)g_{\alpha\beta,\nu}U^\alpha U^\beta U^\nu + pU^\alpha g^{\nu\beta} g_{\alpha\beta,\nu}\right] \Big/ (-U_0), \tag{51}$$

where ρ, p, n, T, and S are, respectively, energy density, pressure, conserved number density, temperature, and specific entropy. The first term in (51) is the analog of $v_j F^j$. The second term reminds us that the particle-conservation law is also affected to first order in $h_{\mu\nu}$, so that locally a Newtonian observer would reckon that the self-field causes creation of particles, $(nU^\nu)_{,\nu} \neq 0$. We will return to this in a moment, but note here that this never contributes to the averaged energy change at the lowest reaction order. The third term is the energy change for nonadiabatic flows, but since entropy is a scalar it is conserved regardless of $h_{\mu\nu}$ and therefore contributes nothing to the energy loss. The final terms are geometrical effects which seem to have no ready interpretation. The division by $(-U_0)$ is simply a red-shift factor. It is in the final terms that the compensating contribution arises which allows the energy-loss rate for (50) to equal that for (48). The final two terms cancel each other when averaged in the standard gauge. It is this "accident" which permits one to use Newtonian language and intuition in that gauge.

E. Nonconservation of particles

The fact that the particle-conservation law is affected by reaction terms is important. When the system of interest consists of disjoint bodies whose internal structure is unimportant—the binary pulsar, for example—then this extra effect is ignorable. But if one is interested in the local effects inside a body—for instance, reaction effects inside a pulsating star—then the equations

$$T^{i\nu}{}_{,\nu} = F^i_{(\text{react})} \tag{52}$$

do not form a determinate system by themselves. In the case of a perfect fluid, they must be supplemented by an equation of state and any two of the following three equations:

$$T^{0\nu}{}_{,\nu} = F^0_{(\text{react})}, \tag{53}$$

$$U^\nu S_{,\nu} = 0, \tag{54}$$

$$(nU^\nu)_{,\nu} = M_{(\text{react})}, \tag{55}$$

where, in the standard radiation-reaction gauge,

$$\begin{aligned} F^0_{(\text{react})} = {} & T^{00}(\tfrac{1}{30}\,{}^{(6)}_{(4)}J - \tfrac{2}{3}\,{}^{(4)}_{(2)}L^l{}_l) + T^{0i}\partial_i(-\tfrac{2}{3}\,{}^{(3)}_{(2)}L^l{}_l) \\ & + T^{ij}(-{}^{(4)}\not{I}_{ij} + \tfrac{1}{3}\,{}^{(4)}_{(2)}L_{ij} - \tfrac{1}{6}\,{}^{(4)}_{(2)}L^l{}_l\delta_{ij} + \tfrac{1}{120}\,{}^{(6)}_{(4)}J\delta_{ij} - \tfrac{1}{3}\partial_i\,{}^{(3)}_{(2)}K_j - \tfrac{1}{3}\partial_j\,{}^{(3)}_{(2)}K_i) \end{aligned} \tag{56}$$

and

$$M_{(\text{react})} = n(\partial_t + v^i\partial_i)(\tfrac{1}{60}\,{}^{(6)}_{(4)}J - \tfrac{1}{3}\,{}^{(4)}_{(2)}L^l{}_l). \tag{57}$$

The third equation of (53)–(55) follows from the other two and (52) by using (51).

V. DISCUSSION AND SUMMARY

The main conclusions of this paper may be summarized as follows.

(i) To derive radiation-reaction formulas, it is not necessary to make explicitly time-asymmetric assumptions about the radiation field, such as purely outgoing radiation or zero incoming radiation. Indeed, the use of such assumptions has certain drawbacks.

(ii) It is possible instead to derive the standard results from a statistical approach in which radiation reaction emerges as the most likely evolution from an ensemble of initial conditions. This has the added advantage of allowing the problem to be posed as a local one, involving only a compact region of spacetime. Weak-field and slow-motion approximations, therefore, should be well behaved and uniform in the region of interest.

(iii) The standard radiation-reaction formalism permits one to treat the problem as a nonrelativistic one, in which one pretends that the radiation-reaction force is just another Newtonian force. This is permissible only in a very specially chosen gauge, and even here one should not ignore the modification in the mass-conservation equation caused by the radiation-reaction terms.

Perhaps the most uncomfortable feature of the present approach is that the solution for *earlier* times involves advanced potentials, a necessary consequence of the time-symmetric initial conditions. In one respect this is a positive advantage, as described below. But the fact that the solution for $t<0$ is clearly wrong warrants discussion.

