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On form factors in $\mathcal{N} = 4$ SYM

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Abstract: In this paper we study the form factors for the half-BPS operators $\mathcal{O}_I^{(n)}$ and the $\mathcal{N} = 4$ stress-tensor supermultiplet T^{AB} up to the second order of perturbation theory and for the Konishi operator $\mathcal{K}$ at first order of perturbation theory in the $\mathcal{N} = 4$ SYM theory at weak coupling. For all the objects we observe the exponentiation of the IR divergences with two anomalous dimensions: the cusp anomalous dimension and the collinear anomalous dimension. For the IR finite parts we obtain a similar situation as for the gluon scattering amplitudes, namely, apart from the case of T^{AB} and $\mathcal{K}$ the finite part has some remainder function which we calculate up to the second order. It involves the generalized Goncharov polylogarithms of several variables. All the answers are expressed in terms of the integrals related to the dual conformal invariant ones which might be a signal of integrable structure standing behind the form factors.

Keywords: Supersymmetric gauge theory, Supersymmetry and Duality, Extended Supersymmetry

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Contents

1 Introduction

Much attention in the past few years has been dedicated to the study of the planar limit for the scattering amplitudes in $\mathcal{N} = 4$ SYM theory. It is believed that the hidden symmetries responsible for integrability properties of $\mathcal{N} = 4$ SYM completely fix the structure of the amplitudes (the S-matrix of the theory) [1–3]. One of the possible views on this subject is that the answers for the amplitudes are expressed in terms of the scalar integrals which are pseudo-conformal invariant in momentum space [4, 5] which appear in the unitarity-based calculation of the scattering amplitudes pioneered in papers [6, 7].

The (dual)conformal symmetry at weak coupling regime can be extended to $\mathcal{N} = 4$ supersymmetric version and can be fused with the original $\mathcal{N} = 4$ superconformal symmetry to the so-called Yangian symmetry [8] which is governed by Yangian infinite dimensional algebra. The Yangian like symmetries are common features of the integrable systems [1].

At strong coupling the computation of the amplitudes in $\mathcal{N} = 4$ SYM can be reduced via AdS/CFT to the computation of the open string scattering amplitudes in AdS_5, with strings ending on $D3$-brane positioned at some fixed value of the radial AdS_5 coordinate z, in the quasi-classical regime [9] which in turn can be formulated as the problem of finding the minimal surface in AdS_5 with special boundary condition (see [10] for review). This problem has recently been reduced to that of solving the set of functional equations for the conformal invariant cross ratios as functions of the spectral parameters– the so-called Y-system [11]. The Y-systems usually appear in integrable systems [12] which is another hint that the $\mathcal{N}=4$ amplitudes have some underling integrable structure.

In strong coupling regime the natural generalization of the Y-system for the amplitudes is the Y-system for the form factors [13]: the matrix elements of the form

$$\langle 0|\mathcal{O}|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle. \tag{1.1}$$

where $\mathcal{O}$ is some gauge invariant operator which acts on vacuum and produces some state $|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle$ with momenta $p_1\dots p_n$ and helicities $\lambda_1\dots\lambda_n$. In the dual string theory this matrix element can be described via the open string scattering amplitudes, with strings ending on $D3$-brane positioned at some fixed value of the radial AdS_5 coordinate z, in the presence of some closed string state [13].

One can wonder whether these objects at weak coupling possess similar features as the amplitudes or in other words whether form factors are influenced by the Yangian symmetry (or some analog of it) and whether they are fixed by it. Also the general structure of the form factors at weak coupling should be understood.

Being inspired by the two-loop calculation of the form factor associated with the operator $\mathcal{V}_X$ from the stress-tensor superconformal multiplet of $\mathcal{N}=4$ SYM theory performed long ago by van Neerven [14] we would like to study systematically some types of form factors in planar $\mathcal{N}=4$ SYM at weak coupling for half-BPS operators $\mathcal{O}_I^{(n)}$ and the Konishi operator $\mathcal{K}$. For the former type of operators there was recently an interest in studying the correlation functions and their connection to the amplitudes and the Wilson loops [15, 16]. A new kind of relation has been proposed between the logarithm for such correlation functions and twice the logarithm of the MHV gluon scattering amplitudes.

The dual-conformal symmetry plays an important role in another remarkable property of the $\mathcal{N}=4$ SYM — the Wilson loop/Amplitudes duality [5, 17–19]. In this duality (we restrict ourselves to the most studied case of the MHV amplitude[1] sector) the dual-conformal symmetry is understood as conformal symmetry of light-like Wilson loop constructed of the segments which satisfy the following property:

$$x^\mu_{i,i+1}=x^\mu_i-x^\mu_{i+1}=p^\mu_i, \tag{1.2}$$

where p^μ_i are external momenta of the dual MHV amplitude. The dual-conformal symmetry is broken on-shell for the amplitudes due to the presence of the IR divergences (these IR divergences correspond to the UV divergences for dual Wilson loops); however, the violation of the dual-conformal symmetry is controlled by the 1-loop exact anomaly, which in turn can be used to make constraints for the finite part of the corresponding MHV amplitude,

[1]MHV (maximally helicity violating) amplitudes by definition are called the amplitudes with all particles being treated as outgoing and the net helicity λ_Σ being equal to $n-4$ where n is the number of particles.

i.e. one can write the anomalous Ward identities allowing one to constrain the finite parts (pioneered in [18, 20], see, for example, review [10] for details):

$$\sum_{i=1}^{n}(2x_i^\nu x_i\partial_i - x_i^2\partial_i^\nu)Fin[\log \mathcal{W}_n] = \Gamma_{\text{cusp}}\sum_{i=1}^{n}\log\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}x_{i,i+1}^\nu, \tag{1.3}$$

where Γ_{cusp} is the so-called cusp anomalous dimension [21–23] known from a solution of the integral equation [24, 25]. These identities can fix the finite part for a small number of legs/cusps (namely, one can fix it for Wilson loops with $n = 4$ and $n = 5$ cusps) [10]. In fact, the famous BDS conjecture was the simplest possible ansatz of these identities, which is not precisely correct for a number of external legs > 5.

One may wonder if there is a similar duality for the form factors/Wilson loops and one can use similar arguments to obtain information on the finite parts of the form factors. We hope that our calculation sheds some light on the possibility of such duality.

The paper is organized as follows. In section 2, we present the general considerations of the form factors in the $\mathcal{N} = 4$ SYM theory and introduce the operators to be discussed later. In section 3, we study the form factors for both protected and non-protected operators with naive conformal dimension $\Delta_0 = 2$ and confirm the results obtained long time ago in [14]. In section 4, we study the half-BPS operators $\mathcal{O}_I^{(n)}$ for arbitrary conformal dimension $\Delta_0 = n$ and present the one- and two-loop calculations of the corresponding form factors which suggest the exponentiation of the IR divergences. Also, in the same section, we discuss the collinear limit for which the finite parts take a simple form. In section 5, we discuss in more detail the dual conformal invariance of the integrals contributing to the calculation of the form factors. We conclude with some remarks concerning the form factors and Wilson loop duality. In the appendices we give the details of our calculations. Appendix A contains the Lagrangian of the $\mathcal{N} = 4$ SYM theory together with the Feynman rules. In appendix B we present the analytic expressions for the integrals entering into our calculations with their ϵ-expansion. The results of our work have been reported at the international conference devoted to the memory of A.N. Vasiliev held in Saint-Petersburg on 18-21 October 2010.

2 General considerations

2.1 Form factors in $\mathcal{N} = 4$ SYM

Consider the Lagrangian $\mathcal{L}_{\mathcal{N}=4}(\mathcal{W})$ for the $\mathcal{N} = 4$ SYM theory coupled to some external classical current J through some gauge invariant local operator $\mathcal{O}[\mathcal{W}]$ (for all the details concerning the explicit expression for the Lagrangian together with Feynman rules we refer to appendix A)

$$\mathcal{L}_{\mathcal{N}=4}(\mathcal{W}) \to \mathcal{L}_{\mathcal{N}=4}(\mathcal{W}) + \mathcal{O}[\mathcal{W}]J, \tag{2.1}$$

where we collectively refer to the whole $\mathcal{N} = 4$ on-shell multiplet as $\mathcal{W}$ which consists of the physical gluon A^μ states with positive and negative helicities, four gauginos λ_α^N with positive and negative helicities and also six real scalar states ϕ_{NM}, where N and M are the $\mathrm{SU}(4)_R$ indices, which can also be re-arranged into 3 complex ϕ^I scalars, I is an index of SU(3) subgroup of $\mathrm{SU}(4)_R$. By default we assume everywhere the planar limit.

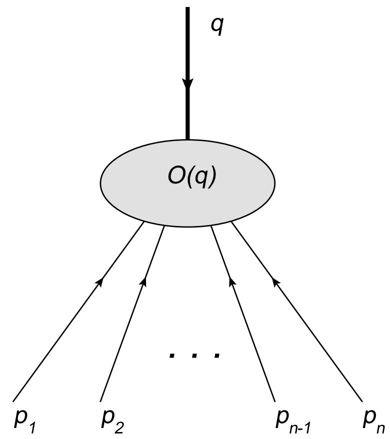

Figure 1. Feynman diagram for the matrix element of the operator $\mathcal{O}$.

Then one can study the following processes where the operator $\mathcal{O}$ acts on the vacuum and produces some state $|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle$ with momenta $p_1 \dots p_n$ and helicities $\lambda_1 \dots \lambda_n$

$$\langle 0|\mathcal{O}|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle. \tag{2.2}$$

Schematically, it is shown in figure 1.

This is a general situation in QFT and one can keep in mind, for example, $\gamma^* \to$ jets process [26] where we take into account all orders in α_s but the first order in $\alpha_{\rm em}$. In perturbation theory the latter type of processes can be thought of as the matrix elements of the following form:

$$\langle 0|j_{\rm em}^{\rm QCD}|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle, \tag{2.3}$$

where $j_{\rm em}^{\rm QCD}$ is the QCD quark electromagnetic current.

The matrix element $\langle 0|\mathcal{O}|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle$ in some sense can also be viewed as the generalization of the scattering amplitudes, which in "all ingoing" notation can schematically be written as $\langle 0|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle$.

In the language of the dual string theory, in the $\mathcal{N}=4$ SYM case this process can be described as an insertion of some close string state (which corresponds to $\mathcal{O}$ local operator) on the worldsheet in addition to n open string states (which corresponds to $|p_1^{\lambda_1}\dots p_n^{\lambda_n}\rangle$ state in the dual theory).

For the construction of particular examples of the objects discussed above we choose the following set of the gauge invariant operators (we use the component notation of the $\mathcal{N}=4$ SYM), the lowest stress tensor supermultiplet members:

$$\begin{aligned} \mathcal{C}_{IJ} &= \mathrm{Tr}(\phi_I\phi_J), \\ \mathcal{V}_I^J &= \mathrm{Tr}(\bar{\phi}^J\phi_I), \end{aligned} \tag{2.4}$$

with naive mass dimension $\Delta_0 = 2$ which coincides with the conformal dimension due to the lack of quantum corrections. These operators can be viewed as the lowest members of the stress-tensor multiplet

$$T^{AB} = \mathrm{Tr}\left(W^A W^B - \frac{1}{6}\delta^{AB} W^C W_C\right), \tag{2.5}$$

where $A, B, \ldots = 1, \ldots, 6$ are the $\mathrm{SO}(6)_R \simeq \mathrm{SU}(4)_R$ indices, $I, J, \ldots = 1, 2, 3$ are the indices of SU(3) subgroup of $\mathrm{SU}(4)_R$, and W^A is some constrained chiral superfield in $\mathcal{N} = 4$ superspace containing all components of the $\mathcal{N} = 4$ supermultiplet.

Other objects are the so-called half-BPS operators

$$\mathcal{O}_I^{(n)} = \mathrm{Tr}(\phi_I^n), \tag{2.6}$$

whose naive mass dimension coincides with conformal dimension $\Delta_0 = n$ being protected from the quantum corrections, and the lowest component of the Konishi supermultiplet

$$\mathcal{K} = \sum_{I=1}^{3} \mathrm{Tr}(\bar{\phi}^I \phi_I), \tag{2.7}$$

with naive mass dimension $\Delta_0 = 2$ and has nonvanishing anomalous dimension due to the presence of the UV divergences. The calculation of this anomalous dimension has been intensively discussed during the last few years [27–29].

