Reconstruction of Deligne Classes and Cocycles

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⋙The day will surely come when you see the true believers, men and women, with their lights shining before them and on their right hands, and a voice saying to them: 'Rejoice this day. You shall enter gardens watered by flowing streams. in which you shall abide for ever.' This is the supreme triumph.
On that day, the hypocrites, men and women, will say to the true believers: 'Wait for us, that we may borrow some of your light.' But they will be told: 'Go back, and seek some other light!'
A wall with a gate shall be set before them. Inside there shall be mercy, and without, to the fore, the scourge of Hell. They will call out to them, saying: 'Were we not on your side?' 'Yes,' they will reply, 'but you tempted yourselves, you wavered, you doubted, and were deceived by vain desires, until His judgement came. You were diverted from Him by illusions. The Fire shall be your home: you have justly earned it, an evil end!'.
Know that the life of this world is but a sport and a diversion, a show and an empty boast among you, a quest for greater riches and more childern. It is like plants that flourish after rain and pleases the disbelievers. But then they wither and turn yellow, into useless hay, and are blown away by the wind. In the life to come a grievous scourge awaits you- or the forgiveness of God and His pleasure.
This worldly life is no more than a temporary illusion.⋘

(57:13-15, 57:20)

Abstract

In this thesis we mainly generalize two theorems from Mackaay-Picken and Picken ([19], [24]). In the first paper, Mackaay and Picken show that there is a bijective correspondence between Deligne 2-classes $\xi \in \check{H}^2(M, \mathcal{D}^2)$ and holonomy maps from the second thin-homotopy group $\pi_2^2(M)$ to $U(1)$. In the second one, a generalization of this theorem to manifolds with boundaries is given: Picken shows that there is a bijection between Deligne 2-cocycles and a certain variant of 2-dimensional topological quantum field theories.

In this thesis we show that these two theorems hold in every dimension. We consider first the holonomy case, and by using simplicial methods we can prove that the group of smooth Deligne d-classes is isomorphic to the group of smooth holonomy maps from the d^{th} thin-homotopy group $\pi_d^d(M)$ to $U(1)$, if M is $(d-1)$-connected.

We contrast this with a result of Gajer ([10]). Gajer showed that Deligne d-classes can be reconstructed by a different class of holonomy maps, which not only include holonomies along spheres, but also along general d-manifolds in M. This approach does not require the manifold M to be $(d-1)$-connected. We show that in the case of flat Deligne d-classes, our result differs from Gajers, if M is not $(d-1)$-connected, but only $(d-2)$-connected. Stiefel manifolds do have this property, and if one applies our theorem to these and compare the result with that of Gajers theorem, it is revealed that our theorem reconstructs too many Deligne classes. This means, that our reconstruction theorem cannot live without the extra assumption on the manifold M, that is our reconstruction needs less informations about the holonomy of d-manifolds in M at the price of assuming M to be $(d-1)$-connected.

We continue to show, that also the second theorem can be generalized: By introducing the concept of Picken-type topological quantum field theory in arbitrary dimensions, we can show that every Deligne d-cocycle induces such a d-dimensional field theory with two special properties, namely thin-invariance and smoothness. We show that any d-dimensional topological quantum field theory with these two properties gives rise to a Deligne d-cocycle and verify that this construction is surjective and injective, that is both groups are isomorphic.

Zusammenfassung

In der vorliegenden Arbeit verallgemeinern wir im Wesentlichen zwei Theoreme von Mackaay-Picken und Picken ([19], [24]). Im ersten Artikel zeigen Mackaay und Picken, dass es eine bijektive Korrespodenz zwischen Deligne 2-Klassen $\xi \in \check{H}^2(M, \mathcal{D}^2)$ und Holonomie Abbildungen von der zweiten dünnen Homotopiegruppe $\pi_2^2(M)$ in die abelsche Gruppe $U(1)$ gibt. Im zweiten Artikel wird eine Verallgemeinerung dieses Theorems bewiesen: Picken zeigt, dass es eine Bijektion gibt zwischen Deligne 2-Kozykeln und gewissen 2-dimensionalen topologischen Quantenfeldtheorien.

In dieser Arbeit zeigen wir, dass diese beiden Theoreme in allen Dimensionen gelten. Wir betrachten zunächst den Holonomie Fall und können mittels simplizialen Methoden nachweisen, dass die Gruppe der glatten Deligne d-Klassen isomorph ist zu der Gruppe der glatten Holonomie Abbildungen von der d-ten dünnen Homotopiegruppe $\pi_d^d(M)$ nach $U(1)$, sofern M eine $(d-1)$-zusammenhängende Mannigfaltigkeit ist.

Wir vergleichen dieses Resultat mit einem Satz von Gajer ([10]). Gajer zeigte, dass jede Deligne d-Klasse durch eine andere Klasse von Holonomie-Abbildungen rekonstruiert werden kann, die aber nicht nur Holonomien entlang von Sphären, sondern auch entlang von allgemeinen d-Mannigfaltigkeiten in M enthält. Dieser Zugang benötigt dann aber nicht, dass M hoch-zusammenhängend ist. Wir zeigen, dass im Falle von flachen Deligne d-Klassen unser Rekonstruktionstheorem sich von Gajers unterscheidet, sofern M nicht als $(d-1)$, sondern nur als $(d-2)$-zusammenhängend angenommen wird. Stiefel Mannigfaltigkeiten besitzen genau diese Eigenschaft, und wendet man unser Theorem auf diese an und vergleicht das Resultat mit dem von Gajer, so zeigt sich, dass es zuviele Deligne Klassen rekonstruiert. Dies bedeutet, dass unser Rekonstruktionsthreorem ohne die Zusatzbedingungen an die Mannigfaltigkeit M nicht auskommt, d.h. unsere Rekonstruktion benötigt zwar weniger Informationen über die Holonomie entlang von d-dimensionalen Mannigfaltigkeiten, aber dafür muss M auch $(d-1)$-zusammenhängend angenommen werden.

Wir zeigen dann, dass auch das zweite Theorem verallgemeinert werden kann: Indem wir das Konzept einer Picken topologischen Quantenfeldtheorie in beliebigen Di-

mensionen einführen, können wir nachweisen, dass jeder Deligne d-Kozykel eine solche d-dimensionale Feldtheorie mit zwei besonderen Eigenschaften, der dünnen Invarianz und der Glattheit, induziert. Wir beweisen, dass jede d-dimensionale topologische Quantenfeldtheorie nach Picken mit diesen zwei Eigenschaften auch eine Deligne d-Klasse definiert und prüfen nach, dass diese Konstruktion sowohl surjektiv als auch injektiv ist. Demzufolge sind beide Gruppen isomorph.

Contents

Introduction

It is well known that a G-principal fibre bundle with connection on a smooth manifold M induces a map called its holonomy map. Loosely speaking, this map, being regarded as a map $\mathrm{hol} : \pi_1(M) \longrightarrow G$, measures how an element in the fibre changes under parallel transport along a closed loop. Barrett showed in his thesis [2], that in some sense every such holonomy map gives rise to a G-principal fibre bundle with connection and that this correspondence is bijective. For this to be true, a holonomy has to be defined as a smooth map from the so called thin homotopy group of M to $U(1)$. This thin homotopy group, $\pi_1^1(M)$, is essentially the group of loops in M modulo reparametrization.

On the other hand, $U(1)$-principal fibre bundles have always played a major role in mathematics and especially in differential geometry. With the advent of quantum and gauge theory, one was led to believe that $U(1)$-prinicipal fibre bundles are some kind of »basic« objects in physics. String theory on the other hand developed a many-dimensional theory, and it emerged that the role of $U(1)$-principal fibre bundles has to be taken by higher dimensional objects, the so called B-fields. Mathematically these have been described much earlier by Giraud, but have been forgotten soon after. Only with String theory these objects have received reviving interest. The more abstract, algebraic-geometric description of gerbes via stacks has been replaced by many different differential-geometric descriptions. Examples of these are the gerbes in physics of Gawedzki, the book on gerbes of Brylinski, the direct approach of Hitchin and finally the n-vector bundles of Baez et al, which also includes hints to the non-abelian theory. It also occured to many people that instead of working with the geometric object one can also work with just the cocycles, the so called Deligne cocycles, as it is common for the 1-gerbes, the $U(1)$-principal fibre bundles.

This led to the question whether there is an equivalent correspondence between $U(1)$-gerbes and holonomy maps. Following the lines of Barrett, Mackaay and Picken showed the existence of a bijective correspondence between $U(1)$-gerbes with connections on the one side (actually they use the rather geometric definition of a $U(1)$-gerbe, given by Hitchin and Chatterjee [16], but the theorems can be easily rephrased in the cocycle language) and smooth holonomy maps $\pi_2^2(M) \longrightarrow U(1)$ on the other. Here $\pi_2^2(M)$ is the thin 2-homotopy group of X, analogously defined as $\pi_1^1(X)$.

In the first part of this thesis we generalize the construction by Mackaay and Picken to arbitrary degree. By choosing canonical integral curves we can use a kind of simplicial approach which simplifies the calculations. In this way we also can get rid of Barrett's Lemma ([19, 8.2]). The theorem we prove is

THEOREM: *The map* $\mathrm{HOL} : \check{H}^d(M, \mathcal{D}^d) \longrightarrow \mathrm{Hom}^\infty(\pi_d^d(M), U(1))$*, which maps every Deligne* d*-class to its smooth holonomy map is an isomorphism of groups, if* M *is* $(d-1)$*-connected.*

This has to be compared with a result of Gajer ([9]), who proves a related theorem, but which does not require the manifold M to be more than just connected, at the expense of having to consider the holonomy of all d-dimensional closed manifolds, and not just closed d-loops. To perform the comparison, we consider the case of flat Deligne classes. The above theorem reduces then to a much easier one, and with the help of Stiefel manifolds we are able to show that our theorem is not true, if M is only $(d-2)$-conntected. Therefore, our reconstruction is not as general as Gajers, but to reconstruct, one needs less information.

Holonomy only deals with closed loops and closed manifolds. If one replaces these by paths and manifolds with boundary, the concept of holonomy has to be replaced by a more refined one. Picken showed in his paper ([24]) that the right concept is that of a topological quantum field theory. His definition differs from the usual one, given by Atiyah ([1]), for it is not a pair of a vector space and an element in it, but a pair of two scalars. The main result of Pickens paper is the theorem, that there is a bijective correspondence between Deligne 2-cocycles and 2-dimensional topological quantum field theories.

We will generalize this result to arbirtary dimension in the second part of this paper, and prove the following theorem:

THEOREM: *The map* $\mathrm{PT} : \check{Z}^d(M, \mathcal{D}^d) \longrightarrow$ *TQFT*$^{d,\infty}_{Picken}(M)$*, assigning to a Deligne* d*-cocycle its* d*-dimensional, smooth, thin-invariant Picken-type topological quantum field theory is an isomorphism of groups.*

The thesis is organized as follows: We begin in Chapter 1 with recapitulating the concept of Deligne classes and their holonomy. This is mainly done to fix the notations and objects we will deal throughout this thesis. In Chapter 2, we begin with considering the 1-dimensional case, which gives us the motivation and the right insight for the generalization of the correspondence. We proceed to introduce the notion of a holonomy map and the notion of thin homotopy, which will be the key objects (and definitions) for the reconstruction in the holonomy case. We then outline the 2-dimensional case, that is Mackaay and Pickens construction. This leads us directly to the definition of the simplices we need for the reconstruction of Deligne classes. With these simplices, we are able to show that every smooth d-holonomy map gives rise to a Deligne d-class, proving the surjectivity of the reconstruction. Furthermore we can show that if the holonomy map vanishes, the reconstructed Deligne class is zero, providing the injectivity of the construction. As already noted, our reconstruction differs from the one by Gajer, and to show this, we continue to consider the special case of flat Deligne classes. This gives us a reconstruction theorem, that is easier to handle. Assuming, that our theorem is also true, if M is not highly connected, we show that it produces a contradiction to the results of Gajer. This we do by introducing manifolds, which have the property that their holonomy and homotopy differ. The easiest examples of such are spheres, and indeed even these show the contradiction for all cases $d > 2$. For $d = 2$ we cannot use spheres and introduce the broader class of Stiefel manifolds, which provide counterexamples for all even dimensions. This completes the analysis of the reconstruction of holonomy maps.

We proceed to the concept of parallel transport and Deligne cocycles in Chapter 3. Again we start by considering the case $d = 1$, which already captures a large amount of the concept behind parallel transport. After defining the central objects, which are now triangulated manifolds, we outline the general definition of a Picken-type topological quantum field theory. We show that every Deligne cocycle induces such a field theory, and that it has special properties, namely thin-invariance and smoothness. With these remarks in mind, we can start the reconstruction, with is basically equivalent to the one for the holonomy, so we first define the Deligne cocycle corresponding to a thin-invariant, smooth Picken-type topological quantum field theory which shows the surjectivity of the reconstruction. After that we establish the injectivity. This proves our main theorem.

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General assumptions

Throughout the whole text we will only work with oriented and smooth manifolds. These may or may not have a boundary. Let us collect here the more important notations. The base manifold will always be called M, and will (during the reconstruction) be an N-dimensional smooth, oriented manifold and connected. It will always be equipped with a choice of a good open cover $\mathcal{U} = \{U_i\}_{i \in I}$. Also the index-set I will be ordered, and whenever we refer to a string of indices $(i_1, i_2, \ldots, i_n)$ we will assume that these indices are ordered.

Deligne classes will often be denoted by greek letters, namely ξ, ω and η. With ξ^p, ω^p etc. we mean the p-form of ξ. Also as above, we will always use *smooth* Deligne classes..

Whenever we call a manifold S, it is usually a $(d-1)$-dimensional, closed, oriented smooth manifold in M, while Σ will denote a (d)-dimensional, oriented smooth manifold in M with boundary $\partial\Sigma$. Sometimes we will assume that $\partial\Sigma = S$. The maps of S and Σinto M will usually be called φ_S and φ_Σ respectively.

We will deal with a lot of simplices, and will built chains out of these. These we will *still* call simplex, even if it is not only a chain, but a loop. This should not lead to any confusion, since it should be clear from the context, what object we actually consider.

Chapter 1

Deligne cohomology and Holonomy

The central objects of this thesis are Deligne cocycles and classes. These have been introduced 1972 by Deligne in the algebraic-geometric context (see [7]) and have subsequently been used widely in algebraic geometry as well as in other branches of geometry and topology. For the sake of completeness, in this chapter we remind the reader of the definition of Deligne classes and their holonomy. We will not work with the more advanced definitions of Deligne cocycles and classes, often used in the algebraic-geometric context, which make use of concepts like sheaves and categories. Our approach here is a more computational approach [6], which can be regarded as some kind of watered-down version of these more complicated definitions. For our purposes, the reconstruction theorems we will present later on, this approach is the most convenient. This chapter, partly being a reminder and partly fixing the notations, is far from being a full treatise of the subject. Neither it does provide full proofs, except in some more or less pedagogical cases. Introductions into the subject, including proofs, can be found in the standard literature (e.g. [4], [6]).

1.1 Deligne cohomology

In the following let M be a smooth oriented, d-dimensional manifold, on which we eventually will define a Deligne class. For this we need a **good cover**, i.e. an open cover $\mathcal{U} = \{U_i\}_{i\in I}$, with I some ordered index set, where all intersections $U_{i_1,\dots,i_j} := U_{i_1} \cap \dots \cap U_{i_j}$ are contractible. Particularly each set U_i is so. Fix for now a good cover $\mathcal{U}$ on M.

We introduce now the **Čech-deRham double-complex** $C^{p,r}(\mathcal{U})$ with differentials d and δ. This is the double complex $C^{p,r}(\mathcal{U})$ consisting of p-forms defined on the disjoint union of all $(r+1)$-intersections with values in $U(1)$ and $i\mathbb{R}$ for $p = 0$ and $p > 0$

respectively. That is we have

$$C^{p,r}(\mathcal{U}) := \Omega^p(\coprod_{i_1,\dots,i_{r+1}} U_{i_1,\dots,i_{r+1}}, i\mathbb{R})$$

for $p > 0$ and $C^{0,r}(\mathcal{U}) := \Omega^0(\coprod_{i_1,\dots,i_{r+1}} U_{i_1,\dots,i_{r+1}}, U(1))$. Note the kind of index-shift present in the double-complex. Because of this there are no global forms, but only local ones.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\Omega^0(\coprod_{i,j,k} U_{ijk}, U(1)) & \xrightarrow{d\log} & \Omega^1(\coprod_{i,j,k} U_{ijk}, i\mathbb{R}) & \xrightarrow{d} & \Omega^2(\coprod_{i,j,k} U_{ijk}, i\mathbb{R}) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\Omega^0(\coprod_{i,j} U_{ij}, U(1)) & \xrightarrow{d\log} & \Omega^1(\coprod_{i,j} U_{ij}, i\mathbb{R}) & \xrightarrow{d} & \Omega^2(\coprod_{i,j} U_{ij}, i\mathbb{R}) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\Omega^0(\coprod_{i} U_{i}, U(1)) & \xrightarrow{d\log} & \Omega^1(\coprod_{i} U_{i}, i\mathbb{R}) & \xrightarrow{d} & \Omega^2(\coprod_{i} U_{i}, i\mathbb{R}) & \xrightarrow{d} & \cdots
\end{array}$$

Figure 1.1: The Čech-deRham complex

We have two differentials, the deRham differential d and the Čech differential δ, where $d : C^{p,r} \longrightarrow C^{p+1,r}$ is given by taking the exterior derivative and $\delta : C^{p,r} \longrightarrow C^{p,r+1}$ is given by restricting a p-form to intersections $U_{i_1,\dots,i_{r+1}}$ and summing up alternatively, i.e. if $c \in C^{p,r} = \Omega^p(\coprod_{i_1,\dots,i_{r+1}} U_{i_1,\dots,i_{r+1}}, U(1))$, then $\delta c \in C^{p,r+1}$ is given by

$$(\delta c)_{i_1,\dots,i_{r+2}} = \sum_{j=1}^{r+1} (-1)^{j-1} c_{i_1,\dots,\hat{i}_j,\dots,i_{r+1}}. \tag{1.1.1}$$

One can show, that δ is a differential, though it is a little arkward to do so.

The Deligne cohomology is build from the Čech-deRham complex by truncating it. For this, fix some d and set $C^{p,r}(\mathcal{U}) = 0$ whenever $p > d$. To distinguish it notationally from the untruncated one, we will call this double complex $C^{p,r}(\mathcal{U}, \mathcal{D}^d)$. In the following we will only work with this double complex.

To $C^{p,r}(\mathcal{U}, \mathcal{D}^d)$ we may consider the associated total complex

$$\check{C}^i(\mathcal{U}, \mathcal{D}^d) := \bigoplus_{i=p+r} C^{p,r}(\mathcal{U}, \mathcal{D}^d).$$

Any $c \in \check{C}^d(\mathcal{U}, \mathcal{D}^d)$ is a $(d+1)$-tupel $(c^0_{i_1,\dots,i_{d+1}}, c^1_{i_1,\dots,i_d}, \dots, c^d_{i_1})$, which we call **Deligne d-cochain**. Note again that though e.g. $c^0 \in C^{0,d}(\mathcal{U}, \mathcal{D}^d)$ has degree $(0, d)$, it has $(d+1)$ indices.

By the description above, a Deligne cochain is simply a family of locally defined p-forms and one $U(1)$-valued function. For example, in degree $d = 1$ a Deligne 1-cochain is a tupel (c^0_{ij}, c^1_i), with c^0_{ij} a $U(1)$-valued function, defined on $U_i \cap U_j$, and c^1_i an $i\mathbb{R}$-valued 1-form on each U_i. It is obvious, that these data resemble the data of a principal $U(1)$-bundle with connection. Only the cocycle conditions are missing. These conditions are encoded by a differential on the double complex: We define the differential D on $C^{p,r}(\mathcal{U}, \mathcal{D}^d)$ to be

$$D := \delta + (-1)^{r-1} d,$$

where r refers to the intersection degree of the component of the Deligne cochain. This is given by $r = d + 1 - p$ in case of the p-form c^p, e.g. c^0 has intersection degree $r = d + 1 - 0 = d + 1$, c^1 has $r = d + 1 - 1 = d$, while c^d has $r = d + 1 - d = 1$. Explicitely we have:

$$D(c^0_{i_1,\dots,i_{d+1}}, c^1_{i_1,\dots,i_d}, \dots, c^d_{i_1}) = (\delta c^0, \delta c^1 + (-1)^d d \log c^0, \dots, \delta c^d + (-1)^1 d c^{d-1}),$$

where we dropped the intersection indices for cleairity. Every element c of the total complex $\check{C}^d(\mathcal{U}, \mathcal{D}^d)$ with $Dc = 0$ will be called a **Deligne d-cocycle**. The condition for a cochain c to be a cocycle is then

$$\delta c^p_{i_1,\dots,i_r} + (-1)^r d \log c^{p-1}_{i_1,\dots,i_{r+1}} = 0$$

at »position« p. Note that the intersection index r is the one for c^p, so we should have written $c^{p-1}_{i_1,\dots,i_{r(p)+1}}$ instead. We will continue to use this notation, that is r always depends on p and not on $p+1$ or $p-1$.

Before we continue, let us write down what D actually boils down to in low degrees. The Deligne total complex starts like

$$\check{C}^0(\mathcal{U}, \mathcal{D}^d) \xrightarrow{\quad D \quad} \check{C}^1(\mathcal{U}, \mathcal{D}^d) \xrightarrow{\quad D \quad} \check{C}^2(\mathcal{U}, \mathcal{D}^d) \longrightarrow \cdots$$

with $D^0(c^0) = (\delta c^0, d \log c^0)$ and $D^1(c^0, c^1) = (\delta c^0, \delta c^1 - d \log c^0, dc^1)$. and $D^2(c^0, c^1, c^2) = (\delta c^0, \delta c^1 + d \log c^0, \delta c^2 - dc^1, dc^2)$. Coming back to the example of a 1-Deligne cocycle, where we truncate at $d = 1$, we see that additionally to the data of a principal $U(1)$-fibre bundle with connection, i.e. (c^0_{ij}, c^1_i), we have also the conditions $\delta c^0_{ij} = 1$ and $\delta c^1_i = d \log c^0_{ij}$. (As said, there is no condition $dc^1_i = 1$, which would correspond to flat fibre bundles.) These are exactly the cocycle conditions, so we can identify the set of 1-Deligne cocycles with cocycle data of $U(1)$-principal fibre bundles (with connection).

A very simple observation is, that all cocycles form a group, just by adding two cocycles componentwise, and that the trivial cocycle $(1, 0, \ldots, 0)$ is the neutral element. Now let us show that D is indeed a differential, that is $D^2 = 0$:

1.1.1 LEMMA: *Let $\check{C}^i(\mathcal{U}, \mathcal{D}^q)$ be associated the total complex as above. Then $D = \delta + (-1)^{r-1}d$ is a differential on $\check{C}^i(\mathcal{U}, \mathcal{D}^q)$.*

Proof. We only need to compute

$$D(D(c^0, c^1, \ldots, c^d)) = D(\delta c^0, \delta c^1 + (-1)^d d \log c^0, \ldots, \delta c^d - dc^{d-1}, dc^d).$$

Now $\delta c^0 \in C^{0,d+1}(\mathcal{U}, \mathcal{D}^q), \delta c^1 + (-1)^d d \log c^0 \in C^{1,d}(\mathcal{U}, \mathcal{D}^q)$ etc., so $D^2(c) = (\delta\delta c^0, \delta(\delta c^1 + (-1)^d d \log c^0) + (-1)^{d+1} d \log(\delta c^0), \ldots, \delta(dc^d) - d(\delta c^d - dc^{d-1}), ddc^d)$. Using the properties of d and δ, namely $d^2 = 0$, $\delta^2 = 0$ and $d\delta = \delta d$ we see directly $D^2 = 0$. □

1.1.2 DEFINITION: The homology $\check{H}^d(\mathcal{U}, \mathcal{D}^d)$ of this complex is called the **Deligne cohomology** of M with regard to the cover $\mathcal{U}$. Any homology class of $\check{H}^d(\mathcal{U}, \mathcal{D}^d)$ will be called a **Deligne class** for the (fixed) cover $\mathcal{U}$.

Let us come back to the case $d = 1$. As we have seen, a Deligne 1-cochain c with respect to some cover $\mathcal{U}$ boils down to the data of a $U(1)$-principal fibre bundle with connection. Roughly, we have seen the equivalence of $U(1)$-bundles with connection and Deligne 1-classes.

Coming back to the general case, any Deligne cochain can be written as $\xi = (\omega^0, \omega^1, \ldots, \omega^d)$. For ξ to represent a homology class we need $D\xi = 0$, which means $\delta\omega^0 = 1$, $d \log \omega^0 = \pm\delta\omega^1$, …, $d\omega^{q-1} = \pm\delta\omega^q$, and if η is another family of p-forms such that η and ξ do only differ by a coboundary, that there is a $(d-1)$-cochain c with $\omega^0 - \eta^0 = \delta c^0$, $\omega^1 - \eta^2 = \pm d \log c^0 + \delta c^1$, ..., $\omega^d - \eta^d = dc^d$, then ξ and η represent the same Deligne class.

Up to now we have not discussed the dependence of the whole construction upon the cover $\mathcal{U}$. We will not need this, for we will always choose a good cover $\mathcal{U}$, and work with this, but let us comment on the choice of the cover.

By introducing refinements of the cover one gets corresponding maps between the Deligne cohomologies. Taking the inverse limit over all coverings of M one can define the Deligne cohomology of M itself. Though this is the right definition for us, it is much more convenient to define a Deligne class to be a family like above for some good cover. This poses no problem, because one can show that every Deligne class for a good cover $\mathcal{U}$ defines uniquely a Deligne class for M.

1.2 Holonomy of Deligne classes

In this section we will present the central definition for the next chapter, the holonomy of a Deligne class. In order to give the proper definition, analogously to the fibre bundle case, we have to say, when a Deligne class is trivial. Usually, a fibre bundle is trivial, if its associated Chern class vanishes. This is true also for higher Deligne classes, and thus we start with the definition of the Chern class. Having done so, we can immediately define the holonomy of a submanifold S in M. Finally we will show that this value can also be computed locally by choosing a triangulation of S, giving rise to a neat formula. This will be a starting point for the reconstruction of Deligne cocycles via their parallel transport in chapter 3.

Before we discuss these topics, let us review the following isomorphism between Čech-cohomology and deRham cohomology (a reference for a proof is [3]).

1.2.1 PROPOSITION: *For any manifold M there is an isomorphism of the form*

$$\check{H}^d(M, U(1)) \cong H^{d+1}(M, \mathbb{Z}),$$

where the Čech-cohomology has the sheaf $U(1)$ as coefficients.

Utilizing this isomorphism, we may define the Chern class, and also the curvature of a Deligne class ω:

1.2.2 DEFINITION: Let $\omega = [(\omega^0, \ldots, \omega^d)]$ be some Deligne d-class. The **Chern class** $c(\omega)$ of ω is

$$c(\omega) = [\omega^0] \in \check{H}^d(M, U(1)) \cong H^{d+1}(M, \mathbb{Z}),$$

that is the Chern class is defined to be the image of $[\omega^0]$ under the above isomorphism. The **curvature** of ω is defined to be $d\omega^d$, which is a closed, global $(d+1)$-form.

Both concepts are well-defined, for if we have two representatives $\omega = [(\omega^0, \ldots, \omega^d)] = [(\eta^0, \ldots, \eta^d)]$, then we know that they differ by $\omega^0 - \eta^0 = \delta\xi^0$ with $\xi^0 \in C^{0,d-1}(\mathcal{U}, \mathcal{D}^d)$ and hence we have $[\omega^0] = [\eta^0 + \delta\xi^0] = [\eta^0] = c(\eta)$. This shows that the Chern class is well-defined. On the other hand also the curvature is well-defined for again we have $\omega^d = \eta^d + d\xi^{d-1}$, and hence $d\omega^d = d\eta^d$. It is obvious that the curvature of any ω is a closed d-form. It is also a global form on M, for $\delta d\omega^d = d\delta\omega^d = \pm dd\omega^{d-1} = 0$.

