

## SYNERGISM BETWEEN NUMERICAL AND ANALYTIC RELATIVITY

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**Abstract.** The relation between numerical and analytical calculations is explored in a number of ways. I begin with brief comments on a number of simple examples, many of which are detailed elsewhere in this volume. Then I describe in detail a specific example: a numerical test of the validity of the analytic approximation known as the "quadrupole formula" for calculating the gravitational radiation emitted by a pulsating neutron star. In this case, the analytic approximations revealed unsuspected errors in early numerical calculations, and subsequent numerical calculations not only verified the quadrupole formula but also delimited its range of validity. As another example, I show how simple model-problem calculations suggest that there are as yet undiscovered strongly-damped normal modes of pulsation of neutron stars, and even that such modes may be unstable in very compact stars. I follow this with a survey of some features that present analytic calculations suggest may be seen in three-dimensional gravitational collapse problems: (i) Their results should be qualitatively different from two-dimensional axisymmetric collapse calculations. (ii) They will show much stronger fluxes of gravitational wave emission. (iii) They may require much more numerical precision to produce believable results than current two-dimensional collapse calculations have needed. This leads to the conclusion that in the immediate future, computations limited by machine resources should give priority to the accurate calculation of the hydrodynamics, rather than devote large numbers of grid points to the exterior gravitational-wave calculation. To this end, many workers have suggested that one might be able to read off the dominant quadrupolar contribution to the outgoing radiation from the near-zone field, effectively solving the exterior-wave problem analytically and matching this to numerical data near the star. I give a new, gauge invariant way to do this, by

integrating certain components of the Riemann tensor over a sphere of radius  $r_0$  near the star:

$$I_{ik} = \frac{15r_0^3}{64\pi} \oint R_{ijkl} n^j n^l d\Omega + a \delta_{ik},$$

where  $a$  is a coefficient which disappears when one takes the TT-projection of the distant wave amplitude, and where  $n^j$  is the unit radial vector. An error analysis suggests that this will be reasonably accurate (factor of 2) for strong sources of gravitational waves. This is a crude but simple way to avoid calculating the waves numerically. More sophisticated approaches are possible and are being investigated by others.

### 1. INTRODUCTION

As the increasing power of supercomputers makes interesting and realistic numerical calculations possible in general relativity, the balance between analytic and numerical calculations will inevitably change, and the interaction between the two will increase. There are many examples of this mutual feedback in the lectures in this volume. It is interesting to try to list some of the ways this happens.

Analytic calculations feed into the numerical work in a number of ways:

(i) They provide tests. Exact solutions, secure analytic approximations and specially devised test problems all help to debug and to verify the validity of numerical schemes. The articles by Evans, Piran & Stark, Shapiro & Teukolsky, and Isaacson, Welling & Wiricour in this volume all contain explicit examples of this.

(ii) They provide support. First there are analytic studies of numerical schemes to determine their accuracy, efficiency, and stability, for which numerical relativity can draw on a vast amount of development in other fields. Analytic relativity contributes in addition such assistance as studies of the initial value problem (Choquet-Bruhat & York 1980), alternative computational approaches (such as variational methods: Bardeen 1970, Detweiler & Ipser 1973, Nahmad-Achar & Schutz 1986), properties of different gauges (Bardeen 1983), ways of slicing spacetime (Isenberg & Nester 1980), the treatment of coordinate singularities (Bardeen 1983), and (still in its infancy) the treatment and identification of genuine physical singularities (Tipler *et al.* 1980).

(iii) They can be incorporated into mixed analytic-numerical schemes. For example, one can use an analytic approximation to calculate the spacetime exterior to a star, reserving the numerical calculation for the interior hydrodynamics. Anderson & Hobill describe in this volume the implementation of such a scheme in a model calculation, and I describe a crude version for general relativity below.

Numerical calculations similarly feed back into analytic ones. For

example:

(i) They can test analytic approximations. I will describe such a test of the quadrupole formula by Balbinski *et al* (1985) below.

(ii) More generally, they help to develop our physical intuition. This is, after all, the goal of numerical explorations of physical theories. One way of doing this is to change or remove certain aspects of the physics from a numerical code to see how the results change. This is a useful way to develop an understanding of which aspects of a complicated physical theory are responsible for which physical effects. Numerical code builders often do this to solidify their own understanding of their code, but they do not usually tell the analyticians about these experiments. Maybe it is embarrassing to admit that one got the same result after leaving out 90% of one's code; but this would be very important information to an analytic approximator! Numerical calculations also provide a body of "experimental" data which can lead to new analytic approximations designed to exhibit the same results, or they may give one confidence to use existing approximations in new territory. The original calculations of Eppley & Smarr (see Smarr 1979) showed that the head-on collision of two black holes gave gravitational radiation fluxes not very different from the predictions of test-particle calculations; for this reason, we may be more sanguine about extrapolating to black holes the test-particle calculations for particles spiralling into black holes reported by Oohara & Nakamura in this volume. Eardley in this volume describes an analytic attempt to calculate the results of that Eppley-Smarr computation. If successful it might be extendable to more complicated situations, but such an extrapolation would be inconceivable without the confidence that it reproduces well the numerical results of Eppley and Smarr. Another example of this is the stellar cluster collapse calculation reported by Shapiro & Teukolsky in this volume, on the basis of which they have been able to devise useful analytic estimates of the size of the black hole finally formed by the collapse. We even have a mention of computer-algebraic examples of this in the article by MacCallum in this volume where long algebraic calculations by computer have resulted in simple answers which were then re-derived by hand, with hindsight. And finally, and most obviously, we develop physical intuition by using the computer to explore parameter space, to calculate many examples of a phenomenon with different values of relevant parameters. Smarr in this volume gives examples, in some of which the computer was used in preference even to performing the physical experiments.

