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On fundamental domains and volumes of hyperbolic Coxeter-Weyl groups

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We present a simple method for determining the shape of fundamental domains of generalized modular groups related to Weyl groups of hyperbolic Kac-Moody algebras. These domains are given as subsets of certain generalized upper half planes, on which the Weyl groups act via generalized modular transformations. Our construction only requires the Cartan matrix of the underlying finite-dimensional Lie algebra and the associated Coxeter labels as input information. We present a simple formula for determining the volume of these fundamental domains. This allows us to re-produce in a simple manner the known values for these volumes previously obtained by other methods.

I. INTRODUCTION

Constructions of fundamental domains of generalized modular groups usually rely on geometric considerations. By considering the different possible symmetry transformations acting on some generalized upper-half plane, the precise shape of the fundamental domain is narrowed down step-by-step until one arrives at its final shape. Especially for higher rank groups (such as $SL_n(\mathbb{Z})$) this poses a considerable computational and combinatorial problem since one has to consider a large number of possible successive symmetry transformations (already the determination of the fundamental domain of the standard modular group $PSL_2(\mathbb{Z})$ along these lines takes more than two pages of computations, see e.g. [1]). Although one can show that the precise shape of the fundamental domain can be determined within a finite number of steps, in the actual computation of a domain it is not always clear how many steps are actually necessary.

In this paper we show that, at least for modular groups arising as (even) Weyl groups of certain hyperbolic Kac–Moody algebras, such cumbersome constructions can be altogether avoided. More specifically, we present an easy method for obtaining the complete geometric information about the associated fundamental domains. All we require as information for determining the explicit shape and volume is the Cartan matrix of the corresponding Kac-Moody algebra and its Coxeter labels. As we will demonstrate this construction works for *all* hyperbolic Kac–Moody algebras [1] $\mathfrak{g}^{++}$ of over-extended type, which are generally obtained by extending a given finite dimensional simple Lie algebra $\mathfrak{g}$ via its affine extension $\mathfrak{g}^{+}$ by adding two nodes to the Dynkin diagram in a specified way. Likewise it applies to the *twisted* algebras obtained by inverting the arrows in the Dynkin diagram, because their Weyl groups are the same (but note that these twisted algebras, while being indefinite Kac–Moody algebras, in general are not of over-extended type). In particular, our construction also applies to those hyperbolic Kac–Moody algebras whose even Weyl groups can be identified with generalized modular groups defined over rings of integers in division algebras [3]. The first example of such an identification was given in [4] where it was shown that the rank-3 hyperbolic Kac–Moody algebra A_1^{++} (also denoted AE_3 or $\mathcal{F}$ in the literature) has the usual modular group $PSL_2(\mathbb{Z})$ as its even Weyl group, the full Weyl group being $W(A_1^{++}) = PGL_2(\mathbb{Z})$. In [3] more complicated examples were given, involving for instance the quaternionic integers (Hurwitz numbers), and admitting a Möbius-like realization [5]. The most interesting (and most complicated) example is the even Weyl group $W^+(E_{10})$ which can be identified with the arithmetic group $PSL_2(\mathtt{O})$ (where $\mathtt{O}$ are octonionic integers, also called *octavians*). For this example we will explicitly display the coordinates of the vertices of the fundamental domain of the Weyl group.

Knowledge of the shape of the fundamental domain allows one to compute its volume. In the non-linear realization of the hyperbolic Weyl group on some generalized upper half plane [5] (a hyperbolic space of constant negative curvature) the fundamental domains are realized as higher dimensional simplices. We present a very simple general formula for the volume of the domain in terms of integrals involving a quadratic form which contains all the information about the Lie algebra $\mathfrak{g}^{++}$ (see (32) below). We note that our considerations would also apply to cases where analogs of the so-called congruence subgroups of $PSL_2(\mathbb{Z})$ can de defined: the volume is then simply a multiple of the original volume, with the factor equal to the index of the congruence subgroup in the given generalized modular group. Such congruence subgroups presumably do exist for the generalized arithmetic groups studied in [3], but we are not aware of any concrete results along these lines.

As an historic aside, we mention that the first computation of hyperbolic volumes in terms of the dihedral angles of the simplex under consideration is due to one of the inventors of hyperbolic geometry, N.I. Lobachevsky [6]. His results were extended by Schläfli and Coxeter [7], see also Vinberg [8]. Further work on this problem can

[1] An indefinite Kac–Moody algebra is called *hyperbolic* if the removal of any one node from its Dynkin diagram leaves an algebra which is either affine or finite [2].

be found in [9] which gives a list of numerical values for the volumes of hyperbolic Coxeter simplices, as well as analytical expressions for some special cases. Using (32) these values can be easily reproduced.

II. HYPERBOLIC ROOTS AND WEIGHTS

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. We denote the simple roots of $\mathfrak{g}$ by $\mathbf{a}_i \in \mathbb{R}^n$ and their associated fundamental weights by $\boldsymbol{\lambda}_i$, where $i = 1, \ldots, n$ with $n = \text{Rank}(\mathfrak{g})$ (see e.g. [10] for details). With the Cartan matrix of $\mathfrak{g}$

$$A_{ij} = \langle \mathbf{a}_i | \mathbf{a}_j \rangle \equiv \frac{2\mathbf{a}_i \cdot \mathbf{a}_j}{\mathbf{a}_j \cdot \mathbf{a}_j} \tag{1}$$

we define the *symmetrized Cartan matrix* B_{ij} as

$$B_{ij} \equiv (AD)_{ij} = 2\mathbf{a}_i \cdot \mathbf{a}_j = A_{ij}\, \mathbf{a}_j^2 \ , \tag{2}$$

where $\mathbf{a}_j^2 \equiv \mathbf{a}_j \cdot \mathbf{a}_j$ and there is no summation over double indices. Unlike A_{ij}, the matrix B_{ij} and the symmetrizing matrix $D_{ij} = \delta_{ij}\mathbf{a}_j^2$ depend on the normalization of $\mathbf{a}_j$. Following [3] we choose this normalization such that always $\boldsymbol{\theta}^2 = 1$ for the highest root

$$\boldsymbol{\theta} = \sum_{j=1}^{n} m_j \mathbf{a}_j \tag{3}$$

with the Coxeter labels m_j. When $\boldsymbol{\theta}$ is a *long* root we therefore have $\mathbf{a}_j^2 = 1$ for the long roots.