1.2.3 DEFINITION: We call a Deligne d-class ω **trivial**, iff $c(\omega)$ vanishes.

Especially, if $H^{d+1}(M, \mathbb{Z}) = 0$, then any Deligne class on M is trivial. This is the main feature of Deligne classes, which we will exploit to introduce the concept of holonomy.

Notice, that this definition of triviality coincides with the one for principal bundles: Any such is trivial, iff the cohomology class of its transition functions vanishes.

We have the following proposition, which can be easily shown (see e.g. section 3.2 of [6] for a discussion):

1.2.4 PROPOSITION: *There is an exact sequence*

$$0 \longrightarrow \Omega^d(M)_{c,0} \longrightarrow \Omega^d(M) \xrightarrow{i} \check{H}^d(M, \mathcal{D}^d) \xrightarrow{c} H^{d+1}(M, \mathbb{Z}) \longrightarrow 0$$

Here $\Omega^d(M)_{c,0}$ denotes the group of closed d-forms with $2\pi i\mathbb{Z}$ periods, and the maps above are defined as follows: The first map is the usual inclusion, while the second one, $i : \Omega^d(M) \longrightarrow \check{H}^d(M, \mathcal{D}^d)$, maps a d-form ρ to $[(1, 0, \ldots, \delta(\rho))]$. Finally c associates to any Deligne d-class its Chern class.

1.2.5 COROLLARY: *Any trivial Deligne d-class ω can be represented by a family of the form $(1, 0, \ldots, 0, \delta(\rho))$, where ρ is a global d-form.*

The central definition of this section is the holonomy of a Deligne class along a submanifold S in M:

1.2.6 DEFINITION: Let $\omega \in \check{H}^d(\mathcal{U}, \mathcal{D}^d)$ be a Deligne class, and let $\varphi_S : S^d \longrightarrow M$ be a map from a closed d-dimensional manifold $S = S^d$ into M. The **holonomy** $\mathrm{hol}_\omega(S) = \mathrm{hol}_\omega(\varphi_S : S \longrightarrow M)$ of ω along S (more precisely along φ_S) is defined to be

$$\mathrm{hol}_\omega(S) := \exp(\int_S \rho) \in U(1).$$

Here ρ is a global d-form on S, defined in corollary 1.2.5, corresponding to the pulled back Deligne class $\varphi_S^*(\omega)$. Note that indeed $H^{d+1}(S) = 0$, since S is only d-dimensional, and therefore every Deligne d-class on S is by definition trivial, so that we can apply the corollary to choose such a trivialization form ρ.

Let us verify that this definition does not depend on the trivialization form ρ. First observe that the zero class has trivial holonomy along every submanifold S. Second, the holonomy is additive:

1.2.7 LEMMA: *The holonomy is a group-morphism from the Deligne d-classes to $U(1)$, i.e. we have for any d-manifold S in M:*

$$\mathrm{hol}_{\xi+\eta}(S) = \mathrm{hol}_\xi(S) \cdot \mathrm{hol}_\eta(S).$$

Proof. One sees this either directly from the local holonomy formula below, or by the following argument: Since ${\varphi_S}^*(\xi+\eta) = {\varphi_S}^*(\xi)+{\varphi_S}^*(\eta)$, with ${\varphi_S}^*(\xi) = [(1, 0, \dots, 0, \rho)]$ and ${\varphi_S}^*(\eta) = [(1, 0, \dots, 0, \kappa)]$ we have $\int_S {\varphi_S}^*(\xi + \eta) = \int_S \rho + \int_S \kappa$. □

Now if $\varphi_S^*(\omega) = [(1, 0, \dots, 0, \rho)] = [(1, 0, \dots, 0, \eta)]$, then we know that $\rho - \eta \in \Omega^d(M)_{c,0}$ by proposition 1.2.4, that is we know

$$\exp \int_S \rho - \exp \int_S \eta = \exp \int_S (\rho - \eta) = 1,$$

since $\rho - \eta$ integrated over a closed manifold S is $2\pi i\mathbb{Z}$-valued. This proves that the definition does not depend on the representative of ω.

One could also deduce this result by the local formula below, if one takes it as the primary definition of the holonomy, for if we have $\xi = \eta + Dc$, then Dc does not contribute to the local formula.

The reasoning above more or less shows, that the map, assigning to a Deligne d-class ω its holonomy hol_ω, is a group morphism with respect to a natural group structure on a certain set of »holonomy maps«to be defined in the next chapter. This map is the main object of the next chapter, where we will show that (under certain assumptions) it is bijective. On the other hand we know $\mathrm{hol}_\omega(\mathrm{pt}) = 1$ for any Deligne d-class ω. We also have some kind of inverse in this slot, for if $\bar{S}$ is S with the orientation reversed, then by the properties of the integral, one can immediately see that the holonomy behaves good:

$$\mathrm{hol}_\omega(S) = -\,\mathrm{hol}_\omega(\bar{S}).$$

Furthermore we have for any two d-dimensional manifolds $S, \tilde{S}$ in M

$$\mathrm{hol}_\omega(S \amalg \tilde{S}) = \mathrm{hol}_\omega(S)\,\mathrm{hol}_\omega(\tilde{S}).$$

Finally let us write down a local formula for the holonomy: Assume, that the closed d-manifold S in M is equipped with a triangulation T_S, that is S is triangulated into faces k^d, subfaces k^{d-1} etc. and a **labeling** $\ell : T_S \longrightarrow I$ is chosen, such that each d-face $f_d \in k^d$ lies within a certain open set $U_{\ell(f_d)}$ of the open cover, each $(d-1)$-subface $f_{d-1} \in k^{d-1}$ lies within some $U_{\ell(f_{d-1})}$, etc. Then the holonomy can be computed as follows:

1.2.8 PROPOSITION: *The holonomy of a closed, triangulated d-manifold S in M with respect to a Deligne d-class ω, an open cover $\mathcal{U}$ of M and a labeling ℓ is given by*

$$\mathrm{hol}_\omega(S) = \exp\left(\sum_{f_d \in k^d} \int_{f_d} \varphi_S^*(\omega^d_{\ell(f_d)}) + \sum_{f_{d-1}\subset f_d} \int_{f_{d-1}} \varphi_S^*(\omega^{d-1}_{\ell(f_d)\ell(f_{d-1})}) + \cdots\right) \cdot \prod_{f_0 \subset f_1 \subset \cdots \subset f_d} \varphi_S^*(\omega^0_{\ell(f_d)\cdots\ell(f_0)})(f_0).$$

In the next section we will often drop the cumbersome notation $\varphi_S^*(\omega^d_{\ell(f_d)})$ and write just $\omega^d_{\ell(f_d)}$, that is we pretend that S is a submanifold of M. This should lead to no confusion.

Chapter 2

Holonomy

In the last chapter we have seen how to assign to any Deligne d-class ξ its holonomy, hol_ξ, which maps to any d-dimensional closed manifold S in M a number in $U(1)$. Considering principal $U(1)$-fibre bundles, that is looking at Deligne 1-classes, it is a well-known fact that isomorphism classes of *flat* bundles are the same as holonomy maps $\pi_1 M \longrightarrow U(1)$ (see theorem 2.1.1). This motivates the search for similar results for higher Deligne classes.

The goal of this chapter is to generalize this result not only to flat but also to arbitrary Deligne d-classes.

We start by analyzing the above isomorphism for fibre bundles. This gives us a good hint on how to proceed in the general case. It turns out that in order to have such an isomorphism, we have to replace the homotopy groups by a finer topological invariant, the thin homotopy groups. We introduce this concept in section 2.2 and use it to find the proper generalization of the homomorphisms from $\pi_1(M)$ to $U(1)$. These objects we will call holonomy maps.

After this, we outline the basic notations and define the recentering homotopies we will need for the reconstruction.

Having introduced all these notations, we are in the position to start the reconstruction of Deligne classes via their holonomy. This reconstruction is done in three steps: First, from a given holonomy map, a family of p-forms is reconstructed (section 2.5.2). This family is then shown to obey the cocycle relations, which shows the surjectivity of the map, that assigns to any Deligne class its holonomy (section 2.5.3). In the third step we show that any Deligne class with trivial holonomy map must be trivial itself, establishing injectivity. During this reconstruction, we basically follow the lines of [19].

Finally we come back to the flat case in section 2.6, and show that every flat Deligne d-class induces a special kind of holonomy map, namely a map of the form $\pi_d(M) \longrightarrow$

$U(1)$. This result is then the main ingredient to compare our reconstruction with that of Gajer (section 2.7). While we can do our reconstruction with d-loops alone, Gajer has to consider the holonomy along all closed d-manifolds in M. But our result is wrong, if the manifold M is not highly connected: Looking only at d-loops alone is not enough, if the first d homotopy groups of M do not vanish. We show this by considering Stiefel manifolds, which are not d-connected, and show that Gajers and our theorem give different results, thus proving that our initial assumption is indeed necessary for the reconstruction.

2.1 Motivation

The following theorem makes the reconstruction theorem for $U(1)$-bundles (or Deligne 1-classes) precise, and is well-known (see e.g. theorem 6.60 in [20]):

2.1.1 THEOREM: *For any smooth manifold M the group of isomorphism classes of flat $U(1)$-bundles and maps $\rho : \pi_1 M \longrightarrow U(1)$ are isomorphic via the map assigning to any flat $U(1)$-bundle its holonomy.*

Isomorphism classes of principal fibre bundles are (nearly) the easiest case of Deligne classes. Given this, a natural question is: How can we generalize the theorem to higher Deligne classes? Can we drop the flatness assumption and still obtain a reasonable isomorphism? If so, how do these holonomy maps look like?

These questions will be answered in the following sections, but let us first look at $U(1)$-bundles, that is the case $d = 1$. Suppose we are given some 1-holonomy map, assigning to any loop in M a number, and want to reconstruct an $U(1)$-bundle. As always, to construct such a bundle, one glues trivial line bundles via transition functions. To do so, choose a good open cover $\mathcal{U}$ of M. Consider the local, trivial bundles $U_i \times U(1)$, together with charts $U_i \longrightarrow D^N \subset \mathbb{R}^N$. We need to identify $V_i := U_i \times U(1)$ and $V_j := U_j \times U(1)$ by some transition function g_{ij}, i.e. we need to tell how the unit of the fibre $\{y\} \times U(1) \subset V_i$ is identified with unit of the corresponding fibre of V_j. Since knowing the holonomy means that we know how to parallel transport elements along a closed loop. The simplest idea is therefore to choose two canonical paths from y to the basepoint »via« U_i and U_j respectively, and to parallel transport one of the unit elements e via the loop this path defines. In other words, we transport $e \in \{y\} \times U(1) \subset V_i$ to x_i, the »center« of U_i, along a straight (radial) path (since by assumption U_i is contractible, this can be done), then along a chosen path p_{x_i} into the basepoint, further along $p_{x_j}^{-1}$ into x_j and finally along another straight path back to y.

Denoting this loop $y \longrightarrow x_i \longrightarrow \star \longrightarrow x_j \longrightarrow y$ by s_{ij}^y, we obtain a transition function by letting

$$g_{ij}(y) := H(s_{ij}^y).$$

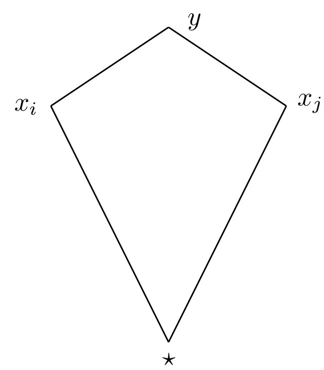


Figure 2.1: Reconstrucion of the transition functions g_{ij}

Note, that this loop s_{ij}^y can be understood as closing the open simplex $(-\langle y, x_i \rangle) \star \langle y, x_j \rangle$.

In order to reconstruct the connection A_i from the holonomy, we have to define a 1-loop from a given point $y \in U_i$ and a given tangential vector v in y. As above, we denote this loop by $\tilde{s}_i^{y,v}$. The main idea is again to start with a vector in the basepoint of the manifold, transport it along fixed paths to y, follow then along the integral curve of the given tangent vector and transport the element back to the basepoint, where we can measure how the element has changed. Differentiating this gives us the desired 1-form:

$$(A_i)_y(v) := \frac{d}{dt} \log H(\tilde{s}_i^{y,v(t)})|_{t=0}.$$

By definition (see definition 2.5.6) this will not depend on the choice of the integral curve, therefore we may as well choose the straight line from y to $v(t)$, for we may pretend to be in $\mathbb{R}^N$. That is, in the special case $d = 1$ the whole construction amounts to the boundary of the simplex with the endpoints x_i, y and $v(t)$ (and a recentering which we drop here, see Figure 2.2).

2.2 Thin Homotopy and Holonomy maps

For the reconstruction of higher Deligne classes we must know, what kind of object will correspond to a Deligne class. We have seen, that the holonomy of a Deligne class is a map that assigns to any closed d-dimensional manifold in M a number, so it is reasonable to call any such map a holonomy map. As we already noted, we will restrict ourselves to d-loops, that is we only look at maps $S^d \longrightarrow M$. Furthermore, since we want our constructions all to be smooth, we consider only so-called sitting d-loops. So a holonomy

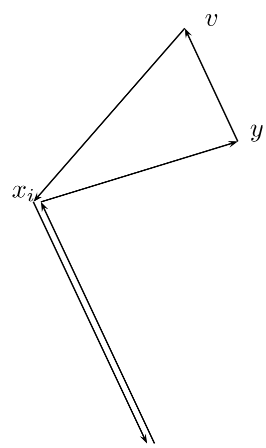


Figure 2.2: Reconstrucion of the 1-form A_i

map might be a map from the group of sitting d-loops to $U(1)$. But actually we still can do better, by the following observation: Two loops in M, that just differ by reparametrization, have also the same holonomy (with respect to a Deligne 1-class). The question is now: Can we divide out a certain subgroup that corresponds to reparametrization? Indeed, this is possible. Vaguely we say that two d-loops are thin homotopic if they are homotopic, but the homotopy does not sweep out any volume. Dividing out the group of d-loops by this different notion of homotopy, one obtains the thin homotopy group. An element of the thin homotopy group can be thought of as a d-loop up to parametrization.

Altogether we have sketched, what a holonomy map will be: A smooth group morphism from the group of (smooth, sitting) thin invariant loops to $U(1)$. Let us now give the proper definitions.

2.2.1 DEFINITION: A (smooth, based) **d-loop** in M is smooth map $\gamma : [0,1]^d = I^d \longrightarrow M$ with $\gamma(\partial[0,1]^d) = \star$. We call it **sitting**, if $\gamma(t_1, \ldots, t_d) = \star$ whenever one of the t_i is smaller than some ϵ (with $\epsilon > 0$) or bigger than $1 - \epsilon$. Let $\Omega_d^\infty(M)$ be the set of sitting smooth d-loops in M.

2.2.2 REMARK: Actually this definition is somewhat misleading. Initially we want to consider *closed* manifolds in M, but the unit cube I^d certainly not closed. But by identifying the boundary, as it is required by the conditions of a sitting loop, it is a closed manifold in M. It would have been better to work with S^d throughout, but the unit cube is easier to handle, which is why we stick with it.

By dropping the condition that a sitting d-loop is mapped to $\star$ and just requiring that $\gamma(t_1, \ldots, t_d)$ is constant for $t_i < \epsilon$ and $t_i > 1-\epsilon$ we get **d-paths**.

Since we will only work with sitting (smooth) d-loops, we often call these just d-loops. Technically we need this definition to avoid the (slightly) more arkward concept of piecewise smooth maps. It was Barrett who first came up with this definition [2].

On $\Omega_d^\infty(M)$ we do have a group-structure coming from composition along the first coordinate, just as we have it for usual loops.

Next we make precise the idea of 'up to reparametrization':

2.2.3 DEFINITION: Two d-loops γ and γ' are **thin homotopic** to each other, $\gamma \sim^d \gamma'$, iff there is a smooth homotopy between γ and γ' that has everywhere deficient rank and maps to the basepoint along an ϵ-neighborhood of the sides of the cube (roughly we say, that each slice in time direction is a sitting loop). We use $\pi_d^d(M)$ to denote the thin homotopy classes of d-loops in M.

With the composition in $\Omega_d^\infty(M)$ also $\pi_d^d(M)$ becomes a group, which is abelian for $d > 1$. We have the projection $\mathrm{pr} : \Omega_d^\infty \longrightarrow \pi_d^d(M)$, that maps any d-loop to its thin homotopy classes. Since thin homotopy is much finer than normal homotopy, we have a second projection map, which we also denote by pr:

2.2.4 DEFINITION: Let $\mathrm{pr} : \pi_d^d(M) \longrightarrow \pi_d(M)$ the projection, which maps a d-dimensional thin homotopy class to its homotopy class.

The next proposition makes clear, why thin homotopy fits into the concept of holonomy of Deligne classes.

2.2.5 PROPOSITION: *If two d-loops γ and γ' are thin homotopic, then their holonomy is the same, i.e.*

$$\mathrm{hol}_\omega(\gamma) = \mathrm{hol}_\omega(\gamma')$$

for any Deligne d-class ω.

Proof. By definition there is some smooth homotopy $W : I^{d+1} \longrightarrow M$ having everywhere deficient rank. Therefore we get

$$1 = \exp\Big(\int_{I^{d+1}} W^* \operatorname{curv}(\xi)\Big) = \exp\Big(\int_{I^{d+1}} d\hat{\rho}\Big) = \exp\Big(\int_{I^d} \rho - \int_{I^d} \tilde{\rho}\Big)$$

by virtue of Stokes Theorem and the fact, that W restricts to γ and γ' respectively. Furthermore, we have used $W^*(\operatorname{curv}(\omega)) = \operatorname{curv}(W^*(\omega)) = \operatorname{curv}(i(\hat{\rho})) = d\hat{\rho}$ in the second equality, and denoted the trivialization of $\gamma^*(\omega)$ by ρ and of $\gamma'^*(\omega)$ by $\tilde{\rho}$. Note that the sides of the cube I^{d+1} are all mapped to $\star$ so that they do not contribute. □

Especially we see that the holonomy of a Deligne d-class factorizes over $\pi_d^d(M)$:

$$\begin{array}{ccc} \mathrm{hol}_\xi : \Omega_d^\infty(M) & \longrightarrow & U(1) \\ & \searrow & \uparrow \\ & & \pi_d^d(M). \end{array}$$

Maps of this type we will call smooth holonomy maps. The precise meaning of smooth is the following one:

2.2.6 DEFINITION: A **family of d-loops** is a map $\psi : U \longrightarrow \Omega_d^\infty(M)$ with U an open subset of $\mathbb{R}^r$. We call ψ **smooth** iff the induced map $\tilde{\psi} : U \times I^d \longrightarrow M$ with $\tilde{\psi}(x, t_1, \ldots, t_d) := \psi(x)(t_1, \ldots, t_d)$ is smooth.

This definition transfers the problem of saying what smooth is on $\Omega_d^\infty(M)$ back to smoothness of maps between finite dimensional spaces. It enables us to define holonomy maps, which will always be smooth for us:

2.2.7 DEFINITION: A **d-holonomy** (or holonomy map) H is a group morphism

$$H : \pi_d^d(M) \longrightarrow U(1)$$

such that for every smooth family $\psi : U \longrightarrow \Omega_d^\infty(M)$ of d-loops the composition

$$U \xrightarrow{\psi} \Omega_d^\infty(M) \xrightarrow{\mathrm{pr}} \pi_d^d(M) \xrightarrow{H} U(1)$$

is smooth. Here the second map is the projection pr of a d-loop to its thin homotopy class. Let $\mathrm{Hom}^\infty(\pi_d^d(M), U(1))$ be the group of d-holonomies over M, where the group structure is induced by the group structure of $U(1)$.

The upper construction provides us with a map:

2.2.8 COROLLARY: *There is a map*

$$\mathrm{HOL} : \check{H}^d(M, \mathcal{D}^d) \longrightarrow \mathrm{Hom}^\infty(\pi_d^d(M), U(1)),$$

mapping any $\xi \in \check{H}^d(M, \mathcal{D}^d)$ to its holonomy hol_ξ.

The main issue of this chapter is to show that this map is an isomorphism. A basic observation is that HOL is a group-morphism. This follows at once from the fact that the holonomy itself is a group-morphism (see 1.2.7):

2.2.9 Corollary: HOL *is a group morphism.*

To reach our goal we have to show that HOL is surjective and injective, which we will do in the next section. Before we delve into this, let us remark that hol_ξ is smooth, because it is basically given by integration over smooth forms. Hence in the following we will only consider smooth maps $\mathrm{Hom}^\infty(\pi_d^d(M), U(1))$. Any such smooth map we will call a d-holonomy, as defined below.

2.3 Motivation for the case $d = 2$

To make the mechanism behind the reconstruction clearer, we analyze the construction of Mackaay-Picken, so we deal with Deligne 2-classes. Here, to reconstruct the transition functions g_{ijk} one builds an open 2-path $\tilde{s}^y_{ijk}$, as seen in Figure 2.3, and close it using canonical paths. These are chosen homotopies from the »sides«of the simplex to the basepoint, and will be defined below (but for clearity we drop these in this motivation). The path itself is given by a product of simplices of the form $\langle y, \hat{x}_{ijk}, x_{jk}, x_k\rangle$, where the hat means that the vertex has to be dropped. Let us write down, how these simplices look like:

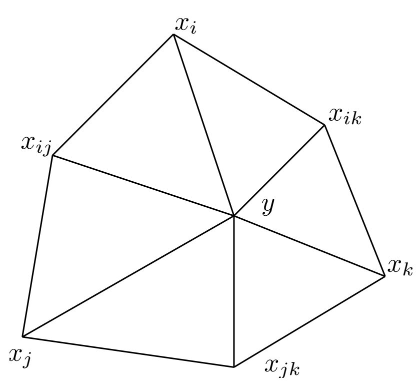


Figure 2.3: Reconstruction of the 0-form g_{ijk}

We have 6 simplices, each one containing the midpoint y, one of the midpoints x_i and one of the »mid-midpoints« x_{ij}: $\langle y, x_{ij}, x_i\rangle$, $\langle y, x_{ij}, x_j\rangle$, $\langle y, x_{ik}, x_i\rangle$, $\langle y, x_{ik}, x_k\rangle$, $\langle y, x_{jk}, x_j\rangle$, $\langle y, x_{jk}, x_k\rangle$.

Analogously one obtains the corresponding loop $\tilde{s}^{y,v(t)}_{ij}$ (with fixed $t \in \mathbb{R}$) for the 1-connection A_{ij} by taking the boundary of the two simplices $\langle y, v(t), x_{ij}, x_j\rangle$ and

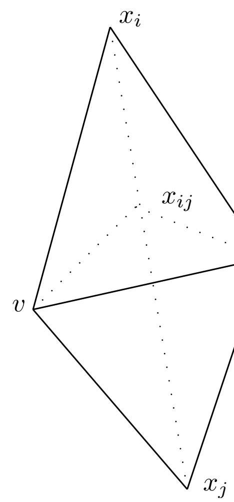


Figure 2.4: Reconstruction of the 1-form A_{ij}

$-\langle y, v(t), x_{ij}, x_i\rangle$ (see Figure 2.4)). Indeed, the boundary of the 2-simpliex for the 1-form is hence given by the 8 simplices $\langle v, x_{ij}, x_j\rangle$, $-\langle y, x_{ij}, x_j\rangle$, $\langle y, v, x_j\rangle$, $-\langle y, v, x_{ij}\rangle$ and $-\langle v, x_{ij}, x_i\rangle$, $\langle y, x_{ij}, x_i\rangle$, $-\langle y, v, x_i\rangle$, $\langle y, v, x_{ij}\rangle$. Note that the simplex $\langle y, v, x_{ij}\rangle$ occurs twice with different sign and hence vanish. This corresponds to the definition of Mackaay-Picken (figure 9 in [19]).

Finally the 2-form F_i, as it is called in the paper of Mackaay-Picken, is simply the boundary of the 3-simplex $\langle y, v, w, x_i\rangle$., that is it consists of the simplices $\langle v, w, x_i\rangle$, $-\langle y, w, x_i\rangle$, $\langle y, v, x_i\rangle$ and $\langle y, v, w\rangle$.

2.4 Prerequisites

2.4.1 Notations and Definitions

Our general assumptions will be the following: First, M is a highly connected manifold of dimension N. Highly connected means here that to construct a Deligne d-class we assume M to be $(d-1)$-connected, i.e. $\pi_1(M) = \cdots = \pi_{d-1}(M) = 0$. Second, on M we fix a good cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M. As always we will assume that the index-set is ordered.

Since $\mathcal{U}$ is chosen to be a good cover, every U_i and every intersection of these are assumed to be contractible. Hence we can choose diffeomorphisms $\phi_{i_1,\ldots,i_r} : U_{i_1,\ldots,i_r} \longrightarrow D^N \subset \mathbb{R}^N$ from $U_{i_1,\ldots,i_r}$ onto the unit ball D^N in $\mathbb{R}^N$. Denote by $x_{i_1,\ldots,i_r}$ the 'midpoint' of $U_{i_1,\ldots,i_r}$, that is $x_{i_1,\ldots,i_r} := \phi^{-1}_{i_1,\ldots,i_r}(0)$. (Notice that reordering the index-set does not change the point, i.e. $x_{ij} = x_{ji}$, simply because intersections $U_i \cap U_j = U_{ij}$ do not depend on the ordering of the indices, but we will always assume the indexed to be ordered.)

For any other point x in $U_{i_1,\dots,i_r}$ we can now find a canonical path from the midpoint $x_{i_1,\dots,i_r}$ to x, which we will call $r_{x;i_1,\dots,i_r}$, because we can use the chosen diffeomorphism $\phi_{i_1,\dots,i_r}$ to lift the straight path from 0 to $\phi_{i_1,\dots,i_r}(x)$.

Later on we will work with simplices and will adopt the usual notation, so that $\langle x_1, \dots, x_l \rangle$ will denote the simplex spanned by the points $x_1, \dots, x_l$. Constructing the simplex spanned by some points poses no problem in $\mathbb{R}^N$, but we have to give it a concrete meaning for points in M.

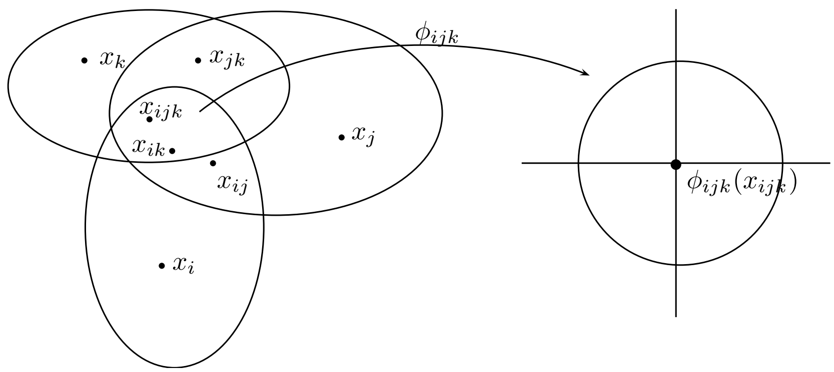


Figure 2.5: Midpoints x_{ij} and the diffeomorphism ϕ_{ij}

But since in the following all the points $x_1, \dots, x_l$ will be contained in some U_i (or some $U_{i_1,\dots,i_r}$), which is diffeomorphic to the open ball in $\mathbb{R}^N$ we may use the map $\phi_i : U_i \longrightarrow \mathbb{R}^N$ to lift the standard simplex spanned by $\phi_i(x_1), \dots, \phi_i(x_l)$ in $\mathbb{R}^N$ to U_i. By abuse of notation, we can define the simplex $\langle x_1, \dots, x_l \rangle$ in M to be the map that maps the standard-simplex $\Delta_l := \langle e_1, e_2, \dots, e_l \rangle$ to $\langle \phi_i(x_1), \dots, \phi_i(x_l) \rangle$ and further via ϕ_i^{-1} to $\langle x_1, \dots, x_l \rangle \subset U_i$. If we need to refer to this map explicitly we will call it φ_{Δ_l}, where we drop the dependency on the points x_i.