## 2. A NUMERICAL TEST OF THE QUADRUPOLE FORMULA

In the limit of very weak gravitational fields (linearized theory) and slow motions, the gravitational radiation generated by systems is quadrupolar in character and has its source in the second time-derivative of the system's quadrupole moment, as we shall show in the final section of this lecture. The appreciation of this goes back to Einstein (1918), and the demonstration that there is a corresponding quadrupolar radiation-reaction force inside the source was first given

by Eddington (1924). It is a remarkable consequence of the equivalence principle that these formulae, suitably expressed, apply unchanged in nearly-Newtonian systems, i.e. when the system's self-gravitational field exerts a significant influence on its motions. But the demonstration of this fact has proved to be very subtle, and its history is marked by controversy and considerable confusion. (For reviews, see Walker & Will 1980 and Damour 1983). Even though the controversy has died away and the result is regarded now as secure, there is still uncertainty about the range of validity of these quadrupole approximations: how relativistic may a system be and still generate radiation governed by these formulae? Analytically it is possible to give only rough estimates, but numerical calculations offer the possibility of a quantitative test.

Small-amplitude pulsations of stars offer an attractive testbed for this problem. On the one hand it is possible to solve the normal-mode pulsation problem to essentially arbitrary accuracy for Newtonian stellar models, and then the quadrupole radiation-reaction formula may be applied to estimate the damping times of these modes. On the other hand, one can similarly construct relativistic stellar models for the same equation of state and calculate their outgoing-wave normal mode eigenfrequencies. Each eigenfrequency will have a real part, which we expect to be close to the Newtonian eigenfrequency if the star is not too relativistic, and an imaginary part, which is the standard against which the Newtonian reciprocal damping time is to be compared. The Newtonian problem can be done to essentially arbitrary accuracy, as it involves only ordinary differential equations. Its relativistic counterpart also involves only ode's, but the eigenfrequency problem is potentially a source of errors, especially in weakly relativistic stars. The imaginary part of the frequency may be only  $10^{-4}$  of the real part, and its value depends on the outgoing-wave boundary condition that is applied in the asymptotic region far from the star.

The first comparisons of this type were performed by Balbinski & Schutz (1982), with very surprising results. Using the early relativistic calculations of Thorne (1969) and Detweiler (1973) as the standard, Balbinski & Schutz calculated the Newtonian frequencies and quadrupolar damping times and found discrepancies of a factor of three in damping times in very weakly Newtonian models (surface redshifts of 3%). Was this a failure of the quadrupole formula, a sign that some of the early relativistic calculations had large errors, or a result of some subtle physics?

Of the many possible explanations, there was a real possibility that the discrepancy was a result of the particular equation of state used for the models (the HWW equation of state: see Harrison *et al* 1965). Because of neutronization, this has a very soft regime at densities that are important for the low-redshift neutron stars, and there were indeed significant differences in the masses and radii of the Newtonian and relativistic models compared by Balbinski & Schutz. (They chose to compare models with the same central density, since these gave very

close agreement for the real parts of the frequencies.) In order to eliminate such non-uniformities in the equation of state and focus just on the quadrupole approximations, Balbinski, Detweiler, Lindblom & Schutz (1985) completely redid the comparison for models obeying the simple polytropic equation of state

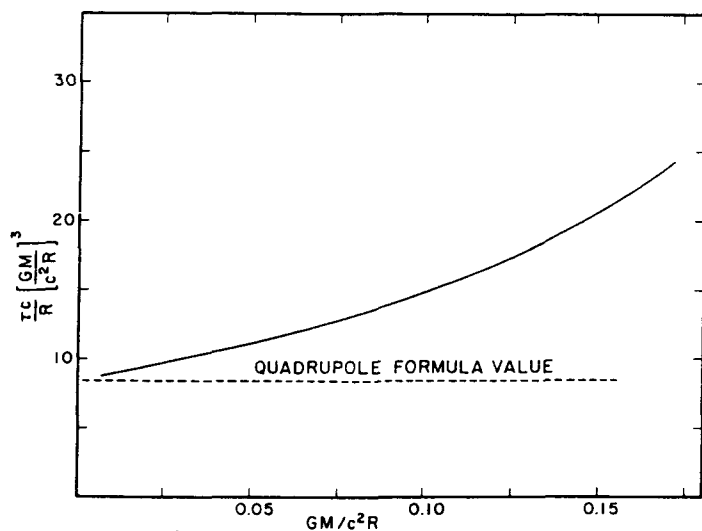
$$p = K\rho^{1+n}, \quad (2.1)$$

where  $n$  is the polytropic index and  $\rho$  is interpreted as the density of total mass-energy in the relativistic stars.

The relativistic eigenfrequencies were calculated by the improved techniques of Detweiler & Lindblom (1985), and the principal results are illustrated in Figs. 1 and 2, taken from Balbinski *et al.* (1985). The results are presented in terms of the numbers  $c_\tau(n)$ , where for the Newtonian polytropes the damping time  $\tau$  is given by

$$\tau = c_\tau(n) \frac{R}{C} \left[ \frac{GM}{c^2 R} \right]^{-3}. \quad (2.2)$$