The associated fundamental weights $\boldsymbol{\lambda}_j$ constitute a basis dual to the simple roots [10]

$$\langle \boldsymbol{\lambda}_i | \mathbf{a}_j \rangle \equiv \frac{2\boldsymbol{\lambda}_i \cdot \mathbf{a}_j}{\mathbf{a}_j \cdot \mathbf{a}_j} = \delta_{ij} \tag{4}$$

implying

$$\boldsymbol{\lambda}_i \cdot \mathbf{a}_j = \frac{1}{2}\, \delta_{ij}\, \mathbf{a}_j^2 \ . \tag{5}$$

With the inverse Cartan matrix A^{-1} we thus have

$$\boldsymbol{\lambda}_i = \sum_k (A^{-1})_{ik} \mathbf{a}_k \ , \tag{6}$$

from which we deduce

$$\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j = \frac{1}{2} (A^{-1})_{ij}\, \mathbf{a}_j^2 \ . \tag{7}$$

or

$$\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j = \frac{1}{2} \mathbf{a}_i^2\, (B^{-1})_{ij}\, \mathbf{a}_j^2 \ . \tag{8}$$

Next we consider the *hyperbolic extension* $\mathfrak{g}^{++}$ of the finite-dimensional algebra $\mathfrak{g}$ obtained by adjoining to the Dynkin diagram of $\mathfrak{g}$ the affine node (labeled '0') and the over-extended node (labeled '-1'). This entails extending the Euclidean root space $\mathbb{R}^n$ to the *Lorentzian* space $\mathbb{R}^{1,n+1} = \mathbb{R}^{1,1} \oplus \mathbb{R}^n$. We denote the roots of $\mathfrak{g}^{++}$ by α_I, $I = -1, 0, 1, \ldots, n$, and define them according to

$$\alpha_{-1} \equiv -\delta - \bar{\delta} \ , \quad \alpha_0 \equiv \delta - \boldsymbol{\theta} \ , \quad \alpha_i \equiv \mathbf{a}_i \tag{9}$$

with the affine null vector $\delta \in \mathbb{R}^{1,1}$ and the conjugate null vector $\bar{\delta} \in \mathbb{R}^{1,1}$ obeying $\delta \cdot \bar{\delta} = \frac{1}{2}$. In this way we obtain the Cartan matrix of $\mathfrak{g}^{++}$ as

$$A_{IJ} = \langle \alpha_I | \alpha_J \rangle \equiv \frac{2\alpha_I \cdot \alpha_J}{\alpha_J \cdot \alpha_J} \tag{10}$$

with the *Lorentzian* inner product

$$\alpha_I \cdot \alpha_J \equiv \eta_{\mu\nu} \alpha_I^\mu \alpha_J^\nu \tag{11}$$

where the signature of $\eta_{\mu\nu}$ is $(- + \cdots +)$. Notice that the affine and over-extended simple roots are also normalized as $\alpha_{-1}^2 = \alpha_0^2 = 1$. The normalization $\boldsymbol{\theta}^2 = 1$ is necessary to obtain a single line between the affine and the hyperbolic node (connecting α_0 and α_{-1}).

The fundamental weights Λ_I for the hyperbolic extension $\mathfrak{g}^{++}$ are defined in analogy with (4)

$$\langle \Lambda_I | \alpha_J \rangle \equiv \frac{2\Lambda_I \cdot \alpha_J}{\alpha_J \cdot \alpha_J} = \delta_{IJ} \ . \tag{12}$$

By a standard construction (see e.g. [11]), the fundamental weights of $\mathfrak{g}^{++}$ can be expressed in terms of the null vectors δ and $\bar{\delta}$ and the finite weights $\boldsymbol{\lambda}_j$ as

$$\Lambda_{-1} = -\delta \ , \quad \Lambda_0 = \bar{\delta} - \delta \ , \quad \Lambda_j = n_j \Lambda_0 + \boldsymbol{\lambda}_i \ . \tag{13}$$

The coefficients n_j are fixed by requiring $\alpha_0 \cdot \Lambda_j = 0$ (cf. (12)), which gives

$$n_j = m_j \mathbf{a}_j^2 \ , \tag{14}$$

The fundamental Weyl chamber $\mathcal{C}_0 \subset \mathbb{R}^{1,n+1}$ is

$$\mathcal{C}_0 := \left\{ X \in \mathbb{R}^{1,n+1} \,|\, X \cdot \alpha_I \geq 0 \ \text{ for } I = -1, 0, 1, ..., n \right\}$$

With the fundamental weights Λ_I one obtains a more convenient representation of $\mathcal{C}_0$

$$\mathcal{C}_0 = \left\{ X \in \mathbb{R}^{1,n+1} \,|\, X = \sum_I s_I \Lambda_I \ \text{ with } s_I \geq 0 \text{ for all } I \right\} \tag{15}$$

The null vector δ lies on the forward light-cone in root space. The Weyl chamber itself is the convex hull of the hyperplanes orthogonal to the simple roots of the algebra. The fundamental weights are vectors pointing along the edges of the Weyl chamber. In other words, $\mathcal{C}_0$ is a 'wedge' in $\mathbb{R}^{1,n+1}$. For the hyperbolic algebras $\mathfrak{g}^{++}$ of over-extended type considered here this wedge lies inside the forward lightcone, always touching it with the lightlike weight vector Λ_{-1}, while all other fundamental

