

# Dynamical clustering in oscillator ensembles with time-dependent interactions

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(Dated: February 2, 2008)

We consider an ensemble of coupled oscillators whose individual states, in addition to the phase, are characterized by an internal variable with autonomous evolution. The time scale of this evolution is different for each oscillator, so that the ensemble is inhomogeneous with respect to the internal variable. Interactions between oscillators depend on this variable and thus vary with time. We show that as the inhomogeneity of time scales in the internal evolution grows, the system undergoes a critical transition between ordered and incoherent states. This transition is mediated by a regime of dynamical clustering, where the ensemble recurrently splits into groups formed by varying subpopulations.

PACS numbers: 05.45.Xt, 89.75.Fb, 05.70.Fh

Synchronization phenomena in interacting dynamical systems have attracted a great deal of attention in the last two decades [1]. The spontaneous appearance of coherent evolution as a result of interactions is ubiquitous in a vast class of natural and artificial systems, and can be successfully reproduced by relatively simple models. Most studies of this kind of phenomena have focused on patterns of highly correlated evolution, such as full and phase synchronization. Though these forms of synchronization are essential to certain artificial systems—for instance, secure-communication and chaos-control devices—they are expected to play a more restricted role in such objects as biological populations, where the complexity of their collective function requires a delicate balance between behavioral coherence and diversity [2]. A clear-cut example is the human brain, where highly coherent functional patterns are only realized under pathological states—notably, during epileptic seizures. Neural tissues, as well as many natural systems ranging from intracellular molecular complexes to social populations, are normally found in segregated configurations, where the system splits into groups of elements with distinct functions. In these clustered states, highly coherent evolution occurs inside each group, while the collective dynamics of different groups are much less correlated. This is typically a dynamical phenomenon, since—as a consequence of the evolution of the internal state of each element—clusters may temporarily disperse to be reconstituted later, and individual elements or small groups can migrate between clusters.

Clustering has been observed in ensembles of coupled dynamical systems with specific individual dynamics [3, 4], or subject to sufficiently complex interaction functions [5, 6]. Disintegration of clusters and migration of elements is typically observed under the action of noise

[3, 7, 8]. In this Letter, we show that such kind of dynamical clustering can occur in an ensemble of coupled periodic oscillators, in the absence of external fluctuations, when the interaction between oscillators depends on individual internal state variables with autonomous evolution. The time scale of this internal evolution differs between oscillators. A critical transition between ordered and incoherent evolution, mediated by dynamical clustering, takes place as the inhomogeneity of such time scales grows.

Consider  $N$  coupled phase oscillators, each of them characterized by a phase variable  $\phi_i(t)$ , obeying by the equations

$$\dot{\phi}_i = N^{-1} \sum_{j=1}^N J_{ij}(t) \sin(\phi_j - \phi_i) \quad (1)$$

( $i = 1, \dots, N$ ). When the interaction weights  $J_{ij}(t)$  are constant,  $J_{ij}(t) \equiv J$  for all  $i$  and  $j$ , these equations reduce to Kuramoto's model for identical phase oscillators [9], which exhibits full synchronization for any positive value of  $J$ . Time-independent non-identical interaction weights have also been considered, disclosing typical features of disordered systems, such as glassy-like behavior, frustration, and algebraic relaxation towards equilibrium [10, 11]. Here, we analyze the case where the interaction weights depend on the internal state of each oscillator, which is specified by a variable  $\theta_i(t)$  with autonomous evolution. Specifically, we consider that  $\theta_i$  is itself a phase variable evolving according to  $\dot{\theta}_i = \Omega_i$ , where the frequency  $\Omega_i$  is chosen at random for each element, from a fixed distribution  $g(\Omega)$ . The interaction weight is given by

$$J_{ij}(t) = J \cos[\theta_j(t) - \theta_i(t)], \quad (2)$$

where  $J > 0$  is a constant factor that can be fixed to  $J = 1$  by rescaling time. With this choice for  $J_{ij}$ , the interaction between the phase  $\phi_i$  and  $\phi_j$  is attractive when  $\cos(\theta_j - \theta_i)$  is positive, and repulsive otherwise. The sign of  $J_{ij}$  changes in a time scale of order  $|\Omega_j - \Omega_i|^{-1}$ , giving

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rise—at the level of the whole ensemble—to a complex time-dependent interaction pattern. This evolving connection network makes possible the appearance of several dynamical regimes, including clustering.

