THE USE OF PERTURBATION AND APPROXIMATION METHODS IN GENERAL RELATIVITY

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1. Why Study Perturbation and Approximation Theory?

Let me begin by saying what a pleasure it is to give these lectures in such beautiful surroundings, and I hope that this meeting will stimulate many more contacts between us.

I want to preface my lectures with some remarks on the role that approximation theory plays in general relativity. Relativists seem to take a perverse pride in the fact that Einstein's equations are hard to solve. We are generally fond of blaming most of our problems on the fact that there are few exact solutions that describe physically interesting situations, especially dynamical ones. In fact, however, this is a circumstance that we share with most of rest of physics. Most important physical equations are nonlinear, and in most branches of physics exact solutions are hard to obtain except in the simplest situations. Although some field equations are linear, such as Schrödinger's equation or Maxwell's equations,

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as soon as they are coupled to dynamical sources the system becomes nonlinear. What really distinguishes general relativity from other branches of physics is that it is not an experimental science. Even when we can observe situations in astrophysics where general relativity is important, we cannot control or even observe all the variables that one would normally control in a well-designed laboratory experiment.

The difficulty of solving equations means that in all branches of physics progress can normally be made only by using idealizations and approximations. In most of physics, what happens is that someone makes an assumption or approximation, and that leads to some prediction. Others may make different approximations and derive different predictions. The question of which approximation is justified is resolved by experiment. Nobody worries too much about the rigor of the methods: their justification is in the answers they give. If the methods are really difficult to justify mathematically, they can even be incorporated into the 'axioms' of the physical theory, as has happened with renormalization in quantum field theory. Some of the most respected physicists are precisely those who have the insight and intuition to develop successful approximations and idealizations. The BCS theory of superconductivity is a case in point: the fundamental equations —Schrodinger's and Maxwell's equations for N-body systems —were already known but impossible to solve. The Nobel-prize-winning achievement was to know what extraneous degrees of freedom one should throw away in order to arrive at a tractable idealization that worked, i.e., that explained the experimental results.

In general relativity we also have to make approximations, and we also often disagree about what approximation methods are appropriate. But we cannot resolve the questions experimentally, at least not with quite the finality that other branches of physics can. It is therefore harder to resolve disagreements, and questions of mathematical rigor play a greater role.

The debate over the validity of the so-called quadrupole formula for gravitational radiation illustrates this difficulty. The initial derivations of the quadrupole formula used approximation methods adapted from ones which have worked successfully in other branches of physics; they were guided by considerable physical intuition. But when looked at with a more rigorous eye, they were flawed: there were hidden divergences, unspoken assumptions, incomplete analysis. The question was, do we believe the results despite these flaws, relying instead on physical intuition? There was only one observed fact which supported the quadrupole formula: The rate of change of the orbital period of the binary pulsar system.[1] But this is only an observation, not an experiment. We cannot change some parameters to test the theory. If the quadrupole formula were wrong, we could undoubtedly find another explanation for the changing period.

It has taken considerable time to achieve a consensus on the issue of the quadrupole formula. It has proved possible to add rigor to the earlier derivations, a step which has not only cleared up some problems but has also taught us new things about the Newtonian approximation. It has also proved possible to derive the quadrupole formula from a number of independent points of view. I will return to this subject in §7.

This example suggests that we in relativity should recognize that approximation theory is one of the most important tools we have for getting physical predictions from Einstein's equations. Our mathematical education tends to stress the 'exact' disciplines, such as geometry, group theory, and functional analysis. We know, as a whole, far less about approximations than do our colleagues in other branches of physics. But in order to make contributions to the interesting dynamical problems that astrophysics will throw our way in the coming decade — the decade of the Space Telescope and the first gravitational wave detectors with interesting sensitivity—we will have to become much more comfortable with approximation theory. Approximating general relativity, both analytically and numerically, is one of the 'growth areas' of our dicipline.

The plan of my lectures is as follows. After a short review of approximation theory for simple functions (§2), we will study the simplest 'approximate' solutions: waves on fixed backgrounds, such as stars and black holes (§3). Then in §4 we will study linear perturbations of Einstein's equations and meet some puzzles regarding second-order conserved quantities. To solve these we need to define higher-order perturbation theory more carefully (§5) and then apply our new methods to practical problems, like the gravitational-wave instabilities in rotating stars (§6). Finally we put these methods to work on the more difficult problem of the (singular) Newtonian limit, and establish the validity of the quadrupole formula (§7).

Footnotes §1

1. J. H. Taylor and J. M. Weisberg, Astrophys. J. 253, 908 (1982); and V. Boriakoff, D. C. Ferguson, M. P. Haugan, Y. Terzian, and S. A. Teukolsky, Astrophys. J. 261, L101 (1982).

2. Approximations in Physical Theories

Because most physical problems are too hard to solve exactly, most physical predictions emerge from various types of approximations. There are three main types, which I list in ascending order of their usefulness: convergent Taylor expansions, asymptotic approximations, and physical idealizations. It is useful to begin our study by reviewing what these are and how they are used.

2.1 Taylor Expansions. An analytic function $f(x)$ has the representation

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0) \qquad , \tag{2.1}$$

which converges for all complex x in some neighborhood N of the origin, $|x| < x_0$, say. The very existence of such an expansion for f may be useful for proving analytic theorems about f, but for physical predictions the usefulness of (2.1) is that for any $x \varepsilon N$ we may compute $f(x)$ to any desired accuracy by taking enough terms. Equation (2.1) may therefore be regarded as giving an arbitrarily accurate approximation to the value of f at some (any) point $x \varepsilon N$.

2.2 Asymptotic Approximations. In most physical problems we do not try to use a large number of terms in (2.1) even when we know f is analytic, either because that would be too cumbersome or because we do not know more than the first few terms of the expansion. We therefore deduce from (2.1) an asymptotic approximation to f, such as

$$f(x) = f(o) + xf'(o) + O(x^2) \tag{2.2}$$

where $O(g(x))$ stands for any function $h(x)$ such that $\lim_{x\to 0}(h(x)/g(x))$ exists. If this limit is zero then we can use the lower-case symbol $o(g(x))$. Equation (2.2) is not guaranteed to be arbitrarily accurate at any $x \varepsilon N$. Instead, if we are given some accuracy limit δ, then we are guaranteed that we can find some sufficiently small x_1 such that $|f(x) - (f(o) + xf'(o))| < \delta$ for all $|x| < x_1$. It is best, therefore, to regard Eq. (2.2) as an approximation to the function f rather than to its value at any x, since to get increased accuracy we have to change x.

Asymptotic approximations are used in the overwhelming majority of cases where physicists need numerical answers. For example, the approximation $J_0(x) = 1 - x^2/4$ for the Bessel function is very useful if x is small. Its somewhat more accurate extension, $J_0(x) = 1 - x^2/4 + x^4/64$, provides only a small extension to the useful range of values of x. And if we want $J_0(x)$ for x near 10, taking more and more terms of its expansion about $x = 0$ is not the way to calculate it: one looks up an asymptotic approximation[1] suitable for this x.

There are other ways of generating asymptotic approximations than simply truncating Taylor expansions. By Taylor's theorem any C^n function whose (n+1)-st derivative exists has the asymptotic approximation

$$f(x) = f(o) + xf'(o) + \ldots + \frac{x^n}{n!} f^{(n)}(o) + R_{n+1} \tag{2.3}$$

where the remainder term R_{n+1} is

$$R_{n+1} = \frac{x^{n+1}}{(n+1)!} \int_0^1 (1-\ell)^{n+1} \frac{d^{n+1}f}{dx^{n+1}} (\ell x)d\ell = o(x^n) \quad . \tag{2.4}$$

Notice that $f^{(n+1)}$ is needed over the whole range (o,x) in order to calculate R_{n+1}, but R_{n+1} may be bounded by a bound on $f^{(n+1)}$. This is the sort of approximation generated by iteration schemes, such as the slow-motion approximation in general relativity, which do not á-priori know that f is analytic but simply generate one term after another of the series (2.3). We will discuss this in detail in the final section.

Another useful asymptotic approximation is that by continued fractions. These can be convergent but are most often used asymptotically. They are very useful in numerical analysis[2,3] but are far less useful than power-series methods in approximations to operators and matrices, because they involve inverses that are hard to compute.

The various methods of numerical analysis can also be viewed as asymptotic approximations. For example, finite-difference methods get better as the grid is refined. Most such methods can be viewed as generalized approximations by polynomials or rational fractions (equivalently continued fractions), whose order and whose coefficients change as the 'smallness parameter' (grid size) changes.

Perturbation theory, which is the study of small changes away from known solutions of whatever field equations one has, is a form of asymptotic approximation, valid when the perturbation is sufficiently small. We shall see in §5 how to view it as a special case of Eq. (2.3).

2.3 Idealizations. Perhaps the most widely used form of approximation in physics is idealization. It is so ubiquitous that it is easy to forget that it is there, but it often plays a crucial role. An idealization reduces the complexity of a problem by throwing away elements that are regarded as unimportant. Thermodynamics and other continuum theories of physics are idealizations that throw away the variables associated with individual atoms in favor of macroscopic averages. The successful BCS theory of superconductivity throws away most electron-electron interactions, focussing on the spin correlations called Cooper pairs. Solutions in general relativity which apply an asymptotically-flat boundary condition throw away the rest of the universe surrounding the system of interest. The primary objective of astrophysical modelling is to find a successful and simple idealization for observed phenomena.

The difficulty with idealizations is that it is sometimes hard to estimate their errors. This is particularly true in astrophysics, where the physical system cannot be manipulated to test an idealization. I will return to this problem when we discuss the binary pulsar system in the final section.

Footnotes 2

1. For example, M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, Washington, D.C., 1964). Even the polynomial approximations given here and elsewhere are asymptotic approximations in the sense of numerical calculations, described below.
2. F. S. Acton, Numerical Methods that Work (Harper and Row, N. Y. 1970).
3. A nice application in relativity is J. M. Cohen and M. W. Kearney, Astrophys. Sp. Sci. 70, 295 (1980).

3. Normal Modes of Wave Fields

The simplest introduction to perturbation theory in general relativity is to study scalar wave fields on fixed backgrounds. All massless, integer-spin vacuum wave equations have basically the same character, particularly when the wavelength is short compared to the mean radius of curvature of the spacetime, so a study of the scalar wave equation provides a good introduction to the behavior of electromagnetic and gravitational wave fields on fixed backgrounds.

3.1 Modes of the Schwarzschild Metric. Because the Schwarzschild metric is a vacuum metric, all small perturbations (that do not change its mass to first order) are gravitational waves, so the study of scalar waves should help us understand the metric's dynamical properties, especially its stability. Because of its spherical symmetry, the scalar wave equation[1]

$$\nabla_\mu \nabla^\mu \chi = 0 \tag{3.1}$$

separates with the assumption

$$\chi(x^\mu) = \frac{1}{r}\,\psi(r) Y_{\ell m}(\theta,\phi) e^{i\omega t} \tag{3.2}$$

to give a radial equation for ψ:

$$r(r-2M)\psi'' + 2M\psi' + \left[\frac{r^3\omega^2}{r-2M} - \frac{2M}{r} - \ell(\ell+1)\right]\psi = 0 \quad .$$

We can eliminate the first-derivative term by defining a new radial coordinate r_* by

$$dr_*/dr = r/(r-2M) \ , \quad r_* = r + 2M \,\ell n\left|\frac{(r-2M)}{M}\right| \tag{3.3}$$

We obtain

$$\frac{d^2\psi}{dr_*^2} + [\omega^2 - V(r_*)]\psi = 0 \tag{3.4}$$

with the 'potential' V given implicitly in terms of $r(r_*)$ by

$$V(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right) \quad . \tag{3.5}$$

This equation is of a type familiar from one-dimensional wave-scattering problems. As $r \to \infty$, r_* approaches r with a small logarithmic correction (similar to the Coulomb wave equation). Near the horizon 2M, things are different: $r \to 2M \Longrightarrow r_* \to -\infty$. (For this reason r_* is sometimes called the tortoise coordinate: a step in r_* is a very small step in r.) Plotted against r_*, the potential looks like Fig. (3.1).

Anyone who has studied the null geodesics of the Schwarzschild metric will recognize that for large ℓ, V(r) becomes the effective potential governing massless particles, which is reassuring. Our interest here, however, is in wave motion. It is clear from Fig. (3.1) that $V(r_*)$ vanishes fast enough near $r_* = \pm\infty$ that the asymptotic solutions are $\exp(\pm i\omega r_*)$ in both regions. The physical boundary condition is that the waves be ingoing at the horizon.

For our convention on ω this means

$$\psi \sim Z_H e^{i\omega r_*} \quad , \qquad r_* \to -\infty \quad , \tag{3.6}$$

for some complex horizon amplitude Z_H. Far away the wave has the form

$$\psi \sim Z_{out} e^{-i\omega r_*} + Z_{in} e^{i\omega r_*} \quad , \quad r_* \to \infty \quad . \tag{3.7}$$

We know from the Schrodinger equation that when ω^2 is real, $|Z_{in}|^2 = |Z_{out}|^2 + |Z_H|^2$.

Figure 3.1. The effective potential for the scalar wave problem in Schwarzschild. As $r_* \to \infty$, $V(r_*) \sim \ell(\ell+1)/r_*^2$, as in flat spacetime. As $r_* \to -\infty$ (toward the horizon), $V(r_*) \sim \exp(r_*/M)$.

