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# Quantum spinning strings in $\mathrm{AdS}_4 \times \mathbb{CP}^3$: testing the Bethe Ansatz proposal

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Abstract: Recently, an asymptotic Bethe Ansatz that is claimed to describe anomalous dimensions of "long" operators in the planar $\mathcal{N}=6$ supersymmetric three-dimensional Chern-Simons-matter theory dual to quantum superstrings in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ was proposed. It initially passed a few consistency checks but subsequent direct comparison to one-loop string-theory computations created some controversy. Here we suggest a resolution by pointing out that, contrary to the initial assumption based on the algebraic curve considerations, the central interpolating function $h(\lambda)$ entering the BMN or magnon dispersion relation receives a non-zero one-loop correction in the natural string-theory computational scheme. We consider a basic example which has already played a key role in the $AdS_5 \times S^5$ case: a rigid circular string stretched in both $\mathrm{AdS}_4$ and along an $S^1$ of $\mathbb{CP}^3$ and carrying two spins. Computing the leading one-loop quantum correction to its energy allows us to fix the constant one-loop term in $h(\lambda)$ and also to suggest how one may establish a correspondence with the Bethe Ansatz proposal, including the non-trivial one-loop phase factor. We discuss some problems which remain in trying to match a part of world-sheet contributions (sensitive to compactness of the worldsheet space-like direction) and their Bethe Ansatz counterparts.

Keywords: AdS-CFT Correspondence, Integrable Field Theories.

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## Contents



JHEP11(2008)069

## 1. Introduction

The duality [1] between planar $\mathcal{N} = 6$ supersymmetric three-dimensional Chern-Simons-matter theory and free type IIA superstring theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ ($\mathrm{AdS/CFT}_3$ for short) has attracted much attention recently. This is for a good reason, as both the perturbative gauge theory and the dual free string theory appear to be integrable (as was partially verified at two-loop level in gauge theory — namely in the scalar sector [2] (see also [3]) — and at the classical level in string theory [4, 5]). If so, this correspondence may be providing us with a second example of integrable gauge-string duality, in addition to the by now well understood canonical one relating the $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) and the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring (or $\mathrm{AdS/CFT}_4$).

Being less than maximally supersymmetric, this new duality is useful as it reveals various seemingly obvious assumptions that were made (and eventually shown to be correct in the maximally supersymmetric context) in the construction of the solution for the spectrum of AdS/CFT$_4$ based on the Bethe Ansatz (see [6] and references therein). Bearing in mind possible future studies of less supersymmetric dualities in both three and four dimensions this is an important step forward.

One crucial change compared to the AdS/CFT$_4$ case is that now the BMN or magnon dispersion relation is no longer protected and receives nontrivial corrections both in the weak and strong coupling expansions [7 − 9] (see also [10 − 12]). For example, the dispersion relation for the "lighter" magnon and its BMN limit are given by

$$\epsilon(p) = \frac{1}{2}\sqrt{1 + 16h^2(\lambda)\sin^2\frac{p}{2}} \qquad \to \qquad \frac{1}{2}\sqrt{1 + 16\pi^2 h^2(\lambda)\frac{k^2}{J^2}}\,, \tag{1.1}$$

with $p = \frac{2\pi k}{J}$ in the BMN limit and where

$$h(\lambda \ll 1) = \lambda[1 + c_1\lambda^2 + c_2\lambda^4 + \cdots]\,, \qquad h(\lambda \gg 1) = \sqrt{\frac{\lambda}{2}} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \cdots\,. \tag{1.2}$$

Incorporating this new interpolating function, the authors of [10] made a remarkable proposal for the corresponding Bethe Ansatz which has (somewhat surprisingly at first sight) essentially the same structure as in the AdS/CFT$_4$ case. It was suggested in [10] that the leading strong coupling (one-loop in the world sheet theory) correction to $h(\lambda)$ should vanish, i.e. $a_1 = 0$; this was apparently confirmed in [13] where the fluctuation spectrum near the giant magnon solution was computed using the algebraic curve technique [14 − 16] (the conjecture also passed a few other consistency checks see [11, 17]).

However, the subsequent direct string theory computations [18 − 20] of the one-loop correction to the universal scaling function, i.e. the coefficient of the $\ln S$ term in the folded spinning string energy [21, 22], led to the result that was different from the Bethe Ansatz prediction of [10] based on the assumption that $a_1 = 0$.

It was suggested in [23] that this disagreement was due to different regularizations used, or rather to different ways of combining fluctuation frequencies in the calculation of the one-loop correction to the string energy. The proposed prescription, argued to be intrinsic to the algebraic curve description of the classical string solutions in the Bethe Ansatz context, favored the $a_1 = 0$ choice.

While the string theory sigma model is manifestly one-loop finite in the ultraviolet, separate terms in one-loop corrections contain logarithmic divergences. Hence results obtained by regularizing separate terms in different ways, e.g. using different cutoffs, may differ by finite terms (for an example, see [25]). On general grounds, however, in the string theory calculation one should regularize the world-sheet action or the path integral; any acceptable regularization should be independent of the fine structure of the spectrum of fluctuations around a specific solution[1] and should preserve the basic (global and local) symmetries of the theory. Within the class of *acceptable* world-sheet regularizations all choices should be equivalent.

Our aim here will be to provide a resolution to the apparent contradiction between the world-sheet [18 − 20] and the Bethe Ansatz [10, 23] calculations while staying within a

[1] Ideally, the classical solution should be constructed in the presence of the regulator.

natural and consistent world-sheet regularization scheme. We will be led to the conclusion that, in this context, the coefficient $a_1$ in equation (1.2) has a non-zero value

$$a_1 = -\frac{\ln 2}{2\pi} \quad . \tag{1.3}$$

Using this value in the Bethe Ansatz prescription of [10] restores the agreement between the string theory result and the Bethe Ansatz result for the one-loop term in the universal scaling function. A non-zero value for the constant term $a_1$ may be accounted for by a redefinition of the 't Hooft coupling[2], suggesting that the world-sheet and the Bethe Ansatz calculations effectively employ different regularization schemes. While anomalous dimensions at renormalization group fixed points are scheme-independent, for conformal field theories parameterized by free parameters the scheme dependence may, in fact, arise as the freedom of redefining these parameters. Such may be the case here, in contrast with the world-sheet theory in $AdS_5{\times}S^5$ where no such redefinitions appear to be necessary.

A possible way of avoiding such an ambiguity is to define the coupling constant of the theory in terms of an observable, e.g. a particular anomalous dimension. Perhaps a natural choice for such an observable is the universal scaling function $f(\lambda)$. Eliminating the 't Hooft coupling in favor of $f$ effectively removes all scheme ambiguities related to coupling constant redefinitions. Such a proposal was put forward in QCD [26] to systematically account for the scheme dependence in the running of the coupling constant. Since at weak coupling $f(g(\lambda)) \sim \lambda$, the resulting expressions are necessarily analytic in $f$. This analyticity property holds also (despite a different dependence on the 't Hooft coupling) for the gauge theory dual of the world-sheet theory in $AdS_4 \times \mathbb{CP}^3$.

It is worth noting that in the all-loop Bethe Ansatz proposal of [10] the 't Hooft coupling appears only through the function $h(\lambda)$ and consequently if we express all other anomalous dimensions in terms of the scaling function $f(h(\lambda))$ any trace of the function $h$ will be removed, demonstrating that it is unphysical. However, this is only true for the Bethe Ansatz of [10]; for the perturbative calculation in the gauge or string theory we must work with $\lambda$ and thus need the explicit weak or strong coupling expansion of $h(\lambda)$ in whatever regularization scheme we choose to work in.

In addition, below we will be able to provide a non-trivial test of the proposal of [10] by directly computing the one-loop correction to the energy of the circular $(S, J)$ string [54, 27, 28] from the $AdS_4 \times \mathbb{CP}^3$ string theory action and then trying to match the result with the prediction of the Bethe Ansatz of [10]. We will find that the two answers are remarkably similar, indicating that the Bethe Ansatz proposal of [10] may indeed be correct at strong coupling. However, few issues remain, warranting a further more systematic study on the Bethe Ansatz side.

The computation of the one-loop correction [28 − 30] to the energy of the simplest rigid circular $(S, J)$ string in $AdS_5{\times}S^5$ played a key role in discovering the presence of the one-loop term [29, 32] in the phase in the strong-coupling (or "string") form of the

[2]Here we consider the coupling as it appears on the string worldsheet and thus have in mind redefinition of the form $\frac{1}{\sqrt{\lambda'}} = \frac{1}{\sqrt{\lambda}} - a_1 \frac{1}{\lambda} + \ldots$. It is not *a priori* clear that such a redefinition will be consistent with a similar weak coupling redefinition of the form $\lambda' = \lambda + c_1 \lambda^2 + \ldots$.

Bethe Ansatz [33]. Our plan here will be to follow the same logic as in [28, 29], i.e. carry out the analogous computation in the $AdS_4 \times \mathbb{CP}^3$ case and then compare to the Bethe Ansatz prediction.

For later use let us define the rescaled coupling constant, $\bar{\lambda}$, in terms of the 't Hooft coupling, $\lambda$ (equal to $\frac{N}{k_{cs}}$, where $k_{cs}$ is the level of the Chern-Simons action), [3] as well as the function $\bar{h}$ as

$$\bar{\lambda} = 2\pi^2\lambda\,, \qquad\qquad \bar{h}(\bar{\lambda}) = 2\pi h(\lambda) \quad . \tag{1.4}$$

The role of $\bar{\lambda}$ is to emphasize the close analogy between the $AdS_5$ and the $AdS_4$ string-theory expressions.

Let us first recall the story in the $AdS_5\times S^5$ case. The $(S, J)$ string solution of [27] has a spiral-like shape, with projection to $AdS_3$ being a constant radius circle (with winding number $k$), and projection to $S^5$ — a big circle (with winding number $m$). The corresponding spins are, respectively, $S$ and $J$ with the Virasoro condition implying that $u \equiv \frac{S}{J} = -\frac{m}{k}$. The classical string energy has the following expansion in large semiclassical parameters $\mathcal{S}$ and $\mathcal{J}$ with fixed $k$ and fixed $u = \frac{\mathcal{S}}{\mathcal{J}}$ [27, 28] ($E_0 = \sqrt{\lambda}\mathcal{E}(\mathcal{S}, \mathcal{J}, k)$, $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$, $\frac{\sqrt{\lambda}}{2\pi}$ is the string tension)

$$E_0 = S + J + \frac{\lambda}{J}e_1(u, k) + \frac{\lambda^2}{J^3}e_3(u, k) + \frac{\lambda^3}{J^5}e_5(u, k) + \cdots\,, \tag{1.5}$$

where $e_1 = \frac{k^2}{2}u(1 + u)$, $e_3 = -\frac{k^4}{8}u(1 + u)(1 + 3u + u^2)$, $e_5 = \frac{k^6}{16}u(1 + u)(1 + 7u + 13u^2 + 7u^3 + u^4)$, etc. In the limit when $u \to 0$ or $S \ll J$ this takes the familiar BMN form

$$E_0 = J + \sqrt{1 + \frac{\lambda k^2}{J^2}}\ S\ + O(S^2)\ . \tag{1.6}$$

Computing the one-loop correction $E_1 = \mathcal{E}_1(\mathcal{S}, \mathcal{J}, k)$ to the energy gives [28, 29]

$$E_1 = E_1^{\rm even} + E_1^{\rm odd}, \quad E_1^{\rm even} = \frac{\lambda}{J^2}g_2(u, k) + \frac{\lambda^2}{J^4}g_4(u, k) + \cdots\,, \quad E_1^{\rm odd} = \frac{\lambda^{5/2}}{J^5}g_5(u, k) + \cdots \tag{1.7}$$

The absence of the $\frac{1}{J}$ and $\frac{1}{J^3}$ terms here implies the non-renormalization of the BMN-type part of the classical energy (1.6) which is consistent with the non-renormalization of the BMN dispersion relation in the $AdS_5\times S^5$ case. This also suggests that the two leading $\frac{\lambda}{J}$ and $\frac{\lambda^2}{J^3}$ terms are protected and their coefficients should directly match the corresponding one-loop and two-loop perturbative gauge theory results.

Indeed, the coefficient $g_2$ of the "even" $\frac{1}{J^2}$ term[4] in (1.7) can be reproduced as a leading $\frac{1}{J}$ (finite spin chain length) correction from the one-loop Bethe Ansatz in the $sl(2)$ sector of the $\mathcal{N} = 4$ SYM theory [36]. An extension to higher orders was discussed in [31]. The same should apply to the coefficient of the other analytic even $\frac{\lambda^2}{J^4}$ term — i.e. it should match the two-loop gauge theory result.

[3] Since we are interested in the strict 't Hooft limit when $N \to \infty$, $k_{cs} \to \infty$ with $\lambda$ being fixed we can treat $\lambda$ as a continuous parameter.

[4] Its value is $g_2 = -\frac{1}{2}M^2 + \sum_{n=1}^{\infty}[n\sqrt{n^2 + 4M^2} - n^2 - 2M^2]$, $\quad M^2 \equiv k^2u(1 + u)$.



JHEP11(2008)069

At the same time, the presence of the non-analytic in $\lambda$ and "odd" in $\frac{1}{J}$ term $\frac{\lambda^{5/2}}{J^5}$ in (1.7) (with $g_5 = \frac{k^6}{3} u^3 (1+u)^3$) implies that a similar $\frac{1}{J^5}$ term in the classical energy (1.5) is not protected so that its coefficient cannot be directly compared to three-loop result on the gauge theory side. This resolves the infamous "three-loop disagreement" [24] and implies [29] that the corresponding "string" Bethe Ansatz [33] should be modified to contain a non-trivial one-loop correction to the phase.[5]

The circular $(S, J)$ string solution in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is essentially the same as that in $\mathrm{AdS}_5 \times \mathrm{S}^5$, with the classical energy having again the form (1.5) (modulo some numerical factors due to the different definition of string tension). However, as we shall find below, the expression for the one-loop correction is drastically changed: the expansion of $E_1^{\mathrm{odd}}$ in (1.7) starts already with $\frac{1}{J}$ and $\frac{1}{J^3}$ terms. This implies that the corresponding leading terms in the classical energy (1.5) are no longer protected.[6] Indeed, considering the $S \ll J$ limit, i.e. comparing to equation (1.6), these odd one-loop corrections can be unambiguously interpreted as a one-loop renormalization of the coefficient of the $\frac{k^2}{J^2}$ term under the square root in the BMN dispersion relation (1.6), leading to the value of the one-loop shift in $h(\lambda)$ given in (1.3) (cf. equations (1.1), (1.6)).