Since the Konishi operator is not protected, the corresponding form factors a priori do contain the UV divergences and hence must be UV renormalized. It means that one has to consider the renormalized form factor

$$\langle 0|\mathcal{K}_R|p_1^{\lambda_1} \ldots p_n^{\lambda_n}\rangle, \tag{2.8}$$

where

$$\mathcal{K}_R = Z_K^{-1}\mathcal{K}_B. \tag{2.9}$$

Here Z_K is the renormalization constant which appears due to the UV divergences and which should be calculated to the same order of perturbation theory as the form factors. After such UV renormalization we are left only with the IR divergences. All the statements concerning the Konishi operator are valid for the renormalized one.

We choose for simplicity the state $|p_1^{\lambda_1} \ldots p_n^{\lambda_n}\rangle$ produced by the operator $\mathcal{O}$ to consist of scalars only and then we can write it as $|p_1 \ldots p_n\rangle$ without helicities. We also restrict ourselves to the states with the number of particles equal to the naive mass dimension of the operator $\mathcal{O}$, i.e. we consider the states consisting of Δ_0 scalars.

2.2 Calculation strategy

For the calculation it is convenient to use the $\mathcal{N} = 1$ formulation of $\mathcal{N} = 4$ SYM and perform an explicit computation in terms of the $\mathcal{N} = 1$ superfields in momentum space. Our computation is familiar, from a diagrammatic point of view, to perturbative computations of anomalous dimensions [29]. However, there is a significant difference: each of our supergraphs is UV finite except for one. So all the divergences that appear throughout the calculation have the IR nature.

The operators $\mathcal{O} = \{\mathcal{C}_{IJ}, \mathcal{V}_I^J, \mathcal{K}, \mathcal{O}_I^{(n)}\}$ can be considered as the lowest components of the following $\mathcal{N}=1$ local operators:

$$\begin{aligned}
\mathcal{C}_{IJ} &= \mathrm{Tr}(\Phi_I\Phi_J), I \neq J \\
\mathcal{V}_I^J &= \mathrm{Tr}(e^{-gV}\bar{\Phi}^J e^{gV}\Phi_I), I \neq J \\
\mathcal{O}_I^{(n)} &= \mathrm{Tr}\left(\Phi_I^n\right), \\
\mathcal{K} &= \sum_I \mathrm{Tr}(e^{-gV}\bar{\Phi}^I e^{gV}\Phi_I),
\end{aligned} \tag{2.10}$$

where Φ_I are chiral $\mathcal{N}=1$ superfields, and V is $\mathcal{N}=1$ real vector superfield (see appendix A for details). The operators $\mathcal{C}_{IJ}$, $\mathcal{O}_I^{(n)}$ are chiral and $\mathcal{V}_I^J$, $\mathcal{K}$ are non-chiral from the $\mathcal{N}=1$ supersymmetric point of view.

We use the following notation for the form factor of the corresponding operator

$$\mathcal{F}(p_1 \dots p_n) = \langle p_1 \dots p_n|\mathcal{O}(q)|0\rangle. \tag{2.11}$$

We expect the following factorization property for $\mathcal{F}$ to hold:

$$\mathcal{F}(p_1 \dots p_n) = \mathcal{F}_{\text{tree}}(p_1 \dots p_n)(1+\text{loops}), \tag{2.12}$$

where $\mathcal{F}_{\text{tree}}$ stands for the tree level contribution, and "loops" schematically denote the contributions of the next orders of PT. It is convenient to consider the ratio

$$\mathcal{M} = \frac{\mathcal{F}}{\mathcal{F}_{\text{tree}}} = (1+loops) = \sum_{l=0} \lambda^l \mathcal{M}^{(l)},$$

where $\lambda \equiv g^2 N_c$ is the 't Hooft coupling which stays fixed when $N_c \to \infty$.

Consider first the chiral case. To calculate the form factor it is convenient to consider the generating functional for the one-particle irreducible super diagrams $\Gamma[\Phi^{cl}, J]$ in $\mathcal{N}=1$ superspace. It can be obtained from the generating functional

$$Z[j,J] = \int \mathcal{D}(\Phi_I, V, \dots)\exp[S^{\mathcal{N}=4} + \int d^6 z J(z)\mathcal{O}(z) + \int d^6 z \mathrm{Tr}(j(z)\Phi(z))],$$

after Legendre transformation with respect to external chiral sources j (note that the source J is untouched). After performing the D-algebra each supergraph gives a local contribution in θ's, and $\Gamma[\Phi^{cl}, J]$ can be written as (we imply the mass shell condition $p_i^2=0$ when performing the D-algebra)

$$\begin{aligned}
\Gamma[\Phi^{cl}, J] &= \sum_{l=0} \lambda^l \Gamma^{(l)}[\Phi^{cl}, J] \\
&= \sum_{l=0} \lambda^l \int d^4p_1 \dots d^4p_n \, d^6 z \, J(-q,\theta) \mathrm{Tr}\left(\Phi^{cl}(-p_1,\theta)\dots\Phi^{cl}(-p_n,\theta)\right) \mathcal{M}^{(l)}(p_1, \dots p_n) \\
&\quad + O(J^2)\,,
\end{aligned} \tag{2.13}$$

where $d^6z = d^4q d^2\theta$, $\mathcal{M}^{(l)}$ is given by the sum of scalar integrals. Thus,

$$\mathcal{M}^{(l)}(p_1 \dots p_n) = \frac{\delta^{n+1}\Gamma^{(l)}}{\delta\Phi^{cl}\dots\delta\Phi^{cl}\delta J}\bigg|_{p_i^2=0,\ \theta=0, \Phi^{cl}=0, J=0}. \tag{2.14}$$

We stress that on-shell condition $p_i^2 = 0$ and momenta conservation $q + p_1 + \ldots + p_n = 0$ are implemented to obtain the latter expression.

The situation is a bit more involved in the nonchiral case. All the integrals in $\Gamma[\Phi^{cl}, \bar{\Phi}^{cl}, \mathcal{J}]$ ($\mathcal{J}$ is a non-chiral source) are now in full $\mathcal{N} = 1$ superspace $\int d^8 z$, where $d^8 z = d^4 q d^4 \theta$ and the expression for $\Gamma[\Phi^{cl}, \bar{\Phi}^{cl}, \mathcal{J}]$ contains extra terms

$$\begin{aligned}
\Gamma[\Phi^{cl}, \bar{\Phi}^{cl}, \mathcal{J}] &= \sum_{l=0} \lambda^l \Gamma^{(l)}[\Phi^{cl}, \bar{\Phi}^{cl}, \mathcal{J}] = \\
&= \sum_{l=0} \lambda^l \int d^4 p_1 \ldots d^4 p_n \; d^8 z \; \mathcal{J}(-q, \theta, \bar{\theta}) \\
&\quad \times \Big[\mathrm{Tr}\left(\bar{\Phi}^{cl}(-p_1, \bar{\theta}) \ldots \Phi^{cl}(-p_n, \theta) \right) \mathcal{M}^{(l)}(p_1, \ldots p_n) \\
&\qquad + \mathrm{Tr}\left(\bar{D}^{\dot{\beta}} \bar{\Phi}^{cl}(-p_1, \bar{\theta}) \ldots D^{\alpha} \Phi^{cl}(-p_n, \theta) \right) \mathcal{M}^{(l)}_{\dot{\beta}\alpha}(p_1, \ldots p_n) \\
&\qquad + \mathrm{Tr}\left(\bar{D}^2 \bar{\Phi}^{cl}(-p_1, \bar{\theta}) \ldots D^2 \Phi^{cl}(-p_n, \theta) \right) \mathcal{M}^{(l)}_2(p_1, \ldots p_n) \Big] + O(\mathcal{J}^2) \,.
\end{aligned} \tag{2.15}$$

From the point of view of $\mathcal{N} = 1$ superspace the additional terms correspond to the operators of higher dimension and one actually has a mixing of several operators. However, from the point of view of components, one can always consider a projection on a particular component of a superfield and we choose the scalar component insofar. Then, the last terms of eq. (2.15) are irrelevant for our calculation and can be dropped.

We perform all the calculations in the formalism of $\mathcal{N} = 1$ superspace and at the end take the projection to $\theta = \bar{\theta} = 0$. There are pluses and minuses of this approach. The big advantage is the drastic reduction of the number of diagrams compared to the component case together with the simplified form of the scalar integrals. Its disadvantage is that we do not use the power of the on-shell $\mathcal{N} = 4$ covariant methods used in perturbative studies of the amplitudes [30, 31] (see also recent [32]). The application of this method for the calculation of the form factors when some legs are off-shell requires some modification.

2.3 IR finite observables based on form factors

As the amplitudes, the form factors $\langle 0|\mathcal{O}|p_1^{\lambda_1} \ldots p_n^{\lambda_n}\rangle$ with on-shell momenta are, strictly speaking, ill-defined in $D = 4$-dimensional space-time due to the presence of the IR divergences, and, hence, some IR regulator must be introduced — in our case it is the parameter μ coming from the dimensional regularization which also breaks the conformal symmetry. In other words, one may say that $\langle 0|\mathcal{O}|p_1^{\lambda_1} \ldots p_n^{\lambda_n}\rangle$ are the intermediate objects, and the true physical quantities are the IR safe observables constructed of $\langle 0|\mathcal{O}|p_1^{\lambda_1} \ldots p_n^{\lambda_n}\rangle$ and which are free from the IR regulator (see, for example, the discussion of the IR finite observables for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA in [33–35]). Indeed, as in QCD for $\gamma^* \to$ jets processes we are really interested in the total cross section $\sigma_{\mathrm{tot}}(\gamma^* \to \text{jets})$ or some differential distributions rather than in the matrix elements $\langle 0|j_{\mathrm{em}}^{\mathrm{QCD}}|p_1^{\lambda_1} \ldots p_n^{\lambda_n}\rangle$ themselves. This kind of observables are IR finite due to the *Kinoshita-Lee-Nauenberg (KLN) theorem* which states that it is not sufficient to consider only the processes with the fixed number of final particles. To get the physical result, one has to include all the processes allowed by conservation laws in the same order of perturbation theory with emission of extra soft quanta and integrate over their momenta. Practically, if the dimensional regularization is used (the IR divergences manifest themselves through the appearance of the $1/\epsilon$ poles), part of the poles cancel between the loop integrals from the virtual contributions and the phase space integrals from the real contributions coming from the processes with additional particles, while others are absorbed in the functions describing probability distributions of the initial and final states (in [33–35] we call them initial-collinear and final-collinear divergences which appear as a collinear configuration of initial and final particles).

Consider, for instance, the total cross section σ_{tot} for the process

$$J \to \text{anything from } \mathcal{N} = 4 \text{ supermultiplet}$$

for classical current J coupled to $\mathcal{N} = 4$ through some local gauge invariant operator $\mathcal{O}$. Due to the optical theorem

$$\sigma_{\text{tot}}(s) \sim \frac{1}{s} Im_s \left[\int d^D x \exp(-iqx) \langle \mathcal{O}(x)\mathcal{O}(0)\rangle \right], \; q^2 = -s, \tag{2.16}$$

The two-point function for the operators $\mathcal{O}$ apart from the canonical mass dimension Δ_0 can have anomalous dimension $\gamma = \gamma(\lambda)$ being a function of the coupling constant

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{(x^2)^{(\Delta_0(1-\epsilon)+\gamma)}}, \tag{2.17}$$

After some calculation this gives the total cross-section

$$\sigma_{\text{tot}}(s) \sim \frac{1}{\Gamma(\Delta_0+\gamma)\Gamma(\Delta_0+\gamma-1)} \frac{1}{s^{3-\Delta_0-\gamma}}, \tag{2.18}$$

and its asymptotic at weak and strong couplings can be studied (compare this with C.3 from [36]).

