A CLASS OF DUST-LIKE SELF-SIMILAR SOLUTIONS OF THE MASSLESS EINSTEIN-VLASOV SYSTEM.

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Abstract

In this paper the existence of a class of self-similar solutions of the Einstein-Vlasov system is proved. The initial data for these solutions are not smooth, with their particle density being supported in a submanifold of codimension one. They can be thought of as intermediate between smooth solutions of the Einstein-Vlasov system and dust. The motivation for studying them is to obtain insights into possible violation of weak cosmic censorship by solutions of the Einstein-Vlasov system. By assuming a suitable form of the unknowns it is shown that the existence question can be reduced to that of the existence of a certain type of solution of a four-dimensional system of ordinary differential equations depending on two parameters. This solution starts at a particular point P_0 and converges to a stationary solution P_1 as the independent variable tends to infinity. The existence proof is based on a shooting argument and involves relating the dynamics of solutions of the four-dimensional system to that of solutions of certain two- and three-dimensional systems obtained from it by limiting processes.

1 INTRODUCTION

It is well known that solutions of the Einstein equations coupled with suitable models of matter can yield singularities in finite time. The unknowns in these equations are the spacetime metric and some matter fields. The exact nature of the latter depends on the physical situation being considered. The usual terminology in general relativity is that there is said to be a singularity if the metric fails to be causally geodesically complete, i.e. if there are timelike or null geodesics which in at least one direction are inextendible and of finite affine length. The singularity is said to be in the future or the past according to the incomplete direction of the geodesics. It is expected on the basis of physical intuition, and known to be true in some simple cases, that the geodesic incompleteness is associated with the energy density or some curvature invariants blowing up. For background on this subject see textbooks such as [13], [28] and

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[25]. One of the best known types of singularities in general relativity are those which occur inside black holes. When a black hole is formed by the collapse of matter it is known that under suitable circumstances an event horizon is formed which ensures that the singularity can have no influence on distant observers.

Mathematical relativity is the study of the properties of solutions of the Einstein equations coupled to various matter equations. One of the main questions in the field is the cosmic censorship hypothesis. There are two versions of this conjecture called weak and strong cosmic censorship, both of which were proposed by Roger Penrose. It should be noted that, contrary to what the names might suggest, the strong version does not imply the weak one. The results proved in what follows are motivated by weak cosmic censorship and strong cosmic censorship will not be discussed further here. Weak cosmic censorship is a statement which concerns isolated systems in general relativity. Mathematically this means considering solutions of the Einstein equations which evolve from asymptotically flat initial data. Initial data for the Einstein equations consist of a Riemannian metric h_{ab}, a symmetric tensor k_{ab} and some matter fields which for the moment will be denoted generically by F_0, all defined on a three-dimensional manifold S. Solving the Cauchy problem for the Einstein-matter equations means embedding the manifold S into a four-dimensional manifold M on which are defined a Lorentzian metric $g_{\alpha\beta}$ and matter fields F such that h_{ab} and k_{ab} are the pull-backs to S of the induced metric and second fundamental form of the image of the embedding of S while F_0 is the pullback of the matter fields. The metric $g_{\alpha\beta}$ and the matter fields F are required to satisfy the Einstein-matter equations. A comprehensive treatment of the Cauchy problem for the Einstein equations can be found in [26]. Initial data on $\mathbb{R}^3$ are called asymptotically flat if the metric h_{ab} tends to the flat metric at infinity in a suitable sense while k_{ab} and F_0 tend to zero. Physically this corresponds to concentrating attention on a particular physical system while ignoring the influence of the rest of the universe.

A solution of the Einstein-matter equations evolving from initial data is said to be a *development* of that data if each inextendible causal curve intersects the initial hypersurface precisely once. When this property holds the initial hypersurface is said to be a Cauchy hypersurface for that solution. In general, a solution is called globally hyperbolic if it admits a Cauchy hypersurface. For prescribed data there is a development which is maximal in the sense that any other development can be embedded into it. It is unique up to a diffeomorphism which preserves the initial hypersurface.

In a spacetime evolving from asymptotically flat data it is often possible to define future null infinity $\mathcal{I}^+$ as a set of ideal endpoints of complete future-directed null geodesics. We can say that any singularity occurring does not influence events near infinity if there is no inextendible causal curve to the future of the initial hypersurface which is incomplete in the past while intersecting a future-complete null geodesic. The first of these properties means intuitively that this curve represents a signal which comes out of a singularity while the second property means that it reaches a region which can communicate with infinity. If a curve of this type does exist it is said that a globally naked singularity exists. The past of null infinity, $J^-(\mathcal{I}^+)$, is the set of points for which there is a future-directed causal curve starting there and going to null infinity. The complement of $J^-(\mathcal{I}^+)$ is called the black hole region. Its boundary is called the event horizon and is a null hypersurface in M.

There is a notion of completeness of null infinity. A precise definition will not be given here but roughly speaking it corresponds to the situation where there are timelike curves contained in $J^-(\mathcal{I}^+)$ which exist for a infinite time towards the future. Physically this means that there are observers which can remain outside the black hole for an unlimited amount of time. If the maximal globally hyperbolic development of asymptotically flat initial data always has a complete null infinity then this ensures the absence of globally naked singularities. For any inextendible causal curve to the future of the initial surface which goes to null infinity must intersect the initial hypersurface. Hence it cannot be incomplete in the past. The completeness of $\mathcal{I}^+$ ensures that the solution is large enough to represent the whole future of a system evolving from the initial data under consideration. The intuitive content of the weak cosmic censorship hypothesis is that in the time evolution corresponding to initial data for the Einstein equations coupled to reasonable (non-pathological) matter the existence of a singularity implies that of an event horizon which covers the singularity and hides it from distant observers. Often this is weakened to the requirement that a horizon exists in the case of generic initial data. Up to now this intuitive picture has only been developed into a precise mathematical formulation under special circumstances. In general finding the correct formulation is part of the problem to be solved.

Due to the mathematical complexity of the Einstein equations many of the studies related to singularity formation for these equations have been carried out for spherically symmetric solutions. In spherical symmetry the Einstein vacuum equations are non-dynamical due to Birkhoff's theorem, which says that any spherically symmetric vacuum solution is locally isometric to the Schwarzschild solution and, in particular, static. Thus it is essential to include matter of some kind. A matter model which has proved very useful for this task is the scalar field. This is a real-valued function ϕ which satisfies the wave equation $\nabla^\alpha\nabla_\alpha\phi = 0$. In this case the Einstein equations take the form $R_{\alpha\beta} = 8\pi\nabla_\alpha\phi\nabla_\beta\phi$, where $R_{\alpha\beta}$ is the Ricci curvature of $g_{\alpha\beta}$. The spherically symmetric Einstein-scalar field equations were studied in great detail in a series of papers by Demetrios Christodoulou. This culminated in [7] and [8]. In [7] it was shown that in this system naked singularities can evolve from regular asymptotically flat initial data. This represents a problem for the weak cosmic censorship hypothesis but the conjecture can be saved by a genericity assumption since it was shown in [8] that generic initial data do not lead to naked singularities.

For the spherically symmetric Einstein-scalar field equations it is known from the work of Christodoulou [6] that small asymptotically flat initial data lead to a solution which is geodesically complete and hence free of singularities. (In fact this small data result has recently been extended to the case without symmetry [15].) On the other hand there are certain large initial data for which it is known that a black hole is formed. The threshold between these two types of behaviour was studied in influential work by Choptuik [3] and many other papers since. This area of research is known as critical collapse and is surveyed in [12]. It is entirely numerical and heuristic and unfortunately mathematically rigorous results are not yet available.

The scalar field provides a simple and well-behaved matter model. At the same time no such field has been experimentally observed and the matter fields of importance for applications to astrophysics are of other kinds. One astrophysically relevant matter field which has good mathematical properties is collisionless matter described by the Vlasov equation. The necessary definitions are given in the next section. For the moment let it just be noted that the unknown in the Vlasov equation is a non-negative real-valued function $f(t, x^a, v^b)$ depending on local coordinates (t, x^a) on M and velocity variables v^b. Analogues of a number of the results proved for the scalar field have been proved for the Einstein-Vlasov system. For small initial data the solutions are geodesically complete [22]. There are certain large initial data for which a black hole is formed [1]. The threshold between these two types of behaviour has been investigated numerically in [23] and [18]. A closely related matter model which has been very popular in theoretical general relativity is dust, a fluid with vanishing pressure. It is equivalent to consider distributional solutions of the Vlasov equation of the form $f(t, x^a, v^b) = \rho(t, x^a)\delta(v^b - u^b(t, x^a))$ where the δ is a Dirac distribution. From many points of view dust is relatively simple to analyse. Unfortunately it has a strong tendency to form singularities where the energy density blows up, even in the absence of gravity. For this reason it must be regarded as pathological and of limited appropriateness for the investigation of cosmic censorship. A detailed mathematical study of formation of singularities in the Einstein equations coupled to dust was given in [5]. In spherical symmetry dust particles move as spherical shells. It can easily happen that shells including a strictly positive total mass come together at one radius and this causes the density to blow up. This effect is known as shell-crossing.

The motivation for this paper is the wish to understand cosmic censorship better for spherically symmetric solutions of the Einstein-Vlasov system. Is it true that in asymptotically flat spherically symmetric solutions of the Einstein-Vlasov system there are no naked singularities for generic data so that collisionless matter is as well-behaved as the scalar field? Could it even be that the Vlasov equation is better-behaved and that there are no naked singularities at all? No answers to these questions, positive or negative, are available although considerable effort has been invested into obtaining a positive answer. In what follows we try to obtain new insights by approaching a negative result through an interpolation between dust and smooth solutions of the Vlasov equation and looking for self-similar solutions. There are some results on related equations which give some hints. In the case of the Vlasov-Poisson system, the non-relativistic analogue of the Einstein-Vlasov system, global existence for general data, not necessarily symmetric, was proved by Pfaffelmoser [20] and Lions and Perthame [16]. The relativistic Vlasov-Poisson system, which is in some sense intermediate between the Vlasov-Poisson and Einstein-Vlasov systems, (but not in all ways) has been shown to have solutions which develop singularities in finite time. Rather precise information is available about the nature of these singularities [14].

As a side remark, we mention a paper [27] where it was suggested that naked singularities are formed in solutions of the Einstein-Vlasov system. The solutions concerned were axially symmetric but not spherically symmetric. The work is purely numerical but trying to understand what it means for the analytical problem leads to the conclusion that the solutions computed in [27] were dust solutions rather than smooth solutions of the Einstein-Vlasov system. This is discussed in [24]. There are also reasons for doubting that the numerical results really show the formation of a naked singularity [29].

A class of distributional solutions of the Einstein-Vlasov system intermediate between smooth solutions and dust is given by the Einstein clusters [11]. These are spherically symmetric and static, i.e. there exists a timelike Killing vector field which is orthogonal to spacelike hypersurfaces. It is supposed that the support of f consists of v^a such that the geodesics with these initial data are tangent to the spheres of constant distance from the centre of symmetry on these spacelike hypersurfaces. This means that the radial velocity and its time derivative in the geodesic equation are zero. These are in general two independent conditions on the data at a given time. A wider class, the generalized Einstein clusters [9], [2], is obtained as follows. In the case of the Einstein clusters taking the union of the spheres at a fixed distance from the centre defines a foliation of the spacetime by timelike hypersurfaces and the condition on the support means that the four-velocity of a particle with the given initial data is everywhere tangent to these timelike hypersurfaces. The generalized Einstein clusters are obtained by dropping the condition of staticity and replacing the family of timelike hypersurfaces invariant under the timelike Killing vector field by another foliation by timelike hypersurfaces which intersect any Cauchy surface in spheres and whose equation of motion follows from the Vlasov equation. Once again the four-velocity of a particle in the support of f is tangent to these hypersurfaces at all times. An analytical formulation of this definition will be given in the next section. It should be noted that the generalized Einstein clusters exhibit shell-crossing singularities and thus can still be thought of as pathological. We are interested in them as an intermediate step towards better-behaved matter models.

There are two major differences between the generalized Einstein clusters and the solutions studied in this paper. In the case of Einstein clusters the value of the angular momentum of the particles F is uniquely determined by the distance r to the centre of symmetry. By contrast, in the solutions studied in this paper the angular momentum takes a continuous range of values for each value of r. The second difference is that in the case of Einstein clusters at each spacetime point the component of the velocity vector v^a of a particle in the direction of the vector ∂_r takes on only one value. In the case of the solutions obtained in this paper the component along the direction ∂_r of the velocity vector takes on two different values at most spacetime points. This only fails at some exceptional values of r at a given time. This difference in the structure of the generalized Einstein clusters and the solutions considered in this paper is what gives some plausibility to the idea that the solutions described here could be a big step towards better-behaved matter models. From the physical point of view, in the case of the generalized Einstein clusters the material particle with the smallest value of r would not experience any gravitational field, and therefore could not approach the centre $r = 0$ unless its angular momentum vanished. In the solutions studied in this paper, since two radial velocities are allowed at each spacetime point, the material particle with the smallest value of r changes in time. This allows the occurrence of a collective collapse of the whole distribution of particles towards the origin with some of them coming closer and closer to the center as the value of some suitable time coordinate t increases.

Self-similar solutions of the massless Einstein-Vlasov system have also been considered in the paper [17]. There are several differences between the approach in [17] and the one considered in this paper. The first one is the choice of the rescaling group under which the solutions are invariant. The massless Einstein-Vlasov system is invariant under a two-dimensional group of rescalings. The choice of a particular one-dimensional rescaling group has been made in this paper by imposing that the distribution function f for the particles remains always of order one (see Sections 2, 3). This condition is natural, because the function f is invariant along characteristic curves. On the contrary, the choice of one-dimensional rescaling group for the solutions in [17] imposes that f becomes unbounded near the singularity for the particles within the self-similar region, something that can be achieved assuming that the distribution of matter is singular near the light-cone. The second difference between the solutions in [17] and those in this paper is that the solutions in [17] can be thought of as self-similar perturbations of the flat Minkowski space. As a matter of fact they have been computed by means of a perturbative iteration procedure that takes flat space as a starting point and where the terms in the resulting series have been computed numerically. By contrast, the solutions of this paper are obtained by means of a shooting procedure in which a parameter that measures the amount of energy in the self-similar region is of order one. The approach in this paper uses purely analytical methods and does not rely on numerical computations. On the other hand, in order to simplify the arguments, we have restricted the analysis in this paper to the study of dust-like solutions, an assumption that was not made in [17].

The plan of the paper is as follows. We will first reduce the problem of finding self-similar solutions of the Einstein-Vlasov system to an ODE problem that can be transformed into a four-dimensional system using suitable changes of variables. Using these transformations it will be seen that the construction of the desired self-similar solutions reduces to finding a particular orbit in the corresponding four-dimensional space connecting a certain point with a steady state that has a three-dimensional stable manifold. The existence of such an orbit will be shown by adjusting a parameter that measures the density of particles in a particular perturbative limit. The precise limit under consideration, which has the goal of making the problem feasible using analytical methods, corresponds to assuming that the radius of the region empty of particles, measured in the natural self-similar variables, is small.

2 THE EINSTEIN-VLASOV SYSTEM IN SCHWARZSCHILD COORDINATES.

We do not use exactly the classical Schwarzschild coordinates, but a slight modification of them that normalizes the time to be the proper time at the center $r = 0$. The metric is given by (cf. [21]):

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2.1}$$

If we restrict our attention to spherically symmetric solutions it is convenient to use the quantities (cf. [21]):

$$r = |x| \ , \ w = \frac{x \cdot v}{r} \ , \ F = |x \wedge v|^2$$

to parametrize the velocity variables. In particular F is constant along characteristics. Writing the particle density as

$$f = f(r, w, F, t)$$

the Einstein-Vlasov system for spherically symmetric solutions in these coordinates becomes:

$$\partial_t f + e^{\mu-\lambda}\frac{w}{E}\partial_r f - \left(\lambda_t w + e^{\mu-\lambda}\mu_r E - e^{\mu-\lambda}\frac{F}{r^3 E}\right)\partial_w f = 0 \tag{2.2}$$

where:

$$E = \sqrt{1 + w^2 + \frac{F}{r^2}} \tag{2.3}$$

and the functions λ, μ that characterize the gravitational field satisfy:

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2\rho, \tag{2.4}$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p \tag{2.5}$$

with boundary conditions:

$$\mu(0) = 0 \ , \ \lambda(0) = 0, \tag{2.6}$$

$$\lambda(\infty) = 0. \tag{2.7}$$

On the other hand ρ and p are given by:

$$\rho = \rho(r, t) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\left[\int_0^{\infty} E f dF\right] dw, \tag{2.8}$$

$$p = p(r, t) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\left[\int_0^{\infty} \frac{w^2}{E} f dF\right] dw. \tag{2.9}$$

With these basic equations in hand it is possible to give some details concerning generalized Einstein clusters, as promised in the introduction. These are not required to understand the main results of the paper but help to put those results into a wider context. A distributional solution of the Vlasov equation whose support is a smooth submanifold Σ has the property that Σ is a union of characteristics of the equation. A simple example is that of dust where the support is the graph of a function $u^a(t,x)$ of the form $W(t,r)\frac{x^a}{r}$. When expressed in terms of polar coordinates this becomes the graph of a function $W(t,r)$ augmented by the condition $F=0$. Here the function W solves the equations

$$\frac{dR}{dt}=e^{\mu(t,R)-\lambda(t,R)}\frac{W}{E}, \tag{2.10}$$

$$\frac{dW}{dt}=-(\dot{\lambda}(t,R)W+e^{\mu(t,R)-\lambda(t,R)}\mu'(t,R)E) \tag{2.11}$$

where $E=\sqrt{1+W^2}$.