Several of the $\frac{1}{J^5}$ terms can similarly be interpreted as arising from the one-loop shift in $h(\lambda)$; the remaining term happens to be essentially the same as in $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, which is in perfect agreement with the Bethe Ansatz proposal of [10] where the S-matrix dressing phase has the same form as in the AdS/CFT$_4$ case (up to the replacement of $\sqrt{\lambda}$ by $2\bar{h}(\bar{\lambda}) = 4\pi h(\lambda)$).

The even $\frac{1}{J^2}$ and $\frac{1}{J^4}$ terms do not appear to be the same as in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, but can be formally related to their $\mathrm{AdS}_5 \times \mathrm{S}^5$ counterparts by restricting the sum over mode numbers to odd integers and making some re-identification of parameters. While the results of our computation appear to be in agreement with the general structure of the Bethe Ansatz of [10] with $h(\lambda)$ given by (1.2), (1.3) there are still remaining subtle issues related to $\frac{1}{J^{2n}}$ terms which require further clarification.

The rest of this paper is organized as follows. In section 2 we discuss, following closely the model of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [27, 28], the structure of the classical circular string solution in $\mathrm{AdS}_4 \times \mathbb{CP}^3$. In section 3 we present the spectrum of quadratic fluctuations near this solution derived directly from the Green-Schwarz superstring action. In section 4 we sum up these frequencies to derive the one-loop correction to the string energy. We then compare it to the similar expression in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, determining in the process the one-loop term in the $h(\lambda)$ function and discussing correspondence with the Bethe Ansatz result implied by the proposal of [10]. Some computational details and special cases are collected in five appendices.

[5] The one-loop term in the S-matrix dressing phase can be completely determined by including higher order terms in the expansion of $E_1$ [32].

[6] This is of course not surprising given that the leading gauge-theory correction here is the two-loop one [2], i.e. proportional to $\lambda^2$, while the leading term in the classical string energy still scales as $\lambda$.



JHEP11(2008)069

## 2. The circular rotating string in $\mathbf{AdS}_4 \times \mathbb{CP}^3$

As was recently pointed out [1], the closed superstring (type IIA) background which describes holographically the $\mathrm{U}(N) \times \mathrm{U}(N)$ $\mathcal{N} = 6$ Chern-Simons theory at levels $(k_{\rm cs}, -k_{\rm cs})$ is (we follow the notation of [18])

$$
\begin{aligned}
ds^2 &= \frac{R^3}{4k_{\rm cs}}\left(ds^2_{{\rm AdS}_4} + 4ds^2_{\mathbb{CP}^3}\right) , \qquad & e^{2\phi} &= \frac{R^3}{k^3_{\rm cs}} \\
F_2 &= k_{\rm cs}\, \mathbb{J}_{\mathbb{CP}^3} , \qquad & F_4 &= \frac{3}{8}R^3 {\rm Vol}_{{\rm AdS}_4}
\end{aligned}
\tag{2.1}
$$

Here

$$
ds^2_{{\rm AdS}_4} = -\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{2.2}
$$

$$
\begin{aligned}
ds^2_{\mathbb{CP}^3} = d\zeta_1^2 + \sin^2\zeta_1 \Big[ & d\zeta_2^2 + \cos^2\zeta_1 \left(d\tau_1 + \sin^2\zeta_2 \left(d\tau_2 + \sin^2\zeta_3 d\tau_3\right)\right)^2 \\
& + \sin^2\zeta_2 \left(d\zeta_3^2 + \cos^2\zeta_2 \left(d\tau_2 + \sin^2\zeta_3 d\tau_3\right)^2 + \sin^2\zeta_3\cos^2\zeta_3 d\tau_3^2\right)\Big] .
\end{aligned}
\tag{2.3}
$$

The radii of curvature of the $\mathrm{AdS}_4$ and of $\mathbb{CP}^3$ factors are

$$
R^2_{\rm AdS} = \frac{R^3}{4k_{\rm cs}} , \qquad R^2_{\mathbb{CP}^3} = 4R^2_{\rm AdS} . \tag{2.4}
$$

At the world-sheet tree level, the relation between the radius of curvature and the gauge theory 't Hooft coupling arises from simply matching the charges of the supergravity soliton describing the relevant stack of branes, and to leading order in the strong coupling expansion one finds [1]

$$
R^2_{\rm AdS} = \sqrt{\bar\lambda} . \tag{2.5}
$$

Due to the non-maximal supersymmetry of the space this relation may, in principle, receive world-sheet quantum corrections (see footnote 8 below).[7] We have used here the notation $\bar\lambda$ introduced in (1.4) to maintain a formal similarity with string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$, where the radius of the space is the 't Hooft coupling of the dual gauge theory.

While not entering in the interactions of the world-sheet bosons, the flux fields govern the interactions of the bosons and the Green-Schwarz fermions. In that context, their tangent space components are relevant. For the field strengths in (2.1) these components read

$$
(F_2)_{\mu\nu} = 2\frac{k^2_{\rm cs}}{R^3}\mathbb{J}_{\mu\nu} , \qquad (F_4)_{abcd} = 6\frac{k^2_{\rm cs}}{R^3}\epsilon_{abcd} , \tag{2.6}
$$

or

$$
e^{\phi}(F_2)_{\mu\nu} = \frac{1}{R_{\rm AdS}}\mathbb{J}_{\mu\nu} , \qquad e^{\phi}(F_4)_{abcd} = \frac{3}{R_{\rm AdS}}\epsilon_{abcd} . \tag{2.7}
$$

[7] We shall ignore this possibility here. One way to determine if this relation is modified would be to study possible renormalization of 3-point functions of chiral primary operators both on the gauge theory and string theory sides.

An important property of $\mathrm{AdS}_4 \times \mathbb{CP}^3$ space, largely similar to that of $\mathrm{AdS}_5{\times}\mathrm{S}^5$ space, is that all relevant tangent space tensors are constant. Indeed, here $\mathbb{J}$ and $\epsilon$ are numerical tensors with entries $\pm 1$ and 0. They are, respectively, the entries of the Kähler form and of the volume form on unit $\mathbb{CP}^3$ and $\mathrm{AdS}_4$.

All classical spinning string solutions with sufficiently few charges are common between string theory in $\mathrm{AdS}_5{\times}\mathrm{S}^5$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$, since $\mathrm{AdS}_4 \subset AdS_5$ and, up to a change of radius, a single isometry direction looks the same in $\mathrm{S}^5$ and $\mathbb{CP}^3$ (for a discussion of related classical string solutions which excite more fields in $\mathbb{CP}^3$ see [37]). Like the spinning folded string, the circular rotating string is also in this class of common solutions. They, in fact, excite the same fields, which makes them ideal to identify potential conceptual differences between strings in $\mathrm{AdS}_5{\times}\mathrm{S}^5$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

The world-sheet action is

$$\begin{aligned} S &= S_{\mathrm{AdS}_4} + S_{\mathbb{CP}^3} \\ &= \frac{R_{\mathrm{AdS}}^2}{4\pi} \int d\tau \int_0^{2\pi} d\sigma\ \sqrt{g} g^{ab} \left( G_{\mu\nu}^{\mathrm{AdS}} \partial_a X^\mu \partial_b X^\nu + 4 G_{\mu\nu}^{\mathbb{CP}^3} \partial_a X^\mu \partial_b X^\nu \right) . \end{aligned} \tag{2.8}$$

We express the string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ in terms of the radius of the AdS space as in the $\mathrm{AdS}_5{\times}\mathrm{S}^5$ case.[8] We will be using the conformal gauge and thus take the worldsheet metric to be flat, $g_{ab} = \eta_{ab}$.

All conserved quantities derived from this action are related to the corresponding charge densities by factors of the string tension:

$$(E,\, S,\, J) = \sqrt{\lambda}\, (\mathcal{E},\, \mathcal{S},\, \mathcal{J}) \ , \tag{2.9}$$

where $(\mathcal{E},\, \mathcal{S},\, \mathcal{J})$ are given in terms of the momenta conjugate to isometry directions from the Lagrangian

$$L = \frac{1}{2} \eta^{ab} \left( G_{\mu\nu}^{\mathrm{AdS}} \partial_a X^\mu \partial_b X^\nu + 4 G_{\mu\nu}^{\mathbb{CP}^3} \partial_a X^\mu \partial_b X^\nu \right) \quad . \tag{2.10}$$

The rotating string solution we are interested in lies in an $\mathrm{AdS}_3{\times}\mathrm{S}^1$ subspace of $\mathrm{AdS}_4{\times}\mathbb{CP}^3$. The choice of the circle $\mathrm{S}^1 \subset \mathbb{CP}^3$ should be such that it corresponds to the BMN vacuum state chosen on the gauge theory side, i.e. a gauge-invariant combination $\mathrm{Tr}(Y^1 Y_4^\dagger)^J$ of the scalar field bilinear $Y^1 Y_4^\dagger$ [7, 9].

It is useful to discuss in more detail how the $\mathbb{CP}^3$ coordinates in (2.3) are related to the scalar fields of the dual gauge theory of [1]. It is natural to start with the form of the metric written in projective coordinates (see e.g. [34]). Given an eight-dimensional flat space $ds^2 = dZ^A d\bar{Z}_A$ with complex coordinates $Z^A$ with $A = 1, 2, 3, 4$, restricting to the

[8] This relation may, in fact, receive quantum corrections. Indeed, since the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ geometry is not maximally supersymmetric, it may be corrected at the world-sheet quantum level. Type IIA supergravity action is known to receive higher derivative corrections; modifications of the classical geometry arise from requiring that the geometry solves the modified equations of motion. Such higher derivative corrections, however, first arise at order $\mathcal{O}(\alpha'^3)$, i.e. they would be suppressed by an additional factor of $\lambda^{-3/2}$. They are thus of too high an order to be relevant to the one-loop calculation we will be interested in here.

7-sphere $\sum_A |Z^A|^2 = 1$, choosing $Z^4 = e^{i\tau_4}|Z^4|$ and then introducing $\xi^m = Z^m/Z^4$ with $m = 1,2,3$ one ends up with the $S^7$ metric written as a circle fibration over $\mathbb{CP}^3$. Rewriting $\xi^m$ in terms of its norm and a unit vector $u^m$ as $\xi^m = \tan\zeta_1\, u^m$ one may then repeat this construction recursively.

The isometric directions of the resulting metric denoted by $\tau_{1,2,3}$ correspond to the phases of the analogs of $Z^4$ at each step of the recursion, i.e. $Z^4 = e^{i\tau_4}|Z^4|$, $Z^3 = e^{i(\tau_3+\tau_4)}|Z^3|$, $Z^2 = e^{i(\tau_2+\tau_3+\tau_4)}|Z^2|$, $Z^1 = e^{i(\tau_1+\tau_2+\tau_3+\tau_4)}|Z^1|$.
To identify the spinor representation of the SO(6) $\subset$ SO(8) R-symmetry of the gauge theory (the scalars $Y^A$ are transforming as a spinor) let us define a new set of angles

$$\tau_1 = \varphi_2 - \varphi_1\,,\ \ \tau_2 = \varphi_3 - \varphi_2\,,\ \ \tau_3 = \varphi_2 + \varphi_1\,,\ \ \tau_4 = \tau_0 + \varphi_4\,,\ \ \varphi_4 \equiv -\frac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3) \tag{2.11}$$

getting

$$\begin{aligned} Z^1 &= e^{i[\tau_0 + \frac{1}{2}(+\varphi_3+\varphi_2-\varphi_1)]}|Z^1|\,, \qquad & Z^2 &= e^{i[\tau_0 + \frac{1}{2}(+\varphi_3-\varphi_2+\varphi_1)]}|Z^2|\,, \\ Z^3 &= e^{i[\tau_0 + \frac{1}{2}(-\varphi_3+\varphi_2+\varphi_1)]}|Z^3|\,, \qquad & Z^4 &= e^{i[\tau_0 + \frac{1}{2}(-\varphi_3-\varphi_2-\varphi_1)]}|Z^4|\,. \end{aligned} \tag{2.12}$$

Observing that the shift by $\tau_0$ does not affect the $\mathbb{CP}^3$ coordinates, the homogeneous coordinates $Z^A$ are in one-to-one correspondence with the four gauge theory scalar fields $Y^A$ in the spinor representation of SO(6), provided one identifies the three Cartan generators of SO(6) as represented by shifts of $\varphi_{1,2,3}$, i.e. $J_i = -i\frac{\partial}{\partial\varphi_i}$. Then

$$J_1(Z^A) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right),\ J_2(Z^A) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right),\ J_3(Z^A) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right). \tag{2.13}$$

Thus, the SO(6) charges of the operator $\mathrm{Tr}[(Y^1 Y_4^\dagger)^J]$ are matched by the charge of the product of $J$ bilinears, $[Z^1(Z^4)^\dagger]^J$. Since $Z^1(Z^4)^\dagger = e^{i(\varphi_2+\varphi_3)}|Z^1||Z^4|$ this means that $\varphi_2 + \varphi_3$ should have nontrivial background. Then $J_1(Z^1(Z^4)^\dagger) = 0$, $J_2([Z^1(Z^4)^\dagger]^J) = J_3([Z^1(Z^4)^\dagger]^J) = J$. To guarantee that the vacuum contains no other fields it is necessary to require that $\varphi_2 - \varphi_3$ and $\varphi_1$ have trivial background.[9] In terms of the original coordinates $\tau_{1,2,3}$ this translates into

$$\tau_2 = 0\,, \qquad \tau_1 = \tau_3\,, \tag{2.14}$$

which may be realized if the coordinates $\zeta_i$ in (2.3) take the background values

$$\bar\zeta_1 = \frac{\pi}{4}, \qquad \bar\zeta_2 = \frac{\pi}{2}, \qquad \bar\zeta_3 = \frac{\pi}{2}\,. \tag{2.15}$$