In at least one case, there is no reason for concern: if $t=0$ is the moment of formation of the system then (as discussed in Sec. II A) the simple equations we are using do not apply for $t<0$ anyway. But if $t=0$ is the moment the system is first observed, then there is an apparent problem. But this same problem is present in the statistical version of the second law of thermodynamics and it has the same resolution. If a certain isolated thermodynamic system is observed at $t=0$ to have low entropy, then the prediction is that it will probably have higher entropy for $t>0$, and—because the ensemble of initial conditions is time-symmetric[17]—that the entropy was also higher for $t<0$. We know that for most systems this retrodiction is likely to be wrong. Even if we have no knowledge of the history of the given system, we would conclude that its entropy was probably even lower in the past than now. The reason is that we have much experience with other systems and know that most systems had lower entropy in the past. (The origin of this global entropy famine must be traceable to the expansion of the universe, but this is irrelevant to the present discussion.) The initial conditions used to derive the second law do not take this other knowledge into account; to do so would require putting in correlations among the velocities and positions of particles so as to decrease the entropy at earlier times.[18] Given that these correlations have been omitted from the initial ensemble, why do we believe the future prediction of the calculation? The answer is that we do not expect these correlations to matter. If in some circumstance they should matter, then the second law would not apply.

The analogy with the statistical radiation-reaction formula is very close. In order to get the correct behavior for $t<0$—by "correct" we mean the evolution by retarded potentials that we expect of our system by analogy with all other observed systems—we would need to include correlations in our initial data. These would put just the right amount of outgoing radiation onto the initial hypersurface. But we can omit this radiation and still believe the future prediction, since the radiation leaves the system. Our only reservation is when there is backscatter of radiation, and even here it would be very surprising if the backscattered radiation was sufficiently well correlated with the system's future motion to make much difference.

The time symmetry of the problem has one distinct advantage: there are no runaway solutions. The reaction force derived in Sec. IV C would, if used from $t=0$ onward, require initial data not just for $T_{\mu\nu}$ but also for several time derivatives of $T_{\mu\nu}$. These extra, physically unnecessary data would allow some runaway solutions, just as in electromagnetism.[15] But recall that the reaction force in Sec. IV does *not* apply from $t=0$. It is valid only after one light-crossing time. As t goes to zero, the retarded solution and consequently the reaction terms also go to zero, because of the cutoff of the retarded integral at $t=0$. This is an inevitable consequence of the time-symmetric radiation data. At $t=0$, then, there is no radiation reaction and no need for initial data for the higher time derivatives of $T_{\mu\nu}$. It follows that there will be no runaway solutions to the radiation-reaction problem when formulated in the present way.

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[2]See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, second ed. (Addison-Wesley, Reading, Mass., 1962), Sec. 104; or C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 36.

[3]S. Chandrasekhar and P. Esposito, Astrophys. J. 160, 153 (1970); W. Burke, J. Math. Phys. 12, 401 (1971); K. S. Thorne, Astrophys. J. 158, 997 (1969); Misner *et al.* (Ref. 2).

[4]J. Ehlers, A. Rosenblum, J. N. Goldberg, and P. Havas, Astrophys. J. 208, L77 (1976).

[5]A. Rosenblum, Phys. Rev. Lett. 41, 1003 (1978).

[6]See P. C. W. Davies, *The Physics of Time Asymmetry* (Intertext, Scranton, 1974).

[7]Many textbooks treat radiation reaction in electromagnetism. The approach here is partly modeled on that found in W. Pauli, *Electrodynamics: Pauli Lectures on Physics* (MIT, Cambridge, Mass., 1973), Vol. 1; or in J. D. Jackson, *Classical Electrodynamics*, second ed. (Wiley, New York, 1975). Our units are chosen to set $c=G=1$.

[8]Throughout this paper Latin indices run from 1 to 3, Greek for 0 to 3, and the summation condition on repeated indices is employed. Other conventions follow Misner *et al.* (Ref. 2). In Eqs. (4)–(7) the tensors are regarded as Cartesian. Time derivatives will always be denoted by a bracketed number in the superscript.

[9]Null infinity is described in the article by R. Penrose, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964); and in S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).

[10]R. A. Hulse and J. H. Taylor, Astrophys. J. 206, L53 (1976).

[11]The exception to this is when there is significant backscatter of radiation off the spacetime curvature. Though correlated in frequency with the system's subsequent behavior, it is essentially random in phase and may be ignorable. If this is not the case, no *local* formulation of radiation reaction is possible. This is an aspect of the problem which has so far received very little attention.

[12]See the derivation in Chap. 36 of Misner *et al.* (Ref. 2).

[13]The asymptotic nature of the linearized-theory approximation is probably guaranteed by the linearization stability of Einstein's equations. See V. Moncrief, J. Math. Phys. 16, 493 (1975); 17, 1893 (1976). (I know of no proof that the second approximation is likewise close to the second derivative of a sequence of nonlinear solutions.)

[14]Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds, and Physics* (North-Holland, Amsterdam, 1977), p. 433.

[15]See, e.g., Jackson (Ref. 7).

[16]Jackson (Ref. 7), Chap. 36.

[17]R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, New York, 1975), Sec. 17.5. From the viewpoint developed in this text, the usual kinetic equations, and hence the entropy theorem (*H* theorem), are obeyed *exactly* by systems whose initial conditions satisfy a certain constraint, Eq. (17.5.10) of this reference. This constraint involves only time-independent operators and is time-reversible. From a statistical point of view, this is a time-unbiased constraint on the microcanonical ensemble for the initial data.

[18]There is a good discussion of this in the article by R. Penrose, in *General Relativity: An Einstein Centenary Volume*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979). Penrose also stresses the statistical element in the fact that radiation from sources is outgoing. A preprint of this article was in fact one of the stimuli for the present research.

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