Now consider the generalized Einstein clusters. They are only defined under the condition of spherical symmetry. They can be thought of as defining a matter model which can be used in the spherically symmetric Einstein-matter equations. Here they will be described in terms of Schwarzschild coordinates. The basic unknown is a function $R(t,r)$ which satisfies $R(0,r)=r$. It is the area radius at time t of the shell which had area radius r at time 0. As input we require a function $F(r)$ which is the angular momentum of the particles on the shell which was at radius r at time zero and $N(r)$ which is the density of particles per shell evaluated on the shell which had area radius r at $t=0$. For some purposes it is more convenient to use R as a radial coordinate instead of r and this is what was done in the original papers [9] and [2]. For a given shell at a given time the angular momentum and radial velocity of the particles are fixed and so the intersection of the support of the solution with the fibre of the mass shell over the point with coordinates (t,r) has codimension two. The following equations should be satisfied:

$$\frac{dR}{dt}=e^{\mu-\lambda}\frac{W}{E}, \tag{2.12}$$

$$\frac{dW}{dt}=-\left(\dot{\lambda}W+e^{\mu-\lambda}\mu'E-e^{\mu-\lambda}\frac{F}{R^3E}\right) \tag{2.13}$$

where $E=\sqrt{1+W^2+\frac{F}{R^2}}$ and the functions λ and μ are to be evaluated at the point (t,R). These are the full characteristic equations for the Vlasov equation. The difference in the coupled system comes from the fact that the expressions for the components of the energy-momentum tensor are different in the two cases. In the case of Einstein clusters the characteristics of interest have $W=0$ and $\frac{dW}{dt}=0$. It follows immediately that $\frac{dR}{dt}=0$. In that case the angular momentum is related to the geometry by the relation $F=\frac{r^3\mu'}{1-r\mu'}$.

The equations which have been written up to now describe particles of unit mass. We are interested in the construction of solutions of (2.2)-(2.9) supported in a region where $\left(w^2 + \frac{F}{r^2}\right)$ takes large values near the formation of the singularity. This suggests replacing (2.3) by:

$$E = \sqrt{w^2 + \frac{F}{r^2}}. \tag{2.14}$$

The system (2.2), (2.4)-(2.9), (2.14) is invariant under the rescaling:

$$r \to \theta r \ \ , \ \ t \to \theta t \quad \text{for } t < 0 \ , \ \ w \to \frac{1}{\sqrt{\theta}} w \ \ , \ \ F \to \theta F \tag{2.15}$$

for any $\theta > 0$. It is then natural to look for solutions of (2.2), (2.4)-(2.14) invariant under the rescaling (2.15). They will be the self-similar solutions in which we will be interested in this paper.

The system obtained when (2.3) is replaced by (2.14) can be interpreted as describing particles of zero rest mass. The rationale for this assumption is that near the singularity the derived solution will satisfy $w^2 + \frac{F}{r^2} >> 1$, and therefore it could be expected that it is possible to treat the whole Einstein-Vlasov system with massive particles as a perturbation of the massless problem.

In what follows we will consider solutions of (2.2), (2.4)-(2.14) where f is not a bounded function, but a measure concentrated on some hypersurfaces that will be described in detail later. As was mentioned in the introduction there is a class of distributional solutions of the Einstein-Vlasov system which are equivalent to what is usually known in the literature as dust. From this point of view the solutions considered in this paper are intermediate between dust and smooth solutions and hence will be called dust-like solutions. Note, however, that in contrast to dust they do have some velocity dispersion. The dimension of the support of f in the tangent space at a given spacetime point is zero for dust, one for generalized Einstein clusters, two for the solutions in this paper and three for smooth solutions. For the solutions here it will be possible to describe the distribution of velocities for the particles at a given point using a function depending on one coordinate, while a general distribution of velocities compatible with the assumption of spherical symmetry would depend on two coordinates.

3 SELF-SIMILAR SOLUTIONS

In this section we formulate the system of equations satisfied by the solutions of (2.2), (2.4)-(2.9), (2.14) that are invariant under the transformation (2.15). We will call these self-similar solutions in what follows. It is convenient, as a first step, in order to transform (2.2), (2.4)-(2.14) to a more convenient form to define a new variable:

$$v = \frac{w}{\sqrt{F}}. \tag{3.1}$$

We will assume in the rest of the paper that $f = 0$ for $(r, v, F, t) = (r, v, 0, t)$ in order to avoid singularities in (3.1). Moreover, we can even assume a more stringent condition on f, namely $f = 0$ for $0 \leq F \leq \delta_0$ for some $\delta_0 > 0$. Concerning the support in the r coordinate, the solutions constructed in this paper will vanish for $r \leq y_0(-t)$ for some $y_0 > 0$.

Making the change of variables $(r, w, F, t) \to (r, v, F, t)$ and denoting the new distribution function by f with a slight abuse of notation we can transform the system (2.2), (2.8)-(2.14) into:

$$\partial_t f + e^{\mu-\lambda}\frac{v}{\tilde{E}}\partial_r f - \left(\lambda_t v + e^{\mu-\lambda}\mu_r \tilde{E} - e^{\mu-\lambda}\frac{1}{r^3\tilde{E}}\right)\partial_v f = 0, \tag{3.2}$$

$$\tilde{E} = \sqrt{v^2 + \frac{1}{r^2}}, \tag{3.3}$$

$$\rho = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\tilde{E}\left[\int_0^{\infty} fFdF\right]dv, \tag{3.4}$$

$$p = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\frac{v^2}{\tilde{E}}\left[\int_0^{\infty} fFdF\right]dv. \tag{3.5}$$

Notice that the change of variables (3.1) eliminates the dependence on the variable F for the characteristic curves associated to the Vlasov equation (cf. (3.2)). Moreover, the functions ρ and p and therefore the functions λ, μ characterizing the gravitational fields depend on f only through the reduced distribution function:

$$\zeta(r, v, t) \equiv \int_0^{\infty} fFdF. \tag{3.6}$$

In particular, it is possible to write a closed problem for the reduced distribution function that can be obtained multiplying (3.2) by F and integrating with respect to this variable:

$$\partial_t \zeta + e^{\mu-\lambda}\frac{v}{\tilde{E}}\partial_r \zeta - \left(\lambda_t v + e^{\mu-\lambda}\mu_r \tilde{E} - e^{\mu-\lambda}\frac{1}{r^3\tilde{E}}\right)\partial_v \zeta = 0, \tag{3.7}$$

$$\tilde{E} = \sqrt{v^2 + \frac{1}{r^2}}, \tag{3.8}$$

$$\rho = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\tilde{E}\zeta dv, \tag{3.9}$$

$$p = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\frac{v^2}{\tilde{E}}\zeta dv. \tag{3.10}$$

The system (3.7)-(3.10) complemented with (2.4), (2.5) is a closed system of equations.

We will now study the class of self-similar solutions of the system (2.4), (2.5), (3.2)-(3.5). These are the functions having the functional dependence:

$$f\left(r,v,F,t\right)=G\left(y,V,\Phi\right)\quad,\quad\mu\left(r,t\right)=U\left(y\right)\quad,\quad\lambda\left(r,t\right)=\Lambda\left(y\right), \tag{3.11}$$

$$y=\frac{r}{\left(-t\right)}\quad,\quad V=\left(-t\right)v\quad,\quad\Phi=\frac{F}{\left(-t\right)}. \tag{3.12}$$

The solutions of (2.4), (2.5), (3.2)-(3.5) with this functional dependence satisfy:

$$\begin{aligned}&yG_y-VG_V+\Phi G_\Phi+e^{U-\Lambda}\frac{V}{\hat{E}}G_y\\&-\left(y\Lambda_yV+e^{U-\Lambda}U_y\hat{E}-e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)G_V\\&=0\end{aligned} \tag{3.13}$$

where:

$$\hat{E}=\sqrt{V^2+\frac{1}{y^2}} \tag{3.14}$$

and

$$e^{-2\Lambda}\left(2y\Lambda_y-1\right)+1=8\pi y^2\tilde{\rho}, \tag{3.15}$$

$$e^{-2\Lambda}\left(2yU_y+1\right)-1=8\pi y^2\tilde{p} \tag{3.16}$$

with boundary conditions:

$$U=0\ \ ,\ \Lambda=0\quad\text{at}\quad y=0. \tag{3.17}$$

Here:

$$\tilde{\rho}=\frac{\pi}{y^2}\int_{-\infty}^{\infty}\hat{E}\left[\int_0^\infty G\Phi d\Phi\right]dV, \tag{3.18}$$

$$\tilde{p}=\frac{\pi}{y^2}\int_{-\infty}^{\infty}\frac{V^2}{\hat{E}}\left[\int_0^\infty G\Phi d\Phi\right]dV. \tag{3.19}$$

The function G which is a solution of (3.13)-(3.19) is constant along the characteristic curves of (3.13) which are given by:

$$\frac{dy}{d\sigma}=y+e^{U-\Lambda}\frac{V}{\sqrt{V^2+\frac{1}{y^2}}}=y+e^{U-\Lambda}\frac{Vy}{\sqrt{V^2y^2+1}}, \tag{3.20}$$

$$\frac{dV}{d\sigma}=-V-\left(y\Lambda_yV+\frac{e^{U-\Lambda}U_y}{y}\sqrt{V^2y^2+1}-e^{U-\Lambda}\frac{1}{y^2\sqrt{V^2y^2+1}}\right), \tag{3.21}$$

$$\frac{d\Phi}{d\sigma}=\Phi. \tag{3.22}$$

In these equations σ is just a parameter that is used to parametrize the characteristic curves. Its precise definition will be given later in some specific cases.

The equations (3.20)-(3.22) can be integrated explicitly for any pair of functions $U = U(y)$, $\Lambda = \Lambda(y)$. Indeed, the first two equations can be rewritten as:

$$\frac{dy}{d\sigma} = e^{-\Lambda}\frac{\partial H}{\partial V}, \tag{3.23}$$

$$\frac{dV}{d\sigma} = -e^{-\Lambda}\frac{\partial H}{\partial y} \tag{3.24}$$

where:

$$H \equiv \frac{e^U}{y}\sqrt{V^2y^2+1} + yVe^{\Lambda}. \tag{3.25}$$

The trajectories in the (y, V)-plane associated to the solutions of (3.20), (3.21) are contained in the level sets:

$$H = h. \tag{3.26}$$

We will also need the self-similar formulation of the integrated form of the equation (3.7). In this case the function ζ in (3.6) has the functional dependence:

$$\zeta(r, v, t) = (-t)^2\,\Theta(y, V).$$

Notice that:

$$\Theta(y, V) = \int_0^\infty G\Phi d\Phi. \tag{3.27}$$

The function Θ satisfies:

$$\begin{aligned} & y\Theta_y - V\Theta_V - 2\Theta + e^{U-\Lambda}\frac{V}{\hat{E}}\Theta_y \\ & - \left(y\Lambda_y V + e^{U-\Lambda}U_y\hat{E} - e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)\Theta_V \\ & = 0 \end{aligned} \tag{3.28}$$

and:

$$\tilde{\rho} = \frac{\pi}{y^2}\int_{-\infty}^{\infty}\hat{E}\Theta dV, \tag{3.29}$$

$$\tilde{p} = \frac{\pi}{y^2}\int_{-\infty}^{\infty}\frac{V^2}{\hat{E}}\Theta dV. \tag{3.30}$$

The characteristic curves associated to (3.28) are (3.20), (3.21) and:

$$\frac{d\Theta}{d\sigma} = 2\Theta. \tag{3.31}$$

4 SINGULAR SELF-SIMILAR SOLUTIONS: GENERAL PROPERTIES.

The main goal of this paper is to construct a family of distributional solutions of (3.13)-(3.19) for which $G = G(y, V, \Phi)$ is a measure supported on some surfaces in the three-dimensional space with coordinates (y, V, Φ). In this section we will describe in a heuristic manner the argument yielding the construction of such solutions. The arguments will be made rigorous in the rest of the paper. The key idea behind the argument is that the problem can be transformed into a system of ordinary differential equations for the particular class of solutions described in this section.

Taking into account that the singularities of the distribution G might be expected to be propagated by characteristics it is natural to look for solutions of (3.13)-(3.19) of the form:

$$G(y, V, \Phi) = A(y, V, \Phi)\, \delta(H(y, V) - h) \tag{4.1}$$

satisfying (3.13) in the sense of distributions. Let us assume that $A,\ H$ have the differentiability properties required for all the following formal computations. Plugging (4.1) into (3.13) we obtain:

$$\begin{aligned} &(a(y,V)\, A_y + b(y,V)\, A_V + \Phi A_\Phi)\, \delta(H-h) \\ &+ A\,(a(y,V)\, H_y + b(y,V)\, H_V)\, \delta'(H-h) \\ &= 0 \end{aligned}$$

where:

$$a(y,V) \equiv y + e^{U-\Lambda}\frac{V}{\hat{E}} = e^{\Lambda}\frac{\partial H}{\partial V}, \tag{4.2}$$

$$b(y,V) \equiv -V - \left(y\Lambda_y V + e^{U-\Lambda} U_y \hat{E} - e^{U-\Lambda}\frac{1}{y^3 \hat{E}} \right) = -e^{\Lambda}\frac{\partial H}{\partial y}. \tag{4.3}$$

Notice that $a(y,V)\, H_y + b(y,V)\, H_V = 0$. Then:

$$(a(y,V)\, A_y + b(y,V)\, A_V + \Phi A_\Phi)\, \delta(H-h) = 0.$$

This equation is satisfied if:

$$a(y,V)\, A_y + b(y,V)\, A_V + \Phi A_\Phi = 0 \tag{4.4}$$

on the surface $\{H = h\} \times \mathbb{R}^+$. Let us assume that the curve $\{H = h\}$ can be parametrized, at least locally, using a parameter σ satisfying:

$$\begin{aligned} y &= y(\sigma) \quad , \quad V = V(\sigma), \\ \frac{dy(\sigma)}{d\sigma} &= a(y(\sigma), V(\sigma)) \quad , \quad \frac{dV(\sigma)}{d\sigma} = b(y(\sigma), V(\sigma)). \end{aligned} \tag{4.5}$$

Then the function A can be written on the surface $\{H = h\} \times \mathbb{R}^+$ as a function of the variables (σ, Φ). We can write:

$$A(y(\sigma), V(\sigma), \Phi) = \bar{A}(\sigma, \Phi) \quad \text{for} \quad (y, V, \sigma) \in \{H = h\} \times \mathbb{R}^+ \tag{4.6}$$

and using (4.5) we can rewrite (4.4) as:

$$\bar{A}_\sigma + \Phi \bar{A}_\Phi = 0. \tag{4.7}$$

Since the curves $\{H = h\}$ can be determined in terms of Θ alone it is convenient to compute this distribution explicitly. If G has the form (4.1) the distribution Θ defined in (3.27) is given by:

$$\Theta(y, V) = \beta \delta (H - h) \tag{4.8}$$

where:

$$\beta = \int_0^\infty A \Phi d\Phi.$$

Since A is given by (4.6) it follows that:

$$\beta = \beta(\sigma) = \int_0^\infty \bar{A}(\sigma, \Phi) \Phi d\Phi \quad \text{for} \quad (y, V) \in \{H = h\}. \tag{4.9}$$

We can compute $\beta(\sigma)$ along the curve $\{H = h\}$. To this end we multiply (4.7) by Φ and integrate in the Φ variable in the interval $[0, \infty)$. Then:

$$\beta_\sigma = 2\beta.$$

The function β then takes the form:

$$\beta(\sigma) = \beta_0 e^{2\sigma} \tag{4.10}$$

for some $\beta_0 \geq 0$.

In the rest of the paper we prove that there exist functions $\bar{A}(\sigma, \Phi)$ as in (4.6) and curves $\{H = h\}$ with Λ, U solving (3.15)-(3.17) and $\tilde{\rho}$, $\tilde{p}$ as in (3.18), (3.19) such that (4.1) solves (3.13) in the sense of distributions.