It is important to notice that each of the simplices $\langle x_1, \dots, x_l \rangle$ can be regarded as an l-path. For this we take the usual diffeomorphism of the $(l-1)$-cube to the l-simplex, by squeezing along the diagonal and concatenate it with the given simplex. In case of $l = 3$ for example this map is given by $(t_1, t_2) \mapsto (t_1 - t_2, \frac{1}{2}(t_1 + t_2 + |t_1 - t_2|) - 1)$. One can find similar maps for higher l. To obtain an l-path one must reparametrize this map to be constant at its boundaries, but which is always possible. Also observe that these simplices carry an orientation, defined by the order of the indices of the vertices. This means that whenever we change two vectors, the simplex changes its orientation. Any subsimplex then obtains the induced orientation. We will denote the orientation of the simplex by a sign *in front* of the simplex, though we write the union of two simplices as a product.

2.4.2 Recentering

For the construction we need to say how to recenter d-loops. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be the chosen good open cover. Given any ordered subset $\sigma_r := (i_1, \ldots, i_r)$ of the index-set I, for each $1 \leq j \leq r$ let σ_j be any ordered subset of σ_r with j elements, such that $\sigma_j \subset \sigma_{j+1}$. To any such choice use the midpoints $x_{\sigma_1}, \ldots, x_{\sigma_r}$ to define a simplex $\langle x_{\sigma_r}, \ldots, x_{\sigma_1} \rangle$ which we call $\Delta_{\sigma_r,\ldots,\sigma_1}$. This can be done, since all midpoints lie in U_{σ_1}.

By assumption on the manifold M we know that $\Delta_{\sigma_r,\ldots,\sigma_1}$, being an $(r-1)$-dimensional path and if $r \leq d$, is contractible to $\star$, so let $p_{\sigma_r,\ldots,\sigma_1}$ be a homotopy from $\Delta_{\sigma_r,\ldots,\sigma_1}$ to $\star$. We deform $p_{\sigma_r,\ldots,\sigma_1}$ to be constant along its boundary, so that it becomes a sitting r-path. Note that composing $p_{\sigma_r,\ldots,\sigma_1}$ with its inverse is thin homotopic to the constant path at $\star$.

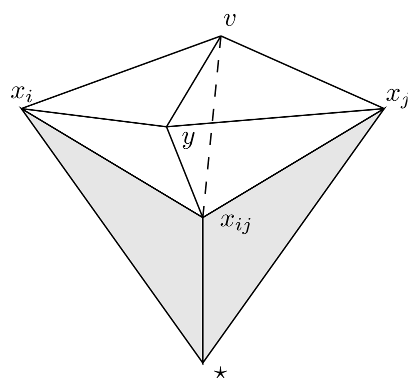


Figure 2.6: Recentering a closed loop

2.5 Reconstruction

In this subsection we will associate a Deligne class to a given d-holonomy map $H : \pi_d^d(M) \longrightarrow U(1)$.

Before we start our construction note that a Deligne d-class is a family $(\omega^0, \omega^1, \ldots, \omega^d)$, where every $\omega^p_{i_1,\ldots,i_r}$ is a p-form that lives on $(d-p+1)$-fold intersections, i.e. on $\coprod U_{i_1,\ldots,i_r}$, where we set r to be $d-p+1$. It has to obey the conditions

$\delta\omega^p = \pm d\omega^{p-1}$ for $p > 0$ and $\delta\omega^0 = 1.$ for $p = 0$. We will take care of this by treating these cases separately. Also notice that because we have truncated the Čech-deRham complex we do not have the condition $d\omega^d = 0$.

2.5.1 The p-loops $s^{v_0,\ldots,v_p}_{i_1,\ldots,i_r}$

Let us turn to the general definition. First we assume that $p > 0$ since, as we saw in the last subsection, there is a small difference between the case $p = 0$ and $p > 0$.

For the p-forms ω^p, which are defined on intersections $U_{i_1,\ldots,i_r}$, we consider the boundary of the *all* simplices of the form

$$(-1)^{s+rp}\langle y, v_1, \ldots, v_p, x_{\sigma_r}, x_{\sigma_{r-1}}, \ldots, x_{\sigma_1}\rangle.$$

Here $v_1, \ldots, v_p$ are points in $U_{i_1,\ldots,i_r}$, which will later on represent the tangent vectors, $\sigma_r := (i_1, \ldots, i_r)$ is an ordered set of indices and every $\sigma_{j-1} \subset \sigma_j$ is an ordered subset with $j - 1$ elements. One can obtain every σ_{j-1} from σ_j by dropping an element at some position, say $k(j)$. The sign $s = s(\sigma_r, \ldots, \sigma_1)$ is then given $(-1)^{\sum_{j=1}^{r}(k(j)-1)}$ and reflects the orientation of the simplex (e.g. if $\sigma_4 = (i,j,k,l)$, $\sigma_3 = (i,k,l)$, $\sigma_2 = (k,l)$ and $\sigma_1 = (l)$, then we have dropped the second, first and again the first element, giving sign $(-1)^{1+0+0} = -1$). The boundary of a simplex is given by dropping the vertices successively (and by assigning the right sign). If we drop any vertex, we will denote this by $\hat{\cdot}$. Applying the definition of the differential ∂ upon all these simplices, we see that we have three types of simplices: Simplices of the form $(-1)^{s+rp}\langle y, v_1, \ldots, v_p, x_{\sigma_r}, x_{\sigma_{r-1}}, \ldots \hat{x}_{\sigma_j}, \ldots, x_{\sigma_1}\rangle$ vanish, if $j < r$, since they occur twice with different sign, so we are left only with simplices of the form

$$(-1)^{i+s+rp}\langle y, v_1, \ldots, \hat{v}_i, \ldots, v_p, x_{\sigma_r}, x_{\sigma_{r-1}}, \ldots x_{\sigma_1}\rangle, \tag{2.5.1}$$

where i runs from 0 to p, on the one hand and

$$(-1)^{(p+1)+s+rp}\langle y, v_1, \ldots, v_p, \hat{x}_{\sigma_r}, x_{\sigma_{r-1}}, \ldots x_{\sigma_1}\rangle. \tag{2.5.2}$$

on the other. Here and in the following we set $v_0 := y$. With this understood, we easily see that we have $(p+1)\cdot r!$ and $r!$ simplices, so altogether $(p+2)r!$ simplices (see the numbers in figure 2.7).

2.5.1 EXAMPLE: We have already seen this mechanism at work in the motivating examples. Here, for more clearity, we write down the simplices of the 3-form $\omega^3_{ij} = C_{ij}$ in case of $d = 4$. The $(3+2)\cdot 2! = 10$ simplices are given by $\langle v, w, u, x_{ij}, x_j\rangle$, $-\langle y, w, u, x_{ij}, x_j\rangle$, $\langle y, v, u, x_{ij}, x_j\rangle$, $-\langle y, v, w, x_{ij}, x_j\rangle$, $\langle y, v, w, u, x_j\rangle$, -$\langle v, w, u, x_{ij}, x_i\rangle$, $\langle y, w, u, x_{ij}, x_i\rangle$, $-\langle y, v, u, x_{ij}, x_i\rangle$, $\langle y, v, w, x_{ij}, x_i\rangle$, $-\langle y, v, w, u, x_i\rangle$.

5	120	360	480	600	720
4	24	72	96	120	144
3	6	18	24	30	36
2	2	6	8	10	12
1	1	3	4	5	6
r/p	0	1	2	3	4

Figure 2.7: Number of simplices

In the same way $\omega^2_{ijk} = B_{ijk}$ is built out of $-\langle v, w, x_{ijk}, x_{ij}, x_i\rangle$, $\langle y, w, x_{ijk}, x_{ij}, x_i\rangle$, $-\langle y, v, x_{ijk}, x_{ij}, x_i\rangle$, $\langle y, v, w, x_{ij}, x_i\rangle$ together with the same set of simplices , but with x_{ij} replaced by x_{jk} and x_{ki}, x_i replaced by x_j and x_j respectively and decorated with the correct sign. B_{ijk} consists of 24 simplices.

2.5.2 DEFINITION: Suppose $v_1, ..., v_p$ are points in $U_{i_1,...,i_r}$, and $v_0 := y$. Then let $\tilde{s}^{v_0,...,v_p}_{i_1,...,i_r}(t_1, ..., t_p)$ for some ordered index set $\sigma_r = (i_1, \ldots, i_r)$ be given by

$$\prod_{\substack{\sigma_{r-1}\subset(i_1,...,i_r)\\|\sigma_{r-1}|=r-1}} \cdots \prod_{\substack{\sigma_1\subset\sigma_2\\|\sigma_1|=1}} (-1)^{s+rp}\Big[(-1)^{p+1}\langle v_0, v_1, ..., v_p, \hat{x}_{\sigma_r}, x_{\sigma_{r-1}}, \ldots, x_{\sigma_1}\rangle \star \prod_{0\le i\le p} (-1)^i \langle v_0, ..., \hat{v}_i, ..., v_p, x_{\sigma_r}, ..., x_{\sigma_1}\rangle\Big]$$

Being a boundary of a $(d + 1)$-simplex, this is indeed a d-loop. But we still have to recenter this d-loop by using the recentering homotopies from the last subsection. By doing this we finally arrive at

2.5.3 DEFINITION: The recentered d-loop is defined to be

$$s^{v_0,v_1,...,v_p}_{i_1,...,i_r} := \tilde{s}^{v_0,v_1,...,v_p}_{i_1,...,i_r} \star \prod_{\substack{\sigma_{r-1}\subset(i_1,...,i_r)\\|\sigma_{r-1}|=r-1}} \cdots \prod_{\substack{\sigma_1\subset\sigma_2\\|\sigma_1|=1}} p_{\sigma_r,...,\sigma_1}$$

Again we do not explicitly say that we still have to make these simplices constant at their boundary, in order to make the d-loop smooth.

Let us now turn to the case $p = 0$. This case is more or less the same as the case $p > 0$. As we already saw in the definition of the transition functions g_{ij} and g_{ijk}, the main difference is that we do not include the 'highest' midpoint $x_{i_1,...,i_r}$ into the construction, for we need $\delta\omega^0 = 1$. We will show in the next section that this relation is satisfied.

2.5.4 DEFINITION: Let $\tilde{s}^{v_0}_{i_1,\dots,i_r}$ be given by

$$\prod_{\substack{\sigma_{r-1}\subset(i_1,\dots,i_r)\\|\sigma_{r-1}|=r-1}} \cdots \prod_{\substack{\sigma_1\subset\sigma_2\\|\sigma_1|=1}} (-1)^{s+1}\langle v_0, \hat{x}_{\sigma_r}, x_{\sigma_{r-1}}, \dots, x_{\sigma_1}\rangle$$

Notice that there are $r! = (p+1)r!$ simplices, so in case of $p=0$ this number differs from the case $p>0$. Recentering this n-loop we arrive at

$$s^{v_0}_{i_1,\dots,i_r} := \tilde{s}^{v_0}_{i_1,\dots,i_r} \star \prod_{\substack{\sigma_{r-1}\subset(i_1,\dots,i_r)\\|\sigma_{r-1}|=r-1}} \cdots \prod_{\substack{\sigma_1\subset\sigma_2\\|\sigma_1|=1}} p_{\hat{\sigma}_r,\sigma_{r-1},\dots,\sigma_1}.$$

2.5.5 EXAMPLE: For the 0-form g_{ijklm} in case of $n=4$ we have $5! = 120$ simplices. These are of the form $\pm\langle y, x_{ijkl}, x_{jkl}, x_{kl}, x_l\rangle$.

2.5.2 The p-Forms

Having said how the simplices will look like, we are now ready to define to a given thin-invariant holonomy map H a family of p-forms, which will be shown to form a Deligne class in the next section. The construction of the p-forms can be seen as a simple total derivative. To accomplish this, we define a map $h^p : (\mathbb{R}^N)^p \longrightarrow i\mathbb{R}$ (for $p>0$) by

$$h^p_{v_0} : (v_1,\dots,v_p) \mapsto \log H(s^{v_0,v_1,\dots,v_p}_{i_1,\dots,i_r}),$$

with $v_1,\dots,v_p \in U_{i_1,\dots,i_r} \cong \mathbb{R}^N$. Moreover h^p is a smooth function, as s depends smoothly on the vertices and H is smooth by assumption. Looking at the p-th total derivate, we obtain

$$D^p_{(v_0,\dots,v_0)} h^p : (\mathbb{R}^N)^p \times \cdots \times (\mathbb{R}^N)^p \longrightarrow i\mathbb{R}.$$

2.5.6 DEFINITION: Let d,p be given, and set $r = d-p+1$. Suppose $v_1,\dots,v_p$ are tangent vectors in $y=v_0$. Then we define the p-form ω^p to be

$$(\omega^p_{i_1,\dots,i_r})_{v_0}(v_1,\dots,v_p) := D^p_{(v_0,\dots,v_0)} h^p((v_1,0,\dots,0),\dots,(0,\dots,0,v_p)).$$

For $p=0$ we set $\omega^0_{v_0} = H(s^{v_0}_{i_1,\dots,i_r})$.

This definition does only depend on the tangent vectors, which we will write also with small letters subsequently. Because of the properties of the total derivative, ω^p is multilinear and anti-symmetric, so it is a p-form.

We may choose integral curves for the tangential vectors, and rewrite this definition. If we take the canonical integral curves (since we may pretend to be in $\mathbb{R}^N$, these do exist), we have

$$(\omega^p_{i_1,\ldots,i_r})_{v_0}(v_1,\ldots,v_p) = \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \log H(s^{v_0,v_1(t_1),\ldots,v_p(t_p)}_{i_1,\ldots,i_r})\Big|_{(t_1,\ldots,t_p)=0}.$$

In the following calculations we will use this definition, which is easier to handle.

2.5.3 Relations

The construction leaves us with a family $(\omega^0,\ldots,\omega^p)$ of differential forms on M. We have to verify that we really defined a Deligne class, i.e. we have to show

$$(-1)^{r-1} d\omega^p_{i_1,\ldots,i_r} + (\delta\omega^{p+1})_{i_1,\ldots,i_r} = 0.$$

Let us first see how δ is defined (see equation (1.1.1)). $\delta\omega^{p+1}$ is nothing else than

$$(\delta\omega^{p+1})_{i_1,\ldots,i_r}(v_1,\ldots,v_{p+1}) = \sum_{j=1}^{r} (-1)^{j-1} \omega^{p+1}_{i_1,\ldots,\hat{i_j},\ldots,i_r}(v_1,\ldots,v_{p+1})$$

If we plug the definition of ω^p into this formula, and drag the sum into H we see that to obtain the form $\delta\omega^{p+1}$ geometrically we have to construct the simplices for every set of midpoints of the form $x_{i_1},\ldots,\hat{x}_{i_j},\ldots,x_{i_r}$, and glue these with the right orientation (Here we use the explicit expression of ω^{p+1} as a partial derivative, see the end of last section). Since there are r ordered subsets with $r-1$ elements of $(i_1,\ldots,i_r)$ we get r times more simplices than we had before. Let us first give a simple example:

2.5.7 EXAMPLE: Consider the case $d = 2$. As above, B_i is given by $\langle v,w,x_i\rangle$, $-\langle y,w,x_i\rangle$, $\langle y,v,x_i\rangle$, $-\langle y,v,w\rangle$. Hence $(\delta B)_{ij}$ is given by the simplices $\langle v,w,x_j\rangle$, $-\langle y,w,x_j\rangle$, $\langle y,v,x_j\rangle$, $-\langle y,v,w\rangle$ and $-\langle v,w,x_i\rangle$, $\langle y,w,x_i\rangle$, $-\langle y,v,x_i\rangle$, $\langle y,v,w\rangle$. Notice that the simplex $\langle y,v,w\rangle$ occurs twice with different sign and hence vanish, so we are left with 6 simplices. We will see further examples later on.

Before explaining the general construction it is better to examine the case $d = 2$ and directly verify the relation $dA_{ij} = (\delta B)_{ij}$. There are two main observations that will eventually lead to the relation: The boundary of the simplex $\tilde{s}^{y,v,w}_{ij}$ (which is at »position« $r = 2$ and $p = 2$, so belongs inofficially to dimension $d = 3$), consists of the simplices of $(\delta B)_{ij}$ as well as of simplices which can be interpreted as an integral of A_{ij} over $\partial\langle v_0,v_1,v_2\rangle$. Using Stokes theorem and differentiating the equation gives us the upper relation. Let us write this down explicitly:

2.5.8 Example: Let us consider the boundary of the 3-simplex $s_{ij}^{y,v,w}$ (Actually this is $\tilde{s}_{ij}^{y,v,w}$, but since the recentering will not play any role in the computation to follow, as we will see, we pretend that it is enough to work with $s_{ij}^{y,v,w}$ instead). The 3-loop consists of 8 simplices $\langle v, w, x_{ij}, x_j \rangle$, $-\langle y, w, x_{ij}, x_j \rangle$, $\langle y, v, x_{ij}, x_j \rangle$, $-\langle y, v, w, x_j \rangle$ and $-\langle v, w, x_{ij}, x_i \rangle$, $\langle y, w, x_{ij}, x_i \rangle$, $-\langle y, v, x_{ij}, x_i \rangle$, $\langle y, v, w, x_i \rangle$ and its boundary is given by:

$$\begin{array}{cccc} \langle w, x_{ij}, x_j \rangle, & -\langle v, x_{ij}, x_j \rangle, & \langle v, w, x_j \rangle, & -\langle v, w, x_{ij} \rangle, \\ -\langle w, x_{ij}, x_j \rangle, & \langle y, x_{ij}, x_j \rangle, & -\langle y, w, x_j \rangle, & \langle y, w, x_{ij} \rangle, \\ \langle v, x_{ij}, x_j \rangle, & -\langle y, x_{ij}, x_j \rangle, & \langle y, v, x_j \rangle, & -\langle y, v, x_{ij} \rangle, \\ -\langle v, w, x_j \rangle, & \langle y, w, x_j \rangle, & -\langle y, v, x_j \rangle, & \langle y, v, w \rangle, \end{array}$$

$$\begin{array}{cccc} -\langle w, x_{ij}, x_i \rangle, & \langle v, x_{ij}, x_i \rangle, & -\langle v, w, x_i \rangle, & \langle v, w, x_{ij} \rangle, \\ \langle w, x_{ij}, x_i \rangle, & -\langle y, x_{ij}, x_i \rangle, & \langle y, w, x_i \rangle, & -\langle y, w, x_{ij} \rangle, \\ -\langle v, x_{ij}, x_i \rangle, & \langle y, x_{ij}, x_i \rangle, & -\langle y, v, x_i \rangle, & \langle y, v, x_{ij} \rangle, \\ \langle v, w, x_i \rangle, & -\langle y, w, x_i \rangle, & \langle y, v, x_i \rangle, & -\langle y, v, w \rangle. \end{array}$$

Note, that line four and eight contain $-\delta s_i^{y,v,w}$. The 24 simplices in the other lines make up a complex we will call $\Delta_w s_{ij}^{y,v}$. By examining these simplices a bit closer, one sees directly that the first and the fifth line is just $s_{ij}^{v,w}$. Analogously the other lines are identified, giving $\Delta_w s_{ij}^{y,v} = s_{ij}^{y,v} \star s_{ij}^{v,w} \star s_{ij}^{w,y}$. Altogether we arrive at a decomposition $\partial s_{ij}^{y,v,w} = (-\delta s_i^{y,v,w}) \star \Delta_w s_{ij}^{y,v}$.

Suppose for a moment we had proven a relation like $\int_{\langle y,v \rangle} A_{ij} = \log H(s_{ij}^{y,v})$. Using this we have (note that we have excluded the dependency of v on k in our notation, but we display it here, for we will differentiate with respect to k):

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial k \partial l} \log H(\partial s_{ij}^{y,v(k),w(l)}) \Big|_{k,l=0} \\ &= \frac{\partial^2}{\partial k \partial l} (\log H(-\delta s_i^{y,v(k),w(l)}) + \log H(\Delta_{w(l)} s_{ij}^{y,v(k)})) \Big|_{k,l=0} \\ &= -(\delta B)_{ij}(v,w) + \frac{\partial^2}{\partial k \partial l} \log H(\Delta_{w(l)} s_{ij}^{y,v(k)}) \Big|_{k,l=0} \\ &= -(\delta B)_{ij}(v,w) + \frac{\partial^2}{\partial k \partial l} \log[H(s_{ij}^{y,v(k)}) H(s_{ij}^{v(k),w(l)}) H(s_{ij}^{w(l),y})] \Big|_{k,l=0} \\ &= -(\delta B)_{ij}(v,w) + \frac{\partial^2}{\partial k \partial l} [\int_{\langle y,v(k) \rangle} A_{ij} + \int_{\langle v(k),w(l) \rangle} A_{ij} + \int_{\langle w(l),y \rangle} A_{ij}] \Big|_{k,l=0} \end{aligned}$$

$$= -(\delta B)_{ij}(v,w) + \frac{\partial^2}{\partial k \partial l} \int_{\partial\langle y,v(k).w(l)\rangle} A_{ij}\Big|_{k,l=0}$$
$$= -(\delta B)_{ij}(v,w) + \frac{\partial^2}{\partial k \partial l} \int_{\langle y,v(k),w(l)\rangle} dA_{ij}\Big|_{k,l=0}$$
$$= -(\delta B)_{ij}(v,w) + dA_{ij}(v,w),$$

where we used the fact $s_{ij}^{y,v,w}$ is a 3-loop (so $\partial s_{ij}^{y,v,w} = \emptyset$) and, in the third equation, Stokes theorem. Furthermore we regarded the forms A_{ij} and B_i to be given by partial derivatives instead of a total derivative. So the only thing left to prove is the relation $\int_{\langle y,v(t)\rangle} A_{ij} = \log H(s_{ij}^{y,v(t)})$. But this is more or less nothing else than the fundamental theorem of calculus, as we will prove in general later on.

The above example gives rise to a guess which we will now specify and prove. We mimic the steps of the example in the general case. First we define $\Delta_{v_{p+1}} s_{i_1,\ldots,i_r}^{v_0,\ldots,v_p}$:

2.5.9 DEFINITION: For any vertices $v_0, \ldots, v_{p+1} \in M$ we define

$$\Delta_{v_{p+1}} s_{i_1,\ldots,i_r}^{v_0,\ldots,v_p} := \prod_{j=0}^{p+1} (-1)^j s_{i_1,\ldots,i_r}^{v_0,\ldots,\hat{v}_j,\ldots,v_{p+1}}.$$

Next we need a lemma, how the boundary of $s_{i_1,\ldots,i_r}^{v_0,\ldots,v_{p+1}}$ decomposes.

2.5.10 LEMMA: *The boundary of $s_{i_1,\ldots,i_r}^{v_0,\ldots,v_{p+1}}$ can be rewritten as a sum of the form*

$$-(\delta s^{v_0,\ldots,v_{p+1}}))_{i_1,\ldots,i_r} \star (-1)^r (\Delta_{v_{p+1}} s_{i_1,\ldots,i_r}^{v_0,\ldots,v_p}).$$

Proof. We have to be careful about the sign, so in the definition of the simplex $s_{i_1,\ldots,i_r}^{v_0,\ldots,v_{p+1}}$ we will not write just $(-1)^s$, but exhibit its dependency on the choice of $\sigma_r, \ldots, \sigma_1$.

Using the definitions we obtain

$$\partial(s_{i_1,\ldots,i_r}^{v_0,\ldots,v_{p+1}}) = \partial\partial[\prod_{\sigma} (-1)^{s(\sigma_r,\ldots,\sigma_1)+r(p+1)} \langle v_0, \ldots, v_{p+1}, x_{\sigma_r}, \ldots, x_{\sigma_1}\rangle],$$

and also

$$\delta(s_{i_1,\ldots,i_{r-1}}^{v_0,\ldots,v_{p+1}}) = \delta\partial[\prod_{\sigma} (-1)^{s(\sigma_{r-1},\ldots,\sigma_1)+(r-1)(p+1)} \langle v_0, \ldots, v_{p+1}, x_{\sigma_{r-1}}, \ldots, x_{\sigma_1}\rangle]$$

$$\Delta_{v_{p+1}}(s_{i_1,\ldots,i_r}^{v_0,\ldots,v_p}) = \prod_{j=0}^{p+1} (-1)^j \partial \prod_{\sigma} (-1)^{s(\sigma_r,\ldots,\sigma_1)+pr} \langle v_0, \ldots, \hat{v}_j, \ldots, v_{p+1}, x_{\sigma_r}, \ldots, x_{\sigma_1}\rangle,$$

where we use just a single $\prod_\sigma$ to denote the product over all subsets $\sigma_j \subset \sigma_{j-1}$, as in the definition of $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$. By expanding the boundary-operator ∂ in $\partial(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_r})$, we obtain

$$\partial\partial(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_r}) = \prod_{j=0}^{p+r+2} (-1)^j \partial \prod_\sigma (-1)^{s(\sigma_r,\dots,\sigma_1)+r(p+1)} \langle v_0,\dots,\hat{v}_j,\dots,v_{p+1},x_{\sigma_r},\dots,x_{\sigma_1}\rangle.$$

The first $p+2$ factors are exactly $\Delta_{v_{p+1}}(s^{v_0,\dots,v_p}_{i_1,\dots,i_r})$ up to the sign $(-1)^r$. Therefore we have identified one part of the boundary of $s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_r}$ as $(-1)^r\Delta_{v_{p+1}}(s^{v_0,\dots,v_p}_{i_1,\dots,i_r})$. Let us consider now $\delta(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}})$. We would like to identify this with the next, i.e. the $(p+3)^{rd}$ factor of the boundary of $\partial(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}})$, since this factor is just

$$(-1)^{p+2}\partial \prod_\sigma (-1)^{s(\sigma_r,\dots,\sigma_1)+r(p+1)} \langle v_0,\dots,v_{p+1},\hat{x}_{\sigma_r},x_{\sigma_{r-1}},\dots,x_{\sigma_1}\rangle.$$

For clearity let us expand $\delta(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}})$:

$$\delta(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}}) = \prod_{j=1}^{r}(-1)^{j-1}\partial \prod_\sigma (-1)^{s(\sigma_{r-1},\dots,\sigma_1)+(r-1)(p+1)} \langle v_0,\dots,v_{p+1},\hat{x}_{\sigma_r},x_{\sigma_{r-1}},\dots,x_{\sigma_1}\rangle.$$

The last product is now over all $\sigma_1 \subset \cdots \subset \sigma_{r-1} \subset \sigma_r$, by the very definition of δ. Therefore the vertices in δ and the $(p+2)$ component are the same. The only problem is the sign. To see how the signs are related, consider any fixed choice of $\sigma_r,\dots,\sigma_1$. Suppose that σ_{r-1} is obtained by σ_r by dropping the $k(r)^{th}$ index. In this case the signs are correlated by $k(r)+1+s(\sigma_{r-1},\dots,\sigma_1) = s(\sigma_r,\dots,\sigma_1)$. Taking now the product over all these, each possible choice of $k(r)$ occurs exactly once, hence we have the overall correction sign $\prod_{j=1}^r(-1)^{j+1}$. So, δs has the sign $(-1)^{s(\sigma_r,\dots,\sigma_1)+rp+r-p+1}$, while ∂s has the sign $(-1)^{s(\sigma_r,\dots,\sigma_1)+rp+p+r}$. This shows, that $-\delta s$ is just the $(p+2)$ component of ∂s, and we have nearly completed our calculations. There is only the rest of the boundary $\partial(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}})$ left. But it consists of

$$\partial\prod_{j=1}^{r-1}(-1)^{p+2+r-j}\partial \prod_\sigma (-1)^{s(\sigma_r,\dots,\sigma_1)+(r+1)(p+1)} \langle v_0,\dots,v_{p+1},x_{\sigma_r},\dots,\hat{x}_{\sigma_j},\dots,x_{\sigma_1}\rangle].$$

and these simplices vanishes, as we already noted in the definition of the loops $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$. □

2.5.11 PROPOSITION: *For any set of vertices $\{w_1,\dots,w_{p+1}\}$ in $\mathbb{R}^N$ we have*

$$\int_{\langle w_1,\dots,w_{p+1}\rangle} \omega^p_{i_1,\dots,i_r} = \log H(s^{w_1,\dots,w_{p+1}}_{i_1,\dots,i_r}).$$

Proof. Let us first examine the case $p = 2$. Since $D^1_y(D^1_y h^2(v,0))(0,w) = D^2_{(y,y)}h((v,0),(0,w))$ and everything in sight is smooth, we can interchange the integral over t_1 with D^1_y and have to consider $\int_{t_1=1}^0 D^1_y h^2(u_1,-)du_1$. Using the theorem of Stokes and the relation $D^1_y f(v) = df_y(v)$, the integral is given by $h^2(1,-) - h^2(0,-)$. Now evaluating the integral over t_2 leaves us with $h^2(1,1) - h^2(0,1) - h^2(1,0) + h^2(0,0)$. Noting, that every term except $h^2(1,1)$ is zero, since the simplex at hand is degenerated, one obtains the theorem. The general case works identically. □

As a direct corollary we have:

2.5.12 COROLLARY: *There is the following relation:*

$$\int_{\partial\langle v_0,\ldots,v_{p+1}\rangle} \omega^p_{i_1,\ldots,i_r} = \log H(\Delta_{v_{p+1}} s^{v_0,v_1,\ldots,v_p}_{i_1,\ldots,i_r}).$$

2.5.13 COROLLARY: *We have*

$$(-1)^{r-1} d\omega^p_{i_1,\ldots,i_r} + \delta(\omega^{p+1})_{i_1,\ldots,i_r} = 0.$$

Proof. Following exactly the example and using the above lemma, one gets:

$$\begin{aligned}
0 &= -\frac{\partial^{p+1}}{\partial t_1 \ldots \partial t_{p+1}} \log H(\partial s^{v_0,v_1,\ldots v_{p+1}}_{i_1,\ldots,i_r})\Big|_{t_i=0} \\
&= -\frac{\partial^{p+1}}{\partial t_1 \ldots \partial t_{p+1}} (\log H(-\delta(s^{v_0,v_1,\ldots,v_{p+1}}_{i_1,\ldots,i_{r-1}})))\Big|_{t_i=0} \\
&\qquad - \frac{\partial^{p+1}}{\partial t_1 \ldots \partial t_{p+1}} \log H((-1)^r \Delta_{v_{p+1}} s^{v_0,\ldots,v_p}_{i_1,\ldots,i_r})\Big|_{t_i=0} \\
&= (\delta\omega^{p+1})_{i_1,\ldots,i_r}(v_1,\ldots,v_{p+1}) + (-1)^{r-1} \frac{\partial^{p+1}}{\partial t_1 \ldots \partial t_{p+1}} \log H(\Delta_{v_{p+1}} s^{v_0,\ldots,v_p}_{i_1,\ldots,i_r})\Big|_{t_i=0} \\
&= (\delta\omega^{p+1})_{i_1,\ldots,i_r}(v_1,\ldots,v_{p+1}) + (-1)^{r-1} \frac{\partial^{p+1}}{\partial t_1 \ldots \partial t_{p+1}} \int_{\partial\langle v_0,\ldots,v_{p+1}\rangle} \omega^p_{i_1,\ldots,i_r}\Big|_{t_i=0} \\
&= (\delta\omega^{p+1})_{i_1,\ldots,i_r}(v_1,\ldots,v_{p+1}) + (-1)^{r-1} d\omega^p_{i_1,\ldots,i_r}(v_1,\ldots,v_{p+1}).
\end{aligned}$$

Here, as before, we used the last corollary and the theorem of Stokes. □

Up to now we neglected the case $p = 0$, for which we have:

2.5.14 PROPOSITION: *For $p = 0$ we have $\delta\omega^0 = 1$.*

Proof. This is easy to see, for we can rewrite $\tilde{s}^y_{i_1,\dots,i_r}$ as

$$\tilde{s}^y_{i_1,\dots,i_r} = \delta \left(\prod_{\substack{\sigma_{r-1}\subset(i_1,\dots,i_{r-1}) \\ |\sigma_{r-1}|=r-1}} \prod_{\substack{\sigma_{r-2}\subset\sigma_{r-1} \\ |\sigma_{r-2}|=r-2}} \cdots \prod_{\substack{\sigma_1\subset\sigma_2 \\ |\sigma_1|=1}} (-1)^s \langle y, x_{\sigma_{r-1}}, \dots x_{\sigma_1} \rangle \right).$$

Since $\delta^2 = 0$ and the recentering homotopies cancel (being also a product over certain indices i_j), the proposition is shown.