Our interest here is the normal modes of this equation, by which we mean soutions which are not 'driven' by an incoming wave but which have $Z_{in} = 0$. These are impossible if ω is real but may exist for complex ω, and the condition $Z_{in}/Z_{out} = 0$ may be regarded as an eigenvalue condition that determines ω. The idea (which can easily be shown to be correct by the method of Laplace transforms) is that the evolution of Φ will generally involve a superposition of these normal modes. If any of the modes has $\mathrm{Im}(\omega) < 0$ then it will grow exponentially in time and the metric will be unstable. It is easy to show from the positive-definiteness of $V(r_*)$ that no solutions of Eq. (3.1) grow unboundedly in time, so Schwarzschild is in fact stable.[2]

The modes of Schwarzschild are nevertheless of interest, both because they can be expected to give the characteristic frequencies of gravitational waves that reach us just after the hole has been formed in gravitational collapse, and because they are important in the study of quantized massless fields on the black-hole background. If we use some 'WKBJ-intuition' when we look at Fig. 3.1, we will see that we can only hope to find a solution in which $Z_{in} = 0$ if we choose $\mathrm{Re}(\omega^2) \approx V_{max}$. For $\mathrm{Re}(\omega^2) > V_{max}$, Z_H matches to Z_{in}, not Z_{out}. For $\mathrm{Re}(\omega^2) \ll V_{max}$, there is too much tunnelling to permit Z_H to match to just Z_{out}. For $\mathrm{Re}(\omega^2) \approx V_{max}$ we can approximate the top of the potential by a parabola, obtaining a problem that can be solved exactly by parabolic cylinder functions. Although this exact solution is not valid far away from the peak of V, it suffices to determine the mode condition, because it can distinguished a wave approaching the peak from one moving away from it. The algebra gives, in the limit of large ℓ, the sequence of modes $\omega_{\ell n}$,

$$M\omega_{\ell n} = [\ell + i(n+1/2)]/3\sqrt{3} \quad , \tag{3.8}$$

where n is a nonnegative integer.[3] Table 3.1 compares this method with the numerical integrations of the Regge-Wheeler[2] equation by Chandrasekhar and Detweiler.[4] (The R-W equation governs gravitational wave motion and differs from our scalar wave equation only in that the last term in Eq. (3.5) is $-6M/r^3$. This makes no difference for large ℓ.) Agreement is excellent for the fundamental mode n = 0, especially for its imaginary part, and gets worse as n increases. All the modes decay exponentially, as expected. In the limit as $\ell \to \infty$ all the fundamental modes have the same imaginary part, $\mathrm{Im}(\omega) = 1/6\sqrt{3} = 0.09623$, a fact observed numerically by Detweiler.[5] The approximation (3.8) gives an infinite number of modes of increasing imaginary part, but numerical searches have only found a few modes, and an approximate potential which can be solved analytically has only a finite number.[4] This leads to our first suggested research problem.

Problem 3.1. Determine whether Eq. (3.4) has a finite number of modes for each ℓ and devise an efficient numerical scheme for searching for those with large imaginary parts. The techniques of Kearney, et al.[6] may be helpful here.

Table 3.1. Comparison of Regge-Wheeler modes calculated numerically by Chandrasekhar and Detweiler[4] and by our parabolic approximation to the top of the potential[3] (not assuming ℓ is large).

ℓ	n	$M\omega$ (numerical)	$M\omega$ (parabolic approx.)
2	0	0.3737 + 0.0889i	0.3988 + 0.0882i
	1	0.3484 + 0.2747i	0.4534 + 0.2329i
3	0	0.5994 + 0.0927i	0.6166 + 0.0923i
	1	0.5820 + 0.2812i	0.6619 + 0.2580i
4	0	0.8092 + 0.0941i	0.8223 + 0.0939i
	1	0.7965 + 0.2844i	0.8601 + 0.2694i

3.2 Modes of Metrics Representing Compact Stars. When the Schwarzschild horizon is replaced by a compact star, the character of the problem changes dramatically. Supposing the star to be transparent to our scalar field, then if the star is not very compact, the metric is nearly that of flat spacetime and the wave equation has the form

$$\psi'' + [\omega^2 - \ell(\ell+1)/r^2]\psi = 0 \quad ,$$

where primes denote derivatives with respect to r. This has the characteristic angular momentum barrier at $r = 0$ (Fig. 3.2a). It can be shown that there is no value of ω for which $Z_{in}/Z_{out} = 0$.

If, on the other hand, the star is so compact that its radius is less than 3M, then the wave equation will have an effective potential that looks like Fig. 3.2b.

Problem 3.2. For the Schwarzschild interior solution[1] for a uniform-density star of radius $R_s < 3M$, show that the local minimum of $V(r)$ in Fig. 3.2b corresponds to the existence of stable null circular geodesics inside the star. Calculate (in a WKBJ approximation or with more accuracy) the normal modes the wave field has for $\mathrm{Re}(\omega^2)$ near this local minimum, and show that they decay exponentially in time. Show that there are only a finite number of modes.

If we allow R_s to increase for fixed M, eventually the star gets so diffuse that the minimum and maximum in Fig. 3.2b disappear and we recover Fig. 3.2a. The potential $V(r)$ changes continuously with R_s (though it is not analytic in either r or R_s), and we would like to think that the normal-mode eigenfrequencies change smoothly with R_s. But clearly they all have to disappear as R_s gets large enough. If, as suggested in Problem 3.2, there are only a finite number of them, then presumably they disappear one-by-one.

Problem 3.3. Study the disappearance of the modes of Problem 3.2 as R_s increases. What happens to their role in the description of the evolution of an arbitrary initial perturbation?

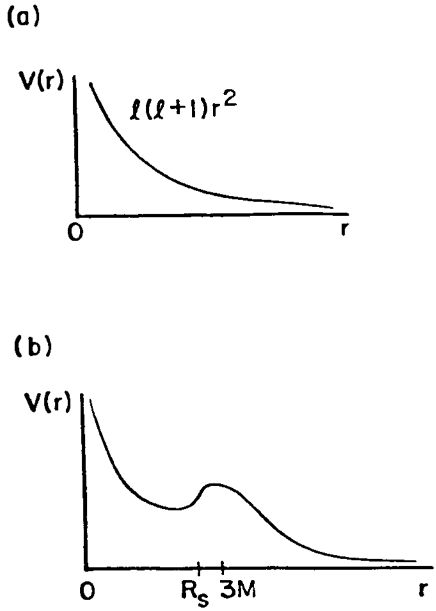

Figure 3.2. (a) The effective potential of flat spacetime is just the angular-momentum term. (b) Quantitative character of V(r) for a star of radius $R_S < 3M$. The potential for $r > R_S$ is the same as in Figure 3.1, so it has a maximum near 3M. Near $r = 0$ spacetime is flat and the angular-momentum rise dominates.

3.3 Modes of Rotating Stars with Ergospheres. Things get even more interesting when we allow our stars and black holes to rotate. Once a star rotates, spherical symmetry is broken and the wave equation no longer separates. But we can get a qualitative understanding of the effects of rotation by studying waves on the following background metric:

$$ds^2 = - e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta (d\phi - \sigma(r) dt)^2 \quad . \tag{3.9}$$

This turns out to be a good approximation for compact rotating stars.[7] The scalar wave equation separates, using spherical harmonics, but now because the metric is not invariant under the reflection $t \to -t$ but is still invariant under $\{t \to -t$ and $\phi \to -\phi\}$, the radial equation has terms not just in ω^2 but also in $m\omega$, where m is the axial eigenvalue of $Y_{\ell m}$. Separating the equation as before and defining

$$dr_*/dr = \exp(\Lambda - \Phi) \quad , \tag{3.10}$$

we obtain the radial equation

$$\frac{d^2\psi}{dr_*^2} + \left[(\omega + m\sigma)^2 + \frac{\Lambda' - \Phi'}{r} e^{2(\Lambda-\Phi)} - \frac{\ell(\ell+1)}{r^2} e^{2\Phi} \right] \psi = 0 \ . \tag{3.11}$$

Since σ is a function of r, this contains linear terms in ω in an essential way. Such cases do not arise in the Schrödinger equation, but we can generalize our intuition by factoring the potential term:

$$\frac{d^2\psi}{dr_*^2} + (\omega - V_+)(\omega - V_-)\psi = 0 \tag{3.12}$$

where

$$V_{\pm}(r_*) = -m\sigma(r) \pm \left[\frac{\ell(\ell+1)}{r^2} e^{2\Phi} - \frac{\Lambda' - \Phi'}{r} e^{2(\Lambda-\Phi)}\right]^{\frac{1}{2}} \quad (3.13)$$

again depending on r_* implicitly through r. In this case r_* is a regular function of r and we can set $r_* = 0$ where $r = 0$, while as $r \to \infty$, r_* approaches r. The potentials $V_{\pm}$ are sketched in Fig. 3.3a for a slowly-rotating star (σ small) and in Fig. 3.3b for a very rapidly rotating star. In the limit of large ℓ and $m = \ell$ we should recover the equation for null geodesics in the equatorial plane, and (3.13) becomes

$$V_{\pm} \simeq m[-\sigma \pm e^{\Phi}/r] \quad . \quad (3.14)$$

We can see that V_+ is negative, as in Fig. 3.3b, where $e^{\Phi} - r\sigma$ is negative. From the metric (3.9), this is where g_{00} in the equatorial plane is positive: this is an ergosphere, like that outside the Kerr horizon. Because there is no horizon, it is toroidal in shape.

How do we 'read' factored-potential diagrams like Fig. 3.3? In the standard picture, Fig. 3.2, the wave has oscillatory character where $\omega^2 > V(r_*)$, because there $\psi^{-1}\, d^2\psi/dr_*^2$ is negative. This is the classically allowed region, in quantum-mechanical language. Conversely, the classically forbidden, tunnelling region is where $\omega^2 < V(r_*)$, i.e., where $\psi^{-1}\, d^2\psi/dr_*^2 > 0$. In our factored potentials, $\psi^{-1}\, d^2\psi/dr_*^2$ will be negative where ω exceeds both V_+ and V_- or is less than both, so the wave-like region, marked 'allowed' in Fig. 3.3b, is above V_+ and below V_-. Similarly, the tunnelling, 'forbidden' region is between them, where $\psi^{-1}\, d^2\psi/dr_*^2 > 0$. In the limit of large m these potentials govern equatorial null orbits. In Fig. 3.2b there was a single stable photon orbit at the minimum of V(r). This is really two orbits, since a photon can go around in either direction. In the rotating case, Fig. 3.3a, these two orbits occur, respectively, at the minimum of V_+ and the maximum of V_-: they have different radii and one frequency is no longer the negative of the other. Since the wave solution is proportional to $\exp(im\phi+i\omega t)$, and we assume without loss of generality that $m>0$, a wave for which $\omega>0$ has $d\phi/dt<0$: it is counterrotating. So V_+ governs counterrotating orbits and V_- corotating ones. When there is an ergospere, as in Fig. 3.3b, both circular orbits have become corotating: no particle can remain at rest.

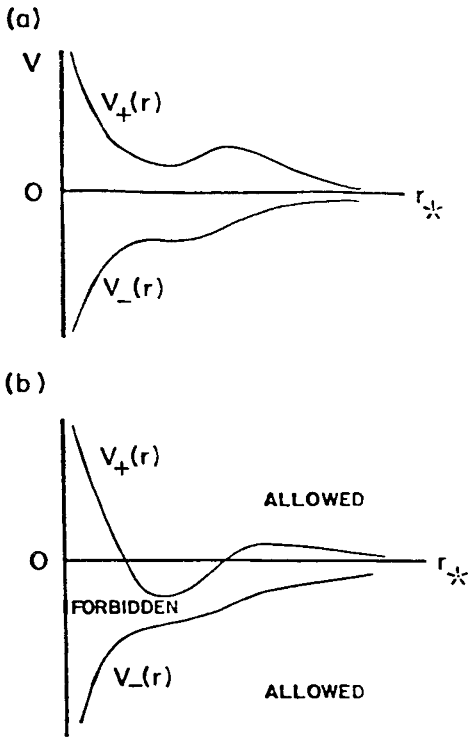

Figure 3.3. (a) The factored potentials for a slowly rotating version of the star in Fig. 3.3b. They are basically tne square roots of $V(r)$ for the spherical star. (b) If $\sigma(r)$ is sufficiently large, V_+ can dip below the axis.

In Problem 3.2 we pointed out that there would be normal modes of the scalar field associated with the stable photon orbit of Fig. 3.2b. Comins and I[8] have calculated these for Fig. 3.3 in the WKBJ approximation. In Fig. 3.3a the potentials give rise to exponentially damped normal modes. But when, as in Fig. 3.3b, V_+ dips below the axis, there are exponentially growing modes: the star with an ergosphere is unstable. This is the ergosphere instability discovered by Friedman.[9] On physical grounds the instability is easy to understand. The mode in question corresponds to a photon orbit of negative energy in the ergosphere. When the wave tunnels through the barrier to large r it radiates positive energy. This can only be compensated by decreasing the already negative energy in the ergosphere. The wave amplitude grows there, sending more waves to infinity, thereby amplifying itself further.

3.4 Modes of the Kerr Metric. Although stars with ergospheres may be rare in nature,[7] we believe that black holes with the Kerr metric may be all over the place. Does the ergosphere cause a similar instability? This question still has no definitive answer, although Detweiler[10] has recently suggested that a subtle kind of instability does exist.

In contrast to the case of the rotating star, we need make no approximations to separate the wave equation on the Kerr background: the equations for scalar, electromagnetic, and gravitational waves all separate[11] using spin-weighted spheroidal harmonics. For scalar waves, these are the ordinary spheroidal harmonics whose properties are catalogued in a number of books.[12] Our discussion of the Kerr problem will be qualitative; the most complete study to date is in a recent book by Chandrasekhar.[13]

The factored potentials for the scalar wave equation are sketched in Fig. 3.4. There is only one feature of this diagram that our previous discussion has not prepared us for: the fact that both potentials limit to $-m\Omega_H$ as we approach the horizon, $r_* \to -\infty$, where Ω_H is the angular velocity of the horizon itself. This means that the wave behaves like a free wave in the two asymptotic regions $r_* \to \pm\infty$, but with a different zero point for the frequency. Moreover, the correct boundary conditions at the horizon depend on the frequency. For $\omega > 0$ and $\omega < -m\Omega_H$, the condition is the usual one that the wave be ingoing. But for $0 > \omega > -m\Omega_H$ (the 'superradiant' frequency range) the wave carries negative energy and its phases are <u>outgoing</u> near the horizon, as measured by the unphysical observer at rest with respect to infinity. (As measured by an observer on a timelike world line near the horizon, all three frequency ranges have ingoing phases and ingoing energy with these boundary conditions.)