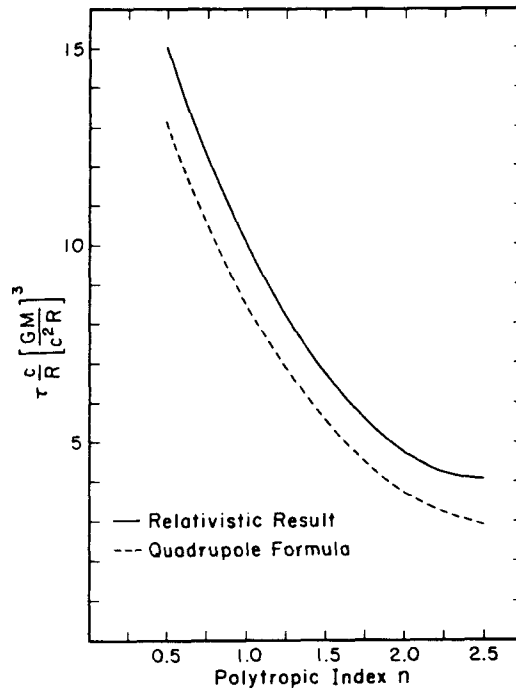
Figure 1. Gravitational radiation damping times for relativistic  $n=1$  polytropes.



Because of the scaling properties of Newtonian polytropes,  $c_\tau(n)$  depends only on the polytropic index. In relativity, the equivalent coefficient depends on central density as well. In Fig.1 we display the Newtonian and relativistic coefficients for  $n = 1$  polytropes. The most striking feature is the smooth approach of the relativistic curve to the Newtonian value as the surface redshift gets smaller. This shows at once that the quadrupole approximation is an asymptotic approximation at

least in this case, and also that the numerical calculations of the relativistic eigenfrequencies are accurate even for remarkably low redshifts (1%). The quadrupole formula is in error by no more than a factor of two even at fairly high redshifts (12%).

Figure 2. Gravitational radiation damping times for polytropes with  $M=1.0M_{\odot}$  and  $GM/c^2R=0.03$ .



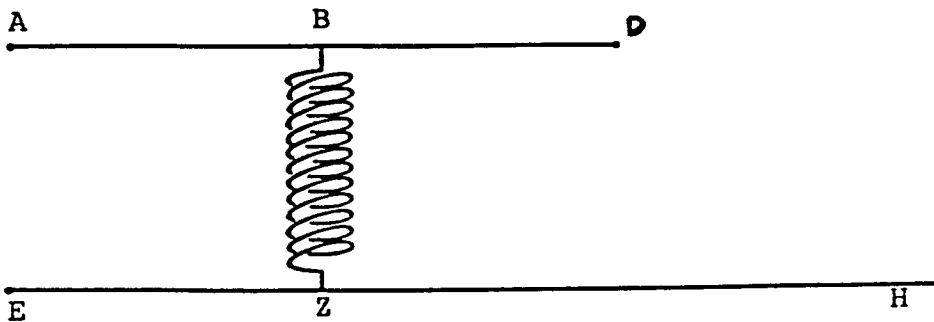
Why, then, did the first comparison give a factor of 3 error? Part of the error can be ascribed to equation-of-state effects (discussed by Balbinski et al. 1985), but mostly it seems that the early relativistic calculations had larger errors than were appreciated at the time. With hindsight, given the delicacy of the calculation, in which for low-redshift stars one has to determine a tiny imaginary part of the frequency, these errors are not hard to understand. But this example illustrates the two-way nature of analytic/numerical comparisons. In the first comparison the analytic approximations revealed errors in the numerical calculation. In the second comparison, the asymptotic validity of the quadrupole formula was established by numerical means, and its range of validity delineated.

Note that the error the quadrupole formula makes is to overestimate the amount of radiation produced by the relativistic star. Presumably that is because, in this case, the effects of wave interference in the relativistic star dominate the higher  $l$ -poles of radiation which the relativistic star produces. It will be interesting to see if collapse calculations follow this pattern, too. (See the final section of this lecture.)

### 3. NEW FEATURES OF THE STELLAR PULSATION PROBLEM?

Because the problem of the linear pulsations of a relativistic star must be solved numerically, it is useful to have model calculations which can be treated analytically and which can guide our calculations in the full problem. Even the simplest model calculations turn out to reveal features which have not yet been seen (or even looked for) in the stellar problem. I shall discuss two models: a very simple one examined by Kokkotas & Schutz (1985), and a more realistic one due originally to Aichelburg & Beig (1976) and treated in detail by Dyson (1980) and by Anderson & Hobill in this volume.

Figure 3. The coupled system consists of a finite string of length  $2\ell$  and a semi-infinite string, coupled as shown by a spring with spring constant  $k$ .



The Kokkotas and Schutz problem is of two strings coupled by a massless spring, as in Figure 3. String 1 is of finite length  $2\ell$ ; string 2 is semi-infinite. Both have wave speed  $c$  and tension  $T$ . When the spring constant  $k$  is set to zero, the strings are uncoupled. Then string 1's normal mode eigenfrequencies are the usual ones (with the convention  $e^{i\sigma t}$ ),

$$\sigma_n = \pi n c / \ell \quad (3.1)$$

whereas the semi-infinite string has no modes that satisfy an outgoing-wave boundary condition. For general  $k$ , the eigenfrequency equation is (in terms of  $k' = k\ell/2T$ )

$$z(e^{-z} + e^z) = k' (e^{-z} - e^z)(2 + e^{-2z}), \quad z = i\sigma\ell/c. \quad (3.2)$$

For small  $k'$ , the solutions of Eq.(3.2) fall into two families of modes. One family is close to the modes of the uncoupled finite string. The even-order modes of this family have nodes at the attachment point of the spring, so they do not couple to the second string for any  $k$ ; their frequencies are unchanged. The odd-order modes damp slowly, as one would expect, with (for any integer  $n \geq 0$ )

$$\sigma_n = \pi(n+\frac{1}{2})c/\ell + k'c/\pi(n+\frac{1}{2})\ell + i2k'^2c/(n+\frac{1}{2})^2\pi^2\ell. \quad (3.3)$$

The amplitudes of these modes are larger in the finite string than in the semi-infinite one. The second family of modes is strongly damped under an outgoing-wave boundary condition, with

$$\sigma_n = \pi(n+\frac{1}{2})(1+l/2a)c/l + iac/l, \quad (3.4)$$

where the dimensionless imaginary part  $a$  is the larger of the two solutions of the transcendental equation

$$a = k'e^{2a}. \quad (3.5)$$

The amplitude of this family is larger in the semi-infinite string. As  $k' \rightarrow 0$  these modes acquire infinitely large imaginary parts, accounting for their absence in the uncoupled strings.