5 SINGULAR SELF-SIMILAR SOLUTIONS: DESCRIBING THEIR SUPPORT.

In this section we describe in a precise manner the form of the curved surface containing the support of the distribution G for the self-similar solutions constructed in this paper. Such a surface is contained in the surface $S = \gamma \times \mathbb{R}^+$, where $\gamma \subset \{(y, V) : y > 0 \, , \, V \in \mathbb{R}\}$ is an unbounded curve, at a strictly positive distance from the line $\{y = 0\} \equiv \{(y, V) : y = 0 \, , \, V \in \mathbb{R}\}$ with a discontinuity in its curvature at the point $(y_0, V_0) \in \gamma$ placed at the minimum distance from the line $\{y = 0\}$. In order to avoid such irregular curves it is more convenient to assume that the curve γ is the union of two analytic curves γ_1 and γ_2 that can be parametrized in the form:

$$\gamma_i = \{(y, V) : y_0 < y < \infty \ , \ V = V_i(y)\} \quad , \quad i = 1, 2 \tag{5.1}$$

where the functions $V_i(y)$ are analytic and satisfy:

$$\lim_{y \to y_0^+} V_1(y) = \lim_{y \to y_0^+} V_2(y) = V_0 = -\frac{1}{\sqrt{1 - y_0^2}}, \tag{5.2}$$

$$V_1(y) < V_2(y) \quad \text{for} \quad y_0 < y < \infty \tag{5.3}$$

for some $y_0 \in (0, 1)$. Since the curves γ_i are contained in the curve $\{H = h\}$ it follows that the functions $V_i(y)$ are the two roots of the equation:

$$\frac{e^U}{y} \sqrt{V^2 y^2 + 1} + y V e^\Lambda = h \tag{5.4}$$

assuming that such roots exist. Then:

$$V_1(y) = \frac{1}{(e^{2U} - y^2 e^{2\Lambda})} \left[-y e^\Lambda h - \sqrt{(y e^\Lambda h)^2 - (e^{2U} - y^2 e^{2\Lambda}) \left(\frac{e^{2U}}{y^2} - h^2 \right)} \right], \tag{5.5}$$

$$V_2(y) = \frac{1}{(e^{2U} - y^2 e^{2\Lambda})} \left[-y e^\Lambda h + \sqrt{(y e^\Lambda h)^2 - (e^{2U} - y^2 e^{2\Lambda}) \left(\frac{e^{2U}}{y^2} - h^2 \right)} \right]. \tag{5.6}$$

Notice that for such solutions the support of G in (4.1) is contained in the half-plane $\{y \geq y_0\}$. Therefore, $\rho(y) = p(y) = 0$ for $y < y_0$. Then (3.15)-(3.17) imply $U(y) = \Lambda(y) = 0$ for $y < y_0$.

Under suitable regularity assumptions for the curves γ_i near the point (y_0, V_0) that will be made precise below the functions U and Λ are continuous at the point $y = y_0$. In such a case (5.4) implies:

$$h = \frac{\sqrt{V_0^2 y_0^2 + 1}}{y_0} + y_0 V_0 = \frac{\sqrt{1 - y_0^2}}{y_0}. \tag{5.7}$$

We will prove later that it is possible to construct the desired curves γ_i, $i = 1, 2$, defined by means of (5.1) with the property that the following limits exist:

$$\lim_{y \to y_0^+} \frac{V_i(y) - V_0}{\sqrt{y - y_0}} = K_i \quad , \quad K_i \in \mathbb{R} \quad , \quad i = 1, 2 \ , \ K_1 < K_2. \tag{5.8}$$

Moreover, the quotients of the functions Λ and U by $\sqrt{y-y_0}$ also tend to finite limits. Let:

$$\lim_{y\to y_0^+}\frac{\Lambda(y)}{\sqrt{y-y_0}}=\theta_1\in\mathbb{R}, \tag{5.9}$$

$$\lim_{y\to y_0^+}\frac{U(y)}{\sqrt{y-y_0}}=\theta_2\in\mathbb{R}. \tag{5.10}$$

We parametrize the curve $\gamma=\{H=h\}$ as in the previous section using a parameter σ. We will denote a parameter of this kind on the curves $\gamma_1,\ \gamma_2$ by $\sigma_1,\ \sigma_2$ respectively. Due to (4.2), (4.5), (5.1) it follows that:

$$\frac{d\sigma_i}{dy}=\frac{1}{a(y,V_i(y))}=\frac{1}{y+e^{U-\Lambda}\frac{V_i(y)y}{\sqrt{(V_i(y))^2y^2+1}}}\ ,\quad i=1,2. \tag{5.11}$$

We will normalize the parameters $\sigma_i=\sigma_i(y)$ by means of the condition:

$$\sigma_i(y_0)=0\ ,\quad i=1,2. \tag{5.12}$$

Finally we remark that in order to obtain the functions U and Λ we need to prescribe the distribution Θ defined by (3.27). Using (4.8), (4.10) it then follows that:

$$\begin{aligned}\Theta(y,V)=&\frac{\beta_0\chi_{\{y>y_0\}}e^{2\sigma_1(y)}}{\left|\frac{\partial H}{\partial V}(y,V_1(y))\right|}\delta(V-V_1(y))\\ &+\frac{\beta_0\chi_{\{y>y_0\}}e^{2\sigma_2(y)}}{\left|\frac{\partial H}{\partial V}(y,V_2(y))\right|}\delta(V-V_2(y))\end{aligned} \tag{5.13}$$

where $\chi_{\{y>y_0\}}$ is the characteristic function of the half-plane $\{y>y_0\}$. Using (3.29), (3.30) it follows that:

$$\begin{aligned}\tilde{\rho}(y)=&\frac{\pi\beta_0\chi_{\{y>y_0\}}}{y^3}\left[\frac{e^{2\sigma_1(y)}}{\left|\frac{\partial H}{\partial V}(y,V_1(y))\right|}\sqrt{(V_1(y))^2y^2+1}\right.\\ &\left.+\frac{e^{2\sigma_2(y)}}{\left|\frac{\partial H}{\partial V}(y,V_2(y))\right|}\sqrt{(V_2(y))^2y^2+1}\right],\end{aligned} \tag{5.14}$$

$$\begin{aligned}\tilde{p}(y)=&\frac{\pi\beta_0\chi_{\{y>y_0\}}}{y}\left[\frac{e^{2\sigma_1(y)}}{\left|\frac{\partial H}{\partial V}(y,V_1(y))\right|}\frac{(V_1(y))^2}{\sqrt{(V_1(y))^2y^2+1}}\right.\\ &\left.+\frac{e^{2\sigma_2(y)}}{\left|\frac{\partial H}{\partial V}(y,V_2(y))\right|}\frac{(V_2(y))^2}{\sqrt{(V_2(y))^2y^2+1}}\right]\end{aligned} \tag{5.15}$$

and the functions U and Λ can then be obtained using the equations (3.15), (3.16).

Due to the dust-like character of the solutions considered in this paper, they exhibit a singular behaviour for $\tilde{\rho}$ and $\tilde{p}$ at the radius $y = y_0$. This singularity is due to the fact that at this point the radial velocity of the particles, in self-similar variables, vanishes. However, since the motion of the trajectories after they reach the singularity continues in a smooth way, and since $\tilde{\rho}$ and $\tilde{p}$ are integrable near this radius, this singularity can be expected to disappear if the dust-like assumption is relaxed and some thickness is given to the support of the distribution function in the phase space.

The main result of this paper is the following:

Theorem 1 *There exists $\varepsilon_0 > 0$ small such that, for any $y_0 \in (0, \varepsilon_0)$ there exist a value of $\beta_0 > 0$ and two curves $\gamma_1,\ \gamma_2$ that can be parametrized as in (5.1) with the functions $V_1(y),\ V_2(y)$ as in (5.5), (5.6) satisfying (5.2), (5.3), (5.8), the functions $U,\ \Lambda$ satisfying (3.15), (3.16) and (5.9), (5.10) with $\tilde{\rho},\ \tilde{p}$ as in (5.14), (5.15) and $\sigma_1,\ \sigma_2$ solving (5.11), (5.12).*

Using Theorem 1 it is possible to obtain distributional solutions of the problem (3.13)-(3.19). In order to make the definition of the distribution G in (4.1) precise we use (4.6), (4.7). Let us prescribe a smooth function $\bar{A}_0(\Phi)$ in $\Phi \in (0, \infty)$. Taking into account (4.7) we can then define:

$$\bar{A}(\sigma, \Phi) = \bar{A}_0\left(e^{-\sigma}\Phi\right).$$

Using the structure of the curves $\gamma_1,\ \gamma_2$ it would then follow that the distribution G in (4.1) would be given by:

$$\begin{aligned} G(y, V, \Phi) = & \frac{\bar{A}_0\left(e^{-\sigma_1(y)}\Phi\right)\chi_{\{y>y_0\}}}{\left|\frac{\partial H}{\partial V}(y, V_1(y))\right|}\delta(V - V_1(y)) \\ & + \frac{\bar{A}_0\left(e^{-\sigma_2(y)}\Phi\right)\chi_{\{y>y_0\}}}{\left|\frac{\partial H}{\partial V}(y, V_2(y))\right|}\delta(V - V_2(y)). \end{aligned} \tag{5.16}$$

We then have the following result:

Theorem 2 *Suppose that the function $\bar{A}_0(\cdot) \in C_0^1(0, \infty)$ satisfies*

$$\int_0^\infty \bar{A}_0(\Phi)\,\Phi d\Phi = \beta_0. \tag{5.17}$$

Let us define a Radon measure $G \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+)$ by means of (5.16) with the functions $V_1(\cdot),\ V_2(\cdot),\ \sigma_1(\cdot),\ \sigma_2(\cdot)$ as in Theorem 1. Then the functions $\tilde{\rho},\ \tilde{p}$ defined (3.18), (3.19) belong to the spaces $L^p_{loc}(0, \infty)$ for $1 \leq p < 2$. The functions $\Lambda,\ U$ defined by means of (3.15)-(3.17) belong to $W^{1,p}_{loc}(0, \infty)$ for $1 \leq p < 2$. The measure G satisfies (3.13) in the sense of distributions.

Remark 3 *The space* $C_0^1(0,\infty)$ *is the space of compactly supported continuously differentiable functions and the space* $\mathcal{M}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+)$ *is the space of Radon measures on* $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$. *It is not necessary to require* $A_0(\cdot)$ *to be compactly supported. Actually this condition could be replaced by assumptions of fast enough decay near the origin and infinity.*

Remark 4 *It is worth noticing that the functions* $\tilde{\rho}$, $\tilde{p}$ *associated to the distribution* G *have an integrable singularity as* $y\to y_0^+$.

In the rest of this section we will prove Theorem 2. Theorem 1 will be proved in the remaining sections of the paper using a shooting argument and refined asymptotics of the solutions for y_0 small. The following auxiliary result will be used in the proof of Theorem 2 and it will be proved in Section 6. We remark that Theorem 2 will not be used in either the proof of Theorem 1 or that of Proposition 5 below.

Proposition 5 *The curves* γ_1, γ_2 *whose existence has been proved in Theorem 1 satisfy the following conditions:*

$$\lim_{y\to y_0^+}\frac{\frac{\partial H}{\partial V}(y,V_1(y))}{\sqrt{y-y_0}}=L_1\ \ ,\ \ \lim_{y\to y_0^+}\frac{\frac{\partial H}{\partial V}(y,V_2(y))}{\sqrt{y-y_0}}=L_2 \tag{5.18}$$

for some constants $L_1<L_2$.

Proof of Theorem 2. Using (3.27), (5.16) and (5.17) we obtain:

$$\begin{aligned}\Theta(y,V)=\int_0^\infty G\Phi d\Phi&=\frac{\beta_0 e^{2\sigma_1(y)}\chi_{\{y>y_0\}}}{\left|\frac{\partial H}{\partial V}(y,V_1(y))\right|}\delta(V-V_1(y))\\&+\frac{\beta_0 e^{2\sigma_2(y)}\chi_{\{y>y_0\}}}{\left|\frac{\partial H}{\partial V}(y,V_2(y))\right|}\delta(V-V_2(y)).\end{aligned} \tag{5.19}$$

We can then compute $\tilde{\rho}$, $\tilde{p}$ using (3.29), (3.30):

$$\tilde{\rho}(y)=\frac{\pi\beta_0\chi_{\{y>y_0\}}}{y^3}\left[\frac{e^{2\sigma_1(y)}\sqrt{1+y^2(V_1(y))^2}}{\left|\frac{\partial H}{\partial V}(y,V_1(y))\right|}+\frac{e^{2\sigma_2(y)}\sqrt{1+y^2(V_2(y))^2}}{\left|\frac{\partial H}{\partial V}(y,V_2(y))\right|}\right], \tag{5.20}$$

$$\begin{aligned}\tilde{p}(y)=\frac{\pi\beta_0\chi_{\{y>y_0\}}}{y}&\left[\frac{e^{2\sigma_1(y)}(V_1(y))^2}{\left|\frac{\partial H}{\partial V}(y,V_1(y))\right|\sqrt{1+y^2(V_1(y))^2}}\right.\\&\left.+\frac{e^{2\sigma_2(y)}(V_2(y))^2}{\left|\frac{\partial H}{\partial V}(y,V_2(y))\right|\sqrt{1+y^2(V_2(y))^2}}\right].\end{aligned} \tag{5.21}$$

Using (5.8)-(5.10), (5.18), we obtain:

$$|\tilde{\rho}(y)| + |\tilde{p}(y)| \leq \frac{C\chi_{\{y>y_0\}}}{\sqrt{y-y_0}} \tag{5.22}$$

whence the estimate $\tilde{\rho},\ \tilde{p} \in L^p_{loc}(0,\infty),\ 1 \leq p < 2$, in the theorem follows. On the other hand (3.15)-(3.17) imply:

$$\Lambda = -\frac{1}{2}\log\left(1 - \frac{8\pi}{y}\int_{y_0}^{y} \xi^2 \tilde{\rho}(\xi)\, d\xi\right), \tag{5.23}$$

$$U = \int_{y_0}^{y} \frac{\left[\left(8\pi\xi^2\tilde{p}(\xi)+1\right)e^{2\Lambda(\xi)} - 1\right]}{2\xi} d\xi. \tag{5.24}$$

Due to Theorem 1 the functions $\Lambda,\ U$ are bounded for any finite value $y > 0$. On the other hand, (5.23), (5.24) imply $\Lambda,\ U \in W^{1,p}_{loc}(0,\infty),\ 1 \leq p < 2$.

In order to conclude the proof of Theorem 2 it only remains to prove that G solves (3.13) in the sense of distributions. This is equivalent to showing that:

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+}\Bigg[-(y\varphi)_y + (V\varphi)_V - (\Phi\varphi)_\Phi - \left(e^{U-\Lambda}\frac{V}{\hat{E}}\varphi\right)_y \\ &+ \left(\left(y\Lambda_y V + e^{U-\Lambda}U_y\hat{E} - e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)\varphi\right)_V\Bigg] G\, dy\, dV\, d\Phi \\ &= 0\end{aligned} \tag{5.25}$$

for any $\varphi = \varphi(y,V,\Phi) \in C_0^\infty(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+)$. Using (5.16) we can rewrite (5.25) as:

$$\begin{aligned}J \equiv \sum_{i=1}^{2}\int_{y_0}^{\infty}\int_0^{\infty}&\Bigg[-(y\varphi)_y + (V\varphi)_V - (\Phi\varphi)_\Phi - \left(e^{U-\Lambda}\frac{V}{\hat{E}}\varphi\right)_y + \\ &\left(\left(y\Lambda_y V + e^{U-\Lambda}U_y\hat{E} - e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)\varphi\right)_V\Bigg]\Bigg|_{(y,V_i(y),\Phi)} \frac{\bar{A}_0\left(e^{-\sigma_i(y)}\Phi\right)}{\left|\frac{\partial H}{\partial V}(y,V_i(y))\right|} d\Phi\, dy \\ &= 0\end{aligned} \tag{5.26}$$

and making the change of variables $e^{-\sigma_i(y)}\Phi \to \Phi$ we can transform J into:

$$\begin{aligned}J \equiv \sum_{i=1}^{2}\int_{y_0}^{\infty}\int_0^{\infty}&\Bigg[-(y\varphi)_y + (V\varphi)_V - (\Phi\varphi)_\Phi - \left(e^{U-\Lambda}\frac{V}{\hat{E}}\varphi\right)_y + \\ &\left(\left(y\Lambda_y V + e^{U-\Lambda}U_y\hat{E} - e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)\varphi\right)_V\Bigg]\Bigg|_{\left(y,V_i(y),\Phi e^{\sigma_i(y)}\right)} \frac{\bar{A}_0(\Phi)\, e^{\sigma_i(y)}}{\left|\frac{\partial H}{\partial V}(y,V_i(y))\right|} dy\, d\Phi\end{aligned} \tag{5.27}$$