Then the relevant part of the full 10-d metric becomes

$$ds^2 = R^2_{\mathrm{AdS}}\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2) + d(\varphi_2+\varphi_3)^2\right]\,. \tag{2.16}$$

The values of the remaining coordinates on the solution of [28] here are ($\boldsymbol{\sigma} = (\tau, \sigma)$)

$$\bar t = \kappa\tau = \hat{\mathrm{n}}\cdot\boldsymbol{\sigma}\,, \qquad \bar\rho = \rho_*\,, \qquad \bar\theta = \frac{\pi}{2}\,, \qquad \bar\phi = \mathrm{w}\tau + k\sigma = \tilde{\mathrm{n}}\cdot\boldsymbol{\sigma}\,, \tag{2.17}$$

$$\bar\varphi_1 = 0\,, \qquad\qquad \bar\varphi_2 = \bar\varphi_3 = \frac{1}{2}(\omega\tau + m\sigma) = \frac{1}{2}\mathrm{m}\cdot\boldsymbol{\sigma}\,, \tag{2.18}$$

[9] Then in terms of homogeneous coordinates $Z^A$, the phases of only $Z^1$ and $Z^4$ will be nonvanishing which is consistent with having a bilinear combination $(Y^1 Y_4^\dagger)$ in the spin chain vacuum [2].

where the constant vectors are

$$\hat{\mathrm{n}} = (\kappa, 0)\,, \qquad \tilde{\mathrm{n}} = (\mathrm{w}, k)\,, \qquad \mathrm{m} = (\omega, m)\ . \tag{2.19}$$

Here $k$ and also $m$ are arbitrary integers ($\sigma$-coordinate is $2\pi$ periodic). Indeed, as one can show by considering the flat space limit of the metric (2.3), the combinations of angles $\tau_3 = \varphi_2 + \varphi_1$, $\tau_2 + \tau_3 = \varphi_3 + \varphi_1$ and $\tau_1 + \tau_2 + \tau_3 = \varphi_2 + \varphi_3$ should have $2\pi$ periodicity, while each $\varphi_i$ is $\pi$-periodic.

Written in terms of $\varphi \equiv \varphi_2 + \varphi_3$ with $\bar{\varphi} = 2\bar{\varphi}_2 = \mathrm{m} \cdot \boldsymbol{\sigma}$ this solution becomes the same as in $AdS_5 \times S^5$:[10] in particular, the relations between the parameters following from the equations of motion and the Virasoro constraints are the same as those in the string theory in $AdS_5 \times S^5$ case (cf. [28]):[11]

$$\begin{aligned} &\mathrm{w}^2 - (\kappa^2 + k^2) = 0\,, \qquad \mathrm{r}_1^2 \mathrm{w} k + \omega m = 0 \\ &- \mathrm{r}_0^2 \kappa^2 + \mathrm{r}_1^2(\mathrm{w}^2 + k^2) + \omega^2 + m^2 = 0\ , \end{aligned} \tag{2.20}$$

$$\mathrm{r}_0 \equiv \cosh \rho_*\,, \qquad\qquad \mathrm{r}_1 \equiv \sinh \rho_*\ . \tag{2.21}$$

From these constraints one may find, e.g. the expression of $(\kappa, \mathrm{r}_1^2, \mathrm{w})$ in terms of $(m, k, \omega)$. The explicit relations look rather complicated and not very enlightening; below we will only need their series expansion in a certain limit.

The charge densities are given by

$$\mathcal{E} = \int_0^{2\pi} \frac{d\sigma}{2\pi} r_0^2 \kappa = \mathrm{r}_0^2 \kappa, \quad \mathcal{S} = \int_0^{2\pi} \frac{d\sigma}{2\pi} r_1^2 \mathrm{w} = \mathrm{r}_1^2 \mathrm{w}, \qquad \mathcal{J}_2 = \mathcal{J}_3 = \int_0^{2\pi} \frac{d\sigma}{2\pi} \omega = \omega, \tag{2.22}$$

so that the classical energy, spin and the charges under the second and third Cartan generators of SO(6) are

$$E_0 = \sqrt{\lambda}\, r_0^2 \kappa\,, \qquad S = \sqrt{\lambda}\, \mathrm{r}_1^2 \mathrm{w}\,, \qquad J \equiv J_2 = J_3 = \sqrt{\lambda}\, \omega\,, \tag{2.23}$$

while the Virasoro constraint in (2.20) implies that

$$kS + Jm = 0\ . \tag{2.24}$$

As already mentioned these are exactly the same as the $AdS_5 \times S^5$ case.

Similarly to the $AdS_5 \times S^5$ case, a (technically) useful limit is that of large spin $\mathcal{S}$ and large angular momentum $\mathcal{J}$ with their ratio $u$ (and also $k$) held fixed, i.e.

$$\mathcal{S},\ \mathcal{J} \to \infty\,, \qquad u = -\frac{m}{k} = \frac{\mathcal{S}}{\mathcal{J}} = \frac{S}{J} = \text{fixed}\ . \tag{2.25}$$

---

[10] Let us mention that the definition of R-charges used here is different from the one used in [18]; there the charge $J$ was given by the momentum conjugate to the field $\varphi$ and thus is twice as large as the R-charges used here.

[11] These relations imply certain useful identities between the seven parameters entering the solution; one of them, which will be useful later in the calculation of the fermionic characteristic frequencies is [28]:

$$\frac{\mathrm{r}_1(k\omega - \mathrm{w}m)}{\sqrt{m^2 + \mathrm{r}_1^2 k^2}} = \frac{\omega}{k \mathrm{r}_1} \sqrt{m^2 + \mathrm{r}_1^2 k^2}\ .$$

In this limit it is possible to solve perturbatively the constraints (2.20)

$$
\begin{aligned}
\kappa &= \omega + \frac{k^2}{2\omega^2}u(2+u) - \frac{k^4}{8\omega^3}u(4+12u+8u^2+u^3) + \mathcal{O}\left(\frac{1}{\omega^5}\right), \\
\mathrm{r}_1^2 &= u - \frac{k^2}{2\omega^2}u(1+u)^2 + \frac{k^4}{8\omega^4}u(1+u)^2(3+10u+3u^2) + \mathcal{O}\left(\frac{1}{\omega^6}\right), \\
\mathrm{w} &= \omega - \frac{k^2}{2\omega}(1+u)^2 - \frac{k^4}{8\omega^3}(1+u)^2(1+6u+u^2) + \mathcal{O}\left(\frac{1}{\omega^5}\right).
\end{aligned} \tag{2.26}
$$

Using these expressions, the expansion of the classical energy at large $\mathcal{J}$ and thus large angular momentum $J = \sqrt{\bar\lambda}\mathcal{J} = \sqrt{\bar\lambda}\,\omega$ is given by

$$
\begin{aligned}
E_0 &= S + J + \frac{\bar\lambda}{2J}k^2 u(1+u) - \frac{\bar\lambda^2}{8J^3}k^4 u(1+u)(1+3u+u^2) \\
&\quad + \frac{\bar\lambda^3}{16J^5}k^6 u(1+u)(1+7u+13u^2+7u^3+u^4) + \mathcal{O}\left(\frac{1}{J^7}\right).
\end{aligned} \tag{2.27}
$$

This result is essentially the same as in the $\mathrm{AdS}_5\times\mathrm{S}^5$ case (1.5) provided one identifies the two tensions, i.e. $\sqrt{\lambda_{\mathrm{AdS}_5}} \to \sqrt{\bar\lambda}$.

A formally alternative prescription that also relates the $\mathrm{AdS}_5\times\mathrm{S}^5$ and $\mathrm{AdS}_4\times\mathbb{CP}^3$ results for the classical string energy, is (i) to replace $\sqrt{\lambda_{\mathrm{AdS}_5}} \to 2\sqrt{\bar\lambda}$ in (1.5), and, simultaneously, (ii) to replace $E$, $S$ and $J$ in $\mathrm{AdS}_5\times\mathrm{S}^5$ result by $2E$, $2S$ and $2J$ (i.e. $S \to 2S$, $J \to 2J$ and add an extra overall 1/2 factor in the energy). At the classical level this is obviously equivalent to no rescaling at all: changing the string tension by 2 is compensated by rescaling of charges by 2 so that classical parameters remain the same.

As we shall see, it is a generalization (with $2\sqrt{\bar\lambda} \to 2\bar h(\bar\lambda)$) of the second prescription that will actually extend to the quantum level. This should not be too surprising since the two quantum string theories appear to be quite different.

It is an analog of this generalized second prescription that was proposed, from the Bethe Ansatz perspective, in [10] as a relation between the universal scaling functions (or leading terms in the folded string energies) in $\mathrm{AdS/CFT}_4$ and $\mathrm{AdS/CFT}_3$ cases. As we shall demonstrate below, quite remarkably, this prescription applies also to the non-trivial quantum circular string case as well as to the generalized folded string case with non-zero orbital momentum $J$ discussed in [18, 19].

## 3. The spectrum of quadratic fluctuations

### 3.1 Bosons

It is not hard to expand the string action (2.8) around the solution (2.18). This is, however, largely unnecessary since, using the close connection to the circular string solution in $\mathrm{AdS}_5\times\mathrm{S}^5$, we can quickly write down the characteristic frequencies for the bosonic fluctuations. The six fluctuations from the $\mathbb{CP}^3$ split into one massless, four "light" degrees of freedom

$$
p_0 = \sqrt{p_1^2 + \frac{1}{4}(\omega^2 - m^2)}\,, \tag{3.1}
$$

and one "heavy" fluctuation

$$p_0 = \sqrt{p_1^2 + (\omega^2 - m^2)}\ . \tag{3.2}$$

From the AdS space one finds one massless degree of freedom, one massive one

$$p_0 = \sqrt{p_1^2 + \kappa^2}\,, \tag{3.3}$$

and two fluctuations whose dispersion relation is given by the roots of the quartic equation

$$(p_0^2 - p_1^2)^2 + 4r_1^2\kappa^2 p_0^2 - 4\left(1 + r_1^2\right)\left(\sqrt{\kappa^2 + k^2}\ p_0 - kp_1\right)^2 = 0\ . \tag{3.4}$$

As in $AdS_5 \times S^5$ [28], the explicit solution to this equation looks complicated, but may be constructed perturbatively in the limit (2.25). Furthermore, one can determine the appropriate signs with which these modes contribute to the energy correction in a similar fashion to [28] by considering the behavior of the frequencies at large $\omega$.

### 3.2 Fermions

Since the solution has non-zero angular momentum along $\mathbb{CP}^3$, the spectrum of fermionic fluctuations could be constructed by starting with the coset superstring action of [4, 5]. This is, however, not necessary here; instead, we will use the standard form of the quadratic part of the $\kappa$-symmetric Green-Schwarz action

$$L_{2F} = i(\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I \not{e}_a D_b^{JK}\theta^K\ . \tag{3.5}$$

Here $s^{IJ} = \mathrm{diag}(1, -1)$ and $e_a^A = \partial_a X^M E_M^A$, where $X$ denote generic coordinates and $E_M^A$ is the vielbein. In the string frame the type IIA covariant derivative is (see e.g. [38] for a choice of field variables with nice transformation properties under T-duality)

$$\begin{aligned} D_a^{JK} = &\left(\partial_a + \frac{1}{4}\partial_a X^M \omega_M{}^{AB}\Gamma_{AB}\right)\delta^{JK} - \frac{1}{8}\partial_a X^M E_M^A H_{ABC}\Gamma^{BC}(\sigma_3)^{JK} \\ &+\frac{1}{8}e^{\phi}\left[F_{(0)}(\sigma_1)^{JK} + \not{F}_{(2)}(i\sigma_2)^{JK} + \not{F}_{(4)}(\sigma_1)^{JK}\right]\not{e}_a \end{aligned} \tag{3.6}$$

The spin connection components in the AdS directions are:

$$\begin{aligned} \omega^{01} &= -\omega^{10} = \sinh\rho\ dt\,, & \omega^{21} &= -\omega^{12} = \cosh\rho\ d\theta\,, \\ \omega^{31} &= -\omega^{13} = \cosh\rho\ \sin\theta\ d\phi\,, & \omega^{32} &= -\omega^{23} = \cos\theta\ d\phi\ . \end{aligned} \tag{3.7}$$

The spin connection components along $\mathbb{CP}^3$ are more complicated but, due to our choice of coordinates, they will not be needed in this leading-order calculation.

To find the fermionic spectrum we evaluate the fermionic action (3.5) on the background solution (2.18) and then impose a gauge-fixing condition which is adapted to the resulting kinetic operator: one needs to make sure that the resulting operator is invertible.

The features of the resulting kinetic operator may be exposed through a series of constant field redefinitions which map the background vielbein to a scalar multiple of a single

Dirac matrix. Then, after combining the two type IIA fermions of opposite chirality into a single unconstrained 32-component spinor $\psi$ and also using the symmetry properties of the ten-dimensional Dirac matrices, the fermionic kinetic operator (3.5) becomes manifestly proportional to the projector

$$\mathcal{P}_+ = \frac{1}{2}\left(1 + \Gamma_0\Gamma_3\Gamma_{-1}\right) \quad . \tag{3.8}$$

The natural $\kappa$-symmetry gauge then is

$$\mathcal{P}_+\psi = \psi \quad . \tag{3.9}$$

We relegate the details of this calculation, as well as the construction of the eigenvalues of the resulting quadratic operator, to appendix A and record here only the conclusions. The spectrum contains four different frequencies, each being doubly-degenerate. Two such pairs have frequencies

$$(p_0)_{\pm 12} = \pm\frac{\mathrm{r}_0^2 k\kappa m}{2(m^2+\mathrm{r}_1^2k^2)} + \sqrt{(p_1 \pm b)^2 + (\omega^2 + k^2\mathrm{r}_1^2)}\,, \qquad b = -\frac{\kappa m}{\mathrm{w}}\frac{\mathrm{w}^2-\omega^2}{2(m^2+\mathrm{r}_1^2k^2)} \tag{3.10}$$

while the frequencies of the other two pairs are solutions of the equation

$$(p_0^2 - p_1^2)^2 + \mathrm{r}_1^2\kappa^2 p_0^2 - \left(1+\mathrm{r}_1^2\right)\left(\sqrt{\kappa^2+k^2}\; p_0 - kp_1\right)^2 = 0 \quad . \tag{3.11}$$

The latter equation may be mapped to a similar one in the bosonic case (3.4) by replacing $k$ and $\kappa$ with $2k$ and $2\kappa$ (or equivalently by replacing $p_0$ and $p_1$ with $\frac{1}{2}p_0$, $\frac{1}{2}p_1$). The constant shifts of several of the fermionic frequencies are similar to those found in the case of the folded string and, in fact, even for the short and fast BMN string. They may be removed (at least at the level of the quadratic action) by a further time-dependent redefinition of the fermions. We will not, however, do this here: as is easily seen, they simply cancel among themselves when we consider the sum over all frequencies and so these constant shifts do not contribute to the one-loop correction to the energy.

