The Newtonian Limit

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Abstract

We discuss in detail the development of the Newtonian and post-Newtonian approximations to general relativity. By using an initial-value approach, we are able to show that the post-Newtonian hierarchy through gravitational-radiation-reaction order is an asymptotic approximation to general relativity, thereby verifying the validity of the quadrupole formula for radiation reaction. We also show with equal rigor that the radiation from nearly-Newtonian systems obeys the far-field quadrupole formula (Landau-Lifshitz formula). There are no divergent terms in these approximations at any order, although logarithmic terms in the expansion parameter do appear at high order. We discuss the relationships of observables to post-Newtonian quantities by the method of osculating Newtonian orbits. Finally we discuss the role exact solutions may play in shedding light on some of these questions.

1. Introduction

The Newtonian limit of general relativity is what its name says: a limit. No solution of Einstein's equations is itself perfectly Newtonian; rather, in some limit solutions are dominated by certain Newtonian characteristics. In this lecture I will attempt to make the nature of this limit precise and to show how one can also describe gravitational radiation and its effects in the same limit.

All serious textbooks on general relativity extract the Newtonian limit in one way or another. One method (see §2 below) is to assume (in 'natural' units, $c = 1$) that a characteristic velocity v of a material system goes to zero and that the density ρ and the stress-to-density ratio T^{ij}/ρ both go to zero as v^2 . One then finds that the conservation laws involve only $\nabla_\alpha g_{\alpha\beta}$ and that $g_{\alpha\beta}$ is dominated by its 'Newtonian' part,

$$g_{\alpha\beta}(x^i) = -1 + 2 \int \rho(y^i) |x^i - y^i|^{-1} d^3y + o(v^2), \quad (1)$$

where I have set G to unity as well.¹ We can draw a number of conclusions from such simple but non-rigorous approaches.

1. The Newtonian limit is a limit of a coupled matter-gravity system: pure vacuum solutions do not have Newtonian limits.

2. It is a limit in which weak fields and slow motion are linked. That is, initial data or some other conditions must be invoked to ensure that, say, T^{ij}/ρ is small everywhere in spacetime. Weak-field solutions that do not have this property fall within the province of linearized theory, but are not Newtonian.

3. Obviously, to discuss a limit we need to consider *sequences* of solutions of Einstein's equations, not just individual exact solutions. The sequences I define below have Newtonian limits, but they are not unique in this respect. In particular it would be interesting (and difficult) to find sequences of exact solutions having a Newtonian limit. I will come back to this in the final section.

There are at least two reasons why the simple textbook extractions of the Newtonian limit are not rigorous:²

(a) they do not examine the higher-order terms in, say, $(T^{ij}/\rho)^2$, to ensure that they really are negligible; and

(b) they do not tell us whether solutions having the assumed limiting character (e.g. $T^{ij}/\rho \sim v^2$) actually exist, and if they do whether the limit is uniformly valid everywhere or only in some compact region of spacetime.

There have been many contributions to a more rigorous formulation of the Newtonian limit. It is well known that Cartan³ showed that Newtonian gravity can be formulated geometrically in terms of a connection and a degenerate metric. This suggests that we should be able to express the limit in geometrical terms as well, not just as an exercise in differential equation theory. But Cartan's manifold admits no gravitational radiation, so a limit to it is only part of the story. Indeed, the recent renewal of interest in the Newtonian limit arises precisely because we wish to find a mathematically acceptable formulation of the limit which incorporates gravitational radiation at some level. We want a mathematical framework in which it is meaningful to say that a nearly Newtonian system radiates gravitational waves. The aim of this paper is to construct such a framework.

Looking for radiation terms means, at least in part, considering those terms that vanish in the limit to a Cartan spacetime. This means studying the Newtonian approximation, not just the limit. The most important systematic study of these terms was by Chandrasekhar and colleagues⁴, who showed that there was indeed an ascending hierarchy of approximations in the Newtonian spirit, which they called the post-Newtonian approximations. At sufficiently high order they found radiation-reaction terms that ensured the near-zone 'quadrupole formula'⁵, that the Newtonian energy decreased with time at a rate equal to the

energy carried away by waves, as calculated by Landau and Lifshitz⁶ (the far-zone quadrupole formula). Chandrasekhar's method was also a step toward eliminating difficulty (a) mentioned above. But it unfortunately was only a first step, because they and subsequent workers⁷ were unable to eliminate completely the occurrence of infinite integrals at some order in the approximation scheme. This of course casts doubt on the validity of the lower-order approximations obtained by discarding these higher-order terms².

It turns out that these divergent terms arise from integrals which express the boundary condition imposed on these calculations, that there be no incoming gravitational radiation. But since such a condition must be imposed in the infinite past (on $\mathcal{I}^-$), we immediately face difficulty (b) above: what is the nature of nearly Newtonian solutions in the distant past?

A little thought will convince one that the limit to a single Newtonian system cannot be uniform over the whole spacetime. Suppose a relativistic solution is closely approximated at some time by a Newtonian one, say for two binary stars. We expect the relativistic solution to be emitting radiation, leading to a contraction of the orbit in the future and an expansion in the past. But Walker and Will⁸ have shown that if the quadrupole formula is correct, this expansion can continue only for a finite time into the past. Earlier than this the stars are unbound and flying toward each other on marginally hyperbolic orbits, eventually to be captured as a result of the energy they lose to radiation on their first encounter. So the approximation of the system by the original bound binary cannot be uniformly valid for all time. Rather, as Walker and Will point out, it is best to regard the Newtonian approximation as an *osculating* one: at any time there is a good Newtonian approximation, but its properties change with time. This in turn suggests that the approximation involves two time-scales, a short timescale in which a single Newtonian approximation is valid, and a long timescale during which this approximation smoothly changes. I will return to this point later.

Evidently, if we wish to describe radiation in the same framework as the Newtonian limit, we must abandon any notion of globally valid Newtonian approximations. We shall instead construct an approximation scheme that is uniform only for a finite time. We will see that this will automatically regularize the divergent integrals of global approaches. Full details of this picture may be found in a series of papers by T. Futamase and myself⁹⁻¹².

The stress I have placed on the need to formulate the Newtonian *approximation* carefully may seem out of place in a meeting on *exact* solutions, but exact solutions do in fact have a useful role to play. The need for such care in our approximations would be considerably relieved if we had an exact solution for a non-singular, radiating fluid of compact support in an asymptotically flat spacetime, but this seems too much to hope for at present. But even an exact stationary

solution for, say, a family of rotating stars could be useful in testing many of the assumptions underlying the approximation methods. I will make some remarks about this at the end.

The plan of the paper is as follows. In §2 I will review the usual textbook derivation of the Newtonian limit, and then in §3 show that these methods work because they take advantage of an exact scale-invariance of the Newtonian equations. This leads in §4 to a definition of an asymptotically Newtonian sequence of solutions of Einstein's equations in terms of initial data having the Newtonian scaling. In §5 I sketch the derivation of the Newtonian and higher post-Newtonian equations, leading to the near-zone quadrupole formula. Then in §6 I examine the far zone and the radiation of energy and angular momentum. The two-time-scale point of view and osculating orbits are described in §7, and I conclude with some remarks about stationary exact solutions in §8.

2. The nonrigorous Newtonian limit

For our later discussion it will help to summarize the usual textbook derivation of the Newtonian limit¹. It is nonrigorous because it does not ask whether there are any solutions which have this limit, and if there are then whether the limit is uniform everywhere or just in some region; and because it does not examine whether the terms it neglects are genuinely small.