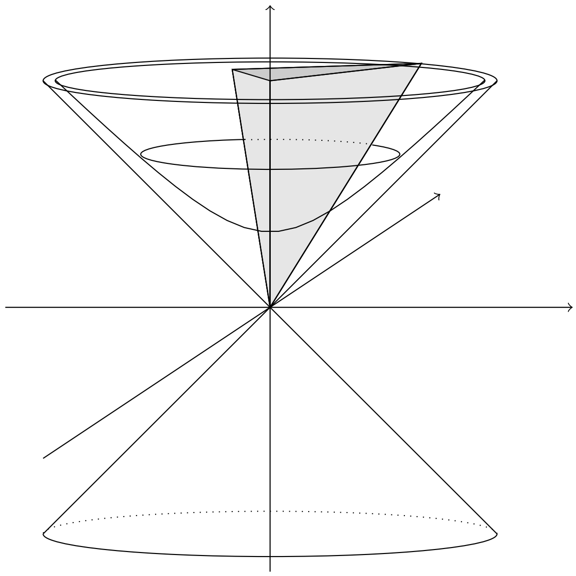

Figure 1: Sketch of the Weyl chamber as a wedge inside the forward light cone that is intersected by the unit hyperboloid.

weights obey $\Lambda_j^2 \leq 0$. By contrast, for general indefinite (Lorentzian) $\mathfrak{g}^{++}$ the Weyl chamber may stretch beyond the lightcone and also contain space-like vectors. A schematic picture of the Weyl chamber for hyperbolic $\mathfrak{g}^{++}$ is shown in Fig. 1. We have included the forward light-cone and the intersecting unit hyperboloid.

As it turns out the assumptions made suffice to cover all cases of interest. This concerns in particular the *twisted* algebras: as these are obtained by inverting the arrows in the relevant Dynkin diagrams, the associated Coxeter Weyl groups, not being sensitive to the direction of the arrows, coincide with those of the untwisted diagrams. We therefore note the following isomorphisms of Weyl groups using Kac' notation [2]:

$$\begin{aligned} W(G_2^{(1)+}) &\cong W(D_4^{(3)+}) \\ W(B_n^{(1)+}) &\cong W(A_{2n-1}^{(2)+}) \\ W(C_n^{(1)+}) &\cong W(D_{n+1}^{(2)+}) \\ W(F_4^{(1)+}) &\cong W(E_6^{(2)+}) \end{aligned} \tag{16}$$

where the superscript $^+$ on the r.h.s. indicates the extension of the affine algebra by another node. But note that the twisted algebras, though perfectly well-defined as indefinite Kac–Moody algebras, are not necessarily of over-extended type. In the notation of Fuchs and Schweigert [12], the later three isomorphisms are

$$\begin{aligned} W(B_n^{(1)+}) &\cong W(C_n^{(2)+}) \\ W(C_n^{(1)+}) &\cong W(B_n^{(2)+}) \\ W(F_4^{(1)+}) &\cong W(F_4^{(2)+}) \end{aligned} \tag{17}$$

The corresponding volumes of the fundamental domains therefore also coincide.

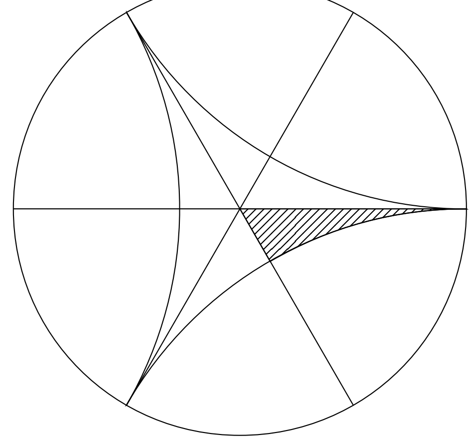

Figure 2: Example of a fundamental domain on the Poincaré disk obtained by intersecting the Weyl chamber with the (compactified) unit hyperboloid, here for the algebra A_1^{++}.

III. VOLUME FORMULA

The linear action of the Weyl group in $\mathbb{R}^{1,n+1}$ preserves the (Lorentzian) length, and therefore induces a non-linear *modular action* on the forward unit hyperboloid

$$X \cdot X \equiv -x^+x^- + \mathbf{x}^2 = -1 \ , \quad x^\pm > 0 \tag{18}$$

with light-cone coordinates $x^\pm \equiv (x^0 \pm x^{n+1})/\sqrt{2}$ in $\mathbb{R}^{1,1}$ and $\mathbf{x} \in \mathbb{R}^n$. For the cases $n = 1, 2, 4$ and 8 studied in [3], where the root space of $\mathfrak{g}$ can be endowed with the structure of a division algebra, the induced non-linear action takes the form of a generalized Möbius transformation over a (possibly non-commutative and non-associative) ring of integers.

The intersection of the fundamental Weyl chamber $\mathcal{C}_0$ with the unit hyperboloid defines a corresponding fundamental domain $\mathcal{F}_0$ on the unit hyperboloid. The corresponding domain on the (compactified) unit hyperboloid (*alias* the Poincaré disk) is depicted in Fig. 2. In the remainder, however, we will study this domain as a subset of the *generalized (Poincaré) upper half plane* $\mathcal{H}$ rather than the unit hyperboloid [2]. This upper half plane is defined as

$$\mathcal{H} \equiv \mathcal{H}_{n+1} := \big\{ (\mathbf{u}, v) \,|\, \mathbf{u} \in \mathbb{R}^n \, , \, v > 0 \big\} \tag{19}$$

and is thus of dimension $n+1$. $\mathcal{H}_{n+1}$ is isometric to the forward unit hyperboloid in $\mathbb{R}^{1,n+1}$ by means of the standard coordinate transformation

$$x^- = \frac{1}{v} \ , \quad x^+ = v + \frac{\mathbf{u}^2}{v} \ , \quad \mathbf{x} = \frac{\mathbf{u}}{v} \tag{20}$$

[2] Note that the fundamental domain $\mathcal{F}_0$ is half of the fundamental domain $\mathcal{F}$ of the ordinary modular group. The latter corresponds to the *even* subgroup of the Weyl group.