□

The only very last step in the proof of the surjectivity, is to verify that the forms ξ are indeed mapped to H under HOL, so that we really constructed a preimage of H. But this one can verify easily using the local formula for the holonomy.

2.5.4 Injectivity of HOL

The last sections showed that HOL is indeed surjective. Let us now establish the injectivity of this map. This can be done in several ways, e.g. by utilizing the isomorphism between Deligne classes and differential characters or by contemplating about the injectivity of the transgression map. Though these rather homological proofs may be more elegant, we follow Mackaay-Picken and show with the methods already used, the injectivity of HOL, i.e. if ω is a Deligne class with trivial holonomy, then $\omega = D\eta$ for some Deligne class η. The proof involves cumbersome formulas, but the main idea is to use the very simple fact that $s^{v_0,v_1,\dots,v_p}_{i_1,\dots,i_r}$ is composed out of simplices of the form

$$\langle v_0, v_1, \dots, v_p, \hat{x}_{\sigma_r}, x_{\sigma_{r-1}}, \dots, x_{\sigma_1} \rangle$$

on the one hand (see (2.5.2)) and

$$\langle v_0, v_1, \dots, \hat{v}_i, \dots, v_p, x_{\sigma_r}, \dots, x_{\sigma_1} \rangle$$

on the other (see (2.5.1)). The first of these sets can be rather directly seen to yield the contribution of $\delta\eta^p$, the second will be seen to represent $d\eta^{p-1}$.

Before we start, we need to make more precise the local formula for the holonomy of a Deligne class ω (see 1.2.8) in case of a simplex like $\langle v_0, \dots, v_p, x_{\sigma_r}, \dots, x_{\sigma_1} \rangle$. In the local formula we need to choose some arbitrary labeling. For simplicity we choose the following one:

2.5.15 CONVENTION: Whenever we have a subface f of a simplex $\langle v_0, \dots, v_p, x_{\sigma_r}, \dots, x_{\sigma_1} \rangle$, we take the *highest* index j such that the subface f is contained in U_j.

For example, $\langle v_0, \dots, v_p, x_{ijkl}, x_{jk}, x_j \rangle$ takes the label j for all vertices $v_0, \dots, v_p$ and midpoints x_{ijkl}, x_{jk}, x_j are contained in U_j. If we drop x_j and consider (up to sign) $\langle v_0, \dots, v_p, x_{ijkl}, x_{jk} \rangle$, this subface carries the label k, for all points lie in $U_{jk} = U_j \cap U_k$, and k is the highest index.

Let us now examine the holonomy of a simplex $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$. By its very definition the simplex is a product of faces, and each face, say $\langle v_0, \dots, v_m, x_{\sigma_r}, \dots, x_{\sigma_j} \rangle$, contributes to the holonomy via an integral of the form

$$\int_{\langle v_0,\dots,v_m,x_{\sigma_r}\dots,x_{\sigma_j}\rangle} \omega^{m+r-j+2}_{\sigma_j}.$$

Note, that the dimension of the simplex is indeed $m + r - j + 2$, and therefore we have to integrate over $\omega^{m+r-j+2}_{\sigma_j}$. To understand the index of $\omega^{m+r-j+2}_{\sigma_j}$, it is better to take the viewpoint that σ_j keeps track of the indices of the faces that $\langle v_0, \dots, v_m, x_{\sigma_r}, \dots, x_{\sigma_j} \rangle$ is part of. The reason for this is our labeling convention and the anti-symmetry of the forms ω^p. Let us consider a simple example to understand why this is so, before we formulate the corresponding proposition:

2.5.16 EXAMPLE: Assume we try to compute the local holonomy of a simplex that contains a face like $\langle v_0, v_1, x_{\sigma_{ijk}}, x_{\sigma_{ij}}, x_{\sigma_i} \rangle$. Our convention 2.5.15 above assigns the (only) label i to this face, so it's the form ω_i that has to be integrated over it. The subfaces are $\langle v_0, v_1, x_{\sigma_{ij}}, x_{\sigma_i} \rangle$, $\langle v_0, v_1, x_{\sigma_{ijk}}, x_{\sigma_{ij}} \rangle$, $\langle v_0, v_1, x_{\sigma_{ijk}}, x_{\sigma_i} \rangle$, having labels i, j and i (we do not bother with the signs here and have dropped the two boundary simplices $-\langle v_0, x_{\sigma_{ijk}}, x_{\sigma_{ij}}, x_{\sigma_i} \rangle$ and $\langle v_1, x_{\sigma_{ijk}}, x_{\sigma_{ij}}, x_{\sigma_i} \rangle$, which obviously do not contribute by our convention). Now the local holonomy formula tells us that we have to integrate $\omega^2_{ii}, \omega^2_{ij}$ and ω^2_{ii} over these three faces respectively. Obviously, because of the anti-symmetry of the forms ω^2, the only one contribution that survives is just the integral over $\langle v_0, v_1 x_{\sigma_{ijk}}, x_{\sigma_{ij}} \rangle$, and we have just to integrate ω^2_{ij}. This makes clear, why the last midpoint of any simplex keeps track of the indices of the labeling.

We will be only interested in the logarithm of the holonomy, Then, if we also take into account the anti-symmetry of the forms ω^p, the holonomy takes the following form:

2.5.17 PROPOSITION: *Let $f = \langle v_0, \dots, \hat{v}_j, \dots, v_p, x_{\sigma_r}, \dots, x_{\sigma_1} \rangle$ be a general simplex of the d-loop $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$. Then the contribution of f to $\log \mathrm{hol}(s^{v_0,\dots,v_p}_{i_1,\dots,i_r})$ is given by:*

$$\int_{\langle v_0,\dots,\hat{v}_j,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_1}\rangle} \omega^d_{\sigma_1} + \int_{\langle v_0,\dots,\hat{v}_j,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_2}\rangle} \omega^{d-1}_{\sigma_2} + \dots + \int_{\langle v_0,\dots,\hat{v}_j,\dots,v_p,x_{\sigma_r}\rangle} \omega^{d-r+1}_{\sigma_r}.$$

The lemma is also true for a simplex of the form $\langle v_0, \dots, v_p, \hat{x}_{\sigma_r}, x_{\sigma_{r-1}}, \dots, x_{\sigma_1} \rangle$, with the obvious modifications. (Notice again, that we do pretend f being a submanifold of M)

Proof. We have only to check that every subface apart from those in the formula vanish. Any subface of $\langle v_0, \ldots, \hat{v}_j, \ldots, v_p, x_{\sigma_r}, \ldots, x_{\sigma_1}\rangle$ of the form $\langle v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_p, x_{\sigma_r}, \ldots, x_{\sigma_j}\rangle$ does not contribute, for by the convention 2.5.15 its label is just the same as of the face itself. Also, any subface $\langle v_0, \ldots, \hat{v}_j, \ldots, v_p, x_{\sigma_r}, \ldots, \hat{x}_{\sigma_k}, \ldots, x_{\sigma_j}\rangle$ has the same label as $\langle v_0, \ldots, \hat{v}_j, \ldots, v_p, x_{\sigma_r}, \ldots, x_{\sigma_j}\rangle$. This follows from the fact that the label is governed by the label of σ_j, or more precise, from $\sigma_j \subset \sigma_k$, and so $U_{\sigma_k} \subset U_{\sigma_j}$ and only the label σ_j is important. By repeating this argument, one obtains this formula. Note, that since we have only r indices at our disposal, this procedure stops after r steps. □

Even for a general subsimplex, not all the summands of the proposition may yield a contribution. For clearification of this we give another easy example:

2.5.18 EXAMPLE: Suppose we want to compute the contribution of the simplex $f := \langle v_0, x_{ijklm}, x_{ijkm}, x_{ijk}, x_{jk}, x_j\rangle$ to the holonomy. With the convention above we would assign to $\langle v_0, x_{ijklm}, x_{ijkm}, x_{ijk}, x_{jk}, x_j\rangle$ the index j, to $\langle v_0, x_{ijklm}, x_{ijkm}, x_{ijk}, x_{jk}\rangle$ the index k and to $\langle v_0, x_{ijklm}, x_{ijkm}, x_{ijk}\rangle$ also the index k. Thus, the last simplex, and also every other subface, does not yield any contribution. The contribution to the holonomy is therefore just

$$\int_{\langle v_0, x_{ijklm}, x_{ijkm}, x_{ijk}, x_{jk}, x_j\rangle} \omega_j^5 - \int_{\langle v_0, x_{ijklm}, x_{ijkm}, x_{ijk}, x_{jk}\rangle} \omega_{jk}^4.$$

With this in mind, we can tackle the injectivity. We first start by a simple example to make clear the idea behind the construction that follows.

2.5.19 EXAMPLE: Suppose $d = 2$. To verify injectivity, we compute for a given Deligne 2-class its holonomy hol_ω along the 2-loop $s_{ij}^{y,v}$. Following the proposition 2.5.17 and dropping all vanishing contributions, we are led to the formula

$$\log \mathrm{hol}(s_{ij}^{y,v}) = \int_{\substack{-\langle v, x_{ij}, x_i\rangle \\ \langle y, x_{ij}, x_i\rangle \\ -\langle y, v, x_i\rangle}} B_i + \int_{\substack{\langle v, x_{ij}, x_j\rangle \\ -\langle y, x_{ij}, x_j\rangle \\ \langle y, v, x_j\rangle}} B_j + \int_{\substack{-\langle v, x_{ij}\rangle \\ \langle y, x_{ij}\rangle \\ -\langle y, v\rangle}} A_{ij} + R,$$

Some remarks are due: First, R is the contribution from the recentering. We do not have the need to specify it further, for it does not contain all the vertices v_i, but at most only a subset of these (actually it does not contain any v_i by definition). Since we are going to differentiate the holonomy with respect to all directions t_i, this extra-term will vanish, so R will play no role in the computations to come, and will be neglected.

Now we may define two differential forms $(\eta^0)_{ij}$ and $(\eta^1)_i(v)$ by

$$(\eta^0)^y_{ij} := \exp\left(\int_{\langle y,x_{ij},x_i\rangle} B_i + \int_{-\langle y,x_{ij},x_j\rangle} B_j + \int_{\langle y,x_{ij}\rangle} A_{ij}\right)$$

and

$$(\eta^1)^y_i(v) := \frac{d}{dt}\int_{\langle y,v,x_i\rangle} B_i\Big|_{t=0}.$$

Let us calculate $d\log(\eta^0)^y_{ij}$. By definition this is nothing else than

$$d\log(\eta^0)^y_{ij}(v) = \frac{d}{dt}\left(\int_{\langle v(t),x_{ij},x_i\rangle} B_i + \int_{-\langle v(t),x_{ij},x_j\rangle} B_j + \int_{\langle v(t),x_{ij}\rangle} A_{ij}\right)\Bigg|_{t=0}.$$

Assume now, that the holonomy of ω does vanish, so that $\log\mathrm{hol}(s^{y,v}_{ij})$ is zero. We may differentiate the equation for hol and use the above calculations to directly obtain

$$0 = \frac{d}{dt}\log\mathrm{hol}(s^{y,v}_{ij})\big|_{t=0} = -d\log(\eta^0)^y_{ij}(v) + \delta(\eta^1)_i(v) + \frac{d}{dt}\int_{-\langle y,v\rangle} A_{ij}\Big|_{t=0}.$$

which is nothing else than

$$A_{ij} = -d\log\eta^0 + \delta\eta^1.$$

This shows that if the holonomy vanishes, A_{ij} can be realized as $D\eta$. The other two remaining relations can be proven in the same fashion.

Compare this with the definitions of Mackaay-Picken, part 3 of theorem 8.1 [19]. First of all, in their proof, Mackaay and Picken do not assume the holonomy map corresponding to the given Deligne d-class ω to vanish, but to induce another Deligne class ω'. This alters the formulas a bit. Still one can find the same definition of the form η^1, which they call B_i. It is being integrated over the 2-path $C_i(k)$, which is simply $\langle y, v, x_i\rangle$ (see figure 14 of [19]). The holonomy formula above can be found on the end of the page 333, and one immediately recognizes the formula of η^0 in it, which we introduced as an explicit form.

Generalizing this example, we first have to reexpress the holonomy of the d-loop $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$. Using the local formula above we find:

$$\log\mathrm{hol}_\omega(s^{v_0,\dots,v_p}_{i_1,\dots,i_r}) = (-1)^{rp}(\sum_{\substack{\sigma_{r-1}\subset\sigma_r=(i_1,\dots,i_r)\\|\sigma_{r-1}|=r-1}}\left[\sum_{\substack{\sigma_{r-2}\subset\sigma_{r-1}\\|\sigma_{r-2}|=r-2}}\left[\dots\left[\sum_{\substack{\sigma_2\subset\sigma_1\\|\sigma_1|=1}} A^1_{p,r}\right]+\dots\right]+A^{r-1}_{p,r}\right]+A^r_{p,r}),$$

where this time we dropped the contribution R that corresponds to the recentering, and which depends only on the indices i_j.

In light of the proposition 2.5.17 one might say that $A^q_{p,r}$ is (the integral of) the collection of all simplices having q midpoints attached to them. The concrete formula for $A^q_{p,r}$ is given by a sum, reflecting the two different kinds of simplices in $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$:

$$\begin{aligned}
A^q_{p,r} &:= D^q_{p,r} + \tilde{D}^q_{p,r}\\
D^q_{p,r} &:= \int_{(-1)^{(q-1)(p+r+q)}(-1)^s \sum_{j=0}^p (-1)^j \langle v_0,\dots,\hat{v}_j,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_q}\rangle} \omega^{d-q+1}_{\sigma_q}\\
\tilde{D}^q_{p,r} &:= \int_{(-1)^{(q-1)(p+r+q)}(-1)^s(-1)^{p+1}\langle v_0,\dots,v_p,\hat{x}_{\sigma_r},x_{\sigma_{r-1}}\dots,x_{\sigma_q}\rangle} \omega^{d-q+1}_{\sigma_q}.
\end{aligned}$$

Notice, that the formula does not involve any ω^j for $j < d-r+1$, hence the last term is $A^r_{p,r}$, which only involves $\omega^{d-r+1} = \omega^p$. This follows directly from the anti-symmetry, as already seen in the example, since we only have $r = d-p+1$ indices at hand. Furthermore we pulled the global sign $(-1)^{rp}$ out of the integrals, which will slightly ease the calculations to come.

To understand the signs in $D^q_{p,r}$ and $\tilde{D}^q_{p,r}$ respectively, one needs to understand from which simplex $\langle v_0,\dots,v_p,\hat{x}_{\sigma_r},\dots,x_{\sigma_q}\rangle$ comes from. We do this only for $\tilde{D}^q_{p,r}$, for the argument for $D^q_{p,r}$ is quite the same. Suppose $\sigma_q = (j_1,\dots,j_q)$. Now, the only simplex in question is $\langle v_0,\dots,v_p,\hat{x}_{\sigma_r},\dots,x_{\sigma_q},x_{\tilde{\sigma}_{q-1}},\dots,x_{\tilde{\sigma}_1}\rangle$, where we define $\tilde{\sigma}_t$ as $\tilde{\sigma}_t := (j_1,\dots,j_t)$, with $0 < t < q$. Every other has vanishing contribution, because of the convention we have adopted. For if $\tilde{\sigma}_{q-1}$ were different from $(j_1,\dots,j_{q-1})$, it must contain the index j_q, whence the label of $\tilde{\sigma}_{q-1}$ would be the same as that of σ_q, and the contribution of the simplex would indeed vanish. Repeating this argument proves our claim.

Denote now the sign of $(\sigma_r,\dots,\sigma_q)$ by $s = s(\sigma_r,\dots,\sigma_q)$. Then the sign of the simplex $\langle v_0,\dots,v_p,\hat{x}_{\sigma_r},\dots,x_{\sigma_q},x_{\tilde{\sigma}_{q-1}},\dots,x_{\tilde{\sigma}_1}\rangle$ is given by $(-1)^{(p+1)+s} \cdot (-1)^{q-1}(-1)^{q-2}\cdots(-1)^1$. Here $(p+1)$ comes from having dropped x_{σ_r}, s from the sign of the family and the product $(-1)^{q-1}\dots(-1)^1$ represents the sign of the family $(\tilde{\sigma}_{q-1},\dots,\tilde{\sigma}_1)$. Altogether the simplex has the sign $(-1)^{(p+1)+s+\sum_{m=1}^{q-1} q-m}$.

From this, let us determine the sign of the simplex $\langle v_0,\dots,v_p,\hat{x}_{\sigma_r},\dots,x_{\sigma_q}\rangle$. Since it is obtained from the simplex above by dropping the rear midpoints, the sign gets multiplied by $(-1)^{p+r-1}\cdots(-1)^{p+r+q-1}$, that is by $(-1)^{\sum_{m=1}^{q-1} p+r-m}$, so the correct sign for the simplex is

$$(-1)^{(p+1)+s+(q-1)(p+r+q)},$$

which is the reason for the sign in $\tilde{D}^q_{p,r}$. Note again, that the global sign $(-1)^{rp}$ of $s^{v_1,\dots,v_p}_{i_1,\dots,i_r}$ has been pulled outside the holonomy and does not play any role yet.

Since we will have to differentiate the holonomy p times, we need the following, most trivial lemma:

2.5.20 LEMMA: *The p-th derivative of $D^q_{p,r}$ is given by*

$$\frac{\partial^p}{\partial t_1 \dots \partial t_p} D^q_{p,r} = \frac{\partial^p}{\partial t_1 \dots \partial t_p} \int_{(-1)^{(q-1)(p+q+r)+s}\langle v_1,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_q}\rangle} \omega^{d-q+1}_{\sigma_q}.$$

According to our example above we define differential forms η^p by the same formula as the holonomy,

$$(\eta^p)_{i_1,\dots,i_r}(v_0,\dots,v_p) := (-1)^{rp+1} \frac{\partial^p}{\partial t_1 \dots \partial t_p} \eta_{p,r}$$

with

$$\eta_{p,r} := \left(\sum_{\substack{\sigma_{r-1}\subset\sigma_r=(i_1,\dots,i_r)\\ |\sigma_{r-1}|=r-1}} \left[\dots \left[\sum_{\substack{\sigma_1\subset\sigma_2\\ |\sigma_1|=1}} \tilde{A}^1_{p,r} \right] + \dots \right] + \tilde{A}^r_{p,r} \right) \Bigg|_{t_i=0},$$

but put in some extra sign $(-1)^{rp+1}$, which we will need later on, and replace $A^q_{p,r}$ with the term $\tilde{A}^q_{p,r}$, which is given by

$$\tilde{A}^q_{p,r} := \int_{(-1)^{(q-1)(p+q+r+1)+s}\langle v_0,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_q}\rangle} \omega^{p+r-q+1}_{\sigma_q}.$$

Notice that the dimension of the simplex $\langle v_0,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_1}\rangle$ is indeed $p+r-q+1$, and so does differ from the dimension in the formula for $D^q_{p,r}$ and $\tilde{D}^q_{p,r}$. We will only need these for $\tilde{A}^q_{p,r-1}$ and $\tilde{A}^q_{p-1,r}$, where the dimensions match again.

Next we need to take the exterior derivative of η^{p-1}. Because partial derivatives commute, its enough to just consider $\tilde{A}^q_{p-1,r}$:

2.5.21 LEMMA: *We have*

$$\begin{aligned} d\left(\frac{\partial^{p-1}}{\partial t_1 \cdots \partial t_{p-1}} \tilde{A}^q_{p-1,r}\right)(v_1,\dots,v_p) &= \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \int_{(-1)^{(q-1)((p-1)+q+r+1)+s}\langle v_1,\dots,v_p,x_{\sigma_r},\dots,x_{\sigma_q}\rangle} \omega^{(p-1)+r-q+1}_{\sigma_q} \\ &= \frac{\partial^p}{\partial t_1 \cdots \partial t_p} D^q_{p,r}. \end{aligned}$$

Proof. We have $(p-1)+r-q+1 = d-q+1$, and if we use the definition of d to compute the left hand side, we get

$$\frac{1}{p}\left\{\sum_{j=1}^{p}(-1)^{j-1}\partial_{v_j}\frac{\partial^{p-1}}{\partial t_1\ldots,\hat{\partial} t_j\ldots\partial t_p}\int_{(-1)^{(q-1)((p-1)+q+r+1)+s}\langle v_0,v_1,\ldots,\hat{v}_j,\ldots,v_p,x_{\sigma_r},\ldots,x_{\sigma_q}\rangle}\omega_{\sigma_r}^{d-q+1}\right\}$$
$$=\frac{\partial^p}{\partial t_1\ldots,\partial t_p}\frac{1}{p}\left\{\sum_{j=1}^{p}(-1)^{j-1}\int_{(-1)^{(q-1)(p+q+r)+s}\langle v_j,v_1,\ldots,\hat{v}_j,\ldots,v_p,x_{\sigma_r},\ldots,x_{\sigma_q}\rangle}\omega_{\sigma_r}^{d-q+1}\right\}$$
$$=\frac{\partial^p}{\partial t_1\ldots,\partial t_p}\int_{(-1)^{(q-1)(p+q+r)+s}\langle v_1,\ldots,v_j,\ldots,v_p,x_{\sigma_r},\ldots,x_{\sigma_q}\rangle}\omega_{\sigma_r}^{d-q+1},$$

where in the last step we used the fact that moving v_j to the right cancels exactly the sign $(-1)^{j-1}$. But this is just the only one term that survives, if we differentiate $D^q_{p,r}$ in all p directions, for the other terms do miss at least one v_j and hence vanish, as we have seen in the lemma above. □

Let us split the holonomy into two parts, so that we have $\log\operatorname{hol}(s) = (-1)^{rp}(D_{p,r} + \tilde{D}_{p,r})$. These two parts we will be able to identify shortly. Notice, that we do not include the global sign into $D_{p,r}$ and $\tilde{D}_{p,r}$.

2.5.22 DEFINITION: Denote by $D_{p,r}$ and $\tilde{D}_{p,r}$, the sum

$$D_{p,r} = \sum_{\substack{\sigma_{r-1}\subset\sigma_r=(i_1,\ldots,i_r)\\|\sigma_{r-1}|=r-1}}\left[\sum_{\substack{\sigma_{r-2}\subset\sigma_{r-1}\\|\sigma_{r-2}|=r-2}}\left[\ldots\left[\sum_{\substack{\sigma_2\subset\sigma_1\\|\sigma_1|=1}}D^1_{p,r}\right]+\ldots\right]+D^{r-1}_{p,r}\right]+D^r_{p,r},$$

and analogously for $\tilde{D}_{p,r}$, so that we have a splitting

$$\log\operatorname{hol}(s^{v_0,\ldots,v_p}_{i_1,\ldots,i_r}) = (-1)^{rp}(D_{p,r}+\tilde{D}_{p,r}).$$

2.5.23 COROLLARY: $d(\eta^{p-1})(v_1,\ldots,v_p) = (-1)^{r(p-1)+1}\frac{\partial^p}{\partial t_1\cdots\partial t_p}D_{p,r}$.