Let us consider a system of mass M , typical size R , velocity v , density ρ , and pressure p . Newtonian systems have low redshifts, so we want a limit in which $M/R \rightarrow 0$. Let us choose coordinates that keep the size of the system fixed. Then we want $M \rightarrow 0$, and consequently $\rho \rightarrow 0$. But a Newtonian system is one in which gravity supplies a significant force, so by the virial theorem we must have p/ρ and v^2 of the same order as M/R . Thus, as v goes to zero we want $\rho \sim v^2$ and $p \sim v^4$. In terms of the stress-energy tensor we want $|T^{00}| \gg |T^{0i}| \gg |T^{ij}|$, and $T^{00} \rightarrow 0$. Since the field is weak we may write in quasi-Lorentz coordinates

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad (2)$$

and expect $|h_{\alpha\beta}| \ll 1$. If we define

$$\bar{h}^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g} g^{\alpha\beta} \quad (3)$$

$$= h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h_{\mu}^{\mu} + O(h^2) \quad (4)$$

(raising indices on $h_{\mu\nu}$ with $\eta^{\alpha\beta}$), and adopt the harmonic gauge condition

$$\bar{h}^{\alpha\beta}_{,\beta} = 0 \quad (5)$$

then the field equations become ($\square = -\partial_t^2 + \nabla^2$)

$$\square h^{\mu\nu} = -16\pi T^{\mu\nu} + O(h^2), \quad (6)$$

which in turn imply that $|\bar{h}^{00}| \gg |\bar{h}^{0i}| \gg |\bar{h}^{ij}|$. In the limit of slow motion ($v \rightarrow 0$) the dominant part of Eq. (6) to survive is

$$\nabla^2 \bar{h}^{00} = -16\pi\rho, \quad (7)$$

which leads to the identification of $-\bar{h}^{00}/4$ with the Newtonian potential ϕ . The equations of motion have similarly two implications. The energy equation $T^{0\alpha}_{;\alpha} = 0$ implies the continuity equation at order v^2 ,

$$\rho_{,0} + (\rho v^i)_{,i} = 0, \quad (8)$$

while the momentum equation $T^{i\alpha}_{;\alpha} = 0$ has its dominant terms at order v^4 (for a perfect fluid)

$$\rho v^i_{,0} + \rho v^j v^i_{,j} + p^{,i} + \rho(-\bar{h}^{00}/4)^{,i} = 0. \quad (9)$$

This is the Newton-Euler equation.

3. Newtonian scale-invariance

Why should the ordering of v , ρ and p in the previous section lead to Newton's equations? The answer is that Newton's equations are themselves invariant under changes in the variables that strictly preserve that ordering. Specifically, if $\rho(x^i, t)$, $p(x^i, t)$, $v^j(x^i, t)$, and $\bar{h}^{00}(x^i, t)$ satisfy Eq. (7)-(9) then so do the following re-scaled functions for arbitrary ϵ :

$$\begin{aligned} \rho(x^i, t) &\rightarrow \epsilon^2 \rho(x^i, \epsilon t) \\ p(x^i, t) &\rightarrow \epsilon^4 p(x^i, \epsilon t) \\ v^j(x^i, t) &\rightarrow \epsilon v^j(x^i, \epsilon t) \\ \bar{h}^{00}(x^i, t) &\rightarrow \epsilon^2 \bar{h}^{00}(x^i, \epsilon t). \end{aligned} \quad (10)$$

The factors of ϵ are what we expect from our previous discussion. The scaling of t is equally important: since velocities are changing with ϵ , the time it takes things to happen must likewise scale. The limit $v \rightarrow 0$ of general relativity corresponds to the limit $\epsilon \rightarrow 0$ in Eq. (10). If we are describing a binary system, for example, then the masses of the stars would decrease, their sizes and orbits remain the same, and their orbital period increase.

4. Framework for a careful Newtonian limit

In order to get around the twin problems of existence and domain of uniformity of solutions having Newtonian limits, it seems safest to incorporate the scaling of Eq. (10) into a careful definition of a sequence of relativistic solutions. Since such sequences are most conveniently defined by giving initial data, it seems natural to define a *regular, asymptotically Newtonian sequence* of

solutions of Einstein's equations to be a sequence parametrized by ϵ and developing from the following sequence of initial data:

$$\begin{aligned}\rho(t=0, x^i, \epsilon) &= \epsilon^2 a(x^i) \\ p(t=0, x^i, \epsilon) &= \epsilon^4 b(x^i) \\ v^j(t=0, x^i, \epsilon) &= \epsilon c^j(x^i) \\ \bar{h}^{ij}(t=0, x^i, \epsilon) &= \bar{h}^{ij}_0(t=0, x^i, \epsilon) = 0 ,\end{aligned}\tag{11}$$

where a , b , and c^j are functions of compact support. The three-velocity v^j is defined as U^j/U^0 . These data suffice to determine a solution, since initial data for $\bar{h}^{0\alpha}$ are determined by the constraint equations. If there is an equation of state it should contain ϵ in such a way as to permit the scaling of ρ and p in Eq. (11). We shall discuss possible variants of these data that also lead to a Newtonian limit below.

For $t > 0$ the nonlinearities of Einstein's equations will add higher-order terms in ϵ to the various functions in Eq. (11). We thus expect an asymptotic expansion of the form

$$\rho(t, x^i, \epsilon) = \epsilon^2 f(t, x^i) + \epsilon^3 g(t, x^i) + \dots ,$$

where $g(0, x^i) = 0$. But to the extent that we have a Newtonian limit, the leading term in ρ will behave like the Newtonian density, so we expect it to be a function only of the Newtonian *dynamical time*

$$\tau = \epsilon t .\tag{12}$$

So we should look instead for an expansion of ρ of the form

$$\rho(t, x^i, \epsilon) = \epsilon^2 {}_2\rho(\tau, x^i) + \epsilon^3 {}_3\rho(\tau, x^i) + \dots ,\tag{13}$$

in an obvious notation. Now we can identify ${}_2\rho(\tau, x^i)$ as the Newtonian density, and higher-order corrections are called post-Newtonian terms. The post-Newtonian approximation to general relativity is the asymptotic expansion in ϵ at fixed τ and x^i of the regular asymptotically Newtonian sequence. See Figure (1).

How important is it that the initial data have the form given in Eq. (11)? From our discussion it will be clear that the leading orders of at least ρ , p , and v^j should be the same as in Eq. (11), but it is possible to add higher-order terms in ϵ to the initial data without destroying the Newtonian limit. These terms will simply serve as initial data for the post-Newtonian equations of motion. This might be a natural relaxation of Eq. (11) in the search for an exact solution, especially if one is determined not by initial data but, at least partly, say, by asymptotic data on its gravitational field.

Figure 1. For each ϵ we have drawn only the t -dimension of each solution vertically. Since the $\epsilon = 0$ manifold is Minkowski spacetime (see Eq.11), t is a proper-time coordinate for small ϵ . Lines of constant τ are hyperbolae which connect points with similar physical configuration in different manifolds. As $\tau \rightarrow 0$ these hyperbolae go to $t = \infty$, because weak-field solutions take longer times to evolve. The collection of manifolds may be regarded as a fiber bundle over the base space R^1 parameterized by ϵ . We will develop various limiting boundaries to this fiber bundle below, the manifolds OM, NM and FM.