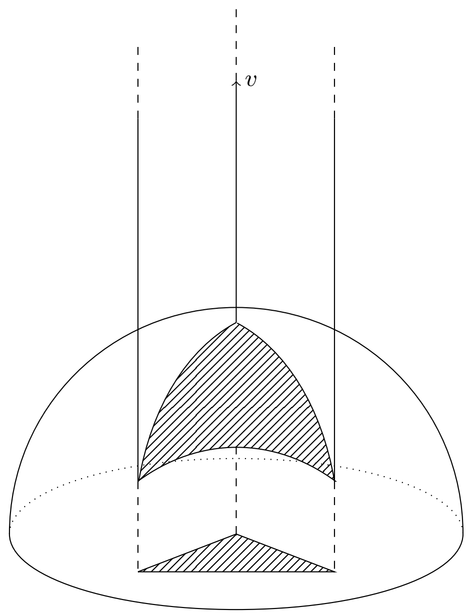

Figure 3: Schematic depiction of a Weyl chamber on the UHP, corresponding in this case to A_2^{++}.

The Minkowskian line element is transformed to

$$\mathrm{d}s^2 = \frac{\mathrm{d}\mathbf{u}^2 + \mathrm{d}v^2}{v^2} \tag{21}$$

where, of course, $\mathrm{d}\mathbf{u}^2 \equiv \mathrm{d}u_1^2 + \cdots + \mathrm{d}u_n^2$. The fundamental domain $\mathcal{F}_0 \subset \mathcal{H}$ is now rather easy to determine from the representation (15) by identifying the points where the rays along Λ_I 'pierce' the unit hyperboloid, and then mapping these points to $\mathcal{H}$ by means of (20). We first notice that the over-extended fundamental weight Λ_{-1} (*alias* the affine null vector δ) corresponds to the 'cusp' at infinity in $\mathcal{H}$ with coordinates $v = \infty$, $\mathbf{u} = 0$ [5], while Λ_0 corresponds to the point $v = 1$, $\mathbf{u} = 0$ in $\mathcal{H}$. From (13) we see that the remaining fundamental weights are mapped to the points

$$v_j = \sqrt{1 - \frac{\boldsymbol{\lambda}_j^2}{n_j^2}} \quad , \quad \mathbf{u}_j = \frac{\boldsymbol{\lambda}_j}{n_j} \tag{22}$$

on the unit hemisphere $v^2 + \mathbf{u}^2 = 1$, $v > 0$ in $\mathcal{H}_{n+1}$. If $|\mathbf{u}_j| = 1$ for some j we have another cusp in addition to the cusp at infinity, but now lying on the boundary $v = 0$ of $\mathcal{H}$. Therefore, the fundamental region always has the shape of a 'skyscraper' that extends to infinite height over the simplex $\Sigma \subset \mathbb{R}^n$ defined by the points 0 and $\mathbf{u}_j$, and whose 'bottom' is cut off by the unit hemisphere. See Fig. 3 for an artist's view; the 'bottom' of the skyscraper is the excised shaded region on the unit sphere.

Using the above formulas we obtain

$$\mathbf{u}_i \cdot \mathbf{u}_j \equiv S_{ij} = \frac{1}{2 m_i m_j} (B^{-1})_{ij} \ . \tag{23}$$

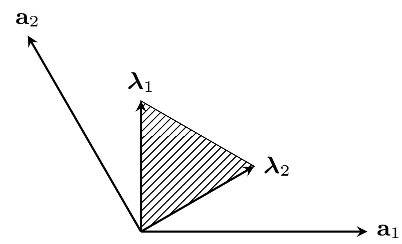

Figure 4: Schematic example for the projection of the fundamental domain for the Weyl chamber onto the hypersurface $v = 0$ for A_2^{++}. In accordance with (8) the roots and weights here are normalized as $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $\boldsymbol{\lambda}_1 = (0, \frac{1}{\sqrt{3}})$, $\boldsymbol{\lambda}_2 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$.

The matrix S_{ij} encodes all the Lie algebraic information about the over-extended algebra $\mathfrak{g}^{++}$ via the inverse symmetrized Cartan matrix B^{-1} and the Coxeter labels m_j. By a general result valid for all finite $\mathfrak{g}$ [10] the matrices B^{-1} are positive definite; furthermore their individual entries B_{ij}^{-1} are also positive. It thus follows that

$$S > 0 \ \text{(as a matrix) and} \ \ S_{ij} > 0 \ \text{ for all } i, j \tag{24}$$

Note that this formula holds for simply-laced as well as non-simply-laced (untwisted) algebras. In particular, in the non-simply laced case one has to distinguish between the Coxeter/dual Coxeter labels of the untwisted and the Coxeter/dual Coxeter labels of the twisted version of the over-extension of the algebra.

As we just explained the fundamental domain $\mathcal{F}_0 \subset \mathcal{H}$ rises over the simplex $\Sigma \subset \mathbb{R}^n$ defined by

$$\Sigma := \left\{ \mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} = \sum_{i=1}^n t_i \mathbf{u}_i \, ; \ t_i \geq 0, \ \sum_{i=1}^n t_i \leq 1 \right\} . \tag{25}$$

With the above definitions we get

$$\mathbf{x}(t)^2 = \sum_{i,j=1}^n S_{ij} t_i t_j \ . \tag{26}$$

From the positivity properties (24) we deduce the following chain of inequalities valid for all points $\mathbf{x}(t) \in \Sigma$

$$0 \leq \sum_{i,j} S_{ij} t_i t_j \ \leq \ \max_{i,j} S_{ij} \left(\sum_k t_k \right)^2 \leq \max_{i,j} S_{ij} \tag{27}$$

Therefore $\mathbf{x}(t)^2 < 1$ as long as all matrix entries satisfy $S_{ij} < 1$. From (13) it is straighforward to see that

$$\Lambda_j^2 = n_j^2 \left(\mathbf{u}_j^2 - 1 \right) \tag{28}$$

and it therefore follows that $S_{ii} = 1$ when the corresponding hyperbolic weight Λ_i becomes null; for spacelike weights ($\Lambda_j^2 > 0$) we have $|\mathbf{u}_j| > 1$, and the corresponding point $(v_j, \mathbf{u}_j)$ no longer lies in the generalized upper half-plane. This happens when $\mathfrak{g}^{++}$ is Lorentzian, but no longer hyperbolic, as is for instance the case for all A_n^{++} with $n \geq 8$ and B_n^{++} and D_n^{++} for $n \geq 9$.