Notice that we can write:

$$
\begin{aligned}
F \equiv & -(y\varphi)_y + (V\varphi)_V - (\Phi\varphi)_\Phi - \left(e^{U-\Lambda}\frac{V}{\hat{E}}\varphi\right)_y \\
& + \left(\left(y\Lambda_y V + e^{U-\Lambda}U_y\hat{E} - e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)\varphi\right)_V \\
& = -y\varphi_y + V\varphi_V - (\Phi\varphi)_\Phi - \left(e^{U-\Lambda}\frac{Vy}{\sqrt{1+V^2y^2}}\varphi\right)_y \\
& + \left(\left(y\Lambda_y V + \frac{e^{U-\Lambda}U_y}{y}\sqrt{1+V^2y^2} - e^{U-\Lambda}\frac{1}{y^2\sqrt{1+V^2y^2}}\right)\varphi\right)_V
\end{aligned}
$$

and, using Leibniz's rule:

$$
\begin{aligned}
F = & -y\Lambda_y\varphi - ye^{\Lambda}\left(e^{-\Lambda}\varphi\right)_y + V\varphi_V - \Phi\varphi_\Phi - \varphi - U_y e^{U-\Lambda}\frac{Vy}{\sqrt{1+V^2y^2}}\varphi \\
& - e^{U-\Lambda}\frac{V}{\sqrt{1+V^2y^2}}\varphi + e^{U-\Lambda}\frac{V^3y^2}{(1+V^2y^2)^{\frac{3}{2}}}\varphi - e^{U}\frac{Vy}{\sqrt{1+V^2y^2}}\left(e^{-\Lambda}\varphi\right)_y \\
& + \left(y\Lambda_y + U_y e^{U-\Lambda}\frac{Vy}{\sqrt{1+V^2y^2}} + e^{U-\Lambda}\frac{V}{(1+V^2y^2)^{\frac{3}{2}}}\right)\varphi \\
& + \left(y\Lambda_y V + \frac{e^{U-\Lambda}U_y}{y}\sqrt{1+V^2y^2} - e^{U-\Lambda}\frac{1}{y^2\sqrt{1+V^2y^2}}\right)\varphi_V .
\end{aligned}
$$

After some cancellations:

$$
\begin{aligned}
F(y,V,\Phi) = & -\left(ye^{\Lambda} + e^{U}\frac{Vy}{\sqrt{1+V^2y^2}}\right)\left(e^{-\Lambda}\varphi\right)_y - \Phi e^{\Lambda}\left(e^{-\Lambda}\varphi\right)_\Phi - \varphi \\
& + \left(y\Lambda_y Ve^{\Lambda} + Ve^{\Lambda} + \frac{e^{U}U_y}{y}\sqrt{1+V^2y^2} - e^{U}\frac{1}{y^2\sqrt{1+V^2y^2}}\right)\left(e^{-\Lambda}\varphi\right)_V .
\end{aligned}
\tag{5.28}
$$

Then (5.27) can be rewritten as:

$$
\sum_{i=1}^{2}\int_{y_0}^{\infty}\int_0^{\infty} F\left(y, V_i(y), \Phi e^{\sigma_i(y)}\right)\frac{\bar{A}_0(\Phi)\, e^{\sigma_i(y)}}{\left|\frac{\partial H}{\partial V}(y, V_i(y))\right|}d\Phi dy = 0.
$$

Due to Proposition 5 as well as the fact that the curves $\gamma_1,\ \gamma_2$ are globally defined it follows that:

$$
\left|\frac{\partial H}{\partial V}(y, V_i(y))\right| = (-1)^{i-1}\frac{\partial H}{\partial V}(y, V_i(y)).
$$

Then:

$$
J = \sum_{i=1}^{2}(-1)^{i-1}\int_{y_0}^{\infty}\int_0^{\infty} F\left(y, V_i(y), \Phi e^{\sigma_i(y)}\right)\frac{\bar{A}_0(\Phi)\, e^{\sigma_i(y)}}{\frac{\partial H}{\partial V}(y, V_i(y))}d\Phi dy. \tag{5.29}
$$

Using (3.25) and (5.28) we obtain:

$$\frac{F(y,V,\Phi)}{ye^{\Lambda}+e^{U}\frac{Vy}{\sqrt{V^2y^2+1}}} = -\left(e^{-\Lambda}\varphi\right)_y - \frac{\Phi}{y+e^{U-\Lambda}\frac{Vy}{\sqrt{V^2y^2+1}}}\left(e^{-\Lambda}\varphi\right)_{\Phi}$$
$$-\frac{1}{y+e^{U-\Lambda}\frac{Vy}{\sqrt{V^2y^2+1}}}\left(e^{-\Lambda}\varphi\right)$$
$$+\frac{1}{y+e^{U-\Lambda}\frac{Vy}{\sqrt{V^2y^2+1}}}\left(y\Lambda_y V+V+\frac{e^{U-\Lambda}U_y}{y}\sqrt{1+V^2y^2}\right)\left(e^{-\Lambda}\varphi\right)_V$$
$$-\frac{e^{U-\Lambda}}{y+e^{U-\Lambda}\frac{Vy}{\sqrt{V^2y^2+1}}}\frac{1}{y^2\sqrt{1+V^2y^2}}\left(e^{-\Lambda}\varphi\right)_V.$$

Equations (4.3), (4.4) and (5.11) give:

$$\frac{dV_i}{dy}(y) = -\frac{1}{y+e^{U-\Lambda}\frac{Vy}{\sqrt{V^2y^2+1}}}\left(y\Lambda_y V+V+\frac{e^{U-\Lambda}U_y}{y}\sqrt{1+V^2y^2}\right.$$
$$\left.-e^{U-\Lambda}\frac{1}{y^2\sqrt{1+V^2y^2}}\right). \tag{5.30}$$

Therefore

$$F\left(y,V_i(y),\Phi e^{\sigma_i(y)}\right)\frac{e^{\sigma_i(y)}}{\frac{\partial H}{\partial V}(y,V_i(y))}$$
$$= e^{\sigma_i(y)}\left[-\left(e^{-\Lambda}\varphi\right)_y - \Phi\frac{d\sigma_i}{dy}\left(e^{-\Lambda}\varphi\right)_{\Phi}\right.$$
$$\left.\left.-\frac{d\sigma_i}{dy}\left(e^{-\Lambda}\varphi\right)-\frac{dV_i}{dy}(y)\left(e^{-\Lambda}\varphi\right)\right]\right|_{V\left(y,V_i(y),\Phi e^{\sigma_i(y)}\right)}.$$

It then follows, using the chain rule that:

$$\frac{d}{dy}\left(e^{\sigma_i(y)}e^{-\Lambda(y)}\varphi\left(y,V_i(y),\Phi e^{\sigma_i(y)}\right)\right) = -F\left(y,V_i(y),\Phi e^{\sigma_i(y)}\right)\frac{e^{\sigma_i(y)}}{\frac{\partial H}{\partial V}(y,V_i(y))}.$$

Formula (5.29) then becomes:

$$J=\sum_{i=1}^{2}(-1)^{i-1}\int_{y_0}^{\infty}\int_0^{\infty}\bar{A}_0(\Phi)\frac{d}{dy}\left(e^{\sigma_i(y)}e^{-\Lambda(y)}\varphi\left(y,V_i(y),\Phi e^{\sigma_i(y)}\right)\right)d\Phi dy$$

or, equivalently:

$$J=\sum_{i=1}^{2}(-1)^{i-1}\int_0^{\infty}\bar{A}_0(\Phi)\left(e^{\sigma_i(y_0)}e^{-\Lambda(y_0)}\varphi\left(y_0,V_i\left(y_0^+\right),\Phi e^{\sigma_i(y_0)}\right)\right)d\Phi$$

and using (5.9), (5.10), (5.12):

$$J \equiv \sum_{i=1}^{2} (-1)^{i-1} \int_0^\infty \bar{A}_0 (\Phi) \varphi (y_0, V_i (y_0), \Phi) d\Phi.$$

Due to the fact that $\varphi \left(y_0, V_i \left(y_0^+\right), \Phi\right) = \varphi (y_0, V_0, \Phi)$ for $i = 1, 2$ we have $J = 0$ and (5.26) follows. This concludes the proof of the theorem. ■

We now remark that it is possible to derive some detailed information about the behaviour of the curves $\gamma_1,\ \gamma_2$ as $y \to \infty$.

Theorem 6 *Suppose that the curves $\gamma_1,\ \gamma_2$ are as in Theorem 1. Then, the following asymptotic formulas hold:*

$$\begin{aligned}
U &= \log\left(\frac{y}{y_0}\right) + \log\left(\sqrt{1-y_0^2}\right) + o(1) \ \ as \ \ y \to \infty, \\
\Lambda &\to \log\left(\sqrt{3}\right) \ \ as \ \ y \to \infty, \\
V_1 &= -\frac{2y_0\sqrt{3\left(1-y_0^2\right)}}{\left(1-4y_0^2\right)y}(1+o(1)) \ \ as \ \ y \to \infty, \\
V_2 &= -\frac{\sqrt{1-y_0^2}}{\sqrt{3}y_0}\frac{C_1}{y}\left(\frac{y_0}{y}\right)^2 (1+o(1)) \ \ as \ \ y \to \infty
\end{aligned}$$

for a suitable constant $C_1 \in \mathbb{R}$.

Notice that the asymptotic behaviour of the solutions in Theorem 6 shows that the support of these solutions approaches the line $\{V = 0\}$ away from the self-similar region (i.e. for $y \to \infty$). This is the one of the main differences between the solutions described in this paper and the ones in [17].

It is relevant to notice that the spacetime described by the solutions in Theorem 6 exhibits curvature singularities and not just coordinate singularities. To this end we use Kretschmann scalar (cf. [25]):

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 4K^2 + \frac{16m^2}{r^6} + 12r^{-2}\nabla_a \nabla_b r \nabla^a \nabla^b r$$

where K is the gaussian curvature of the quotient of the spacetime by the symmetry group and m is the Hawking mass that can be computed by means of:

$$m = \frac{r}{2}\left(1 - \partial_a r \partial^a r\right).$$

Combining (2.1), (3.11), (3.12) we obtain the following self-similar form for the Hawking mass:

$$m = \frac{r}{2}\left(1 - e^{-2\Lambda\left(\frac{r}{(-t)}\right)}\right)$$

and therefore, it follows from Theorem 6 that:

$$m \sim \frac{r}{3} \quad \text{for} \quad \frac{r}{(-t)} \text{ sufficiently large.}$$

On the other hand, the last term in the Kretschmann scalar can be written as (cf. [10], Appendix A):

$$24r^{-2}\left(\frac{1}{2r}\left(k - \nabla_b r \nabla^c r\right) + 2\pi r \operatorname{tr} T\right)^2 + 96\pi^2 \left(T_{ab} - \frac{\operatorname{tr} T}{2} g_{ab}\right)\left(T^{ab} - \frac{\operatorname{tr} T}{2} g^{ab}\right).$$

The last term turns out to be positive for any matter model satisfying the dominant energy condition, which includes in particular the case of Vlasov matter. Therefore $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \geq \frac{16m^2}{r^6}$ and so the curvature becomes singular as $r \to 0$ for a fixed large value of $\frac{r}{(-t)}$.

We remark that the solutions which have been derived do not provide an example of violation of the cosmic censorship hypothesis for Vlasov matter, because the spacetimes concerned are not asymptotically flat as $r \to \infty$. Moreover, it turns out that the region contained inside the light cone reaching the singular point at $r = 0,\ t = 0^-$ in the spacetime described by Theorem 6 is dependent on the data on the whole region with $0 \leq r < \infty$. This implies that a gluing of this spacetime with another one causally disconnected from the singular point is not possible, because this would require doing some gluing along regions where $r = \infty$. In order to check these statements it is convenient to rewrite the metric (2.1) in double null coordinates. Notice that (2.1), (3.11) and (3.12) yield the following self-similar structure for the metric:

$$ds^2 = -e^{2U\left(\frac{r}{(-t)}\right)} dt^2 + e^{2\Lambda\left(\frac{r}{(-t)}\right)} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

The double null coordinates are then just the constants of integration associated to the pair of differential equations:

$$\begin{aligned} -e^{U\left(\frac{r}{(-t)}\right)} dt + e^{\Lambda\left(\frac{r}{(-t)}\right)} dr &= 0, \\ e^{U\left(\frac{r}{(-t)}\right)} dt + e^{\Lambda\left(\frac{r}{(-t)}\right)} dr &= 0. \end{aligned}$$

The solutions of these equations can be written in terms of two integration constants u and v that will define the double null coordinates. The particular choice of coordinates has been made in order to obtain u and v taking values in compact sets:

$$\begin{aligned} \operatorname{arctanh}(u) &= \log(-t) + \int_0^y \frac{e^{\Lambda(\xi) - U(\xi)}}{1 + \xi e^{\Lambda(\xi) - U(\xi)}} d\xi \\ \operatorname{arctanh}(v) &= \log(-t) - \int_0^y \frac{e^{\Lambda(\xi) - U(\xi)}}{1 - \xi e^{\Lambda(\xi) - U(\xi)}} d\xi \end{aligned}$$

In the region close to the centre (i.e. $y << 1$) the structure of the metric is similar to Minkowski. On the other hand, Theorem 6 yields the following asymptotics for $r >> (-t)$:

$$\operatorname{arctanh}(v) \sim \log(-t) + \frac{\sqrt{3}y_0}{\sqrt{1-y_0^2}} \log\left(\frac{r}{(-t)}\right),$$
$$\operatorname{arctanh}(u) \sim -\log(-t) + \frac{\sqrt{3}y_0}{\sqrt{1-y_0^2}} \log\left(\frac{r}{(-t)}\right).$$

The light cone approaching the singular point is described in these coordinates by the line $u = 1$. Notice along such a line, for v of order one we would have $r = \infty$, whence the assertion above follows.

For these reasons a spacetime behaving asymptotically as Minkowski cannot be obtained gluing the self-similar solution obtained in this paper with a spacetime causally disconnected from the singular point. This kind of gluing might be possible for non self-similar solutions of the Einstein equations behaving asymptotically near the singular point like those described in this paper. However, such an analysis is beyond the scope of this paper.

6 PROOF OF THEOREM 1.

The strategy used to prove Theorem 1 is the following. We first transform the original problem (3.15), (3.16), (5.2), (5.3), (5.5), (5.6), (5.8)-(5.12), (5.14), (5.15) into a family of four-dimensional autonomous systems depending on the parameter β_0 by means of a change of variables. It will be shown that proving Theorem 1 is equivalent to finding an orbit for this system connecting two specific points P_0, P_1 of the four-dimensional phase space. The point P_1 is a unstable saddle point with an associated three-dimensional stable manifold $\mathcal{M} = \mathcal{M}(\beta_0)$ that can be described in detail in the limit $y_0 \to 0$. A shooting argument will show that for a suitable choice of the parameter β_0 the manifold $\mathcal{M}(\beta_0)$ contains the point P_0. In the rest of this section we give the details of this argument.