The physical interpretation of these modes becomes clearer if we imagine posing initial data for the wave problem on both strings. The weakly-damped family, being related to the modes of the uncoupled string 1, are a complete set for representing the excitation of string 1, but they have no freedom left over for representing the independent data on string 2. For this we need to use the strongly-damped modes. They damp rapidly because the initial excitation simply moves down the semi-infinite string at the wave speed  $c$ . We can see from Eq.(3.5) that the modes' damping rate is independent of the order  $n$  of the mode, which is reasonable in light of the fact that all waves leave the system at speed  $c$ , regardless of wavelength.

This interpretation makes it plausible that these two families of modes ought to be present in any system consisting of a bounded wave system coupled to a 'radiative' wave system, and in particular that of a star coupled to gravitational waves. The weakly damped modes are clearly analogous to the modes that have been calculated for relativistic stars, as discussed in the previous section. No analogues of the strongly-damped modes have been seen in numerical calculations of relativistic stars, but no searches for modes have been made far from the real axis. It seems certain they should be there.

The second model problem is of a spring with a scalar charge coupled to a scalar field, and it too shows both families of modes. (For the equations, see the article by Anderson & Hobill in this volume.) The first discovery of strongly-damped modes was by Dyson (1980) in this problem. Here the remarkable added feature is that as the coupling between the charge and the field increase, some modes of the strongly-damped family move toward the real frequency axis and even go unstable. This is impossible in the two-string problem, where the energy is positive-definite. But in the Aichelburg-Beig problem there is an interaction-energy term which is of indefinite sign, and this allows (but does not explain) the unstable modes. This instability was discovered by Dyson (1980).

The energy of a pulsating relativistic star (Friedman & Schutz 1975) is



not positive-definite either, so it is at least possible that very relativistic stars (where the coupling between star and waves is strong) also have this instability. The question is open, but amendable to numerical investigation.

#### 4. PRIORITIES AND EXPECTATIONS

When we look ahead to the next five years of numerical relativity, the challenge in everyone's mind is to compute a realistic three-dimensional collapse. Despite the increasing power of machines (see the lecture by Smarr in this volume), the vastly increased complexity of 3-D calculations may force a choice in allocating available machine resources between doing the hydrodynamics accurately, which requires a fine grid, and computing the outgoing gravitational radiation accurately, which requires a large grid if it is to be done numerically. If such a choice has to be made, it is arguable that the hydrodynamics should be done accurately, while approximations should be found for the calculation of the radiation. These approximations ought at least to be able to estimate the radiation to within a factor of 2 or so with negligible numerical effort. In this section, I will first consider what new features we expect to see in 3-D hydrodynamical calculations and how they will impact on the computing resources required for the problem. Then I will consider what information may be needed about the gravitational waves in such calculations in the near future, and I will conclude with some remarks on how Newtonian 3-D hydrodynamics can give us some hints about the outcome of relativistic calculations. In the next section (§5) I will describe a possible method for calculating the gravitational wave emission by doing integrals in the "near zone", outside the star but inside one gravitational wavelength.

##### a. *What new features will 3-D collapse reveal?*

(i) *Global rotational instabilities.* As the rotating fluid collapses, conservation of angular momentum will increase its angular velocity and rotational/kinetic energy. Two kinds of global non-axisymmetric instabilities may appear if the rotation is fast enough.

The first are called gravitational-wave-induced instabilities, otherwise known as radiation-reaction "secular" instabilities. These were first discovered by Chandrasekhar (1970) and their theory developed by Friedman & Schutz (1978) and by Friedman (1978). Detailed calculations of instability points for various systems have been made by Bardeen *et al.* (1972), Comins (1979a,b), Durisen & Imamura (1981), and Friedman (1983). The strongest instabilities may grow on a timescale of several rotation periods.

For the numerical calculations it is important to appreciate that, despite the fact that these instabilities appear only because the star is coupled to gravitational radiation, the instability is a near-zone effect caused by the speed-of-light retardation in the interactions between different parts of the star. To see why this should be, first consider Newtonian theory. Here, the equality of action and reaction

guarantees that a body's net self-force is zero. In a relativistic field theory, however, action and reaction cannot always balance out; although the action of element A on element B may cause B to react, the effect of this reaction on A may be different if, in the intervening delay, A has moved, say, further away. Relativistic field theories therefore allow net self-forces, which can do work and change the body's energy. Since all fundamental physical field theories are conservative, the energy lost by the body must turn up somewhere, and it does so in the radiation. This leads to the term "radiation reaction" for such net self-forces, but this name can be misleading. In a non-conservative theory such self-forces would still be present but the work they did would not tally with the radiated energy. The "gravitational radiation induced" instability (a similar misnomer) is caused by the retardation self-forces. If a numerical calculation correctly treats general relativity inside the fluid but fails to calculate the radiation field correctly (for example by imposing the wrong boundary condition or by fixing the boundary in the near zone) it becomes in effect a nonconservative theory, but there is no reason to expect that it will not exhibit the correct "radiation reaction" effects. This will be true even if, say, the boundary reflects the waves back inwards (a standing-wave boundary condition). This is because, in order for the reflected waves to have a significant effect on the interior dynamics, they will have to be well correlated in frequency and phase with the material motions. Since the relative phase of the reflected wave and the material motion depends on where the boundary is placed and where the collapsing matter is when the wave finally catches up with it, and since the frequency of the reflected wave is a function of the dynamics at a considerably earlier time, any such correlations are unlikely, and those that occur by chance will be short-lived.