This corollary finishes the identification of (one part of) the holonomy and $d\eta^{p-1}$. We still have to identify the other part. Looking closer at $\tilde{D}_{p,r}$, we see that it is nearly the same as $\delta\eta^p$. Apart from sign-issues, there is only one summand more in $\tilde{D}_{p,r}$, which is the contribution from $\langle v_0,\ldots,v_p\rangle$. Let us determine its sign, before we pour this fact into a lemma. By definition, we have to look at $\tilde{D}^r_{p,r}$, so the simplex with its correct sign is $(-1)^{(r-1)(p+r-r)+s+(p+1)}\langle v_0,\ldots,v_p\rangle = (-1)^{rp+1}\langle v_0,\ldots,v_p\rangle$. Here we used that the sign s of the simplex $\langle v_0,\dots,v_p\rangle$ is just 1 for there are no midpoints involved. Let us now compare $\delta\eta^p$ and $\tilde{D}_{p,r}$. Both involve the same simplex $\langle v_0,\dots,v_p,x_{\sigma_{r-1}},\dots,x_{\sigma_q}\rangle = \langle v_0,\dots,v_p,\hat{x}_{\sigma_r},x_{\sigma_{r-1}},\dots,x_{\sigma_q}\rangle$. But they differ in signs. To see this, notice that the sign of this simplex in $\tilde{A}^q_{p,r-1}$ is $(-1)^{(q-1)+(p+q+r)+s(\sigma_{r-1},\dots,\sigma_q)}$, where we exhibited the dependence of the sign s on the various σ_i. Contrary, the sign in $\tilde{D}^q_{p,r}$ is $(-1)^{(q-1)+(p+q+r)+s(\sigma_r,\dots,\sigma_q)+(p+1)}$. Assume for a moment, that σ_{r-1} is obtained from σ_r by dropping the j^{th} index. Then we have the relation $s(\sigma_{r-1},\dots,\sigma_q) = s(\sigma_r,\dots,\sigma_q)+j-1$. Hence both simplices differ by the sign $(-1)^{(p+1)+(j-1)}$. While in $\tilde{D}_{p,r}$ the sum is over all possible $\sigma_{r-1}\subset\sigma_r$, in $\tilde{A}^q_{p,r-1}$ we have only one summand, with $i_1,\dots,i_{r-1}$. But taking δ of $\tilde{A}^q_{p,r-1}$ with respect to $i_1,\dots,i_r$, on the one hand we get all the same simplices as in $\tilde{D}^q_{p,r}$, together with an extra sign $(-1)^{j-1}$, coming from the definition of δ. Altogether, apart from the global sign in $\delta\eta^{p-1}$, the expressions for $\delta\eta^{p-1}$ and $D_{p,r}$ differ only by the sign $(-1)^{p+1}$. Taking this global sign into account, these considerations prove the following lemma:

2.5.24 COROLLARY:

$$(-1)^{(p+1)+(r-1)p+1}\delta\eta^p(v_1,\dots,v_p)+(-1)^{rp+1}\frac{\partial^p}{\partial t_1\cdots\partial t_p}\int_{\langle v_0,\dots,v_p\rangle}\omega^p_{i_1,\dots,i_r} = \frac{\partial^p}{\partial t_1\cdots\partial t_p}\tilde{D}_{p,r}.$$

Now take the p-th derivative of the holonomy formula. We can directly use the upper two lemma to obtain:

$$\begin{aligned}
0 &= \frac{\partial^p}{\partial t_1\cdots\partial t_p}|_{t_i=0}\log\mathrm{hol}(s^{v_0,\dots,v_p}_{i_1,\dots,i_r})\\
&= \frac{\partial^p}{\partial t_1\cdots\partial t_p}|_{t_i=0}(-1)^{rp}(D_{p,r}+\tilde{D}_{p,r})\\
&= (-1)^{rp}[(-1)^{r(p-1)+1}d\eta^{p-1}+(-1)^{rp}\delta\eta^p+(-1)^{rp+1}\omega^p](v_1,\dots,v_p)\\
&= (-1)^{r-1}d\eta^{p-1}(1_0,\dots,v_p)+\delta\eta^p(v_1,\dots,v_p)-\omega^p(v_1,\dots,v_p)
\end{aligned}$$

where we again used that

$$\frac{\partial^p}{\partial t_1\cdots\partial t_p}\int_{\langle v_0,\dots,v_p\rangle}\omega^p_{i_1,\dots,i_r}\Big|_{t_i=0} = \omega^p(v_1,\dots,v_p).$$

2.5.25 COROLLARY: *We have* $\omega^p = (-1)^{r-1}d\eta^{p-1}+\delta\eta^p$.

Altogether by means of a direct construction we have fully established the following fact:

2.5.26 THEOREM: $\mathrm{HOL}:\check{H}^d(M,\mathcal{D}^d)\longrightarrow \mathrm{Hom}^\infty(\pi^d_d(M),U(1))$ *is an isomorphism of groups.*

2.6 Flat Deligne classes

Let us come back to the flat case. We have seen, that there is a bijective correspondence between flat $U(1)$-bundles over M and homomorphisms from the first homotopy group of M to the group $U(1)$, see theorem 2.1.1 (see also for example Brylinski's book [4] for an overview)

$$\left\{\begin{matrix}\text{Isomorphism classes of} \\ U(1)\text{-principal fibre bundles with flat connection}\end{matrix}\right\} \cong \mathrm{Hom}^\infty(\pi_1(M), U(1)).$$

We would like to extend this bijection to higher Deligne classes by using the results of the last sections. For this we first have to find a generalization of flatness:

2.6.1 DEFINITION: Let ξ be a Deligne d-class over a manifold M. We call ξ **flat**, iff the curvature $\mathrm{curv}(\xi)$ of ξ is zero.

One could also define a flat Deligne class to be a cocycle of the non-truncated Čech-Deligne complex, but it's easier to adopt the extra condition above.

As any trivial Deligne d-class ξ has a special form, $\xi = [(1, 0, \dots, 0, \delta(\rho)]$ for some $\rho \in \Omega^d(M)$ (see lemma 1.2.5), any flat Deligne d-class also has a special representative:

2.6.2 LEMMA: *If ξ is a flat Deligne d-class on M, then ξ has a representative of the form*

$$\xi = [(\xi^0_{i_1,\dots,i_{d+1}}, 0, \dots, 0)].$$

Obviously any ξ of this form has vanishing curvature, and hence is flat.

Proof. Consider first the easiest case $d = 1$. Then $\xi = [(g_{ij}, A_i)]$. Since we know that $dA_i = 0$, by Poincare's Lemma and the assumption that U_i is contractible, we can find some $h_i \in \Omega^0(U_i)$ such that $d \log h_i = A_i$. Subtract this Deligne 0-class from ξ to obtain $\xi = [(g_{ij}, A_i)] - [(\delta h_i, d \log h_i)] = [(g_{ij} - \delta h_i, 0)] =: [(\omega^0_{ij}, 0)]$. Hence ξ has a representative of the desired form. If ξ is a Deligne d-class with d arbitrary, one applies Poincare's Lemma again and again to arrive at a representative of the form as above: If ξ is of the form $\xi = [(\xi^0_{i_1,\dots,i_{d+1}}, \dots, \xi^{d-1}_{ij}, \xi^d_i)]$, then we can find h^d_i with $dh^d_i = \xi_i$. Subtracting it from ξ gives us $\xi = [(\xi^0_{i_1,\dots,i_{d+1}}, \dots, \xi^{d-1}_{ij} - \delta h_i, 0)]$. To apply this step again, we have to know that $d(\xi^{d-1}_{ij} - \delta h^d_i) = 0$. But

$$d(\xi^{d-1}_{ij} - \delta h^d_i) = d\xi^{d-1}_{ij} - \delta dh^d_i = d\xi^{d-1}_{ij} - \delta \xi^d_i.$$

Since ξ is a cocycle, this sum is zero, and we can indeed repeat the step above (Note that we have made two sign errors, which cancel). □

Suppose now that we are given a flat Deligne d-class over M. We can still define its holonomy, and obtain a holonomy map from $\pi_d^d(M)$ to $U(1)$. Looking at the case of prinicipal fibre bundles, one can guess, that this holonomy map factors over $\pi_d(M)$. Indeed, this is true and quite easy to show:

2.6.3 PROPOSITION: *For any flat Deligne d-class ξ over M the holonomy map* $\mathrm{hol}_\xi : \pi_d^d(M) \longrightarrow U(1)$*, factors over $\pi_d(M)$, e.g. the following diagram commutes:*

$$\begin{array}{ccc} \pi_d^d(M) & \xrightarrow{\quad \mathrm{hol}_\xi \quad} & U(1). \\ & {\scriptstyle \mathrm{pr}} \searrow \qquad \nearrow & \\ & \pi_d(M) & \end{array}$$

Here the projection maps a thin homotopy class to its homotopy class (see definition 2.2.4).

Proof. We need to show that any two d-loops $s, s' \in \pi_d^d(M)$, which are homotopic (relative to the basepoint), that is $\mathrm{pr}(s) = \mathrm{pr}(s')$, have the same holonomy with respect to ξ. Let H be such a homotopy $H : I^d \times I \longrightarrow M$. By the very definition H is a map, that is equal to s and s' at the top and the bottom of I^{d+1} and to the constant map $I^d \mapsto \star$ at the sides of I^{d+1}. The definition of the holonomy (see definition 1.2.6) together with the fact, that ξ is a flat Deligne class, now leads directly to:

$$1 = \exp(\int_{I^{d+1}} H^* \operatorname{curv}(\xi)) = \exp(\int_{I^d} s^* \rho - \int_{I^d} s'^* \tilde{\rho}) = \mathrm{hol}_\xi(s) \, \mathrm{hol}_\xi(s')^{-1},$$

where we used the same steps as in the proof of proposition 2.2.5. Hence any homotopic d-loops s and s' have the same holonomy map and the proposition is proved. □

So we have defined a map from the flat Deligne d-classes to the smooth homomorphisms of the form $\pi_d(M) \longrightarrow U(1)$. We can now easily show the following theorem.

2.6.4 THEOREM: *Suppose that M is a $(d-1)$-connected, oriented manifold. Then there is a group isomorphism between flat Deligne d-classes and smooth homomorphisms of the form $\pi_d(M) \longrightarrow U(1)$.*

Proof. We only have to show that any smooth homomorphism $h : \pi_d(M) \longrightarrow U(1)$ defines a flat Deligne d-class. Observe that h induces a well-defined map $\tilde{h} : \pi_d^d(M) \longrightarrow U(1)$ by composing it with the projection $\mathrm{pr} : \pi_d^d(M) \longrightarrow \pi_d(M)$. Using the reconstruction theorem (see theorem 2.5.26), we obtain a Deligne d-class ξ. We claim that this class is flat. By the lemma above it's enough to show that ξ has the form $\xi = [(\xi_{i_1,\ldots,i_{d+1}}, 0, \ldots, 0)]$.

For this reconsider the definition of the p-forms ω^p, with $p > 0$ (see remark after definition 2.5.6):

$$(\omega^p_{i_1,\dots,i_r})_{v_0}(v_1,\dots,v_p) = \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \log \tilde{h}(s^{v_0,v_1(t_1),\dots,v_p(t_p)}_{i_1,\dots,i_r})\Big|_{(t_1,\dots,t_p)=0}.$$

Now any loop of the form $s^{v_0,v_1(t_1),\dots,v_p(t_p)}_{i_1,\dots,i_r}$ can be first shrinked and then drawn back to some arbitrary, but fixed d-loop at the basepoint $\star$. This shriking and moving does not change the homotopy of the d-loop, hence we know, that $\tilde{h}$ applied to any such loop is constant, because $\tilde{h}$ factors over $\pi_d(M)$. Since this does neither depend on the tangential vectors used in the construction of ω^p nor at the point $y \in M$, the p-forms ω^p are constant (for all $p > 0$). Since we know that $\tilde{h}$ of the constant loop at $\star$ is $1 \in U(1)$, we further know that all ω^p-forms for $p > 0$ are 0. Hence ξ has the form as above, that is, ξ is flat. □

2.7 Connectedness and Gajer's Theorem

2.7.1 Gajer's Theorem

In our reconstruction we have assumed that the manifold M is highly connected, to be more precise, to reconstruct a Deligne d-class, we needed M to be $(d-1)$-connected. Gajer, in his paper [10], has a very similar result. He proves an analogue of our reconstruction within the piecewise, not the smooth realm. By doing so, he also does not assume M to be more than just connected. Consequently one might ask in how far our reconstruction theorem does differ from Gajers. We will show that there is indeed a difference. The price for the more general theorem of Gajer is that to reconstruct the Deligne class one has to know *all* holonomies, not just those of based loops. In our approach, by assuming the connectedness of the manifold M, based loops already provide us the information needed to reconstruct the corresponding Deligne class.

Let us give a very easy example that connectedness is necessary, for the case $d = 1$. Consider the disjoint union of two circles, $M = S^1 \amalg S^1$, and choose a basepoint in the first S^1 component. Then the thin homotopy of M is given by $\pi_1^1(S^1 \amalg S^1) = \pi_1^1(S^1) \cong \pi_1(S^1) \cong \mathbb{Z}$, since the homotopy sees only the component of the basepoint, and thin d-homotopy is the same as d-homotopy, if the dimension of M is less or equal to d. If our reconstruction theorem was true, we would expect that the isomorphism classes of $U(1)$-fibre bundles with connection over M correspond to $\mathrm{Hom}^\infty(\pi_1^1(M), U(1)) = \mathrm{Hom}^\infty(\mathbb{Z}, U(1)) \cong U(1)$. But obviously, this just measures the isomorphism classes of bundles on the one S^1-component. Since we know our theorem is true for $M = S^1$, we know that there are $U(1) \oplus U(1)$ Deligne 1-classes over $S^1 \amalg S^1$. This shows, that our reconstruction theorem fails in the case $d = 1$, if M is not connected.

In the same spirit one can construct non-connected counterexamples for any d, just by taking the disjoint union of two manifolds which carry non-trivial Deligne d-classes.

We would like to have better counterexamples, where M just slightly fails to be $(d-1)$-connected, i.e. M will be $(d-2)$-connected, but not $(d-1)$ connected. Such manifolds are known in algebraic topology, and a well-known example is the class of Stiefel manifolds, which we will deal with in the next sections. We will see later on that spheres are also Stiefel manifolds, and since these are the very basic objects, we begin with these.

2.7.2 Connectedness: Spheres

The counterexamples will be based upon the flat case (see theorem 2.6.4) for one can work with this much better than with the general case, because one does not have to deal with the full thin homotopy group, which is often infinite-dimensional: As we have seen, the flat case only involves the homotopy groups of the base manifold M, which are often already computed in the literature. In our case we will start with the class of spheres, which already provide counterexamples, and then analyze the Stiefel manifolds, whose homotopy groups as well as the homology and cohomology groups are quite well known. These provide a more general class of counterexamples than the spheres.

Now, if we show that the reconstruction fails in the flat case, by providing a non-trivial flat Deligne classes whose holonomy vanishes, obviously also the more general reconstruction result is proven to be wrong for M not highly-connected enough. This is why we can restrict ourselves just to the flat case.

First, let us state the theorem of Gajer that generalizes (at the price of having to calculate more maps) our reconstruction theorem for flat Deligne-classes (see theorem 2.6.4):

2.7.1 THEOREM: *([10, p. 198]) Let M be any oriented, connected, smooth manifold. Then there is a group isomorphism*

$$\check{H}^d_{flat}(M, \mathcal{D}^d) \cong \mathrm{Hom}^\infty(H_d(M, \mathbb{Z}), U(1)).$$

It's important to note that our result follows in the flat case directly from Gajers, if we assume M to be $(d-1)$-connected. This is a consequence of the Hurewicz theorem, which states that for any $(d-1)$-connected manifold the d^{th} homotopy and homology groups are isomorphic, $H_d(M, \mathbb{Z}) \cong \pi_d(M)$. Gajers result then is equivalent to ours.

Because of this, to find a counterexample, the easiest way is to use Gajers result to disprove our theorem: If we know that M is $(d-2)$-connected, but not $(d-1)$-connected and fails to have $H_d(M) \cong \pi_d(M)$, then by Gajers theorem the group of flat Deligne d-classes is not $\mathrm{Hom}^\infty(\pi_d(M), U(1))$, but $\mathrm{Hom}^\infty(H_d(M), U(1))$.

Remark, that if M is only $(d-2)$-connected, we still have an isomorphism $\mathrm{Hom}^\infty(H_{d-1}(M), U(1)) \cong \mathrm{Hom}^\infty(\pi_{d-1}(M), U(1))$. If M has the additional property

$H_d(M) \cong H_{d-1}(M)$, then we have $\mathrm{Hom}^\infty(\pi_{d-1}(M), U(1)) \cong \mathrm{Hom}^\infty(H_d(M), U(1))$ and our theorem would be true even if M is only $(d-2)$-connected.

The easiest case of such a manifold is the $(d-1)$-dimensional sphere S^{d-1}. This manifold is indeed $(d-2)$-connected, as we know from algebraic topology, since $\pi_i S^{d-1} \cong 0$ for all $i < d-1$, but $\pi_{d-1} S^{d-1} \cong \mathbb{Z}$. By the above considerations, we need to get a grip on the homotopy $\pi_d(S^{d-1})$. Luckily this problem has already been solved in stable homotopy theory, and the result is as follows:

2.7.2 PROPOSITION: *(e.g. [25, 1.1.6]) For every $d > 3$ the homotopy groups $\pi_d(S^{d-1})$ are isomorphic to $\mathbb{Z}_2$.*

Hence, for $d \geq 4$ the counterexample is simply the sphere S^{d-1}, with homotopy groups $\pi_{d-2}(S^{d-1}) \cong 0$, $\pi_{d-1}(S^{d-1}) \cong \mathbb{Z}$ and $\pi_d(S^{d-1}) \cong \mathbb{Z}_2$, for $H_d(S^{d-1}) = 0$.

Even for $d = 3$ the sphere S^2 does provide a counterexample, since the unstable homotopy $\pi_3 S^2$ is isomorphic to $\mathbb{Z}$, but $H_3 S^2 \cong 0$.

Unluckily, since the higher homotopy groups of the circle vanish, the result is not true for the case $d = 2$, and we have to find other highly-connected manifolds with the needed property. One class are the Stiefel manifolds, which we will discuss in the next section.

2.7.3 Connectedness: Stiefel manifolds

As we have seen, a counterexample for the case $d = 2$ cannot be constructed by considering spheres alone. To remedy this, we introduce the Stiefel manifolds $V_{n,k}$. We will be only interested in the case $k = 2$, that is we will only consider the Stiefel manifolds $V_{n,2}$, because these manifolds have all the properties we want, and will finally provide counterexamples for all even cases, including the open case $d = 2$. Unluckily for d odd we cannot use these manifolds, and have to refer back to the spheres (these are also Stiefel manifolds, namely $V_{n,1}$).

Now let us describe the Stiefel manifolds $V_{n,k}$ in detail. We begin by reminding the reader of the definition of these manifolds:

2.7.3 DEFINITION: For any $n \in \mathbb{N}$ and $k \leq n$ define $V_{n,k}$ to be the set of orthonormal k-tupels of vectors in $\mathbb{R}^n$, the **Stiefel-manifold** of k-frames in $\mathbb{R}^n$.

One can easily show, that $V_{n,k}$ with the induced topology of $\mathbb{R}^{nk}$ is a smooth, compact manifold of dimension $\dim V_{n,k} = nk - \frac{1}{2}k(k+1)$ (see for example [8, Ch.3, §2]). Before we begin computations, let us give two easy examples:

2.7.4 EXAMPLE: If $k = 1$, then $V_{n,1}$ consists of all vectors with length 1, so $V_{n,1} \cong S^n$. For $k = 2$ we can identify $V_{n,2}$ with the tangent space of S^{n-1}. This is a $2n-3$ dimensional manifold. Especially $V_{3,2}$ is the tangent space $TS^2 \cong \mathbb{R}P^3$, a three-dimensional manifold.

The homotopy, homology and cohomology of $V_{n,2}$ can be computed (at least partially), which is why these manifolds are interesting for us (the basic reference for this is [26]). First we look at the homology:

2.7.5 PROPOSITION: *[26, Ch. IV, 10.14] The homology groups of the Stiefel-manifolds $V_{n,2}$ are*

$$H_i(V_{r+2,2},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if } i = 0, r, r+1, 2r+1 \\ 0 & \textit{else} \end{cases}$$

for r even and

$$H_i(V_{r+2,2},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if } i = 0, 2r+1 \\ \mathbb{Z}_2 & \textit{if } i = r \\ 0 & \textit{else} \end{cases}$$

for r odd. Note, that these isomorphisms are not canonical.

2.7.6 EXAMPLE: The homology of $V_{3,2}$ is zero except for the groups $H_0(V_{3,2}) = H_3(V_{3,2}) \cong \mathbb{Z}$ and $H_1(V_{3,2}) \cong \mathbb{Z}_2$.

For completeness, we also write down the cohomology groups, which can be easily deduced from the homology by the universal coefficient theorem:

2.7.7 COROLLARY: *The homology groups of the Stiefel-manifolds $V_{n,2}$ are*

$$H^i(V_{r+2,2},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if } i = 0, r, r+1, 2r+1 \\ 0 & \textit{else} \end{cases}$$

for r even and

$$H^i(V_{r+2,2},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if } i = 0, 2r+1 \\ \mathbb{Z}_2 & \textit{if } i = r+1 \\ 0 & \textit{else} \end{cases}$$

for r odd. Note, that these isomorphisms are not canonical.

2.7.8 EXAMPLE: The cohomology of $V_{3,2}$ is zero except for the groups $H^0(V_{3,2}) = H^3(V_{3,2}) \cong \mathbb{Z}$ and $H^2(V_{3,2}) \cong \mathbb{Z}_2$.

2.7.9 PROPOSITION: *[26, Ch. IV, 10.12] The Stiefel-manifolds $V_{n,2}$ are $(n-3)$-connected for $n > 1$.*

2.7.10 PROPOSITION: *[26, Ch. IV, 10.13] The first non-trivial homotopy group of the Stiefel-manifolds $V_{n,2}$ is $\pi_{n-2}V_{n,2}$, which is (non-canonically) isomorphic to $\mathbb{Z}$ if n is even and to $\mathbb{Z}_2$ if n is odd.*

From this proposition it is clear, that these manifolds do have exactly the homotopy we need, if we want to construct counterexamples. Unluckily there is a little index shift, so to construct a counterexample for a Deligne 2-class, we have to consider $V_{3,2} = V_{2+1,2}$. This is now a connected, but not simply-connected manifold, and its homotopy group $\pi_1 V_{3,2}$ is isomorphic to $\mathbb{Z}_2$.

To use our reconstruction for flat Deligne-classes we need also to compute $\pi_{n-1}V_{n,2}$. By utilizing the remark that $V_{n,2}$ is homotopy equivalent to TS^{n-1}, we can easily compute these. First we need to relate the homotopy of the tangent space of S^n to the homotopy of S^n:

2.7.11 PROPOSITION: *The homotopy of the tangent space of S^n is isomorphic to the homotopy of S^n, $\pi_i(TS^n) \cong \pi_i(S^n)$ for all $i > 0$.*

Proof. Since $\mathbb{R}^n \longrightarrow TS^n \longrightarrow S^n$ is a fibration, we have by the long exact homotopy sequence (see [26, IV, (8.7)]) and $\pi_i(\mathbb{R}^n) = 0$ for all $i > 0$ the isomorphisms $\pi_i(TS^n) \cong \pi_i(S^n)$. □

In particular we have an isomorphism $\pi_{n-1}(V_{n,2}) \cong \pi_{n-1}(TS^{n-1}) \cong \pi_{n-1}(S^{n-1})$. Note, that the higher homotopies of $V_{n,2}$ are the homotopies of the sphere, which are unknown in general. Luckily we do not need them.

With these results, we can directly show that our reconstruction fails. Let us do this first in the missing case $d = 2$. Gajers result tells us, that the isomorphism classes of flat Deligne 2-classes are in bijection to $\mathrm{Hom}^\infty(H_2(V_{3,2}), U(1))$. But, as we have calculated, the second homology of $V_{3,2}$ vanishes. Hence there are no flat, non-trivial Deligne 2-classes on $V_{3,2}$. But we have seen, that $\pi_2(V_{3,2}) \cong \mathbb{Z}$, and hence $\mathrm{Hom}^\infty(\pi_2(V_{3,2}), U(1)) \cong U(1)$. If our reconstruction theorem was right, then there should exist non-trivial, flat Deligne 2-classes on $V_{3,2}$, which is in contradiction to Gajers theorem.

In the same manner we can deduce this for any d even. The corresponding manifold is then $V_{d+1,2}$ and we need $\pi_d(V_{d+1,2})$. By the above proposition, we know that this is $\pi_d(S^d) \cong \mathbb{Z}$. Therefore our result would affirm the existence of non-trivial flat Deligne d-classes on $V_{d+1,2}$. The d^{th} homology group of $V_{d+1,2}$ is zero by proposition 2.7.5 (To use the proposition, set $r = d-1$, which is odd now. This gives $H_d(V_{d+1,2}) = H_{r+1}(V_{r+2,2}) = 0$). It follows from Gajers theorem, that there are no non-trivial flat Deligne d-classes on $V_{d+1,2}$, which shows that our correspondence does not work.

Finally remark that for odd d, as already mentioned, the above manifolds $V_{d+1,2}$ do not give counterexamples anymore, because though they are $(d-2)$-connected as we need it, they do not have the essential property $\pi_{d+1}(V_{d+1,2}) \ncong H_{d+1}(V_{d+1,2})$.

With this we have found counterexamples for every case.

Chapter 3

Parallel Transport

In physics one often has to deal with the concept of parallel transport, that is one extends the notion of holonomy to d-dimensional surfaces *with* boundary. In this chapter we will generalize the reconstruction theorem of the last chapter to parallel transport. We will show that parallel transport, unlike holonomy, cannot be captured by a single scalar anymore. The right concept is that of a topological quantum field theory. Motivated by the local formulas for the holonomy, we introduce a variant of such a topological quantum field theory, which consists, very loosely, not of a pair of a vector space V and an element in V, but of a tupel of two scalars in $U(1)$. Such a topological quantum field theory has been considered first by Picken for the case $d = 2$ [24]. We continue to show that any Deligne cocycle defines a very special kind of a Picken-type topological quantum field theory, and that any such gives rise to a Deligne cocycle. Indeed, both concepts are the same, and we show that there is a group isomorphism from the group of Deligne cocycles to the group of thin-invariant, smooth Picken-type topological quantum field theories. This kind of result has already been obtained by Picken for the case $d = 2$.

3.1 Motivation

To understand, why a single number does not capture parallel transport, which can be thought of »holonomy along submanifolds with boundary«, let us try to mimic the construction of holonomy for the easiest example, a Deligne 1-class[1] $\xi = [(g_{ij}, A_i)]$ over a smooth manifold M along a non-closed path p in M (see figure 3.1).

As always we choose a good cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M, and consider Deligne classes

[1] In this section we work with Deligne classes though our reconstruction result will be formulated for Deligne cocycles, thereby providing a much stronger result.

with respect to this covering.

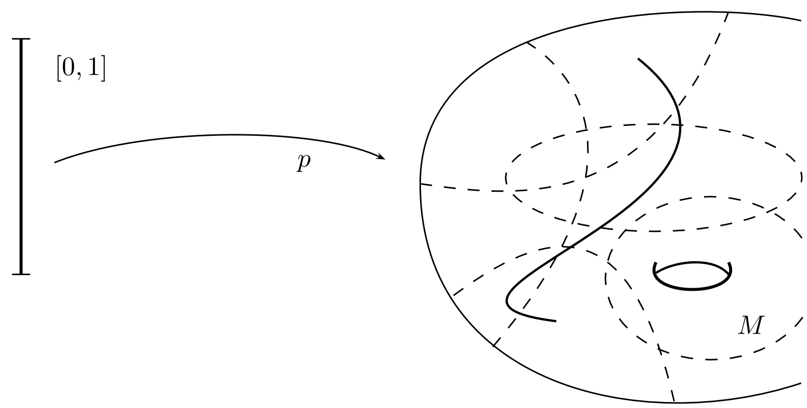


Figure 3.1: A path p in M

The holonomy of p was computed by pulling back the Deligne class ξ via p and trivializing it over $[0,1]$. This class has the form $p^*(\xi) = [1, \delta(\rho)]$, with ρ a global 1-form on p. The holonomy was then defined as

$$\mathrm{hol}_\xi(p) := \exp \int_{[0,1]} \rho.$$

We saw that the trivialization is not unique, and we could have chosen another global 1-form $\tilde{\rho}$ instead. Pretending that p is a closed path for a second, we know that the holonomy of p did not depend on this choice, as $\exp(\int_{[0,1]} \rho) \exp(\int_{[0,1]} \tilde{\rho})^{-1} = \exp \int_{[0,1]}(\rho - \tilde{\rho}) = 1$, since by the exact sequence (see proposition 1.2.4) the global 1-form $\rho - \tilde{\rho}$ is closed with $2\pi i\mathbb{Z}$ periods, so the integral over $[0,1]$ vanishes. The problem is now, that this argument fails, if p is not closed. We simply cannot claim that any choice of a trivialization of the pullback class $\varphi_p^*(\xi)$ gives the same holonomy.