6.1 Reduction of the problem to an autonomous system.

Instead of the set of variables $(y, U, \Lambda, V_i, \sigma_i)$ it is more convenient to use the set of variables $(s, u, \Lambda, \zeta_i, Q_i)$ where:

$$s = \log\left(\frac{y}{y_0}\right) \ , \ U = \log\left(\frac{y}{y_0}\right) + u \ , \ \zeta_i = yV_i \ , \ Q_i = \frac{y_0}{y} e^{\sigma_i} \ , \ i = 1, 2. \quad (6.1)$$

Then, the evolution equations (3.15), (3.16), (5.4), (5.11) become:

$$e^u\sqrt{\zeta_i^2+1}+y_0\zeta_i e^{\Lambda}=\sqrt{1-y_0^2}\ \ ,\ \ i=1,2\ \ ,\ \ \zeta_1<\zeta_2, \tag{6.2}$$

$$\frac{dQ_i}{ds}=-\frac{e^uQ_i\zeta_i}{\left[y_0e^{\Lambda}\sqrt{(\zeta_i)^2+1}+\zeta_ie^u\right]}\ \ ,\ \ i=1,2, \tag{6.3}$$

$$e^{-2\Lambda}\left(2\Lambda_s-1\right)+1=\frac{\theta}{2}\left[\frac{Q_1^2\left[\zeta_1^2+1\right]}{\left|\zeta_1e^u+y_0e^{\Lambda}\sqrt{\zeta_1^2+1}\right|}+\frac{Q_2^2\left[\zeta_2^2+1\right]}{\left|\zeta_2e^u+y_0e^{\Lambda}\sqrt{\zeta_2^2+1}\right|}\right], \tag{6.4}$$

$$e^{-2\Lambda}\left(2u_s+3\right)-1=\frac{\theta}{2}\left[\frac{Q_1^2\left(\zeta_1\right)^2}{\left|\zeta_1e^u+y_0e^{\Lambda}\sqrt{\zeta_1^2+1}\right|}+\frac{Q_2^2\left(\zeta_2\right)^2}{\left|\zeta_2e^u+y_0e^{\Lambda}\sqrt{\zeta_2^2+1}\right|}\right] \tag{6.5}$$

where

$$\theta=\frac{16\pi^2\beta_0}{y_0}. \tag{6.6}$$

The initial conditions (3.17), (5.12) imply:

$$u=0\ \ ,\ \ \Lambda=0\ \ ,\ \ Q_i=1\ \ ,\ \ i=1,2\ \ \text{at}\ \ s=0. \tag{6.7}$$

Notice that the system (6.3)-(6.5) with ζ_i as in (6.2) is a four-dimensional autonomous system of equations for the unknown functions (Q_1,Q_2,Λ,u). Notice however that the system seems to becomes singular if the variables (Q_1,Q_2,Λ,u) approach the values in (6.7) due to the vanishing of the denominators in (6.4), (6.5). To treat these singularities we rewrite the terms $\left[y_0e^{\Lambda}\sqrt{(\zeta_i)^2+1}+\zeta_ie^u\right]$. Notice that (6.2) implies:

$$\zeta_i=\frac{1}{\left(1-y_0^2e^{2(\Lambda-u)}\right)}\left[-y_0\sqrt{1-y_0^2}e^{\Lambda-2u}\mp Z\right]\ ,\ i=1,2, \tag{6.8}$$

$$Z=\sqrt{\left(e^{-2u}\left(1-y_0^2\right)-1\right)\left(1-y_0^2e^{2(\Lambda-u)}\right)+y_0^2\left(1-y_0^2\right)e^{2(\Lambda-2u)}}. \tag{6.9}$$

Then:

$$y_0e^{\Lambda}\sqrt{(\zeta_i)^2+1}+\zeta_ie^u=\mp e^uZ\ \ ,\ i=1,2$$

and the system of equations (6.3)-(6.5) becomes:

$$\frac{dQ_1}{ds} = \frac{Q_1\zeta_1}{Z}, \tag{6.10}$$

$$\frac{dQ_2}{ds} = -\frac{Q_2\zeta_2}{Z}, \tag{6.11}$$

$$e^{-2\Lambda}\left(2\Lambda_s - 1\right) + 1 = \frac{\theta e^{-u}}{2}\left[\frac{Q_1^2}{Z}\left[\zeta_1^2+1\right] + \frac{Q_2^2}{Z}\left[\zeta_2^2+1\right]\right], \tag{6.12}$$

$$e^{-2\Lambda}\left(2u_s + 3\right) - 1 = \frac{\theta e^{-u}}{2}\left[\frac{Q_1^2\zeta_1^2}{Z} + \frac{Q_2^2\zeta_2^2}{Z}\right]. \tag{6.13}$$

We now eliminate the variables Λ, u in (6.3)-(6.5) and replace them by the functions Z and G where Z is as in (6.9) and G is defined by means of:

$$G = e^{-2\Lambda}. \tag{6.14}$$

Then (6.12) becomes:

$$G_s = 1 - G - \frac{\theta e^{-u}}{2}\left[\frac{Q_1^2}{Z}\left[\zeta_1^2+1\right] + \frac{Q_2^2}{Z}\left[\zeta_2^2+1\right]\right]. \tag{6.15}$$

On the other hand (6.9) implies:

$$e^{-2u} = \frac{Z^2+1}{\left[\left(1-y_0^2\right) + y_0^2 e^{2\Lambda}\right]} = \frac{\left(Z^2+1\right)G}{\left[G + y_0^2\left(1-G\right)\right]} \tag{6.16}$$

whence:

$$u = -\frac{1}{2}\log\left(\frac{\left(Z^2+1\right)G}{\left[G + y_0^2\left(1-G\right)\right]}\right).$$

Differentiating this formula we obtain:

$$u_s = -\frac{ZZ_s}{\left(Z^2+1\right)} - \frac{y_0^2}{2}\frac{G_s}{\left[G + y_0^2\left(1-G\right)\right]G}.$$

Eliminating u_s from this formula using (6.13), (6.15) we obtain:

$$ZZ_s = \left(\frac{3}{2} - \frac{1}{2G} - \Delta\right)\left(Z^2+1\right) \tag{6.17}$$

where:

$$\Delta \equiv \frac{y_0^2}{2}\frac{G_s}{\left[G + y_0^2\left(1-G\right)\right]G} + \frac{\theta e^{-u}}{4G}\left[\frac{Q_1^2\left(\zeta_1\right)^2}{Z} + \frac{Q_2^2\left(\zeta_2\right)^2}{Z}\right].$$

Using (6.15) it then follows, after some computations, that:

$$\begin{aligned} 4GZ\left[G + y_0^2\left(1-G\right)\right]\Delta &= 2\left(1-G\right)y_0^2 Z \\ &+ \theta e^{-u}\left[-y_0^2\left[Q_1^2\left[\zeta_1^2+1\right] + Q_2^2\left[\zeta_2^2+1\right]\right]\right. \\ &\left.+\left[Q_1^2\left(\zeta_1\right)^2 + Q_2^2\left(\zeta_2\right)^2\right]\left[G + y_0^2\left(1-G\right)\right]\right]. \end{aligned} \tag{6.18}$$

The last bracket in (6.18) can be rewritten as:

$$\begin{aligned}&\left[-y_0^2\left[Q_1^2\left[\zeta_1^2+1\right]+Q_2^2\left[\zeta_2^2+1\right]\right]+\left[Q_1^2\left(\zeta_1\right)^2+Q_2^2\left(\zeta_2\right)^2\right]\left[G+y_0^2\left(1-G\right)\right]\right]\\&=Q_1^2\left[\zeta_1^2-y_0^2\left(\zeta_1^2+1\right)\right]+Q_2^2\left[\zeta_2^2-y_0^2\left(\zeta_2^2+1\right)\right]\\&+\left(1-y_0^2\right)\left[Q_1^2\left(\zeta_1\right)^2+Q_2^2\left(\zeta_2\right)^2\right]\left(G-1\right).\end{aligned}\tag{6.19}$$

Using (6.8) we obtain:

$$\begin{aligned}\left[\zeta_i^2-y_0^2\left(\zeta_i^2+1\right)\right]&=\frac{\left(1-y_0^2\right)Z^2}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\pm\frac{2y_0\left(1-y_0^2\right)^{\frac{3}{2}}e^{\Lambda-2u}}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}Z\\&+y_0^2\left[\frac{\left(1-y_0^2\right)^2e^{2(\Lambda-2u)}}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}-1\right]\quad,\quad i=1,2.\end{aligned}\tag{6.20}$$

Plugging (6.20) into (6.19) it then follows that:

$$\begin{aligned}&\left[-y_0^2\left[Q_1^2\left[\zeta_1^2+1\right]+Q_2^2\left[\zeta_2^2+1\right]\right]+\left[Q_1^2\left(\zeta_1\right)^2+Q_2^2\left(\zeta_2\right)^2\right]\left[G+y_0^2\left(1-G\right)\right]\right]\\&=\frac{\left(1-y_0^2\right)Z^2}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left(Q_1^2+Q_2^2\right)+\frac{2y_0\left(1-y_0^2\right)^{\frac{3}{2}}e^{\Lambda-2u}}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left(Q_1^2-Q_2^2\right)Z\\&+y_0^2\left[\frac{\left(1-y_0^2\right)^2e^{2(\Lambda-2u)}}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}-1\right]\left(Q_1^2+Q_2^2\right)\\&+\left(1-y_0^2\right)\left[Q_1^2\left(\zeta_1\right)^2+Q_2^2\left(\zeta_2\right)^2\right]\left(G-1\right)\end{aligned}$$

and using (6.18) we arrive at:

$$\begin{aligned}\Delta&=\frac{\left(1-G\right)y_0^2}{2G\left[G+y_0^2\left(1-G\right)\right]}\\&+\frac{\theta e^{-u}}{4G\left[G+y_0^2\left(1-G\right)\right]}\left[\frac{\left(1-y_0^2\right)Z}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left(Q_1^2+Q_2^2\right)\right.\\&\left.+\frac{2y_0\left(1-y_0^2\right)^{\frac{3}{2}}e^{\Lambda-2u}}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left(Q_1^2-Q_2^2\right)+\frac{1}{Z}\Phi\right]\end{aligned}\tag{6.21}$$

where:

$$\begin{aligned}\Phi&=y_0^2\left[\frac{\left(1-y_0^2\right)^2e^{2(\Lambda-2u)}}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}-1\right]\left(Q_1^2+Q_2^2\right)\\&+\left(1-y_0^2\right)\left[Q_1^2\left(\zeta_1\right)^2+Q_2^2\left(\zeta_2\right)^2\right]\left(G-1\right).\end{aligned}\tag{6.22}$$

In order to obtain analytic solutions it is convenient to introduce the change of variables:

$$ds=2GZd\chi\quad,\quad\chi=0\ \text{ at }\ s=0.\tag{6.23}$$

Then the system (6.10), (6.11), (6.15), (6.17) becomes:

$$\frac{dQ_1}{d\chi} = 2GQ_1\zeta_1, \tag{6.24}$$

$$\frac{dQ_2}{d\chi} = -2GQ_2\zeta_2, \tag{6.25}$$

$$\frac{dG}{d\chi} = 2G\left[Z(1-G) - \frac{\theta e^{-u}}{2}\left[Q_1^2\left[\zeta_1^2+1\right] + Q_2^2\left[\zeta_2^2+1\right]\right]\right], \tag{6.26}$$

$$\frac{dZ}{d\chi} = (3G - 1 - 2G\Delta)\left(Z^2+1\right) \tag{6.27}$$

with the initial conditions:

$$Q_1 = Q_2 = 1 \ \ , \ \ G = 1 \ \ , \ \ Z = 0 \ \ , \text{ at } \chi = 0. \tag{6.28}$$

We can further simplify Φ in (6.22) using (6.8):

$$\begin{aligned} \Phi &= y_0^2\left[\frac{\left(1-y_0^2\right)^2 e^{2(\Lambda-2u)}}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2} - 1\right]\left(Q_1^2+Q_2^2\right) \\ &+ \frac{\left(1-y_0^2\right)(G-1)}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2}\left[\left[y_0^2\left(1-y_0^2\right)e^{2(\Lambda-2u)} + Z^2\right]\left(Q_1^2+Q_2^2\right)\right. \\ &\left. +2y_0\sqrt{1-y_0^2}Ze^{\Lambda-2u}\left(Q_1^2-Q_2^2\right)\right]. \end{aligned} \tag{6.29}$$

In order to identify the behaviour of Φ as $Z \to 0$ we write the terms in brackets on the right-hand side of (6.29) as:

$$\begin{aligned} &\left[\frac{\left(1-y_0^2\right)^2 e^{2(\Lambda-2u)}}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2} - 1\right] \\ &= \frac{1}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2}\left[\left(1-y_0^2\right)^2\left(e^{2(\Lambda-2u)}-1\right) + 2\left(1-y_0^2\right)y_0^2\left(e^{2(\Lambda-u)}-1\right)\right. \\ &\left. -y_0^4\left(e^{2(\Lambda-u)}-1\right)^2\right]. \end{aligned}$$

Then (6.29) becomes:

$$\begin{aligned} \Phi &= \frac{y_0^2\left(1-y_0^2\right)\left(Q_1^2+Q_2^2\right)}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2}\left[\left(1-y_0^2\right)\left(e^{2(\Lambda-2u)}-1\right) + 2y_0^2\left(e^{2(\Lambda-u)}-1\right)\right. \\ &\left. + \left(1-y_0^2\right)(G-1)\,e^{2(\Lambda-2u)}\right] \\ &+ \frac{\left(1-y_0^2\right)(G-1)}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2}\left[Z^2\left(Q_1^2+Q_2^2\right) + 2y_0\sqrt{1-y_0^2}Ze^{\Lambda-2u}\left(Q_1^2-Q_2^2\right)\right] \\ &- \frac{y_0^6\left(Q_1^2+Q_2^2\right)}{\left(1-y_0^2 e^{2(\Lambda-u)}\right)^2}\left(e^{2(\Lambda-u)}-1\right)^2. \end{aligned} \tag{6.30}$$

In order to simplify this formula we write, using (6.14), (6.16):

$$\left[\left(1-y_0^2\right)\left(e^{2(\Lambda-2u)}-1\right)+2y_0^2\left(e^{2(\Lambda-u)}-1\right)+\left(1-y_0^2\right)(G-1)\,e^{2(\Lambda-2u)}\right]$$
$$=-\left(1-y_0^2\right)\left(1-e^{-4u}\right)+2y_0^2\left(e^{2(\Lambda-u)}-1\right)$$
$$=-\left(1-y_0^2\right)\left(1-\left(\frac{G}{\left[G+y_0^2\left(1-G\right)\right]}\right)^2\right)+2y_0^2\left(\frac{1}{\left[G+y_0^2\left(1-G\right)\right]}-1\right)$$
$$+\left(1-y_0^2\right)\left(2Z^2+Z^4\right)\left(\frac{G}{\left[G+y_0^2\left(1-G\right)\right]}\right)^2+2y_0^2\left(\frac{Z^2}{\left[G+y_0^2\left(1-G\right)\right]}\right),$$

$$\left(e^{2(\Lambda-u)}-1\right)=\frac{Z^2+(1-G)\left(1-y_0^2\right)}{\left[G+y_0^2\left(1-G\right)\right]}.$$

Plugging these formulas into (6.30) we obtain, after some computations:

$$\begin{aligned}
\frac{\Phi}{Z}&=\frac{y_0^2\left(1-y_0^2\right)\left(Q_1^2+Q_2^2\right)}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left[\left(1-y_0^2\right)\left(2Z+Z^3\right)\left(\frac{G}{\left[G+y_0^2\left(1-G\right)\right]}\right)^2\right.\\
&\left.+2y_0^2\left(\frac{Z}{\left[G+y_0^2\left(1-G\right)\right]}\right)\right]\\
&+\frac{\left(1-y_0^2\right)(G-1)}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left[Z\left(Q_1^2+Q_2^2\right)+2y_0\sqrt{1-y_0^2}\,e^{\Lambda-2u}\left(Q_1^2-Q_2^2\right)\right]\\
&-\frac{y_0^6\left(Q_1^2+Q_2^2\right)}{\left(1-y_0^2e^{2(\Lambda-u)}\right)^2}\left[\frac{2\left(1-y_0^2\right)Z\left(1-G\right)+Z^3}{\left[G+y_0^2\left(1-G\right)\right]^2}\right].
\end{aligned}\tag{6.31}$$

Summarizing, we have transformed the original problem (3.15), (3.16), (5.4), (5.11) into the system of equations (6.24)-(6.27) with Δ as in (6.21), $\frac{\Phi}{Z}$ as in (6.31), ζ_i as in (6.8) and Λ, u given by (6.14), (6.16). The initial data for (Q_1, Q_2, G, Z) are as in (6.28).

Some of the forms that we have derived for the ODE problems above are more convenient for describing the solutions in different regions of the phase space. We will change freely between the different groups of equivalent variables in the following.

6.2 Local existence of the curves $\gamma_1,\ \gamma_2$.

With the reformulation of the problem obtained in the previous subsection the existence of the curves $\gamma_1,\ \gamma_2$ in a neighbourhood of the point (y_0, V_0) can be obtained using standard ODE theory.

Proposition 7 *For any $y_0 \in (0,1)$ and any $\beta_0 > 0$ there exist $\delta > 0$ and two curves γ_1, γ_2 that can be parametrized as*

$$\gamma_i = \{(y, V) : y_0 < y < y_0 + \delta \ , \ V = V_i(y)\} \quad , \quad i = 1, 2 \tag{6.32}$$

with the functions $V_1(y)$, $V_2(y)$ as in (5.5), (5.6) satisfying (5.2), (5.3), (5.8) the functions U, Λ satisfying (3.15), (3.16) and (5.9), (5.10) with $\tilde{\rho}$, $\tilde{p}$ as in (5.14), (5.15) and σ_1, σ_2 solving (5.11), (5.12).

Proof. The arguments in Subsection 6.1 show that the proposition follows from proving local existence and uniqueness for (6.24)-(6.27) with initial data (6.28). Since the right-hand side of (6.24)-(6.27) is analytic in a neighbourhood of $(Q_1, Q_2, G, Z) = (1, 1, 1, 0)$ it follows that there exists a unique solution of (6.28), (6.24)-(6.27) on an interval of the form $0 < \chi < \delta_0$ for some $\delta_0 > 0$. Moreover, for such a solution $\Delta \to 0$ as $\chi \to 0^+$, whence $Z \sim 2\chi$ as $\chi \to 0^+$. Therefore (6.23) yields:

$$s \sim 2\chi^2 \text{ as } \chi \to 0^+ \quad , \quad \chi \sim \sqrt{\frac{s}{2}} \text{ as } s \to 0^+,$$

$$Z \sim \sqrt{2s} \text{ as } s \to 0^+. \tag{6.33}$$

Using (6.1) it follows that:

$$s \sim \frac{y - y_0}{y_0} \text{ as } y \to y_0^+. \tag{6.34}$$

Combining then (6.1) and (6.8) we obtain (5.8). The asymptotics (5.9), (5.10) follows from the asymptotics for G, Z in an analogous way. ■

Moreover, we can prove Proposition 5 in a similar way.