We start our study of the above model by performing extensive numerical realizations in ensembles ranging in size from  $N = 10^2$  to  $10^6$ . The internal frequencies  $\Omega_i$  are drawn at random from a Gaussian distribution,  $g(\Omega) = \exp[-(\Omega - \Omega_0)^2/2\sigma^2]/\sqrt{2\pi\sigma^2}$ . Since the dynamics is invariant under a constant shift in these frequencies, we fix  $\Omega_0 = 0$ . It turns out that the collective behavior of the ensemble is sensible to the frequency dispersion  $\sigma$ . For small  $\sigma$  interactions evolve slowly, over long time scales. Consequently, the phase  $\phi_i$  can adiabatically follow the evolution of the interaction weights. For long times, the system is found in a state where the phases  $\phi_i$  are homogeneously distributed over the interval  $[0, 2\pi)$ . A strong correlation is observed in such state between  $\phi_i$  and the internal variable  $\theta_i$ , namely,  $\phi_i \approx \pm\theta_i + \phi_0$ , where  $\phi_0$  is a constant. The sign factor of  $\theta_i$  and the value of  $\phi_0$  are the same for *all* oscillators, and depend on the initial condition; moreover,  $\phi_0$  can slowly change as time elapses. For large  $\sigma$ , on the other hand, many of the interaction weights exhibit large changes over very short times. Each oscillator is thus subject to rapidly fluctuating forces, which again results in a state where the phases  $\phi_i$  are homogeneously distributed over  $[0, 2\pi)$ . Now, however, there is no correlation between  $\phi_i$  and  $\theta_i$ .

To characterize the transition between the regimes of small and large frequency dispersion we introduce, for each oscillator, the two-dimensional complex vector  $\mathbf{m}_i = (\exp[i(\phi_i - \theta_i)], \exp[i(\phi_i + \theta_i)])$ , and define the average

$$\mathbf{m} = N^{-1} \sum_{j=1}^N \mathbf{m}_j \equiv (\mu_+ \exp(i\psi_+), \mu_- \exp(i\psi_-)). \quad (3)$$

The time average  $\mu$  of  $\sqrt{\mu_+^2 + \mu_-^2}$  is a suitable order parameter for the transition. If the system spends most of the time in the state where  $\phi_i \approx \pm\theta_i + \phi_0$  for all  $i$ , we have  $\mu \approx 1$ , while  $\mu \sim N^{-1/2}$  in the incoherent state. Figure 1 shows numerical results for  $\mu$  as a function of  $\sigma$  for various system sizes. The dependence with  $N$  suggest the presence of a critical phenomenon for  $\sigma \approx 0.3$ . Within a few assumptions supported by numerical evidence, this critical behavior can be analytically studied in the limit  $N \rightarrow \infty$ , as follows. First of all, we note that for  $\sigma \neq 0$  and at sufficiently long times, the internal variables  $\theta_i$  are homogeneously distributed over  $[0, 2\pi)$ . If, in the limit  $\sigma \rightarrow 0$  (but  $\sigma \neq 0$ ), the temporal variation of  $\theta_i$  is disregarded,  $\phi_i = \pm\theta_i + \phi_0$  is a stationary state of our system, as direct substitution in Eq. (1) shows. This conclusion holds for any distribution of frequencies highly concentrated around  $\Omega = 0$ . For larger values of  $\sigma$ , we write  $\phi_i = \pm(\theta_i - \delta_i) + \phi_0$ , where  $\delta_i(t)$  measures the deviation of each oscillator with respect to the small-dispersion state. This deviation satisfies

$$\dot{\delta}_i = \Omega_i - \frac{\mu}{2} \sin \delta_i, \quad (4)$$

where we have written the order parameter  $\mu$  in terms of the distribution of the deviations  $\delta_i$  over the ensemble,  $p(\delta)$ , as

$$\mu = \int_{-\pi/2}^{\pi/2} p(\delta) \cos \delta \, d\delta. \quad (5)$$

Here, we have assumed that  $p(\delta)$  is an even function; as shown below, this amounts to postulate that  $g(\Omega)$  is itself even.