Although bad boundary treatment should not suppress the gravitational-wave-induced instability, viscosity will do so (Lindblom & Detweiler 1977, Comins 1979a,b). Ordinary kinematical viscosities in neutron matter are too small to matter (Friedman 1983), but small-scale turbulence may contribute an effective viscosity orders of magnitude greater, as might MHD effects. If the gravitational-wave-induced instability manifests itself, it will be important to model such viscosities and to include them in the calculations if necessary. Conversely, numerical viscosity may have the effect of artificially suppressing the onset of the instability. This point deserves careful consideration in the design and testing of collapse codes. Since this instability may be the principal source of the nonaxisymmetry of collapses in a wide range of physical situations, the correct treatment of viscosity places a stringent requirement on numerical codes.

The second kind of instability is the so-called "dynamical" or "bar-mode" instability. This is seen in purely Newtonian stars at rotation rates some 50% larger than those needed to produce the radiation-induced instability. (See Tassoul 1978 for a review.) Because it is driven by Newtonian hydrodynamical and gravitational forces, it grows much faster than the radiation-induced variety,

typically with a timescale of one rotation period. If collapsing rotating stars reach this sort of rotation rate despite the radiation-induced instability, then the nonaxisymmetry will grow much more dramatically, and perhaps the star might even fission. The accurate calculation of this instability should not be particularly difficult.

The accurate calculation of the development of these instabilities is important not just in order to predict the gravitational radiation emitted, but also because the nonaxisymmetric reaction forces remove angular momentum from the collapsing star, and this can have a significant effect on its subsequent dynamics.

(ii) *Frame-dragging effects.* The angular momentum of a collapsing object exerts a gravitational force (the Lense-Thirring effect) that tends to drag other bodies in the same sense. The effect of this is to reduce the effective centrifugal forces on material, perhaps allowing the collapse to proceed to higher densities than would be seen in Newtonian theory. This may make black holes more likely, and even ergoregions (Schutz & Comins 1976), if only as a stage on the way to a black hole.

(iii) *Nonaxisymmetric shear instabilities.* One should distinguish the rotation-induced instabilities discussed in (i) above from those that arise only if there is differential rotation (shear). Much recent work (Papaloizou & Pringle 1984, 1985; Balbinski 1984, 1985; Blaes & Glatzel 1986; Goldreich & Narayan 1985) has shown that many differentially rotating systems with free boundaries have instabilities that are not manifested by laboratory fluids, whose boundaries are usually fixed. Growth times are typically a few rotation periods. These may well arise in collapse, where differential rotation typically increases as collapse proceeds. They would have the likely effect of transporting angular momentum outwards from the core, driving the core to higher densities and to either a black hole or a nearly uniformly rotating neutron star. A reasonably fine mesh inside the star would be needed to see these effects.

(iv) *More sophisticated physics.* While the stiffness of the equation of state is already important in axisymmetric collapse, there are reasons for believing that nonaxisymmetric collapse may be sensitive to a variety of other physical parameters. These include:

- Realistic nuclear equation of state. Since shear-induced instabilities would cause mixing, one might have to take into account the effects of composition gradients and nuclear reaction rates.

- Neutrino transport. Present axisymmetric collapse calculations suggest that neutrinos are trapped, but in the nonaxisymmetric situation this may change for two reasons. First, gravitational radiation may be emitted over a longer timescale (perhaps 10 rotation periods), making diffusion more important. Second, if the core fissions or goes into an extreme nonaxisymmetric state, there will

be an increase in the ratio of surface area to volume and a decrease in the mean density. Both of these will make trapping less efficient.

- Magnetohydrodynamics. If the fluid-shear-induced instabilities carry away significant angular momentum, then it will be natural to ask whether MHD with realistic initial fields can offer competing mechanisms.

The net effect of the new features described in (i) - (iv) will probably make a three-dimensional collapse qualitatively different from an axisymmetric collapse that begins with the same angular momentum. It will be a much more efficient generator of gravitational waves. And it will require much more computational power per unit physical volume: besides the extra dimension, one will be likely to need a finer grid, more physics per grid point, and greater numerical accuracy than in the axisymmetric collapse.

b. *What do we want to know about the gravitational waves?*

Ten years ago, when the "quadrupole formula" was in some doubt, a high priority for a collapse calculation would have been to test the formula: to verify that a real system in relativity radiated an amount well approximated by the quadrupole formula. Today, however, this is not such a strong motivation. There is more confidence in the formula, not only from observations of the reaction effects in the binary pulsar system (Weisberg & Taylor 1984) but also from numerical tests (Balbinski *et al.* 1985) as described earlier, and from a large body of analytic work (Anderson *et al.* 1982; Kates 1980; Damour 1983; Futamase & Schutz 1985; Walker & Will 1980; and many others).

The priority today must be to predict the observable features of astrophysical radiation, and to be in a position to interpret any observations that may be made five or ten years from now. Over that period of time, I would argue that accuracy to a factor of two in amplitude is likely to be adequate. At present, gravity-wave experimenters want that sort of accuracy as reassurance that observable radiation may well be arriving at the Earth regularly. But to insist on better accuracy than this from any collapse calculation would be to ignore the other uncertainties in the astrophysics: initial conditions for collapse, the right equation of state, the influence of magnetic fields, even the distance to likely sources (which depends on the Hubble constant).