Proof of Proposition 5. It follows from (3.25), (6.1), (6.33), (6.34). ■

We notice for further reference that we have also proved the following result:

Proposition 8 *There exists a unique solution of the system (6.3)-(6.5) with ζ_i as in (6.2) and initial data $(Q_1, Q_2, \Lambda, u) = (1, 1, 0, 0)$ as $s \to 0^+$.*

6.3 Steady states for the system (6.24)-(6.27).

In order to study the steady states of (6.24)-(6.27) it is more convenient to use the form of the equations in (6.2)-(6.5). Then the steady states are characterized by:

$$Q_i\zeta_i = 0 \;\; i = 1, 2, \tag{6.35}$$

$$-e^{-2\Lambda} + 1 = \frac{\theta}{2}\left[\frac{Q_1^2}{\left|\zeta_1 e^u + y_0 e^\Lambda \sqrt{\zeta_1^2+1}\right|}\left[\zeta_1^2+1\right]\right.$$
$$\left. + \frac{Q_2^2}{\left|\zeta_2 e^u + y_0 e^\Lambda \sqrt{\zeta_2^2+1}\right|}\left[(\zeta_2)^2+1\right]\right], \tag{6.36}$$

$$3e^{-2\Lambda} - 1 = \frac{\theta}{2}\left[\frac{Q_1^2(\zeta_1)^2}{\left|\zeta_1 e^u + y_0 e^\Lambda \sqrt{\zeta_1^2+1}\right|} + \frac{Q_2^2(\zeta_2)^2}{\left|\zeta_2 e^u + y_0 e^\Lambda \sqrt{\zeta_2^2+1}\right|}\right]. \tag{6.37}$$

The first and third equations imply:

$$3e^{-2\Lambda} - 1 = 0. \tag{6.38}$$

Then, the second equation reduces to:

$$\frac{2}{3} = \frac{\theta}{2}\left[\frac{Q_1^2}{\left|\zeta_1 e^u + y_0 e^\Lambda \sqrt{\zeta_1^2+1}\right|} + \frac{Q_2^2}{\left|\zeta_2 e^u + y_0 e^\Lambda \sqrt{\zeta_2^2+1}\right|}\right]. \tag{6.39}$$

Notice that (6.39) implies that at least one of the variables $Q_1,\ Q_2$ is different from zero at the steady state. Suppose that both of them are different from zero. Then $\zeta_1 = \zeta_2 = 0$, whence, using

$$e^u\sqrt{\zeta_i^2+1} + y_0\zeta_i e^\Lambda = \sqrt{1-y_0^2} \;\;,\;\; i = 1,2 \;\;,\;\; \zeta_1 \le \zeta_2$$

it follows that:

$$e^u = \sqrt{1-y_0^2} \tag{6.40}$$

and (6.39) reduces to:

$$\left(Q_1^2 + Q_2^2\right) = \frac{4y_0 e^\Lambda}{3\theta} = \frac{4y_0\sqrt{3}}{3\theta}.$$

This defines a family of steady states. Local analysis near these solutions indicates that they are reached for finite values of y. Since we are interested in solutions defined for arbitrarily large values of $y > y_0$ a more detailed analysis of these solutions will not be pursued here. We will then restrict our analysis to the solutions for which $Q_1Q_2 = 0$.

Suppose that $Q_1 \neq 0$. Then $\zeta_1 = 0$. (6.40) implies:

$$\sqrt{\zeta_2^2+1} + \frac{y_0\zeta_2 e^\Lambda}{\sqrt{1-y_0^2}} = 1,$$

$$\zeta_2 = \frac{\sqrt{1-y_0^2}}{y_0 e^\Lambda}\left[1 - \sqrt{\zeta_2^2+1}\right] < 0.$$

This contradicts $\zeta_1 \leq \zeta_2$. Therefore for solutions with $Q_1 Q_2 = 0$ we must have $Q_2 \neq 0$ whence $\zeta_2 = 0$. Then (6.40) is satisfied and (6.39) yields:

$$Q_2 = \sqrt{\frac{4\sqrt{3}}{3\theta} y_0} = \frac{2\sqrt{y_0}}{3^{\frac{1}{4}}\sqrt{\theta}}.$$

We remark that for this solution:

$$\zeta_1 = -\frac{2he^{\Lambda_\infty}}{(h^2 - e^{2\Lambda_\infty})} = -\frac{2y_0\sqrt{3\left(1-y_0^2\right)}}{1-4y_0^2}.$$

In order to have $\zeta_1 < \zeta_2 = 0$ we need $y_0 \in \left(0, \frac{1}{2}\right)$.

Summarizing, for each $y_0 \in \left(0, \frac{1}{2}\right)$ the system (6.24)-(6.27) has the following steady state:

$$Q_1 = Q_{1,\infty} = 0, \tag{6.41}$$

$$Q_2 = Q_{2,\infty} = \frac{2\sqrt{y_0}}{3^{\frac{1}{4}}\sqrt{\theta},} \tag{6.42}$$

$$\Lambda = \Lambda_\infty = \frac{\log(3)}{2}, \tag{6.43}$$

$$u = u_\infty = \log\left(\sqrt{1-y_0^2}\right). \tag{6.44}$$

We also introduce the following notation for further reference:

$$\zeta_{1,\infty} = -\frac{2he^{\Lambda_\infty}}{(h^2 - e^{2\Lambda_\infty})} = -\frac{2y_0\sqrt{3\left(1-y_0^2\right)}}{1-4y_0^2}, \tag{6.45}$$

$$\zeta_{2,\infty} = 0. \tag{6.46}$$

6.4 Linearization near the equilibrium.

The main result that we prove in this subsection is the following:

Theorem 9 *For each $y_0 \in \left(0, \frac{1}{2}\right)$ the point $P_1 = (Q_{1,\infty}, Q_{2,\infty}, \Lambda_\infty, u_\infty)$ defined by (6.41)-(6.44) is an unstable hyperbolic point of the system (6.2)-(6.5). The corresponding stable manifold of the point $(Q_{1,\infty}, Q_{2,\infty}, \Lambda_\infty, u_\infty)$ that will be denoted by $\mathcal{M}_\theta$ is three-dimensional and it is tangent at this point to the subspace generated by the vectors*

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{\left(1-y_0^2\right)}{3^{\frac{5}{4}} y_0^{\frac{3}{2}} \sqrt{\theta}} \\ -\frac{2}{3} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{2\sqrt{1-y_0^2}}{3^{\frac{5}{4}} y_0^{\frac{3}{2}} \sqrt{\theta}} \\ -\frac{\sqrt{1-y_0^2}}{3y_0} \\ 1 \end{pmatrix} \right\}. \tag{6.47}$$

Proof. The key ingredient in the proof of this theorem is the linearization of the system (6.2)-(6.5) around the point $(Q_{1,\infty}, Q_{2,\infty}, \Lambda_\infty, u_\infty)$. Let us write:

$$\begin{aligned}
\Lambda &= \Lambda_\infty + L, \\
u &= u_\infty + \nu, \\
Q_1 &= Q_{1,\infty} + q_1 = q_1, \\
Q_2 &= Q_{2,\infty} + q_2.
\end{aligned}$$

Neglecting terms quadratic in $|L|+|\nu|+|q_1|+|q_2|$ we obtain, after some tedious, but mechanical computations, the following linearized problem:

$$\frac{dq_1}{ds} = -\frac{2h^2}{(h^2-3)}q_1 = -\frac{2\left(1-y_0^2\right)}{\left(1-4y_0^2\right)}q_1, \tag{6.48}$$

$$\frac{dq_2}{ds} = \frac{2\left(1-y_0^2\right)}{3^{\frac{5}{4}}\sqrt{\theta}y_0^{\frac{3}{2}}}\nu, \tag{6.49}$$

$$L_s = \frac{3^{\frac{1}{4}}\sqrt{\theta}}{\sqrt{y_0}}q_2 + \frac{\left(1-y_0^2\right)}{3y_0^2}\nu - 2L, \tag{6.50}$$

$$\nu_s = 3L. \tag{6.51}$$

Looking for solutions of the linearized problem with the form:

$$e^{\gamma s}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

we obtain the following possible values of γ with their corresponding eigenvectors:

$$\gamma_1 = -\frac{2\left(1-y_0^2\right)}{\left(1-4y_0^2\right)} \leftrightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\gamma_2 = -2 \leftrightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\left(1-y_0^2\right)}{3^{\frac{5}{4}}y_0^{\frac{3}{2}}\sqrt{\theta}} \\ -\frac{2}{3} \\ 1 \end{pmatrix},$$

$$\gamma_3 = -\frac{\sqrt{\left(1-y_0^2\right)}}{y_0} \leftrightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2\sqrt{1-y_0^2}}{3^{\frac{5}{4}}y_0^{\frac{3}{2}}\sqrt{\theta}} \\ -\frac{\sqrt{1-y_0^2}}{3y_0} \\ 1 \end{pmatrix},$$

$$\gamma_4 = \frac{\sqrt{(1-y_0^2)}}{y_0} \leftrightarrow \begin{pmatrix} 0 \\ \frac{2\sqrt{1-y_0^2}}{3^{\frac{5}{4}} y_0^{\frac{3}{2}} \sqrt{\theta}} \\ \frac{\sqrt{1-y_0^2}}{3y_0} \\ 1 \end{pmatrix}.$$

The theorem then follows from standard results for stable manifolds (cf. for instance [4], [19]). ■

6.5 Reformulation of the solution in the original variables.

Our goal now is to obtain a trajectory connecting the point $(Q_1, Q_2, \Lambda, u) = (1, 1, 0, 0)$ at $s = 0$ with the point P_1 at $s = \infty$ for a suitable value of θ (or equivalently β_0). Let us remark that such a trajectory would satisfy the requirements in Theorem 1. Indeed, notice that such a trajectory behaves near the point (y_0, V_0) as stated in Theorem 1 due to Proposition 7. On the other hand, such a trajectory would belong to the stable manifold of the point P_1 and therefore its asymptotic behaviour as $s \to \infty$ would be given by:

$$\begin{pmatrix} Q_1 \\ Q_2 \\ \Lambda \\ u \end{pmatrix} \sim \begin{pmatrix} Q_{1,\infty} \\ Q_{2,\infty} \\ \Lambda_\infty \\ u_\infty \end{pmatrix} + C_1 e^{-2s} \begin{pmatrix} 0 \\ \frac{\left(1-y_0^2\right)}{3^{\frac{5}{4}} \sqrt{\theta} y_0^{\frac{3}{2}}} \\ \frac{2}{3} \\ -1 \end{pmatrix} + C_2 e^{-\frac{2\left(1-y_0^2\right)}{\left(1-4y_0^2\right)} s} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \ldots$$

for sufficiently small y_0 (cf. [4]). Notice that the smallness of y_0 guarantees that the last term yields a contribution larger for $s \to \infty$ than the first quadratic corrections if $C_2 \neq 0$.

Using (6.1) we obtain the following asymptotics for the original set of variables $U,\ \Lambda,\ \sigma_i,\ V_i,\ i = 1, 2:$

$$U = \log\left(\frac{y}{y_0}\right) + u \sim \log\left(\frac{y}{y_0}\right) + \log\left(\sqrt{1-y_0^2}\right) + o(1) \text{ as } y \to \infty,$$

$$\Lambda \to \log\left(\sqrt{3}\right) \text{ as } y \to \infty,$$

$$e^{\sigma_1} \sim C_2 \left(\frac{y}{y_0}\right)^{-\frac{1+2y_0^2}{\left(1-4y_0^2\right)}} \text{ as } y \to \infty,$$

$$e^{\sigma_2} \sim Q_{2,\infty} \left(\frac{y}{y_0}\right) \text{ as } y \to \infty,$$

$$V_1 \sim \frac{\zeta_{1,\infty}}{y} = -\frac{2y_0\sqrt{3\left(1-y_0^2\right)}}{\left(1-4y_0^2\right) y} \text{ as } y \to \infty,$$

$$V_2 \sim -\frac{\sqrt{1-y_0^2}}{\sqrt{3} y_0} \frac{C_1}{y} \left(\frac{y_0}{y}\right)^2 \text{ as } y \to \infty.$$

in particular these formulas prove Theorem 6.

6.6 The shooting argument: Approximation of the stable manifold $\mathcal{M}_\theta$ for small y_0.

Since the stable manifold $\mathcal{M}_\theta$ is three-dimensional we cannot expect the point $(Q_1, Q_2, \Lambda, u) = (1, 1, 0, 0)$ to belong to $\mathcal{M}_\theta$ for generic values of θ. The intuitive idea of the proof which follows is to show that the manifold $\mathcal{M}_\theta$ divides the set $\{0 < G < 1,\ Z > 0,\ Q_i > 0\ ,\ i = 1, 2\}$ into two different regions. If the point $(1, 1, 0, 0)$ lies on different sides of $\mathcal{M}_\theta$ for different values of θ then by continuity there must exist a value θ^* of θ such that $(1, 1, 0, 0) \in \mathcal{M}_\theta$. In the rest of the paper we will obtain approximations to the manifold $\mathcal{M}_\theta$ for y_0 small that will show that the point $(1, 1, 0, 0)$ lies on different sides of $\mathcal{M}_\theta$ for large positive values of θ and small positive values of θ. More precisely, the main result of this subsection is the following:

Theorem 10 *There exists $\bar{y}_0$ small enough such that, for any y_0 in the interval $[0, \bar{y}_0]$ there exists $\theta^* = \theta^*(y_0) > 0$ such that $(1, 1, 0, 0) \in \mathcal{M}_{\theta^*}$.*

Proof. In order to prove Theorem 10 it is convenient to use the coordinates (Q_1, Q_2, G, Z) (cf. (6.9), (6.14)). These variables satisfy the system of equations (6.24)-(6.27). The steady state $P_1 = P_1(y_0)$ is given in these coordinates by:

$$P_1 = (Q_{1,\infty}, Q_{2,\infty}, G_\infty, Z_\infty) = \left(0, \frac{2\sqrt{y_0}}{3^{\frac{1}{4}}\sqrt{\theta}}, \frac{1}{3}, \sqrt{\frac{3y_0^2}{(1-y_0^2)}}\right). \tag{6.52}$$

The point P_1 depends continuously on y_0 if $y_0 \in \left[0, \frac{1}{2}\right]$. If $y_0 = 0$ the system (6.24)-(6.27) becomes:

$$\frac{dQ_1}{d\zeta} = -2GZQ_1, \tag{6.53}$$

$$\frac{dQ_2}{d\zeta} = -2GZQ_2, \tag{6.54}$$

$$\frac{dG}{d\zeta} = 2G\left[Z(1-G) - \frac{\theta\left[Z^2+1\right]^{\frac{3}{2}}}{2}\left(Q_1^2 + Q_2^2\right)\right], \tag{6.55}$$

$$\frac{dZ}{d\zeta} = \left(3G - 1 - \frac{\theta e^{-u}}{2} Z\left(Q_1^2 + Q_2^2\right)\right)\left(Z^2+1\right). \tag{6.56}$$

Theorem 9 shows that the point $P_1(y_0)$ is hyperbolic for $y_0 \in \left(0, \frac{1}{2}\right]$ with a three-dimensional stable manifold $\mathcal{M}_\theta = \mathcal{M}_\theta(y_0)$. On the other hand two of the eigenvalues associated to the linearization around P_1 of the system (6.24)-(6.27) degenerate for $y_0 = 0$. More precisely, let us write $G = \frac{1}{3} + g$. Since $P_1(0) = \left(0, 0, \frac{1}{3}, 0\right)$ we obtain the following linearization of (6.53)-(6.56) near $P_1(0)$:

$$\frac{dQ_1}{d\zeta} = 0 \quad , \quad \frac{dQ_2}{d\zeta} = 0 \quad , \quad \frac{dG}{d\zeta} = \frac{4Z}{9} \quad , \quad \frac{dZ}{d\zeta} = 3g.$$

The corresponding eigenvalues are $\left\{0, 0, -\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right\}$ and the corresponding eigenvectors are $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{2\sqrt{3}}{9} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2\sqrt{3}}{9} \\ 1 \end{pmatrix}\right\}$. Standard results (cf. [4]) show the existence of a centre-stable manifold that will be denoted by $\mathcal{M}_\theta(0)$ that is invariant under the flow defined by the system (6.53)-(6.56) and is tangent at $P_1(0)$ to the plane spanned by $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{2\sqrt{3}}{9} \\ 1 \end{pmatrix}\right\}$.