Let us note that the superconformal algebra supercharges — and thus the Green-Schwarz fermions — transform in the $\mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}$ of the SU(4) R-symmetry group. In the presence of the rotating string background the SU(4) $\simeq$ SO(6) breaks to SO(4) = SU(2) $\times$ SU(2). This breaking pattern and the spectrum listed above are consistent if we associate the degenerate fermion pairs with the self-dual and anti-self-dual representations of SO(4) or pairs of singlets related by charge conjugation. One may test this in the BMN limit: the fermion spectrum splits in two sets of four modes of equal masses; since the R-symmetry group is SO(4) and two modes are R-symmetry singlets, it follows that in this limit the spectrum decomposes as $\mathbf{4}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}$.

## 4. One-loop correction to the string energy

The expression for the correction to the string energy can be found by summing the frequencies over all flavours and mode numbers

$$E_1 = E_1^{(0)} + \bar{E}_1\,, \tag{4.1}$$

where $E_1^{(0)}$ is the contribution of the zero modes and $\bar{E}_1$ involves the infinite sum over all non-zero modes (we set $p_1 \equiv n = 0, \pm 1, \ldots$)

$$E_1^{(0)} = \frac{1}{2\kappa} e(0)\,, \qquad\qquad \bar{E}_1 = -\frac{1}{2\kappa}\, e(0) + \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} e(n) \quad . \tag{4.2}$$

The summand $e(n)$ is simply the weighted sum of the bosonic and fermionic frequencies found in the previous section:[12]

$$\begin{aligned} e(n) \,=\, & \frac{1}{2}\left[(p_0)_1^B + (p_0)_2^B - (p_0)_3^B - (p_0)_4^B\right] + \sqrt{n^2+\kappa^2} + \sqrt{n^2 + (\omega^2 - k^2 u^2)} \\ & +4\sqrt{n^2 + \frac{1}{4}(\omega^2 - k^2 u^2)} - 2\sqrt{(n-b)^2 + (\omega^2 + k^2 r_1^2)} - 2\sqrt{(n+b)^2 + (\omega^2 + k^2 r_1^2)} \\ & - \ \left[(p_0)_1^F + (p_0)_2^F - (p_0)_3^F - (p_0)_4^F\right]\,, \end{aligned} \tag{4.3}$$

where $(p_0)_i^B$ and $(p_0)_i^F$ stand for solutions of the quartic equations (3.4) and (3.11).

Before proceeding, let us make few comments about the derivation of (4.2). This expression for the one-loop correction to the energy of the rotating string may be arrived at in several different ways. One can use the expression for the string energy in conformal gauge in terms of the fluctuation fields derived in appendix A of [22] (in that paper this was in the context of the folded spinning string):

$$E_1 = \frac{1}{\kappa}\langle\Psi|H_2|\Psi\rangle\,, \tag{4.4}$$

with $H_2 = \int \frac{d\sigma}{2\pi}\mathcal{H}_2(\tilde{t}, \tilde{\phi}, \ldots)$ being the quadratic worldsheet Hamiltonian corresponding the fluctuation action at this order.

As here we are interested only in the one-loop result, this Hamiltonian approach is sufficient and practical. However, certain conceptual issues are perhaps clearer in the path-integral approach. In the $\mathrm{AdS}_5\times\mathrm{S}^5$ theory where two-loop calculations have been performed it has been found useful to extract the correction to the string energy from the sigma model partition function. It was argued in [39 − 41] and in greater detail in [42] that for a homogeneous string solution like the one we consider here $E_1$ may be defined as the one-loop effective action divided by the two-dimensional time interval. Moreover, the (quantum-corrected) charges of the background solution are also determined by the one-loop effective action and, similarly to the energy, are finite at this order.

In a path integral approach the frequency sum appearing in the Hamiltonian formalism arises in the process of evaluating the logarithm of the regularized determinants of the operators of quadratic fluctuations around the classical solution. Though the final result is finite, as may be seen by inspecting the large mode-number behavior of frequencies, each determinant taken separately is divergent. As a consequence of the path integral approach, all determinants are regularized in the same way.

An advantage of this approach is that field/fluctuation redefinitions are systematically accounted for the path integral evaluation of the effective action or free energy. Such

[12]The contribution of two massless degrees of freedom cancels against the contribution of the diffeomorphism ghosts.

redefinitions (e.g. the ones equivalent to changing the original coset representative) may effectively lead to constant shifts in the frequencies of various modes. While *a priori* such shifts may lead to (power-like) divergences in the free energy, their contribution is, in fact, canceled exactly by the Jacobian due to the change in the measure of the path integral and thus it does not change the expression for the energy shift.

### 4.1 Large spin expansion of one-loop correction to the energy

While computing exactly the sum over frequencies in (4.3) is difficult, there is one particular region of the parameter space that is amenable to explicit evaluation: this is the scaling region (2.25), i.e. that of large angular momentum $\mathcal{J}$ or large $\omega$, and large spin $\mathcal{S}$ with the ratio $u = -\frac{m}{k} = \frac{\mathcal{S}}{\mathcal{J}}$ (and also $k$) fixed. As discussed in [29] in the context of string theory in $AdS_5 \times S^5$, in this limit the sum over modes receives contributions from two distinct regions:

(I) $n \ll \omega$: here the sum remains discrete

(II) $n/\omega = x$ =fixed: here the sum may be replaced by an integral over $x$

These two regimes are compatible; while each regime exhibits singularities, it is possible to see that the singular part of one regime is captured by the regular part of the other. Thus, the complete result as an expansion in $1/\omega$ is the sum of the regular parts of the two regimes,

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty} e(n) = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty} e^{\rm sum}_{\rm reg}(n) + \frac{\omega}{2\kappa}\int_{-\infty}^{\infty} dx\, e^{\rm int}_{\rm reg}(x) = E_1^{(0)} + \bar{E}_1^{\rm even} + \bar{E}_1^{\rm odd}\,. \quad (4.5)$$

It is an interesting question whether the zero-mode part $E_1^{(0)}$ should be kept separate or whether it effectively belongs to $\bar{E}_1^{\rm even}$ or $\bar{E}_1^{\rm odd}$. As we will argue shortly, it belongs to $\bar{E}_1^{\rm odd}$ part, i.e. $E_1^{\rm odd} = E_1^{(0)} + \bar{E}_1^{\rm odd}$.

It is not difficult to solve perturbatively the quartic equations (3.4) and (3.11) and find the most non-trivial bosonic and the fermionic frequencies at large $\omega$:

$$\begin{aligned}
(p_0)^B_{1,3} &= \frac{p_1}{2\omega}\left[2k(1+u) \pm \sqrt{p_1^2 + 4k^2u(1+u)}\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right) \\
(p_0)^B_{2,4} &= \pm 2\omega \pm \frac{1}{2\omega}\left[p_1^2 \mp 2kp_1(1+u) + 2k^2(1+u(3+u))\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right)
\end{aligned} \quad (4.6)$$

$$\begin{aligned}
(p_0)^F_{1,3} &= \frac{p_1}{\omega}\left[k(1+u) \pm \sqrt{p_1^2 + k^2u(1+u)}\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right) \\
(p_0)^F_{2,4} &= \pm\omega \pm \frac{1}{\omega}\left[p_1^2 \mp kp_1(1+u) + \tfrac{k^2}{2}(1+u(3+u))\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right)\,.
\end{aligned} \quad (4.7)$$

Then, the summand $e(n)$ in equation (4.5) as a function of the momentum mode number $n$ takes the form

$$\begin{aligned}
e^{\rm sum}(n) = \frac{1}{2\omega}\Bigg[ & n\left(3n - 4\sqrt{n^2 + k^2u(1+u)} + \sqrt{n^2 + 4k^2u(1+u)}\right) \\
& - k^2(1+u)(1+3u)\Bigg] + \mathcal{O}\left(\frac{1}{\omega^3}\right)\,.
\end{aligned} \quad (4.8)$$

The sum over $n$ is singular with a divergence arising from the constant term which also gives rise to the zero mode piece of the energy. This occurs at one order lower in the $1/\omega$ expansion than for the rotating string in $\mathrm{AdS}_5\times\mathrm{S}^5$. Continuing to higher orders in $1/\omega$ one finds the same splitting into regular and singular parts, $e^{\mathrm{sum}} = e^{\mathrm{sum}}_{\mathrm{reg}} + e^{\mathrm{sum}}_{\mathrm{sing}}$.

The contribution of large mode numbers, $n = \omega x$ with fixed $x$, may be accounted for by replacing the sum over $n$ with an integral over $x$. To leading order in the large-$\omega$ expansion the summand becomes

$$e^{\mathrm{int}}(x) = \frac{k^2(1+u)}{2\omega}\left[\frac{1+u(3+2x^2)}{(1+x^2)^{3/2}} - 2\frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}}\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right) \quad , \tag{4.9}$$

where one can see that $e^{\mathrm{int}}_{\mathrm{reg}}(n/\omega) = e^{\mathrm{sum}}_{\mathrm{sing}}(n)$. It is interesting to note that, while capturing the singular part of the sum over $e^{\mathrm{sum}}(n)$ in equation (4.8), it also correctly captures the zero-mode contribution:

$$e^{\mathrm{int}}_{\mathrm{reg}}(0) = e^{\mathrm{sum}}_{\mathrm{sing}}(0) \quad . \tag{4.10}$$

Thus, we may simply combine the zero-mode contribution $e^{\mathrm{sum}}(0)$ together with the contribution of large mode numbers. It is possible to extend the comparison above to $e^{\mathrm{sum}}_{\mathrm{reg}}$ and $e^{\mathrm{int}}_{\mathrm{sing}}$ to higher orders in the $1/\omega$ expansion, which we carry out explicitly in appendix B and show that indeed $e^{\mathrm{int}}_{\mathrm{sing}}(x) = e^{\mathrm{sum}}_{\mathrm{reg}}(\omega x)$ to all orders we checked. This is exactly analogous to the recombination which takes place in $\mathrm{AdS}_5\times\mathrm{S}^5$ case.

Since the sum is absolutely convergent, the coefficients in the $1/J$ expansion of the discrete part of the correction to the energy may be computed as formal power series in $k$

$$\begin{aligned}
\bar{E}_1^{\mathrm{even}} &= \frac{1}{\kappa}\sum_{n=1}^{\infty} e^{\mathrm{sum}}_{\mathrm{reg}}(n) \\
&= -\frac{\bar{\lambda}k^4(1+u)^2u^2}{2^3J^2}\left(6\zeta(2) - 15k^2u(1+u)\zeta(4) + \frac{315}{8}k^4u^2(1+u)^2\zeta(6) + \dots\right) \\
&\quad + \frac{\bar{\lambda}^2k^6(1+u)^2u^2}{2^6J^4}\Big(24(1+2u-u^2)\zeta(2) + 15k^2u^2(1+u)(5+13u)\zeta(4) \\
&\qquad\qquad - \frac{63}{2}k^4u^2(1+u)^2(5+22u+27u^2)\zeta(6) + \dots\Big) \\
&\quad - \frac{\bar{\lambda}^3k^8(1+u)^2u^2}{2^9J^6}\Big(48(3+18u+26u^2+10u^3+7u^4)\zeta(2) \\
&\qquad\qquad - 60k^2u^2(1+u)(7+27u+53u^2+49u^3)\zeta(4) \\
&\qquad\qquad + 63k^4u^2(1+u)^2(5-20u-183u^2-382u^3-264u^4)\zeta(6) + \dots\Big) \\
&\quad + \mathcal{O}\left(\frac{1}{J^8}\right).
\end{aligned} \tag{4.11}$$

Using the expression for $e^{\mathrm{int}}$ listed in appendix B to go to higher orders in the $1/J$ expansion, the continuum contribution to the energy reads:

$$\begin{aligned}
E_1^{\mathrm{odd}} &= \frac{\omega}{2\kappa}\int_{-\infty}^{\infty} dx\; e^{\mathrm{int}}_{\mathrm{reg}}(x) \\
&= -\frac{\bar{\lambda}^{1/2}k^2}{J}\ \ln 2\ u(1+u) + \frac{\bar{\lambda}^{3/2}k^4}{2J^3}\ \ln 2\ u(1+u)(1+3u+u^2)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\bar{\lambda}^{5/2}k^6}{8J^5}u(1+u)\left[3(1+7u+13u^2+7u^3+u^4)\ln 2 - \frac{4}{3}u^2(1+u)^2\right] \\
&+ \mathcal{O}\left(\frac{1}{J^7}\right) \ .
\end{aligned} \tag{4.12}
$$

While anticipated by the existence of divergences in the discrete contribution to leading nontrivial order, the appearance of such low odd powers of $1/J$ with "non-analytic" factors of $\bar{\lambda}$ may at first look surprising. It is possible to test numerically that the expressions above are indeed accurate (see appendix D).