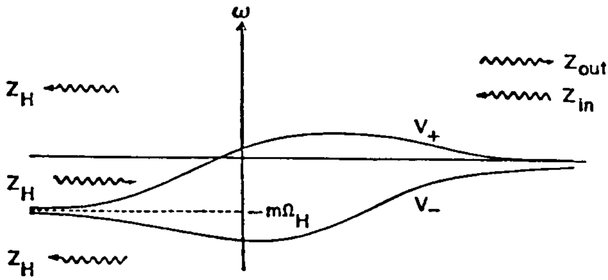

Figure 3.4. Factored potentials for the Kerr problem. The ergosphere causes V_+ to dip below the axis, and it stays below as $r_* \to -\infty$ because the ergosphere continues to horizon. Both $V_\pm$ limit to $-m\Omega_H$ at the horizon, where Ω_H is the angular velocity of the horizon, $a/2r_+M$. The appropriate wave conditions near the horizon are shown for the different frequency regimes.

Our study of Schwarzschild suggests to us that there will be modes at the peaks of $V_\pm$. If Fig. 3.4 is qualitatively correct for all possible parameters in the problem (ℓ,m, and a) then Re(ω) will be outside the superradiant regime for these modes. They will carry positive energy to the horizon and to infinity, and they will be damped. But are there any analogues of the ergosphere modes of Fig. 3.3b here? The long tunnelling region for a mode with $0 > \omega > -m\Omega_H$ in Fig. 3.4 suggests not: WKBJ intuition argues that it would be impossible to match a purely outgoing wave on the left to a purely outgoing wave on the right unless the tunnelling region were small.

Detweiler[10] has given a delicate asymptotic analysis which indicates that, for a nearly equal to M all Re(ω) near $-m\Omega_H$, such modes do exist and are unstable. Is it possible that our picture in Fig. 3.4 is wrong in some important detail for a near M? It is hard to know what detail to look for, but the following properties of V can be shown to hold for all $a < M$: (i) V_- decreases away from the horizon, so it has a minimum at some $\omega < -m\Omega_H$; (ii) V_+ increases away from the horizon and has a maximum at some positive value of ω. Perhaps $V_\pm$ have secondary maxima or minima that we do not know about, or perhaps they even intersect somewhere. Unlikely as these possibilities seem, the reason that they cannot yet be discounted is that we do not yet know enough about $V_\pm$. Their form is:

$$V_\pm = -\frac{2mar}{(r^2+a^2)^2} \pm \frac{(r^2-2r+a^2)^{\frac{1}{2}}}{r^2+a^2}\left[\lambda + 2\frac{r-a^2}{r^2} - m^2a^2\,\frac{r^2+2r+a^2}{(r^2+a^2)^2}\right]^{\frac{1}{2}}. \tag{3.15}$$

The problem is with λ, which is the separation constant from the spheroidal harmonic equation. For the spherical case ($a = 0$) we have $\lambda = \ell(\ell+1)$. But when $a \neq 0$, λ is a function of $a^2\omega^2$. That means that Fig. 3.4 is somewhat deceptive: $V_\pm$ have a (hopefully weak) dependence on ω. Even in the limit $m \simeq \ell \to \infty$, $\omega \approx -m\Omega_H \to -\infty$, there are few analytic results for λ to help us.

Problem 3.4. Obtain asymptotic expressions for λ in the limit $m \to \infty$, ℓ/m fixed, ω/m fixed, perhaps by the methods of Thorne.[14] Use these to investigate Detweiler's[10] modes.

Footnotes §3

1. Our conventions on notation follow C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973). In particular we set c = G = 1 and let Greek indices range from 0 to 3.
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4. Linear Perturbations of General-Relativistic Solutions

In the last section we argued on physical grounds that a massless scalar field was not a bad approximation to a perturbed gravitational field, but that argument clearly fails if the wave has to interact with a matter perturbation. In this section we study the full perturbed Einstein equations for a self-gravitating perfect fluid.[1] The equations have some beautiful symmetry properties which enable powerful theorems to be proved with relative ease. Nevertheless, when we investigate them more closely we find puzzles that we will not be in a position to unravel until §6.

4.1 Relativistic Fluid Dynamics.

A perfect fluid in general relativity is characterized by the stress-energy tensor

$$T^{\mu\nu} = (\rho+p)U^{\mu}U^{\nu} + pg^{\mu\nu} \quad , \qquad (4.1)$$

where ρ is the total energy density and p the pressure in each fluid element's momentary rest frame. If we assume a particle conservation law

$$(n\,U^{\mu})_{;\mu} = 0 \qquad (4.2)$$

then the equation of motion

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad (4.3)$$

implies that the specific entropy is also conserved:

$$U^{\mu}S_{,\mu} = 0 \quad . \qquad (4.4)$$

There is in fact another conservation law, that of vorticity conservation. If we define the specific momentum to be

$$V_{\mu} = (\rho+p)U_{\mu}/n \qquad (4.5)$$

then if the specific entropy is uniform, $\nabla_\mu S = 0$, the curl of the specific momentum (the vorticity) is convected with the fluid,[2]

$$\mathcal{L}_u(\nabla_\mu V_\nu - \nabla_\nu V_\mu) = 0 \quad , \tag{4.6}$$

where $\mathcal{L}_u$ denotes the Lie derivative[3] along the vector field U^μ. This is the relativistic form of the well known Helmholtz vorticity-conservation theorem. Less well known, but just as important for our purposes in §6, is its generalization to fluids in which $\nabla_\mu S \neq 0$. This was first discovered for nonrelativistic fluids by Ertel[4] and generalized to relativity by Friedman:[5]

$$\mathcal{L}_u(\nabla_{[\alpha} S \nabla_\beta V_{\gamma]}) = 0 \quad . \tag{4.7}$$

4.2 <u>Perturbation Equations.</u> Suppose the set $(g_{\alpha\beta}, \rho, p, U^\alpha)$ satisfies Einstein's equations on some manifold M. How are we to describe a perturbation of this solution, by which we mean another solution $(\bar{g}_{\alpha\beta}, \bar{\rho}, \bar{p}, \bar{U}^\alpha)$ on the same manifold M which is not very different from the first solution? The most naive way is simply to assume that the differences between the solutions at the same coordinate points are small. These are called the Eulerian perturbations:

$$\begin{aligned} h_{\alpha\beta}(x^\mu) &= \bar{g}_{\alpha\beta}(x^\mu) - g_{\alpha\beta}(x^\mu) \\ \delta\rho(x^\mu) &= \bar{\rho}(x^\mu) - \rho(x^\mu) \\ \delta p(x^\mu) &= \bar{p}(x^\mu) - p(x^\mu) \\ \delta U^\alpha(x^\mu) &= \bar{U}^\alpha(x^\mu) - U^\alpha(x^\mu) \end{aligned} \tag{4.8}$$

are all small in some sense. This is the prescription we shall adopt here. In §5 we will refine it to make it at once more geometrical and more amenable to asymptotic analysis. We regard $h_{\alpha\beta}$, $\delta\rho$, δp, and δU^{α} as scalar, vector, and tensor fields on the manifold M.

It is customary, however, to insist that the perturbed solution represent the same physical system as the unperturbed one, just in a different state of motion. This leads to the idea that there exists a way of indentifying a fluid element at x^{μ} of the unperturbed configuration with the 'same' one at a perturbed position $x^{\alpha} + \xi^{\alpha}(x^{\mu})$ in the perturbed configuration. This quantity ξ^{α} is called the Lagrangian displacement vector field. The identification only makes sense if the perturbed state could be obtained by moving the fluid elements around while preserving the conserved quantities of mass, entropy, and vorticity. How do we express these conditions?

One can think of ξ^{α} as the tangent vector field to a congruence which represents an increasing deformation of the fluid. The perturbed fluid element at $x^{\mu} + \xi^{\mu}$ has a four-velocity $\bar{U}^{\alpha}$ which differs from the unperturbed field U^{α} at $x^{\mu} + \xi^{\mu}$ by $\delta U^{\alpha}(x^{\mu}+\xi^{\mu})$. But in the unperturbed state the element's four-velocity was $U^{\alpha}(x^{\mu})$, not $U^{\alpha}(x^{\mu} + \xi^{\mu})$. So the change in its four velocity involves the difference between $U^{\alpha}(x^{\mu}+\xi^{\mu})$ and $U^{\alpha}(x^{\mu})$. On a manifold with a connection we could represent this as $(\xi\cdot\nabla)U^{\alpha}$, but it is more convenient to take it as $\mathcal{L}_{\xi}U^{\alpha}$. This <u>defines</u> the Lagrangian change

$$\Delta U^{\alpha} \equiv \delta U^{\alpha} + \mathcal{L}_{\xi} U^{\alpha} \qquad (4.9)$$

where we can evaluate δU^{α} at x^{μ} rather than $x^{\mu}+\xi^{\mu}$, the correction being of second order. Equation (4.9) can be applied, of course, to any tensor field on M.

Now we can formulate our conditions on the perturbation. We demand that ΔS, $\Delta(nU^{0}\sqrt{-g})$, and $\Delta(\nabla_{[\alpha} S \, \nabla_{\beta} V_{\gamma]})$ all vanish. Normally only the first two constraints are imposed. The necessity of using the third was first understood by Friedman and Schutz.[6] With just the first two constraints it is possible to prove the following results for the first-order perturbation of an arbitrary solution of Einstein's equation with perfect fluid.[1] The perturbed Einstein tensor has the compact form

$$(-g)^{-\frac{1}{2}}\delta[G^{\mu\nu}(-g)^{\frac{1}{2}}] = -\frac{1}{2}\epsilon^{\mu\alpha\lambda\sigma}\,\epsilon^{\nu\beta\gamma}{}_{\sigma}\,\nabla_{(\alpha}\nabla_{\beta)}h_{\lambda\gamma} + G^{\mu\nu\alpha\beta}h_{\alpha\beta} \tag{4.10}$$

where

$$G^{\mu\nu\alpha\beta} = \frac{1}{2}R^{\mu(\alpha\beta)\nu} - \frac{1}{2}\left[\frac{3}{2}R^{\mu(\alpha}g^{\beta)\nu} + \frac{3}{2}R^{\nu(\alpha}g^{\beta)\mu} - R^{\mu\nu}g^{\alpha\beta} - R^{\alpha\beta}g^{\mu\nu}\right]$$

$$+\frac{1}{4}R(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}) \quad . \tag{4.11}$$

The tensor $G^{\mu\nu\alpha\beta}$ has the properties

$$G^{\mu\nu\alpha\beta} = G^{(\mu\nu)(\alpha\beta)} = G^{\alpha\beta\mu\nu} \quad , \quad G^{\mu\nu\alpha}{}_{\alpha} = 0 \quad . \tag{4.12}$$

Similarly the perturbed stress-energy tensor is

$$(-g)^{-\frac{1}{2}}\,\Delta[T^{\mu\nu}(-g)\;] = W^{\mu\nu\alpha\beta}\,\Delta g_{\alpha\beta} \tag{4.13}$$

where $\Delta g_{\alpha\beta}$ is the Lagrangian change in $g_{\alpha\beta}$,

$$\Delta g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \quad , \tag{4.14}$$

and with

$$W^{\mu\nu\alpha\beta} = \frac{1}{2}(\rho+p)U^\alpha U^\beta U^\mu U^\nu + \frac{1}{2}p(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu})$$

$$-\frac{1}{2}\gamma p(g^{\alpha\beta} + U^\alpha U^\beta)(g^{\mu\nu} + U^\mu U^\nu) \quad , \tag{4.15}$$

where γ is the adiabatic index $(\partial\, \ell n\, p/\partial\, \ell n\, n)_S$. Again we have the symmetries

$$W^{\mu\nu\alpha\beta} = W^{(\mu\nu)(\alpha\beta)} = W^{\alpha\beta\mu\nu} \quad . \tag{4.16}$$

The symmetries enable us to prove the following 'hermiticity' relation for two independent perturbations (barred and unbarred), which do not necessarily satisfy any field equations:

$$\begin{aligned} &16\pi\, \bar{\xi}_\alpha\, \Delta(\nabla_\beta T^{\alpha\beta}) + \bar{h}_{\alpha\beta}\, \delta(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) \\ &= 16\pi\, \xi_\alpha\, \bar{\Delta}(\nabla_\beta T^{\alpha\beta}) + h_{\alpha\beta}\, \bar{\delta}(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) + \nabla_\mu R^\mu \end{aligned} \tag{4.17}$$

where R^μ is some vector field bilinear in barred and unbarred quantities.

4.3 <u>A Stability Theorem</u>. Suppose now the unperturbed metric is stationary and asymptotically flat and that t is the Killing time coordinate. We imagine we are perturbing a rotating star. Then we suppose we have a solution of the first-order perturbed field equations of the form

$$\xi^\mu = \xi^\mu(x^i)e^{i\omega t} \quad , \qquad h^{\mu\nu} = h^{\mu\nu}(x^i)e^{i\omega t} \tag{4.15}$$

where x^i denotes the three spatial coordinates. Since the barred terms were arbitrary in Eq. (4.17), let

$$\bar{\xi}^\mu = \xi^\mu(x^i)^*\, e^{-i\omega t} \quad , \qquad \bar{h}^{\mu\nu} = h^{\mu\nu}(x^i)^*\, e^{-i\omega t} \tag{4.19}$$

where stars denote complex conjugates. The barred quantities are <u>not</u> overall complex conjugates of the unbarred ones because ω is not necessarily real. The perturbed field equations imply the l.h.s. of Eq. (4.17) is zero, and since it involves second time derivatives it has the form

$$\omega^2 A \ -i\omega B \ - C + \nabla_j \ R^j = 0 \tag{4.20}$$

for quadratic functionals A, B, C, and R^j whose form is listed in ref. 1. The key point is that A, iB, and C are all hermitian forms and hence have real values when integrated over all space. If we therefore integrate Eq. (4.20) over an asymptotically future-directed null hypersurface, a remarkable thing happens. The divergence in (4.20) becomes a surface integral at future null infinity and is just $-4i\omega$ times the modulus squared of the Bondi news function of the perturbation $h_{\mu\nu}$. This leads directly to the stability theorem: along a sequence of unperturbed models, a mode which involves gravitational radiation can change from stable to unstable only through zero frequency, $\omega = 0$. This is because the mode will have real frequency at the transition, and if ω is real the integral of the first three terms of Eq. (4.20) is real, while the final term is pure imaginary. The equation can hold only if this imaginary term vanishes. Since the news function will not vanish for a radiative, time-dependent motion ($\omega \neq 0$), we must have $\omega = 0$. This is a powerful restriction on where we need to look for marginally stable modes, and is certainly not true for, say, Newtonian perfect-fluid stars.[7]

<u>4.4 The Energy Criterion</u>. From Eq. (4.20) it is clear that a necessary condition for marginal stability ($\omega = 0$) is that the integral of C vanish. This can be shown to be sufficient as well, in the sense that if $C \geqslant 0$ but $C \not> 0$ on perturbations satisfying the initial-value constraints, then there is a mode with $\omega = 0$. What is C?