But in a decade or so, when (if?) gravitational wave observations become of reasonable quality, then accurate gravitational wave calculations will become necessary in order to use the observations to help unravel the other astrophysical uncertainties. This is then the timescale over which a gravitational wave astronomer might hope for the development of very accurate 3-D codes.

c. *What can we learn from 3-D Newtonian hydrodynamics?*

Obviously, the development that is now taking place, of

three-dimensional collapse codes in Newtonian gravity, is an important first step toward the relativistic codes. Many of the new 3-D effects mentioned earlier can be studied here: the rotation-induced dynamical instability, shear-induced instabilities, and whatever extra physics is needed.

A first attempt can also be made here at taking into account the radiation-reaction effects, such as the development of the 'secular' nonaxisymmetric instabilities in more slowly rotating collapses, and the calculation of the loss of angular momentum and energy through gravitational radiation. This can be done by using the near-zone quadrupole Newtonian reaction potential, as given in Misner *et al.* (1973). Although this gives reasonably good results for pulsating neutron stars (the calculation of Balbinski *et al.* (1985) described earlier), it must be used with care. The potential depends on five time derivatives of the quadrupole moment; done carelessly (e.g. handled as a fifth-order initial-value problem), these can feed back into the dynamical equations to cause runaway solutions, as in (but worse than) electromagnetism. To my knowledge, this has not been adequately studied.

Even the frame-dragging effects can be included to first order in the angular momentum by adding into the Newtonian equations the Lense-Thirring "gravito-magnetic" terms.

But Newtonian theory, even with corrections, will not generate results we can have confidence in when the collapse produces compact objects. Will we need then to go to a full implementation of general relativity, including its wave field, or is there a useful halfway stage? That is the subject of the next section.

##### 5. NEAR-ZONE CALCULATION OF GRAVITATIONAL WAVE EMISSION

As I shall describe in some detail below, it turns out that for a radiating, nearly-Newtonian system, the quadrupole moment that determines the radiation also determines the quadrupole part of the near-zone Newtonian potential. Is it therefore possible to read off this quadrupole moment from the near-zone field and thereby to predict the radiation emitted without even extending the grid into the far zone? This possibility was first suggested by Ipser (1970), and it has been repeated in various circumstances by Thorne (1980), Anderson & Hobill (this volume), Futamase (1985), and others. I will suggest here a method of doing this, at least approximately, for nearly-Newtonian systems, and argue that the method may not be much worse even for some highly relativistic systems.

###### a. *How it works in Linearized Theory*

In Linearized Theory (see Misner *et al.* 1973), we have a metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (5.1)$$

in which it is possible to treat  $h_{\mu\nu}$  as a tensor field on flat

spacetime. Defining the trace-reversed potential

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^\alpha{}_\alpha, \quad (5.2)$$

and adopting the gauge condition (Lorentz gauge)

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0, \quad (5.3)$$

we arrive at a simple form for Einstein's equations, correct to linear order in  $h_{\mu\nu}$ :

$$\square \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \quad (5.4)$$

in units in which  $c = G = 1$ . (Other conventions follow Misner *et al.* 1973.) This has the retarded solution

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int |\mathbf{x}-\mathbf{y}|^{-1} T_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|, \mathbf{y}) d^3\mathbf{y}. \quad (5.5)$$

Now suppose that we want the field of a bounded source at some very large distance (far zone), and that the source is in slow motion. Letting  $|\mathbf{x}|$  be denoted by  $r$ , and defining  $u = t-r$ , then the dominant term in Eq.(5.5) is

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) \approx \frac{4}{r} \int T_{\mu\nu}(u, \mathbf{y}) d^3\mathbf{y}. \quad (5.6)$$

To see what this says about the radiation, we change from Lorentz gauge to the TT gauge. Defining the projection on to a sphere,

$$P_{ij} = \delta_{ij} - n_i n_j, \quad n_i = x_i/r, \quad (5.7)$$

we have  $\bar{h}_{\mu 0}^{\text{TT}} = 0$  for all  $\mu$  and

$$\bar{h}_{ij}^{\text{TT}} = P^k{}_i P^l{}_j \bar{h}_{kl} - \frac{1}{2} P_{ij} (P^{kl} \bar{h}_{kl}). \quad (5.8)$$

The spatial components of  $\bar{h}_{\mu\nu}$  can be found from Eq.(5.6) if we define

$$I_{ij}(u) = \int T_{00}(u, \mathbf{y}) y_i y_j d^3\mathbf{y} \quad (5.9)$$

and use the identity (for bounded sources)

$$\int T_{ij}(u, \mathbf{y}) d^3\mathbf{y} = \frac{1}{2} \ddot{I}_{ij}(u), \quad (5.10)$$

where dots denote derivatives with respect to  $u$ . Then we have the familiar result that

$$\bar{h}_{ij}^{\text{TT}}(t, \mathbf{x}) \approx = \frac{2}{r} \ddot{I}_{ij}^{\text{TT}}(u). \quad (5.11)$$

How to turn this into a flux and the quadrupole formula is described in Misner *et al.* (1973) or Schutz (1985).