Setting the free-field initial data $\bar{h}^{ij}$ and $\bar{h}^{ij}_{,0}$ in Eq.(11) to zero is the simplest way of ensuring that, at least after one light-crossing time, the metric has a retarded-type solution⁹, but this condition can be relaxed considerably. Since real astrophysical systems are subject to essentially random amounts of incoming gravitational radiation from entirely unrelated sources (other binaries, a cosmic background, etc.) it is attractive to allow $\bar{h}^{ij}$ and $\bar{h}^{ij}_{,0}$ to be random variables and to find the expected evolution over the ensemble of systems with these data^{9,11}. It can be shown¹¹ that if the expectation values of $\bar{h}^{ij}$ and $\bar{h}^{ij}_{,0}$ are both zero, then the random data may be given amplitudes of order ϵ^4 without changing our conclusions below about radiation reaction or outgoing radiation in the Newtonian limit. These are large enough amplitudes to affect the first post-Newtonian equations (stochastically).

5. The near-zone limit: the post-Newtonian hierarchy and the near-zone quadrupole formula

Our goal is to make an asymptotic approximation to $\bar{h}^{\mu\nu}$ and $T^{\mu\nu}$ in ϵ for fixed τ and x^i . In terms of the wavelength of gravitational waves we expect from the system, this is a *near-zone* approximation: the period of such waves will scale as the dynamical time of the system, i.e. as ϵ^{-1} , and since the speed of light is unity the gravitational wavelength also scales as ϵ^{-1} . Any point at fixed x^i , no matter how far away, is within the near zone for sufficiently small ϵ . Since the Newtonian limit is one at fixed x^i , it is a near-zone limit.

For each ε we have a solution of Einstein's equations. With the definition of $\bar{h}^{\mu\nu}$ in Eq. (3) and the harmonic gauge condition, Eq. (5), the full field equations generalize Eq. (6) to

$$\square \bar{h}^{\mu\nu} = -16\pi \Lambda^{\mu\nu}$$

$$\Lambda^{\mu\nu} = (-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}) + (16\pi)^{-1} (\bar{h}^{\mu\alpha} \bar{h}^{\nu\beta} - \bar{h}^{\mu\nu} \bar{h}^{\alpha\beta})_{,\alpha\beta}$$

where $t_{LL}^{\mu\nu}$ is the Landau-Lifshitz pseudotensor⁶. We introduce the notation

$$\tilde{\Lambda}^{\mu\nu}(\tau, y^j, \varepsilon) \equiv \Lambda^{\mu\nu}(\tau/\varepsilon, y^j, \varepsilon) \quad (15)$$

to take explicit account of our rescaling the time variable. Then Eq. (14) has the implicit solution given by Kirchhoff's formula

$$\begin{aligned} \bar{h}^{\mu\nu}(\tau, x^j, \varepsilon) = & 4 \int_{C(\tau, x^j, \varepsilon)} \tilde{\Lambda}^{\mu\nu}(\tau - \varepsilon r, y^j, \varepsilon) r^{-1} d^3y \\ & + \frac{\tau}{4\pi} \int_{S(\tau, x^j, \varepsilon)} \bar{h}^{\mu\nu}_{,\tau}(\tau=0, y^j, \varepsilon) d\Omega y \\ & + \frac{1}{4\pi} \frac{\partial}{\partial \tau} \left\{ \tau \int_{S(\tau, x^j, \varepsilon)} \bar{h}^{\mu\nu}(\tau=0, y^j, \varepsilon) d\Omega y \right\}, \end{aligned} \quad (16)$$

where $r = |y^j - x^j|$, and where the light-cone C and sphere S over which the integrals are taken are shown in Fig. (2). Notice that C is a *truncated* light-cone: it does not extend back to the infinite past. Equation (16) gives the unique solution for $\bar{h}^{\mu}$ in terms of its initial data (integrals over S) and source (integral over C). Therefore, provided the harmonic coordinate condition does not break down, integrating over the 'flat-space' light-cone C does not

Figure 2. The past coordinate-cone of P_1 is C and intersects $\tau = 0$ at the sphere S . At early times (P_2) the retarded integral does not cover the whole region where $\rho \neq 0$ (dashed lines), but at later times (P_1) it is essentially the usual retarded integral. As $\varepsilon \rightarrow 0$, a point of fixed (τ, x^i) moves upwards and the intersection S moves outwards.

introduce errors or acausal behaviour. Any such contributions from C must be compensated by the free-wave solutions (integrals over S). By contrast, global approaches using conditions on $\mathcal{J}^-$ have usually omitted the integrals over S and continued C to the infinite past^{7,13}. This is an approximation, since these flat-space light cones do not end up at $\mathcal{J}^-$. The higher-order divergences encountered in such schemes⁷ may be traceable to this.

Now, Eq. (16) is an implicit expression for $\bar{h}^{\mu\nu}$, but it enables us to develop an asymptotic approximation in ε . Since we are holding τ and not t constant, we must convert all t -indices in Eq. (16) to τ -indices, using Eq. (12) as a coordinate transformation. Then our strategy is to use Taylor's theorem for n -times differentiable functions

$$f(\varepsilon) = f(0) + \varepsilon f'(0) + \frac{1}{2!} \varepsilon^2 f''(0) + \dots + \frac{1}{(n-1)!} \varepsilon^{n-1} f^{(n-1)}(0) + R_n \quad (17)$$

where¹⁴

$$R_n = \frac{1}{(n-1)!} \int_0^1 (1-\ell)^{n-1} \frac{d^n}{d\ell^n} f(\varepsilon\ell) d\ell = o(\varepsilon^{n-1}). \quad (18)$$

When we differentiate Eq. (16) with respect to ε at fixed τ and x^i , we find that the n th derivative of $\bar{h}^{\mu\nu}$ is given by integrals that depend only upon lower-order derivatives of $\bar{h}^{\mu\nu}$. This enables us to develop an expansion like Eq. (17) recursively. The various post-Newtonian approximations emerge as derivatives of Eq. (16) and the conservation equation

$$\Lambda^{\mu\nu}_{,\nu} = 0 \quad (19)$$

with respect to ε at $\varepsilon = 0$. In order to do these calculations one must assume certain properties of the functions $\bar{h}^{\mu\nu}(\tau, x^i, \varepsilon)$, relating to their differentiability and behaviour for small ε . These are described in ref. (11), but their proof awaits a stronger existence/uniqueness theorem for Einstein's equations than we now possess, one which can deal with fluids of compact support.

The transformation from t to τ means that $T^{\tau\tau} = \varepsilon^2 T^{tt}$ and $T^{\tau i} = \varepsilon T^{ti}$, so that the initial data for all components of $T^{\mu\nu}$ begin at ε^4 . It is not surprising, therefore, that all derivatives of $\bar{h}^{\mu\nu}$ up to and including third order in ε are zero at $\varepsilon = 0$. (Given that we are in τ - x^j coordinates, these derivatives are the same as Lie derivatives along the congruence shown in Fig. (1).) The first non-trivial terms are therefore at fourth order, where the initial-value equations imply

$$\frac{\partial^4}{\partial \varepsilon^4} \bar{h}^{\mu\tau}(\tau=0, x^j, \varepsilon) \Big|_{\varepsilon=0} \equiv {}_4\bar{h}^{\mu\tau}(\tau=0, x^j) = -16\pi \nabla^{-2} [{}_4T^{\mu\tau}(\tau=0)], \quad (20)$$

where ∇^{-2} denotes the inverse Laplacian regular at infinity. The other terms in $\Lambda^{\mu\tau}(\tau=0)$ do not contribute at this order. For example, in this gauge we have

$$16\pi(-g)t_{LL}^{\tau\tau} = \left[\frac{1}{2} g^{\tau\tau} g_{\nu\sigma} \delta_{\gamma}^{\mu} \delta_{\sigma}^{\lambda} - 2g^{\tau\mu} g_{\rho\gamma} \delta_{\nu}^{\tau} \delta_{\sigma}^{\lambda} + g_{\gamma\sigma} g^{\lambda\mu} \delta_{\nu}^{\tau} \delta_{\rho}^{\tau} \right. \\ \left. + \frac{1}{8} (2g^{\tau\lambda} g^{\tau\mu} - g^{\tau\tau} g^{\lambda\mu}) (2g_{\nu\rho} g_{\sigma\gamma} - g_{\rho\sigma} g_{\nu\gamma}) \right] \bar{h}^{\nu\gamma}_{,\lambda} \bar{h}^{\rho\sigma}_{,\mu} . \quad (21)$$