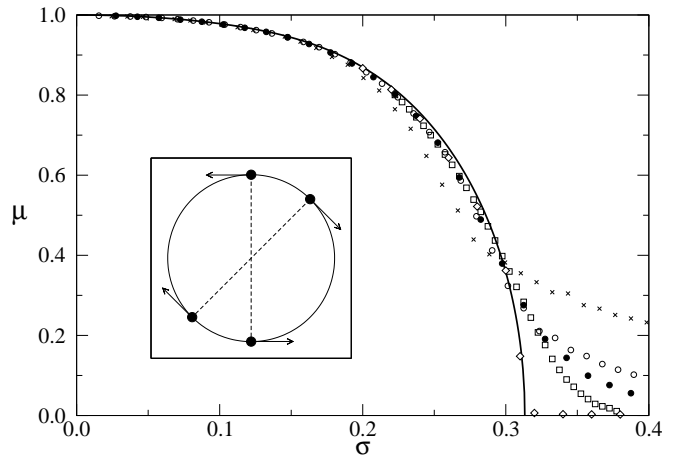


FIG. 1: The order parameter  $\mu$  as a function of the frequency dispersion  $\sigma$  for ensembles of various sizes,  $N = 10^2$  ( $\times$ ),  $10^3$  ( $\circ$ ),  $10^4$  ( $\bullet$ ),  $10^5$  ( $\square$ ), and  $10^6$  ( $\diamond$ ). The curve stands for  $r$  as calculated from Eq. (6). Inset: Schematic representation of the motion of the four clusters observed for intermediate frequency dispersions, in  $\phi$ -space (see text).

Equation (4) has been extensively discussed in connection with phase-oscillator ensembles [12]. For  $|\Omega_i| < \mu/2$  it has a stable fixed point at one of the solutions of  $\sin \delta_i = 2\Omega_i/\mu$ . For  $|\Omega_i| > \mu/2$ , on the other hand, there are no fixed points and  $|\delta_i|$  grows indefinitely with time. The ensemble can therefore be thought of as consisting of two subpopulations. Those oscillators for which  $|\Omega_i| < \mu/2$  attain, at asymptotically large times, a stationary deviation  $\delta_i$ , which depends on the value of  $\Omega_i$ . For the oscillators with  $|\Omega_i| > \mu/2$ , in contrast, the phase  $\phi_i$  does not reach a stationary value with respect to  $\theta_i$ . We refer to these two subpopulations as subensembles I and II, respectively. The distribution  $p(\delta)$  can be split into contributions from each subensemble,  $p = p_I + p_{II}$ . The first contribution is found from the distribution of frequencies, taking into account the identity  $p_I(\delta)d\delta = g(\Omega)d\Omega$ . As for  $p_{II}$ , it can be shown that its contribution to the integral of Eq. (5) vanishes. This equation becomes, then,

$$\mu = \int_{-\mu/2}^{\mu/2} g(\Omega) \sqrt{1 - 4\Omega^2/\mu^2} \, d\Omega, \quad (6)$$

making it possible to find  $\mu$  self-consistently for any (even) distribution of frequencies. Under quite general

conditions on  $g(\Omega)$ , this equation describes a second-order transition between disordered ( $\mu = 0$ ) and ordered ( $\mu \neq 0$ ) states. For  $\mu \neq 0$ , two states are possible, either with  $\mu_+ \neq 0$  or with  $\mu_- \neq 0$ , which demonstrates the occurrence of symmetry breaking at the transition. The curve in Fig. 1 shows the solution to Eq. (6) for the Gaussian distribution of frequencies considered in our numerical realizations. It vanishes as  $\mu \sim |\sigma - \sigma_c|^{1/2}$  at  $\sigma_c = \sqrt{\pi/32} \approx 0.313$ .

Note that Eq. (1) can be written, in terms of the average quantity introduced in Eq. (3), as

$$\dot{\phi}_i = \frac{\mu_+}{2} \sin(\psi_+ + \theta_i - \phi_i) + \frac{\mu_-}{2} \sin(\psi_- - \theta_i - \phi_i). \quad (7)$$

This form of the equation of motion for  $\phi_i$  emphasizes the mean-field nature of interactions. The fact that, as  $N \rightarrow \infty$ ,  $\mu$  vanishes for  $\sigma > \sigma_c$ , implies that the two terms in the right-hand side of Eq. (7) vanish as well. For finite system sizes, they are of order  $N^{-1/2}$ . Hence, beyond the critical dispersion  $\sigma_c$ , the evolution of phases can be thought of as driven by fluctuating forces of order  $N^{-1/2}$ . In the thermodynamical limit, for  $\sigma > \sigma_c$ , the dynamics is frozen. Below the transition, evolution becomes progressively slower as the critical point is approached.