### 4.2 Relation to the energy of the circular rotating string in $\mathbf{AdS_5\times S^5}$

Motivated by the similarity of the classical solution we started with to the one in $AdS_5\times S^5$ and also by the fact that the proposed Bethe Ansatz of [10] has a structure similar to that of the $AdS/CFT_4$ case let us now compare the result for $E_1$ to the corresponding expression in $AdS_5\times S^5$ string theory.[13]

Collecting the results of the previous section, the total one-loop corrected energy of the circular rotating string in $AdS_4 \times \mathbb{CP}^3$ is

$$
E = E_0 + E_1 = E_0 + \bar{E}_1^{\rm even} + E_1^{\rm odd} \ , \tag{4.13}
$$

with $E_0$, $\bar{E}_1^{\rm even}$ and $E_1^{\rm odd}$ are given by equations (2.27), (4.11) and (4.12), respectively.

From equation (2.27) we note that the classical energy of the circular rotating string in the scaling (large spin (2.25)) limit is a series in inverse odd powers of the angular momentum $J$. One may then contemplate that $E_0$ and $E_1^{\rm odd}$ might naturally combine together. This is indeed the case as we may write their sum as

$$
\begin{aligned}
E_0 + E_1^{\rm odd} &= S + J + \frac{\bar{h}^2(\bar{\lambda})k^2}{2J}u(1+u) - \frac{\bar{h}^4(\bar{\lambda})k^4}{8J^3}u(1+u)(1+3u+u^2) \\
&+ \frac{\bar{h}^6(\bar{\lambda})k^6}{16J^5}u(1+u)(1+7u+13u^2+7u^3+u^4) \\
&+ \frac{\bar{h}^5(\bar{\lambda})k^6}{6J^5}u^3(1+u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right) \ .
\end{aligned} \tag{4.14}
$$

Here we introduced the function

$$
\bar{h}(\bar{\lambda}) = \sqrt{\bar{\lambda}} - \ln 2 + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right) \ . \tag{4.15}
$$

The powers of $\bar{h}(\bar{\lambda})$ in equation (4.14) are understood to be truncated to the two leading terms in $1/\sqrt{\bar{\lambda}}$ expansion except for the last term, proportional to $\bar{h}^5(\bar{\lambda})/J^5$ which is understood to be truncated to the leading term. This is indeed the correct prescription, as the $\bar{\lambda}$ dependence of the next-to-leading term identifies it as a two-loop correction.

[13] Note that the fluctuation frequencies in the two theories are not directly related (corresponding to the two superficially quite different 2d quantum theories), but their respective sums representing $E_1$'s happen to be similar as we describe below.

For comparison, let us recall the analogous part of the expression for the one-loop energy of the same circular rotating string in the $AdS_5 \times S^5$ case [28, 29] (see (1.5), (1.7) and the discussion in the introduction)

$$
\begin{aligned}
(E_0 + E_1^{\rm odd})_{AdS_5\times S^5} &= J + S + \frac{\lambda k^2}{2J} u(1+u) - \frac{\lambda^2 k^4}{8J^3} u(1+u)(1+3u+u^2) \\
&+ \frac{\lambda^3 k^6}{16 J^5} u(1+u)(1+7u+13u^2+7u^3+u^4) \\
&+ \frac{\lambda^{5/2} k^6}{3J^5} u^3(1+u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right) . \qquad (4.16)
\end{aligned}
$$

We then observe that the $AdS_4 \times \mathbb{CP}^3$ expression (4.14) can be obtained from the $AdS_5 \times S^5$ one (4.16) by the prescription mentioned earlier at the end of section 2:

$$
E^{\rm odd}_{AdS_4\times\mathbb{CP}^3}(S, J, k;\ \sqrt{\bar\lambda}\ ) \ = \ \frac{1}{2} E^{\rm odd}_{AdS_5\times S^5}(2S, 2J, k;\ 2\bar h(\bar\lambda)\ ) , \qquad (4.17)
$$

with the function $\bar h(\bar\lambda)$ given by (4.15). It is important to note that the replacement $\sqrt{\lambda} \mapsto 2\bar h(\bar\lambda) = 2\sqrt{\bar\lambda} - 2\log 2 + \cdots$ is to be implemented *after* the energy is expressed in terms of the conserved charges $(S, J)$ (which are also the parameters on the gauge theory side).

Notice that what selects between the simple replacement $\lambda \to h(\bar\lambda)$ with no change to the charges and the prescription (4.17) (which were equivalent at the classical level) is the matching of the last "quantum phase" term in (4.14) and in (4.16).

As was already mentioned in the introduction (see eq. (1.6) and discussion below it) the one-loop renormalization of the leading "analytic" terms in the $AdS_4 \times \mathbb{CP}^3$ string energy implies that the BMN spectral relation here gets a one-loop renormalization, i.e. the function (4.15) should be identified with the function $\bar h(\bar\lambda)$ entering the magnon dispersion relation (cf. (1.1), (1.4))

$$
\epsilon(p) = \frac{1}{2}\sqrt{1 + \frac{4}{\pi^2}\bar h^2(\bar\lambda)\sin^2\frac{p}{2}} \ . \qquad (4.18)
$$

Notice that (4.18) is related to the familiar $AdS_5 \times S^5$ expression $\epsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$ by the same prescription (4.17) (cf. also (1.6)).

One useful way to understand the relation between the renormalization of the magnon dispersion relation and the above function $\bar h(\bar\lambda)$ is to consider the analog of the effective Landau-Lifshitz (LL) model description of the large $J$ limit as was done in the $AdS_5 \times S^5$ case in [43, 44]. The LL model may be viewed as an effective 2d field theory which describes the "fast string" or large $J$ expansion on both the string and spin chain side and thus interpolates between the two descriptions. Considering for illustrative purposes the analog of the SU(2) sector action parameterized by a unit 3-vector $\vec n$ the corresponding LL action is $S = J\int dt \int \frac{d\sigma}{2\pi}\, L$ where [44]

$$
\begin{aligned}
L = C(\vec n)\cdot\partial_0 \vec n - \vec n\left[\sqrt{1 - \frac{4\bar h^2(\bar\lambda)}{J^2}\partial_1^2} - 1\right]\vec n - \frac{a(\bar\lambda)}{J^4}(\partial\vec n)^4 \\
-\frac{1}{J^6}\left[b_1(\bar\lambda)(\partial_1\vec n)^2(\partial_1^2\vec n)^2 + b_2(\bar\lambda)(\partial_1\vec n\cdot\partial_1^2\vec n)^2 + b_3(\bar\lambda)(\partial_1\vec n)^6\right] + \dots \ . \qquad (4.19)
\end{aligned}
$$

In general, $\bar{h}$, $a$, $b_i$, etc., are interpolating functions parameterizing this low-energy effective action. In the $AdS_5\times S^5$ case the first three functions are simple: $2\bar{h} \to \sqrt{\lambda}$, $a \to \frac{3}{128}\lambda^2$, $b_1 \to -\frac{7}{4}\lambda^3$. The functions $b_2, b_3$ are non-trivial, having the same $r\lambda^3$ behaviour at weak and strong coupling but with different numerical coefficients (reflecting the "3-loop disagreement").

All of these functions are expected to be nontrivial in the present $AdS_4 \times \mathbb{CP}^3$ case.[14] By comparing the energy of the rotating string as described by the LL action with the explicit string theory computations one observes that the $u \to 0$ limit of (4.14) should be essentially captured by the leading quadratic in $\vec{n}$ terms in (4.19), thus identifying $\bar{h}$ in (4.15) with the function that governs the magnon dispersion relation (4.18).

Remarkably, the same prescription (4.17) also relates the folded string energies in $AdS_5\times S^5$ and $AdS_4 \times \mathbb{CP}^3$. Indeed, ignoring first the $J$-dependence, starting with the $AdS_5\times S^5$ one-loop result [22]

$$E_{_{AdS_5\times S^5}} = S + \frac{1}{\pi}(\sqrt{\lambda} - 3\log 2)\ln S + \cdots \,, \tag{4.20}$$

and making the replacements in (4.17) one finds (for $S \gg 1$)

$$E_{_{AdS_4\times \mathbb{CP}^3}} = S + \frac{1}{2\pi}[2\bar{h}^2(\bar{\lambda}) - 3\log 2]\ln S + \cdots \,. \tag{4.21}$$

Using the expression (4.15) for $\bar{h}$ found here we end up with

$$E_{_{AdS_4\times \mathbb{CP}^3}} = S + \frac{1}{\pi}(\sqrt{\bar{\lambda}} - \frac{5}{2}\log 2)\ln S + \cdots \,, \tag{4.22}$$

which is the expression found by direct string computation in [18–20]. Moreover, by including the dependence on $J$ (in the limit of large $\mathcal{S}$ with $\frac{\mathcal{J}}{\ln \mathcal{S}}$ fixed) one finds that the equations (4.17), (4.15) directly relate the $AdS_5\times S^5$ result of [39] to the one in the $AdS_4 \times \mathbb{CP}^3$ case as found in [18, 19]. This provides a nontrivial consistency check between currently available one-loop results in the $AdS_4 \times \mathbb{CP}^3$ superstring.

It should be noted, however, that the prescription (4.17) is so far rather heuristic (or empirical, on the string theory side) and need not a priori apply to the whole expression for the one-loop string correction.[15]

Returning to the circular string solution, the relation between the equation (4.14) and the corresponding result in $AdS_5\times S^5$ (4.16) via the equation (4.17) suggests to compare also the terms containing even powers of $1/J$ in (4.11) with the analogous terms in the

[14] In particular, due to the structure of perturbation theory in the $\mathcal{N} = 6$ CS theory the function $a(\bar{\lambda})$ should start at weak coupling with a 4-loop $\bar{\lambda}^4$ term.

[15] It does seem to apply to the "non-analytic" part of the one-loop correction, which comes from the "integral" term in the one-loop calculation and is not sensitive to the compactness of the worldsheet; this is also the only term that determines the leading one-loop shift in the folded string case. It is, in principle, possible that a different prescription is necessary to map the "analytic" part of the one-loop correction to the corresponding $AdS_5\times S^5$result.

$AdS_5 \times S^5$ case. The part of the $AdS_5 \times S^5$ one-loop energy which is proportional to the even inverse powers of the $S^5$ angular momentum is (cf. [28])[16] [17]

$$\begin{aligned}
(\bar{E}_1^{\rm even})_{\rm AdS_5\times S^5} &= \frac{1}{\kappa}\sum_{n=1}^{\infty} e^{\rm sum}_{\rm reg,\,AdS_5\times S^5}(n) \\
&= -\,\frac{\lambda k^4(1+u)^2u^2}{2^2J^2}\left(4\zeta(2) - 8k^2u(1+u)\zeta(4) + 20k^4u^2(1+u)^2\zeta(6) + \ldots\right) \\
&\quad + \frac{\lambda^2k^4(1+u)^2u^2}{2^5J^4}\Big(16k^2(1+2u-u^2)\zeta(2) + 8k^4u^2(1+u)(5+13u)\zeta(4) \\
&\qquad\qquad - 16k^6u^2(1+u)^2(5+22u+27u^2)\zeta(6) + \ldots\Big) \\
&\quad - \frac{\lambda^3k^4(1+u)^2u^2}{2^8J^6}\Big(32k^4(3+18u+26u^2+10u^3+7u^4)\zeta(2) \\
&\qquad - 32k^6u^2(1+u)(7+27u+53u^2+49u^3)\zeta(4) \\
&\qquad + 32k^8u^2(1+u)^2(5-20u-183u^2-382u^3-264u^4)\zeta(6) + \ldots\Big) \\
&\quad + \mathcal{O}\left(\frac{1}{J^8}\right) .
\end{aligned} \tag{4.23}$$

Comparing this with equation (4.11) we note that, while not exactly the same, the two expressions may be mapped into each other by again replacing $\sqrt{\lambda} \mapsto 2\bar{h}(\bar{\lambda})$, $S \to 2S$, $J \to 2J$ (i.e. $u \to u$) and $E \to 2E$ as in (4.17) but in addition also by replacing $\zeta(n)$ in the $AdS_5 \times S^5$ result (4.23) by

$$\zeta(n) \mapsto 2\Big(1 - \frac{1}{2^n}\Big)\zeta(n) \quad . \tag{4.24}$$

This modification of the $\zeta$-constants in the $AdS_5 \times S^5$ calculation may be formally interpreted as replacing the sum over even mode numbers $n$ in (4.11) by a sum over odd mode numbers,

$$\sum_n \omega_n = \sum_n \omega_{2n} + \sum_n \omega_{2n+1} \;\;\mapsto\;\; 2\sum_n \omega_{2n+1} \quad . \tag{4.25}$$

### 4.3 Comments on comparison to the Bethe Ansatz proposal

In a finite two-dimensional quantum field theory, loop corrections to the conserved charges (such as the target space energy) of classical solitons may be computed using the standard perturbative approach, either in the Hamiltonian or in the path integral setting. If this two-dimensional theory is dual, through gauge/string duality, to some planar gauge theory,

[16] In the $AdS_5 \times S^5$ case the zero-mode contribution to the energy is also accounted for by the contribution of large mode numbers. Note also that here we are assuming that one can interchange summation with expansion in $1/J$, but otherwise there is no regularization ambiguity (as would be present in the Landau-Lifshitz model approach) as we start with the full UV finite expression for the sum.

[17] Here we record only the regular contributions to the discrete sum from the $AdS_5 \times S^5$ case. The divergent contribution starts at order $\frac{1}{J^6}$ and corresponds to the non-analytic contribution coming from the dressing phase at order $\frac{1}{J^5}$ [29]. Keeping only the regular contributions is equivalent to evaluating the summation using the zeta-function regularization as was done in [31].

JHEP11(2008)069

then the target space energies obtained this way in an acceptable (in the sense defined in the introduction) regularization scheme should yield the strong coupling expansion of the anomalous dimensions of certain gauge theory operators.

If this two-dimensional theory is also integrable, then its semiclassical states can be described using the algebraic curve techniques [35], which also determines the fluctuation frequencies [14, 52, 15] near the solitonic solutions and thus, effectively, the 1-loop corrections to their charges. Furthermore, there may exist a set of (discrete) Bethe equations that should provide the exact description of quantum corrections to all loop orders. The results of the algebraic curve approach and the Bethe Ansatz approach should of course agree with the results found by the direct worldsheet computations, and this should be, in fact, a test of the validity of the algebraic curve and Bethe Ansatz approaches.