In $\mathcal{N} = 4$ SYM, as in any conformal theory, if the operator $\mathcal{O}$ is protected, which means that it does not receive quantum corrections, then $\gamma = 0$. Then the cross section is independent of the coupling constant and behaves like $\sim C/s^{3-\Delta_0}$. From the latter expression it might seem that we get violation of unitarity since we can get increasing cross sections for protected operators with conformal dimension greater than 3. But it is not the case since the statement about the unitarity holds only for the operators which give rise to renormalizable interactions, i.e. with conformal dimension less than 3.

If one is interested not in σ_{tot} but in some differential distributions, then the optical theorem is not very useful any more, and direct computations must be done. The form factors discussed here can be viewed as the building blocks in the same sense as the amplitudes for the inclusive cross sections.

3 Form factors with $\Delta_0 = 2$

In the following two sections we give explicit results for the direct diagrammatic computation of the form factors of the operators introduced above in the planar limit.[2] More

[2] $g \to 0$ and $N_c \to \infty$ so that $\lambda = g^2 N_c$ =fixed.

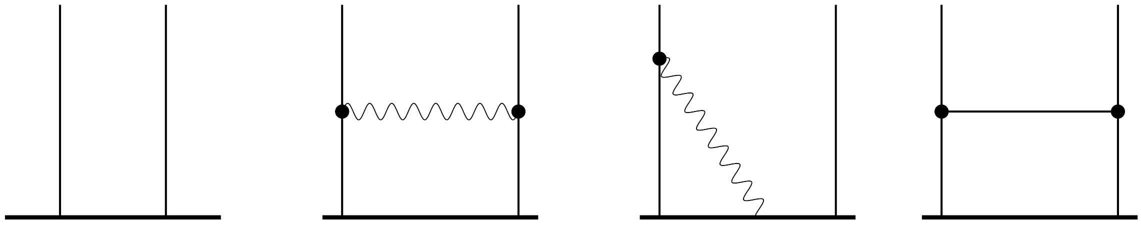

Figure 2. The relevant supergraphs. The internal black lines correspond to chiral propagators $\langle\bar{\Phi}^a_I\Phi^b_J\rangle$, wavy lines correspond to vector $\langle V^aV^b\rangle$ propagator (see appendix A). $C0$ is the tree level diagram, and the rest are one-loop ones. External lines are Φ or $\bar{\Phi}$, and the lower bold line represents the insertion of the corresponding operator in modern notation. For the chiral operator $\mathcal{C}_{IJ}$ only the diagrams $C0$ and $C1$ contribute, while for non-chiral operators $\mathcal{V}^J_I$ and $\mathcal{K}$ the other two ($B1$ and $B2$) are also relevant.

concretely we present the results for non-chiral operators $\mathcal{V}^J_I$, $\mathcal{K}$ in the leading order in λ and for the chiral operators $\mathcal{C}_{IJ}$, $\mathcal{O}^{(n)}_I$ in the next-to-leading order. The dimensional regularization (dimensional reduction to be precise) with $D = 4-2\epsilon$ is used. All the divergences except for the specially mentioned cases have the IR (both soft and collinear) nature. The Feynman rules for the supergraphs are given in appendix A. The complete list of all the necessary scalar integrals is given in appendix B. The results for $\mathcal{C}_{IJ}$, $\mathcal{V}^J_I$ coincides with those presented in [14] because $\mathcal{V}_X$ which in $\mathcal{N}=1$ superspace notations takes the form $2\mathrm{Tr}\Phi_1\overline{\Phi}_1 - \mathrm{Tr}\Phi_2\overline{\Phi}_2 - \mathrm{Tr}\Phi_3\overline{\Phi}_3$ and $\mathcal{C}_{IJ}$, $\mathcal{V}^J_I$ lie in the same $\mathcal{N}=4$ supermultiplet.

3.1 $\mathcal{C}_{IJ}$, $\mathcal{V}^J_I$ and $\mathcal{K}$ form factors at 1-loop

The corresponding tree level and one-loop Feynman diagrams are shown in figure 2.

For the chiral operator $\mathcal{C}_{IJ}$, after performing the D-algebra for the supergraph $C1$, the resulting expression is

$$C1 = \mathrm{Tr}\left(\Phi^{cl}_I\Phi^{cl}_J\right)\ 2s_{12}\ G_1(s_{12}), \tag{3.1}$$

where $s_{ij} = (p_i+p_j)^2$, so that

$$\mathcal{M}^{(1)} = 2s_{12}\ G_1(s_{12}), \tag{3.2}$$

where the scalar integral $G_1(s_{12})$ is given in appendix B Hereafter we will suppress index cl in Φ and $\bar{\Phi}$.

For the non-chiral operator $\mathcal{V}^J_I$ after performing the D-algebra one has

$$\begin{aligned}
C1 &= 2\left((-G_0(p_1^2) - G_0(p_2^2) + G_0(s_{12}) + (s_{12}-p_1^2-p_2^2)G_1(s_{12}))\ \mathrm{Tr}\left(\bar{\Phi}^J\Phi_I\right)\right.\\
&\qquad \left. -G_1^{\alpha\dot{\beta}}(s_{12})\ \mathrm{Tr}\left(\bar{D}^{\dot{\beta}}\bar{\Phi}^J D^\alpha\Phi_I\right) + G_1(s_{12})\ \mathrm{Tr}\left(\bar{D}^2\bar{\Phi}^J D^2\Phi_I\right)\right),\\
B1 &= 2G_0(p_i^2)\ \mathrm{Tr}\left(\bar{\Phi}^J\Phi_I\right),\\
B2 &= 2\left(-G_0(s_{12})\mathrm{Tr}\left(\bar{\Phi}^J\Phi_I\right) + G_1^{\alpha\dot{\beta}}(s_{12})\mathrm{Tr}\left(\bar{D}^{\dot{\beta}}\bar{\Phi}^J D^\alpha\Phi_I\right) + G_1(s_{12})\mathrm{Tr}\left(\bar{D}^2\bar{\Phi}^J D^2\Phi_I\right)\right),
\end{aligned} \tag{3.3}$$

where all the scalar integrals are also given in appendix B. So keeping only the terms that are relevant for our discussion, which are proportional to $\mathrm{Tr}\left(\bar{\Phi}^J\Phi_I\right)$, one gets

$$\mathcal{M}^{(1)} = C1(s_{12}, p_1^2, p_2^2) + B1(p_1^2) + B1(p_2^2) + B2(s_{12}) = 2(s_{12} - p_1^2 - p_2^2)G_1(s_{12}), \tag{3.4}$$

The integral G_1 is UV finite which reflects the fact that $\mathcal{V}_I^J$ is a protected operator.

For the non-chiral Konishi operator $\mathcal{K}$, after performing the D-algebra one has

$$\begin{aligned}
C1 &= 6\Bigg(\left(-G_0(p_1^2) - G_0(p_2^2) + G_0(s_{12}) + (s_{12} - p_1^2 - p_2^2)G_1(s_{12})\right)\sum_I^3 \mathrm{Tr}\left(\bar{\Phi}^I\Phi_I\right) \\
&\qquad - G_1^{\alpha\dot{\beta}}(s_{12})\sum_I^3 \mathrm{Tr}\left(\bar{D}^{\dot{\beta}}\bar{\Phi}^I D^\alpha \Phi_I\right) + G_1(s_{12})\sum_I^3 \mathrm{Tr}\left(\bar{D}^2\bar{\Phi}^I D^2\Phi_I\right)\Bigg), \\
B1 &= 6G_0(p_i^2)\sum_I^3 \mathrm{Tr}\left(\bar{\Phi}^I\Phi_I\right), \\
B2 &= 6\Bigg(-G_0(s_{12})\sum_I^3 \mathrm{Tr}\left(\bar{\Phi}^I\Phi_I\right) + G_1^{\alpha\dot{\beta}}(s_{12})\sum_I^3 \mathrm{Tr}\left(\bar{D}^{\dot{\beta}}\bar{\Phi}^I D^\alpha\Phi_I\right) \\
&\qquad + G_1(s_{12})\sum_I^3 \mathrm{Tr}\left(\bar{D}^2\bar{\Phi}^I D^2\Phi_I\right)\Bigg),
\end{aligned} \tag{3.5}$$

and again selecting the proper structures, prior to the application of the on-shell conditions, gives

$$\mathcal{M}^{(1)} = C1(s_{12}, p_1^2, p_2^2) + B1(p_1^2) + B1(p_2^2) + 2B2(s_{12}) = 6(s_{12} - p_1^2 - p_2^2)G_1(s_{12}) - 6G_0(s_{12}), \tag{3.6}$$

The UV divergent part of the answer is given by $6G_0$ and extracting the coefficient of the $1/\epsilon$ pole, which is the first coefficient in the anomalous dimension expansion $\gamma_{\mathcal{K}}(\lambda) = \gamma_{\mathcal{K}}^{(1)}\lambda + \dots$, we obtain the well-known result

$$\gamma_{\mathcal{K}}^{(1)} = \frac{3}{8\pi^2}. \tag{3.7}$$

We see that up to one loop all the form factors for the operators $\mathcal{C}_{IJ}, \mathcal{V}_I^J, \mathcal{K}, \mathcal{O}_I^{(n)}$ are proportional to G_1, the scalar triangle function (see appendix B).

3.2 $\mathcal{C}_{IJ}$ form factors at 2-loops

We see that the form factors associated with $\mathcal{C}_{IJ}$ and $\mathcal{V}_I^J$ are equal to each other at the one-loop level. This is because $\mathcal{C}_{IJ}$ and $\mathcal{V}_I^J$ are different components of the $\mathcal{N} = 4$ conserved stress tensor. In what follows we compute the λ^2 contribution to $\mathcal{M}$ for $\mathcal{C}_{IJ}$ since for the chiral operator the D-algebra is essentially simpler. The corresponding diagrams are shown in figure 3.

Their contribution to the form factor are summarized in table 1. All the relevant integrals are given in appendix B.

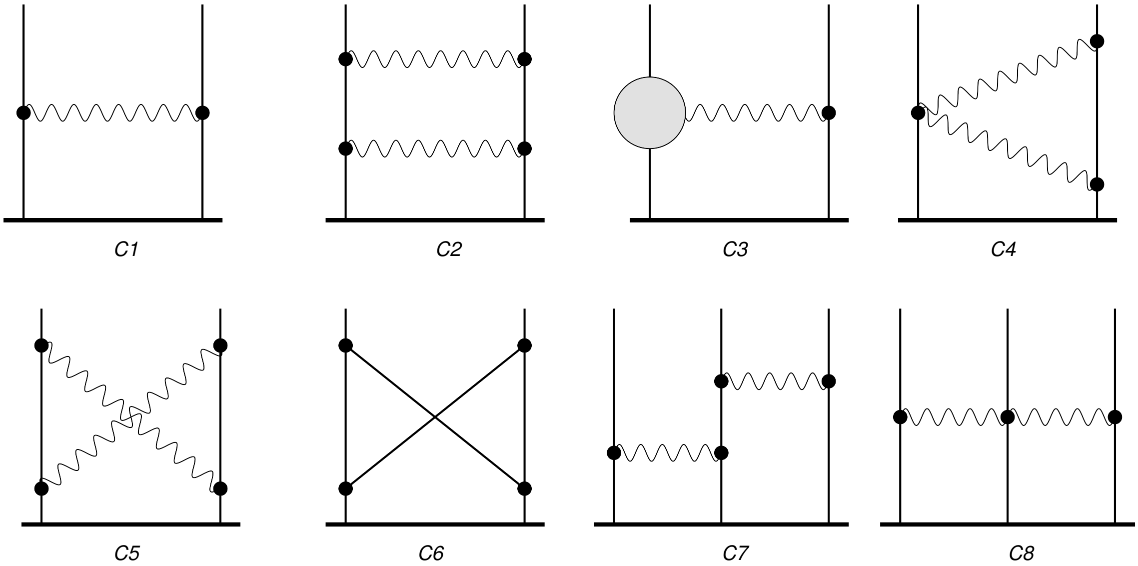

Figure 3. The relevant supergraphs in the chiral case. $C1$ is the one-loop diagram, and the rest are two-loop ones. For the chiral operator $\mathcal{C}_{IJ}$ with two legs the last two diagrams $C7$ and $C8$ do not exist, they are only relevant for the operator $\mathcal{O}_n$ with $n \geq 3$. A grey circle is the one-loop effective vertex.