With the hyperbolic volume element

$$\mathrm{dvol}(\mathbf{u}, v) = \frac{\mathrm{d}^n u \, \mathrm{d}v}{v^{n+1}} \tag{29}$$

we thus obtain

$$\mathrm{vol}(\mathcal{F}_0) = \sqrt{\det S} \int\limits_{\Delta_n} \mathrm{d}t_1 \cdots \mathrm{d}t_n \int\limits_{\sqrt{1-\mathbf{x}(t)^2}}^{\infty} \frac{\mathrm{d}v}{v^{n+1}} \tag{30}$$

where Δ_n is the *standard simplex* in $\mathbb{R}^n$

$$\Delta_n := \left\{ (t_1, \ldots, t_n) \,|\, t_i \geq 0 \,, \sum t_i \leq 1 \right\} \tag{31}$$

Performing the integral over v we arrive at

$$\boxed{\mathrm{vol}(\mathcal{F}_0) = \frac{1}{n} \int_{\Delta_n} \mathrm{d}t_1 \cdots \mathrm{d}t_n \frac{\sqrt{\det S}}{(1 - \sum t_i S_{ij} t_j)^{\frac{n}{2}}} \ .} \tag{32}$$

This simple formula is our main result: it expresses the hyperbolic volume as an integral over a standard simplex Δ_n in $\mathbb{R}^n$ with the single matrix S_{ij} encoding all the Lie algebraic information about the hyperbolic Weyl group. The integral is manifestly convergent if all $S_{ij} < 1$. When $S_{ii} = 1$ the corresponding point has $|\mathbf{u}_i| = 1$ and $v_i = 0$ and thus lies on the boundary of $\mathcal{H}$, but the integral is still convergent (see below for examples when this happens). For non-hyperbolic Lorentzian algebras the integral diverges and therefore $\mathrm{vol}(\mathcal{F}_0) = \infty$.

When evaluating this formula it may be convenient to diagonalize the quadratic form in terms of new integration variables ξ_i such that

$$\sum_{i,j} S_{ij} t_i t_j = \xi_1^2 + \cdots + \xi_n^2 \tag{33}$$

and the determinant factor $(\det S)^{1/2}$ is cancelled by the Jacobian. The variables ξ_i always exist by the positivity properties of the matrix S. However, the (still simplicial) domain of integration is then more complicated to parametrize.

IV. ANALYTIC RESULTS

We now show how our formula (32) immediately yields the volumes for various hyperbolic reflection groups corresponding to over-extended hyperbolic algebras $\mathfrak{g}^{++}$ of low rank. It is straightforward to check that for A_1 (corresponding to the rank-3 Feingold-Frenkel algebra A_1^{++}) we have $S = \frac{1}{2}$, and one easily recovers the well known result $\mathrm{vol}(\mathcal{F}_0[A_1^{++}]) = \frac{\pi}{6}$. For this reason we proceed right away to the case of rank 4.

The rank-4 algebras of over-extended type are A_2^{++}, C_2^{++} and G_2^{++}. For $\mathfrak{g}^{++} = A_2^{++}$ we have $m_1 = m_2 = 1$ and thus the matrix S_{ij} is $1/2$ the inverse of the A_2 Cartan matrix

$$B^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix} \tag{34}$$

Transforming to new coordinates $\xi_1 = \frac{1}{2}(t_1 + t_2)$ and $\xi_2 = (1/2\sqrt{3})(t_1 - t_2)$ such that the Jacobi determinant cancels the factor $(\det S)^{1/2} = 1/2\sqrt{3}$ and

$$\frac{1}{3}\big(t_1^2 + t_1 t_2 + t_2^2\big) = \xi_1^2 + \xi_2^2 \tag{35}$$

we obtain

$$\begin{aligned} \mathrm{vol}(\mathcal{F}_0[A_2^{++}]) \ &= \ \frac{1}{2} \int_0^{\frac{1}{2}} \mathrm{d}\xi_1 \int_{-\frac{\xi_1}{\sqrt{3}}}^{\frac{\xi_1}{\sqrt{3}}} \frac{\mathrm{d}\xi_2}{1 - \xi_1^2 - \xi_2^2} \\ &= \frac{1}{2} \int_0^{\frac{1}{2}} \frac{\mathrm{d}\xi}{\sqrt{1-\xi^2}} \ln \left(\frac{\sqrt{1-\xi^2} + \frac{1}{\sqrt{3}}\xi}{\sqrt{1-\xi^2} - \frac{1}{\sqrt{3}}\xi} \right) \end{aligned} \tag{36}$$

The substitution $\xi = \sin\theta$ leads to

$$\begin{aligned} \mathrm{vol}(\mathcal{F}_0[A_2^{++}]) &= \frac{1}{2} \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \ln \left(\frac{\cos\theta + \frac{1}{\sqrt{3}} \sin\theta}{\cos\theta - \frac{1}{\sqrt{3}}\sin\theta} \right) \\ &= \frac{1}{2} \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \ln \left(\frac{2\sin(\theta + \frac{\pi}{3})}{2\sin(\frac{\pi}{3} - \theta)} \right) \end{aligned} \tag{37}$$

After a suitable shift of integration variables and using the definition and properties of the Lobachevsky function, this reduces to

$$\begin{aligned} &\mathrm{vol}(\mathcal{F}_0[A_2^{++}]) && (38) \\ &\quad = \frac{1}{2} \left[Л\left(\frac{\pi}{3}\right) - Л\left(\frac{\pi}{6}\right) - Л\left(\frac{\pi}{2}\right) + Л\left(\frac{\pi}{3}\right) \right] \\ &\quad = \frac{1}{4} Л\left(\frac{\pi}{3}\right) && (39) \end{aligned}$$