So, the »holonomy«[2] of an open path is not well-defined anymore. How to remedy this? One idea would be to simply include the choice of a trivialization, so instead of trying to compute the holonomy of an open path p, we could instead equip p with a choice of a trivialization ρ and talk about the holonomy of the tupel (p, ρ). This seems a bit unnatural, though it is still possible to do so. Another way to remedy the problem, is to drop the concept of holonomy altogether, that is, one does not expect an open path to give a single number anymore. Instead, one replaces it with the concept of a topological

[2] We will still continue to use the word holonomy sometimes in this section though it would be much better to speak of parallel transport.

quantum field theory, thereby assigning to p not a number, but a vector space V together with an element in V But working with vector spaces and elements in these, is not nice anymore. Computations with vector spaces are not as simple as calculations with scalars in $U(1)$.

In order to find a better approach, let us work out in how far two trivializations differ: Let ρ and $\tilde{\rho}$ be two such choices. As the holonomy is $U(1)$-valued, we know that there must be a scalar $c(\rho, \tilde{\rho}) \in U(1)$ that measures the difference of the holonomies given by ρ and $\tilde{\rho}$, so $c(\rho, \tilde{\rho})$ is defined to be

$$c(\rho, \tilde{\rho}) := \exp(\int_{[0,1]} \rho - \int_{[0,1]} \tilde{\rho}).$$

Now we may reinterprete the right hand side. If we run the path p backwards, the integral $\int_p \tilde{\rho}$ simply switches its sign: $\int_{p^{-1}} \tilde{\rho} = -\int_p \tilde{\rho}$. If we assume furthermore, that both ρ and $\tilde{\rho}$ do coincide at the boundary of p, that is at a and b, then we can rewrite the right hand side: $c(\rho, \tilde{\rho}) = \exp(\int_{p \amalg p^{-1}} (\rho + \tilde{\rho}))$. Obviously $\rho + \tilde{\rho}$ is now the trivialization of the pullback of ξ along $p \amalg p^{-1}$ and can be regarded as the holonomy of the *closed* path $p^{-1} \circ p$. This expression, being the holonomy of a path that is thin homotopic to a constant path, is 1, so also the correction factor is $c(\rho, \tilde{\rho}) = 1$. Viewing this from another angle, altogether we have shown, that the correction term does only depend on the values of ρ and $\tilde{\rho}$ on the boundary of p.

This observation leads to another possibility of defining the parallel transport, if one takes into consideration that the above way is not the only one to compute the holonomy. Instead, as we have seen in the discussion of the holonomy, one can choose a triangulation of p and obtain the holonomy via some local formula. Let us recapitulate this for the case at hand. Pretending that p is closed, choosing a triangulation of p into edges k^1 and vertices k^0, together with a labeling ℓ, the local formula takes the form

$$\mathrm{hol}_\xi(p) = \exp(\sum_{f_1 \in k^1} \int_{f_1} A_{\ell(f_1)}) \prod_{f_0 \subset f_1} g_{\ell(f_1)\ell(f_0)}(f_0).$$

Obviously this expression was only independent of the triangulation, because we showed that if p were closed, it can be regarded as an expansion of the holonomy defined via $\exp \int_p \rho$, thereby proving the desired independence. But p is not closed, so the formula is not well-defined in general.

As we have seen that two trivializations differ only on their boundary, given a triangulation of p, we are tempted to redefine the »holonomy« of ξ along p to be the integral of the internal faces only, thereby dropping the troubling contributions which come from the boundary. Let us pursue this idea. Denote by T_p any triangulation of p with edges

k^1 and vertices k^0. In the following we will always consider such pairs (p, T_p) of a path $p : [a, b] \longrightarrow M$ and a triangulation T_p. For any such (p, T_p) we define

3.1.1 DEFINITION: Let (p, T_p) be a path $p : [a, b] \longrightarrow M$ in M with a triangulation T_p and a labeling $\ell : T_p \longrightarrow I$. The **parallel transport** of p is defined as

$$\mathrm{pt}_\xi(p, T_p) = \exp\Big(\sum_{f_1 \in k^1} \int_{f_1} A_{\ell(f_1)}\Big) \prod_{\substack{f_0 \subset f_1 \\ f_0 \notin \partial p}} g_{\ell(f_1)\ell(f_0)}(f_0).$$

Basically we exchange the dependency of the »holonomy« on the choice of a trivialization with the dependency on the choice of a triangulation. But triangulations are much easier to handle than differential forms, and are indeed often used for computations of the holonomy in the physical literature. We need to show that this definition makes sense, by further analyzing it.

The first question is: How do two different choices of a triangulation compare? It is reasonable to expect, that two triangulations T_p and $\tilde{T}_p$ are correlated to each other. So we simply choose another such triangulation $\tilde{T}_p$ and compare the terms $\mathrm{pt}_\xi(p, T_p)$ and $\mathrm{pt}_\xi(p, \tilde{T}_p)$. Both will differ by a scalar $\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p)$ (of course we will have to show that it only depends on the boundary). Before we give some general formulas, let us look at easy examples, to get a better grip on this correction term.

3.1.2 EXAMPLE: Let $p : [0, 1] \longrightarrow M$ be an open path in $U_{ij} \subset M$. First choose T_p and $\tilde{T}_p$ both the trivial triangulation with the same labeling $\ell([0, 1]) = i$. Obviously $\mathrm{pt}_\xi(p, T_p) = \exp \int_{[0,1]} A_i = \mathrm{pt}_\xi(p, \tilde{T}_p)$, as $[0, 1]$ is the only inner edge, and both vertices are not internal. So we have shown that $\mathrm{pt}_\xi(\partial[0, 1], T_p, T_p) = 1$.

Choose T_p to be the trivial triangulation with $\ell([0, 1]) = i$ and $\tilde{T}_p$ also to be the trivial triangulation, but with $\ell([0, 1]) = j$. By definition we have $\mathrm{pt}_\xi(p, T_p) = \exp \int_{[0,1]} A_i$ and $\mathrm{pt}_\xi(p, \tilde{T}_p) = \exp \int_{[0,1]} A_j$. But since ξ is a Deligne 1-cocycle, we know $A_j - A_i = d \log g_{ij}$, so we can write $\mathrm{pt}_\xi(p, \tilde{T}_p) = \exp \int_{[0,1]} (A_i + d \log g_{ij}) = \mathrm{pt}_\xi(p, T_p) g_{ij}^{-1}(0) g_{ij}(1)$, so the correction factor is $\mathrm{pt}_\xi(\partial[0, 1], i, j) = g_{ij}^{-1}(0) g_{ij}(1)$, where we abbreviated the triangulations by i and j respectively. Note that $\partial[0, 1] = \{1\} - \{0\}$, so one might guess that the definitions $\mathrm{pt}_\xi(p, i, j) = g_{ij}(p)$ and $\mathrm{pt}_\xi(p \amalg q, i, j) = g_{ij}(p) g_{ij}(q)$ is reasonable.

To compare slightly more complicated triangulations, one needs to go over to a common refinement. Let us show that refining a triangulation does not change the parallel transport: Suppose we are given the above path $p : [0, 1] \longrightarrow M$ with $\ell(p) = i$. Introducing any vertex x in $[0, 1]$ with arbitrary label $\ell(x)$, and denoting this triangulation by $\tilde{T}_p$, we obtain

$$\mathrm{pt}_\xi([0,1],\tilde{T}_p) = \exp\Big(\int_{[0,x]} A_i + \int_{[x,1]} A_i\Big) g_{i\ell(x)}(x) g^{-1}_{i\ell(x)}(x) = \exp\Big(\int_{[0,1]} A_i\Big).$$

Thus, we may refine any triangulation without changing its parallel transport.

Consider now two triangulations as seen in figure 3.2. For this example we adopt the following rule: We specify the labeling of the edges, but assign to any vertex the highest index of the neighboring edges. For example, the point x in the first triangulation in figure 3.2 would be labeled k, as does z.

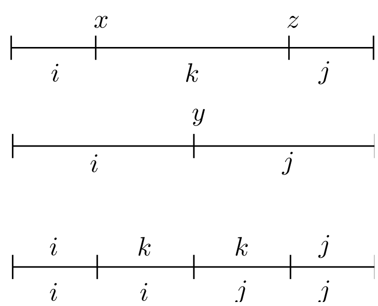


Figure 3.2: Two possible triangulations and their refinement

The parallel transport of the first triangulation T_p is then given by

$$\mathrm{pt}_\xi(p,T_p) = \exp\Big(\int_{I_1} A_i + \int_{I_2} A_k + \int_{I_3} A_k + \int_{I_4} A_j\Big) g_{ik}(x) g^{-1}_{jk}(z),$$

where we denoted by I_1, I_2, I_3 and I_4 the four intervals of the refined triangulation (the third one in the figure). One immediately sees that our refinement of the triangulation did not change the parallel transport, as we claimed. On the other hand, we have

$$\mathrm{pt}_\xi(p,\tilde{T}_p) = \exp\Big(\int_{I_1} A_i + \int_{I_2} A_i + \int_{I_3} A_j + \int_{I_4} A_j\Big) g_{ij}(y).$$

By using the cocycle relations, e.g. in the form $\int_{I_2} A_k = \int_{I_2}(A_i + d\log g_{ik})$ we see that changing the labels of two integrals over I_2 and I_3 has the effect of multiplying the formula with $g^{-1}_{ik}(x) g_{ik}(y) g^{-1}_{jk}(y) g_{jk}(z)$. Combining all these vertex factors of $\mathrm{pt}_\xi(p,T_p)$, we get $g_{ij}(y)$, which is exactly the vertex factor in the second formula above, thereby showing again that the correction factor $\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p)$ is trivial, if T_p and $\tilde{T}_p$ agree on the boundary.

Now let us change the labels i and j of the second triangulation in the figure. Let us call this triangulation T'_p. Its holonomy takes the form

$$\mathrm{pt}_\xi(p, T'_p) = \exp(\int_{I_1} A_j + \int_{I_2} A_j + \int_{I_3} A_i + \int_{I_4} A_i) g_{ij}^{-1}(y).$$

Rewriting the parallel transport of the first triangulation by using the cocycle relations on every I_j, the vertex part becomes a product of $g_{ik}(x)g_{jk}^{-1}(z)$ (these are part of the parallel transport), $g_{ji}(x)g_{ji}^{-1}(0)$ (from $I_1 = [0, x]$) $g_{jk}(y)g_{jk}^{-1}(x)$ (from I_2) $g_{ik}(z)g_{ik}^{-1}(y)$ (from I_3) and $g_{ij}(1)g_{ij}^{-1}(z)$ (from $I_4 = [z, 1]$). Using the cocycle relation $\delta g = 1$ again, we see that only $g_{ji}^{-1}(a)g_{ij}(b)g_{ij}^{-1}(y)$ survive. Therefore the correction factor is indeed $\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p) = g_{ji}^{-1}(a)g_{ij}(b)$.

Pathes can also be glued (or cut), so a natural question is how these behave with respect to pt_ξ. Suppose we are given two pathes $p : [a, b] \longrightarrow M$ and $\tilde{p} : [b, c] \longrightarrow M$ with $p(b) = \tilde{p}(b)$, and two triangulations T_p and $\tilde{T}_p$ (see figure 3.3). We may glue these two pathes to obtain a path $\tilde{p} \circ p : [a, c] \longrightarrow M$ (We must assume that these path do fit smoothly together, e.g. we may adopt the sitting path convention from last chapter). It can be endowed with a canonical triangulation, denoted by $T_{\tilde{p}} \cdot T_p$, given by simply taking T_p on $[a, b]$ and $\tilde{T}_p$ on $[b, c]$ (actually one have to choose which label the point b will obtain, but we will see in a moment that result does not depend on this choice). Assume that the triangulation is as in figure 3.3. Let us compare $\mathrm{pt}_\xi(p, T_p)\mathrm{pt}_\xi(\tilde{p}, T_{\tilde{p}})$ with $\mathrm{pt}_\xi(\tilde{p} \circ p, T_{\tilde{p}} \cdot T_p)$. The product $\mathrm{pt}_\xi(p, T_p)\mathrm{pt}_\xi(\tilde{p}, T_{\tilde{p}})$ accounts for all contributions except from the vertex b, for this is an external vertex in both T_p and $T_{\tilde{p}}$. In contrast to this, b is an internal vertex in $\tilde{p} \circ p$, so it contributes to $\mathrm{pt}_\xi(\tilde{p} \circ p, T_{\tilde{p}} \cdot T_p)$.

Then the contribution of b is $g_{i\ell(b)}(b)g_{j\ell(b)}^{-1}(b) = g_{ij}(b)$. Note that this is indeed independent of the label of b, as we expected. But the contribution can be identified with $\mathrm{pt}_\xi(b, i, j)$, so we have shown the relation

$$\mathrm{pt}_\xi(p, T_p)\mathrm{pt}_\xi(\tilde{p}, T_{\tilde{p}})\mathrm{pt}_\xi(b, i, j) = \mathrm{pt}_\xi(\tilde{p} \circ p, T_{\tilde{p}} \cdot T_p).$$

Immediately one sees, that if on both sides the labeling is the same, then we have some sort of gluing formula:

$$\mathrm{pt}_\xi(p, T_p)\mathrm{pt}_\xi(\tilde{p}, T_{\tilde{p}}) = \mathrm{pt}_\xi(\tilde{p} \circ p, T_{\tilde{p}} \cdot T_p),$$

affirming the fact that $\mathrm{pt}_\xi(b, T_p, T_p)$ is always 1.

Finally consider p with three triangulations T_p, $\tilde{T}_p$ and T'_p. We can consider $\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p) \cdot \mathrm{pt}_\xi(\partial p, \tilde{T}_p, T'_p)$. Assume, that T_p is given by i and i' on the boundary at a and b respectively, T'_p by j and j' and $\tilde{T}_p$ by k and k'. Our example showed that

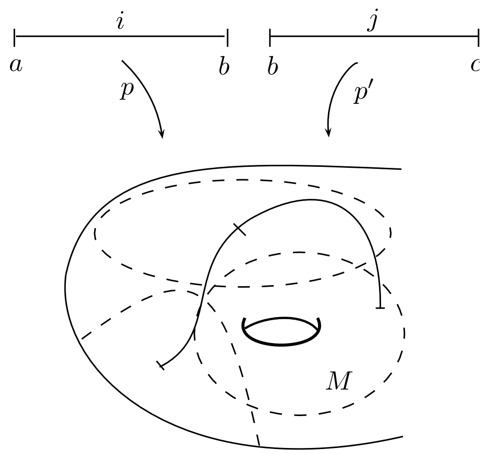


Figure 3.3: Gluing of two triangulated pathes

$\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p) = g_{ij}^{-1}(a) g_{i'j'}(b)$ as well as $\mathrm{pt}_\xi(p, \tilde{T}_p, T'_p) = g_{jk}^{-1}(a) g_{j'k'}(b)$. By virtue of the cocycle relations, the product is $g_{ik}^{-1}(a) g_{i'k'}(b)$, which is the same as $\mathrm{pt}_\xi(p, T_p, T'_p)$. We verified the rule

$$\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p) \cdot \mathrm{pt}_\xi(\partial p, \tilde{T}_p, T'_p) = \mathrm{pt}_\xi(\partial p, T_p, T'_p)$$

for this specific example.

After seeing that our initial guess works in these situations, let us deduce these results in general (for our 1-dimensional case). These calculations will work as a base for the higher dimensional cases.

Before we state these properties in general, let us fix the correction factor:

3.1.3 DEFINITION: Let (p, T_p) and $(p, \tilde{T}_p)$ be two triangulations of a path $p : [a, b] \longrightarrow M$ and $\xi = [(g_{ij}, A_i)]$ be a Deligne 1-class on M. Suppose that T_p at the boundary is labelled by i and j, and $\tilde{T}_p$ by i' and j'. Then the correction factor $\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p)$ is defined to be

$$\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p) := g_{ii'}^{-1}(a) g_{jj'}(b).$$

Moreover, for any *oriented* 0-dimensional manifold y (also known as a point) with two labels i and j we set $\mathrm{pt}_\xi(y, i, j) := g_{ij}(y)$.

3.1.4 PROPOSITION: *For any three triangulations of y we have*

$$pt_\xi(y, T_y, \tilde{T}_y) \cdot pt_\xi(y, \tilde{T}_y, T'_y) = pt_\xi(y, T_y, T'_y)$$

Using the definition of $\mathrm{pt}_\xi(y, T_y, \tilde{T}_y)$ and the cocycle relation, one directly verifies this proposition. Furthermore:

3.1.5 PROPOSITION: *Suppose p is a path in M and (p, T_p), $(p, \tilde{T}_p)$ are two triangulations of p. Then*

$$pt_\xi(p, T_p) pt_\xi(\partial p, T_p, \tilde{T}_p) = pt_\xi(p, \tilde{T}_p).$$

Proof. By switching to a common refinement of both triangulations, we may assume that T_p and $\tilde{T}_p$ differ only by their labeling (We saw how this works in the example above). Then we have

$$\mathrm{pt}_\xi(p, T_p) = \exp\Big(\sum_{f_1 \in k^1} \int_{f_1} A_{\ell(f_1)}\Big) \prod_{\substack{f_0 \in f_1 \\ f_0 \notin \partial p}} g_{\ell(f_1)\ell(f_0)}(f_0).$$

and

$$\mathrm{pt}_\xi(p, \tilde{T}_p) = \exp\Big(\sum_{f_1 \in k^1} \int_{f_1} A_{\tilde{\ell}(f_1)}\Big) \prod_{\substack{f_0 \in f_1 \\ f_0 \notin \partial p}} g_{\tilde{\ell}(f_1)\tilde{\ell}(f_0)}(f_0).$$

Using the relation $A_{\ell(f_1)} = A_{\ell(\tilde{f}_1)} + d \log g_{\tilde{\ell}(f_1)\ell(f_1)}$ we see that we only have to compare the vertex factors. Suppose we consider a vertex y, being start and endpoint of the edges f_L and f_R respectively (hence we also assume y to be an inner vertex). The vertex factors coming from T_p are given by $g_{\ell(f_L)\ell(y)}(y) g^{-1}_{\ell(f_R)\ell(y)}(y)$, the vertex factors from the cocycle relations on the other hand are $g_{\tilde{\ell}(f_L)\ell(f_L)}(y) g^{-1}_{\tilde{\ell}(f_R)\ell(f_R)}(y)$. By rearranging these and using $g_{ij}^{-1} = g_{ji}$ as well as $g_{ij} g_{jk} = g_{ik}$, we see that these can be written as $g_{\tilde{\ell}(f_L)\tilde{\ell}(f_R)}(y)$– but this is exactly the same as $g_{\tilde{\ell}(f_L)\tilde{\ell}(y)}(y) g^{-1}_{\tilde{\ell}(f_R)\tilde{\ell}(y)}(y)$, the vertex factor of the triangulation $\tilde{T}_p$. The only contributions we have not considered are those coming from external vertices, i.e. we have to compare the integrals on the first and last interval. In $\mathrm{pt}_\xi(p, T_p)$ we find the contribution $g^{-1}_{\ell(f_R)\ell(a)}(a) g_{\ell(f_L)\ell(b)}(b)$. By the cocycle relation we get two more factors $g^{-1}_{\tilde{\ell}(f_R)\ell(f_R)}(a)$ and $g_{\tilde{\ell}(f_L)\ell(f_L)}(b)$. Combing these we have $g^{-1}_{\tilde{\ell}(f_R)\ell(a)}(a) g_{\tilde{\ell}(f_L)\ell(b)}(b)$. The vertex factor in $\mathrm{pt}_\xi(p, \tilde{T}_p)$ is $g^{-1}_{\tilde{\ell}(f_R)\tilde{\ell}(a)}(a) g_{\tilde{\ell}(f_L)\tilde{\ell}(b)}(b)$, and the correction term hence $g^{-1}_{\ell(a)\tilde{\ell}(a)}(a) g_{\ell(b)\tilde{\ell}(b)}(b)$, which is just $\mathrm{pt}_\xi(\partial p, T_p, \tilde{T}_p)$. □

The idea of gluing two intervals is captured in the following proposition:

3.1.6 PROPOSITION: *Let (p, T_p) and $(\tilde{p}, T_{\tilde{p}})$ be two pathes with $p : [a, b] \longrightarrow M$ and $\tilde{p} : [b, c] \longrightarrow M$ and $p(b) = \tilde{p}(b)$. Then we can glue the intervals and obtain a path $\tilde{p} \circ p : [a, c] \longrightarrow M$ with triangulation $T_{\tilde{p}} \cdot T_p$ and the following property:*

$$pt_\xi(p, T_p) pt_\xi(\tilde{p}, T_{\tilde{p}}) pt_\xi(b, T_p, T_{\tilde{p}}) = pt_\xi(\tilde{p} \circ p, T_{\tilde{p}} \circ T_p).$$

Proof. Both sides of the equation do only differ at the vertex b. The product $\mathrm{pt}_\xi(p, T_p)\mathrm{pt}_\xi(\tilde{p}, T_{\tilde{p}})$ does not contain any contribution from b at all, since b is an external vertex. Contrary, b is an inner vertex at the right hand side. The corresponding vertex factor is $g_{\ell(f_L)\ell(b)}(b)g_{\ell(f_R)\ell(b)}(b)^{-1} = g_{\ell(f_L)\ell(f_R)}(b)$. This yields exactly the above equation. □

3.1.7 COROLLARY: *If in the situation of the proposition both triangulations agree on the boundary at* b *(i.e. the labels of* f_L *and* f_R*, the edges to the left and right of* b *are equal), we have*

$$pt_\xi(p, T_p)pt_\xi(\tilde{p}, T_{\tilde{p}}) = pt_\xi(\tilde{p} \circ p, T_{\tilde{p}} \circ T_p).$$

These two statements are referred to as the basic gluing and partial gluing axioms in Pickens work (see definition 4.1 in [24]). The meaning of these will become clearer during the discussion of the axioms for higher dimensions next section.

Next, suppose we have two parametrizations of a path, that is we have $p : [a, b] \longrightarrow M$ and $\tilde{p} : [c, d] \longrightarrow M$ such that $\tilde{p} = p \circ \psi$. If we regard $[a, b]$ and $[c, d]$ as 1-dimensional manifolds, this means that they are diffeomorphic relative to the boundary. And diffeomorphic paths should have the same parallel transport, as long as they coincide at their boundary, giving rise to the guess:

3.1.8 PROPOSITION: *In the situation above we have*

$$pt_\xi(p, T_p)pt_\xi(\partial p, T_p, T_{\tilde{p}}) = pt_\xi(\tilde{p}, T_{\tilde{p}}).$$

To verify this proposition, we need to analyze the diffeomorphism itself. In this situation, this is a simply a triangulated square, such that the upper boundary corresponds to p, the lower to $\tilde{p}$ and the sides gets mapped to $p(a) = p(c)$ and $p(b) = p(d)$. Following Picken ([24]), we show that integrating the pullback of the curvature over this square corresponds to the parallel transport of its boundary, which proves by using the gluing lemma, the proposition. The following lemma itself can be regarded as a generalization of the statement

$$\exp(\int_{I^{d+1}} W^* \operatorname{curv}(\xi)) = \exp(\int_{I^{d+1}} d\hat{\rho}) = \exp(\int_{\partial I^{d+1}} \hat{\rho}).$$

which we used to in the proof of proposition 2.2.5.

3.1.9 LEMMA: *Let* (H, T_H) *be the triangulated square, described above. Then we have*

$$\exp \int_H \varphi_H^*(\operatorname{curv}(\xi)) = pt_\xi(\partial H, \partial T_H)$$

Proof. As said, $H = [0,1] \times [0,1]$ is a 2-dimensional manifold with 4 corners, but because the corners form a null set, we may ignore them. Then by definition of the curvature and its naturality we have (by Stokes theorem) to calculate

$$\exp \sum_{f_2 \in k^2} \int_{\partial f_2} \varphi_H^*(A_{\ell(f_2)}),$$

with k^2 the set of 2-faces of H. Now we can, as in the proof of the formula for the local holonomy, rewrite this as a sum of all 1-faces of all 2-faces, i.e. as

$$\exp \sum_{f_2 \in k^2} (\sum_{f_1 \subset f_2} \int_{f_1} \varphi_H^*(A_{\ell(f_2)})).$$

Using the cocycle relation $A_{\ell(f_2)} = A_{\ell(f_1)} + d \log g_{\ell(f_1)\ell(f_2)}$, we get

$$\exp \sum_{f_2 \in k^2} (\sum_{f_1 \subset f_2} \int_{f_1} \varphi_H^*(A_{\ell(f_1)} + d \log g_{\ell(f_1)\ell(f_2)})).$$

But any internal edge occurs exactly twice, with different sign. Using again Stokes theorem, the formula reduces to

$$\exp \sum_{f_1 \subset \partial H} \int_{f_1} \varphi_H^*(A_{\ell(f_1)}) \exp \sum_{f_1 \subset f_2} \int_{\partial f_1} \varphi_H^*(\log g_{\ell(f_1)\ell(f_2)}).$$

The first term is exactly what we want, so we only have to deal with the second one. We can again rewrite it to obtain

$$\exp \sum_{f_2 \in k^2} \sum_{f_1 \subset f_2} \sum_{f_0 \subset f_1} \log g_{\ell(f_1)\ell(f_2)}(\varphi_H(f_0)).$$

Next we use the cocycle relation for the index triple $(\ell(f_1), \ell(f_2), \ell(f_0))$ to reexpress $g_{\ell(f_1)\ell(f_2)}$ as $g_{\ell(f_1)\ell(f_2)} = g_{\ell(f_0)\ell(f_2)} g_{\ell(f_1)\ell(f_0)}$. Observe first that at any vertex f_0 the term $g_{\ell(f_0)\ell(f_2)}$ vanishes, for in every triangle f_2 at f_0 there is one incoming edge and one outgoing edge (remember that we have to sum over all edges $f_1 \subset f_2$), and these exactly give $g_{\ell(f_0)\ell(f_2)} g_{\ell(f_0)\ell(f_2)} = 1$. Suppose now f_0 is an internal vertex. Since we know that there is a neighboring triangle, we know that $g_{\ell(f_1)\ell(f_0)}$ occurs also twice with different sign, coming from these two faces. This holds for every two faces around f_0, and hence there is no contribution from $g_{\ell(f_0)\ell(f_2)} g_{\ell(f_1)\ell(f_0)}$ at all. We only have to consider external vertices

f_0. The first argument did not depend on the vertex being inner or outer, so only $g_{\ell(f_1)\ell(f_0)}$ survives, leaving the term

$$\exp \sum_{f_2 \in k^2} \sum_{f_1 \subset f_2} \sum_{\substack{f_0 \subset f_1 \\ f_0 \in \partial H}} \log g_{\ell(f_1)\ell(f_0)}(\varphi_H(f_0)) = \prod_{\substack{f_0 \subset f_1 \\ f_0 \in \partial H}} g_{\ell(f_1)\ell(f_0)}(\varphi_H(f_0))$$

So both surviving terms together give exactly $\mathrm{pt}_\xi(\partial H, \partial T_H)$. This proves the formula $\mathrm{pt}_\xi(\partial H, \partial T_H)) = \exp \int_H \varphi_H^*(\mathrm{curv}(\xi))$. □

Especially, if $[a, b]$ and $[c, d]$ are diffeomorphic and their triangulations coincide at the boundary, then their parallel transport is the same. This is a variation of the comparison of two triangulations and the upper situation, where $[a, b] = [c, d]$ is just a special case of this statement.

There are some more obvious properties, which we state for completeness:

3.1.10 PROPOSITION: *If p and $\tilde{p}$ are two pathes in M, then $pt_\xi((p, T_p) \amalg (\tilde{p}, T_{\tilde{p}})) = pt_\xi(p, T_p) pt_\xi(\tilde{p}, T_{\tilde{p}})$. The same goes for the boundary: $pt_\xi((\partial p, T_p, \tilde{T}_p) \amalg (\partial \tilde{p}, T_{\tilde{p}}, \tilde{T}_{\tilde{p}})) = pt_\xi(\partial p, T_p, \tilde{T}_p) pt_\xi(\partial \tilde{p}, T_{\tilde{p}}, \tilde{T}_{\tilde{p}})$. We also have $pt_\xi(\emptyset) = 1$*

The essence of the above discussion is, that to define parallel transport, we have to consider the holonomy not of the path p alone, but of the pair (p, T_p), where T_p is a triangulation of p. Only then we are able to obtain a single, well-defined number, which we call $\mathrm{pt}_\xi(p, T_p)$. Computing the difference of two such triangulations of p, we found that they are related by a correction term $\mathrm{pt}_\xi(p, T_p, \tilde{T}_p)$ that only depended on the triangulations on the boundary of p, and satisfies a compability rule. Let us collect all these properties of pt_ξ in one theorem, so that we can axiomatize these in the next section.