Numerical simulations show that for intermediate values of  $\sigma$ , as the predominance of the ordered small-dispersion state decreases and the critical transition is approached, a complex regime of clustering develops. Those oscillators in subensemble II with frequencies just above their lowest value,  $|\Omega_i| \gtrsim \mu/2$ , become divided into four clusters in  $\phi$ -space. Though the structure of each cluster is not sharply localized, it is still possible to define its phase by averaging  $\phi_i$  over the oscillators in that cluster. Two of the clusters are formed by oscillators with  $\Omega_i > 0$ , and exhibit anti-phase synchronization, so that their phases differ by  $\pi$  at all times. They move monotonically around  $\phi$ -space. The other two move in the opposite direction. They are formed by oscillators with  $\Omega_i < 0$ , and also show anti-phase synchronization (see inset of Fig. 1). Due to the opposite motion of the two pairs of clusters, they recurrently cross each other. At these crossings, their populations are temporarily superimposed in phase and, at the same time, their motion decelerates and the clusters become more compact. As a result, two well-localized and relatively long-lived big clusters, their phases differing by  $\pi$ , are recurrently built up out of the ensemble (see Fig. 2c).

The occurrence of two-cluster states is quantitatively disclosed by the complex quantity

$$z = N^{-1} \sum_{j=1}^N \exp(2i\phi_j). \quad (8)$$

If the whole ensemble splits into two anti-phase well-localized clusters, we have  $|z(t)| \approx 1$ , while for incoherent or higher-order clustered states we have  $|z(t)| \sim N^{-1/2}$ .

Figure 2a shows the evolution of  $|z(t)|$  in the clustering regime, displaying its irregular oscillations between relatively small and large values. For the same realization, Fig. 2b displays the phase difference  $\Delta\phi$  between two oscillators in subensemble II, both with positive values of  $\Omega_i$ , as a function of time. When  $|z(t)|$  is large,  $\cos \Delta\phi \approx 1$  identifies a two-cluster state where the two oscillators belong to the same cluster, while  $\cos \Delta\phi \approx -1$  corresponds to the situation where the oscillators are in different clusters. The intermittent transitions of  $\cos \Delta\phi$  between large positive and negative values reveal the dynamical nature of clustering, where any two oscillators may alternatively belong to the same or to different clusters.

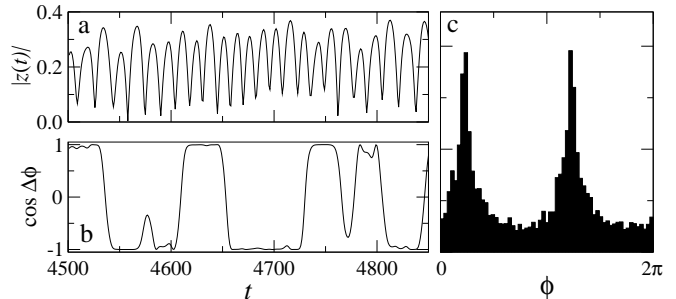


FIG. 2: (a) The quantity  $|z(t)|$  as a function of time in a system of  $10^4$  oscillators with Gaussian frequency distribution of dispersion  $\sigma = 0.31$ . (b) Phase difference of two oscillators with similar frequencies, in the same realization as in the upper plot. (c) Snapshot of the phase distribution at one of the maxima of  $|z(t)|$ , in the same realization.

The time average  $\zeta$  of the quantity  $|z(t)|$  plays the role of an order parameter detecting the two-cluster state. Figure 3 displays the dependence of  $\zeta$  with the frequency dispersion  $\sigma$ , for ensembles of several sizes. A peak near  $\sigma_c$  becomes clearly defined as  $N$  grows. While for  $\sigma > \sigma_c$  the value of  $\zeta$  apparently vanishes for large system sizes, a well-defined profile persists below the critical point. It grows from zero starting at  $\sigma \approx 0.2$  and attains its maximum,  $\zeta \approx 0.25$ , at  $\sigma \lesssim \sigma_c$ .

The formation of the four clusters whose recurrent superimposition gives rise to the two-cluster states detected by  $\zeta$  can be qualitatively understood in terms of previous results for coupled oscillators with disordered interactions. Given an oscillator whose internal variable evolves with frequency  $\Omega_i$ , the most persistent interactions affecting the dynamics of its phase  $\phi_i$  are due to oscillators with similar frequencies,  $\Omega_j \approx \Omega_i$ . In this case, in fact, the interaction weights  $J_{ij}$  are almost constant over very long time scales. These weights, however, are still different from each other, and can take positive and negative values. If the effect of frustration is moderate, the oscillators become entrained into two anti-phase groups of similar sizes [10]. The occurrence of this phenomenon requires that the oscillator phases are not pinned to the internal variables, and that a consistent population with similar frequencies  $\Omega_i$  actually exists to trigger conden-

sation. Such conditions are met by those oscillators of subensemble II with either positive or negative frequencies, close to the limit  $|\Omega_i| = \mu/2$ , where the population density is larger. Note that increasing the frequency dispersion  $\sigma$  contributes in two cooperating ways to the growth of subensemble II, and thus to the formation of clusters: it implies, at the same time, a larger population for high frequencies and—as  $\mu$  decreases—a population depletion in subensemble I. As discussed above, however, clustering is delayed by increasingly long transients as  $\sigma$  approaches the critical point. In the limit  $N \rightarrow \infty$ , the evolution freezes just beyond the critical dispersion, and cluster formation is thus suppressed.