Associated with the symmetry (4.17) is the existence of a conserved quantity when the unperturbed metric has a Killing vector field. In the case of a stationary metric this is an energy-like quantity. The integral of $|\omega|^2 A + C$ is the energy when evaluated for our mode, so the stability criterion is an energy criterion: if the energy of a stationary perturbation vanishes then the star is marginally stable.

4.5 The Trivials. So far the picture looks very nice. We apparently just need to evaluate the energy of stationary perturbations to determine the stability of a star. But not quite. In terms of initial data we can give eight functions of x^i for the fluids's dynamical equations in their Lagrangian form: the four components of each ξ^α and $\dot{\xi}^\alpha$. But in the Eulerian description there are only five quantities: δU^α, ρ, and p. (The normalization $U_\alpha \delta U^\alpha = 0$ means that there are only three independent quantities among the four δU^α.) So it is therefore possible to write down a set $(\xi^\alpha, \dot{\xi}^\alpha)$ which represents no physical perturbation at all. These are called the trivial Lagrangian displacements.[6] Unfortunately, one can find trivials which make the 'energy' negative.[5,6] This means that the energy-like quantity $|\omega|^2 A + C$ is not the physical, second-order energy, since it can be nonzero even when there is no physical perturbation, and the marginal stability criterion is empty.

What, then, does the stability criterion mean? We have to look more closely at the meaning of our perturbations before we answer that.

Footnotes 4

1. Much of this section comes from J. L. Friedman and B. F. Schutz, Astrophys. J. 200, 204 (1975); errata 222, 1119 (1978).
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5. Asymptotic Approximations to Sequences of Operators and Solutions

In order to understand some of the puzzles that first-order perturbation theory leaves us with, we must realize that we use the first-order perturbation as an asymptotic approximation in the sense of Eq. (2.2). We may be interested in approximating a particular solution of the full field equations, say for a star pulsating with surface amplitude $\Delta r/R_s = 0.1$, but we do not know what the error would be if we used only the linear perturbation solution with that amplitude. All we believe is that we can make the error as small as we like if we reduce the amplitude $\Delta r/R_s$ of the full solution sufficiently. The linear perturbation analysis is therefore an approximation to a sequence of full solutions, of which the unperturbed solution is the first member. In this section we will therefore study sequences of operators and their solutions, building up to a picture of perturbation theory involving fiber bundles of solution manifolds (the fibers) over the real line (the sequence's parameter).

5.1 Sequences of Linear Operators.

To fix our ideas, let us study a hypothetical sequence of linear operators $L(\varepsilon;x^\mu)$, $0 \le \varepsilon < 1$, and a sequence of solutions $\phi(\varepsilon;x^\mu)$ of the equation

$$L(\varepsilon)\phi(\varepsilon) = 0 \quad , \tag{5.1}$$

where each ϕ is an element of a vector space S. If we suppose that $L(\varepsilon)$ and $\phi(\varepsilon)$ are differentiable in ε up to some order then we can write (5.1) as

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \ldots)(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots) = 0 \quad , \tag{5.2}$$

where $L_0 = L(0;x^\mu)$, $L_1 = \partial L(\varepsilon;x^\mu)/\partial\varepsilon$ at $\varepsilon = 0$, etc. Since (5.2) holds for all ε we have the hierarchy of equations

$$\begin{aligned} L_0\phi_0 &= 0 \quad , \\ L_0\phi_1 + L_1\phi_0 &= 0 \quad , \\ L_0\phi_2 + L_1\phi_1 + L_2\phi_0 &= 0 \quad , \end{aligned} \tag{5.3}$$

and so on. These are to be solved in sequence: first solve for ϕ_0, then use that solution to solve for ϕ_1, etc. Therefore the nth equation is the inhomogeneous linear equation

$$L_0\phi_n = - L_1\phi_{n-1} - L_2\phi_{n-2} - \dots - L_n\phi_0 \quad . \tag{5.4}$$

So at each step we need only invert the operator L_0. Each inversion requires the application of initial data and/or boundary conditions. If we are solving the hierarchy given by Eq. (5.3) then we must recognize that the data are free at each order.

It often happens that the problem that generates the original equation is invariant under some group G. In general relativity this will at least be the group of coordinate transformations (diffeomorphism group). In electromagnetism or other gauge theories this will be a Lie group. There will be a representation of G acting on the vector space S of solutions ϕ, associating with each element $g \in G$ a transformation $\phi \to g_S\phi$ in S. The idea is that ϕ and $g_S\phi$ are physically equivalent. Suppose we take a one-parameter sequence of elements $g(\varepsilon) \in G$ beginning with the identity element, $g(0) = e$, and apply it to $\phi(\varepsilon)$. Then the asymptotic expansion becomes

$$g_S(\varepsilon)\phi(\varepsilon) = \phi_0 + \varepsilon(\phi_1 + g_S'\phi_0) + \varepsilon^2(\phi_2 + g'_S\phi_1 + \frac{1}{2} g_S'' \phi_0) + \dots , \tag{5.5}$$

where $g_S' = dg_S(\varepsilon)/d\varepsilon$ at $\varepsilon = 0$, etc. Under this kind of transformation ϕ_1 is changed by something which depends on ϕ_0 alone. This is familiar to relativists from studying 'gauge transformations' (infinitesimal coordinate transformations) in linearized theory and in the post-Newtonian hierarchy.

Equation (5.5) applies also if ϕ is not just the solution but any observable property or functional $F(\phi)$, provided G acts naturally in the space H of values of F. If it happens that $F(\phi_0)=0$ then F_1, the first correction to F, will be gauge-invariant, because the gauge change $g_H' \, F(\phi_0)$ will vanish. This applies, for example, to the Riemann tensor of linearized theory, which is invariant under infinitesimal coordinate transformations because the Riemann tensor of flat spacetime vanishes. In general, F will be gauge-invariant at its lowest non-vanishing order. This was first noticed in the context of general relativity by Sachs[1].

5.2 Exact Conservation Laws of Approximate Equations. Of course, Einstein's equations are not linear, but most of the preceding section is easily adapted to this case. In particular, the remarks on invariance groups are unchanged. When Einstein's equations are replaced by some approximation--linear perturbations, Newtonian and post-Newtonian equations--the approximate equations may admit some exact conservation laws. Linear perturbations of a stationary metric have a conserved energy. The Newtonian equations conserve not only energy and momentum, but also the period of an orbit and the position of its perihelion. (Here I use the word 'conserve' just to mean that the time-derivative is zero.) To what extent do these properties carry over to solutions of the full equations? What information do we need in order to compute the extent to which the full theory violates these laws? In this section I follow the arguments of Futamase[2] and of Lapiedra, et al.[3]

We suppose a nonlinear set of field equations

$$G(\varepsilon;\phi(x^\mu;\varepsilon)) = 0 \tag{5.6}$$

for a field ϕ, which is an element of some vector space S. An approximation to these equations of order N is the set of equations $(G)_{\varepsilon=0} = 0$, $(dG/d\varepsilon)_{\varepsilon=0} = 0$, ..., $(d^N G/d\varepsilon^N)_{\varepsilon=0} = 0$. Using our notation for the asymptotic expansion of ϕ introduced in Eq. (5.2), we can write these more explicitly as

$$\begin{aligned}
G(0;\phi_0) &\equiv G_0(\phi_0) = 0 \\
\frac{\delta G}{\delta\phi}(0;\phi_0)(\phi_1) &\equiv G_1(\phi_1) = H_1(\phi_0) \\
&\quad\; G_1(\phi_2) = H_2(\phi_0,\phi_1) \\
&\quad\;\;\vdots \\
&\quad\; G_1(\phi_N) = H_N(\phi_0,\phi_1,\dots,\phi_{N-1}) \quad .
\end{aligned} \tag{5.7}$$

(The main difference from Eqs. (5.3) is that the H_n are nonlinear, but the operator $G_1 = \delta G/\delta\phi$ is still a linear operator.)

Now we define what we mean by a conservation law. It is of course some functional F of $\{\phi_0,\dots,\phi_N\}$, with values in a space H, whose value is constant as $\{\phi_n$, $n=0,\dots,N\}$ evolve according to Eq. (5.7). Since it is constant, it depends only on the initial data for $\{\phi_n\}$. Moreover, since we are interested in how F changes when ϕ satisfies the full field equations, we restrict consideration to functions which depend on $\{\phi_n\}$ only in the combination $\phi_0+\varepsilon\phi_1 + \dots + \varepsilon^N\phi_N$. We assume therefore that we have a function

$$F = F(\varepsilon;\phi_0 + \varepsilon\phi_1 + \dots + \varepsilon^N\phi_N) \quad , \tag{5.8}$$

where dF/dt is identically zero when $\{\phi_n\}$ satisfy Eqs. (5.7). This is to be zero to all orders in ε, since it is an exact conservation law of Eqs. (5.7). Now we define the lowest-order conserved quantity

$$F_0(\psi) \equiv F(0;\psi) \tag{5.9}$$

and assume $F_0(\psi) \not\equiv 0$. (If it is zero then we would simply go up order by order in ε to find the lowest-order conserved quantity.) Then we write

$$F(\varepsilon;\psi) = F_0(\psi) + \Delta F(\varepsilon;\psi) \ , \quad \Delta F(\psi) = O(\varepsilon F_0(\psi)) \quad . \tag{5.10}$$

The conservation law implies that

$$\begin{aligned}
&\frac{d}{dt} F\left(\varepsilon,\phi_0 + \ldots + \varepsilon^N \phi_N\right) = \frac{d}{dt} F_0(\phi_0 + \ldots) + \frac{d}{dt} \Delta F(\varepsilon;\phi_0 + \ldots) \\
&= \frac{\delta F_0}{\delta\psi} \left(\dot\phi_0 + \ldots + \varepsilon^N \dot\phi_N\right) + \frac{\delta \Delta F}{\delta\psi} \left(\dot\phi_0 + \ldots + \varepsilon^N \dot\phi_N\right) \\
&= 0 \quad ,
\end{aligned} \tag{5.11}$$

where $\delta F_0/\delta\psi$ and $\delta\Delta F/\delta\psi$ stand for the linear operators which are the functional derivatives of the functionals F_0 and ΔF defined in Eq. (5.10).

What happens to $F(\varepsilon;\phi(\varepsilon))$ when $\phi(\varepsilon)$ satisfies the full equation? We have

$$\frac{d}{dt} F(\varepsilon;\phi(\varepsilon)) = \frac{\delta F_0}{\delta\psi} (\dot\phi(\varepsilon)) + \frac{\delta\Delta F}{\delta\psi} (\dot\phi(\varepsilon)) \quad .$$

But $\dot\phi(\varepsilon)$ equals $\dot\phi_0 + \ldots + \varepsilon^N \dot\phi_N + \varepsilon^{N+1} \dot\phi_{N+1}$ plus terms of order ε^{N+2}, so the exact law (5.11) implies

$$\frac{d}{dt} F(\varepsilon;\phi(\varepsilon)) = \frac{\delta F_0}{\delta\psi} \left(\varepsilon^{N+1} \dot{\phi}_{N+1} + O(\varepsilon^{N+2})\right) + \frac{\delta \Delta F}{\delta\psi} \left(O(\varepsilon^{N+1})\right) \quad .$$

We have not written the argument of the second term explicitly because ΔF is already of order ε times F_0, so this second term is of the same order as the $O(\varepsilon^{N+2})$ terms from the first part. Therefore we have

$$\frac{d}{dt} F(\varepsilon;\phi(\varepsilon)) = \varepsilon^{N+1} \frac{\delta F_0}{\delta\psi} (\dot{\phi}_{N+1}) + O(\varepsilon^{N+2}) \quad . \tag{5.12}$$

This is our main result. It means that the rate of change of F depends only on the lowest-order part of the conservation law F_0, evaluated on the lowest-order part of the solution ϕ_0, and operating on the first order of $\dot{\phi}$ above that at which the conservation law holds. This is a remarkable result: even though the conservation-law violating terms may be of quite high order, their effect depends only on the lowest order, not on intervening orders.

Examples of this theorem are common. If the sequence of solutions is the one we will study in §7, then the various orders are the Newtonian and post-Newtonian approximations. In the Newtonian equations, the periastron of the orbit of the two-body problem is fixed in space. This is a conservation law: the rate of change of the periastron is zero. The post-Newtonian equations break this law, but it is well known that to compute the periastron shift we need only the solution for the Newtonian orbit, not the full post-Newtonian solutions. Similarly, the radial motion of the two-body problem is periodic through post-post-Newtonian order: the period is a constant of the motion. The 2½-post-Newtonian radiation-reaction terms break this conservation law, but to calculate the period derivative we only need the Newtonian solution for the orbit,[2,3] not the post-post-Newtonian one.

If there is a gauge group for ϕ and the equations $G(\phi) = 0$, as discussed in the previous section, then the sequence of transformations $g(\varepsilon)$ will induce gauge transformations of the various ϕ_n. By an argument similar to the one we have just given one can show that Eq. (5.12) is gauge-invariant to the lowest order: the periastron shift or period change do not depend on gauge transformations that affect post-Newtonian orders, despite the fact that they are post-Newtonian effects.