N	$\mathcal{C}_{IJ}$	$\mathcal{O}_I^{(n)}$
C1	$2s_{12}G_1(s_{12})$	$s_{ii+1}G_1(s_{ii+1})$
C2	$4s_{12}^2G_2(s_{12})$	$s_{ii+1}^2G_2(s_{ii+1})$
C3	$2s_{12}G_3(s_{12}) + 2s_{12}G_4(s_{12})$	$s_{ii+1}G_3(s_{ii+1}) + s_{ii+1}G_4(s_{ii+1})$
C4	$-6s_{12}G_3(s_{12})$	$-2s_{ii+1}G_3(s_{ii+1})$
C5	$2G_5^a(s_{12})$	0
C6	$2G_5^b(s_{12})$	0
C7	0	$(s_{ii+1} + s_{i+1i+2} + s_{ii+2})G_6(s_{ii+1}, s_{i+1i+2}, s_{ii+2})$
C8	0	$s_{i+1i+2}G_7(s_{ii+1}, s_{i+1i+2}, s_{ii+2})$

Table 1. The contributions to the form factors from the individual diagrams.

Adding all together and combining with the leading order one gets

$$\mathcal{M}^{(2)} = C2 + C3 + C4 + C5 + C6 = 2G_5^a + 2G_5^b - 4s_{12}G_3 + 2s_{12}G_4 + 4s_{12}^2G_2. \tag{3.8}$$

Using the identity

$$G_5^a + G_5^b = 2s_{12}G_3 - s_{12}G_4 + \frac{s_{12}^2}{2}G_5$$

one can reduce it to $4s_{12}^2\ G_2 + s_{12}^2\ G_5$ and finally get

$$\mathcal{M} = 1 + \lambda(2s_{12}\ G_1) + \lambda^2(4s_{12}^2\ G_2 + s_{12}^2\ G_5), + O(\lambda^3). \tag{3.9}$$

The general structure of form factors $\mathcal{M}$ "with two external legs" in the gauge theory with zero beta-function has the following form:[3]

$$\log(\mathcal{M}) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{M}(s_{i,i+1}/\mu^2)\right) + O(\epsilon). \tag{3.10}$$

Here we introduced

$$\hat{M}(s_{i,i+1}/\mu^2) = -\frac{1}{2}\sum_{l}\left(\frac{\lambda}{16\pi^2}\right)^{l}\left(\frac{\Gamma^{(l)}_{\rm cusp}}{(l\epsilon)^2} + \frac{G^{(l)}}{l\epsilon} + C^{(l)}\right)\left(\frac{s_{i,i+1}}{\mu^2}\right)^{l\epsilon}, \tag{3.11}$$

where $\Gamma^{(l)}_{\rm cusp}$ are the coefficients of perturbative expansion of the cusp anomalous dimension $\Gamma_{\rm cusp}(\lambda) = \sum_l \Gamma^{(l)}_{\rm cusp}\lambda^l$ which is a universal quantity that governs the IR behavior of gauge theory amplitudes and the UV behavior of the Wilson loops, and some local gauge invariant operators. $G^{(l)}$ are the coefficients of perturbative expansion of the so-called collinear anomalous dimension $G(\lambda) = \sum_l G^{(l)}\lambda^l$ and $C^{(l)}$ are some constants. The quantities $G^{(l)}$ and $C^{(l)}$ are regularization and scheme dependent. Performing the expansion of the integrals G_1, G_2, G_5 in ϵ (see appendix B) and introducing the notation

$$a = \frac{\lambda}{16\pi^2}e^{-\epsilon\gamma_E}, \tag{3.12}$$

where γ_E is the Euler-Mascheroni constant we get the same result as in [14]

$$\log(\mathcal{M}) = a\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\left(\frac{-2}{\epsilon^2} + \zeta_2\right) + a^2\left(\frac{s_{12}}{\mu^2}\right)^{-2\epsilon}\left(\frac{\zeta_2}{\epsilon^2} + \frac{\zeta_3}{\epsilon}\right) + O(a^3) \tag{3.13}$$

where ζ_n are the Riemannian zeta functions

$$\zeta_n = \sum_{k=1}^{\infty}\frac{1}{k^n}.$$

From this answer and comparing with eq. (3.11) we can extract the first two terms of perturbative expansion over a for the cusp and the collinear anomalous dimensions and the finite terms

$$\Gamma^{(1)}_{\rm cusp} = 4, \quad \Gamma^{(2)}_{\rm cusp} = -8\zeta_2, \tag{3.14}$$
$$G^{(1)} = 0, \quad G^{(2)} = -\zeta_3, \tag{3.15}$$
$$C^{(1)} = -\zeta_2, \; C^{(2)} = 0. \tag{3.16}$$

Note that the maximal transcendentality principle [55] holds which in our case means that if we attach to each logarithm and π the level of transcendentality equal to 1 and to polylogarithms $Li_n(x)$ and ζ_n the level of transcendentality equal to n, then at the given order of perturbation theory the coefficient for the n-th pole $1/\epsilon^n$ has the overall transcendentality equal to $2l - n$, where l is the number of loops. For a product of several factors it is given by the sum of transcendentalities of each factor.

[3]The IR exponentiation for two-leg form factors in QCD was established earlier in [37, 38].

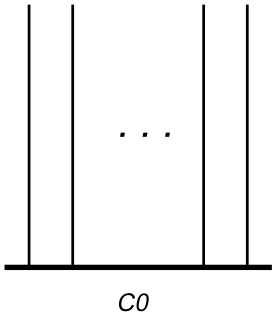

Figure 4. The tree contribution to $\mathcal{O}_I^{(n)}$.

The leading IR behavior of $\mathcal{M}$ in this case can also be captured by considering the Wilson line with one cusp [39, 40] . So in this sense the dual description in terms of Wilson loops for such form factors is well known.

One can see that the finite part for the form factor is given only in one loop and vanishes at two loops. However, this is a scheme dependent result, and, for example, if we choose a different scheme and replace $\exp(l\epsilon\gamma_E)$ for the l-th loop by $\Gamma(1-\epsilon)^l$, we obtain in this scheme:

$$\tilde{C}^{(1)} = 0, \;\; \tilde{C}^{(2)} = -\zeta_2^2, \tag{3.17}$$

while the first two coefficients in the perturbation theory for the cusp anomalous dimension $\Gamma^{(1)}_{\text{cusp}}$ and $\Gamma^{(2)}_{\text{cusp}}$ remain the same, which reflects the fact that they are scheme independent.

The same result is true, and should coincide with [14] for the form factor of a slightly different operator $\mathcal{V}_X$, because it belong to the same stress-tensor superconformal multiplet.

4 Form factors with $\Delta_0 = n, \;\; n > 2$

Here we present the results of calculation of the form factors of the chiral half-BPS operators $\mathcal{O}_I^{(n)}$ introduced earlier. The tree-level contribution for the form factor is presented on figure 4. In the first order of perturbation theory, similar to the form factors of operators with conformal dimension 2, the contribution is given by the triangle type diagram and the corresponding form factor, after performing the D-algebra and the color algebra, is

$$\mathcal{M}^{(1)} = \sum_{i=1}^{n} s_{i,i+1}\, G_1, \tag{4.1}$$

where we assume hereafter $s_{n+i,n+i+1} \;\; = \;\; s_{i,i+1}$.

4.1 $\mathcal{O}_I^{(n)}$, $n=3$ form factors at 2-loops

At the second order of perturbation theory the corresponding diagrams are shown in figure 3 and their contributions are summarized in table 1.

$$
\begin{aligned}
\mathcal{M}^{(2)} &= \sum_{i=1}^{n} \left(s_{ii+1}^2 G_2(s_{ii+1}) - s_{ii+1} G_3(s_{ii+1}) + s_{ii+1} G_4(s_{ii+1})\right) \\
&\quad + \sum_{i=1}^{n} \left(s_{i+1i+2} G_7(s_{ii+1}, s_{i+1i+2}, s_{ii+2}) + s_{ii+1} G_7(s_{i+1i+2}, s_{ii+1}, s_{ii+2})\right) \\
&\quad + \sum_{i=1}^{n} \left(s_{ii+1} + s_{i+1i+2} + s_{ii+2}\right) G_6(s_{ii+1}, s_{i+1i+2}, s_{ii+2}) \qquad (4.2)
\end{aligned}
$$

The next step is to establish the factorization property (3.10), (3.11). Expanding the relevant scalar integrals in ϵ (see appendix B), we obtain for $\log(\mathcal{M})$:

$$
\log(\mathcal{M}) = \sum_{i=1}^{3} a \left(\frac{s_{ii+1}}{\mu^2}\right)^{-\epsilon} \left(-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2}\right) + \sum_{i=1}^{3} a^2 \left(\frac{s_{ii+1}}{\mu^2}\right)^{-2\epsilon} \left(\frac{\zeta_2}{2\epsilon^2} + \frac{7\zeta_3}{2\epsilon}\right) + \text{fin.part.} \quad (4.3)
$$

As in the case of the form factors of the operators with conformal dimension 2, we can extract the first two terms for the cusp and collinear anomalous dimensions. This gives

$$
\Gamma_{\text{cusp}}^{(1)} = 4, \quad \Gamma_{\text{cusp}}^{(2)} = -8\zeta_2, \qquad (4.4)
$$

$$
G^{(1)} = 0, \quad G^{(2)} = -7\zeta_3. \qquad (4.5)
$$

Notice that the values of the cusp anomalous dimension $\Gamma^{(l)}$ are universal and coincide with (3.14), while those of the collinear anomalous dimension depend on the form factor at hand.

We would like to emphasize the highly nontrivial cancelations between the polylogarithms that occurred for $\log(\mathcal{M})$ for the whole set of scalar integrals (the individual contributions to the poles from the scalar integrals are usually complicated polynomials of logarithms and polylogarithms of different weight, see, for example, the expansions in ϵ of G_6 and G_7 in appendix B).

We see that the IR factorization property holds for the form factors like for the amplitudes.

4.2 $\mathcal{O}_I^{(n)}$ form factors for $n > 3$ at 2-loops

The corresponding contribution to the form factor up to λ^2 is similar to the case of $n = 3$ but has an additional term coming from the factorized diagrams

$$\begin{aligned}\mathcal{M}^{(2)} = &\sum_{i=1}^{n} \left(s_{ii+1}^2 G_2(s_{ii+1}) - s_{ii+1} G_3(s_{ii+1}) + s_{ii+1} G_4(s_{ii+1})\right)\\ &+\sum_{i=1}^{n} \left(s_{i+1i+2} G_7(s_{ii+1}, s_{i+1i+2}, s_{ii+2}) + s_{ii+1} G_7(s_{i+1i+2}, s_{ii+1}, s_{ii+2})\right)\\ &+\sum_{i=1}^{n} \left(s_{ii+1} + s_{i+1i+2} + s_{ii+2}\right) G_6(s_{ii+1}, s_{i+1i+2}, s_{ii+2})\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n} s_{ii+1} G_1(s_{ii+1}) s_{jj+1} G_1(s_{jj+1}) \end{aligned} \tag{4.6}$$

Performing the expansion over ϵ we obtain the logarithm of the form factor up to the second order of perturbation theory $\log(\mathcal{M})$

$$\log(\mathcal{M}) = \sum_{i=1}^{n} a \left(\frac{s_{ii+1}}{\mu^2}\right)^{-\epsilon} \left(-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2}\right) + \sum_{i=1}^{n} a^2 \left(\frac{s_{ii+1}}{\mu^2}\right)^{-2\epsilon} \left(\frac{\zeta_2}{2\epsilon^2} + \frac{7\zeta_3}{2\epsilon}\right) + \text{Fin.part.} \tag{4.7}$$

The first two coefficients for the cusp and collinear anomalous dimension which we can extract from the above expression coincide with the coefficients obtained earlier for $n = 3$, (4.4) and (4.5), respectively. As for the finite part

$$\text{Fin.part.} = \lambda F^{(1)}(s_{12}, \ldots, s_{n1}) + \lambda^2 F^{(2)}(s_{12}, \ldots, s_{n1}) + O(\lambda^3), \tag{4.8}$$

at one loop it is trivial $F^{(1)} = 0$, and the two loop expression $F^{(2)}$, contrary to the previous case, is a complicated function containing logarithms, polylogarithms and generalized Goncharov polylogarithms [46–48] of several variables. All the relevant expressions can be found in appendix B. Using the notation from appendix B one can write $F^{(2)}$ in the following schematic way:

$$\begin{aligned} F^{(2)} = &\sum_{i=1}^{n} \left(s_{i+1i+2} \left(G_7\right)_{\text{fin}} (s_{ii+1}, s_{i+1i+2}, s_{ii+2}) + s_{ii+1} \left(G_7\right)_{\text{fin}} (s_{i+1i+2}, s_{ii+1}, s_{ii+2})\right)\\ &+\sum_{i=1}^{n} \left(s_{ii+1} + s_{i+1i+2} + s_{ii+2}\right) \left(G_6\right)_{\text{fin}} (s_{ii+1}, s_{i+1i+2}, s_{ii+2}) + C_n, \end{aligned} \tag{4.9}$$

where C_n is some kinematical independent constant. For example for $n = 2k$ we have $C_n = -\zeta_2^2 k(k\ 21/2 + 227/20)$ Note, the result is still much simpler than in the non-supersymmetric case [45].