Having learned that the radiation field is dominated by the quadrupole tensor  $I_{ij}$ , we turn now to the near zone to see if we can find the same tensor there. The natural place to look is in the analogue of the Newtonian potential,  $\bar{h}_{00}$ :

$$\bar{h}_{00}(t, \mathbf{x}) = 4 \int |\mathbf{x}-\mathbf{y}|^{-1} T_{00}(t-|\mathbf{x}-\mathbf{y}|, \mathbf{y}) d^3y.$$

Here we make the approximation of slow motion, and expand  $T_{00}$  in time about  $t$ :

$$\bar{h}_{00}(t, \mathbf{x}) = 4 \int |\mathbf{x}-\mathbf{y}|^{-1} T_{00}(t, \mathbf{y}) d^3y - 4 \int \dot{T}_{00}(t, \mathbf{y}) d^3y + \dots \quad (5.12)$$

The first term is the Newtonian field, and the second is zero by the law of conservation of energy. We shall drop the higher-order terms: their time derivatives make them small for slow motion. If we expand the Newtonian field for  $r > |y|$  we obtain

$$\begin{aligned} \bar{h}_{00} = \frac{4}{r} \int T_{00}(t, \mathbf{y}) d^3y + 4 \frac{n^i}{r^2} \int T_{00}(t, \mathbf{y}) y_i d^3y \\ - \frac{2}{r^3} (\delta^{ij} - 3n^i n^j) I_{ij} + \dots, \end{aligned} \quad (5.13)$$

where

$$n^i = x^i/r \quad (5.14)$$

is the unit radial vector. These are the monopole, dipole and quadrupole terms. The integral  $I_{ij}$  indeed appears here, and the temptation is to read it off the near-zone field (i.e. as an integral of  $\bar{h}_{00} Y_{2m}$  over a sphere) and use it to predict  $\bar{h}_{ij}$  in the far zone from Eq.(5.11).

There are two difficulties with this idea. The first is one of gauge. None of the currently working numerical codes uses Lorentz gauge, so one would have to transform gauge. This is a nonlocal transformation, involving a solution of the wave equation

$$\square \xi^\mu = \bar{h}^{\mu\nu},{}_{,\nu}$$

for the gauge vector  $\xi^\mu$ , and so it is hard to do. The second difficulty is that two time derivatives of  $I_{ij}$  are needed in Eq.(5.11). Not only does this degrade the accuracy of the numerical result for  $I_{ij}$ , but it also produces a term which is of the same order as the terms that have been neglected in the slow-motion expansion in Eq.(5.12). So the method may not even be self-consistent.

A better way is to look instead at the near-zone  $\bar{h}_{ij}$ , which is given by

$$\bar{h}_{ij}(t, \mathbf{x}) = 4 \int |\mathbf{x}-\mathbf{y}|^{-1} T_{ij}(t, \mathbf{y}) d^3y + \dots \tag{5.15}$$

$$= \frac{2}{r} \ddot{I}(t) + \dots, \tag{5.16}$$

with the same approximations as in Eqs. (5.12)-(5.13). This gives the second time derivative of  $I_{ij}$  directly, and it is the dominant term in the slow-motion limit, so it overcomes the second difficulty mentioned above. Motivated by the other difficulty of gauge dependence, we shall look at a related quantity, the spatial components of the Riemann tensor:

$$R_{ijkl} = \frac{1}{2}(h_{il,jk} + h_{jk,il} - h_{ik,jl} - h_{jl,ik}). \tag{5.17}$$

I shall now prove the following

$$\oint R_{ijkl} n^j n^l r^2 d\Omega = -\frac{64\pi}{15r} \ddot{I}_{ik} + \alpha \delta_{ik}, \tag{5.18}$$

where  $\alpha$  is a constant and the integral is over a sphere of radius  $r$ . Let us begin with the identity

$$\oint |\mathbf{x}-\mathbf{y}|^{-1} n_i n_j d\Omega_x = \frac{4\pi}{5} \frac{r_{<}^2}{r_{>}} (n_i n_j - \frac{1}{3} \delta_{ij}) + \frac{4\pi}{3} \frac{1}{r_{>}} \delta_{ij} \tag{5.19}$$

where  $r_{>}(r_{<})$  is the larger (smaller) of  $|\mathbf{x}|$  and  $|\mathbf{y}|$ , and the integral is over the unit sphere of  $\mathbf{x}$ . (The identity may be proved using the spherical-harmonic expansion of  $|\mathbf{x}-\mathbf{y}|^{-1}$ .) Now, we can find  $h_{ij}$  from Eq.(5.5) in the near zone,

$$h_{ij}(t, \mathbf{x}) = \int |\mathbf{x}-\mathbf{y}|^{-1} f_{ij}(t, \mathbf{y}) d^3y, \quad f_{ij} = 4(T_{ij} - \frac{1}{2}\delta_{ij} T^\alpha_\alpha). \tag{5.20}$$

This means that

$$\begin{aligned} h_{ij,k\ell}(t, \mathbf{x}) &= \int f_{ij}(t, \mathbf{y}) \frac{\partial^2}{\partial x^k \partial x^\ell} |\mathbf{x}-\mathbf{y}|^{-1} d^3y \\ &= \int f_{ij}(t, \mathbf{y}) \frac{\partial^2}{\partial y^k \partial y^\ell} |\mathbf{x}-\mathbf{y}|^{-1} d^3y \\ &= \int f_{ij,k\ell}(t, \mathbf{y}) |\mathbf{x}-\mathbf{y}|^{-1} d^3y. \end{aligned}$$



From this we find, assuming the sphere of radius  $r$  contains all the source (*i.e.* that  $|y| < |x|$ ),

$$\oint h_{ij,kl}(t,x) n_p n_s r^2 d\Omega_x = r^2 \int f_{ij,kl}(t,y) \oint \frac{n_p n_s}{|x-y|} d\Omega_x d^3y.$$

$$= \frac{4\pi}{5r} (\delta_{kp} \delta_{ls} + \delta_{ks} \delta_{lp} - \frac{2}{3} \delta_{kl} \delta_{ps}) \int f_{ij}(t,y) d^3y,$$

(5.21)

where the last step follows from applying Eq.(5.19) and integrating by parts on  $y$ . When this is put into the l.h.s. of Eq.(5.17), the part of  $f_{ij}$  containing  $T^\mu_\mu$  contributes only to the  $\alpha \delta_{ik}$  term on the r.h.s., which will drop out later. The part that involves  $T_{ij}$  introduces  $\bar{I}_{ij}$  from Eq.(5.10), and the result in Eq.(5.18) follows.