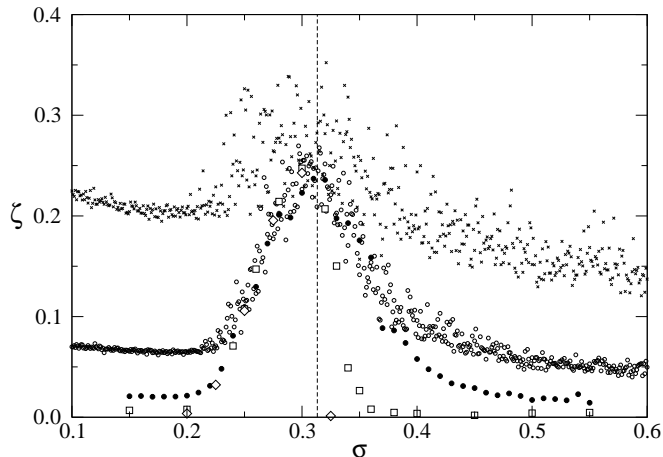


FIG. 3: The order parameter  $\zeta$  as a function of the frequency dispersion  $\sigma$ , obtained from numerical simulations of ensembles of various sizes,  $N = 10^2$  ( $\times$ ),  $10^3$  ( $\circ$ ),  $10^4$  ( $\bullet$ ),  $10^5$  ( $\square$ ), and  $10^6$  ( $\diamond$ ). The vertical dashed line indicates the critical dispersion  $\sigma_c = \sqrt{\pi/32}$ .

It is possible to give an approximate mathematical description of the motion of the four clusters, assuming that one of them is centered around a phase  $\phi^*(t)$  and is formed by oscillators whose internal frequencies are, on the average,  $\Omega^*$ . The phases and frequencies of the other three clusters can be derived from symmetry considerations. If the total fraction of the population involved in clusters is  $n^*$ , the approximate equation of motion for

the cluster phase reads

$$2\dot{\phi}^* = -\mu \sin(\phi^* \mp \Omega^*t) - n^* \cos(2\Omega^*t) \sin(2\phi^*). \quad (9)$$

The two terms in the right-hand side correspond, respectively, to the contributions of subensembles I and II. They have competing dynamical effects, and their relative importance changes as  $\sigma$  and  $n^*$  grow while  $\mu$  decreases. The first term favors the development of correlations between  $\phi^*$  and the internal variable  $\Omega^*t$ . If, on the other hand, the second term prevails, the phase difference between clusters of opposite internal frequencies,  $2\phi^*$ , spends most of the time close to its two fixed points,  $2\phi^* = 0$  or  $\pi$ , with rapid transitions between them when the sign of  $\cos(2\Omega^*t)$  changes. These fixed points correspond, precisely, to the mutual crossing of clusters of opposite frequencies that give origin to the recurrent two-cluster state.

Preliminary numerical results show that dynamical clustering occurs also in ensembles of particles moving in space and coupled through finite-range interactions, with autonomous internal variables as those considered above. In more complex models of interacting dynamical elements with evolving internal states, dynamical clustering may be enhanced if the internal variables themselves undergo segregation into coherent groups. This can be achieved, for example, if interactions between internal states are allowed. Numerical realizations suggest that this is the case when the variables  $\theta_i$  of the model studied above are coupled through higher-harmonics interaction functions [6], which give rise to dynamical clustering if noise is added to the internal evolution. The same effect has recently been illustrated for ensembles of interacting particles moving in space, where the internal dynamics of each element is represented by a chaotic map whose evolution is coupled to those of neighboring particles [13]. In real systems—specifically, in biological populations—the interaction between internal variables is expected to play an important role when such mechanisms as imitation and social influence are in action.

D. H. Z. acknowledges financial support from the Alexander von Humboldt Stiftung, Germany, and is grateful to the Fritz Haber Institut der Max Planck Gesellschaft, Berlin, for hospitality.

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