The quadratic terms in $\bar{h}^{\mu\nu}$ will be of order ε^8 , but we have to take account of the fact that $g^{\tau\tau} \sim \varepsilon^2$ while $g_{\tau\tau} \sim \varepsilon^{-2}$. Inspection of Eq. (21) shows that its lowest-order contribution will be at order ε^6 . Similarly, the second group in Eq. (15) will not contribute until eighth order. The gauge condition Eq. (5) provides the initial data for ${}_4\bar{h}^{\mu\tau}_{,\tau}$. The contribution of the surface integrals over S includes the integral

$$I(\tau, x^j, \varepsilon) = \oint_{S(\tau, x^j, \varepsilon)} \bar{h}^{\mu\tau}(\tau=0, y^j, \varepsilon) d\Omega y , \quad (22)$$

whose fourth-order derivative is

$$\frac{\partial^4}{\partial \varepsilon^4} I(\tau, x^j, \varepsilon) = \oint_{S(\tau, x^j, \varepsilon)} \frac{\partial^4}{\partial \varepsilon^4} \bar{h}^{\mu\tau}(\tau=0, y^j, \varepsilon) d\Omega y \\ + 4 \int_{S(\tau, x^j, \varepsilon)} n^k \nabla_k \left[\frac{\partial^3}{\partial \varepsilon^3} \bar{h}^{\mu\tau}(\tau=0, y^j, \varepsilon) \right] d\Omega y + \dots , \quad (23)$$

where the second and subsequent terms arise from the dependence of the sphere S on ε . Here n^k is the unit outward normal to S . The limit to $\varepsilon = 0$ of $\partial^4 I / \partial \varepsilon^4$ involves the limit to infinity of the sphere S , so the behaviour of the integrals in Eq. (23) depends on how $\partial^4 \bar{h}^{\mu\tau} / \partial \varepsilon^4$ ($\tau=0, \varepsilon$) behaves as $\varepsilon \rightarrow 0$ and $|y^j| \rightarrow \infty$. Since we cannot solve the initial-value problem for $\bar{h}^{\mu\tau}(\tau=0, \varepsilon)$ except iteratively, we are forced to assume that the limit of Eq. (23) is given by

$$\frac{\partial^4}{\partial \varepsilon^4} I(\tau, x^j, \varepsilon) \Big|_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \oint_{S(\tau, x^j, \varepsilon)} {}_4\bar{h}^{\mu\tau}(\tau=0, y^j) d\Omega y . \quad (24)$$

That is, we can replace integrands in the limit by their limiting values. (We have used the fact that ${}_3\bar{h}^{\mu\tau}(\tau=0) = 0$.) This would not be correct if, say, ${}_4\bar{h}^{\mu\tau} \sim r^{-1}$ as $r \rightarrow \infty$ but ${}_5\bar{h}^{\mu\tau} \sim r$. So we are assuming that the behaviour of ${}_4\bar{h}^{\mu\tau}$ at large r is typical of that of $\partial^4 \bar{h}^{\mu\tau} / \partial \varepsilon^4$ for all sufficiently small ε . This is the significance of assumption (iii) of ref. (10). We have to make similar assumptions at each step of the iteration, which is why we have discussed it in some detail here. We will not point it out again.

Since ${}_4\bar{h}^{\mu\tau}(\tau=0)$ is a solution of Laplace's equation with a source of compact support, and it can be shown⁹ that for sufficiently small ε (as soon as S encloses the support of the source in Laplace's equation) the contribution of the surface integrals to Eq. (16) vanishes for ${}_4\bar{h}^{\mu\tau}(\tau, x^j)$. We are left, then, with

$${}_4\bar{h}^{\mu\tau}(\tau, x^j) = 4 \lim_{\epsilon \rightarrow 0} \left[\frac{\partial^4}{\partial \epsilon^4} \int_{C(\tau, x^j, \epsilon)} \tilde{h}^{\mu\tau}(\tau - \epsilon r, y^j, \epsilon) r^{-1} d^3 y \right]. \quad (24)$$

Again, the only contribution to this will be from $T^{\mu\tau}$, which is of compact support. Therefore r in the integrand is bounded, and this means that the retardation may be ignored. We have, finally,

$${}_4\bar{h}^{\mu\tau}(\tau, x^j) = 4 \int_{\text{all } y^j} {}_4T^{\mu\tau}(\tau, y^j) r^{-1} d^3 y, \quad (25)$$

again using the notation introduced in Eq. (13).

More specifically, these are

$${}_4\bar{h}^{\tau\tau}(\tau, x^j) = 4 \int {}_2\rho(\tau, y^j) r^{-1} d^3 y \quad (26)$$

$${}_4\bar{h}^{\tau i}(\tau, x^j) = 4 \int {}_2\rho(\tau, y^j) {}_1v^i(\tau, y^j) r^{-1} d^3 y. \quad (27)$$

This expression for ${}_4\bar{h}^{\tau\tau}$ is the same as the solution of Eq. (7) in our discussion of the nonrigorous Newtonian limit.

For ${}_4\bar{h}^{ij}$ the calculation is simpler, since the zero initial data for $\bar{h}^{ij}$ allow us to ignore the surface integrals in Eq. (16). But here ${}_4t_{LL}^{ij}$ does not vanish: self-gravitational stresses are of the same order as material stresses in the Newtonian limit. We easily deduce that

$${}_4t_{LL}^{ij} = (1/64\pi) ({}_4\bar{h}^{\tau\tau, i} \bar{h}^{\tau\tau, j} - \frac{1}{2} \eta^{ij} {}_4\bar{h}^{\tau\tau, k} \bar{h}^{\tau\tau}_{,k}) , \quad (28)$$

and

$${}_4T^{ij} = {}_2\rho {}_1v^i {}_1v^j + {}_4p \delta^{ij}. \quad (29)$$

Then we have as before

$${}_4\bar{h}^{ij}(\tau, x^k) = 4 \int \left[{}_4T^{ij}(\tau, y^k) + {}_4t_{LL}^{ij}(\tau, y^k) \right] r^{-1} d^3 y. \quad (30)$$

The equation of motion, Eq. (19) implies the conservation equation

$${}_2\rho_{, \tau} + \nabla_i ({}_2\rho {}_1v^i) = 0 \quad (31)$$

and the Newton-Euler equation

$${}_2\rho {}_1v^i_{, \tau} + {}_2\rho {}_1v^j \nabla_j {}_1v^i + \nabla^i {}_4p + \nabla_j {}_4t_{LL}^{ij} = 0, \quad (32)$$

which are the rigorous counterparts of Eqs. (8) and (9).