4.3 Collinear limit

Here we restrict ourselves to the three-leg form factors for which we can study the simplified kinematics and express the finite part in terms of logarithms only without polylogarithms or Goncharov generalized polylogarithms. The most difficult part which appears in our calculation comes from the diagram involving the interaction of three external fields. It reduces to the integral G_7 which is expressed in terms of the Appell function of two variables

$$F_1(1;2\epsilon,1;2+\epsilon|x,y)$$

and after the ϵ-expansion one obtains the generalized Goncharov polylogarithms [46–48]. One can see from integral representation of the Appell function

$$F_1(a;b_1,b_2;c|x,y)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\frac{u^{a-1}(1-u)^{c-a-1}}{(1-ux)^{b_1}(1-uy)^{b_2}}du, \mathrm{Re}a, \mathrm{Re}(c-a)>0 \quad (4.10)$$

that the only way to achieve the desired simplification is to have one of the arguments equal to 0 or to 1. In the first case one gets

$$F_1(a;b_1,b_2;c|x,0) \;=\; {}_2F_1(a,b_1;c|x), \quad (4.11)$$

and similarly in the second case

$$F_1(a;b_1,b_2;c|x,1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b_2)}{\Gamma(c-a)\Gamma(c-b_2)}\,{}_2F_1(a,b_1;c-b_2|x). \quad (4.12)$$

Such a simplification can occur in two-dimensional kinematics when one of the kinematical variables s_{12}, s_{13} or s_{23} equals 0. The other motivation for this kinematics is the recent strong coupling calculations which have been performed for the AdS_3 sub-manifold of AdS_5 which corresponds to the degenerate $1+1$ kinematics in a dual theory [13].

The $1+1$ dimensional kinematics necessarily contains a collinear configuration of the space components $\vec{p}_i$ of momenta p_i. For massless gauge theory it is known that in such collinear limit the factorization of the IR divergencies fails. For the partial color ordered amplitudes in collinear limit when two momentums p_i and p_{i+1} are replaced by zp and $(1-z)p$ the deviation from the factorized form is governed by the so-called "loop splitting functions" $r_s^{(l)}(\epsilon,z,p^2)$, l being the number of loops. In the $\mathcal{N}=4$ SYM theory $r_s^{(l)}(\epsilon,z,p^2)$ have an iterative structure, so one can write the following relation valid in collinear limit (see, for example, the discussion in [73])

$$\log(M_n)\rightarrow\frac{1}{2}\hat{M}_{n-1}+\sum_l\lambda^l\Gamma^{(l)}_{\rm cusp}r_s^{(l)}(l\epsilon,z,p^2)+\sum_l\lambda^lF^{(l),\,\rm coll}_{n-1}+O(\epsilon)$$

$$r_s^{(l)}(l\epsilon,z,p^2)\sim\frac{1}{\epsilon^2}\left(\frac{p^2}{\mu^2}\right)^{\epsilon}\left(-\frac{\pi\epsilon}{\sin(\pi\epsilon)}\left(\frac{1-z}{z}\right)^{\epsilon}+2\sum_{k=0}\epsilon^{2k+1}Li_{2k+1}\left(\frac{-z}{1-z}\right)\right)$$

We expect that similar violation of the IR factorization happens in the case of the form factors. Indeed, in the $s_{23}\rightarrow 0$ limit we have, up to λ^2

$$\begin{aligned}\log(\mathcal{M}) \;=&\; \sum_{i=1}^{2}a\left(\frac{s_{ii+1}}{\mu^2}\right)^{-\epsilon}\left(-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}\right)+\sum_{i=1}^{2}a^2\left(\frac{s_{ii+1}}{\mu^2}\right)^{-2\epsilon}\left(\frac{\zeta_2}{2\epsilon^2}+\frac{\zeta_3}{2\epsilon}\right)\\ &+\sum_{i=1}^{2}a^2\left(\frac{s_{ii+1}}{\mu^2}\right)^{-2\epsilon}\left(\frac{-6\zeta_2+3\log^2\frac{s_{12}}{s_{13}}}{96\epsilon^2}+\frac{19\zeta_3}{8\epsilon}\right)\\ &-\frac{a^2}{2880}\left(75\log^4\frac{s_{12}}{s_{13}}+120\pi^2\log^2\frac{s_{12}}{s_{13}}-317\pi^4\right). \end{aligned} \quad (4.13)$$

We would like to point out that the maximal transcendentality principle holds for this expression.

5 Dual conformal invariance

Here we would like to discuss the property of dual conformal invariance of the integrals which appear in calculation of the amplitudes. The off-shell amplitudes in $\mathcal{N}=4$ SYM obey the dual (super) conformal symmetry as has been pointed out in [4, 5]. When going on-shell, this symmetry is broken due to the presence of the IR regulator but the breaking is controlled by the one-loop anomaly. In the MHV case this symmetry still can be used to constraint the finite parts of the amplitudes due to the anomalous Ward identities. The non-MHV case is sufficiently less studied, but it seems that properly understood dual (super)conformal symmetry can be still used to restrict finite parts [2, 3].

The reflection of the dual (super)conformal symmetry of the amplitudes in $\mathcal{N}=4$ SYM is that scalar integrals appearing in the calculation are pseudo-conformal in momentum space [49], i.e., if one considers these integrals with off-shell external legs $p_i^2=m^2$ (which can be understood as some kind of IR regularization) they are well defined in $D=4$ and are conformal invariant in dual momentum space. It is remarkable that all the scalar integrals appearing in our computation of formfactors for the half-BPS operators $\mathcal{O}_I^{(n)}$ can be obtained from the pseudo-conformal integrals which are present in the amplitudes with the help of some special limiting procedure. This might be some hint that the dual (super)conformal symmetry can also restrict the finite parts of the formfactors similar to the amplitudes.[4]

Consider several examples. At one loop there is a single triangle diagram contributing to all the form factors. The one-loop triangle is the first in a chain of the ladder type diagrams [51, 52] and has the property of dual conformal invariance [53, 54]. This diagram is connected to the box diagram, which is dual conformal, in the following way. Consider the one-loop off-shell box diagram in momentum space which is given by the integral

$$D^{1-\mathrm{loop}}(p_1,p_2,p_3,p_4)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-p_1)^2(k+p_2)^2(k+p_2+p_3)^2}, \tag{5.1}$$

where all external momenta $p_i^2, i=1,2,3,4$ are *off-shell*, i.e. $p_i^2\neq 0$. Introducing the dual coordinates x_i as

$$p_1=x_{12},\ p_2=x_{23},\ p_3=x_{34},\ p_4=x_{41},\ k=x_5,$$

one can rewrite the initial integral in the following form

$$D^{1-\mathrm{loop}}(x_1,x_2,x_3,x_4)=\int\frac{d^4x_5}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}=\frac{1}{x_{13}^2x_{24}^2}\Phi(X,Y), \tag{5.2}$$

[4]Additional hint in this direction is that the formfactors have the dual description in terms of Wilson loops like the amplitudes [74].

where we introduced the notation $x_{ij} = x_i - x_j$ and $\Phi(X,Y)$ is the function given in [51, 52], X and Y are the conformal cross-ratios

$$X = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \; Y = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

To show that the one-loop box diagram is dual conformal invariant [53, 54] perform the inversion of the argument

$$x_\mu \to \frac{x_\mu}{x^2},$$

so that the distance between two points and the measure of integration tranform as

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, d^4x_5 \to \frac{d^4x_5}{x_5^8}.$$

It is easy to see that the integral (5.1) is dual conformal invariant. Here we would like to point out that this is only true in four-dimensional space-time.

If we now multiply (5.1) by x_{12}^2 and take the limit $x_2 \to \infty$ we obtain the one-loop triangle diagram [54] (see also [51, 52])

$$C^{1-\text{loop}} = \lim_{x_2 \to \infty} x_{12}^2 \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \int \frac{d^4x_5}{x_{15}^2 x_{35}^2 x_{45}^2} = \frac{1}{x_{34}^2} \Phi(x,y), \tag{5.3}$$

with

$$x = \frac{x_{34}^2}{x_{13}^2}, \; y = \frac{x_{14}^2}{x_{13}^2}.$$

To justify the limit one can divide the integration region into two parts: $x_5 < x_2$ and $x_5 > x_2$. In the first region we can neglect x_5 in comparison to x_2 and the integral simplifies as

$$\lim_{x_2 \to \infty} x_{12}^2 \int_0^{x_2} \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \sim \int_0^{\infty} \frac{d^4x_5}{x_{15}^2 x_{35}^2 x_{45}^2},$$

while in the second region we can neglect x_2 in comparison to x_5 and by naïve power counting arguments we get

$$x_2^2 \int_{x_2}^{\infty} \frac{d^4x_5}{(x_5^2)^4} \sim \frac{1}{x_2^2},$$

which vanishes in the limit $x_2 \to \infty$. Summing up the two contributions we get (5.3). The same argument works for more complicated cases discussed further.

Schematically, the described procedure of obtaining the one-loop triangle diagram from the one-loop box diagram is represented in figure 5.

On the left hand side one has the one-loop box diagram together with the dual grid, the black lines represent the denominator terms appearing in the integral in x-space. Taking the limit $x_2 \to \infty$ in (5.3) is equivalent to removing the grid line x_{25} from the dual graph and shrinking the crossed line to a point in the initial graph. The resulting initial graph corresponds to the triangle diagram, as is shown on the right hand side. This way

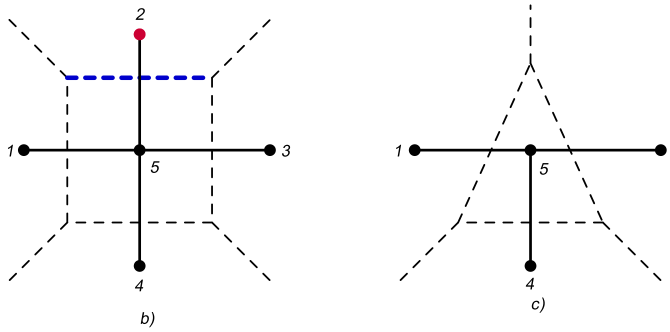

Figure 5. The one-loop triangle diagram from the one-loop box diagram. The red dot should be taken to infinity, and the blue line (propagator in momentum space) should be contracted to a point.

the triangle diagram can be obtained from the box one and inherit its property of dual conformal invariance.

In the same manner one can show how the other diagrams which appear in our calculation can be obtained from the corresponding diagrams entering into the amplitude calculations. Schematically, we present this procedure in figure 6.

For the vertical box diagram one has to take the limit $x_3 \to \infty$. As in the previous case this corresponds to removing the grid line x_{36} (and shrinking the corresponding crossed line) which results in the diagram shown on the right hand side. This is exactly the ladder integral that appears in two-loop calculation of the form factor with $n \geq 2$ legs.