5.3 Fiber-Bundle Picture of a Sequence of Solutions. When we study sequences of solutions of Einstein's equations in the next sections, it will be helpful to look at the sequence as a five-dimensional manifold,[1,4] in fact a fiber bundle over the real line (parameter ε), in which each fiber is the four-dimensional manifold which is the solution of Einstein's equations for that ε (Fig. 5.1a). The fiber-bundle picture is natural because each solution is associated with a unique value of ε but there is no natural map (in general) between different solutions. Nevertheless, in order to describe the tensor fields which solve Einstein's equations, we need to relate points of any manifold $M(\varepsilon_1)$ to those of any other one $M(\varepsilon_2)$. We therefore define an identification map e to be a four-dimensional congruence of curves in the fiber bundle, parametrized by ε, nowhere tangent to the fibers, such that they 'cover' the whole of each $M(\varepsilon)$ (or whatever region of M we are interested in). Then two points P and Q in different manifolds are said to be identified under e if they lie on the same curve of the congruence. Two such maps are shown in Fig. 5.1b: e identifies P and Q while f identifies P with Q'. If two identification maps e and f are 'close', as in Fig. 5.1b, then one may be obtained from the other by moving along a vector field η in each fiber. (This statement can be made more precise by considering a family of identification maps, $e(\mu)$. Then η is $\partial/\partial\mu$ holding ε fixed.)

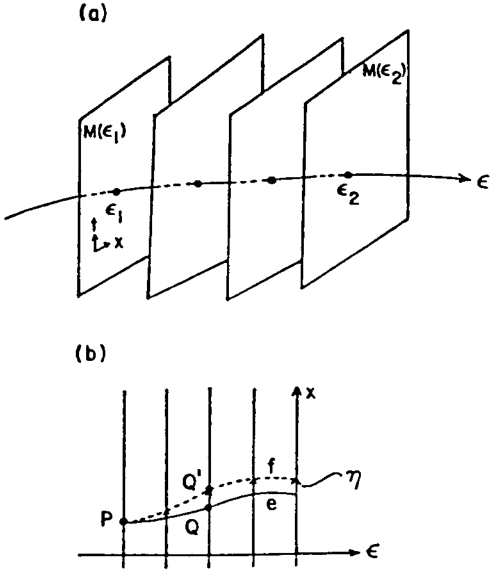

Figure 5.1. (a) Schematic picture of a fiber bundle of solutions. The real line (ε) is the base manifold and each fiber is a solution of Einstein's equations.
(b) One-dimensional fibers with two different identification maps e and f and the vector field η representing their difference in each fiber.

Associated with each identification map is a natural way of differentiating any field $\phi(\varepsilon)$ with respect to ε, in order to develop an asymptotic approximation. This is the Lie derivative of ϕ with respect to the tangent vector field t_e of the identification congruence (see ref. 3 of §4). If ϕ is a field in the fibers, as all physical fields must be, then its Lie derivative will be, too. So the expansion of ϕ will be

$$\phi(\varepsilon;x^{\mu}) = \phi(0;x^{\mu}) + \varepsilon\mathcal{L}_{t_e}\phi + \frac{1}{2}\varepsilon^2(\mathcal{L}_{t_e})^2\phi + \dots \quad . \qquad (5.13)$$

The first-order perturbation in ϕ away from $\phi(0)$ is the field $\pounds_{t_e}\phi$, the second order is $\frac{1}{2}(\pounds_{t_e})^2\phi$, etc., all evaluated at $\varepsilon = 0$.

Usually no particular identification map is preferred. If the manifolds are isometric (as in, say, special relativity) then one might want e to be an isometry. If e identifies points with the same coordinate values in the different manifolds as in §4, it is usually called an Eulerian map and its derivative $\pounds_{t_e}$ is usually just called δ. In fluid dynamics (again as in §4), one sometimes identifies the 'same' fluid element in different manifolds. This is a Lagrangian map: it carries fluid elements' world lines into each other. The derivative $\pounds_{t_e}$ in this case is usually called Δ. If we change from map e to a map f, then Fig. 5.1b makes it clear that the effect to first order is

$$\pounds_{t_f} = \pounds_{t_e} + \pounds_{\eta} \quad . \tag{5.14}$$

Where e is an Eulerian map and f a Lagrangian map, the vector field η is called the Lagrangian displacement vector field, and Eq. (5.14) becomes Eq. (4.9),

$$\Delta = \delta + \pounds_{\eta} \quad . \tag{5.15}$$

When operating on a scalar field this is the more familiar

$$\Delta = \delta + \eta\cdot\nabla \quad . \tag{5.16}$$

Finally we note that an identification map enables us to 'pull back' all the tensors in each $M(\varepsilon)$ to the 'unperturbed' manifold $M(0)$, so that all the fields $\phi(\varepsilon)$ become their images $\phi_*(\varepsilon)$ in the same manifold. This is sometimes easier to work with, but we will keep the general fiber bundle picture because, in the singular Newtonian limit discussed in §7, we can attach various limiting or boundary four-manifolds to the fiber bundle which would be hard to describe on a single manifold.

Footnotes §5

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6. The Instability of Rotating Stars

We can now return to our perturbation problem and look at it in the context of a sequence of solutions. Let the 'unperturbed' rotating stars be the $\varepsilon = 0$ member of the sequence, and let the other members be defined by initial data on a hypersurface which becomes the $t = 0$ hypersurface in the limit $\varepsilon \to 0$. We suppose for convenience that the initial data are analytic functions of ε and that the data which may be regarded as free (e.g. π^{ij}, g_{ij} in the ADM picture) differ from the $\varepsilon = 0$ free data only in a compact region of the initial hypersurface. For this problem one can prove a number of interesting theorems which lead to the conclusion that all rotating stars are unstable. We shall briefly look at the astrophysical consequences of this remarkable result after we show how to arrive at it. In view of the fact that our stability criterion seems to involve a second-order energy-like functional, we begin by studying energy.

6.1 Energy in general relativity. By studying Noether's theorem for classical field theories on fixed background metrics, Sorkin and I[1] found that the usual canonical stress-energy 'tensor' (which is not a tensor at all, but a pseudo-tensor) can be replaced by the truly tensorial Noether operator, which acts (in general as a differential operator) on any vector field ξ to produce a genuine vector density, called the ξ-momentum density of the dynamical field. When ξ has constant components, this gives the usual canonical energy or momentum of the field, depending on whether ξ is taken to be timelike or spacelike. We want a version of this that will work in general relativity, but of course we know[2] that there will be no local vector density describing the energy of a gravitational field. Used with care, however, a pseudotensor can be very helpful. We therefore defined the gravitational Noether operator $\underset{N}{t}^{\mu}{}_{\nu}$ acting on an arbitrary vector field ξ by

$$8\pi \underset{N}{t}^{\mu}{}_{\nu}\cdot\xi^{\nu} = -(-g)^{\frac{1}{2}} G^{\mu}{}_{\nu}\xi^{\nu} + \tfrac{1}{2}\partial_{\alpha}(h^{\mu\alpha\nu\beta}{}_{,\beta}\xi_{\nu}/(-g)^{\frac{1}{2}}) \tag{6.1}$$

where

$$h^{\mu\nu\alpha\beta} \equiv (-g)(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta}). \tag{6.2}$$

It is easy to verify the following important properties:
(i) there are no second derivatives of $g_{\mu\nu}$ in $\underset{N}{t}^{\mu}{}_{\nu}\cdot\xi^{\nu}$; (ii) when ξ^{ν} = const., $\underset{N}{t}^{\mu}{}_{\nu}\cdot\xi^{\nu}$ is the Einstein pseudotensor; and (iii) when ξ_{ν} = const. $x(-g)^{\frac{1}{2}}$ it is the Landau-Lifshitz pseudotensor. To make use of this complex it is necessary to impose certain conditions at spacelike infinity. The reason we took the free initial data for our sequence to be nonstationary only in a region of compact support was that we will now assume that there exists a foliation of spacetime near t = 0 by spacelike hypersurfaces H on which, as $r \to \infty$, coordinates exist such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$
$$h_{\mu\nu} = O(r^{-(1+\alpha)/2}), \qquad h_{\mu\nu,\gamma} = O(r^{-(3+\alpha)/2}) \tag{6.3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and α is some positive number. Then if ξ^{μ} also goes to a constant in these coordinates one can show that for a solution of Einstein's equations the quantity

$$P[\xi,H] \equiv \int (\sqrt{-g}\, T^{\mu}{}_{\nu}\xi^{\nu} + \underset{N}{t}^{\mu}{}_{\nu}\cdot\xi^{\nu})\, d\sigma_{\mu} \tag{6.4}$$

(where $d\sigma_{\mu}$ is the appropriate coordinate volume element) is the usual energy-momentum associated with the asymptotic ξ. The nice thing about (6.4) is that it is an integral measure of the mass which is insensitive to the far-field behaviour of $g_{\mu\nu}$, in the sense that if we violate Einstein's equations by arbitrarily changing the asymptotic behaviour of $g_{\mu\nu}$ sufficiently far away (while remaining consistent with 6.3), we make an arbitrarily small change in $P[\xi,H]$. It is therefore a functional well-suited to variational and perturbation calculations. We define the ξ-momentum of a pair ($g_{\mu\nu}$, $T_{\mu\nu}$) and hypersurface H by Eq.(6.4) regardless of whether $g_{\mu\nu}$ and $T_{\mu\nu}$ satisfy Einstein's equations. Of course, when Einstein's equations are satisfied, Eq.(6.1) shows that

$$P[\xi,H] = \frac{1}{16\pi} \oint h^{\mu\alpha\nu\beta}{}_{,\beta}\xi_{\nu}(-g)^{-\frac{1}{2}}\, d\sigma_{\mu\alpha}, \tag{6.5}$$

the integral being over the 'boundary' of H, i.e. the limit $r \to \infty$ of a 2-sphere of radius r in H. Thus, for solutions, $P[\xi,H]$ depends only on the asymptotic behaviour of ξ and the metric. We will be interested in the case where ξ is timelike at infinity. Then $P[\xi,H]$ is the usual ADM mass of a solution.[2]

6.2 First-order variation in the energy. Now suppose we have a sequence of manifolds ($g_{\mu\nu}$, $T_{\mu\nu}$), not necessarily solutions of Einstein's equations, satisfying (6.3) for every ε. Let there be some identification map, whose first derivative is called Δ (but which we do not yet require to be a Lagrangian map). Let X be a region with boundary ∂X which is preserved under the map, and let ξ also be invariant under the map (Lie dragged). The following equation is an identity on each manifold:

$$-\frac{1}{16\pi}\int_X (G^{\alpha\beta} - 8\pi T^{\alpha\beta})(-g)^{\frac{1}{2}}\pounds_{\xi}g_{\alpha\beta}\,d^4x + \int_X T^{\alpha}{}_{\beta;\alpha}\xi^{\beta}(-g)^{\frac{1}{2}}\,d^4x$$

$$= \int_{\partial X} (\underset{N}{t}{}^{\alpha}{}_{\beta}\cdot\xi^{\beta} + (-g)^{\frac{1}{2}}\, T^{\alpha}{}_{\beta}\xi^{\beta})d\sigma_{\alpha} \tag{6.6}$$

In the important case where ($g_{\alpha\beta}$, $T_{\alpha\beta}$) are invariant under ξ in the $\varepsilon = 0$ manifold, i.e. where

$$\pounds_\xi g_{\mu\nu} = \pounds_\xi T_{\mu\nu} = 0 \quad \text{for } \varepsilon = 0, \tag{6.7}$$

then the first derivative of Eq. (6.6) along the identification map gives

$$\oint_{\partial X} \{-\frac{1}{16\pi}(G^{\alpha\beta} - 8\pi T^{\alpha\beta})(-g)^{\frac{1}{2}}\Delta g_{\alpha\beta}\xi^\mu + 2(-g)^{\frac{1}{2}} n U^{[\alpha}\xi^{\mu]}\Delta V_\alpha$$

$$- V_\alpha \xi^\alpha \Delta[(-g)^{\frac{1}{2}} nU^\mu] + nT(-g)^{\frac{1}{2}} \Delta S\, \xi^\mu$$

$$- \Delta\, [\underset{N}{t}{}^\mu{}_\alpha \cdot \xi^\alpha + (-g)^{\frac{1}{2}}\, T^\mu{}_\alpha \xi^\alpha]\ \}\, d\sigma_\mu = 0 \tag{6.8}$$

This equation has a number of uses; in particular it can form the basis of methods of constructing solutions of Einstein's equations by finding extrema of the ξ-momentum. For our purposes it has one important consequence, which follows from (6.8) by taking ∂X to consist of two hypersurfaces H and H' that approach each other asymptotically and then by choosing our so-far arbitrary sequence for $\varepsilon \to 0$ in such a way that all variations Δ on H' vanish. Then the integrals in (6.8) are restricted to H, and we find that if the $\varepsilon = 0$ member is any solution of Einstein's equations then the first-order change in the ξ-momentum is

$$\Delta P[\xi, H] = \int_H \{(-V_\alpha\xi^\alpha)\Delta[(-g)^{\frac{1}{2}}\, nU^\mu] + nT(-g)^{\frac{1}{2}}\, \Delta S\, \xi^\mu$$

$$+ 2\, n(-g)^{\frac{1}{2}}\, U^{[\alpha}\xi^{\mu]}\, \Delta V_\alpha\}\, d\sigma_\mu\ . \tag{6.9}$$