We can now ask about the geometry of the limiting manifold we reach as $\epsilon \rightarrow 0$ for fixed τ and x^i . This is most conveniently displayed in Fig. (3), a re-scaled version of Fig. (1). I call this manifold NM, the near-zone limiting manifold. The metric in it is degenerate: $g^{\tau\mu} \rightarrow 0$, $g^{ij} \rightarrow \delta^{ij}$. The Christoffel symbols are well-behaved, and the only non-zero one in the limit is

$$\Gamma^i_{\tau\tau} \rightarrow -\frac{1}{4} \nabla^i {}_4\bar{h}^{\tau\tau}, \quad (33)$$

Near zone limit Spaces

Figure 3. This is the same sequence as in Fig. (1), but now use τ as the vertical time coordinate. This illustrates the attachment of two boundaries to the fiber bundle: NM as the limit $\varepsilon \rightarrow 0$ at fixed τ and OM as the limit at fixed t .

which is the Cartan³ connection for Newtonian gravity. Thus, Cartan's geometrical description of Newtonian gravity appears here as a natural limit of the geometry of solutions of Einstein's equations.

At the next order, ε^5 , the initial data all vanish, and one can show¹⁰ that ${}_3\rho(\tau, x^j)$, ${}_2v^i(\tau, x^j)$, and ${}_5\bar{h}^{\mu\tau}(\tau, x^j)$ all vanish. But one limit remains non-zero:

$${}_5\bar{h}^{ij}(\tau, x^k) = -4 \int {}_4\Lambda^{ij}{}_{,\tau}(\tau, y^k) d^3y, \quad (34)$$

which depends only on τ . This will not affect equations of motion in later approximations until radiation-reaction order, ε^9 .

When this procedure is carried to sixth order¹⁰ and beyond¹¹, each time making the assumption that the integrals may be evaluated in the limit by using the limiting values of the corresponding derivatives obtained at lower order, one recovers the usual post-Newtonian (sixth order) and second post-Newtonian (eighth order) equations^{4,14}. At ε^7 order there are again some terms that will contribute to radiation reaction. Once we have examined the solutions at any order and found them to be well-behaved, it follows from Eqs. (17) and (18) that the orders below that constitute a genuine asymptotic approximation to our sequence of solutions for small ε .

We come now to radiation-reaction order, ε^9 . In this gauge the expression for ${}_9\bar{h}^{\tau\tau}$ and for the other terms that contribute to the equations of motion at this order, such as ${}_7\bar{h}^{\tau i}$, are rather complicated and have been written down elsewhere^{7,11}. Futamase¹¹ has examined ε^{10} order to show that these radiation-reaction terms are also asymptotic approximations. By a change of gauge they may be brought into the simpler form derived and used by previous workers^{4,5,9}, that one can

incorporate their effects into the Newtonian equations by adding in a reaction force

$$F_{\text{react}}^i = -2\rho \nabla^i \left(\frac{1}{5} x^j x^k {}_2\ddot{x}_{jk}^{(5)} \right) \quad (35)$$

where ${}_2\ddot{x}_{jk}$ is the reduced or trace-free quadrupole tensor and the (5) above it denotes its fifth derivative with respect to τ :

$${}_2\ddot{x}_{jk}(\tau) = \left(\delta_j^\ell \delta_k^m - \frac{1}{3} \delta_{jk} \delta^{\ell m} \right) \int {}_2\rho(\tau, y^i) y_\ell y_m d^3y. \quad (36)$$

The correct use of Eq. (35) has often been the subject of some confusion. It is often said, for instance, that Eq. (35) causes a secular decrease in the Newtonian energy of the system, because it follows from Eq. (35) that

$$\int v_i F_{\text{react}}^i d^3x = -\frac{1}{5} {}_2\ddot{x}^{jk} {}_2\ddot{x}_{jk}^{(5)}, \quad (37)$$

whose average over one period of an (almost-) periodic motion is

$$\left\langle \int v_i F_{\text{react}}^i d^3x \right\rangle = -\frac{1}{5} \langle {}_2\ddot{x}^{jk} {}_2\ddot{x}_{jk}^{(5)} \rangle. \quad (38)$$

This is negative-definite, indicating a mean loss of energy. But what does this mean? For one thing, one expects post-Newtonian terms to affect the Newtonian energy, yet they have been left out of the calculation leading to Eq. (38), despite their larger size. For another, the concept of energy and its conservation is at best an uncertain one in general relativity: how do we know what observable effects follow from Eq. (38)?

The physicist's intuitive answer to these questions is that the reaction force is the lowest-order term that breaks the conservation laws for energy and angular momentum (the equations up to ϵ^8 order possess such conservation laws). Therefore, if these effects are small we are entitled to use an 'adiabatic' approach, allowing the Newtonian energy, angular momentum, and other quantities which are functions only of them (such as the period and eccentricity in an orbit problem) to change at a rate given by using Eq. (35) without considering other, lower-order terms in the equations of motion. Other details of the motion, such as the movement of the periastron or the non-ellipticity of an orbit, will not be given correctly by just using Eq. (35), since these effects arise at lower order. In this spirit, the reaction formulae have been applied in a number of astrophysical situations^{15,16}, and with particular success to the changing orbital period of the binary pulsar system¹⁷, PSR 1913+16. This intuitive use of the reaction force can be made rigorous by a careful study of the relation between observables and various post-Newtonian quantities^{11,18,19}. We will return to this point in §7 on osculating orbits.

Past radiation-reaction order, some derivatives occur which diverge logarithmically in ε as $\varepsilon \rightarrow 0$. These are related to the divergent terms that were found in global approaches⁷, but they do not signal the end of the approximation scheme. This point is worth describing in some detail.

Earlier approaches looked for terms of the form, say, $A\varepsilon^n$, where A is given as an integral over the analogue of our cone C , only carried back to $\mathcal{J}^-$. This is an infinite domain, and some coefficients A were found to diverge as $\ln R$ or even R as the limit of integration R was taken to infinity⁷. This gave terms in the approximation that were formally infinite for any finite ε , no matter how small, and which were therefore larger than the lower-order terms. Although there was a general feeling that such terms were an artefact of the approach and not physically important, it was clearly impossible to assert that the radiation-reaction terms, for example, were asymptotic approximations to general relativity while such infinities lurked at the next order².

The method described here, however, links the domain of integration C to the parameter ε in such a way that the upper limit R of the previous paragraph is proportional to ε^{-1} . These integrals, therefore, simply change the ε^n term to a term $\varepsilon^n \ln \varepsilon$ or $\varepsilon^{-1}\varepsilon^n = \varepsilon^{n-1}$, respectively, with finite coefficients¹¹. These are *finite* for any ε . From our point of view, the divergent terms of the global approach are terms whose order is mistaken, and which make finite contributions at lower order. Having thus made them finite and smaller than the radiation-reaction order, we thereby show that the radiation-reaction terms are asymptotic.