In the Bethe Ansatz approach one solves directly the algebraic ("discrete") Bethe equations and thus no choice of regularization is required. Such a choice is, however, required in the algebraic curve approach, where, similarly to the worldsheet calculation, one finds the frequencies of small fluctuations near a soliton from an algebraic curve and then uses the standard quantum-mechanical prescription to evaluate the one-loop correction by computing the sum of frequencies (weighted by $(-1)^F$ where $F$ is the fermion number). Since the all-order Bethe Ansatz construction is based on a "discretization" [33] of the classical (integral) Bethe equations [35] and since their solution requires no regularization, it follows that a special choice of regularization is required in the algebraic curve calculation to reproduce the results of the Bethe Ansatz calculation.

This is the case for strings in $AdS_5 \times S^5$, where the Bethe equations and the worldsheet calculation yield the same result which is matched by the algebraic curve calculation [52, 15] provided one chooses a natural regularization which accounts for certain constant shifts in the space-like momenta of fluctuations and essentially amounts to introducing different cutoffs for various partial frequency sums (cf. [15, 53, 36, 31]).

For string theory in $AdS_4 \times \mathbb{CP}^3$ the same three strategies are, in principle, also available. In particular, for the circular rotating string we have already the worldsheet results obtained in the previous subsection. One may also consider the implications of the algebraic curve approach [16, 47] and of the Bethe Ansatz equations proposed in [10] to describe the corresponding set of gauge theory operators with one spin and one R-charge.[18] Below we shall only briefly comment on the corresponding solution to the Bethe Ansatz equations and its comparison with the worldsheet approach.[19] The relevant Bethe Ansatz equations are given by:

$$\left(\frac{x_l^+}{x_l^-}\right)^{2J} = -\prod_{j\neq l=1}^{S} \frac{u_l - u_j + i}{u_l - u_j - i} \left(\frac{x_l^- - x_j^+}{x_l^+ - x_j^-}\right)^2 \sigma^2_{\rm BES}(u_l, u_j) \ , \tag{4.26}$$

[18]It is worth pointing out that this rank one sector, in fact, captures only *some* of the $sl(2)$ sector solutions, namely those with odd Bethe mode number. The other solutions mix with the other sectors, requiring the use of the complete set of nested Bethe equations.

[19]More details and comparison with the algebraic curve approach should appear in ref. [47].

where[20]

$$x^{\pm} + \frac{1}{x^{\pm}} = \frac{1}{h(\lambda)}\left(u \pm \frac{i}{2}\right) , \tag{4.27}$$

and $h(\lambda)$ is the interpolating function in the magnon dispersion relation (cf. (1.1), (1.4)). Here $\sigma_{\rm BES}$ is the same dressing phase as in the context [6], but with $\sqrt{\lambda}$ replaced by $4\pi h(\lambda)$ in the appropriate way [10]. We will consider a class of solutions of (4.26), with the one-cut solution of [46] particularly in mind, vis-à-vis those of the analogous $sl(2)$ Bethe equation in $\mathrm{AdS}_5\times\mathrm{S}^5$ to which it has a great degree of similarity. The total energy of the solutions is given by

$$E - J = 2ih(\lambda)\sum_{l=1}^{S}\left(\frac{1}{x_l^+} - \frac{1}{x_l^-}\right) , \tag{4.28}$$

or in terms of the magnon momenta

$$E - J = \sum_{l=1}^{S}\sqrt{1 + 16h^2(\lambda)\sin^2\frac{p_l}{2}} , \tag{4.29}$$

and the zero-momentum condition is

$$\left[\prod_{l=1}^{S}\left(\frac{x_l^+}{x_l^-}\right)\right]^2 = 1 \ . \tag{4.30}$$

The absence of the factor of 1/2 in the expression for the energy (4.29) and the square in (4.30) are due to the identification of the $u_4$ and $u_{\bar{4}}$ roots [10].

As was mentioned in [10], the only differences between the equations above and the analogous ones in $\mathrm{AdS}_5\times\mathrm{S}^5$ are the replacement of the square-root of the 't Hooft coupling $\lambda$ of $\mathcal{N} = 4$ SYM with $4\pi h(\lambda) = 2\bar{h}(\bar{\lambda})$, the different relation between the R-charge and the spin-chain length and the existence of an additional minus sign on the right-hand-side of (4.26). This additional sign is like a familiar "magnetic field" twist and the corresponding Bethe Ansatz equations are also analogous to those that appear in the $\beta$-deformed SYM theory [48, 49] for a special real value of the deformation parameter $\beta_d = \frac{1}{2J}$. Indeed, the $\beta$-deformed Bethe equations and the zero momentum condition are [50]

$$e^{-2i\pi\beta_d J}\left(\frac{x_l^+}{x_l^-}\right)^{2J} = \prod_{j\neq l=1}^{S}\frac{u_l - u_j + i}{u_l - u_j - i}\left(\frac{x_l^- - x_j^+}{x_l^+ - x_j^-}\right)^2 \sigma^2_{\rm BES}(u_l, u_j) \ . \tag{4.31}$$

This equation becomes the same as equation (4.26) upon choosing $\beta_d = \frac{1}{2J}$. [21]

---

[20] The charge $J$ used here in the Bethe equations is the same as the angular momentum $J$ used in our worldsheet calculation and the gauge theory R-charge $J$ which in turn is half the spin-chain length.

[21] The zero-momentum condition, $e^{-2i\pi\beta_d S}\prod_{l=1}^{S}\frac{x_l^+}{x_l^-} = 1$, is, however, different from eq. (4.30) by a factor of $(-1)^{S/J}$. It is the consequences of the latter equation which we will discuss here.

In the $\beta$-deformed context [49, 50] the only effect of the phase $\beta_d$ is to shift the integer number that appears in the logarithm of the Bethe equations by $\beta_d J$. This may be seen directly by taking the logarithm of equations (4.26) and (4.30):

$$2\pi i\left(\tilde{k}+\frac{1}{2}\right)+2J\ln\frac{x_l^+}{x_l^-} \;=\; \sum_{j\neq l=1}^{S}\ln\left[\frac{u_l-u_j+i}{u_l-u_j-i}\left(\frac{x_l^- - x_j^+}{x_l^+ - x_j^-}\right)^2\sigma^2_{\mathrm{BES}}(u_l,u_j)\right], \quad (4.32)$$

$$2\pi i\tilde{m}+2\sum_{l=1}^{S}\ln\frac{x_l^+}{x_l^-} \;=\; 0\,, \quad (4.33)$$

where $\tilde{k}\in\mathbb{Z}$ is the Bethe mode number and $\tilde{m}$ is an integer. Consistency of the equations (4.32) and (4.33) implies that

$$\tilde{m}J+\left(\tilde{k}+\frac{1}{2}\right)S=0 \quad . \quad (4.34)$$

Compared to the corresponding equations in the $\mathrm{AdS}_5\times\mathrm{S}^5$ case there are five changes:

(1) here $4\pi h(\lambda)$ is in place of $\sqrt{\lambda}$;

(2) here the spin chain length is $2J$ not $J$;

(3) here the energy of a solution is doubled (due to the double number of excitations);

(4) the BPS condition at vanishing coupling requires that spin $S$ be doubled;[22]

(5) here $\tilde{k}+\frac{1}{2}$ is in place of $\tilde{k}$ (due to the additional minus sign in the equation (4.26)).

The square in the equation (4.30) and together with the doubled number of excitations (point (3) above) imply that no change in $\tilde{m}$ is necessary. With these identifications (4.34) is formally the same as the usual constraint in the $\mathrm{AdS}_5\times\mathrm{S}^5$ case: $\tilde{m}(2J)+(\tilde{k}+\frac{1}{2})(2S)=0 \;\rightarrow\; mJ+kS=0$. In the case of the circular (rational) solution we are interested in, $m$ and $k$ are, respectively, the $\mathrm{S}^5$ and the $\mathrm{AdS}_5$ winding numbers.

The solution of these Bethe equations in the strong coupling limit, to the leading and subleading order, proceeds as in the $\mathrm{AdS}_5\times\mathrm{S}^5$ case [36, 31] (see [29, 32] for the inclusion of the one-loop corrections to the dressing phase). To obtain the solution of the Bethe equations (4.26) from that of the $\mathrm{AdS}_5\times\mathrm{S}^5$ Bethe equations with length $J$ and parameters $k, m, S$ with $mJ+kS=0$ one is then to make the following formal replacements as implied by the above discussion:

$$(1)\,\sqrt{\lambda}\rightarrow 4\pi h(\lambda);\quad (2)\,J\rightarrow 2J;\quad (3)\,S\rightarrow 2S;\quad (4)\,E\rightarrow 2E;\quad (5)\,k\rightarrow\tilde{k}+\frac{1}{2};\quad (6)\,m\rightarrow\tilde{m}.$$

Comparing now the solution of the above $\mathrm{AdS/CFT}_3$ Bethe equations to the classical $\mathrm{AdS}_4\times\mathbb{CP}^3$ string solution discussed in section 2 we are led to the following identification:

1. $\tilde{k}+\frac{1}{2} \;\rightarrow k$, where $k$ is the $\mathrm{AdS}_4$ winding number;

2. $\tilde{m} \;\rightarrow m$, where $m$ is the $\mathbb{CP}^3$ winding number.

[22] That is, items (3) and (4) imply that $E=S+J+\ldots \quad\rightarrow\quad 2E=2S+2J+\ldots$

Note that (4.34) becomes then equivalent to the Virasoro constraint in (2.24).

The above replacements reproduce the energy of the classical rotating string in $AdS_4 \times \mathbb{CP}^3$ from the energy of the classical rotating string in $AdS_5 \times S^5$. Moreover, keeping the next to leading term in $h(\lambda)$ reproduces all the non-analytic terms in $(E_0 + E_1^{\rm odd})_{AdS_4 \times \mathbb{CP}^3}$, those related to the classical energy of the string as well as those related to the corrections from the one-loop phase.

In addition, the various identifications of parameters which relate the Bethe Ansatz energy with the result of the worldsheet calculation for the circular rotating string also lead to the correct map for the folded spinning string and the universal scaling function, as may be seen by inspecting the Bethe Ansatz solution in [51].

Unfortunately, the same cannot be said about the relation between the analytic 1-loop terms $(E_1^{\rm even})_{AdS_5 \times S^5}$ in (4.23) and $(E_1^{\rm even})_{AdS_4 \times \mathbb{CP}^3}$ in (4.11). Using the above identifications in $(E_1^{\rm even})_{AdS_5 \times S^5}$, it appears that an additional formal replacement for the $\zeta$-constants is needed. This is exactly the same replacement described earlier, (4.24), and has the same interpretation of replacing the sum over even modes by an additional sum over odd modes. Such a replacement, however, seems unjustified on the basis of the Bethe equations (4.26)–(4.30).

To summarize, we have found that the conjectured all-loop Bethe Ansatz [10] reproduces the general structure of the $AdS_4 \times \mathbb{CP}^3$ superstring calculation.

In particular, the worldsheet approach predicts that the function $h(\lambda)$ that should be used in the Bethe Ansatz proposal of [10] should be given by (1.2), (1.3) (or, equivalently (4.15)). This conclusion (as well as the confirmation that the strong-coupling limit of the phase in the Bethe Ansatz should be, indeed, the same as in $AdS_5 \times S^5$ case) is not sensitive to the compactness of the worldsheet $\sigma$ direction.

However, the matching of the analytic $1/J^{2n}$ terms in the string 1-loop energy (whose coefficients *are* sensitive to the compactness of the $\sigma$ direction) is not immediately clear. It might be that we are missing some subtlety in the identification of the circular string configuration as a Bethe Ansatz solution, or that some details of the Bethe Ansatz proposal of [10] still need to be adjusted. Further analysis is required to settle these issues.

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## A. Details of calculation of the fermionic spectrum

Our starting point is the action in (3.5). We will analyze separately the geometric and the flux part of the covariant derivative. The action has constant coefficients and the kinetic operator may be extracted directly. For the purposes of analytic calculations, however, it is useful to first perform certain field redefinitions.