If we take ξ to be an asymptotically timelike killing vector field at $\varepsilon = 0$, then this has the remarkable consequence that if the 'perturbation' Δ is Lagrangian in the sense of §4 then $\Delta P[\xi, H] = 0$: the first variation in the energy of a stationary solution vanishes if we preserve the conservation laws of §4. That the first two terms in (6.9) vanish is obvious. That the condition $\Delta(\nabla_{[\alpha} S \nabla_\beta V_{\gamma]}) = 0$ implies the third term vanishes requires some work, which we now describe. In the natural coordinates in which $\xi^\mu = \delta^\mu_0$ and $d\sigma_\mu = \delta_\mu{}^0 d^3x$, this final term is $\int n\, U^i \Delta V_i\, (-g)^{\frac{1}{2}}\, d^3x$, clearly just the first-order change in the 'kinetic energy'. (The other two terms are of course the changes in energy caused by the addition of particles and the addition of heat.) The $\varepsilon = 0$ conservation equations $[n\, U^i\, (-g)^{\frac{1}{2}}]_{,i} = 0$ and $U^i\, \nabla_i\, S = 0$ imply, respectively, that $n\, U^i\, (-g)^{\frac{1}{2}} = F^{ij}{}_{,j}$ for some $F^{ij} = F^{[ij]}$ which vanishes outside the star and that $F^{ij}\, \nabla_j\, S = \nabla_j\, \mathcal{H}^{ij}$ for some $\mathcal{H}^{ij} = \mathcal{H}^{[ij]}$ that also vanishes outside the star. The condition above on the variation in the vorticity implies, since $\Delta S = 0$, that $\Delta(\nabla_{[i}V_{j]}) = K_{[i}\nabla_{j]}S$ for some 1-form K_α. But it is also true that Δ commutes with exterior differentiation, i.e. with another curl operation: $\nabla_{[K}\Delta(\nabla_i V_{j]}) = \Delta(\nabla_{[K}\nabla_i V_{j]}) = 0$, which implies $\nabla_{[i}K_{j]}$ can be taken to be zero without loss of generality. The chain of argument then runs as follows:

$$\int n\, U^i\, \Delta V_i (\Delta g)^{\frac{1}{2}}\, d^3x = \int F^{ij}{}_{,j}\, \Delta V_i\, d^3x = - \int F^{ij}\, \Delta\nabla_{[i}V_{j]}\, d^3x =$$

$$- \int F^{ij}\, K_i \nabla_j S\, d^3x = \int H^{ij}{}_{,j}\, K_i\, d^3x = - \int H^{ij}\, \nabla_{[j}K_{i]}\, d^3x = 0\ .$$

<u>6.3 Second-order change in the energy</u>. So the first-order change in $P[\xi,\mathcal{H}]$ is not automatically zero, but it is zero for Lagrangian perturbations. This means that the conserved 'energy' of §4, which is a second-order quantity, probably is at least part of the (now dominant) second-order change in $P[\xi,\mathcal{H}]$. But it is easy to see that it cannot be the whole second-order change. Suppose we choose a sequence in which the first-order perturbations are all actually zero. Then the second-order changes will be the lowest-order perturbation, and if, in an obvious notation, we do not have $\Delta_2 S = 0$, etc., then there will be a second-order change in $P[\xi,\mathcal{H}]$ identical to (6.9) but with Δ replaced by Δ_2. Even if the first-order perturbations are not zero, the second-order perturbations will still contribute such a term to $\Delta_2 P[\xi,\mathcal{H}]$, and this is completely independent of the conserved quantity of §4. I now <u>conjecture</u> that $\Delta_2 P[\xi,\mathcal{H}]$ is simply the <u>sum</u> of the 'energy' of §4 and the explicitly second-order version of Eq. (6.9)

<u>Problem 6.1</u>. Prove this conjecture. It is known to be true for the Newtonian analogues of these sequences. The symplectic product of Friedman[4] probably supplies the key.

We can now solve the puzzle we were left with at the end of §4. In order that a perturbation be genuinely trivial, it must be trivial at second-order, not just at first-order. But if it is trivial at second-order then the second-order Lagrangian changes in S, etc., will not vanish and will, in fact, be rather complicated, involving among other things terms quadratic in the first-order perturbations. Then contributions to the second-order version of (6.9) will not vanish and will just compensate the first-order trivial's value of the 'energy' of §4, giving a net zero change in $P[\xi,H]$ to second-order.

Problem 6.2. Calculate the second-order terms in $\Delta_2 S$, etc., that are quadratic in the first-order trivial and verify that they make $P[\xi,H]$ vanish at this order.

Problem 6.3. Generalize the variational theorems of Schutz and Sorkin[1] to include black holes. Thereby get expressions for the first and second order changes in the mass and angular momentum of the holes produced by arbitrary dynamical perturbations.

Problem 6.4. Determine the asymptotic conditions on the metric and ξ for which $P[\xi,H]$ can be used to calculate the angular momentum of a spacetime, either at spacelike or null infinity. Find variational expressions for angular momentum.

Problem 6.5. Recast relativistic fluid perturbation theory and Ertel's theorem into the rheometric formalism of Carter[5].

Problem 6.6. Develop a full theory of second and higher order perturbations by using second and higher order tangent spaces to the fiber bundle sequence of solutions, or to more general multi-dimensional spaces of solutions. The theory may involve jet bundles[6] or the connections discussed by Gowdy[7]. Formulate and extend the theorems of §5 in this new language.

6.4 The stability criterion rescued. Although we now understand why the original stability criterion was unphysical, since it was not the full second-order change in the energy, we have yet to formulate an acceptable criterion. But it is not hard to see what it should be. If we constrain our perturbations to give not only $\Delta S = 0$ and $\Delta[nU^0(-g)^{\frac{1}{2}}] = 0$ at all orders, but also $\Delta[\nabla_{[\alpha}S\nabla_\beta V_{\gamma]}] = 0$ at all orders, then in particular the first-order change in the energy $P[\xi,H]$ will vanish and the part of the second-order change which is linear in second-order perturbations will also vanish. The constraint on vorticity is natural in view of Ertel's theorem. (Moreover, the condition of triviality $\delta = 0$ now implies six constraints on $(\eta, \dot{\eta})$, leaving only two remaining degrees of freedom. It turns out[4] that these are represented by the transformation $\eta^\mu \to \eta^\mu + f(x^\alpha)U^\mu$, i.e. a change in the Lagrangian vector which preserves the world lines. This transformation has, as we have conjectured, no effect on the energy $P[\xi,H]$.) A displacement which is fully Lagrangian in this sense is called canonical. There are a number of other ways of formulating the same constraints.[8] If the initial data for a perturbation are canonical, the conservation laws guarantee that the displacement will remain canonical. It can therefore be shown[4] that a necessary and sufficient criterion for instablilty is that it is possible to find a canonical pair $(\xi,\dot{\xi})$ which makes the energy of §4 negative.

6.5. All rotating stars are unstable. These technical details about the stability criterion might be of little interest except for the fact that for any rotating perfect-fluid star it is possible to find a canonical displacement which does have negative energy. The proof of this is given by Friedman[4] in the relativistic case and by Friedman and me[9] in the Newtonian radiation-reaction approximation. The fundamental reason is not hard to see. Consider a perturbation of the form $\xi \sim e^{im\phi}$, where ϕ is the azumuthal angle in the rotating star and m is very large. Because the energy will be quadratic in derivatives of ξ, it will involve some terms in m^2, which can be made as large as we like by making m large. These terms come in fact from the term $-W^{\mu\nu\alpha\beta}\,\bar{\Delta g}_{\alpha\beta}\,\Delta g_{\mu\nu}$ contributed by Eq. (4.13), and they are negative definite. One can verify that even when ξ satisfies the canonical constraints, and even when we solve the initial-value constraints for $h^{\alpha\beta}$, the energy can be made negative for sufficiently large m. Every rotating perfect-fluid star is unstable to perturbations of sufficiently large m.

6.6 Astrophysical implications. This instability was first discovered by Chandrasekhar[10] for m = 2 in the Maclaurin sequence, which is an exactly solvable sequence of rotating Newtonian stars of uniform density. It was assumed by everyone that the instability for m > 2 would set in later, i.e. in stars of larger angular momentum. In fact, as we have seen, it sets in earlier. Comins[11] has calculated the growth times for m > 2 for the Maclaurin sequence in the radiation-reaction approximation. While not realistic, these are the only numerical results we have, so we shall use them as a guide.

Problem 6.7. Compute realistic numerical models of compact rotating stars in Newtonian gravity and calculate the instability points for various m and the growth times of the modes in the radiation-reaction approximation.[12]

The instability is damped out by the effects of viscosity[13]: as m gets larger, the effects of gravitational radiation get weaker while those of viscosity get larger because the characteristic length-scale gets smaller. So our result for rotating stars is not valid for realistic stars above some m. Given estimates of the viscosity in neutron stars, Friedman[14] has concluded that this is not significant for $m \leqslant 5$. For m = 4 the growth time of the instability is of the order of one day to 3 years, allowing for uncertainties in the equation of state and estimates based on the Maclaurin sequence, so this has a significant effect on stars. It sets in when the star is rotating at about 700 to 1900 times per second, depending on the same uncertainties.[14] The millisecond pulsar, with a rotation rate near 600Hz, may well be sitting at the m = 4 instability point. The nature of the instability is to radiate away angular momentum: to spin the star down. If all very fast pulsars have a similar history[14] and hence a similar mass, the instability could force them all to the same rotation rate. Future observations will tell us whether this speculation is supported by evidence.

The radiation instability operates in other circumstances, particularly in getting rid of angular momentum after gravitational collapse.[3] Here the time-scales are so short that probably only the m = 2 and 3 instabilities are significant.

Problem 6.8. Our discussion has been based on outgoing-wave-normal modes, but very little is understood of their mathematical properties. Clarify the idea of completeness, perhaps by studying model problems as in Gowdy[15] or Dyson[16]. Also extend the results of Ipser[17] to this case.

Problem 6.9. Detweiler and Ipser[18] have given a variational principle for outgoing-wave modes of spherical stars. Generalize this to rotating stars, perhaps using the Newtonian analogue of Schutz[19].

6.7. Consistency of first-order perturbation theory. We are familiar with the fact[2] that the linearized theory of gravity does not incorporate any coupling of gravitational fields back to the matter that generates them, so that it cannot describe any gravitational 'forces' on matter. It is important to realize that this failing is peculiar to linearized theory and is not shared by linear perturbations of non-trivial solutions of Einstein's equations. The reason for this circumstance in linearized theory is that the unperturbed solution is flat spacetime with no matter. Any matter in the perturbed solution is first-order in the perturbation and generates first-order waves. The coupling of these waves to the equations of motion for the matter involves the product of the first-order waves and the first-order matter, and is thus second-order and negligible in linearized theory. In the cases we have considered in §4 and here, the unperturbed solution contained matter, so the first-order waves can couple to the zero-order matter to produce first-order effects in the equations of motion. Thus, linear perturbations of stationary solutions incorporate 'radiation-reaction' effects within the linear order.

In the next section we will discuss the Newtonian limit, which is a limit essentially to flat spacetime. It shares some of the problems of linearized theory, in particular in putting radiation-reaction effects at a higher order, and it adds some unique singular features of its own. The experience we now have with the regular perturbation theory of stars, and the fiber-bundle framework we have developed for it, will be very helpful in achieving a secure understanding of the Newtonian limit, which is at once the most singular and most important limit of general relativity.

Footnotes §6.

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7. The Newtonian Limit and the Asymptotic Nature of the Quadrupole Formulas

One of the most controversial problems of recent years has been over the validity of the quadrupole formulas for the lowest-order gravitational radiation from nearly-Newtonian systems. The quadrupole formulas (plural because there are two, a far-field formula for the radiation emitted and a near-zone formula for the radiation-reaction effects) have received more attention and sparked more controversy than other approximations in general relativity for a number of reasons. First, the long-standing confusion in some minds about the reality of gravitational radiation itself raised sensitivities higher than normal; and second, with the discovery of the binary pulsar system these formulas became the first predictions of general relativity beyond post-Newtonian order that could be tested observationally[1]. But the main reason is that there were and still are genuine difficulties over the derivations of these formulas, because the Newtonian limit is a singular limit of general relativity. I will begin by outlining the usual derivations of the Newtonian and post-Newtonian equations, and then I will use the techniques we have developed here to formulate the limit in a way that enables one to show that the quadrupole formulas are part of an asymptotic approximation to a sequence of solutions of Einstein's field equations.

7.1 Outline of the usual approaches. Most textbooks extract the Newtonian field equations from Einstein's equations in a slow-motion, weak-field limit, either by formally letting $c \to \infty$ and keeping the leading terms in c^{-1} or by assuming that the velocities v are small, the stresses divided by the densities are $O(v^2)$, and the dominant metric perturbation is $h_{00} \sim O(v^2)$. One can go beyond Newtonian theory by keeping successively higher-order terms in the same approximations. One finds that the next terms after Newtonian order are $O(c^{-2})$ higher, and are called post-Newtonian terms (pN). The post-post-Newtonian terms (p^2N) are another factor of c^{-2} higher. After that are a group of terms only one power of c beyond p^2N, and they are called post-2½-Newtonian order, $p^{2.5}N$.

Up to and including the p^2N terms, the equations of motion for a perfect fluid are conservative. There is a conserved energy-like quantity, and two-body solutions are periodic. The solutions also involve only elliptical operators, which require boundary conditions only at spatial infinity. The $p^{2.5}N$ equations are not of this type. The terms of this order require some conditions on the radiation, such as that it be outgoing far away. Moreover, they break the conservation laws and cause changes in the period and energy in the manner described in §5.2. These are called the radiation-reaction effects, because when one computes the energy appearing at infinity in the manner described by Dr. Walker in his lectures in this volume, it agrees with the energy lost locally to the $p^{2.5}N$ terms:

$$\frac{dE}{dt} = -\frac{1}{5}\,\frac{G}{c^5} < \dddot{I\!\!\!-}_{jk}\, \dddot{I\!\!\!-}^{jk} >, \tag{7.1}$$

where

$$I\!\!\!-_{jk} = I_{jk} - \frac{1}{3}\,\delta_{jk}\, I^{\ell}_{\ \ell}, \tag{7.2}$$

$$I_{jk} = \int \rho\, x_j x_k\, d^3x. \tag{7.3}$$

7.2 Why there are problems. Although the derivations just described give physically plausible, even desirable results, they suffer from a number of problems[2] which arise basically because the limit is a singular one: as $c \to \infty$ the wave operator $\Box = -c^{-2}\,\partial_t^2 + \nabla^2$ goes over to ∇^2, and the whole character of the problem is different. The global meaning of the outgoing-wave boundary conditions that are usually applied is not clear, because we do not know how to connect conditions at $\mathcal{I}^-$, where we wish to exclude incoming radiation, with the metric in the sources. The most serious problem is that if the method is continued beyond $p^{2.5}N$ one finds divergent terms.[3] This is so important that I will describe it in some detail.