For the horizontal box diagram one should take the combined limit $x_2, x_3 \to \infty$ which is schematically shown on the right hand side. This is the new type of integrals which appears only in the case when $n > 2$.

The same procedure is expected to work at higher levels of perturbation theory. Our conjecture is that the integrals appearing at any order of perturbation theory in calculation of form factors for the operators with mass dimension greater than two are obtained from dual conformal invariant diagrams by contraction of n propagators at the n-th loop order. For the crossed ladder type diagrams the above procedure does not work since these diagrams are non-planar and the dual graphs do not exist, but they appear only for the operators of dimension 2.

6 Discussion

In this paper we continue the perturbative study of the form factors at weak coupling for the $\mathcal{N} = 4$ SYM theory which was initiated in [14] where the author considered the form factor for the operator $\mathcal{V}_X$ of conformal dimension 2. The original calculation has been performed in components and the form factor was computed up to the second order of perturbation theory. In our paper, we started with the operators belonging to the stress-tensor superconformal multiplet, namely, with $\mathcal{V}_I^J = \mathrm{Tr}\left(\bar{\phi}^J \phi_I\right)$ and $\mathcal{C}_{IJ} = \mathrm{Tr}\left(\phi_I \phi_J\right)$ and

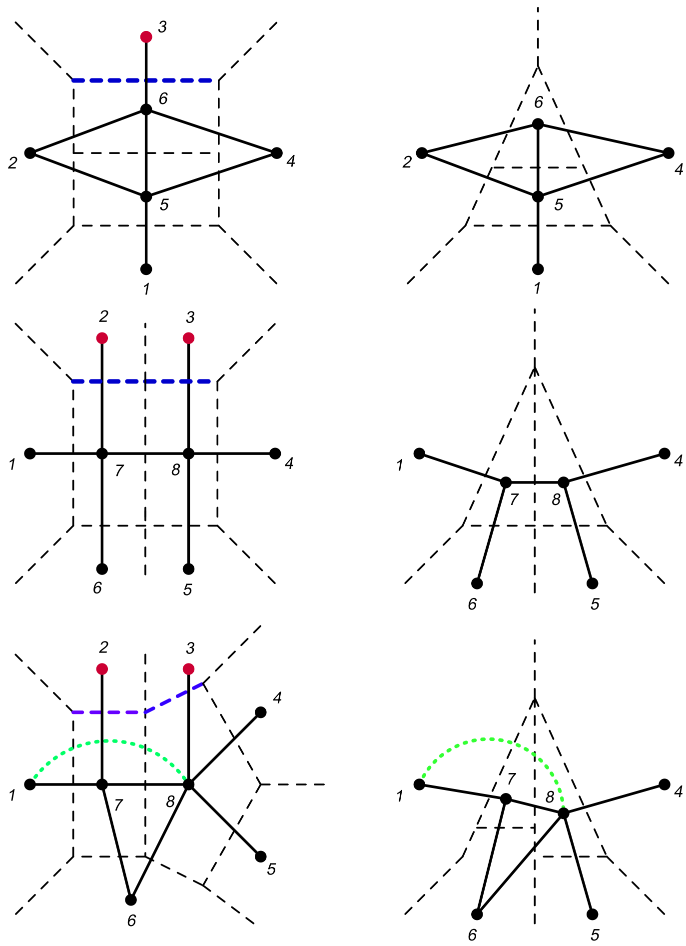

Figure 6. The two-loop ladder type triangle diagrams from the two-loop box diagrams. Green arc corresponds to the presence of a numerator.

calculated them up to the first and second order of perturbation theory, respectively. We obtained the same results as in [14] as it was expected.

Then we considered the Konishi operator $\mathcal{K} = \sum_I \mathrm{Tr}\left(\bar{\phi}^I \phi_I\right)$ with classical conformal dimension 2 in the one loop approximation. Not being protected by supersymmetry this operator has the UV divergences which have to be renormalized.

The main result of our paper is the calculation of the two-loop form factors for the half-BPS operators $\mathcal{O}_I^{(n)} = \mathrm{Tr}\left(\phi_I^n\right), n > 2$. At the one loop level the answer for the form factor is very simple given by triangle diagram while at two-loops it is essentially more complicated. The analytical expressions for the two-loop results are given in terms of the Gauss hypergeometric functions and the Appell function of two variables. Their expansion over ϵ up to $O(\epsilon)$ leads to logarithms, polylogarithms and, because of the Appell function, generalized Goncharov polylogarithms of several variables. However, all of them have the same transcendentality [24, 25, 55].

In the simplified kinematics the answers become much more simple. Thus, in two-

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dimensional (or 1+1-dimensional) kinematics for the form factors of the half-BPS operators $\mathcal{O}_I^{(3)}$ it is possible to get rid of the Appell functions and after expanding over ϵ to get the result in terms of the ordinary logarithms.

For all the considered form factors we observe the factorization of the IR divergences up to the second order of perturbation theory. This allows us to derive the first two terms of expansion for the cusp anomalous dimension in coincidence with the other calculations and for the collinear anomalous dimension, where we obtained the first nontrivial coefficient at two loops $G^{(2)} = -7\zeta_3$. It differs from the collinear anomalous dimension coming from the amplitude calculation but coincides with collinear anomalous dimension for the light-like Wilson loop [10, 39, 40].

The remarkable part of our calculation besides factorization is the fact that the one- and two-loop integrals contributing to the form factors of the operators $\mathcal{O}_I^{(n)}, n > 2$ are related to the dual conformal invariant integrals appearing in the calculation of the amplitudes. One has to look at the "parent" integral which appears while considering the amplitudes and shrink n propagators at the n-th order of perturbation theory. This dual conformal invariance together with the original conformal invariance might lead to a wider algebra eventually constraining the form of the answer and reveal the integrability property of a theory. It is important whether the powerful $\mathcal{N} = 4$ covariant on-shell methods such as recurrence relations (see recent [32] for example) can be generalized for the form-factors studied in our paper.

Note added: while finishing writing the paper we became aware of the paper which is closely connected to the subject studied here [74].

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A Feynman rules in $\mathcal{N} = 1$ superspace

We want to present here the essential elements of $\mathcal{N} = 1$ superspace technique relevant to our computations.

In terms of $\mathcal{N} = 1$ superfields the $\mathcal{N} = 4$ SYM action can be rewritten as (hereafter we use the notation of [56], see recent examples of application of the same technique in [57–60])

$$S^{\mathcal{N}=4} = \int d^8z \mathrm{Tr}\left(e^{-gV}\bar{\Phi}^I e^{gV}\Phi_I\right) + \frac{1}{2g^2}\int d^6z \mathrm{Tr}(W^\alpha W_\alpha) + ig\int d^6z \mathrm{Tr}\left(\Phi_1[\Phi_2, \Phi_3]\right) + c.c., \tag{A.1}$$

where the superfield strength tensor $W_\alpha = \bar{D}^2(e^{-gV}D_\alpha e^{gV})$, $V = V^a T_a$ is the real $\mathcal{N} = 1$ vector superfield and $\Phi_I = \Phi_I^a T_a$ with $I = 1, 2, 3$ are the three chiral superfields (I is the index of the SU(3) subgroup of $\mathrm{SU}(4)_R$), T_a are the generators of the gauge group $\mathrm{SU}(N_c)$ in adjoint representation. For performing $\mathrm{SU}(N_c)$ T-matrix manipulations we used FeynCalc package for Mathematica [61]. The following normalization for T^a is used in which the quadratic Casimir operator

$$\mathrm{Tr}(T^aT^b) = k_2\delta^{ab}, \;\; k_2 = 1/2. \tag{A.2}$$

The relevant Feynman rules for the propagators and vertices are

$$\begin{aligned}
\langle V^aV^b\rangle &= -\frac{1}{k_2}\delta^{ab}\frac{\delta_{12}}{p^2},\\
\langle \bar\Phi^a_I\Phi^b_J\rangle &= \frac{1}{k_2}\delta^{ab}\delta_{IJ}\frac{\delta_{12}}{p^2},\\
V(\bar\Phi V\Phi) &= igk_2 f_{abc}\delta^{IJ}\bar\Phi^a_I V^b\Phi^c_J,\\
V(\Phi\Phi\Phi) &= \frac{-g}{3!}\epsilon^{IJK}f_{abc}\Phi^a_I\Phi^b_J\Phi^c_K,\\
V(\bar\Phi\bar\Phi\bar\Phi) &= \frac{-g}{3!}\epsilon^{IJK}f_{abc}\bar\Phi^a_I\bar\Phi^b_J\bar\Phi^c_K,\\
V(\bar\Phi VV\Phi) &= \frac{g^2}{2}k_2\delta^{IJ}f_{adm}f_{bcm}V^a\Phi^d_I V^b\bar\Phi^c_J.
\end{aligned} \tag{A.3}$$

where $\delta_{12} = \delta^4(\theta_1-\theta_2)$ is the Grassmannian delta function.

The effective one-loop triple vertex is given by

$$V(\bar\Phi V\Phi)_{1-\text{loop}} = ig\frac{\lambda}{4}k_2 f_{abc}\bar\Phi^a_I(-q)\Phi^b_J(-p)\hat{\mathcal{D}}V^c(p+q)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k-q)^2(k+p)^2}, \tag{A.4}$$

where

$$\hat{\mathcal{D}} = 4D^\alpha\bar D^2 D_\alpha + (p-q)^{\alpha\dot\alpha}[D_\alpha,\bar D_{\dot\alpha}]. \tag{A.5}$$

As usual, the vertex with n chiral (anti-chiral) lines requires additional $n-1$ $\bar D^2$ (for anti-chiral D^2) acting on chiral (anti-chiral) lines (or $n-1-m$ $\bar D^2$ (for anti-chiral D^2) if m lines are external). We used SusyMath package for Mathematica [62, 63] for performing D-algebra for supergraphs.

Traces in this case are taken over σ matrices and are evaluated in $D=4$ because dimensional reduction is used. The following set of identities is useful:

$$\begin{aligned}
\sigma^m &= (\sigma^m)_{\alpha\dot\beta}\;\bar\sigma^m = (\bar\sigma^m)^{\alpha\dot\beta}\\
p_{\alpha\dot\beta} &= p_m(\sigma^m)_{\alpha\dot\beta}\;\bar p^{\alpha\dot\beta} = p_m(\bar\sigma^m)^{\alpha\dot\beta}\\
\mathbf{1} &= \delta^\alpha_\beta,\;\; \bar{\mathbf{1}} = \delta^{\dot\alpha}_{\dot\beta}\\
\mathrm{Tr}[\mathbf{1}] &= \mathrm{Tr}[\bar{\mathbf{1}}] = \frac{D}{2},
\end{aligned} \tag{A.6}$$

where $D/2 = 2$ in dimensional reduction and also we have

$$\begin{aligned}
\sigma^m\bar\sigma^n + \sigma^n\bar\sigma^m &= -\eta^{mn}\mathbf{1},\\
\bar{\sigma^m}\sigma^n + \bar{\sigma^n}\sigma^m &= -\eta^{mn}\bar{\mathbf{1}}.
\end{aligned} \tag{A.7}$$

B Scalar integrals and their ϵ expansion

Here we present the list of scalar integrals which we encountered in our computation shown in figure 7. All the integrals are evaluated in $D = 4 - 2\epsilon$ dimensions. For each loop the factor $e^{\epsilon\gamma_E}$ is added in the integration measure, we also do not write 4π which always appear with μ^2.