For $\mathfrak{g}^{++} = G_2^{++}$ we have the Coxeter labels $m_1 = 2 \,, m_2 = 3$ and the relevant matrices are

$$B^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix} \tag{40}$$

Now the substitution to diagonalize the quadratic form is $\xi_1 = \frac{1}{2}(t_1 + t_2) \,, \xi_2 = (1/2\sqrt{3})t_2$, and we get

$$\begin{aligned} \mathrm{vol}(\mathcal{F}_0[G_2^{++}]) \ &= \ \frac{1}{2} \int_0^{\frac{1}{2}} \mathrm{d}\xi_1 \int_0^{\frac{\xi_1}{\sqrt{3}}} \frac{\mathrm{d}\xi_2}{1 - \xi_1^2 - \xi_2^2} \\ &= \frac{1}{2} \mathrm{vol}\big(\mathcal{F}_0[A_2^{++}]\big) = \frac{1}{8} Л\left(\frac{\pi}{3}\right) \end{aligned} \tag{41}$$

Finally, for C_2^{++} we have $m_1 = m_2 = 1$ and

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \tag{42}$$

Note that the corresponding matrix for B_2^{++} is $S = \frac{1}{4}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and yields the same volume.
Now we substitute $\xi_1 = \frac{1}{2}(t_1 + t_2)$ and $\xi_2 = \frac{1}{2}t_2$ to get

$$\begin{aligned} \mathrm{vol}\big(\mathcal{F}_0[C_2^{++}]\big) &= \frac{1}{2}\int_0^{\frac{1}{2}} \mathrm{d}\xi_1 \int_0^{\xi_1} \frac{\mathrm{d}\xi_2}{1-\xi_1^2-\xi_2^2} \\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}} \mathrm{d}\theta \ln\left(\frac{2\sin(\theta+\frac{\pi}{4})}{2\sin(\frac{\pi}{4}-\theta)}\right) \end{aligned} \tag{43}$$

Similar manipulations as before lead to the result

$$\begin{aligned} &\mathrm{vol}\left(\mathcal{F}_0[C_2^{++}]\right) \\ &\quad= \frac{1}{4}\left[\text{Л}\left(\frac{\pi}{4}\right) - \text{Л}\left(\frac{\pi}{12}\right) - \text{Л}\left(\frac{5\pi}{12}\right) + \text{Л}\left(\frac{\pi}{4}\right)\right] \\ &\quad= \frac{1}{6}\,\text{Л}\left(\frac{\pi}{4}\right) \end{aligned} \tag{44}$$

V. HIGHER RANK ALGEBRAS

What about higher rank algebras? Although the integrals (32) look elementary it turns out that calculations become rapidly more complicated with increasing dimension, and we have not been able to derive 'simple' closed form expressions for them when $n > 2$. The complications are mainly due to the integration boundaries which must be analyzed case by case. Although (32) is suggestive of higher order Lobachevsky functions (see appendix), this expectation (as expressed, for instance, in [8]) is not borne out by the concrete calculations, nor have such expressions been explicitly exhibited in the literature, see e.g. [8]. One possibility, to be explored in future work, would be to expand the integrand in (32) whereby the integral is expressed as an infinite sum of terms each one of which involves an integral of a monomial over the standard simplex Δ_n. Such integrals have been studied in the literature [13, 14] but the resulting expressions are still quite involved. Numerically these series converge rapidly, as all terms are of the same sign.

Using (32) one can compute the volume of different fundamental domains numerically. The only input information that is needed is the matrix S, which is calculated from the matrix B^{-1} and the Coxeter labels via (23). As already mentioned above, B^{-1} is the inverse symmetrized Cartan matrix and its form for the different algebras can be found in the standard Lie algebra literature, see e.g. [10]).

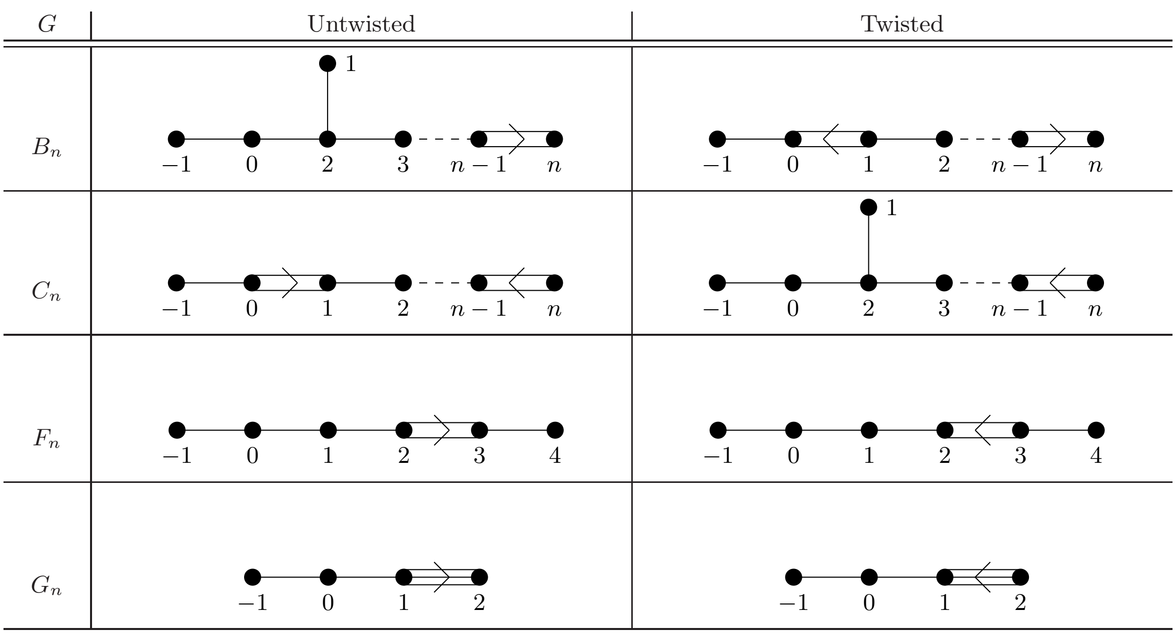

Table I: Dynkin diagrams of the over-extended twisted and untwisted non-simply laced finite-dimensional algebras with Dynkin labeling of nodes