Classical results (cf. [4]) then show that it is possible to obtain a continuously differentiable four-dimensional manifold $\mathcal{M}_{\theta,\text{ext}} \subset \left[0, \frac{1}{2}\right] \times \mathbb{R}^4$, with $(y_0, Q_1, Q_2, G, Z) \in \mathcal{M}_{\text{ext}}$ such that:

$$\mathcal{M}_{\theta,\text{ext}} \cap \{y_0 = b\} = \mathcal{M}_\theta(b) \tag{6.57}$$

for any $b \in \left(0, \frac{1}{2}\right)$. Indeed, the manifold $\mathcal{M}_{\theta,\text{ext}}$ is any centre-stable manifold at the point $(y_0, Q_1, Q_2, G, Z) = \left(0, 0, 0, \frac{1}{3}, 0\right)$ associated to the system (6.24)-(6.27) complemented with the additional equation

$$\frac{dy_0}{d\zeta} = 0. \tag{6.58}$$

More precisely, we make use of the fact that the dynamical system of interest has a smooth extension to an open neighbourhood of the stationary point under consideration. The manifold $\mathcal{M}_{\theta,\text{ext}}$ is the intersection of a centre-stable manifold for the extended system with the subset defined by the inequality $y_0 \geq 0$. The manifold $\mathcal{M}_{\text{ext}}$ contains all the points of the form $(y_0, P_1(y_0))$ with $y_0 \in \left[0, \frac{1}{2}\right]$ since they remain in a neighbourhood of $\left(0, 0, 0, \frac{1}{3}, 0\right)$ for arbitrary times. Moreover, the manifolds $\mathcal{M}_{\theta,\text{ext}} \cap \{y_0 = b\}$ are invariant under the flow (6.24)-(6.27) and since they are formed by points that remain in a neighbourhood of $\left(0, 0, 0, \frac{1}{3}, 0\right)$ for arbitrarily long times, it follows from (6.58) that the points in $\mathcal{M}_{\theta,\text{ext}} \cap \{y_0 = b\}$ are contained in the stable manifold associated to the point $P_1(y_0)$. The uniqueness of the stable manifold then implies $\mathcal{M}_\theta(b) \subset \mathcal{M}_{\theta,\text{ext}} \cap \{y_0 = b\}$. Moreover, the form of the tangent space to $\mathcal{M}_{\theta,\text{ext}}$ at the point $\left(0, 0, 0, \frac{1}{3}, 0\right)$ implies that the dimension of $\mathcal{M}_{\theta,\text{ext}} \cap \{y_0 = b\}$ is three for small b. Since this is also the dimension of $\mathcal{M}_\theta(b)$ the relation (6.57) follows. The continuity of $\mathcal{M}_{\text{ext}}$ then implies that the centre-stable manifold

$\mathcal{M}_\theta\left(0\right)$ can be uniquely obtained as limit of the manifolds $\mathcal{M}_\theta\left(y_0\right)$ as $y_0 \to 0^+$. In particular the manifold $\mathcal{M}_\theta\left(0\right)$ is unique.

The properties of the manifold $\mathcal{M}_\theta\left(0\right)$ can be analysed in more detail. We remark that the curve:

$$\sqrt{(Z^2+1)}\sqrt{G}\left(1-G\right)=\frac{2}{3^{\frac{3}{2}}} \quad , \quad Q_1=Q_2=0 \tag{6.59}$$

belongs to $\mathcal{M}_\theta\left(0\right)$ since the hyperplane $\{Q_1=Q_2=0\}$ is invariant under the dynamics induced by (6.53)-(6.56). On the other hand, the invariance of (6.53)-(6.56) under rotations in the (Q_1,Q_2)-plane allows the problem to be reduced to one with smaller dimensionality. More precisely, defining $Q=\sqrt{\frac{1}{2}(Q_1^2+Q_2^2)}$ leads to the system:

$$\frac{dQ}{d\zeta}=-2GZQ, \tag{6.60}$$

$$\frac{dG}{d\zeta}=2G\left[Z\left(1-G\right)-\theta\left[Z^2+1\right]^{\frac{3}{2}}Q^2\right], \tag{6.61}$$

$$\frac{dZ}{d\zeta}=\left(3G-1-\theta ZQ^2\sqrt{(Z^2+1)}\right)\left(Z^2+1\right). \tag{6.62}$$

We will denote by $\mathcal{N}_\theta$ the (two-dimensional) invariant manifold associated to the system (6.60)-(6.62) that is obtained from $\mathcal{M}_\theta$ by taking the quotient by rotations in the Q_i and which contains the curve (6.59).

Our goal is to show the existence for any y_0 sufficiently small of a value $\theta^*=\theta^*\left(y_0\right)$ of θ such that the manifold $\mathcal{M}_{\theta^*}\left(y_0\right)$ contains the point $Q_1=Q_2=1,\ G=1,\ Z=0$. This will be done by showing that the corresponding statement holds in the case $y_0=0$ and then doing a perturbation argument. The statement about the manifold $\mathcal{M}_{\theta^*}\left(0\right)$ is equivalent to the statement that $\mathcal{N}_{\theta^*}$ contains the point $(1,1,0)$. It will be shown that the latter statement is true and, moreover, that when θ is varied through the value θ^* the manifold $\mathcal{N}_\theta$ moves through $(1,1,0)$ with non-zero velocity. It then follows that $\mathcal{M}_\theta(0)$ moves through $(1,1,1,0)$ with non-zero velocity. Note that the coefficients of the system extend smoothly to an open neighbourhood of the manifold $\mathcal{M}_{\theta^*}\left(0\right)$. As a consequence the manifold $\mathcal{M}_{\theta,\mathrm{ext}}$ extends smoothly to small negative values of y_0. The desired statement concerning $\mathcal{M}_\theta(y_0)$ is a consequence of the implicit function theorem. In more detail, the statement that $\mathcal{M}_\theta$ depends on θ and y_0 in a way which is continuously differentiable means that there is a C^1 mapping Ψ from the product of a neighbourhood of $(0,\theta^*)$ in $\mathbb{R}^2$ with $\mathcal{M}_\theta(0)$ into a neighbourhood of $(1,1,1,0)$ with the properties that its restriction to $y_0=0$ and $\theta=\theta^*$ is the identity and that the image of $\{(y_0,\theta)\}\times\mathcal{M}_{\theta^*}(0)$ under Ψ is $\mathcal{M}_\theta(y_0)$. The condition that the manifold moves with non-zero velocity implies that if x_0 denotes the point of $\mathcal{M}_{\theta^*}(0)$ with coordinates $(1,1,1,0)$ the linearization of Ψ at the point $(0,\theta^*,x_0)$ with respect to the last four variables is an isomorphism. This allows the implicit function theorem to be applied.

In order to check the existence of θ^* it is enough to study the behaviour of the manifolds $\mathcal{N}_\theta$ for $\theta\to 0^+$ and $\theta\to\infty$. These manifolds are two-dimensional manifolds in the three-dimensional space (Q,G,Z). Notice that the structure of the manifolds $\mathcal{N}_\theta$ can be easily understood using the fact that the parameter θ can be rescaled out of the system (6.60)-(6.62) using the change of variables:

$$Q = \frac{1}{\sqrt{\theta}} q. \tag{6.63}$$

Then (6.60)-(6.62) becomes:

$$\frac{dq}{d\zeta} = -2GZq, \tag{6.64}$$

$$\frac{dG}{d\zeta} = 2G\left[Z(1-G) - \left[Z^2+1\right]^{\frac{3}{2}} q^2\right], \tag{6.65}$$

$$\frac{dZ}{d\zeta} = \left(3G - 1 - Zq^2\sqrt{(Z^2+1)}\right)\left(Z^2+1\right). \tag{6.66}$$

Let us denote by $\widetilde{\mathcal{N}}$ the centre-stable manifold at the point $(q,G,Z) = \left(0,\frac{1}{3},0\right)$ for the dynamics (6.64)-(6.66). The manifold $\widetilde{\mathcal{N}}$ contains the curve $\left\{\left(Z^2+1\right)G(1-G)^2 = \frac{4}{3^3},\ q=0\right\}$. Notice that:

$$(Q,G,Z) \in \mathcal{N}_\theta \Longleftrightarrow \left(\sqrt{\theta}Q, G, Z\right) \in \widetilde{\mathcal{N}}.$$

Therefore the family of manifolds $\mathcal{N}_\theta$ can be obtained from the manifold $\widetilde{\mathcal{N}}$ by means of the rescaling (6.63) while keeping the same value of the variables $G,\ Z$. In order to check if $(Q,G,Z) = (1,1,0) \in \mathcal{N}_\theta$ we just need to describe in detail the intersection of the manifold $\widetilde{\mathcal{N}}$ with the line $\{G=1, Z=0\}$. Once the existence of a value θ^* of θ for which the manifold $\mathcal{N}_{\theta^*}$ contains the point $(1,1,0)$ has been shown the statement that the manifold $\mathcal{N}_\theta$ moves through this point with non-zero velocity follows immediately from the rescaling property.

Notice that the plane $\{q=0\}$ is invariant for the system of equations (6.64)-(6.66). The analysis of the trajectories of (6.64)-(6.66) in this plane can be done using phase portrait arguments. There is a unique equilibrium point at $(G,Z) = \left(\frac{1}{3},0\right)$ with stable manifold $\left\{\left(Z^2+1\right)G(1-G)^2 = \frac{4}{3^3}\right\}$. This manifold splits the plane (G,Z) in two connected regions. The trajectories starting their motion in the region that contains the point $(G,Z) = (0,0)$ reach the line $Z=0$ for a finite value of ζ if $Z>0$ initially and eventually develop a singularity where Z approaches $-\infty$ at a finite value of ζ. On the other hand, the trajectories starting their motion in the region containing the point $(G,Z) = (1,0)$ move in the direction of increasing Z towards $Z=\infty,\ G=1$, a value that is achieved for a finite value of ζ.

Notice that the solutions of (6.64)-(6.66) starting their dynamics in the set $\{0 \le G \le 1,\ Z \ge 0\}$ can only evolve in two different ways. Either the trajectory remains in the region where $Z \ge 0$ for arbitrarily large values of ζ or the trajectory enters the region $\{Z<0\}$. In the second case this can only happen through the set $G \le \frac{1}{3}$. Since G is decreasing it remains in the set $\{Z<0\}$ for larger values of ζ and eventually it approaches $Z=-\infty$ for some finite value of ζ.

Suppose otherwise that the trajectory remains in the region where $Z \geq 0$ for arbitrary values of ζ. Then q decreases to zero and the behaviour of the trajectories is then similar to the ones in the plane $\{q=0\}$. We now claim that either this trajectory belongs to the stable manifold $\widetilde{\mathcal{N}}$ or it satisfies $\lim_{\zeta\to\zeta^*} Z(\zeta)=\infty$ for some $\zeta^* \leq \infty$. In order to avoid breaking the continuity of the argument we will prove this result in Lemma 11 in Section 7.

We will show that there exists a point of the line $\{G=1, Z=0\}$ in the manifold $\widetilde{\mathcal{N}}$. The points of this line enter the region $\{0<G<1,\ Z>0\}$ due to the form of the vector field associated to (6.64)-(6.66). If $q(0)>0$ is small, Lemma 12 shows that Z approaches $Z=\infty$ for a finite value of ζ. Suppose now that $q(0)>0$ is sufficiently large. Then the trajectory enters the region $\{Z<0\}$ for a finite value of ζ as the following argument shows. A solution which starts at $(q_0,1,0)$ with q_0 large immediately enters the region $Z>0$, $G<1$. The inequality $Z\leq 1$ will hold for at least a time $\frac{1}{4}$ since $\frac{dZ}{d\zeta}\leq 4$ as long as $Z\leq 1$. The aim is to show that for q_0 sufficiently large Z will become negative within the interval $[0,\frac{1}{4}]$. From now on only that interval is considered. Integrating the equation for q gives the inequality $q(\zeta)\geq e^{-\frac{1}{2}}q_0$. The equation for G then shows that $G(\zeta)\leq e^{-\alpha(q_0)\zeta}$ where $\alpha(q_0)=q_0^2e^{-1}-1$. Choose q_0 large enough so that $e^{-\frac{1}{40}\alpha(q_0)}\leq\frac{1}{6}$. When $\zeta=\frac{1}{40}$ the inequality $Z\leq\frac{1}{10}$ still holds. Under the given circumstances G is decreasing on the whole interval $[0,\frac{1}{4}]$. The equation for Z shows that by the time $\zeta=\frac{9}{40}$ at the latest Z has reached zero.

Let U_1 be the set of positive real numbers q_0 for which the solution starting at $(q_0,1,0)$ is such that $Z\to-\infty$ as $\zeta\to\zeta^*$, where ζ^* denotes the maximal time of existence, and let U_2 be the set of positive real numbers q_0 for which the solution starting at $(q_0,1,0)$ is such that $Z\to+\infty$ as $\zeta\to\zeta^*$. It follows from Lemma 13 that U_2 is open. We also know that U_1 is open. Moreover, it has been proved that both U_1 and U_2 are non-empty. By connectedness of the interval $(0,\infty)$ it follows that there must be a value of q_0 for which the solution starting at $(q_0,1,0)$ is neither in U_1 or U_2. For that solution Z is non-negative and does not tend to infinity and thus, by Lemma 11, it is the desired solution which lies on $\tilde{\mathcal{N}}$.

The equivalence between the existence of the self-similar solution described in Section 5 and the existence of a trajectory connecting the points $(Q_1,Q_2,G,Z)=(1,1,1,0)$ and $(Q_{1,\infty},Q_{2,\infty},\Lambda_\infty,u_\infty)$ proved in Subsection 6.5 concludes the proof of Theorem 10. Theorem 1 is just a Corollary of Theorem 10. ■

7 Some auxiliary lemmas used in the analysis of (6.64)-(6.66).

Lemma 11 *Suppose that a solution of (6.64)-(6.66) is defined for $\zeta\in[\zeta_*,\zeta^*)$, where ζ^* is the maximal time of existence. Suppose that $Z(\zeta)>0$ for any*

$\zeta \in [\zeta_*, \zeta^*)$ and also that $G(\zeta_*) \in (0,1)$, $q(\zeta_*) > 0$. Then, either the curve $\{(q(\zeta), G(\zeta), Z(\zeta)) : \zeta \in (\zeta_*, \zeta^*)\}$ is contained in the stable manifold $\widetilde{\mathcal{N}}$ or $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$.

Proof. The plane $\{G = 0\}$ is invariant under the flow associated to (6.64)-(6.66). On the other hand, the vector field on the right-hand side of (6.64)-(6.66) points into the region $\{G < 1\}$ if $q \neq 0$. Therefore the region $\{0 < G < 1,\ q > 0\}$ is invariant for the flow defined by (6.64)-(6.66) and we can assume that the inequalities $0 < G(\zeta) < 1,\ q(\zeta) > 0$ hold for any $\zeta \in [\zeta_*, \zeta^*)$. We now have two possibilities:

$$\limsup_{\zeta\to\zeta^*} Z(\zeta) < \infty\ , \tag{7.1}$$

$$\limsup_{\zeta\to\zeta^*} Z(\zeta) = \infty. \tag{7.2}$$

Suppose first that (7.1) holds. Then, there exists $M > 0$ such that

$$Z(\zeta) \leq M \quad \text{for any} \quad \zeta \in [\zeta_*, \zeta^*)\,. \tag{7.3}$$

We claim that in this case the trajectory $\{(q(\zeta), G(\zeta), Z(\zeta)) : \zeta \in (\zeta_*, \zeta^*)\}$ is contained in $\widetilde{\mathcal{N}}$. Notice that in this case, the boundedness of $|(q, G, Z)|$ implies that $\zeta^* = \infty$. Since $(GZq)(\zeta) > 0$ for $\zeta \in [\zeta_*, \infty)$ it follows from (6.64) that $q(\zeta)$ is decreasing. Therefore $q_\infty = \lim_{\zeta\to\infty} q(\zeta)$ exists and is non-negative. Suppose that $0 < q_\infty$. Then $0 < q_\infty < q(\zeta)$ for any $\zeta \in [\zeta_*, \infty)$. Integrating (6.64) we obtain $\int_{\zeta_*}^{\infty} (GZq)(\zeta)\, d\zeta < \infty$, whence

$$\int_{\zeta_*}^{\infty} (GZ)(\zeta)\, d\zeta < \infty. \tag{7.4}$$

Since $\frac{dG}{d\zeta}$, $\frac{dZ}{d\zeta}$ are bounded, (7.4) implies $\lim_{\zeta\to\infty} (GZ)(\zeta) = 0$. Then (6.65) implies:

$$\frac{dG}{d\zeta} \leq -q_\infty^2 G$$

for $\zeta \geq \zeta_0$ sufficiently large. Therefore $\lim_{\zeta\to\infty} G(\zeta) = 0$. Equation (6.66) then yields:

$$\frac{dZ}{d\zeta} \leq -\frac{1}{2}$$

for $\zeta \geq \zeta_0$ large enough. Then $Z(\zeta) < 0$ for large ζ, but this contradicts the hypothesis of the lemma. It then follows that $q_\infty = 0$.