One imagines that these approximations form some sort of asymptotic approximation to the full solution, e.g. that for g_{oo} there is an expansion

$$\begin{aligned} g_{oo} &= -1 + c^{-2}\,(\text{Newtonian part}) + c^{-4}(\text{pN part}) \\ &\quad + c^{-6}(p^2N\ \text{part}) + c^{-7}(p^{2.5}N\ \text{part}) \\ &\quad + c^{-8}(p^3N\ \text{part}) + \ldots \end{aligned} \tag{7.4}$$

where each coefficient of c^{-n} is a function of x^{μ} which is found by integrating some differential equation. The problem is that the integral which gives the 'p^3N part' is logarithmically divergent as the upper limit of integration goes to infinity. This means that if one wants to add up the terms in Eq. (7.4) for some value of c, they behave nicely until p^3N order, where one gets an infinite contribution for any non zero c^{-1}. This means that keeping the terms only through $p^{2.5}N$ order seems to be making an arbitrarily large error: our post-Newtonian hierarchy of approximations may not be close to the real world after all!

7.3 One way out: change the idealization. The divergent term came from an integral over an unbounded domain, and a detailed examination of the calculation shows that this integral is related to the attempt to apply a no-incoming-radiation condition at $\mathcal{I}$, which is infinitely far away. Our remedy will be to reformulate the problem in more local terms, so that we have integrals over finite domains.

The global condition that there be no incoming radiation is an idealization as discussed in §2. The real binary pulsar system sits in a bath of gravitational radiation incident on it from all sorts of sources. But we believe that this radiation makes no important difference in the problem, because it is uncorrelated with the dynamics of the system: it is essentially random. The no-incoming radiation condition is one way of idealizing the system as isolated. As with many idealizations, we have no way of knowing exactly how good this idealization is, since we cannot directly measure the radiation incident on the binary pulsar. Since this idealization brings about the unbounded domains of integration that produce the divergent terms, it seems sensible to change it.

Our new idealization allows incoming radiation, but insists that it be random and that we average over its randomness. Specifically, we solve an initial-value problem in which the data for the fluid are fixed but the free data for the gravitational field are allowed to be random. We adopt the coordinate condition

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0 \qquad (7.5)$$

where

$$\bar{h}^{\alpha\beta} = \eta^{\alpha\beta} - (-g)^{\frac{1}{2}}\, g^{\alpha\beta} \qquad (7.6)$$

described by Dr. Walker in his lectures. On the initial hypersurface $t = 0$ one may choose $\bar{h}^{ij}$ and $\bar{h}^{ij}{}_{,0}$ arbitrarily and obtain $\bar{h}^{oo}$ from the constraint equations. The crucial assumption is that the random data average to zero:

$$\langle\bar{h}^{ij}\rangle = \langle\bar{h}^{ij}{}_{,0}\rangle = 0, \qquad (7.7)$$

where < > denotes an ensemble average. This is basically a random-phase condition, and it replaces the usual no-incoming-radiation condition.

To construct a sequence of solutions with a Newtonian limit we have to choose the initial data for the fluid in a particular way. To grasp the idea we must first understand an important scale-invariance of Newton's equations.

7.4. Newtonian scale-invariance. Newton's equations for a perfect fluid are (with $G = 1$)

$$\nabla^2\Phi = 4\pi\rho \tag{7.7}$$

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{7.8}$$

$$\rho\partial_t v^i + \rho v^j \partial_j v^i + \partial^i p + \rho\partial^i\Phi = 0 \tag{7.9}$$

supplemented by an equation of state. These are invariant under the replacement

$$\begin{aligned} \rho(x^i,t) &\to \varepsilon^2\rho(x^i,\varepsilon t) \\ p(x^i,t) &\to \varepsilon^4 p(x^i,\varepsilon t) \\ v^j(x^i,t) &\to \varepsilon v^j(x^i,\varepsilon t) \\ \Phi(x^i,t) &\to \varepsilon^2\Phi(x^i,\varepsilon t) . \end{aligned} \tag{7.10}$$

That is, if the functions $(\rho,\ p,\ v^j,\ \Phi)$ satisfy Eqs. (7.9) then so do the scaled functions in Eq. (7.10), as may be readily verified. A little thought convinces one that this is precisely the scaling that our limit $c^{-1} \to 0$ (with $v^i \to 0$, $\Phi = O(v^2)$, $p/\rho = O(v^2)$) applies to Einstien's equations. Newton's equations are the approximation that scales as in Eqs. (7.10).

In an initial-value problem, therefore, it is clear how we should choose the data for a sequence of solutions of Einstein's equations that should become Newtonian in some limit. We define the data to scale as in (7.10).

7.5. <u>The Newtonian fiber bundle</u>. We define a sequence of solutions by the initial data

$$\rho(t = 0, x^i, \varepsilon) = \varepsilon^2 a(x^i)$$
$$p(t = 0, x^i, \varepsilon) = \varepsilon^4 b(x^i) \tag{7.11}$$
$$v^j(t = 0, x^i, \varepsilon) = \varepsilon\, c^j(x^i)$$

plus Eq. (7.7).

The time-evolution of these data preserves their ordering, though of course higher-order terms arise from the nonlinearity of Einstein's equations. Thus, for $t > 0$ we will still have ρ of order ε^2, but it will have ε^4 (post-Newtonian) pieces as well.

In our fiber bundles of solutions earlier the identification map was a convenience, but it was to a large extent arbitrary. But here it is crucial, because we will use it to implement the scaling of time that our scaled initial data require to resemble Newton's equations (7.10). We define the <u>dynamical time</u>

$$\tau = \varepsilon t \tag{7.12}$$

and map points in the different manifolds into each other at constant (τ, x^i). The name dynamical time is appropriate. If a Newtonian binary system for $\varepsilon = 1$ completes an orbit in a time T, then Eqs. (7.10) show that the $\varepsilon = \frac{1}{2}$ system will complete an orbit in a time $2T$, in other words in the <u>same</u> interval of τ. Our identification map (Fig.7.1) thus identifies points which are at the same dynamical phase in the limit $\varepsilon \to 0$.

With our coordinate choice (7.5) Einstein's equations take the form given in Dr. Walker's lectures

$$\Box\, \bar{h}^{\mu\nu} = -16\pi\, \Lambda^{\mu\nu}, \tag{7.13}$$

where $\Box$ represents the flat-space wave operator in our coordinates and $\Lambda^{\mu\nu}$ is a nonlinear function of $\bar{h}^{\mu\nu}$ and $T^{\mu\nu}$. They have the (formal) solution

$$\bar{h}^{\mu\nu}(x^i,t) = 4 \int_{C(x^i,t)} \Lambda^{\mu\nu}(y^i, t-|x-y|)\, |x-y|^{-1}\, d^3y + \bar{h}^{\mu\nu}_H(x^i,t), \tag{7.14}$$

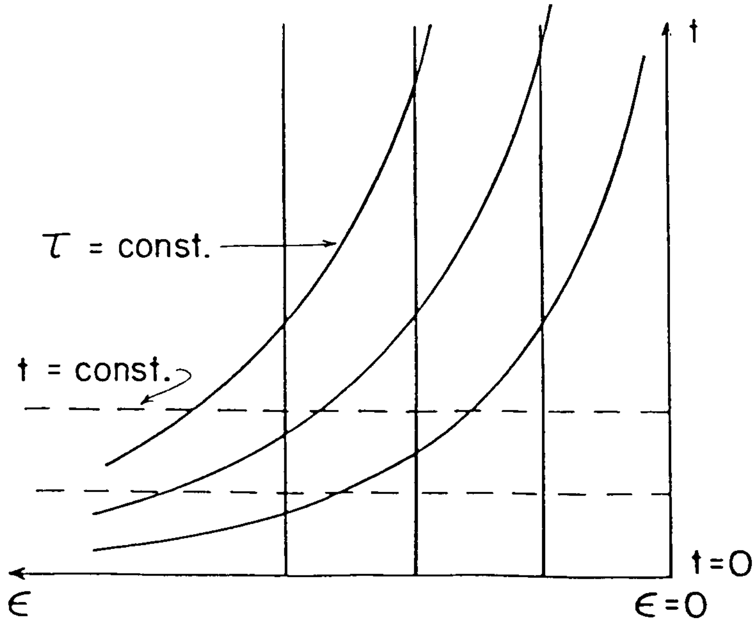


Fig.7.1

The fiber bundle with only the t-coordinate of each $M(\varepsilon)$ displayed. The identification map with τ = const follows the solid hyperbolae. The dashed lines represent the t = const identification map.

where $C(x^i,t)$ is the retarded coordinate light-cone of the event (x^i,t) truncated at t = 0 (see Fig.7.2) and $\bar{h}^{\mu\nu}_H$ is the unique solution of the homogeneous equation $\Box\bar{h}^{\mu\nu} = 0$ that satisfies the initial data we have posed for our field.

A few words are in order about this solution. We presume that the initial data for any ε define a unique solution of Einstein's equation, which is represented by Eq.(7.14). The fact that the integration in (7.14) is over a coordinate light cone and not the true light cone is irrelevant, since the homogeneous solution can and will prevent any faster-than-light propagation of initial data. To assume (7.14) without $\bar{h}^{\alpha\beta}_H$ would be an approximation, and a questionable one. With it, it is exact.

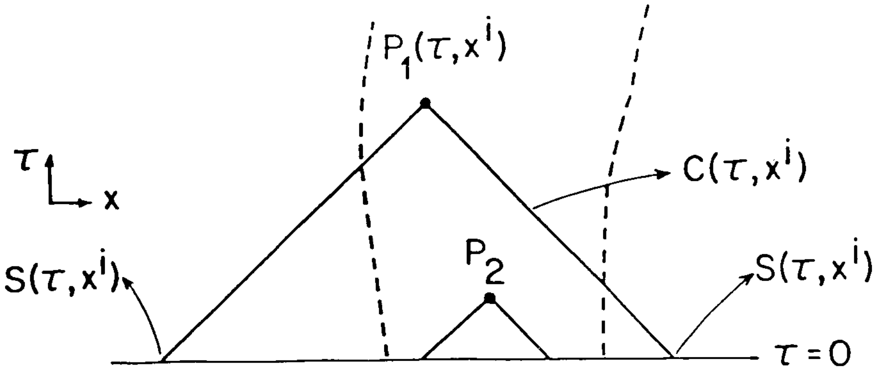


Fig.7.2

As $\varepsilon \to 0$ for fixed τ, the t-coordinate of the point (x^i, τ) we are looking at goes to ∞, so its retarded light-cone C gets larger and larger. It is this which eliminates the divergences at high order.

7.6. How the divergent integrals go away. Given the representation (7.14) and the initial data (7.11), we can in principle differentiate Eq.(7.14) with respect to ε along the τ = const map and evaluate these derivatives at $\varepsilon = 0$ to find successive approximations to the sequence of solutions. In practice these calculations need a number of plausible but unverified assumptions about the differentiability and convergence of the sequence, which must be made in the absence of a theorem on the existence and uniqueness of solutions of Einstein's equations for fluids of compact spatial support.[4]

The lowest non-vanishing derivative is indeed Newtonian theory, and the pN and p^2N equations emerge two and four orders of differentiation higher, respectively. At the next order, the quadrupole radiation-reaction terms come out as in previous calculations. But at the next order, dramatic differences occur: there are no divergent terms. It is easy to see why.

For each ε, the integral domains (light-cone C) are finite. The integrands are the same as the integrands that would arise at p^3N order in the old scheme, but here they are integrated to an upper limit R proportional to ε^{-1}. So an integral that in the old scheme diverged as ℓnR now becomes $\ell n\varepsilon$ times a finite number. Instead of a term $\varepsilon^8 \ell n_\infty$ as in Eq.(7.4). we now have $\varepsilon^8 \ell n\varepsilon$, and (7.4) is replaced by

$$
\begin{aligned}
g_{00} = -1 &+ \varepsilon^2(\text{N part}) + \varepsilon^4(\text{pN part}) + \\
&\varepsilon^6(p^2\text{N part}) + \varepsilon^7(p^{2.5}\text{N part}) + \\
&\varepsilon^8 \ell n\varepsilon(p^{3L}\text{N part}) + \varepsilon^8(p^3\text{N part}) + \ldots
\end{aligned}
\tag{7.15}
$$

where '$p^{3L}N$' part stands for the finite coefficient of $\varepsilon^8\ell n\varepsilon$ and 'p^3N part' stands for the remaining terms that were finite at p^3N order in the old scheme as well as here.

The reason for the divergence is now clear. The old iteration method assumed that the only terms in the asymptotic approximation were powers of ε (or of c^{-1}). This turns out not to be the case, but by trying to force it, the old method put a term into the p^3N terms that was really of lower ($p^{3L}N$) order, so it naturally looked infinitely large compared to genuinely p^3N terms.

The lesson is that above $p^{2.5}N$ order, the sequence of solutions is not differentiable in ε at $\varepsilon = 0$, but it still has an asymptotic approximation in ε that involves logarithms. There are no infinite terms, and in particular the radiation-reaction terms in the equations of motion are good asymptotic approximations.

<u>7.7. Picturing the near-zone map</u>. We have been dealing up to now with the identification map (τ, x^i) = const, which we shall call the near-zone map n because as $\varepsilon \to 0$ the characteristic wavelength λ of radiation gets larger as ε^{-1} (because time-scales get longer as ε^{-1}), so any point x^i = const eventually enters the near zone $|x^i| < \lambda/2\pi$. The effect of this map is displayed pictorially in Fig.(7.3).