The presence of a term like $\varepsilon^{10} \ln \varepsilon$ does, however, mean that the sequence is not ten-times differentiable at $\varepsilon = 0$. But we can continue to use Taylor's theorem, Eq. (17), to develop the approximation past this point. Consider a function $f(\varepsilon)$ which is ε^{n-1} at $\varepsilon = 0$, and suppose that $d^n f/d\varepsilon^n = g(\varepsilon) + h(\varepsilon)$, where $h(0)$ exists but $g(\varepsilon)$ is presumed to be badly behaved near $\varepsilon = 0$. Then the existence of $d^{n-1}f/d\varepsilon^{n-1}$ at zero proves that $g(\varepsilon)$ is integrable near zero, and this means that R_n in Eq. (18) exists. If we define

$$G(\varepsilon) = \frac{1}{(n-1)!} \int_0^1 (1-\ell)^{n-1} g(\varepsilon\ell) d\ell$$

(which generally is also badly-behaved near $\varepsilon = 0$), then Eq. (17) proves that $R_n = \varepsilon^n G(\varepsilon) + a\varepsilon^n + o(\varepsilon^n)$ some constant a . Provided $g(\varepsilon)$ is simple enough for us to evaluate $G(\varepsilon)$, this may be regarded as giving the next term in an asymptotic approximation to $f(\varepsilon)$. Moreover, if we define

$$\tilde{f}(\varepsilon) = f(\varepsilon) - f(0) - \varepsilon f'(0) - \dots - \varepsilon^n G(\varepsilon),$$

then $\tilde{f}(\varepsilon)$ is ε^n at $\varepsilon = 0$. If we can show that the next derivative of $\tilde{f}$ also has the form $\tilde{g}(\varepsilon) + \tilde{h}(\varepsilon)$, for sufficiently simple $\tilde{g}(\varepsilon)$ and regular $\tilde{h}(\varepsilon)$, then we can continue the asymptotic approximation to $f(\varepsilon)$ in this manner. The question of whether we can make such an approximation to our sequence comes down to the

question of whether we can show at every step that a diverging derivative can be decomposed into a simple $g(\epsilon)$ and a regular $h(\epsilon)$. To each order we have explored, we have found only $\ln \epsilon$ and ϵ^{-n} terms for $g(\epsilon)$, both of which are simple enough to work with. Because of the nonlinearity of $\Lambda^{\mu\nu}$ one expects $(\ln \epsilon)^2$ and higher powers eventually, but these are also simple enough to give us $G(\epsilon)$. It seems unlikely that any terms will arise which are not of the form $\epsilon^n (\ln \epsilon)^m$: the divergences come from the integral over C of integrands which always fall off as integer powers of r (or possibly logarithms of r) far away. Thus, the algorithm we have outlined should produce no infinite coefficients of functions of ϵ at any order, and provided the assumptions described in ref. (10) remain valid, this will be an asymptotic approximation to our sequence of solutions.

6. The far-zone limit: radiation in the Newtonian limit

Although Newton's equations do not admit gravitational radiation (so there is none on NM), each of our solutions of Einstein's equations does contain radiation, and here we shall study this as $\epsilon \rightarrow 0$. To see how, we take a hint from our observation in the previous section that any point (τ, x^i) enters the near zone as $\epsilon \rightarrow 0$ because the gravitational wavelength scales as ϵ^{-1} . To stay in the far zone we want to remain a fixed number of wavelengths away from the source. Accordingly, we define a scaled spatial coordinate

$$\eta^I = \epsilon x^i; \quad (39)$$

as $\epsilon \rightarrow 0$ a point at fixed (τ, η^I) will remain in the far zone if it starts out in the far zone. We call η^I the far-zone coordinate and use capital-letter indices to distinguish it from its near-zone counterpart x^i .

We begin again with the implicit formal solution given by Eq. (16), but now take derivatives at fixed τ and η^I . In the scaled far-zone coordinates of Fig. (4), the region occupied by the material system shrinks to zero as $\epsilon \rightarrow 0$, and the light-cone of the origin divides spacetime into two parts. Since we are interested only in the radiation leaving the system, we will confine our discussion to points inside this cone, except for a few remarks at the end of this section. The integrals over S therefore always enclose the source, and are as simple as in the previous section. In particular, they begin to contribute only at ϵ^8 order. But the retarded integral over C now has to be handled carefully (see Fig. 5). At the lowest orders, its main contribution will be from the part of C that cuts through the near zone, which is shrinking to zero in far-zone coordinates. We therefore divide the manifold into two different parts by putting a tube of fixed radius R in far-zone coordinates around the material system. Inside this, we express the integrands in near-zone coordinates and integrate over their near-zone expansions as developed in the previous section. For the part of C for which $|\eta^I| > R$, we proceed iteratively as in the previous section, developing a far-zone approximation

Figure 4. A manifold in far-zone coordinates. As $\varepsilon \rightarrow 0$ the near zone (interior of tube) collapses down, and the light-cone of the origin divides spacetime into regions spacelike and timelike separated from the initial-data region containing the matter ($\rho \neq 0$).

Figure 5. As Fig. (4), but displaying how the retarded cone C of a far-field point must be divided into its far-zone and near-zone pieces. The boundary between them is fixed in far-zone coordinates.

to $\bar{h}^{\mu\nu}$ and using that in subsequent steps wherever it appears in the integrand. The final solution does not depend on R . Fortunately, for our purposes we do not have to carry the approximation very far in ϵ in order to see the radiation. Full details will be given elsewhere¹².

Let us consider the calculation of $\bar{h}^{\tau\tau}$ in the far zone. The two pieces of its integration over C are denoted

$$\bar{h}_N^{\tau\tau}(\tau, \eta^I, \epsilon) = 4\epsilon \int_{|y^I| < R/\epsilon} \bar{\Lambda}^{\tau\tau}(\tau - |\eta^I - \epsilon y^I|, y^I, \epsilon) |\eta^I - \epsilon y^I|^{-1} d^3y \quad (40)$$

and

$$\bar{h}_F^{\tau\tau}(\tau, \eta^I, \epsilon) = 4\epsilon^{-2} \int_{|\xi^I| > R} \bar{\Lambda}^{\tau\tau}(\tau - |\eta^I - \xi^I|, \xi^I, \epsilon) |\eta^I - \xi^I|^{-1} d^3\xi. \quad (41)$$

The factors of ϵ outside come from the conversions to scaled coordinates in the r^{-1} term and the integration element. As in Eq. (15), we have introduced the notation $\bar{\Lambda}^{\tau\tau}$ in Eq. (41) because of the change to scaled coordinates in its argument:

$$\bar{\Lambda}^{\mu\nu}(u, \xi^I, \epsilon) = \tilde{\Lambda}^{\mu\nu}(u, \xi^I/\epsilon, \epsilon). \quad (42)$$

The bar over $\Lambda^{\mu\nu}$ therefore does not have the same meaning as one over $h^{\alpha\beta}$.

We shall now show that $\bar{h}_F^{\mu\nu}$ is necessarily of higher order in ϵ than $\bar{h}_N^{\mu\nu}$. Return for a moment to *unscaled* coordinate indices: the limit to $\epsilon = 0$ of the metric tensor $g^{\mu\nu}$ along any curve through the sequence of solutions in Fig. (1) is $\eta^{\mu\nu}$, because the $\epsilon = 0$ solution is Minkowski spacetime. Therefore in the scaled coordinates (τ, η^I) its limit is $\epsilon^2 \eta^{\mu\nu}$. Since $\bar{h}^{\mu\nu}$ is essentially the perturbation in this metric, it follows that in far-zone coordinates

$$\bar{h}^{\mu\nu} = o(\epsilon^2). \quad (43)$$

Inspection of Eq. (21) for $t_{LL}^{\tau\tau}$ shows that in the far zone it is of order $\bar{h}^2$, which is true of its other components as well. Then Eq. (41) shows that $\bar{h}_F^{\tau\tau}$ is of order $\epsilon^{-2} \bar{h}^2$, which by Eq. (43) is of higher order than $\bar{h}^{\mu\nu}$ itself. This applies to $\bar{h}_F^{I\tau}$ and $\bar{h}_F^{IJ}$ as well. It means that the dominant contribution to $\bar{h}^{\mu\nu}$ is Eq. (40) and its counterparts for other components, which means that the order of $\bar{h}^{\mu\nu}$ in the far zone is determined by the order of its near-zone source.