Due to (7.3) and since $\lim_{\zeta\to\infty} q(\zeta) = 0$ we can approximate the trajectories associated to (6.64)-(6.66) for large values of ζ using the corresponding trajectories associated to (6.64)-(6.66) for $q = 0$. The study of the trajectories associated to (6.64)-(6.66) that are contained in $\{q = 0\} \cap \{0 < G < 1\}$ reduces to a two-dimensional phase portrait. These trajectories can have only three different behaviours. Either they are contained in $\widetilde{\mathcal{N}} \cap \{q = 0\}$, or they reach the plane $\{Z=0\}$, with $G<\frac{1}{3}$, entering $\{Z<0\}$, or they become unbounded. The continuous dependence of the trajectories with respect to the initial values as well as the fact that $\lim_{\zeta\to\infty} q(\zeta)=0$ implies then that either $\lim_{\zeta\to\infty} \operatorname{dist}\left(\left(q(\zeta), G(\zeta), Z(\zeta)\right), \widetilde{\mathcal{N}}\cap\{q=0\}\right) = 0$, or $Z(\zeta)<0$ for some $\zeta<\infty$, or $Z(\zeta)\geq M+1$ for some $\zeta<\infty$. The second alternative contradicts the hypothesis of the lemma. The third alternative contradicts (7.3) and therefore only the first alternative is left. However, in that case $\lim_{\zeta\to\infty} \left(q(\zeta), G(\zeta), Z(\zeta)\right)=\left(0,\frac{1}{3},0\right)$ and the trajectory is contained in $\widetilde{\mathcal{N}}$ as claimed.

Suppose then that (7.2) holds. We claim that in this case $\lim_{\zeta\to\zeta^*} Z(\zeta)=\infty$. Notice that the monotonicity of $q(\zeta)$ implies that $\lim_{\zeta\to\zeta^*} q(\zeta)=q_\infty$ exists. We will first prove that $q_\infty=0$. Suppose that, on the contrary, $q_\infty>0$. Then $q(\zeta)>q_\infty>0$ for any $\zeta\in[\zeta_*,\zeta^*)$. Equation (6.66) as well as $G<1$ yields:

$$\frac{dZ}{d\zeta}<\left(2-Zq_\infty^2\sqrt{Z^2+1}\right)\left(Z^2+1\right)$$

for any $\zeta\in[\zeta_*,\zeta^*)$. This inequality implies $\frac{dZ}{d\zeta}<0$ for $Z>Z_\infty=Z_\infty(q_\infty)$. Therefore $Z(\zeta)<Z_\infty$ for $\zeta\in[\zeta_*,\zeta^*)$ and this contradicts (7.2). From now on take $q_\infty=0$. We can then assume (7.2) and

$$\lim_{\zeta\to\zeta^*} q(\zeta)=0. \tag{7.5}$$

Suppose also that $\liminf_{\zeta\to\zeta^*} Z(\zeta)<\infty$. This is equivalent to the existence of $0<M<\infty$ and a subsequence $\{\zeta_n\}$ with $\zeta_n\to\zeta^*$ as $n\to\infty$ such that:

$$Z(\zeta_n)\leq M. \tag{7.6}$$

We now claim that:

$$\lim_{\zeta\to\zeta^*}\left[Z(\zeta)\,q(\zeta)\right]=0. \tag{7.7}$$

To prove (7.7) we argue as follows. Combining (6.64), (6.66) we obtain:

$$\frac{d}{d\zeta}(Zq)=qZ^2(G-1)+q(3G-1)-Zq^3\sqrt{\left(Z^2+1\right)^3}. \tag{7.8}$$

We now use the inequality $Z\sqrt{\left(Z^2+1\right)^3}\geq Z^4$ for $Z>0$. Then, using also the inequality $G<1$:

$$\frac{d}{d\zeta}(Zq)\leq q^{-1}\left[(3G-1)\,q^2-(Zq)^4\right]. \tag{7.9}$$

It follows from this inequality, as well as (7.5) that for any $\varepsilon>0$, every trajectory satisfying the hypothesis of Lemma 11 and entering any of the regions $\{(q,G,Z): Zq<\varepsilon\}$ for ζ sufficiently close to ζ^* remains in such a region for later times. If $\zeta^*=\infty$, the meaning of sufficiently close is large enough. Due to (7.5) and (7.6), for any $\varepsilon>0$, there exist ζ_n arbitrarily close to ζ^* such that

$(Zq)(\zeta_n) < \varepsilon$. Then $(Zq)(\zeta) < \varepsilon$ for any $\zeta \in (\zeta_n, \zeta^*)$. Since ε is arbitrary we obtain (7.7).

Combining (7.5) and (7.7) it follows that:

$$\lim_{\zeta\to\zeta^*} \delta_1(\zeta) = \lim_{\zeta\to\zeta^*} \delta_2(\zeta) = 0 \tag{7.10}$$

where:

$$\delta_1(\zeta) = Zq^2\sqrt{Z^2+1} \quad , \quad \delta_2(\zeta) = \frac{\left(Z^2+1\right)^{\frac{3}{2}} q^2}{Z+1}.$$

We can then rewrite (6.64), (6.66) as:

$$\frac{dq}{d\zeta} = -2GZq, \tag{7.11}$$

$$\frac{dG}{d\zeta} = 2G\left[Z(1-G) - (Z+1)\delta_2(\zeta)\right], \tag{7.12}$$

$$\frac{dZ}{d\zeta} = (3G - 1 - \delta_1(\zeta))\left(Z^2+1\right). \tag{7.13}$$

We now claim the following. Given any ε_0 belonging to the interval $\left(0, \frac{2}{3}\right)$ suppose that the trajectory under consideration enters the set:

$$\Omega_{\varepsilon_0} = \left\{G \geq \frac{1}{3} + \varepsilon_0 \ , \ Z \geq 1\right\}$$

for some $\zeta < \zeta^*$ sufficiently close to ζ^*. Then $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$ and $\zeta^* < \infty$. The proof as the follows. Due to (7.10) the set Ω_{ε_0} is invariant for (7.11)-(7.13) if ζ is close to ζ^*. Then, for ζ close to ζ^* we have:

$$\frac{dZ}{d\zeta} \geq \varepsilon_0\left(Z^2+1\right)$$

and this implies $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$ and $\zeta^* < \infty$.

Therefore, to complete the proof of Lemma 11 it only remains to prove that the trajectory enters Ω_{ε_0} for values of ζ sufficiently close to ζ^*. Due to (7.2) and (7.6) there exists a sequence $\left\{\bar{\zeta}_n\right\}$ with $\zeta_n < \bar{\zeta}_n < \zeta^*$ such that:

$$Z\left(\bar{\zeta}_n\right) = 2M \quad \text{and} \quad \frac{dZ}{d\zeta}\left(\bar{\zeta}_n\right) \geq 0.$$

Due to (7.13) this implies:

$$\limsup_{n\to\infty} G\left(\bar{\zeta}_n\right) \geq \frac{1}{3}. \tag{7.14}$$

On the other hand, a Gronwall type of argument applied to (7.13) implies the existence of $\alpha_M > 0$, depending only on M such that:

$$0 < \frac{M}{2} \leq Z(\zeta) \leq 4M \quad \text{for} \quad \zeta \in \left[\bar{\zeta}_n, \bar{\zeta}_n + \alpha_M\right]. \tag{7.15}$$

Comparing the solution of the equation (7.12) with the solution of the equation $\frac{dG}{d\zeta} = 2GZ(1-G)$ with the same initial datum at $\zeta = \bar{\zeta}_n$ and taking into account (7.14), (7.15) it then follows that, for n large enough $\left(q\left(\bar{\zeta}_n\right), G\left(\bar{\zeta}_n\right), Z\left(\bar{\zeta}_n\right)\right) \in \Omega_{\varepsilon_0}$. Therefore $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$. This contradicts (7.6) and the lemma follows. ■

Lemma 12 *There exists $\delta > 0$ sufficiently small such that, the solution of (6.64)-(6.66) with initial value $(q(0), G(0), Z(0)) = (q_0, 1, 0)$ and $0 < q_0 < \delta$ satisfies $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$, where ζ^* denotes the maximal time of existence of the trajectory.*

Proof. The trajectory enters the region $\{Z > 0\}$ and as long as it remains there, the function $q(\zeta)$ is decreasing. The inequality $\frac{\partial Z}{\partial \zeta} \leq 4$ holds as long as $Z \leq 1$. It follows that $Z \leq 1$ on the interval $\left[0, \frac{1}{4}\right]$. On that interval the inequality $\frac{\partial(\log G)}{\partial\zeta} \geq -2^{\frac{5}{2}} q_0^2$ holds and hence $G \geq e^{-q_0^2}$. Furthermore

$$\frac{\partial Z}{\partial \zeta} \geq 3e^{-q_0^2} - 1 - \sqrt{2} q_0^2 = \beta(q_0). \tag{7.16}$$

Choose δ sufficiently small that $\beta(\delta) > 1$ and $e^{-\delta^2} > \frac{1}{2}$. Then $Z\left(\frac{1}{4}\right) > \frac{1}{4}$ and $G > \frac{1}{2}$ on $\left[0, \frac{1}{4}\right]$. Choose $\epsilon > 0$ and suppose that $2\delta^2 < \epsilon^4$. Then it follows from (7.9) that the set defined by the inequality $Zq \leq \epsilon$ is invariant. Thus the solution remains in that region on its whole interval of existence. Now $\delta_1(\zeta) \leq \epsilon\sqrt{\epsilon^2 + \delta^2}$ and $\delta_2(\zeta) \leq (\epsilon^2 + \delta^2)$. Let $[0, \zeta_1)$ be the longest interval on which $G \geq \frac{1}{2}$. From what has been shown already $\zeta_1 \geq \frac{1}{4}$. Reduce the size of ϵ if necessary so that $\epsilon\sqrt{\epsilon^2 + \delta^2} < \frac{1}{2}$. Then it follows from (7.13) that Z is increasing on $[0, \zeta_1)$ and hence is greater than $\frac{1}{4}$ for $\zeta \geq \zeta_1$. Putting this information into (7.12) shows that provided $\epsilon^2 + q_0^2 < \frac{1}{16}$ then G cannot decrease. For δ sufficiently small this gives a contradiction unless $\zeta_1 = \zeta^*$. In particular there is a positive lower bound for Z at late times. Furthermore (7.13) implies that $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$ and the lemma follows. ■

Lemma 13 *Suppose that a solution satisfying the hypotheses of Lemma 11 with $\zeta_* = 0$ has the property that $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$. Then any solution starting sufficiently close to the given solution for $\zeta = 0$ also has the property that Z tends to infinity on its maximal interval of existence.*

Proof. To start with a number of further consequences of the hypotheses of Lemma 11 will be derived. The assumption on the initial condition only plays a role towards the end of the proof. It has been shown in the proof of Lemma 11 that $\lim_{\zeta\to\zeta^*} q(\zeta) = 0$. We now claim that (7.7) holds. Suppose that it is not true. Then we claim that the limit $\lim_{\zeta\to\zeta^*} (Zq)(\zeta) = L$ exists and that $L > 0$. Indeed, notice first that $\liminf_{\zeta\to\zeta^*} (Zq)(\zeta) > 0$. Otherwise there would exist a sequence $\{\zeta_n\}$ such that $\lim_{n\to\infty} \zeta_n = \zeta^*$ with $\lim_{n\to\infty} (Zq)(\zeta_n) = 0$. Combining this with the fact that $q \to 0$ and (7.9) we would obtain (7.7), a contradiction. Thus $\liminf_{\zeta\to\zeta^*} (Zq)(\zeta) > 0$. Using again the fact that $q \to 0$

and (7.9) it follows that (Zq) is monotone decreasing for ζ close to ζ^*, whence the limit $\lim_{\zeta\to\zeta^*}(Zq)(\zeta) = L$ exists. Moreover we have obtained also in this case that $(Zq)(\zeta) > L$ for ζ close to ζ^*.

It follows from the proof of Lemma 11 that $\zeta^* < \infty$. By the boundedness of the right hand side of (6.64) it follows by integrating this equation between ζ and ζ^* that $q(\zeta) \le a^{-1}(\zeta^* - \zeta)$ for a positive constant a. Hence $q^{-1}(\zeta) \ge a(\zeta^* - \zeta)^{-1}$. This can be used together with the limiting behaviour of Zq to estimate the right hand side of (7.8) from above. The first term is negative and can be discarded. The second term tends to zero as $\zeta \to \zeta^*$. The third term can be written in a suggestive form as $-q^{-1}[(Zq)\sqrt{((Zq)^2 + q^2)^3}]$. The expression in square brackets tends to a positive limit as $\zeta \to \zeta^*$. Thus the right hand side of (7.8) fails to be integrable, contradicting the fact that Zq is positive. This contradiction completes the proof that $\lim_{\zeta\to\zeta^*}(Zq)(\zeta) = 0$.

We now use some arguments analogous to the ones used in the proof of Lemmas 11 and 12. As a next step we prove that $G(\zeta)$ tends to a limit as $\zeta \to \zeta^*$ and that this limit is greater than $\frac{1}{3}$. We first claim that:

$$S = \limsup_{\zeta\to\zeta^*} G(\zeta) \ge \frac{1}{3}. \tag{7.17}$$

Indeed, suppose first that $S = \limsup_{\zeta\to\zeta^*} G(\zeta) < \frac{1}{3}$. Since $\lim_{\zeta\to\zeta^*}(Zq)(\zeta) = 0$ we can approximate (6.60)-(6.62) by the system (7.11)-(7.13). Using (7.13) it follows that $Z(\zeta)$ is decreasing for ζ close to ζ^*. This contradicts (7.2) and then (7.17) follows. On the other hand (7.12) implies that G is increasing if $G > \frac{1}{4}$ for ζ close to ζ^*. Using (7.17) it then follows that G increases for ζ close to ζ^*. Therefore the limit $\lim_{\zeta\to\zeta^*} G(\zeta)$ exists and:

$$\lim_{\zeta\to\zeta^*} G(\zeta) \ge \frac{1}{3}.$$

Since G is monotonically increasing we can parametrize Z as a function of G. Let us denote the corresponding function by $Z = \tilde{Z}(G)$. Then by (7.12) and (7.13):

$$\frac{d(\log \tilde{Z})}{dG} = \frac{(3G - 1 - \delta_1(\zeta))(1 + \tilde{Z}^{-2})}{2G[(1-G) - (1 + \tilde{Z}^{-1})\delta_2(\zeta)]}. \tag{7.18}$$

If the limit of G were less than one the right hand side of this expression would be bounded and it would follow that Z was bounded, a contradiction. Hence $\lim_{\zeta\to\zeta^*} G(\zeta) = 1$.

To complete the proof the condition on the initial data in the hypotheses of the lemma will be used. Since $\lim_{\zeta\to\zeta^*} Z(\zeta) = \infty$, $\lim_{\zeta\to\zeta^*} q(\zeta) = 0$, $\lim_{\zeta\to\zeta^*}(Zq)(\zeta) = 0$ and $\lim_{\zeta\to\zeta^*} G(\zeta) > \frac{1}{3}$ it follows that for any sufficiently small $\delta > 0$ and for any solution $(\bar{q}, \bar{G}, \bar{Z})$ that is sufficiently close to (q, G, Z) at $\zeta = 0$ we have for some $\zeta_0 < \zeta^*$:

$$\bar{q}(\zeta_0) \le \delta^3 \quad , \quad \bar{G}(\zeta_0) \ge \frac{1}{3} + \delta \quad , \quad (\bar{Z}\bar{q})(\zeta_0) \le \delta \quad , \quad \bar{Z}(\zeta_0) \ge \frac{1}{\delta}. \tag{7.19}$$

It will now be shown that for δ sufficiently small the region defined by these four inequalities is invariant. On the part of the boundary of the region where $\bar{q} = \delta^3$ we have $\frac{d\bar{q}}{d\zeta} < 0$. On the part of the boundary where $\bar{Z}\bar{q} = \delta$ assuming that $\delta < 3^{-\frac{1}{3}}$ suffices to show, using (7.9), that the derivative of $\bar{Z}\bar{q}$ is negative. On the part with $\bar{G} = \frac{1}{3} + \delta$ the following inequality holds:

$$\frac{\partial \bar{G}}{\partial \zeta} \geq \frac{2}{3}\left[\frac{2}{3\delta} - 1 - (\delta^{\frac{1}{2}} + \delta^{\frac{5}{2}})^2 - (\delta + \delta^3)^2\right]. \tag{7.20}$$

Choosing δ sufficiently small implies that the right hand side of this inequality is positive. On the whole region

$$\frac{d\bar{Z}}{d\zeta} \geq \delta(3 - \sqrt{\delta^2 + \delta^6}). \tag{7.21}$$

If δ is small enough then this quantity is positive. Putting these facts together shows that the solution starts in the region of interest when $\zeta = \zeta_0$ and stays there. In particular $\bar{G}(\zeta) \geq \frac{1}{3} + \delta$ for $\zeta \geq \zeta_0$. Therefore $\bar{Z}$ blows up in finite time due to (7.13) and Lemma 13 follows. ■

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