The background value of the slashed vielbein is

$$\not{e}_a = \hat{\mathrm{n}}_a \Gamma_0 + \tilde{\mathrm{n}}_a \Gamma_3 + \mathrm{m}_a \Gamma_9 \tag{A.1}$$

Here as in (2.18) $\hat{\mathrm{n}} = (\kappa, 0)$, $\tilde{\mathrm{n}} = (\mathrm{w}, k)$ and $\mathrm{m} = (\omega, m)$. A sequence of two rotations with constant coefficients

$$\begin{aligned} S = S_{39} S_{09}\,, \qquad & S_{39} = \cos p + \sin p \Gamma_{39}\,, \qquad && S_{09} = \cosh q + \sinh q \Gamma_{09} \\ & \sin 2p = \frac{m}{\sqrt{m^2 + k^2 \mathrm{r}_1^2}}\,, \qquad && \sinh 2q = -\frac{\omega}{k \mathrm{r}_1} \end{aligned} \tag{A.2}$$

transform $\not{e}_0$and $\not{e}_0$ into two Dirac matrices. Then the transformed value of the slashed vielbein becomes:

$$S^{-1} \not{e}_0 S = \sqrt{m^2 + k^2 \mathrm{r}_1^2}\, \Gamma_0\,, \qquad S^{-1} \not{e}_1 S = \sqrt{m^2 + k^2 \mathrm{r}_1^2}\, \Gamma_3\,. \tag{A.3}$$

Due to the choice of isometry direction $\varphi_3$ in the construction of the circular rotating string solution, all components of the spin connection along $\mathbb{CP}^3$ vanish when evaluated on the background. Thus, the geometric part of the (transformed) covariant derivatives is as in [28]:

$$\begin{aligned} D_a &\equiv \partial_a + \frac{1}{4} \omega_a{}^{AB} \Gamma_{AB}\,, \\ D_0^S = S^{-1} D_0 S &= \partial_0 - \frac{\mathrm{r}_0 \mathrm{r}_1 k^2}{2\sqrt{m^2 + \mathrm{r}_1^2 k^2}} \Gamma_{01} - \frac{\mathrm{r}_0 \mathrm{r}_1 k w}{2\sqrt{m^2 + \mathrm{r}_1^2 k^2}} \Gamma_{13} + \frac{\kappa m}{\mathrm{w}} \frac{\mathrm{w}^2 - \omega^2}{2(m^2 + \mathrm{r}_1^2 k^2)} \Gamma_{19}\,, \\ D_1^S = S^{-1} D_1 S &= \partial_1 + \frac{\mathrm{r}_0}{\mathrm{r}_1} \frac{m\omega}{2\sqrt{m^2 + \mathrm{r}_1^2 k^2}} \Gamma_{01} - \frac{\mathrm{r}_0 \mathrm{r}_1 k^2}{2\sqrt{m^2 + \mathrm{r}_1^2 k^2}} \Gamma_{13} + \frac{\mathrm{r}_0^2 k \kappa m}{2(m^2 + \mathrm{r}_1^2 k^2)} \Gamma_{19} \end{aligned} \tag{A.4}$$

In the type IIA theory the fermions $\theta^{1,2}$ are chiral and of opposite chirality

$$\Gamma_{-1} \theta^1 = \theta^1, \qquad \Gamma_{-1} \theta^2 = -\theta^2\,. \tag{A.5}$$

They may be combined into a single 32-component unconstrained spinor $\psi = \theta^1 + \theta^2$. Then,

$$s^{1J} \theta^J + s^{2J} \theta^J = \Gamma_{-1} \psi\,. \tag{A.6}$$

With these observations the geometric part of the action for the fermionic quadratic fluctuations may be written as

$$\begin{aligned} (\eta^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \not{e}_a D_b \theta^J &= \bar{\psi} \not{e}_a D_b (\eta^{ab} \mathbb{1} - \epsilon^{ab} \Gamma_{-1}) \psi \\ &= -\bar{\psi} \not{e}_0 D_0 \psi + \bar{\psi} \not{e}_1 D_1 \psi - \bar{\psi} \not{e}_0 D_1 \Gamma_{-1} \psi + \bar{\psi} \not{e}_1 D_0 \Gamma_{-1} \psi \\ &= -\bar{\psi}' \Gamma_0 (1 + \Gamma_{03} \Gamma_{-1}) D_0^S \psi' + \bar{\psi}' \Gamma_3 (1 + \Gamma_{03} \Gamma_{-1}) D_1^S \psi' \end{aligned} \tag{A.7}$$

where $\psi' = (m^2 + k^2 \mathrm{r}_1^2)^{1/4} S^{-1} \psi$. Opening the parenthesis one finds without difficulty that the terms in $D_i^S$ which do not commute with $(1 + \Gamma_{03} \Gamma_{-1})$ cancel out either among themselves or because of the constraint $\bar{\psi} \Gamma_A \psi = 0$. One is finally left with

$$(\eta^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \not{e}_a D_b \theta^J = \bar{\psi}' (-\Gamma_0 D_0 + \Gamma_3 D_1)(1 + \Gamma_{03} \Gamma_{-1}) \psi' \tag{A.8}$$

JHEP11(2008)069

Let us focus next on the flux-dependent terms in the super-covariant derivative. Their contribution to the action (3.5) as well as the precise expressions for the slashed fluxes are

$$(\eta^{ab}\delta^{IJ}-\epsilon^{ab}s^{IJ})\bar\theta^I \not e_a\left[\not F_2(i\sigma^2)^{JK}+\not F_4(\sigma_1)^{JK}\right]\not e_b\theta^K=\bar\psi\not e_a[-\not F_2\Gamma_{-1}+\not F_4]\not e_b(\eta^{ab}\mathbb{1}+\epsilon^{ab}\Gamma_{-1})\psi$$
$$\not F_2=2(\Gamma_{45}-\Gamma_{67}+\Gamma_{89})\,,\qquad\qquad \not F_4=6\Gamma_{0123}\,. \tag{A.9}$$

To simplify this expression we next split $S^{-1}(-\not F_2\Gamma_{-1}+\not F_4)S$ into a sum of terms

$$S^{-1}(-\not F_2\Gamma_{-1}+\not F_4)S=\mathcal{F}+\mathcal{F}_{03}+\mathcal{F}_0+\mathcal{F}_3\,, \tag{A.10}$$

where $\mathcal{F}$ commutes with $\Gamma_0$ and $\Gamma_3$, $\mathcal{F}_i$ anticommutes with $\Gamma_i$ and $\mathcal{F}_{ij}$ anticommutes with $\Gamma_{ij}$. These properties are sufficient to show that $\mathcal{F}_0$ and $\mathcal{F}_3$ cancel out and that the only relevant terms will be $\mathcal{F}$ and $\mathcal{F}_{03}$ whose expressions are

$$\begin{aligned}\mathcal{F}&=-2(\Gamma_{45}-\Gamma_{67})\Gamma_{-1}-2\cosh 2q\cos 2p\Gamma_{89}\Gamma_{-1}\,,\\ \mathcal{F}_{03}&=6\cosh 2q\cos 2p\Gamma_{0123}\end{aligned} \tag{A.11}$$

Using (A.3) one may rewrite the flux term in the fermionic action as

$$\begin{aligned}&\bar\psi\not e_a\left[-\not F_2\Gamma_{-1}+\not F_4\right]\not e_b(\eta^{ab}\mathbb{1}+\epsilon^{ab}\Gamma_{-1})\psi\\ &\quad=\sqrt{m^2+k^2\mathrm{r}_1^2}\bar\psi'\left[\mathcal{F}-\mathcal{F}_{03}\right](\mathbb{1}+\Gamma_{03}\Gamma_{-1})\psi'\\ &\quad=-\sqrt{m^2+k^2\mathrm{r}_1^2}\bar\psi'\left[2\left((\Gamma_{45}-\Gamma_{67})+\cosh 2q\cos 2p\Gamma_{89}\right)\Gamma_{-1}\right.\\ &\qquad\qquad\left.+6\cosh 2q\cos 2p\Gamma_{0123}\right](\mathbb{1}+\Gamma_{03}\Gamma_{-1})\psi'\,,\end{aligned} \tag{A.12}$$

Then, the complete fermionic quadratic Lagrangian is

$$\begin{aligned}\mathcal{L}&=i(\eta^{ab}\delta^{IJ}-\epsilon^{ab}s^{IJ})\bar\theta^I\not e_aD_b^{JK}\theta^K=i\bar\psi K\psi'\,,\\ K&=\Big\{2(-\Gamma_0D_0^S+\Gamma_3D_1^S)-\frac{1}{4}\sqrt{m^2+k^2\mathrm{r}_1^2}\Big[\,6\cosh 2q\cos 2p\Gamma_{0123}\\ &\qquad+2\left((\Gamma_{45}-\Gamma_{67})+\cosh 2q\cos 2p\Gamma_{89}\right)\Gamma_{-1}\Big]\Big\}\mathcal{P}_+\,,\end{aligned} \tag{A.13}$$

where

$$\mathcal{P}_+=\frac{1}{2}(\mathbb{1}+\Gamma_{03}\Gamma_{-1})\quad. \tag{A.14}$$

The presence of this projector in the quadratic Lagrangian is an indication of the $\kappa$-symmetry of the action. A natural$\kappa$-symmetry gauge choice is then that none of the components of $\psi$ lie in the orthogonal subspace of $\mathcal{P}_+$.

Next, we need to find the frequencies of the fermionic modes described by this Lagrangian. To this end it is useful to consider a general operator of which $K$ is a special case. Such an operator is:

$$\begin{aligned}\mathcal{K}&=-ip_0\Gamma_0+ip_1\Gamma_3+a\Gamma_{013}+b\Gamma_{019}+c\Gamma_{139}\\ &\quad+6A\Gamma_{0123}+2B\Gamma_{89}\Gamma_{-1}+2C(\Gamma_{45}-\Gamma_{67})\Gamma_{-1}\end{aligned} \tag{A.15}$$

where

$$a = 0\,, \qquad A = -\frac{1}{8}\sqrt{\omega^2 + k^2 \mathrm{r}_1^2}\,,$$
$$b = -\frac{\kappa m}{\mathrm{w}}\frac{\mathrm{w}^2 - \omega^2}{2(m^2 + \mathrm{r}_1^2 k^2)}\,, \qquad B = -\frac{1}{8}\sqrt{\omega^2 + k^2 \mathrm{r}_1^2}\,, \tag{A.16}$$
$$c = -\frac{\mathrm{r}_0^2 k \kappa m}{2(m^2 + \mathrm{r}_1^2 k^2)}\,, \qquad C = -\frac{1}{8}\sqrt{m^2 + k^2 \mathrm{r}_1^2}\,.$$

To evaluate the eigenvalues and enforce $\kappa$ gauge fixing $\mathcal{P}_+\psi = \psi$, let us find the eigenvalues of

$$K_g = \mathcal{P}_+^T \Gamma_0 \mathcal{K} \mathcal{P}_+\,. \tag{A.17}$$

The factor of $\Gamma_0$ implies that $p_0$ may be extracted from the zeros of the characteristic polynomial.

To identify the zeros it is useful to note that the operator $K$ commutes with $\Gamma_{4567}$ and that the projectors $P_\pm = \frac{1}{2}(1 \pm \Gamma_{4567})$ commute with the $\kappa$-symmetry projector $\mathcal{P}_+$. Then one may make a further split

$$K_{g+} = P_+^T K_g P_+\,, \qquad K_{g-} = P_-^T K_g P_- \tag{A.18}$$

The characteristic polynomials for these operators may be found without difficulty.

The one for $K_{g+}$ implies that $p_0$ is determined by the equation

$$[-(p_0+c)^2+(p_1-b)^2+4(3A+B)^2]^2[-(p_0-c)^2+(p_1+b)^2+4(3A+B)^2]^2=0\,, \tag{A.19}$$

from which one should keep the positive frequencies. The factorized form means that there are two doubly-degenerate modes with the frequencies:

$$(p_0)_{\pm 12} = \pm c + \sqrt{(p_1 \pm b)^2 + 4(3A+B)^2}\,. \tag{A.20}$$

It is worth noting that these correspond to the "heavy" fermions. If one reduces the solution to the case of the BMN string the mass of these fermions is twice that of the "lighter" fluctuations.

The frequencies of those lighter modes are determined by the characteristic polynomial of $K_{g-}$, i.e. are the roots of the following quartic polynomial:

$$\left[(-p_0^2 + p_1^2)^2 - 2p_0^2 C_{++++} - 2p_1^2 C_{+-+-} - 8\,b\,c\,p_0 p_1 + C_{++--}^2\right]^2 = 0\,, \tag{A.21}$$

where

$$C_{+\alpha\beta\gamma} = b^2 + 4\alpha(3A-B)^2 + \beta c^2 - 16\gamma C^2\,. \tag{A.22}$$

As for $K_{g+}$, there are two doubly-degenerate modes; upon using the expressions (A.16) for the constants appearing above, one finds the equation (3.11) quoted in the text.

## B. Details of comparison of the fixed and large mode number contributions to the one-loop string energy

In this appendix we record the regular and singular terms in the one-loop frequency sum in the large $\omega$ limit in the discrete and continuous regimes (see section 4.1)

$$e^{\rm sum} = e^{\rm sum}_{\rm reg} + e^{\rm sum}_{\rm sing}\,, \qquad\qquad e^{\rm int} = e^{\rm int}_{\rm reg} + e^{\rm int}_{\rm sing}\,, \tag{B.1}$$

and compare their structures. The part of the summand $e^{\rm sum}_{\rm sing}$ that gives rise to a singular contribution in the discrete regime is

$$\begin{aligned}
e^{\rm sum}_{\rm sing}(n) =& \frac{1}{2\omega}\left(-k^2(1+u)(1+3u)\right)\\
&+\frac{1}{4\omega^3}\left(7k^2(1+u)(3+5u)n^2+\frac{1}{8}k^4(1+u(44+u(86+(28-15u)u)))\right)\\
&+\frac{1}{16\omega^5}\Big(-93k^2(1+u)(5+7u)n^4-\frac{1}{4}k^4(1+u)(375+u(2509+u(3157+687u)))n^2\\
&\qquad\qquad -\frac{1}{16}k^6(1+u)(1+u(257+u(1134+u(1006+u(65+33u)))))+\cdots\Big)\\
&+\mathcal{O}\left(\frac{1}{\omega^7}\right)\quad .
\end{aligned}\tag{B.2}$$

The part of the integrand $e^{\rm int}_{\rm reg}(n)$ that leads to a regular contribution in the continuum regime is

$$\begin{aligned}
e^{\rm int}_{\rm reg}(x) =& \frac{1}{2\omega}\left(-k^2(1+u)(1+3u)+\frac{7}{2}k^2(1+u)(3+5u)x^2-\frac{93}{8}k^2(1+u)(5+7u)x^4+\cdots\right)\\
&+\frac{1}{32\omega^3}\Big(k^4(1+u(44+u(86+(28-15u)u)))\\
&\qquad\qquad -\frac{1}{2}k^4(1+u)(375+u(2509+u(3157+687u)))x^2\Big)\\
&-\frac{1}{256\omega^5}\left(k^6(1+u)(1+u(257+u(1134+u(1006+u(65+33u)))))+\cdots\right)\\
&+\mathcal{O}\left(\frac{1}{\omega^7}\right)\quad .
\end{aligned}\tag{B.3}$$

By inspection, it is not hard to notice that

$$e^{\rm sum}_{\rm sing}(n) = e^{\rm int}_{\rm reg}\left(\frac{n}{\omega}\right)\,, \tag{B.4}$$

which shows that the regular part in the continuum regime correctly captures the apparent singularities in the large $\omega$ limit of the discrete regime.