We approximate $\bar{h}_N^{\tau\tau}$ from Eq. (40) in the same manner as in the near zone, only now derivatives with respect to ϵ hold τ and η^I fixed. It is easy to see that since $\bar{\Lambda}^{\tau\tau}$ begins at fourth order, $\bar{h}_N^{\tau\tau}$ and therefore $\bar{h}^{\tau\tau}$ is of order ϵ^5 . If we define

$$u = \tau - |\eta^I|, \quad (44)$$

the scaled retarded time of the far-field point, and use the fact that ${}_4\bar{\Lambda}^{\tau\tau} = {}_2\rho$ is of compact support, so that differential retardation across the source may be neglected to leading order, we find

$${}_5\bar{h}_N^{\tau\tau} = 4|\eta^I|^{-1} \int {}_2\rho(u, y^i) d^3y. \quad (45)$$

But the integral is just the Newtonian mass,

$${}_2M = \int {}_2\rho(u, y^i) d^3y, \quad (46)$$

which is independent of u . Thus we have

$${}_5\bar{h}^{\tau\tau} = {}_5\bar{h}_N^{\tau\tau} = 4 {}_2M/|\eta^I|. \quad (47)$$

This is, not surprisingly, the Newtonian potential far away. Its order, ϵ^5 , is composed of ϵ^2 from the mass, ϵ^2 from the conversion of indices from t to τ , and ϵ from the conversion from $|x^i|^{-1}$ to $|\eta^I|^{-1}$.

Similar calculations show that the leading orders for $\bar{h}^{\tau I}$ and $\bar{h}^{IJ}$ are ϵ^6 and ϵ^7 , respectively:

$${}_6\bar{h}^{\tau I} = 4 {}_3P^i/|\eta^J| \quad (48)$$

and

$${}_7\bar{h}^{IJ} = 2 {}_2I^{ij}_{,uu}(u)/|\eta^K|, \quad (49)$$

where

$${}_3P^i = \int {}_2\rho_1 v^i d^3y \quad (50)$$

is the Newtonian momentum (again independent of u), and

$${}_2I^{ij}(u) = \int {}_2\rho(u, y^k) y^i y^j d^3y \quad (51)$$

is the quadrupole tensor again. (The mixed use of capital and lower-case indices in Eqs. (48) and (49) is deliberate. The calculation of Eq. (49) is more delicate than that of the others, since $\tilde{\Lambda}^{ij}$ is not of compact support.) Since we now know that $\bar{h}^{\mu\nu}$ in far-zone coordinates is of order ϵ^5 , our previous argument shows that $\bar{h}_F^{\mu\nu}$ begins to contribute at order ϵ^8 , which is higher than we need to go to find the radiation.

It is now straightforward to take higher derivatives of Eq. (40) and its counterpart for $\bar{h}_N^{\tau I}$ in order to develop their expansions through ϵ^7 , and we find

$${}_6\bar{h}^{\tau\tau} = 4 \eta^I {}_2P_i/|\eta^J|^2 + 4 \eta^I {}_2D_i/|\eta^J|^3 \quad (52)$$

(sums on repeated indices even when one is capitalized and the other isn't),

$$\begin{aligned} {}_7\bar{h}^{\tau\tau} = & 4 {}_4M/|\eta^I| + 2 \eta^I \eta^J [{}_2I_{ij,uu} + {}_3\mathbb{F}_{ij,u}/|\eta^K| \\ & + {}_3\mathbb{F}_{ij}/|\eta^K|^2] / |\eta^K|^3, \end{aligned} \quad (53)$$

and

$${}_7\bar{h}^{\tau I} = 2 \eta^J [{}_2I^i_{j,uu} + ({}_3M^i_j + {}_2I^i_{j,u})/|\eta^K|] / |\eta^K|^2. \quad (54)$$

Here the expressions

$${}_2D^i(u) = \int {}_2\rho(u, y^i) y^i d^3y \quad (55)$$

and

$${}_3M^{ij} = \int {}_2\rho({}_1v^i y^j - {}_1v^j y^i) d^3y \quad (56)$$

are the Newtonian mass dipole moment and angular momentum, respectively; ${}_2\ddot{x}_{ij}$ was defined in Eq. (36); and ${}_4M$ is the post-Newtonian contribution to the mass,

$${}_4M = \int {}_6\tilde{\Lambda}^{\tau\tau}(u, y^i) d^3y, \quad (57)$$

where ${}_6\tilde{\Lambda}^{\tau\tau}$ is given in Eq. (4.27) of Ref. (10). (It is not of compact support, but differential retardation in Eq. (40) may still be ignored.) By the conservation law Eq. (19), this integral is independent of u .

We therefore have a far-zone expansion for $\bar{h}^{\mu\nu}$ up to and including terms of order ε^7 . The order ε^8 terms can be shown to be well-behaved, so this is again an asymptotic approximation to our far-zone metric. Through this order, it is identical to the far-field metric that would be calculated using a slow-motion approximation in *linearized theory*²⁰ for a body of mass $\varepsilon^5 {}_2M + \varepsilon^7 {}_4M$, momentum $\varepsilon^6 {}_3P^i$, angular momentum $\varepsilon^7 {}_3M^{ij}$, etc. The nonlinearities of the full theory are not apparent at this order. Since the source terms for our metric have the same conservation properties as in linearized theory (e.g. ${}_2M$ and ${}_4M$ independent of u , etc.), the waves may be treated in the far-zone just as waves in linearized theory.

In particular, the energy and angular momentum they carry are as well defined as in linearized theory. This is a considerable simplification over the situation in global approaches, which try to measure the flux at asymptotic null infinity. Notice that we need calculate the flux at a finite (τ, η^I) , not at infinity of any of the spacetimes in our sequence. Using, for example, the Isaacson²¹ measure of flux it is straightforward to calculate that the total luminosity is

$$L = \frac{1}{5} \varepsilon^{10} \langle {}_2\ddot{x}^{ij} {}_2\ddot{x}_{ij} \rangle, \quad (58)$$

which is exactly the flux needed to compensate the energy loss in the near zone, Eq. (38).

Just as the near-zone limit picked out a manifold NM, which was a four-dimensional limit of the sequence of spacetimes, so also is there a four-dimensional far-zone manifold FM, which has coordinates τ and η^I and has a metric which is flat, after a constant conformal factor of ε^2 is removed in the limit. On this manifold the outgoing waves are a linearized-theory perturbation. The relation between FM and NM is shown in Fig. (6).

Location of NM in FM

Figure 6. As $\epsilon \rightarrow 0$ in Fig. (4), all near-zone points are squeezed to the origin $\eta^I = 0$ of FM. The Minkowski manifold OM is squeezed to the point $\eta^I = \tau = 0$.

These calculations have been performed in harmonic gauge, and indeed in a version of harmonic gauge in which $\bar{h}^{ij}$ and $\bar{h}^{ij}_{,0}$ both vanish at $t = 0$. How do our conclusions stand up to a change of gauge? This is discussed elsewhere¹², but it is not hard to show that if we make a different choice of gauge within the family of harmonic gauges which does not affect the near-zone equations at Newtonian order, then they act in the far zone at worst as gauge transformations of order ϵ^7 . This means that they are again within the embrace of linearized theory.

Just as in the near zone, we expect to be able to continue the far-zone approximation indefinitely with terms of the form $\epsilon^n (\ln \epsilon)^m$.

As an approximation to the, say, $\epsilon = 1$ manifold in our sequence, the near-zone approximation should be uniformly valid for only a finite interval $\Delta\tau$ ($= \Delta t$ for $\epsilon = 1$), and for the whole of the near zone. Similarly, the far-zone approximation we have described can be uniformly valid only for the same interval $\Delta\tau$ of scaled retarded time u , but it should be valid for all $r > R$, where R is the boundary of the near zone in the far zone as in Fig. (5). We address the question of approximating larger portions of the $\epsilon = 1$ manifold in the next section.