Here we list the matrices S for the Lie Algebras of $A_n, B_n, C_n, D_n, F_4, E_6, E_7$ and E_8 in the Cartan classification (the matrices for the rank 2 algebras were already given in the previous section). In addition we list the set of Coxeter (or dual Coxeter) labels m_i used in the computation of S. Note that in the case of the non-simply laced algebras it is necessary to distinguish between the labels of the *twisted* and *untwisted* algebra. For these algebras we label the matrix S with a superscript $^{(1)}$ or $^{(2)}$, respectively, indicating whether it corresponds to the *untwisted* or *twisted* over-extension of the algebra. Considering Table I containing Dynkin diagrams of over-extensions of the non-simply laced algebras, we note that the twisted Dynkin diagram of B_n is the same as the untwisted Dynkin diagram of C_n, simply with the direction of the arrows reversed. This tells us that the volumes of the corresponding fundamental domains have to be the same, since the matrix S is the *symmetrized* version the Cartan matrix and therefore contains no information about the direction of the arrows. A similar correspondence holds for the untwisted diagram of B_n and the twisted diagram of C_n, as well as for G_2 and F_4.

The condition for the over-extension of each algebra to be of hyperbolic type is that *all* of the diagonal entries S_{ii} of the underlying finite-dimensional algebra must satisfy $S_{ii} \leq 1$. Geometrically each $S_{ii} = 1$ corresponds to an additional fundamental weight (edge of the Weyl chamber) lying *on* the forward light-cone. For each $S_{ii} > 1$ an additional fundamental weight lies *outside* the light cone and the over-extension is not of hyperbolic type. For each algebra we state the range of n for which the over-extension is hyperbolic.

$\mathfrak{g}^{++} = A_n^{++}$: The Coxeter labels are $m_i = (1, ..., 1)$, and thus the matrix S is

$$S[A_n] = \frac{1}{(n+1)} \times \\ \times \begin{pmatrix} \frac{n}{2} & \frac{n-1}{2} & \frac{n-2}{2} & \frac{n-3}{2} & \cdots & \frac{1}{2} \\ \frac{n-1}{2} & n-1 & n-2 & n-3 & \cdots & 1 \\ \frac{n-2}{2} & n-2 & \frac{3(n-2)}{2} & \frac{3(n-3)}{2} & \cdots & \frac{3}{2} \\ \frac{n-3}{2} & n-3 & \frac{3(n-3)}{2} & 2(n-3) & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \cdots & \frac{n}{2} \end{pmatrix} \tag{45}$$

From the explicit form of the matrix it is obvious that

$$S_{ij} \leq 1 \quad \Leftrightarrow \quad \frac{j(n+1-j)}{2(n+1)} \leq 1 \tag{46}$$

for all $j = 1, \ldots, n$. Hence A_n^{++} is hyperbolic for $n \leq 7$.

$\mathfrak{g}^{++} = B_n^{++}$: the Coxeter labels are $m_i = (1, 2, ..., 2)$ (as untwisted Coxeter labels for B_n, and as twisted dual Coxeter labels for C_n), and the matrix S is

$$S^{(1)}[B_n] = \left(\begin{array}{c|cccccc} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} \\ \hline \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} & \cdots & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{n-1}{8} & \frac{n-1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{n-1}{8} & \frac{n}{8} \end{array}\right) \tag{47}$$

All matrix entries are ≤ 1 for $n \leq 8$, whence B_n^{++} is hyperbolic for $n \leq 8$. Inverting the arrow in the Dynkin diagram we infer that

$$S^{(1)}[B_n] = S^{(2)}[C_n]. \tag{48}$$

$\mathfrak{g}^{++} = C_n^{++}$: the Coxeter labels are $m_i = (1, ..., 1)$ (as twisted Coxeter labels for B_n and untwisted dual Coxeter labels for C_n), so

$$S^{(1)}[C_n] = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \cdots & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \cdots & \frac{n-1}{4} & \frac{n-1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \cdots & \frac{n-1}{4} & \frac{n}{4} \end{pmatrix} \tag{49}$$

Clearly, C_n^{++} is hyperbolic for $n \leq 4$. As before we get

$$S^{(1)}[C_n] = S^{(2)}[B_n] \tag{50}$$

$\mathfrak{g}^{++} = D_n^{++}$: the Coxeter labels are $m_i = (1, 2, ..., 2, 1, 1)$, and therefore

$$S[D_n] = \left(\begin{array}{c|ccccc|cc} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \hline \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} & \cdots & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{n-2}{8} & \frac{n-2}{8} & \frac{n-2}{8} \\ \hline \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{n-2}{8} & \frac{n}{8} & \frac{n-2}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \cdots & \frac{n-2}{8} & \frac{n-2}{8} & \frac{n}{8} \end{array}\right) \tag{51}$$

We see that D_n^{++} is hyperbolic for $n \leq 8$.

$\mathfrak{g}^{++} = F_4^{++}$: the Coxeter labels are $(2, 3, 2, 1)$ for the untwisted dual Coxeter labels as well as for the twisted Coxeter labels:

$$S^{(1)}[F_4] = S^{(2)}[F_4] = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{1}{2} \end{pmatrix} \tag{52}$$

$\mathfrak{g}^{++} = E_6^{++}$: the Coxeter labels are $(1, 2, 3, 2, 1, 2)$, and we have

$$S[E_6] = \begin{pmatrix} \frac{2}{3} & \frac{5}{12} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{5}{12} & \frac{5}{12} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{5}{12} & \frac{5}{12} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{5}{12} & \frac{2}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad (53)$$

$\mathfrak{g}^{++} = E_7^{++}$: the Coxeter labels are $(2, 3, 4, 3, 2, 1)$, and we have

$$S[E_7] = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{5}{12} & \frac{5}{12} & \frac{5}{12} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{5}{12} & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{5}{12} & \frac{1}{2} & \frac{3}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} \end{pmatrix} \quad (54)$$