3.1.11 THEOREM: *Let p be a path in M, and ξ be a Deligne 1-class on M. Define*

$$pt_\xi(p, T_p) := \exp\Big(\sum_{f_1 \in k^1} \int_{f_1} A_{\ell(f_1)}\Big) \prod_{\substack{f_0 \subset f_1 \\ f_0 \notin \partial p}} g_{\ell(f_1)\ell(f_0)}(f_0)$$

for any triangulation T_p of p with edges k^1 and

$$pt_\xi(y, i, j) := g_{ij}(y),$$

for y a (oriented) point with two labels (or triangulations).
Then these quantities possess the following properties:

(i) for any three triangulations $T_y, \tilde{T}_y$ and T'_y of y we have

$$pt_\xi(y, T_y, \tilde{T}_y) pt_\xi(y, \tilde{T}_y, T'_y) = pt_\xi(y, T_y, T'_y).$$

(ii) For any diffeomorphism, that keeps the boundary fixed we have

$$pt_\xi(\tilde{p}, T_{\tilde{p}}) = pt_\xi(\partial p, T_p, T_{\tilde{p}}) pt_\xi(p, T_p).$$

(iii) If $p : [a, b] \longrightarrow M$ and $p' : [b, c] \longrightarrow M$ are two pathes which can be glued at b, and T_p, $T_{p'}$ are two triangulations, then

$$pt_\xi(p, T_p) pt_\xi(p', T_{p'}) pt_\xi(b, T_p|_b, T_p|_b) = pt_\xi(p' \circ p, T_{p'} \cdot T_p).$$

(iv) Any parallel transport of p with respect to ξ can be computed locally, that is we have

$$pt_\xi(p, T_p) = \prod_{f_1 \in k^1} pt_\xi(f_1, T_{f_1}).$$

(v) pt_ξ is compatible with taking disjoint unions and we have $pt_\xi(\emptyset) = 1 \in U(1)$.

Let us come back to the correction factor. Suppose we have $p : [0, 1] \longrightarrow M$ with the two triangulations i and j. Our axioms tell us that $\mathrm{pt}_\xi(p, i)$ and $\mathrm{pt}_\xi(p, j)$ do differ by $\mathrm{pt}_\xi(\partial p = \{1\} - \{0\}, i, j)$. This describes the change from the one triangulation to the other at the boundary of p. But there is another way to achieve this: Let (c_0, ij) be the constant path in 0 with the triangulation i on $[0, \frac{1}{2}]$ and j on $[\frac{1}{2}, 1]$. Obviously we can compose this with (p, j) to obtain $(\tilde{p}, i) := (c_1^{-1}, ij) \circ (p, j) \circ (c_0, ij)$. This new path has the same image as p, but has a new triangulation, namely i, on its boundary. We may guess that $\tilde{p}$ has the same parallel transport as (p, i). Indeed, both paths are diffeomorphic, so proposition 3.1.8 tells us that

$$\mathrm{pt}_\xi(\tilde{p}, i) = \mathrm{pt}_\xi((c_1^{-1}, ij) \circ (p, j) \circ (c_0, ij)) = \mathrm{pt}_\xi(p, j) \mathrm{pt}_\xi(\partial p, i, j).$$

But from the gluing axiom it follows that $\mathrm{pt}_\xi((c_1^{-1}, ij) \circ (p, j) \circ (c_0, ij)) = \mathrm{pt}_\xi(c_1^{-1}, ij) \mathrm{pt}_\xi(p, j) \mathrm{pt}_\xi(c_0, ij)$. Therefore we have an equality between the correction factor and the parallel transport of the trivial path c_0 and c_1^{-1}:

$$\mathrm{pt}_\xi(\partial p, i, j) = \mathrm{pt}_\xi(c_1^{-1}, ij) \mathrm{pt}_\xi(c_0, ij).$$

This example may be further refined by simply considering p to be equipped with the triangulations i, j and i. Then we get $\mathrm{pt}_\xi(\{0\}, i, j) = \mathrm{pt}_\xi(c_0, ij)$. In other words: *The correction factor itself can be reexpressed as a parallel transport*. The path c_0 is just $\{0\} \times [0, 1]$ with the triangulation i on $c_0(0)$ and j on $c_0(1)$. Note also that we could have also choosen any triangulation on c_0 as long as it takes the right values on the boundary of c_0. This observation will be very important for the generalization, for we will not define the correction factor explicitly, but by using this identification.

3.2 Triangulated Manifolds

Generalizing the motivational case of paths, we will now deal with arbitrary manifolds and not just spheres as in the case of holonomy. Therefore, we have to find the correct generalization of the 'constant at boundary'-condition.

3.2.1 DEFINITION: A d-dimensional smooth **collared manifold in** M is a d-dimensional smooth manifold Σ with boundary in M, such that there is a neighborhood of the boundary $\partial\Sigma$ and the map φ_Σ is constant on $\partial\Sigma$ in the transverse direction.

We may glue such collared manifolds as we like, since the extra-condition ensures that the glued manifold is smooth and a collared manifold again.

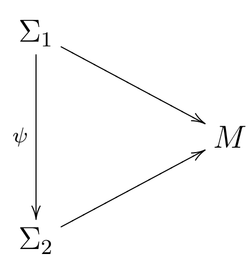
3.2.2 DEFINITION: Let M be a manifold and $\mathcal{U} = \{U_i\}_{i\in I}$ a good covering. Let $\varphi : \Sigma^d \longrightarrow M$ be a d-dimensional, smooth collared manifold. An $\mathcal{U}$**-adapted, triangulated manifold** in M consists of a collared manifold Σ in M together with a triangulation T_Σ of Σ and a labeling ℓ_{T_Σ}, that is a map $\ell_{T_\Sigma} : T_\Sigma \longrightarrow I$, and ℓ assigns to each q-simplex f_q of the triangulation an index $\ell_{T_\Sigma}(f_q)$ with $\varphi(f_q) \subset U_{\ell_{T_\Sigma}(f_q)}$.

We will not write down the labeling of a triangulated manifold, but write only (Σ, T_Σ) instead, and assume that the corresponding map into M is $\varphi_\Sigma : \Sigma \longrightarrow M$ and the labeling is simply $\ell : T_\Sigma \longrightarrow I$. If $\tilde{\Sigma}$ is another triangulation, we also write down $\tilde{\ell}$ instead of $\ell_{\tilde{T}_\Sigma}$.

Because of these technical conditions we are now able to glue two triangulated, $\mathcal{U}$-adapted manifolds (Σ_1, T_{Σ_1}) and (Σ_2, T_{Σ_2}) along a common boundary-component (S, T_S).

Moreover we can take the orientation-reverse of a closed triangulated, $\mathcal{U}$-adapted manifold. If (Σ, T_Σ) is such an manifold, the dual $(\Sigma, T_\Sigma)^*$ is simply $(\bar{\Sigma}, \bar{T}_\Sigma)^*$, e.g. we take the orientation-reversed manifold and equip it with the same triangulation, having different orientation.

A diffeomorphism $\psi : (\Sigma_1, T_{\Sigma_1}) \longrightarrow (\Sigma_2, T_{\Sigma_2})$ between triangulated manifolds in M is a diffeomorphism $\Sigma_1 \longrightarrow \Sigma_2$ such that the following diagram commutes



and the boundary stays fixed, i.e. ψ restricted to the boundary is the identity. We will only consider this type of diffeomorphisms, where we do not care about the triangulation neither on Σ nor on its boundary.

Notice, that our definition of triangulation is not the same as Pickens ([24]). For him a triangulation T_Σ is a $(d+1)$-valent $(d-1)$-graph embedded in M. We will need to extend the topological quantum field theory, which we define on triangulated manifolds, also to such general manifolds. This we do by choosing an arbitrary triangulation of the embedded graph and showing, that the topological quantum field theory is insensitive to the choice of the triangulation.

Note also, that without stating it explicitly, our manifolds are always assumed to be smooth.

3.3 Picken's TQFT

Following Picken [24], we now introduce a variation of the concept of a topological quantum field theory, based on the example in the motivation above (see [1] for the usual definition). Actually this concept is a watered-down version of the more refined picture of topological quantum field theories, which Picken develops in [23]. Though this approach may be elegant in several respects, it is too cumbersome for direct computations. For our purpose, since we are only interested in the abelian $U(1)$-case, we may use a reduced set of axioms, as Picken showed (see the remark before definition 3.2 in [24] as well as the definitions 3.2 and 4.1 therein). Let us write down the parallel transport in an axiomatic way:

3.3.1 Definition: Let M be a manifold with a cover $\mathcal{U}$. A rank-1, d-dimensional topological quantum field theory of Picken-type, adapted to $\mathcal{U}$, is an assignment as follows:

(i) To any $(d-1)$-dimensional closed manifold S and two triangulations T_S and $\tilde{T}_S$ of S an assignment
$$Z'(S, T_S, \tilde{T}_S) \in U(1),$$
(ii) To any d-dimensional manifold Σ with boundary and triangulation T_Σ an assignment
$$Z(\Sigma, T_\Sigma) \in U(1),$$
such that

(iii) For any three triangulations $T_S, \tilde{T}_S$ and T'_S of S we have
$$Z'(S, T_S, \tilde{T}_S) Z'(S, \tilde{T}_S, T'_S) = Z'(S, T_S, T'_S),$$
(iv) For any diffeomorphism $\psi : \Sigma \longrightarrow \tilde{\Sigma}$ (that keeps the boundary fixed as in the last section), we have
$$Z(\tilde{\Sigma}, T_{\tilde{\Sigma}}) = Z'(\partial\Sigma, \partial T_\Sigma, \partial T_{\tilde{\Sigma}}) Z(\Sigma, T_\Sigma),$$

(v) If Σ_1 and Σ_2 are two manifolds with boundary $\partial\Sigma_1 = S_1 \amalg S_2$ and $\partial\Sigma_2 = S_2^* \amalg S_3$, we may glue these along S_2 to obtain $\Sigma_1 \cup_{S^2} \Sigma_2$. Then the following relation holds:

$$Z(\Sigma_1, T_{\Sigma_1})Z(\Sigma_2, T_{\Sigma_2})Z'(S_2, \partial T_{\Sigma_1}, \partial T_{\Sigma_2}) = Z(\Sigma_1 \cup_{S_2} \Sigma_2, T_{\Sigma_1 \cup_{S_2} \Sigma_2}),$$

(vi) Any (Σ, T_Σ) can be computed locally, that is

$$Z(\Sigma, T_\Sigma) = \prod_{f \in T_\Sigma} Z(f, \ell(f)),$$

where f is a d-face of T_Σ, with the trivial triangulation and label $\ell(f)$ coming from T_Σ.

In these axioms we have written ∂T_Σ instead of $T_\Sigma|_{\partial\Sigma}$, and we we will sometimes use this notion in the following sections. In the third axiom, we do not assume that the triangulation on S_2 coming from Σ_1 is the same as the one coming from Σ_2. If these do coincide, the axiom can truly be called *gluing axiom*. This is also true for the last axiom, where we basically glue two manifolds only along a common part of their boundary (here we assume that the triangulations are the same on this part of the boundary).

Special care has to be taken of the first gluing axiom, for it is not clear, how the glued manifold $\Sigma_1 \cup_{S_2} \Sigma_2$ is triangulated, since at the boundary S_2 the triangulations may not be fit together. This one solves by refining both triangulations at the boundary and gluing only afterwards. But for this to work, we have to show that we may refine a triangulation without changing its value under Z. This is indeed what we show next, and it will be shown only using the second gluing axiom, so the problem is thereby solved.

Sometimes we will write such a topological quantum field theory just as Z and not as (Z', Z). The set of Picken-type topological quantum field theories of dimension d on M we denote by $\text{TQFT}^d_{Picken}(M)$, and it can be given the structure of a group, simply by using the fact that $U(1)$ is a group. For this we need a neutral element, which of course will be the trivial topological quantum field theory:

3.3.2 DEFINITION: The **trivial** Picken-type topological quantum field theory in dimension d assigns to all $\mathcal{U}$-adapted manifolds (S, T_S) and (Σ, T_Σ) (as above) the values $Z'(S, T_S) = 1 \in U(1)$ and $Z(\Sigma, T_\Sigma) = 1 \in U(1)$.

This is obviously a Picken-type topological quantum field theory, as it obeys all axioms trivially.

3.3.3 LEMMA: *Let $Z, \tilde{Z} \in TQFT^d_{Picken}(M)$ be two Picken-type topological quantum field theories. Then define $Z \otimes \tilde{Z}$ to be*

$$(Z \otimes \tilde{Z})(S, T_S, \tilde{T}_S) := Z(S, T_S, \tilde{T}_S) \cdot \tilde{Z}(S, T_S, \tilde{T}_S)$$

and

$$(Z \otimes \tilde{Z})(\Sigma, T_\Sigma) := Z(\Sigma, T_\Sigma) \cdot \tilde{Z}(\Sigma, T_\Sigma).$$

*Furthermore, to a given $(Z, Z') \in$ **TQFT**$^d_{Picken}(M)$ let its inverse $(Z^{-1}, Z'^{-1}) \in$ **TQFT**$^d_{Picken}(M)$ be given by taking $Z^{-1}(\Sigma, T_\Sigma) = Z(\Sigma, T_\Sigma)^{-1}$ and*

$$Z'^{-1}(S, T_S, \tilde{T}_S) := Z'(S, T_S, \tilde{T}_S)^{-1}.$$

*Using this structure, the set **TQFT**$^d_{Picken}(M)$ is a group, with the neutral element being the trivial topological quantum field theory.*

Proof. This is trivially the case. □

Actually such a topological quantum field theory is also a functor, but since we do not want to construct the corresponding category of triangulated manifolds, we rather stick with the definition above, and take it for granted that the assignment satisfies the properties of a functor. To be a bit more explicit, e.g. we assume that $Z((\Sigma_1, T_{\Sigma_1}) \amalg (\Sigma_2, T_{\Sigma_2})) = Z(\Sigma_1, T_{\Sigma_1})Z(\Sigma_2, T_{\Sigma_2})$ and that $Z(\emptyset) = 1 \in U(1)$.

Before we continue, let us think about what the axioms of the topological quantum field theory do mean. If we think of Z' and Z to be some kind of general parallel transport, then the first axiom tells us, that going from one triangulation to another is transitive. More interesting, the second axiom tells us how the parallel transport changes, if we change the triangulation of the boundary. Assume for a moment that we only deal with one manifold Σ with two triangulations T_Σ and $\tilde{T}_\Sigma$, so that the diffeomorphism ψ is actually the identity. As we have seen in the motivation, the change of triangulation can be accomplished by simply gluing the special cylinder $\partial\Sigma \times [0, 1]$ along the boundary of Σ, where the cylinder has the given triangulation T_Σ on its bottom and the triangulation $T_{\tilde{\Sigma}}$ on the top. This means, that the correction factor is actually the parallel transport of this special cylinder, so that we may expect that even in the general case $Z'(S, T_S, \tilde{T}_S) = Z(S \times [0, 1], T_S \cdot \tilde{T}_S)$, where we denoted by $(S \times [0, 1], T_S \cdot \tilde{T}_S)$ the cylinder with some kind of triangulation as described above. The only problem with this guess is that the cyclinder does not possess a canonical triangulation that is equal to T_S at its bottom and $\tilde{T}_S$ at its top. We may choose such a triangulation, but then we have to decide how two choices do differ. This is what we will actually do in the next paragraphs. To speak more formally, we extend the topological quantum field theory to include not only triangulated manifolds, but also manifolds with an embedded graph. These will also carry a labelling $\ell : G_\Sigma \longrightarrow I$, where G_Σ denotes the regions and faces, the graph defines. We will denote these manifolds simply by (Σ, G_Σ) (and $(S, G_S, \tilde{G}_S)$ respectively), in order to distinguish them from triangulated manifolds. We will drop this notation as soon as we have shown that any topological quantum field theory can be extended to such objects, and will assume T_S to be either a graph or a triangulation. Any graph, being a collection of vertices, edges, faces etc., can be refined to a triangulation by splitting and adding new q-faces. Whenever we refine a graph to a triangulation, we assume that the labeling stays the same: If a q-face f_q is splitted, both parts get the same label as f_q, and if a new q-face is introduced, it obtains the label of the region it lies in.

3.3.4 DEFINITION: Suppose (Σ, G_Σ) is a d-dimensional manifold with an embedded graph. Then we can extend any given Picken-type topological quantum field theory to manifolds with embedded graphs, by choosing an arbitrary refinement of the graph to a triangulation T_Σ and setting $Z(\Sigma, G_\Sigma) := Z(\Sigma, T_\Sigma)$. If $(S, G_S, \tilde{G}_S)$ is a $(d-1)$-dimensional manifold, we equally define $Z'(S, G_S, \tilde{G}_S) := Z'(S, T_S, \tilde{T}_S)$, where T_S and $\tilde{T}_S$ are refinements of G_S and $\tilde{G}_S$ to triangulations respectively.

3.3.5 COROLLARY: *This extension of a Picken-type topological quantum field theory is well-defined.*

Proof. We simply have to show that if T_Σ is any refinement of G_Σ to a triangulation, then refining it further does not change the value of $Z(\Sigma, T_\Sigma)$. This suffices, because if we choose any other triangulation, say $\tilde{T}_\Sigma$, then we can find a common refinement, say T'_Σ, of T_Σ and $\tilde{T}_\Sigma$. If we have shown that refining any triangulation does not change the value of $Z(\Sigma, T_\Sigma)$, then we know $Z(\Sigma, T_\Sigma) = Z(\Sigma, T'_\Sigma) = Z(\Sigma, \tilde{T}_\Sigma)$ and the claim is verified.

Now, to show that refining a triangulation does not change the value of Z, it is enough to consider just a subdivision of one face, for every refinement can be obtained by splitting finitely many faces. Suppose then $f_d \cong \Delta_d$ is a d-face, that is a d-dimensional simplex. Splitting it gives us two subsimplices, say f_1 and f_2, as shown in figure

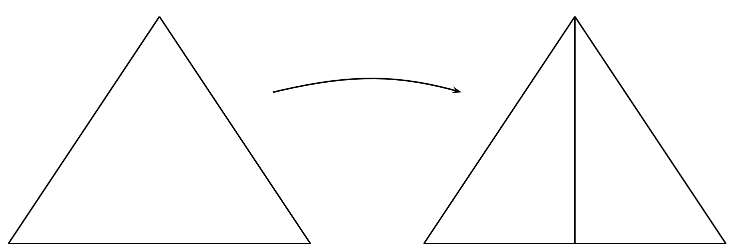


Figure 3.4: Subdividing a 2-simplex

But by the gluing axiom we have $Z(f_d, \ell(f_d)) = Z(f_1, \ell(f_1))Z(f_2, \ell(f_2))$, which shows that nothing changes. Therefore the corollary is proved. □

As we said, with this corollary the first gluing axiom is shown to be well-defined.

Let us come back to the problem of equipping a cylinder with a triangulation that is equal to T_S on the bottom and $\tilde{T}_S$ on its top. By the formal construction above, It is enough to take the graph that is constructed as the union of $T_S \times [0, \frac{1}{2}] \cup S \times \{\frac{1}{2}\} \cup \tilde{T}_S \times [\frac{1}{2}, 1] \subset S \times [0, 1]$, see figure 3.5. This is clearly a d-dimensional graph in $S \times [0, 1]$ (if we take the union of T_S and $\tilde{T}_S$ on $S \times \{\frac{1}{2}\}$).

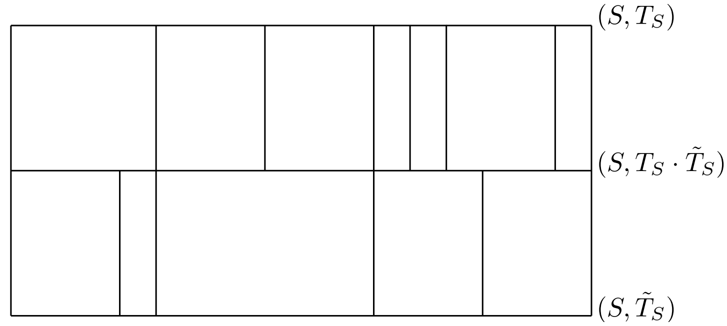


Figure 3.5: The canonical graph of the cylinder

With all these considerations we are finally able to prove the guess we made above:

3.3.6 LEMMA: *Suppose S is a closed, $(d-1)$-dimensional manifold and T_S, $\tilde{T}_S$ are triangulations of S. Consider the cylinder $S \times [0, 1]$ with the graph $T_S \cdot \tilde{T}_S$ defined by T_S and $\tilde{T}_S$ and map $\varphi_{S\times[0,1]} : S \times [0, 1] \longrightarrow M$ by $\varphi_{S\times[0,1]}(t, x) = \varphi_S(x)$. Then we have*

$$Z(S \times [0, 1], T_S \cdot \tilde{T}_S) = Z'(S, T_S, \tilde{T}_S).$$

Proof. First, we claim that $Z(S \times [a, b], T_S \cdot T_S)$ and $Z(S, \times[c, d], T_S \cdot T_S)$ are equal, that is, we take the »constant« graph on $S \times [a, b]$ and $S \times [c, d]$, and claim that they have the same parallel transport. Obviously $S \times [a, b]$ and $S \times [c, d]$ are diffeomorphic, so we know that $Z(S \times [a, b], T_S \cdot T_S) = Z(S \times [c, d], T_S \cdot T_S)Z(\partial S, T_S, T_S)$. But by the first axiom, the correction factor is 1, so the claim follows. Second, we claim that $Z(S \times [a, b], T_S \cdot T_S) = 1$: If we consider $[a, b]$ and $[b, c]$, our gluing axiom tells us, that $Z(S \times [a, c], T_S \cdot T_S) = Z(S \times [a, b], T_S \cdot T_S)Z(S \times [b, c], T_S \cdot T_S)$, since they do have the same triangulation on their common boundary. Rewriting $Z(S \times [b, c], T_S \cdot T_S)$ as $Z(S \times [a, c], T_S \cdot T_S)$ using the first claim, we see that $Z(S \times [a, b], T_S \cdot T_S) = 1$. Finally using the gluing axiom again, we have $Z(S \times [0, 1], T_S \cdot \tilde{T}_S) = Z(S \times [0, \frac{1}{2}], T_S)Z'(S, T_S, \tilde{T}_S)Z(S \times [\frac{1}{2}, 1], T_S) = Z'(S, T_S, \tilde{T}_S)$. □

3.4 The Picken-TQFT induced by a Deligne cocycle

Taking the 1-dimensional case as a guideline, we will show that any Deligne d-cocycle induces a topological quantum field theory in the sense made precise in the last section.

Let us repeat quickly the motivation for using triangulations. Consider ξ a Deligne d-cocycle on M (Note, that we now use cocycles instead of classes. The arguments in the motivation still work). Given any d-dimensional manifold Σ with boundary in M, the expression $\exp \int_\Sigma \rho$, with ρ a trivialization of the pullback of ξ, is generally not well-defined. But two different trivializations differ by a correction term $c(\partial\Sigma)$:

$$\exp(\int_\Sigma \rho) = \exp(\int_\Sigma \tilde{\rho}) \cdot c(\partial\Sigma, \rho, \tilde{\rho}),$$

which *only* depend on the values of ρ on the boundary of Σ.[3] Working with triangulated manifolds (Σ, T_Σ) we can split the contributions in the local holonomy formula into an »internal « and an »external« part, giving a similar formula as above. These two parts then will give rise to a Picken-type topological quantum field theory.

In order to find the right expression for parallel transport, we start again with the local formula for the holonomy, and try to drop all external contributions. Assume again for a moment that Σ is a closed d-dimensional manifold, and T_Σ a triangulation of Σ, having the set k^i as i-faces and ℓ as a labeling, we have

$$\mathrm{hol}_\xi(\Sigma, T_\Sigma) = \exp\left(\sum_{f_d \in k^d} \int_{f_d} \xi^d_{\ell(f_d)} + \sum_{f_{d-1} \subset f_d} \int_{f_{d-1}} \xi^{d-1}_{\ell(f_d)\ell(f_{d-1})} + \cdots\right) \cdot \prod_{f_0 \subset f_1 \subset \cdots \subset f_d} \xi^0_{\ell(f_d)\cdots\ell(f_0)}(f_0).$$

Here we supposed that f_i is an i-face, $f_i \in k^i$. Dropping all external contributions, we get a modified formula for the local holonomy:

3.4.1 DEFINITION: Let (Σ, T_Σ) be a d-dimensional manifold Σ together with a triangulation T_Σ and a labeling ℓ. Define its parallel transport to be

$$\mathrm{pt}_\xi(\Sigma, T_\Sigma) = \exp\left(\sum_{f_d \in k^d} \int_{f_d} \xi^d_{\ell(f^d)} + \sum_{\substack{f_{d-1} \subset f_d \\ f_{d-1} \not\subset \partial\Sigma}} \int_{f_{d-1}} \xi^{d-1}_{\ell(f_d)\ell(f_{d-1})} + \ldots\right) \cdot \prod_{\substack{f_0 \subset f_1 \subset \ldots \subset f_d \\ f_0 \not\subset \partial\Sigma}} \xi^0_{\ell(f_d)\cdots\ell(f_0)}(f_0).$$

[3]This claim one has to show, but we do not need it here. We will derive this formula from our definitions.

Next we want to define the correction factor. Instead of taking a direct approach, which would result in some awkward formula, we will use the relation at the end of section 3.1. So our guess is that

$$\mathrm{pt}_\xi(S, T_S, \tilde{T}_S) := \mathrm{pt}_\xi(S \times [0,1], T_S \cdot \tilde{T}_S)$$

together with $\mathrm{pt}_\xi(\Sigma, T_\Sigma)$ will obey all the axioms of a Picken-type topological quantum field theory (Note that we have not the data for a field theory yet, so we use this as a definition and deduce the axioms from it, just contrary to what we did in the last section). Unluckily up to now we only talked about triangulated manifolds and have to explain the object $(S \times [0,1], T_S \cdot \tilde{T}_S)$, for it is not obvious what triangulation we have to take on it. Again we claim that the choice of the triangulation does not matter, and every choice of a triangulation will yield the same value for $\mathrm{pt}_\xi(S \times [0,1], T_S \cdot \tilde{T}_S)$.

We verify this by the same procedure as before: Suppose we are given a manifold with a labeled graph G_Σ as seen in the last section, and suppose that we are given two ways to subdivide this graph into a proper triangulation. Let us call these T_Σ and $\tilde{T}_\Sigma$. By the theory of triangulations we know, that these two triangulations have a common refinement, $\hat{T}_\Sigma$. If we succeed in verifying, that refining a proper triangulation does not change its parallel transport, we are done, for then we know that $\mathrm{pt}_\xi(\Sigma, T_\Sigma)$ is the same as the parallel transport $\mathrm{pt}_\xi(\Sigma, \hat{T}_\Sigma)$ of the refined triangulation, but the latter one is also the same as $\mathrm{pt}_\xi(\Sigma, \tilde{T}_\Sigma)$, for $\hat{T}_\Sigma$ is also the refinement of $\tilde{T}_\Sigma$, so both triangulations do indeed give the same parallel transport.

3.4.2 PROPOSITION: *Any choices of a proper triangulation for a given labeled graph on Σ yield the same parallel transport. In other words, if we are given two triangulations T_Σ and $\tilde{T}_\Sigma$, refining a graph G_Σ, then*

$$pt_\xi(\Sigma, T_\Sigma) = pt_\xi(\Sigma, \tilde{T}_\Sigma).$$

As we know, every refinement can be realized as a finite chain of subdividing simplices (Notice again, that the labels are not changed while subdividing, each two subdivided faces get the same label, which we will use in a moment). Therefore it is enough to calculate the effect of subdividing one d-simplex. Let us again start by a simple example to get an idea of the general scheme.