The  $\delta_{ik}$  term in Eq.(5.18) disappears when we take the TT-projection, as in Eq.(5.8). It follows, therefore, that an integral of certain components of the Riemann tensor over a sphere containing all the source and located in the near zone produces the radiation amplitude directly.

*b. Can we do the same in the Newtonian limit?*

The main difference between the calculation in Linearized Theory and here is that the stress tensor contains gravitational stresses as well,

$$T_{ij} \rightarrow T_{ij} + (4\pi)^{-1} (\nabla_i \Phi \nabla_j \Phi - \frac{1}{2} \delta_{ij} \nabla^k \Phi \nabla_k \Phi),$$

(5.22)

where  $\Phi$  is the Newtonian potential. The integral for  $\bar{I}_{ij}$  is no longer over a finite domain, so in obtaining the analogue of Eq.(5.21) we need to integrate over regions where  $|y|$  is larger than  $|x|$ . From the identity Eq.(5.19) it can be shown that this adds extra terms to Eq.(5.18) that are of order  $(M/r)^2 \delta_{ik}$  and  $MI_{ij}/r^4$ . The  $\delta_{ik}$  terms goes away when we take the TT part, and the other error can be made small by choosing  $r$  sufficiently large. So on the face of it, the method seems to work here as well.

But the situation is not quite as rosy as this if we consider the effect of numerical errors. The numerical TT-projection will not be perfect, so there will be an error of order  $(M/r)^2 \mu$  in Eq.(5.18), where  $\mu$  is a measure of the relative numerical error in the Riemann tensor. The ratio of this to the term we want in Eq.(5.18) is

$$\text{relative error} \sim \frac{(M/r)^2 \mu}{I/r} \sim \frac{M^2/r}{I} \mu.$$

Now, by the virial theorem,  $\dot{I} \sim \epsilon M v^2 \sim \epsilon M^2/R$ , where  $R$  is the radius of the collapsing body and  $\epsilon$  is a (possibly small) number measuring the nonsphericity of the collapse. We have

$$\text{relative error in radiation} \sim \frac{R}{r} \frac{\mu}{\epsilon} \quad (5.23)$$

Now,  $R/r$  may be small, but not very small, since the whole point of this is to enable one to work with a small grid. Therefore we need  $\mu/\epsilon \lesssim 1$ : the method ought to work in the Newtonian limit if the numerical accuracy  $\mu$  of the Riemann tensor is good and the amount of radiation  $\epsilon$  is not abnormally small, as in a highly symmetrical collapse. Since our interest is in nonsymmetrical collapse, this is an encouraging result.

c. *Will it work in strong-field collapse?*

At first one might expect that a Newtonian result would fail for strong fields, but we are rescued here by the remarkable fact that, at least in slow motion, the equivalence principle applies in general relativity even to "gravitational potential energy". This means that, if motions are slow, there is a region outside any body, no matter how compact, where the field is basically Newtonian and therefore insensitive to the body's compactness. This region is called the near zone. This is borne out by calculations in different contexts by D'Eath (1975), Kates (1980), Futamase (1985), and Damour (1983). Thorne (1980) has stressed that this means that the radiation field of such a body must be the same as the radiation field of a Newtonian body with the same near-zone field, since the radiation field must be determined by the near-zone field. Therefore, we can expect Eq.(5.18) to give the radiation even in strong-field collapse, provided the collapse is slow compared to the speed of light. This might happen if centrifugal effects, for example, hold up the collapse.

In the general strong-field case, however, collapse will be fast, and we can expect Eq.(5.18) to give no more than a rough idea of the radiation. Nevertheless, "rough" might still be within a factor of two of the right result. The only way to learn the accuracy of the formula is to test it on an axisymmetric numerical calculation in which the radiation is calculated directly.

I would strongly urge that such tests ought to be undertaken, because if the formula should prove to be reasonably reliable it will permit a dramatic improvement in the speed with which collapse codes can run and in the accuracy of the first 3-D collapse calculations in relativity.

d. *Are there better ways of doing this?*

The method based on Eq.(5.18) is easy but crude. Much more sophisticated approaches to the problem of treating the radiation field analytically are possible. The lectures by Anderson & Hobill in this volume describe a method of doing this for the scalar wave equation on a Schwarzschild background spacetime, in which one solves the vacuum field equations outside the source as an iterative series in  $M$  (the Schwarzschild mass), and then one uses the solution to set boundary conditions on the numerical interior solution. This in turn provides field variables at the interface between the interior and exterior which determine the outgoing field.

This can be done in principle in general relativity, and of course Anderson and Hobill have this very much in mind. A possible theoretical framework for doing these calculations has recently been worked out by Blanchet & Damour (1986), as an elaboration of slow-motion methods of Bonnor (1959) and Thorne (1980).

Such schemes offer the chance to reduce grid sizes, to make the numerical calculations run either faster or more accurately. But if the exterior analytic solution is so complicated that the matching procedures begin to carry a significant computing cost, then there may be little saving in the end. Nevertheless, such a method could provide a useful alternative to the standard approaches and act as a check on their accuracy.

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