$\mathfrak{g}^{++} = E_8^{++}$: with the Coxeter labels $(2, 3, 4, 5, 6, 4, 2, 3)$ we have

$$S[E_8] = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{5}{12} & \frac{5}{12} & \frac{5}{12} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{5}{12} & \frac{7}{16} & \frac{7}{16} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{5}{12} & \frac{7}{16} & \frac{1}{2} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{5}{12} & \frac{5}{12} & \frac{5}{12} & \frac{2}{9} \end{pmatrix} \quad (55)$$

Using equations (22) one can determine the coordinates of the vertices of the fundamental domain. For example, the vertices of the domain corresponding to the Weyl group of $\mathfrak{e}_{10}$ are given by

$$\begin{aligned}
(v_1, \mathbf{u}_1) &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\mathbf{e}_0\right) \\
(v_2, \mathbf{u}_2) &= \left(\sqrt{\frac{2}{3}}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{6}(\mathbf{e}_1 + \mathbf{e}_5 + \mathbf{e}_6)\right) \\
(v_3, \mathbf{u}_3) &= \left(\sqrt{\frac{5}{8}}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{4}(\mathbf{e}_5 + \mathbf{e}_6)\right) \\
(v_4, \mathbf{u}_4) &= \left(\sqrt{\frac{3}{5}}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{10}(\mathbf{e}_2 + 3\mathbf{e}_5 + 2\mathbf{e}_6 - \mathbf{e}_7)\right) \\
(v_5, \mathbf{u}_5) &= \left(\frac{1}{\sqrt{6}}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{6}(2\mathbf{e}_5 + \mathbf{e}_6 - \mathbf{e}_7)\right) \\
(v_6, \mathbf{u}_6) &= \left(\frac{3}{4}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{8}(\mathbf{e}_3 + 3\mathbf{e}_5 + \mathbf{e}_6 - \mathbf{e}_7)\right) \\
(v_7, \mathbf{u}_7) &= \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{2}\mathbf{e}_5\right) \\
(v_8, \mathbf{u}_8) &= \left(\frac{\sqrt{5}}{3}, \frac{1}{2}\mathbf{e}_0 + \frac{1}{6}(\mathbf{e}_4 + 2\mathbf{e}_5 + \mathbf{e}_6 - \mathbf{e}_7)\right) \quad (56)
\end{aligned}$$

The special feature of this example is that the vectors $\mathbf{u}_j$ now belong to the octonions $\mathbb{O}$, the non-commutative and non-associative maximal division algebra. Accordingly, the unit vectors $\mathbf{e}_j$ (for $j = 1, \ldots, 7$) are just the octonionic imaginary units. The vertex coordinates of the fundamental domains of other Weyl groups are obtained similarly.

By evaluating the integrals in (32) numerically we obtain the volumes of all the fundamental domains of the hyperbolic Weyl groups of the algebras listed above. The values we find agree with those found in [9] where the volumes of all hyperbolic Coxeter simplices were obtained by a different method.

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Appendix A: The Lobachevsky function

The Lobachevsky function Л is defined as

$$Л(\theta) = -\int_0^\theta \log(|2\sin t|)\mathrm{d}t \quad \forall \theta \in \mathbb{R} \quad (A1)$$

and related to the dilogarithm and the Clausen function through the identities

$$Л(\omega) = \frac{1}{2}\,\mathrm{Im}(\mathrm{Li}_2(\mathrm{e}^{2\mathrm{i}\omega})) = \frac{1}{2}\mathrm{Cl}_2(2\omega)\ . \tag{A2}$$

where the dilogarithm is defined as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-w)}{w}\mathrm{d}w \quad \forall z \in \mathbb{C} : z \notin [1,\infty) \tag{A3}$$

The Lobachevsky function fulfills the relations

$$Л(0) = Л\left(\frac{\pi}{2}\right) = 0 \tag{A4}$$
$$Л(\theta+\pi) = Л(\theta) \tag{A5}$$
$$Л(-\theta) = -Л(\theta) \tag{A6}$$

and

$$Л(n\theta) = n\sum_{j=0}^{n-1} Л(\theta + \frac{j\pi}{n}) \quad \forall n \in \mathbb{Z}_+\ , \tag{A7}$$

yielding e.g.

$$\begin{aligned} Л\left(\frac{\pi}{6}\right) &= \frac{3}{2}Л\left(\frac{\pi}{3}\right) \\ Л\left(\frac{\pi}{4}\right) &= \frac{3}{4}\left[Л\left(\frac{\pi}{12}\right) + Л\left(\frac{5\pi}{12}\right)\right]\ . \end{aligned} \tag{A8}$$

The polylogarithm functions

$$\mathrm{Li}_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n} \quad \forall z \in \mathbb{C} \quad \forall n \geq 1 \tag{A9}$$

arise naturally in the computation of hyperbolic volume. They are inductively related through

$$\mathrm{Li}_1(z) = -\log(1-z) \tag{A10}$$
$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(w)\frac{\mathrm{d}w}{w}\ . \tag{A11}$$

It is furthermore common to define the higher Lobachevsky functions through the polylogarithm according to

$$Л_{2m}(\theta) = \frac{1}{2^{2m-1}}\,\mathrm{Im}(\mathrm{Li}_{2m}(\mathrm{e}^{2\mathrm{i}\theta})) \tag{A12}$$
$$Л_{2m+1}(\theta) = \frac{1}{2^{2m}}\,\mathrm{Re}(\mathrm{Li}_{2m+1}(\mathrm{e}^{2\mathrm{i}\theta}))\ . \tag{A13}$$

The higher Lobachevsky functions fulfill the more general relations

$$Л_m(\theta) = Л_m(\theta+\pi) \tag{A14}$$
$$\frac{1}{n^{m-1}}Л_m(n\theta) = \sum_{j=0}^{n-1} Л_m\left(\theta + \frac{j\pi}{n}\right) \tag{A15}$$
$$Л_m(-\theta) = (-1)^{m+1}Л_m(\theta)\ . \tag{A16}$$

However, as we already pointed out, and unlike for rank four, it does not appear that the volumes for the higher rank fundamental domains can be expressed solely in terms of higher Lobachevsky functions.

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