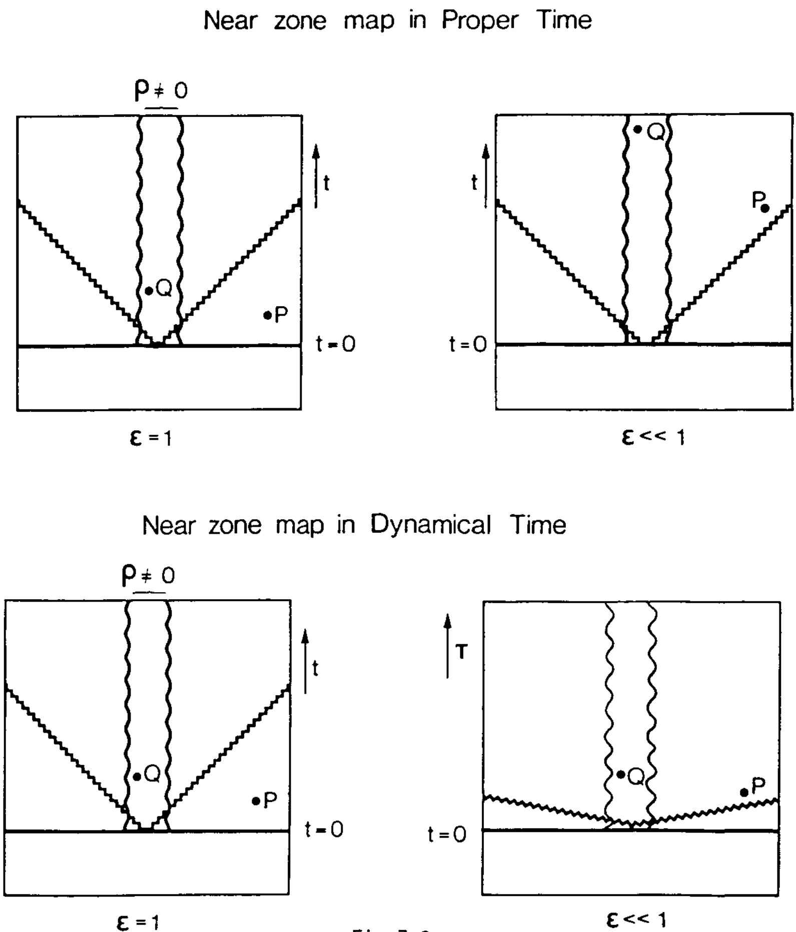


Fig.7.3

The upper left shows the manifold $\varepsilon = 1$, with the $t = 0$ hypersurface, a light cone, the world tube enclosing the matter region, and two events P and Q. These events are mapped into the $\varepsilon = 1/3$ manifold on the upper right by sending $t \to 3t$. The same manifolds are displayed in the lower figures using the coordinate τ as the time coordinate. The left-hand figure is unchanged but in the right-hand figure proper time is 'squashed'. The events P and Q occupy the same position but the light cone gets flattened out. This picture is appropriate for taking the limit $c \to \infty$ rather than $\varepsilon \to 0$.

Now we should ask about the $\varepsilon = 0$ limit of our sequence. Since we know the limit is a singular one for the field equations, we should not be surprised to see unusual behaviour here. From the initial data it is clear that the $\varepsilon = 0$ fiber is flat Minkowski spacetime. This is reached as the $\varepsilon \to 0$ limit of the t = const map in Eq.(7.1). But where do the τ = const lines go? They never reach the $\varepsilon = 0$ fiber. In Fig(7.4). we re-coordinatize the fibers of Fig(7.1) to use τ as the time-coordinate. Then the t = const lines converge on the point in the lower right corner - all of Minkowski spacetime is squashed to this point in this picture. The limit $\varepsilon \to 0$ of the τ = const congruence is a quite different four-dimensional manifold, which I shall call NM, the Newtonian limiting manifold of the fiber bundle $M(\varepsilon)$. (The Minkowski limit is called OM). One can show[4] that it has the Newtonian geometry: a connection which causes the effects of gravity, and a degenerate 4-metric.

Near zone limit Spaces

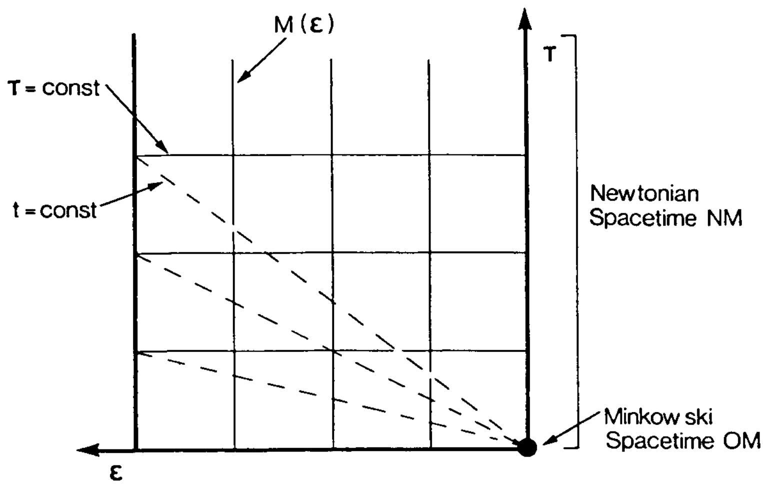


Fig.7.4

7.8 The far zone. The manifolds in our sequence are complete manifolds with outgoing radiation, but our map n has concentrated our attention on the near zone around the fluid bodies. We can study the far zone by defining a new identification map f, which holds not only τ but also $\eta^i = \epsilon x^i$ constant:

$$f : (x^i,t) \mapsto (x^i/\epsilon,\ t/\epsilon). \tag{7.16}$$

The effect of this is to keep a point (η^i,τ) a fixed number of wavelengths from the source as $\epsilon \to 0$, which is to say that if (x^i,t) is in the far zone ($|x^i| >> \lambda/2\pi$) for $\epsilon = 1$ then the corresponding (η^i,τ) will be there for all ϵ. These maps are displayed in Fig(7.5).

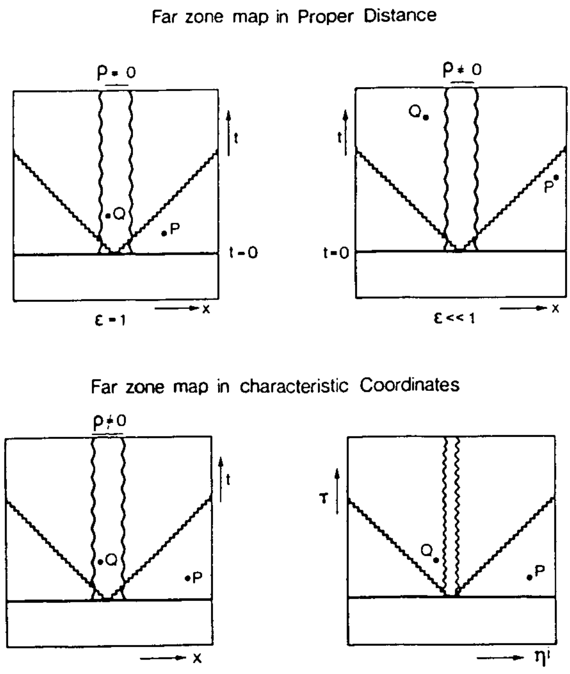
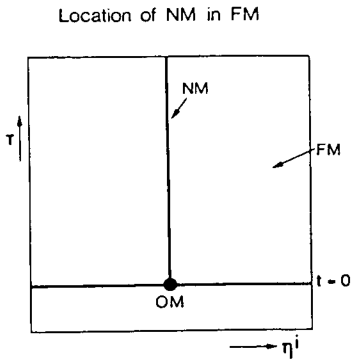
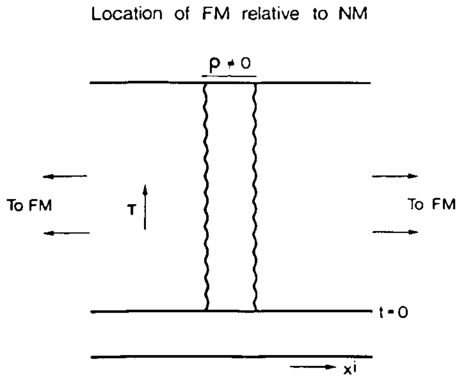


Fig. 7.5

As Fig(7.3) but for the far-zone map f. In the coordinates (η^i,τ) in the lower right, the matter-occupied world tube shrinks.

Since the map f puts us further and further away in proper distance as $\varepsilon \to 0$, the scaled metric in (η^i,τ) space approaches the Minkowski metric with an arbitrarily small perturbation. The limiting manifold FM (far-zone limit of $M(\varepsilon)$) is shown in Fig(7.6) with the manifold NM as the singular line at $|\eta^i| = 0$. The converse picture showing where FM is in relation to NM is shown in Fig(7.7).





Figs. (7.6), (7.7)

By averaging in the sense of Isaacson[5] to obtain the flux in the waves on FM, we find that the 'constant' M/r terms go away and we are left with the wave terms of the form discussed by Dr Walker in his lectures, whose flux in the Isaacson sense is the same as their flux at null infinity, namely the far-field quadrupole formula.

It is important to understand that we have not computed the limit to $\mathcal{I}^+$ of the radiation in any of our manifolds M(ε). Instead we have attached a flat far-field four-dimensional boundary to our sequence, in which the waves behave like linear waves. This is an appropriate way of formulating the problem because the limit to $\mathcal{I}^+$ in any one spacetime does not take into account the fact that as $\varepsilon \to 0$ the mass of the spacetime is changing and thus the structure of $\mathcal{I}^+$ is changing. We have combined these two limits of $r \to \infty$ and $\varepsilon \to 0$ in such a way as to get a boundary which is natural for the Newtonian problem, which is fundamentally a problem of radiation in a sequence of solutions rather than in a single solution. This limit is described further in a forthcoming paper[6].

7.9 What are we approximating? The various identification maps described here have been chosen to facilitate the development of asymptotic approximations in certain regimes. In the near zone we have already described the approximation in some detail. The aim there is to describe the motion of the fluid, and in particular to calculate the rate of change of a binary's period. The results above and in §5 suffice to do this, and the result is in agreement with the observations of the binary pulsar.[1]

In the far field the choice of what to approximate has not usually been clearly defined in discussions in the literature. It seems that most writers have assumed that since $\mathcal{I}^+$ and the Bondi flux give an invariant description of radiation in spacetime, what is wanted is an asymptotic approximation to it. The necessity of this is by no means obvious to me.

In the end, any predictions will have to be tested by observations, and the observer does not really sit at $\mathcal{I}^+$. The notion of $\mathcal{I}^+$ is an idealization of the observer's position. The flux an observer would measure at a fixed number of wavelengths away is presumably approximated to order r^{-2} by quantities on $\mathcal{I}^+$. In our sequence of solutions we are approximating what an observer a fixed number of wavelengths from the source would measure. Since every observer is a finite distance away, such an approximation is appropriate, and it is not necessary to justify it by appealing to concepts developed at I^+ in single spacetimes.

7.10 How accurate are the quadrupole formulas? Now that we know that the quadrupole formulas are asymptotic, we must ask about the size of the errors in using them. We expect them to be quite good for, say, main-sequence stars, moderately good for compact stars, and bad for black holes. But where is the dividing line for a given accuracy? This question is of far greater consequence for astrophysics than many of the technical mathematical problems associated with the Newtonian limit.

Error bounds can in principle be estimated from the next order of the approximation, but this may in practice prove difficult. There seems to be only one area where a fully relativistic calculation can be compared with one using the quadrupole approximation: normal modes of spherical stars. In the late 1960's and early 1970's, Thorne and Detweiler independently integrated the linear perturbation equations of general relativity (described in §4) numerically to obtain normal modes with complex eigenfrequencies. Balbinski and I[7] decided to test the near-zone quadrupole formula by calculating Newtonian models of stars, computing their eigenfrequencies, and using the quadrupole formula to obtain their damping rate (hence the imaginary part of the frequency). On the least relativistic models of Thorne and Detweiler, with surface redshifts of about 5%, we expected agreement of perhaps 50% at worst. We found that we could compute the real parts of the frequencies accurately, but were a factor of three larger in the imaginary parts.

The fully relativistic calculations are delicate for such stars, since the imaginary part of the eigenfrequency is much smaller than its real part and numerical errors may be significant. Recent numerical re-calculations by Detweiler and Lindblom (not yet published) using more accurate techniques, have narrowed the gap to a factor of two, a circumstance that may be regarded as a minor triumph for the quadrupole formula! It remains to be seen whether further numerical improvements will narrow the gap still more. But this is only one example. Much more needs to be done.

Problem 7.1 Either by comparing fully relativistic calculations with quadrupole approximations or by estimating error bounds from higher terms in the approximations, improve our understanding of the applicability of the quadrupole formulas.

Problem 7.2 The quadrupole approximation should certainly fail in gravitational collapse, yet in many collapses the velocities may not exceed 0.1 - 0.3 c. Is there an approximation to general relativity, perhaps neglecting radiation effects, which considerably simplifies the equations that must be integrated numerically?

Footnotes §7

1 See J. H. Taylor and J. M. Weisberg, Astrophys. J. **253**, 908 (1982) for discussion and references. Because of the precision of the pulsar 'clock' in this binary, the observations are accurate enough to fit a number of post-Newtonian effects, which are sufficient to determine the masses of the two stars and the eccentricity, inclination and semi-major axis of the Newtonian orbit. The only higher-order effect measureable is the change of the period of the orbit caused by the radiation-reaction forces. This effect accumulates with time and therefore can be measured despite its intrinsic smallness by simply observing the system for a number of years.

2 J. Ehlers, A. Rosenblum, J. N. Goldberg, and P. Havas Astrophys. J. **208**, L77 (1976). This paper contains references to the original work on the problem by S. Chandrasekhar and collaborators.

3 An important study of them is by G. D. Kerlick, Gen. Rel. and Grav. **12**, 467 (1980) and **12**, 521 (1980). He gives references to work on the problem after that described by Ehlers et al., ibid.

4 Details of the calculations appear in T. Futamase and B. F. Schutz, Phys. Rev. D. (in press) and T. Futamase, Phys. Rev. D. (in press).

5 C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Freeman, San Francisco 1973).

6 T. Futamase and B. F. Schutz, in preparation.

7 E. F. L. Balbinski and B. F. Schutz, Mon. Not. Roy. Astr. Soc. **200**, 43P (1982). References to the papers of Thorne and Detweiler appear here.

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