The reader familiar with other approaches to the radiation-reaction problem will have noticed similarities between the calculations of this section and some of the methods of matched asymptotic expansions that have been brought to bear on this problem, particularly recently²². This relationship is close, but it

has not yet been fully explored. It seems likely that our method of introducing a sequence of solutions to Einstein's equations can provide a more rigorous proof of the validity of some of the heuristic matching methods that have been used so plausibly on this problem.

We have only considered the radiation for retarded time $u > 0$. Similar considerations apply to the region where advanced time $v = \tau + |\eta^I| < 0$ (i.e. to the past). But the details in the region ($u < 0, v > 0$), which is spacelike separated from the compact support of the initial data at $t = 0$, depends sensitively on the initial data chosen for $\bar{h}^{ij}$ outside the matter. Our 'simple' choice, that (the ensemble mean of) $\bar{h}^{ij}$ should vanish, has the disadvantage that for any fixed ϵ the limit to $t = 0$ outside the matter (even in the near zone) is not the same as the limit to $\tau = 0$ of ${}_4\bar{h}^{ij}$: the limits in t and ϵ do not commute. This could presumably be changed by changing the (ensemble mean) data for $\bar{h}^{ij}$, with no effect on our calculations. This might then give the 'Newtonian' data required by Winicour²³ in his null-hypersurface approach to this problem.

7. Osculating orbits and the period of binary systems

As we have just noted, both the near- and far-zone approximations are uniform for only finite intervals of scaled time. Thus, one might start at $t = 0$ with the initial data of Eq. (11) and evolve the Newtonian equations for a time $\Delta\tau$. The approximation will then be in error by a certain amount when compared with the $\epsilon = 1$ relativistic solution, and a new Newtonian approximation should be started, using as initial data the present state of the relativistic solution. The accuracy of the approximation increases as $\Delta\tau$ decreases, so this suggests we can idealize the approximation by defining an instantaneous Newtonian approximation which continuously changes in time. We will describe this in detail for the case of a binary star system, where it is called the method of osculating orbits. It gives us a framework for interpreting the observations of the binary pulsar system¹⁷.

The term 'osculating orbit' arose in celestial mechanics, where it denotes instantaneous Keplerian orbits of solar system bodies whose orbits are subject to perturbations by the planets. To my knowledge, it was first used in the present context by Walker and Will⁸. The Newtonian orbits of a binary system are determined by the initial data: the positions, velocities, and masses of the stars. (Here we neglect tidal interaction of the stars, which would obscure the present discussion. It must, of course, be allowed for in real systems.) According to the correspondence we have already established in Eq. (11) between relativistic systems and their Newtonian approximations, we may take the state of a relativistic ($\epsilon = 1$) binary at any time τ to define an instantaneous Newtonian orbit by taking the relativistic state as initial

data for the Newtonian one. This is the osculating orbit. We may thereby regard all properties of the Newtonian system as continuous variables in t as the relativistic system evolves. These include the period, eccentricity, position and time of periastron, and integrals that are not directly observable, such as the energy and angular momentum. When discussing radiation-reaction effects on the orbit, it is best to define an osculating second-post-Newtonian orbit, i.e. one which evolves according to the approximation up to but not including radiation-reaction effects. It is then possible to show^{18,19} that by taking radiation reaction into account (as in §5), the quadrupole formula gives the rate of change of this second-post-Newtonian energy, to lowest order in ϵ . Moreover, the inferred rate of period change of the binary system is also the lowest-order change in the second-post-Newtonian period.

Parameters of the osculating orbit are exactly what the observers measure when they report, say, a period for the binary pulsar's orbit. They find the period by fitting an orbit to a few months' stretch of data, and find a systematic change in the period from one stretch to the next¹⁷. This is just the situation described at the beginning of this section. The osculating-orbit picture is thus the link between theory and observation.

8. The role of exact solutions

Since exact, nonspherical, nonstationary, nonsingular, asymptotically flat solutions seem unlikely in the near future, it is perhaps more useful to ask about exact stationary sequences with a Newtonian limit. Several sequences are known for static spherically symmetric stars with more-or-less realistic equations of state, so we shall consider here the general case of stationary rotating solutions.

We shall remain within Lorentz gauge, but assume that our time coordinate is tied to the killing time. Then Eq. (14) reduces to

$$\nabla^2 \bar{h}^{\mu\nu} = -16\pi \Lambda^{\mu\nu} \quad (59)$$

and the gauge condition Eq. (5) implies the stationarity condition

$$\Lambda^{\mu i}_{,i} = 0. \quad (60)$$

The time-components of Eq. (59) are the same as the constraints in this case, but the spatial components provide an equation for determining $\bar{h}^{ij}$. It is clear from this that we have to abandon our simple data in Eq. (11). In the dynamical case, the exact nature of the initial radiation data doesn't matter in the weak-field limit (provided the data are not too specially chosen); but in the stationary case, even an unimportant amount of radiation is forbidden. This change affects ${}_4\bar{h}^{ij}$, which is a post-Newtonian term, but it does not affect the Newtonian limit.

The leading-order terms in the solution of Eq. (59) at infinity are

$${}_4\bar{h}^{\tau\tau}(x^j) = 4 \int {}_2\rho(y^j) r^{-1} d^3y, \quad r = |x^j - y^j|, \quad (61)$$

which has the same multipole moments as the Newtonian potential;

$${}_4\bar{h}^{\tau i}(x^j) = 4 \int {}_2\rho(x^j) {}_1v^i(y^j) r^{-1} d^3y; \quad (62)$$

and

$${}_4\bar{h}^{ij}(x^k) = 4 \int {}_4\Lambda^{ij}(y^k) r^{-1} d^3y, \quad (63)$$

which is given by Eqs. (28)-(30) without the time-dependence.

If one is generating solutions by certain methods at infinity (see the lectures by Hoenselaers elsewhere in this volume), there are two points to be borne in mind. First, the asymptotic forms of $\bar{h}^{\mu\nu}$ may differ from Eqs. (61)-(63) by a change of gauge. Second, for our initial data we would have additional terms of all orders in $|x^j|^{-1}$ and higher orders in ε ; it does not seem worthwhile writing these out explicitly here because there may be no real need to match them explicitly. It is probably better to put a simple ε -dependence into the far-field moments, chosen according to convenience according to the requirements of the method. We have already remarked that the post-Newtonian corrections are not unique. What is important is that the dominant (in ε) moments in the far field will have the interpretation conferred on them by Eqs. (61)-(63), modulo gauge transformations.

If such a sequence of solutions could be found, it would help us to answer what is probably the most important unsolved problem of the Newtonian limit: given that the post-Newtonian approximations are asymptotic, what is their accuracy? If a relativistic solution has a typical M/R of 2%, is the Newtonian approximation accurate to 2%? 1%? 10%? This question is hard to answer from the approximation scheme alone, and there are very few existing calculations which enable such a judgement to be made²⁴. A particularly useful insight would be into the effects of compactness on the external gravitational field. The present derivation of radiation reaction, for example, assumes that the stars as well as their orbits are Newtonian, and might not be valid for the binary pulsar system, which consists of neutron stars whose orbits stay in the weak-field region. It is generally assumed that the stars' compactness is not important, and this is reinforced by the analytic-continuation calculations for 'point' masses by Damour and Deruelle²⁵; Futamase¹¹ has described a method by which this might be rigorously established within the approximation method. But an exact solution could considerably help us on this point.

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