$$G_0 = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k+p)^2} = \left(\frac{e^{-\epsilon\gamma_E}}{16\pi^2}\left(\frac{p^2}{\mu^2}\right)^{-\epsilon}\right)\left(\frac{1}{\epsilon} + 2 + O(\epsilon)\right), \tag{B.1}$$

$$\begin{aligned} G_1 &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k+p_1)^2(k+p_2)^2} \\ &= -\left(\frac{e^{-\epsilon\gamma_E}}{16\pi^2}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\right)\frac{1}{s_{12}}\left(\frac{1}{\epsilon^2} - \frac{\zeta_2}{2} - \frac{7\zeta_3}{3}\epsilon - \frac{47\pi^4}{1440}\epsilon^2 + O(\epsilon^3)\right), \end{aligned} \tag{B.2}$$

$$\begin{aligned} G_2 &= \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D} \frac{1}{k^2 l^2 (k-p_1)^2(k+p_2)^2(k+l-p_2)^2(k+l-p_1)^2} \\ &= \left(\frac{e^{-\epsilon\gamma_E}}{16\pi^2}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\right)^2 \frac{1}{s_{12}^2}\left(-\frac{1}{4\epsilon^4} - \frac{5\pi^2}{24\epsilon^2} - \frac{29\zeta_3}{6\epsilon} - \frac{3\pi^4}{32} + O(\epsilon)\right), \end{aligned} \tag{B.3}$$

$$\begin{aligned} G_3 &= \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D} \frac{1}{k^2 l^2 (k-l)^2(k+l-p_1)^2(k+l+p_2)^2(p_2+l)^2} \\ &= \left(\frac{e^{-\epsilon\gamma_E}}{16\pi^2}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\right)^2 \frac{1}{s_{12}}\left(-\frac{1}{2\epsilon^4} + \frac{29\zeta_3}{6\epsilon} + \frac{49\pi^4}{720} + O(\epsilon)\right), \end{aligned} \tag{B.4}$$

$$\begin{aligned} G_4 &= \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D} \frac{1}{k^2 l^2 (l+p_1)^2(k+p_2)^2(k-l+p_2)^2(l-p_2)^2} \\ &= \left(\frac{e^{-\epsilon\gamma_E}}{16\pi^2}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\right)^2 \frac{1}{s_{12}}\left(\frac{1}{4\epsilon^4} - \frac{\pi^2}{24\epsilon^2} - \frac{8\zeta_3}{3\epsilon} - \frac{19\pi^4}{480} + O(\epsilon)\right), \end{aligned} \tag{B.5}$$

$$\begin{aligned} G_5 &= \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D} \frac{1}{k^2 l^2 (k-l)^2(k-p_2)^2(k-l-p_1)^2(l-p_2-p_1)^2} \\ &= \left(\frac{e^{-\epsilon\gamma_E}}{16\pi^2}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\right)^2 \frac{1}{s_{12}^2}\left(-\frac{1}{\epsilon^4} + \frac{\pi^2}{\epsilon^2} + \frac{83\zeta_3}{3\epsilon} + \frac{59\pi^4}{120} + O(\epsilon)\right). \end{aligned} \tag{B.6}$$

The other integrals entering into the calculations are

$$G_5^a = \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D} \frac{s_{12}k^2 - \mathrm{Tr}(\bar{p}_1 p_2 \bar{l} k) + \mathrm{Tr}(\bar{p}_1 k \bar{l} p_2)}{k^2 l^2 (k-l)^2(k-p_2)^2(k-l-p_1)^2(l-p_2-p_1)^2}, \tag{B.7}$$

$$G_5^b = \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D} \frac{s_{12}k^2 - \mathrm{Tr}(\bar{p}_1 p_2 \bar{k} l)}{k^2 l^2 (k-l)^2(k-p_2)^2(k-l-p_1)^2(l-p_2-p_1)^2}, \tag{B.8}$$

$$G_1^{\alpha\dot{\beta}} = \int \frac{d^D k}{(2\pi)^D} \frac{k^{\alpha\dot{\beta}}}{k^2(k+p_1)^2(k+p_2)^2}. \tag{B.9}$$

G_0 is the scalar bubble integral, G_1 is the scalar triangle integral. G_2, G_3 G_4 can be computed by means of the MB representation as series in ϵ or to all orders in ϵ by

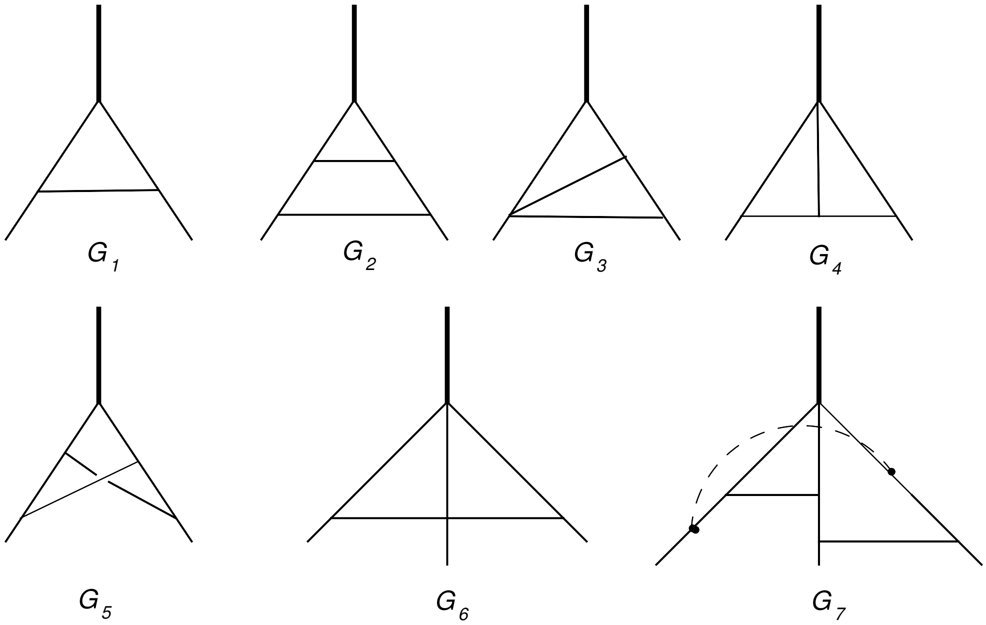

Figure 7. The set of scalar integrals. The arc line in G_7 corresponds to the presence of the numerator $(k-p)^2$. Thick black line corresponds to off-shell leg with momentum q. All the other outer legs are on-shell.

means of the differential equation technique.[5] G_5 can be computed by means of the MB representation as series in ϵ, the answer to all orders in ϵ is given in [66]. G_6 and G_7 can be computed by direct evaluation of integrals over the Feynman parameters in terms of the hypergeometric function ${}_2F_1$ and the Appell function F_1. The formulas from [67] and [68] were useful in verification of our computation. G_6 can also be evaluated by means of the differential equation technique [64, 65], the result coincides with ours after the rearrangement of hypergeometric functions. Using the notation $s_{12} = s, s_{14} = t, s_{13} = u$ the answers for G_6 can be written as:

$$
\begin{aligned}
c_\Gamma &= \frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} \\
G_6 &= \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D}\frac{1}{l^2(l-k)^2(l-p_1)^2(k+p_2)^2\ (k+p_2+p_3)^2} \\
&= \frac{e^{-2\epsilon\gamma_E}c_\Gamma}{(16\pi^2)^2}\ \frac{1}{2\epsilon^3}\frac{1}{(1-2\epsilon)} \\
&\quad \left\{ \left(\frac{\mu^2}{t}\right)^{2\epsilon}\frac{1}{s}\ {}_2F_1\Big(1, 1-2\epsilon, 2-2\epsilon, -\frac{u}{s}\Big) + \left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{1}{t}\ {}_2F_1\Big(1, 1-2\epsilon, 2-2\epsilon, -\frac{u}{t}\Big) \right. \\
&\qquad \left. - \left(\frac{\mu^2}{s+t+u}\right)^{2\epsilon}\frac{s+t+u}{ts}\ {}_2F_1\left(1, 1-2\epsilon, 2-2\epsilon, -\frac{u(s+t+u)}{st}\right) \right\}. \qquad \text{(B.10)}
\end{aligned}
$$

[5]This integrals can be reduced to the set of master topologies presented in [64, 65].

where the hypergeometric function is given by the following expansion:(we used the Nested Sums computational tool [69, 70] and HypExp package for Mathematica [71, 72])

$$ {}_2F_1\left(1-2\epsilon, 1-2\epsilon; 2-2\epsilon; x\right) = \sum_{n=0}^{3} \epsilon^n \; a_n(x) + O(\epsilon^4) \tag{B.11} $$

$$
\begin{aligned}
a_0(x) &= -\frac{\log(1-x)}{x} \\
a_1(x) &= +\frac{(2Li_2(x) - (\log(1-x)-2)\log(1-x))}{x} \\
a_2(x) &= -\frac{2}{3x}\Big(\log^3(1-x) - 3\log(x)\log^2(1-x) - 3\log^2(1-x) + \pi^2\log(1-x) \\
&\quad -6(\log(1-x)-1)Li_2(x) - 6Li_3(1-x) - 6Li_3(x) + 6\zeta(3)\Big) \\
a_3(x) &= -\frac{2}{3x}\Big(\log^4(1-x) - 6\log(x)\log^3(1-x) - 2\log^3(1-x) + 6\log(x)\log^2(1-x) \\
&\quad + 2\pi^2\log^2(1-x) - 6(\log(1-x)-2)Li_2(x)\log(1-x) - 12Li_3(x)\log(1-x) \\
&\quad + 12\zeta(3)\log(1-x) - 2\pi^2\log(1-x) - 12(\log(1-x)-1)Li_3(1-x) + 12Li_3(x) \\
&\quad + 12Li_4\left(\frac{x}{x-1}\right) - 12\zeta(3)\Big)
\end{aligned}
$$

The finite part of G_6 is then given by the following expression:

$$ (G_6)_{\rm fin} = \frac{1}{(16\pi^2)^2}\,\frac{1}{2}\left\{\frac{t}{s}\,a_3\Big(-\frac{u}{s}\Big) + \;a_3\Big(-\frac{u}{t}\Big) - \frac{s+t+u}{s}\,a_3\Big(-\frac{u(s+t+u)}{st}\Big)\right\}. $$

In the case of "$1+1$" dimensional kinematics the following limiting expressions for G_6 are used:

$$
\begin{aligned}
G_6|_{t=0} &= 0, \\
G_6|_{s=0} &= \frac{e^{-2\epsilon\gamma_E}c_\Gamma}{(16\pi^2)^2}\,\frac{1}{4\epsilon^4}\left\{\frac{1}{(t+u)^{2\epsilon}} - \frac{1}{t^{2\epsilon}}\right\} \\
G_6|_{u=0} &= \frac{e^{-2\epsilon\gamma_E}c_\Gamma}{(16\pi^2)^2}\,\frac{-1}{2\epsilon^3}\frac{1}{1-2\epsilon}\left\{\frac{t+s}{s}\,\frac{1}{(t+s)^{2\epsilon}} - \frac{t}{s}\frac{1}{t^{2\epsilon}} - \frac{1}{s^{2\epsilon}}\right\}
\end{aligned}
$$

For G_7 one has

$$
\begin{aligned}
G_7 = &\int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{(k-p_1)^2}{k^2l^2(l-k)^2(l-p_1)^2(k+p_2)^2\;(k+p_2+p_3)^2} = G_6 + \\
&+\frac{e^{-2\epsilon\gamma_E}c_\Gamma}{(16\pi^2)^2}\frac{1}{2\epsilon^4}\frac{1}{t}\left\{\left(\frac{\mu^2}{s}\right)^{2\epsilon}F_{21}\left(1,-\epsilon,1-\epsilon,-\frac{u}{t}\right) + \left(\frac{\mu^2}{t}\right)^{2\epsilon}\left(-1+F_{21}\left(\epsilon,2\epsilon,1+\epsilon,-\frac{s+u}{t}\right)\right)\right. \\
&\left.-\left(\frac{\mu^2}{s+t+u}\right)^{2\epsilon}\frac{\epsilon}{1+\epsilon}\frac{u}{t+u}F_1\left(1,2\epsilon,1,2+\epsilon,\frac{s+u}{s+t+u},\frac{u}{t+u}\right)\right\}
\end{aligned} \tag{B.12}
$$

$$ {}_2F_1(1,-\epsilon,1-\epsilon,x) = \sum_{n=0}^{4}\epsilon^n\;b_n(x) + O(\epsilon^5), \tag{B.13} $$

where

$$
\begin{aligned}
b_0(x) &= 1 \\
b_1(x) &= \log(1-x) \\
b_2(x) &= -Li_2(x) \\
b_3(x) &= -Li_3(x) \\
b_4(x) &= -Li_4(x).
\end{aligned}
$$

and

$$
{}_2F_1(\epsilon, 2\epsilon, 1+\epsilon, x) = \sum_{n=0}^{4} \epsilon^n \; c_n(x) + O(\epsilon^5), \tag{B.14}
$$

where

$$
\begin{aligned}
c_0(x) &= 1 \\
c_1(x) &= 0 \\
c_2(x) &= +2Li_2(x) \\
c_3(x) &= \Bigg(\frac{2\pi^2}{3} \log(1-x) - 2\log^2(1-x)\log(x) - 4\log(1-x)Li_2(x) - 4Li_3(1-x) \\
&\quad -2Li_3(x) + 4\zeta_3 \Bigg), \\
c_4(x) &= \frac{1}{90}\Bigg(4\pi^4 - 90\pi^2 \log(1-x)^2 - 15\log(1-x)^4 + 300\log^3(1-x)\log(x) \\
&\quad +360\log^2(1-x)Li_2(x) + \log(1-x)\big(-360\zeta_3 + 720Li_3(1-x) + 360Li_3(x)\big) \\
&\quad -360Li_4(1-x) - 180Li_4(x) - 360Li_4\Big(\frac{x}{x-1}\Big)\Bigg).
\end{aligned}
$$