Similarly, the singular part in the continuum regime $e^{\rm int}_{\rm sing}(x)$ and the regular part in the discrete regime $e^{\rm sum}_{\rm reg}(n)$ are

$$\begin{aligned}
e^{\rm int}_{\rm sing}(x) =& -\frac{3k^4u^2(1+u)^2}{4x^2\omega^3}+\frac{1}{8\omega^5}\left(\frac{15k^6u^3(1+u)^3}{x^4}-\frac{3k^6u^2(1+u)^2(-1+2u^2)}{x^2}\right)\\
&+\mathcal{O}\left(\frac{1}{\omega^7}\right)\,,
\end{aligned}\tag{B.5}$$

and

$$e_{\rm reg}^{\rm sum}(n) = \frac{1}{2\omega}\left(-\frac{3}{2n^2}k^4u^2(1+u)^2+\frac{15}{4n^4}k^6u^3(1+u)^3+\cdots\right)$$
$$+\frac{1}{4\omega^3}\left(-\frac{3}{2n^2}k^6u^2(1+u)^2(-1+2u^2))+\frac{15}{16n^4}k^8u^4(1+u)^3(13+17u)+\cdots\right)$$
$$+\,\mathcal{O}\left(\frac{1}{\omega^5}\right)\ , \tag{B.6}$$

respectively. Again, it is not hard to see that

$$e_{\rm sing}^{\rm int}(x) = e_{\rm reg}^{\rm sum}(\omega x)\,, \tag{B.7}$$

implying that the regular part of the discrete regime correctly describes the singular part in the continuum regime.

These observations parallel those in $AdS_5\times S^5$ made in [29]. As in that case, they imply that the one-loop correction to the energy of the circular rotating string is given by the equation (4.5).

## C. Higher orders in the $1/\omega$ expansion of $e^{\rm int}(x)$

In this appendix we include the expression of $e^{\rm int}$ (whose leading order was quoted in (4.9)) to higher orders.

$$e^{\rm int}(x) = \frac{k^2(1+u)}{2\omega^2}\left(\frac{1+u(3+2x^2)}{(1+x^2)^{3/2}}-2\frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}}\right)$$
$$-\frac{k^4(1+u)}{32\omega^4x^2}\Bigg[\frac{1}{(1+x^2)^{7/2}}\big(32u^2(1+u)+(7+u(77+u(221+135u)))x^2$$
$$+4(-7+u(-7+u(29+21u)))x^4+16u(1+u(3+u))x^6+16u(1+u)x^8\big)$$
$$-\frac{8}{(1+4x^2)^{7/2}}\big(u^2(1+u)+(1+3u(5+u(11+5u)))x^2$$
$$+8(-1+3u)(2+u(4+u))x^4+64u(2+3u)x^6+256u(1+u)x^8\big)\Bigg]$$
$$+\frac{k^6(1+u)}{256\omega^6x^4}\Bigg[\frac{1}{(1+x^2)^{11/2}}\big(512u^3(1+u)^2+128u^2(1+u)(1+22u+20u^2)x^2$$
$$+(31+u(735+u(3570+u(10418+u(12447+4991u)))))x^4$$
$$+4(-93+u(-596+u(-907+u(373+u(1412+707u)))))x^6$$
$$+8(31+u(93+u(254+u(358+u(201+71u)))))x^8$$
$$+32u(28+u(132+u(146+u(40+u))))x^{10}$$
$$+64u(1+u)(6+u(26+9u))x^{12}$$
$$+32u(1+u)(3+u)(1+3u)x^{14}\big)$$
$$-\frac{32}{(1+4x^2)^{11/2}}\big(u^3(1+u)^2+u^2(1+u)(1+22u+20u)x^2$$
$$+(1+u(31+u(147+u(357+u(391+157u)))))x^4$$

$$
\begin{aligned}
&+4(-12+u(-52+u(-9+u(137+u(179+91u))))))x^6\\
&+16(8+u(64+u(232+u(240+u(67+21u)))))x^8\\
&+128u(32+u(140+u(142+u(32+u))))x^{10}\\
&+1024u(1+u)(7+26u+8u)x^{12}\\
&+2048u(1+u)(3+u)(1+3u)x^{14})\Bigg]\\
&+\mathcal{O}\left(\frac{1}{\omega^8}\right).
\end{aligned} \tag{C.1}
$$

At each order in $1/\omega$ one notices terms which are singular as $x \to 0$. These are the terms contributing to $e^{\rm int}_{\rm sing}$ quoted in the previous appendix.

## D. Numerical checks

The fact that the leading term in the large $\omega$ expansion of the one-loop string energy is proportional to $\omega^{-1}$ contrasts with what happened in the case of the rotating string in $AdS_5 \times S^5$, whose "odd" part starts only at $1/\omega^5$ order. A check of this dependence may be obtained by a numerical evaluation of the sum in the regime leading to (4.12). The main complication is related to the fact that, while the correction to the energy is finite, each of the sums contributing to it is divergent. These divergences are of two types: power-like and logarithmic. While one may directly evaluate the sums with a cutoff, the presence of divergences leads to a quick loss of numerical accuracy.

This may be somewhat improved by separating the sum into two sub-sums and subtracting the divergences in each of them.[23] Concretely, we split the full sum into two sums — over the light and heavy modes; schematically, they are

$$
\begin{aligned}
e(n)^{\rm light} &= 4\times\sqrt{n^2+\frac{1}{4}(\omega^2-k^2u^2)}-2\times\frac{1}{2}\left((p_0)^F_1+(p_0)^F_2-(p_0)^F_3-(p_0)^F_4\right),\\
e(n)^{\rm heavy} &= \sqrt{n^2+\kappa^2}+\sqrt{n^2+(\omega^2-k^2u^2)}+\frac{1}{2}\left((p_0)^B_1+(p_0)^B_2-(p_0)^B_3-(p_0)^B_4\right)\\
&\quad -2\sqrt{(n-b)^2+(\omega^2+k^2{\rm r}_1^2)}-2\sqrt{(n+b)^2+(\omega^2+k^2{\rm r}_1^2)},
\end{aligned} \tag{D.1}
$$

where, as before, $p^{B,F}_{1,2,3,4}$ are solutions of the bosonic and fermionic quartic equations.

Since the subtracted sums are absolutely convergent, one may carry out this subtraction for each mode separately. The subtracted terms add up to zero. In each of them the power-like divergences cancel out. Then, from each of them we may subtract the leading term in the large $n$ expansion for fixed values of the other parameters

$$
\begin{aligned}
\Delta S_{\rm light} &= (\omega^2-k^2-m^2)\frac{1}{2n},\\
\Delta S_{\rm heavy} &= (3(\kappa^2-\omega^2)-m^2-4k^2{\rm r}_1^2)\frac{1}{2n}.
\end{aligned} \tag{D.2}
$$

[23]One may in fact go even further and subtract the divergences of each frequency sum separately.

These subtractions cancel out when the two sums are added together because of the usual mass sum rule

$$\sum_i (-)^{F_i} m_i^2 = 0 \quad \Leftrightarrow \quad \kappa^2 - m^2 - \omega^2 - 2k^2 \mathrm{r}_1^2 = 0\,, \tag{D.3}$$

which here appears as a consequence of the Virasoro constraint.

An unfortunate feature of these sums is that they converge somewhat slowly in their effective parameter which is $n/\omega$. Indeed, since the leading large $n$ behavior is $\sim n^{-3}$, the corrections are of the order $\delta S \sim \frac{1}{2}(\omega/N)^2$. Consequently, for a sufficiently large $\omega$ which probes the asymptotic behavior of the sum, the necessary cutoff $N$ is relatively large.

Numerical evaluation with $\omega = 10^4$ and an estimated error $5 \times 10^{-3}$ (i.e. a cutoff $N = 10^5$) gives

$$\begin{aligned} 2\omega E &= -(1.383 \pm .005) k^2 u(1+u) + \mathcal{O}(\omega^{-1}) \\ &= -(1.995 \pm .01) \ln 2\, k^2 u(1+u) + \mathcal{O}(\omega^{-1}) \end{aligned} \tag{D.4}$$

Clearly, this is consistent with the leading term in the large $\omega = \frac{J}{\sqrt{\lambda}}$ expansion obtained analytically in (4.12).

It is possible, though somewhat cumbersome, to perform similar checks for the subleading terms in the $1/J$ expansion.

## E. Large $J$, large $k$ limit of circular string solution

It is interesting to study the large $J$, fixed $S$, limit of the solution considered in the main text for, as we will see, this limit does not seem to follow the same rules for finding the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ string energies from their $\mathrm{AdS}_5 \times \mathrm{S}^5$ analogues. This limit may, however, be somewhat exceptional, as it requires scaling $\mathrm{AdS}_4$ winding number to infinity. Nonetheless, if for nothing other than completeness, we decided to mention it here.

We will consider the limit where $\mathcal{J} = \omega$ is taken to be large while $\mathcal{S}$ and $m$ are kept fixed with $\frac{m}{\mathcal{S}} < 0$. The constraints on the parameters in section 2 then imply that we must also take the winding $k$ to be large. We will use the notation $k = \beta\omega$ where $\beta = -\frac{m}{\mathcal{S}}$. In this limit the parameters of the solution become

$$\begin{aligned} \kappa &= \omega - \frac{m^2}{\sqrt{m^2+\mathcal{S}^2}} + \mathcal{O}\left(\frac{1}{\omega}\right) \\ r_1^2 &= \frac{\mathcal{S}^2}{\omega\sqrt{\mathcal{S}^2+m^2}} + \mathcal{O}\left(\frac{1}{\omega^2}\right) \\ \mathrm{w} &= \frac{\omega}{\mathcal{S}}\sqrt{\mathcal{S}^2+m^2} + \frac{\mathcal{S}m^2}{\mathcal{S}^2+m^2} + \mathcal{O}\left(\frac{1}{\omega}\right) . \end{aligned} \tag{E.1}$$

Then the energy density, $\mathcal{E}$, is infinite but as for the BMN string or giant magnon the difference $\mathcal{E} - \mathcal{J}$ is finite, and is given simply by

$$E - J = \sqrt{S^2 + m^2\bar{\lambda}}\ \ . \tag{E.2}$$

This classical energy is what we would expect from the analogous $\mathrm{AdS}_5\times\mathrm{S}^5$ result found in [45] where the one-loop correction was also calculated and shown to be zero. Based on the replacement rule, (4.17), we would then expect to find a non-vanishing one-loop contribution proportional to $\ln 2$ in $\mathrm{AdS}_4\times\mathbb{CP}^3$ geometry (coming from the $\bar{\lambda}\mapsto 2\bar{h}(\bar{\lambda})$ replacement in the classical expression (E.2) with $\bar{h}$ given by (4.15)). However, we will see that this is not the case — the one-loop correction found by direct evaluation from string theory actually vanishes.

It is straightforward to find the fluctuation frequencies about this large-$J$ solution from the general frequencies calculated in section 3. From the quartic equation (3.4) we find the characteristic frequencies

$$(p_0)^B_{1,2}=\sqrt{(p+\beta)^2+1}\pm\sqrt{1+\beta^2}\,,\quad (p_0)^B_{3,4}=-\sqrt{(p-\beta)^2+1}\mp\sqrt{1+\beta^2}\,. \tag{E.3}$$

We should note here that we have rescaled the worldsheet coordinate so that the string has infinite length, scaling like $\omega$. Thus the worldsheet momenta, $p$, are now continuous. From the remaining bosonic fluctuation frequencies we have six free massive modes — two with mass 1 and four with mass 1/2. For the fermions we find four with frequencies calculated from the quartic equation (3.11)

$$\begin{aligned}(p_0)^F_{1,2}&=\frac{1}{2}\left(\sqrt{(2p+\beta)^2+1}\pm\sqrt{1+\beta^2}\right)\,,\\ (p_0)^F_{3,4}&=\frac{1}{2}\left(-\sqrt{(2p-\beta)^2+1}\mp\sqrt{1+\beta^2}\right)\end{aligned} \tag{E.4}$$

while the remaining four fermions have frequencies

$$\begin{aligned}(p_0)^F_{5,6}&=\sqrt{\left(p+\frac{1}{2}\beta\right)^2+1}\pm\frac{1}{2}\sqrt{1+\beta^2},\\ (p_0)^F_{7,8}&=\sqrt{\left(p-\frac{1}{2}\beta\right)^2+1}\mp\frac{1}{2}\sqrt{1+\beta^2}.\end{aligned} \tag{E.5}$$

We can now straightforwardly calculate the sum over frequencies which to leading order in $\omega$ can be replaced by an integral.

$$\begin{aligned}E_1\sim\int_0^\infty dp\ \Big[&2\sqrt{1+p^2}+2\sqrt{1+4p^2}+\sqrt{1+(p-\beta)^2}+\sqrt{1+(p+\beta)^2}\\ &-\sqrt{1+(2p-\beta)^2}-\sqrt{4+(2p-\beta)^2}-\sqrt{1+(2p+\beta)^2}-\sqrt{4+(2p+\beta)^2}\Big].\end{aligned} \tag{E.6}$$

If we follow the standard procedure of imposing a cut-off, performing the integral and taking the cut-off to infinity we find that the one-loop correction to the energy of the circle string in this limit is zero.

For comparison, if we follow [23], we can identify the "light" and "heavy" modes as[24]

$$p_0^L=\left\{4\times\sqrt{1/4+p^2},2\times\sqrt{1/4+(p\pm\beta/2)^2}\right\}, \tag{E.7}$$

$$p_0^H=\left\{2\times\sqrt{1+p^2},\sqrt{1+(p\pm\beta)^2},\sqrt{1+(p\pm\beta/2)^2}\right\}, \tag{E.8}$$

[24]This can be done by taking the $\beta\to 0$ which can be viewed as taking $\mathcal{S}\to 0$ but with $m=0$ and which corresponds to the BMN string.

Then we note that the formula [23] for the one-loop energy correction

$$E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[ p_0^H(n) + \frac{1}{2} p_0^L(n/2) \right] \tag{E.9}$$

becomes, in the limit of large $\mathcal{J} = \omega$ (where we again set $n = \omega p$ and replace the sum by an integral), exactly half of the analogous result in $\mathrm{AdS}_5 \times \mathrm{S}^5$ which in this case is also zero.

## References


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