In the integral G_7 there is the Appell function of the first kind defined by the following integral representation:

$$
F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 du u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}, \tag{B.15}
$$

which in our case gives us the one-parametric integral

$$
F_1(1, 2\epsilon, 1, 2+\epsilon; x, y) \;=\; (1+\epsilon)\int_0^1 du(1-u)^{\epsilon}(1-ux)^{-2\epsilon}(1-uy)^{-1}. \tag{B.16}
$$

Expanding the integrand over ϵ and then performing the integration one gets

$$
\begin{aligned}
F_1(1, 2\epsilon, 1, 2+\epsilon; x, y) &= (1+\epsilon)\int_0^1 du \Bigg(1 + \big(\log(1-u) - 2\log(1-ux)\big)\epsilon \\
&\quad + \left(\frac{1}{2}\log^2(1-u) - 2\log(1-ux)\log(1-u) + 2\log^2(1-ux) \right)\epsilon^2 \Bigg) \\
&\quad + \frac{1}{6}\big(\log^3(1-u) - 6\log(1-ux)\log^2(1-u) + 12\log^2(1-ux)\log(1-u) \\
&\quad - 8\log^3(1-ux)\epsilon^3\big) + O(\epsilon^4).
\end{aligned}
$$

Up to the second order in ϵ it is possible to evaluate the integrals in terms of logarithms and polylogarithms; however, in higher orders new functions appear.

Consider, for example, the integral

$$\mathcal{I}_1 \;=\; \int_0^1 du\, \frac{\log^2(1-u)\log(1-ux)}{1-uy}, \tag{B.17}$$

where the parameters satisfy the condition $0 < x < y < 1$. To evaluate this integral we use the integral representation for one of the logarithms and get

$$\int_0^1 du \int_0^1 da \frac{-ux\log^2(1-u)}{(1-uy)(1-uxa)}. \tag{B.18}$$

Now taking the integral over u one has

$$\mathcal{I}_1 = -\frac{2}{y}\log\frac{y-x}{y}Li_3\frac{-y}{1-y} + 2\int_0^1 \frac{Li_3\frac{ax}{ax-1}}{a(ax-y)}da. \tag{B.19}$$

To find the integral

$$\int_0^1 \frac{Li_3\frac{ax}{ax-1}}{a(ax-y)}da \tag{B.20}$$

it is useful to introduce a new variable

$$b \;=\; \frac{ax}{ax-1}$$

then

$$\int_0^1 \frac{Li_3\frac{ax}{ax-1}}{a(ax-y)}da = -\frac{1}{y}\int_0^{\frac{x}{x-1}} \frac{Li_3 b}{b\left(1+\frac{1-y}{y}b\right)}db \tag{B.21}$$

and using the identity

$$\frac{1}{b\left(1+\frac{1-y}{y}b\right)} = \frac{1}{b} - \frac{1-y}{y\left(1+\frac{1-y}{y}b\right)}$$

one comes to the integral

$$-\frac{1}{y}\left(\int_0^{-\frac{x}{1-x}} \frac{Li_3 b}{b}db - \frac{1-y}{y}\int_0^{-\frac{x}{1-x}} \frac{Li_3 b}{1+\frac{1-y}{y}b}db\right). \tag{B.22}$$

The first integral is straightforward and for the second integral one can expand the polylogarithm in power series and get the answer in terms of the function

$$Li_{m,n}(x,y) \;=\; \sum_{j>i>0} \frac{y^j}{j^n}\frac{x^i}{i^m}. \tag{B.23}$$

As a result one gets

$$\int_0^1 \frac{Li_3\frac{ax}{ax-1}}{a(ax-y)}da = -\frac{1}{y}\left(Li_4\left(-\frac{x}{1-x}\right) + Li_{3,1}\left(-\frac{y}{1-y}, \frac{x(1-y)}{y(1-x)}\right)\right). \tag{B.24}$$

Finally putting everything together we obtain

$$\mathcal{I}_1 = -\frac{2}{y}\log\frac{y-x}{y}Li_3\frac{-y}{1-y} - \frac{2}{y}\left(Li_4\left(-\frac{x}{1-x}\right) + Li_{3,1}\left(-\frac{y}{1-y}, \frac{x(1-y)}{y(1-x)}\right)\right). \tag{B.25}$$

Another possibility to expand the Appell function is to use the Nested Sums computational tool [69, 70] which represents the Appell function as some combination of generalized polylogarithms. The ϵ-expansion for the Appell function then takes the form

$$
\begin{aligned}
F_1(1,2\epsilon,1,2+\epsilon;x,y) &= -\frac{\log(1-y)}{y} + \frac{1}{y}\Bigg(-\log^2(1-x) + 2\log(1-y)\log(1-x) \\
&- \frac{1}{2}\log^2(1-y) - \log(1-y) - 2Li_2(x) - 2Li_2\left(\frac{x-y}{x-1}\right) + Li_2(y)\Bigg)\epsilon \\
&+ \frac{1}{y}\Bigg(-\frac{1}{6}\log^3(1-y) + \frac{1}{2}(-\log(y) + 2\log(y-x) - 1)\log^2(1-y) - \frac{1}{6}\pi^2\log(1-y) \\
&+ \log^2(1-x)(\log(x) - \log(y) + \log(y-x) - 1) - 2Li_2(x) - 2Li_2\left(\frac{x-y}{x-1}\right) \\
&+ \log(1-x)\Bigg(\log(1-y)(2-2\log(y-x)) + \frac{1}{3}\Bigg(6Li_2(x) + 6Li_2\left(1-\frac{x}{y}\right) - \zeta(3) \\
&+ 6Li_2\left(\frac{x-y}{x-1}\right) - 6Li_2(y) + \pi^2\Bigg)\Bigg) + Li_2(y) + 2Li_3(x) - 2Li_3\left(1-\frac{x}{y}\right) - Li_3(1-y) \\
&+ 2Li_3\left(\frac{x-y}{x-1}\right) + 2Li_3\left(\frac{y-1}{x-1}\right) + 2Li_3\left(\frac{x-y}{(x-1)y}\right) - Li_3(y)\Bigg)\,\epsilon^2 + \mathtt{Fin}(x,y)\,\epsilon^3 + O(\epsilon^4),
\end{aligned}
\tag{B.26}
$$

where

$$
\begin{aligned}
\mathtt{Fin}(x,y) = \frac{1}{y}\Bigg(& -2Li_{1,1,1,1}\left(1,\frac{x}{y},1,y\right) + 2Li_{1,2,1}\left(\frac{x}{y},1,y\right) - 2Li_{1,2}\left(\frac{x}{y},y\right) + 2Li_{3,1}\left(\frac{x}{y},y\right) \\
& + S_{0,3}(y) + Li_3(y) - S_{0,4}(y) - H_{2,2}(y) - 2Li_{2,1,1}\left(1,\frac{x}{y},y\right) - H_{1,3}(y) + 2Li_{1,1,1,1}\left(\frac{x}{y},1,1,y\right) \\
& - H_{1,2,1}(y) - S_{2,2}(y) - 2Li_{1,1,2}\left(1,\frac{x}{y},y\right) + 2Li_{1,1,2}\left(\frac{x}{y},1,y\right) + 2Li_{2,2}\left(\frac{x}{y},y\right) + H_{1,2}(y) \\
& + 2Li_{1,3}\left(\frac{x}{y},y\right) - S_{1,3}(y) + 2Li_{1,1,1,}\left(\frac{x}{y},y\right) + 2Li_{2,1,1}\left(\frac{x}{y},1,y\right) - Li_4(y) - 2Li_{2,1}\left(\frac{x}{y},y\right) \\
& + S_{1,2}(y) - 2Li_{1,1,1}\left(\frac{x}{y},1,y\right) + 2Li_{1,1,1,1}\left(1,1,\frac{x}{y},y\right) - H_{1,1,2}(y) - 2Li_{1,2,1}\left(1,\frac{x}{y},y\right)\Bigg).
\end{aligned}
\tag{B.27}
$$

Here we use the following definition of the generalized Goncharov polylogarithms [46–48]

$$
Li_{m_1,\ldots,m_k}(x_1,\ldots,x_k) \;=\; \sum_{i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \ldots \frac{x_k^{i_k}}{i_k^{m_k}}. \tag{B.28}
$$

Apart from these functions we have in expansion the so-called Nielsen generalized polylogarithms

$$
S_{n,p} \;=\; Li_{n+1,1,\ldots,1}(x,\underbrace{1,\ldots,1}_{p-1}) \tag{B.29}
$$

and also the harmonic polylogarithms

$$
H_{m_1,\ldots,m_k}(x) \;=\; Li_{m_1,\ldots,m_k}(x,\underbrace{1,\ldots,1}_{k-1}). \tag{B.30}
$$

In the case of "1 + 1" dimensional kinematics the expression for the integral G_7 is simplified and the following limiting cases can be used:

$$G_7 = G_6^{"1+1"} + \frac{e^{-2\epsilon\gamma_E} c_\Gamma}{(16\pi^2)^2} \frac{1}{2\epsilon^4} \left\{ -2\epsilon^2 J - \frac{1}{t^{2\epsilon}} \right\}, \tag{B.31}$$

where J is the integral

$$J = \int_0^1 dxdy \, \frac{ty^{\epsilon-1}}{(tx + sy + uxy)^{1+2\epsilon}} \tag{B.32}$$

$$\begin{aligned}
J|_{t=0} &= 0, && \text{(B.33)}\\
J|_{s=0} &= -\frac{1}{2\epsilon}\int_0^1 dy \frac{ty^{\epsilon-1}}{(t+uy)^{1+2\epsilon}} = -\frac{1}{2\epsilon^2}\frac{1}{t^{2\epsilon}}\, {}_2F_1\left(\epsilon, 1+2\epsilon, 1+\epsilon, -\frac{u}{t}\right) && \text{(B.34)}\\
J|_{u=0} &= -\frac{1}{2\epsilon}\left\{\int_0^1 dy \frac{y^{\epsilon-1}}{(t+sy)^{2\epsilon}} - \int_0^1 dy \frac{y^{\epsilon-1}}{(sy)^{2\epsilon}}\right\}\\
&= -\frac{1}{2\epsilon^2}\frac{1}{s^{2\epsilon}} - \frac{1}{2\epsilon^2}\frac{1}{t^{2\epsilon}}\, {}_2F_1\left(\epsilon, 2\epsilon, 1+\epsilon, -\frac{s}{t}\right) && \text{(B.35)}
\end{aligned}$$

The finite part of G_7 is given by the following expression:

$$(G_7)_{\text{fin}} = (G_6)_{\text{fin}} + \frac{1}{(16\pi^2)^2}\frac{1}{2t}\left\{-1 + b_4\left(-\frac{u}{t}\right) + c_4\left(-\frac{s+u}{t}\right) - \frac{u}{t+u}\texttt{Fin}\left(\frac{s+u}{s+t+u}, \frac{u}{t+u}\right)\right\}. \tag{B.36}$$

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