3.4.3 EXAMPLE: First consider the case $d = 2$ (see figure 3.4). Subdividing the triangle, we see that this is introduces one new edge and one new vertex. Denoting the two faces coming from subdividing f by f_1, f_2, we have for the parallel transport $\int_f \xi^2_{\ell(f)} = \int_{f_1} \xi^2_{\ell(f)} + \int_{f_2} \xi^2_{\ell(f)}$, that is both contributions are equal (The two new faces are obviously internal). Here we agreed (as before) to assign to any subdivided faces the label of its original face, so indeed f_1 and f_2 carry the label $\ell(f)$. Let us look now at the edges.

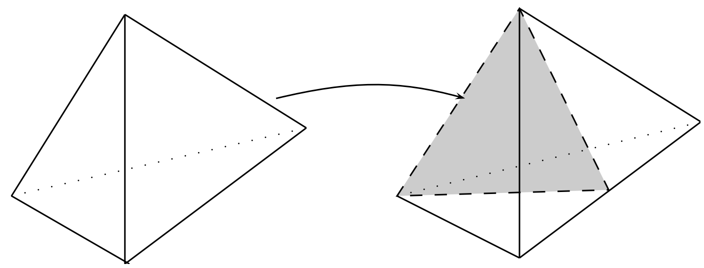


Figure 3.6: Subdividing a 3-simplex

The edge e_0 is again subdivided by the new vertex $\tilde{v}$ into two edges $e_{0,1}$ and $e_{0,2}$, and we get a new edge, let us call it $\tilde{e}$. The refinement assigns to any new face the label of the whole simplex (or better region), so $\tilde{e}$ gets the label $\ell(f)$. Therefore the contribution of the new edge is $\int_{\tilde{e}} \xi^1_{\ell(f_1)\ell(f)} - \xi^1_{\ell(f_2)\ell(f)}$. But as already said, $\ell(f_1) = \ell(f_2) = \ell(f)$, so by the anti-symmetry the contribution vanishes, i.e. the new edge $\tilde{e}$ has no influence on the parallel transport. The other two edges $e_{0,1}$ and $e_{0,2}$ yield as before the contribution of the edge e. Finally we only have to consider the vertices. We get two contributions from the edge $\tilde{e}$, but since $\tilde{e}$ does not contribute, also its subfaces do not. Finally there is a new vertex $\tilde{v}$. But as a subvertex of the edge e_0 it gets the same label as e_0, hence its contributions, e.g. $g_{\ell(f_1)\ell(e_{0,1})\ell(\tilde{v})} = 1$, for $\ell(e_{0,1}) = \ell(e_0) = \ell(\tilde{v})$.

Similarly one can handle the 3-simplex (see figure 3.4). It is subdivided into two 3-simplices v_1 and v_2. Then the 3-dimensional integral is again unchanged. The four faces f_1, f_2, f_3, f_4 partly get subdivided into $f_{2,0}, f_{2,1}, f_{3,0}$ and $f_{3,1}$ and partly stay unaltered. And also one new face, $\tilde{f}_0$ is being introduced. Therefore the subdivided simplex has faces $f_1, f_{2,0}, f_{2,1}, f_{3,0}, f_{3,1}, \tilde{f}_0, f_4$. For the edges we have again that we have two new ones $\tilde{e}_0$ and $\tilde{e}_1$ as well as a subdivided one, into $e_{1,0}$ and $e_{1,1}$. Looking at the vertices we see that there is only one new vertex. One can easily work out the contributions to the parallel transport of all these divided and new faces, and by utilizing the two rules above (assign the same label to subdivided faces, assign the label of the 3-simplex to any new face), one can show that subdividing does not change the parallel transport.

Looking at the numbers of new and subdivided faces, which we collect for the two examples above in figure 3.7, one gets the idea that there is a pattern behind these numbers, and indeed, one can show that generally there are $\binom{d-1}{d-i-1}$ new i-simplices while $\binom{d-1}{i-1}$ i-simplices get split up.

Let us now come to the general case, and proof that subdividing a d-simplex does not change the parallel transport:

3.4.4 PROPOSITION: *Refining any simplex of* (Σ, T_Σ) *does not change its parallel trans-*

	new	split
k_3	0	1
k_2	1	2
k_1	2	1
k_0	1	0

	new	split
k_4	0	1
k_3	1	3
k_2	3	3
k_1	3	1
k_0	1	0

Figure 3.7: Number of simplices

port pt$_\xi(\Sigma, T_\Sigma)$.

Proof. The proof is already outlined in the example. Refining a simplex does only do two things: It splits up q-subfaces f_q into two q-subfaces $f_{q,0}$ and $f_{q,1}$, which obtain both the same label as f_q, and it introduces new q-subfaces with the same label as the face (or region) they do lie in. Comparing both parallel transport formulas for the original simplex and the refined simplex, we see that for the contribution of the q-subface f_q we have

$$\int_{f_q} \xi^q_{\ell(f_d)\ldots\ell(f_q)} = \int_{f_{q,0}} \xi^q_{\ell(f_d)\ldots\ell(f_{q_0})} + \int_{f_{q,1}} \xi^q_{\ell(f_d)\ldots\ell(f_{q_1})},$$

where $f_d, f_{d-1}, \ldots$ is the chain of faces, f_q is part of. This holds by the refinement procedure, for $\ell(f_{q_0}) = \ell(f_{q_1}) = \ell(f_q)$, as we have said. This way we get all the contributions of the parallel transport of the original simplex, for either the face f_q is unaltered, or it gets split, and the above equation shows, that its contribution is not lost.

Let us now consider new q-faces. If $\tilde{f}_q$ is a new q-face, we may assume that it has been introduced by splitting a face f_{q+1}. Its label is then the same as the as the label of f_{q+1}, therefore neither the face itself, nor any of its (new) subfaces does contribute to the parallel transport. This finishes the proof. □

3.4.5 COROLLARY: *Any refinement of* (Σ, T_Σ) *does not change its parallel transport pt*$_\xi(\Sigma, T_\Sigma)$.

This makes it possible to calculate $\text{pt}_\xi(S \times I, T_S \times \tilde{T}_S)$ and take this as the correction factor:

3.4.6 DEFINITION: Let $(S, T_S, \tilde{T}_S)$ be a $(d-1)$-dimensional manifold S with two triangulations T_S and $\tilde{T}_S$. We define

$$\text{pt}_\xi(S, T_S, \tilde{T}_S) := \text{pt}_\xi(S \times [0,1], T_S \cdot \tilde{T}_S).$$

We may now show, that the parallel transport satisfies the two gluing axioms: First consider two manifolds (Σ_1, T_{Σ_1}) and (Σ_2, T_{Σ_2}) with boundaries $\Sigma_1 = S_1 \amalg S_2$ and $\Sigma_2 = S_2^* \amalg S_3$. The glued manifold $\Sigma_1 \cup_{S_2} \Sigma_2$ is smooth (by our assumptions on the collar), but may not be a triangulated manifold anymore. Instead it carries a well-defined graph, and by the reasoning above we may refine this graph to a triangulation, without changing its parallel transport. This is the object we deal with.

3.4.7 PROPOSITION: *In this situation above we have*

$$pt_\xi(\Sigma_1, T_{\Sigma_1}) pt_\xi(\Sigma_2, T_{\Sigma_2}) pt_\xi(S_2, T_{\Sigma_1}|_{S_2}, T_{\Sigma_2}|_{S_2}) = pt_\xi(\Sigma_1 \cup_{S_2} \Sigma_2, T_\Sigma \cdot T_{\tilde{\Sigma}}).$$

Proof. We may assume that Σ_1 and Σ_2 have been subdivided such that their triangulations on the boundary S_2 match up to the labeling. As in the example, the only contributions missing in $\mathrm{pt}_\xi(\Sigma_1, T_{\Sigma_1})\mathrm{pt}_\xi(\Sigma_2, T_{\Sigma_2})$ are those coming from S_2. But comparing these with the parallel transport formula of $\mathrm{pt}_\xi(S_2, T_{\Sigma_1}|_{S_2}, T_{\Sigma_2}|_{S_2})$ we see immediatly that they both agree, that is the upper formula holds. □

Next, to prove the first two axioms, we can introduce $(d+1)$-dimensional objects (H, T_H) (our most prominent $(d+1)$-dimensional object will be $\Sigma \times [0,1]$), and prove a lemma similar to lemma 3.1.9. Notice, again that the object H in general will be a manifold with corners, but these corners can be dropped easily, for they are a null set, so we may again pretend to work with smooth manifolds. Now, if we are going to integrate the curvature of a Deligne cocycle ξ over such an object H, then, being motivated by the same formula for the holonomy, we might expect the following formula:

3.4.8 LEMMA: *Suppose (H, T_H) is a $(d+1)$-dimensional object with a triangulation T_H. Then we have*

$$pt_\xi(\partial H, \partial T_H)) = \exp \int_H \varphi_H^*(\mathrm{curv}(\xi)).$$

Proof. We outline the proof only, which is actually an awkward exercise in repeatedly using Stokes theorem and the cocycle condition. We start with the right hand side and use the definition of the curvature to obtain

$$\exp \int_H \varphi_H^*(\mathrm{curv}(\xi)) = \exp \sum_{f_{d+1} \in k^{d+1}} \int_{f_{d+1}} \varphi_H^*(d\xi^d_{\ell(f_{d+1})}) = \exp \sum_{f_{d+1} \in k^{d+1}} \int_{\partial f_{d+1}} \varphi_H^*(\xi^d_{\ell(f_{d+1})}).$$

Rewrite this and use the cocycle relation to change the labeling: $\xi^d_{\ell(f_{d+1})} - \xi^d_{\ell(f_d)} = d\xi^{d-1}_{\ell(f_d)\ell(f_{d+1})}$ to obtain

$$\exp \sum_{f_d \subset f_{d+1}} \int_{f_d} \varphi_H^* \xi^d_{\ell(f_d)} + \int_{f_d} \varphi_H^* d\xi^{d-1}_{\ell(f_d)\ell(f_{d+1})}.$$

(Actually there is a further sum over al faces f_{d+1}, we drop it here and in the following). Now the first term vanishes, if f_d is an internal face, for we know that every d-face is part of exactly two $(d+1)$-faces (this is obviously not true for the lower-dimensional faces). Contrary to this, on the boundary we get the term $\exp \sum_{f_d \subset \partial H} \int_{f_d} \varphi_H^* \xi^d_{\ell(f_d)}$. Keeping in mind, that φ_H is just φ_Σ and $\varphi_{\Sigma'}$, we have identified the first summand of the parallel transport of Σ and Σ'.

We can repeat this step using the cocycle relation to reexpress the second term. Generally, that is after $d-p$ steps, we have to deal with the term

$$\sum_{f_p \subset \cdots \subset f_{d+1}} \int_{f_p} \varphi_H^* d\xi^{p-1}_{\ell(f_p)\ldots\ell(f_{d+1})}.$$

Note that indeed $d\xi^{p-1}$ is a p-form, and ξ^{p-1} has, by our usual formula, $r+1$ indices, and $d-1+p+1 = d-p+2 = r+1$. Using Stokes theorem we get

$$\sum_{f_{p-1} \subset f_p \subset \cdots \subset f_{d+1}} \int_{f_{p-1}} \varphi_H^* \xi^{p-1}_{\ell(f_p)\ldots\ell(f_{d+1})}.$$

To ease the notation, denote $\ell(f_p)$ by i_1, $\ell(f_{p+1})$ by i_2 etc. until $\ell(f_{d+1})$, which is replaced by i_{r+1}. Furthermore, let $i_0 := \ell(f_{p-1})$. As we saw in lemma 3.1.9, now the idea is to use the cocycle relation, and to replace this term by indices that all do involve i_0 and a exterior differential term, which has the same general formula as above. We know that

$$\delta(\xi^{p-1}_{i_1,\ldots,i_{r+1}})_{i_0,\ldots,i_{r+1}} = (-1)^r d\xi^{p-2}_{i_0,\ldots,i_{r+1}}.$$

Using the definition of $\delta = \sum_{j=0}^{r+1}(-1)^j \xi^{p-1}_{i_0,\ldots,\hat{i}_j,\ldots,i_{r+1}}$ we can write

$$\xi^{p-1}_{i_1,\ldots,i_{r+1}} = \sum_{j=1}^{r+1}(-1)^{j+1}\xi^{p-1}_{i_0,\ldots,\hat{i}_j,\ldots,i_{r+1}} + (-1)^r d\xi^{p-2}_{i_0,\ldots,i_{r+1}}.$$

Now we claim, that each term in the sum on the right hand side that contains the label i_{r+1} is zero, regardless if f_{p-1} is an inner or outer face. This one can see just in the same way as in the »motivating example«, that is lemma 3.1.9. Moreover we claim that the only missing term $\xi^{p-1}_{i_0,\ldots,i_r}$ contributes only to the sum, if f_{p-1} is an outer vertex. This is again proven as dimension 1 case. This contribution gives us a term

$$\sum_{f_{p-1} \subset f_p \subset \ldots f_{d+1}, f_{p-1} \subset \partial H} \int_{f_{p-1}} \varphi_H^* \xi^{p-1}_{\ell(f_{p-1})\ldots\ell(f_d)},$$

which is seen to be next term in the formula for the parallel transport. Now we can repeat this step with the term left over.

We have been very sloppy in this outline of the proof, especially the signs are wrong, and we leave it to the mindful reader as an exercise to fill the gaps in the proof and pick the right signs. □

We can use this lemma to verify the following two propositions:

3.4.9 PROPOSITION: *Let S be a closed d-manifold with three triangulation T_S, $\tilde{T}_S$ and T'_S. Then we have*

$$pt_\xi(S, T_S, \tilde{T}_S) pt_\xi(S, \tilde{T}_S, T'_S) = pt_\xi(S, T_S, T'_S).$$

Proof. By definition we have $\mathrm{pt}_\xi(S, T_S, \tilde{T}_S) = \mathrm{pt}_\xi(S \times [0,1], T_S \cdot \tilde{T}_S)$ and $\mathrm{pt}_\xi(S, \tilde{T}_S, T'_S) = \mathrm{pt}_\xi(S \times [0,1], \tilde{T}_S \cdot T'_S)$. Using the gluing axiom, the product of these is $\mathrm{pt}_\xi(S \times [0,1], T_S \cdot \tilde{T}_S \cdot \tilde{T}_S \cdot T'_S)$. Lemma 3.4.8 yield the equality of this and $\mathrm{pt}_\xi(S \times [0,1], T_S \cdot T'_S)$, for both these are diffeomorphic (and this diffeomorphism, being an $(d+1)$-dimensional object, can be given the graph that squeezes $\tilde{T}_S \cdot \tilde{T}_S$ to $\mathrm{pt} \times [0,1]$). But this is $\mathrm{pt}_\xi(S, T_S, T'_S)$. □

3.4.10 PROPOSITION: *If $\psi : \Sigma \longrightarrow \tilde{\Sigma}$ is a diffeomorphism that keeps the boundary fix, we have for any triangulations of Σ and $\tilde{\Sigma}$*

$$pt_\xi(\tilde{\Sigma}, T_{\tilde{\Sigma}}) = pt_\xi(\partial\Sigma, \partial T_\Sigma, \partial T_{\tilde{\Sigma}}) pt_\xi(\Sigma, T_\Sigma).$$

Proof. Again this is a simple application of lemma 3.4.8, if we take into consideration that φ_H has not full rank by definition, so that $\int_H \varphi_H^*(\mathrm{curv}(\xi)) = 1$. We only have to extend the triangulation on the boundary of the diffeomorphism given by T_Σ and $T_{\tilde{\Sigma}}$, but this is possible by elementary moves. □

Summing up, by replacing the choice of a trivialization with the choice of a triangulation, we have shown:

3.4.11 THEOREM: *Any Deligne d-class on M induces a Picken-type topological quantum field theory on M.*

Denote this assignment $\xi \mapsto \mathrm{pt}_\xi$ by PT, that is

$$\mathrm{PT} : \check{Z}^d(M, \mathcal{D}^d) \longrightarrow \mathbf{TQFT}^d_{Picken}(M).$$

We have already seen that $\mathrm{TQFT}^d_{Picken}(M)$ is a group. Likewise the cocycles $\check{Z}^d(M, \mathcal{D}^d)$ also form a group, and we claim, that the map PT, assigning to any Deligne cocycle the topological quantum field theory above, is indeed a group morphism:

3.4.12 LEMMA: *Let ξ, η be two Deligne cocycles. Then*

$$\mathrm{PT}(\xi + \eta) = \mathrm{PT}(\xi) \otimes \mathrm{PT}(\eta).$$

Proof. This follows directly from the explicit formula for $\mathrm{pt}_{\xi+\eta}$, because the integral is additive and $\exp$ multiplicative, and the tensor product of two topological quantum field theory of Picken type is just given by multiplying, so we have $\mathrm{pt}_\xi \mathrm{pt}_\eta = \mathrm{pt}_{\xi+\eta}$. Note that this is enough, for we can rewrite $\mathrm{pt}_\xi(S, T_S, \tilde{T}_S)$ as $\mathrm{pt}_\xi(S \times [0,1], T_S \cdot \tilde{T}_S)$. □

3.4.13 REMARK: While the definition of $\mathrm{pt}_\xi(\Sigma, T_\Sigma)$ is quite equal to the one of Picken (see theorem 4.3 of [24]), the definition of $Z'(S, T_S, T_S') = \mathrm{pt}_\xi(S, T_S, T_S')$ in Picken's paper looks a bit different (it is called $\exp i \int_{(X,T)}(g, A, F)$ there). Let us derive his formula quickly. Since he only considers the case $d = 2$, assume S to be a circle in M, mapped into M via φ_S. We assume that the triangulations T_S and $\tilde{T}_S$ have a special form and also a special position to each other (see figure 3.8). The definition of $\mathrm{pt}_\xi(S, T_S, T_S')$ tells us to compute the parallel transport of the cylinder seen in 3.8.

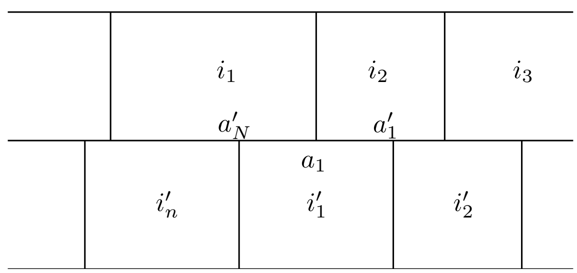


Figure 3.8: A possible labeling of a 1-dimensional submanifold S

The very first thing to notice is, that since the whole cylinder is mapped upon a 1-dimensional manifold in M, neither the faces nor the edges transverse to the core of the cylinder do contribute, for the integral over them vanishes. Next we see, that one possible choice of a labeling is to assign the highest label possible. Therefore the edge between i_1 and i'_1 is labeled i'_1, the vertex between i_1, i'_1, i_2 gets the label i_2 etc. This choice gives the contribution $-g_{i'_1 i_1 i_2}(\ell(a_1)) = g_{i_1 i'_1 i_2}(\ell(a_1))$, the very first term of formula (8) in Picken's Theorem 4.3. Next we have the holonomy along the edge between a_1 and a'_1. Here the only non-vanishing term is given by $\int_{a'_1}^{a_1} \ell^*(A_{i'_1 i_2}) = \int_{a_1}^{a_1} \ell^*(A_{i_2 i'_1})$. Continuing this procedure, one recovers the whole formula (8).

Next we will show any Picken-type topological quantum field theory induced by a Deligne cocycle has very special properties, distinguishing it from a common one:

thin-invariance and smoothness. These two properties already occured in the context of holonomies (see section 2.2), and can be reformulated in the parallel transport case.

3.4.14 DEFINITION: Two triangulated manifolds (Σ, T_Σ) and $(\tilde{\Sigma}, T_{\tilde{\Sigma}})$ are called **thin-homotopic**, if $(\partial\Sigma, T_\Sigma|_{\partial\Sigma}) = (\partial\tilde{\Sigma}, T_{\tilde{\Sigma}}|_{\partial\tilde{\Sigma}})$ and there is a smooth, triangulated relative homotopy (H, T_H) from Σ to $\tilde{\Sigma}$ that fixes the boundary and has $\mathrm{rank}(DH) \leq d$ everywhere.

3.4.15 DEFINITION: A Picken-type topological quantum field theory (Z', Z) is called **thin-invariant**, if for any two thin-homotopic surfaces (Σ, T_Σ) and $(\tilde{\Sigma}, T_{\tilde{\Sigma}})$ we have

$$Z(\Sigma, T_\Sigma) = Z(\tilde{\Sigma}, T_{\tilde{\Sigma}}).$$

The thin-invariance of the topological quantum field theory induced by a Deligne cocycle now follows immediatly from lemma 3.4.8, if we note that the pull-back of the curvature of ξ via H is zero for H is a thin-homotopy.

Smoothness of a topological quantum field theory is similar to the smoothness condition of a holonomy map, but we need to keep in mind that Σ is a manifold.

3.4.16 DEFINITION: Let $U \subset \mathbb{R}^k$ be an open subset. Denote the set of maps from Σ to M by $\mathcal{F}(\Sigma, M)$. Then a **smooth k-dimensional family** of triangulated manifolds is a map $U \longrightarrow \{\varphi_u : \Sigma \longrightarrow M\}$ such that the map $U \times \Sigma \longrightarrow M$, $(u, s) \mapsto \varphi_u(s)$ is smooth.

We call a Picken-type topological quantum field theory (Z', Z) smooth, if for any smooth k-dimensional family, say ψ, the map $U \longrightarrow U(1)$, $u \mapsto Z(\psi(u)) = Z(\Sigma_u, T_{\Sigma_u})$

3.4.17 PROPOSITION: *Let ξ be a Deligne d-class on M and Σ be a d-dimensional manifold with boundary in M. Then the Picken-type topological quantum field theory induced by ξ is thin-invariant and smooth.*

Proof. The smoothness of pt_ξ is clear from its construction, for the local holonomy is a smooth construction, and we have already discussed the thin-invariance. □

Obviously the trivial topological quantum field theory is smooth as well as thin-invariant. Hence also the smooth, thin-invariant topological quantum field theories form a group, which we will denote simply by $\mathrm{TQFT}^{d,\infty}_{Picken}(M)$.

3.5 Reconstruction of the Deligne cocycle

We are now in a position to provide a reconstruction of Deligne cocycles via their parallel transport. The reconstruction is analogous to the holonomy case. As before we start with the construction of certain simplices and define a family of p-forms, which form a Deligne cocycle. We show that the group of Picken topological quantum field theories and Deligne d-cocycles are isomorphic, by showing that the map

$$\mathrm{PT} : \check{Z}^d(M, \mathcal{D}^d) \longrightarrow \mathrm{TQFT}^{d,\infty}_{Picken}(M)$$

is surjective and injective.

Before we start the actual construction, let us lay down some notations and conventions. First, we need a rule on how to triangulate a constant r-simplex $\Delta_r = \langle x_1, \ldots, x_{r+1}\rangle$ in a given point $v_0 \in M$. This is simply done by using the barycentric subdivision of Δ_r.

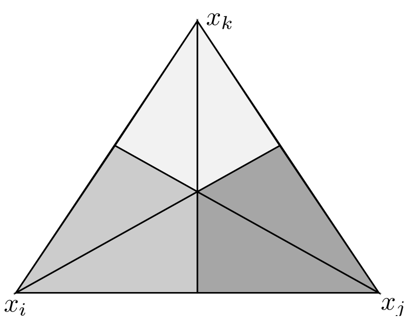


Figure 3.9: Labeling of a barycentrically subdivided simplex

To any face of this subdivision, containing the vertex x_j we assign the label i_j. The subfaces obtain the *highest* label that possible, e.g. in the picture 3.9, showing a 2-simplex, the vertical line below the barycenter is labeled j for it sits between faces with the label i and j. Therefore the barycenter gets the label k. We will always assume that such a standard r-simplex is equipped with this triangulation. As before, we will denote the map of Δ_r into M by φ_{Δ_r}. This may not be the constant map above, but it will always be clear which map we use.

3.5.1 The p-Forms

The easiest component of the Deligne cocycle ξ to be reconstructed is $\xi^0_{i_1,\ldots,i_r}$, since it does not involve any tangent vectors. Note that we, as in the holonomy reconstruction, define the intersection-index r to be $r = d - p + 1$. Again we will be a bit sloppy, and will often disregard the distinction between the simplex $s_{i_1,\ldots,i_r}$ and the map $\varphi^{v_0}_{s_{i_1,\ldots,i_r}}$.

3.5.1 DEFINITION: Suppose $v_0 \in U_{i_1,\ldots,i_r}$. Define $s^{v_0}_{i_1,\ldots,i_r}$ to be the constant $(r-1)$-simplex Δ_{r-1} in v_0 together with the standard triangulation as above, that assigns the label i_j to the vertex x_j.

The other simplices will be slightly more complex. We define a simplex by considering the simplicial complex $\Delta_{r-1} \times [0,1]^p$. We can triangulate it by extending the triangulation on Δ_{r-1} constantly.

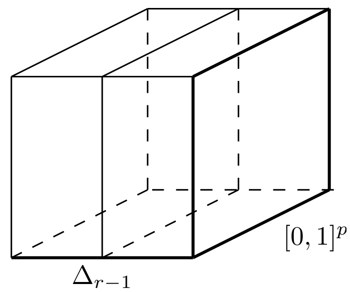


Figure 3.10: The triangulation of the simplicial complex $\Delta_{r-1} \times [0,1]^p$

Given p tangent vectors $v_1, \dots, v_p$ in $v_0 \in U_{i_1,\dots,i_r}$, we may map this simplex into M by mapping the $[0,1]^p$ part onto the »area« spanned by the vectors $v_1, \dots, v_p$ and disregard the simplex part Δ_{r-1}, i.e. the mapping only depends on the »time« coordinates. Obviously here we are using that the intersections are all contractible and the implicit identification of these with an open subset of $\mathbb{R}^N$, enabling us to identify the »area« with the volume spanned by the p tangent vectors in $\mathbb{R}^N$. Altogether we define:

3.5.2 DEFINITION: Let the simplex $s_{i_1,\dots,i_r}^{v_0,\dots,v_p}$ in M given by $\Delta_{r-1} \times [0,1]^p$ together with the map $\varphi_{s_r^p} : (x, t_1, \dots, t_p) \mapsto t_1 v_1 + \cdots + t_p v_p \in U_{i_1,\dots,i_r} \cong \mathbb{R}^N$.

3.5.3 EXAMPLE: We want to be sure that our simplices are just the ones described by Picken, so consider the case $d = 2$. The simplex for g_{ijk} is there defined to be Δ_2 together with the standard triangulation we have described above (compare with figure (9) in the proof of theorem 4.6 in [24]).

Furthermore A_{ij} is $\Delta_1 \times [0,1]$ and F_i is $\Delta_0 \times [0,1]^2$. The maps $\varphi_{s_2^1}$ and $\varphi_{s_1^2}$ from the unit square into M, though they are not depicted in [24], are denoted by Q_t and $Q_{t,u}$ respectively.

Notice that our explicit use of simplices clearifies the distinction between the »space«- and »time«-coordinates, while Picken has to denote these by t, u and later on t, u, s.

With these simplices we can proceed to the definition of p-forms, given a Picken-type topological quantum field theory. Again this will be defined as a total derivative:

3.5.4 DEFINITION: Suppose (Z', Z) is a d-dimensional, thin-invariant, smooth Picken-type topological quantum field theory. Let $h^p : (\mathbb{R}^N)^p \longrightarrow U(1)$ be the map

$$h^p : (v_1, \dots, v_p) \mapsto \log Z(s_{i_1,\dots,i_r}^{v_0,\dots,v_p}),$$

for any $p > 0$. Then the total derivative of h^p defines a p-form ω^p, i.e. for points $v_0, v_1, \dots, v_p$ in $\mathbb{R}^N$ set

$$(\omega_{i_1,\dots,i_r}^p)_{v_0}(v_1, \dots, v_p) := D_{(v_0,\dots,v_0)}^p h^p((v_1, 0, \dots, 0), \dots, (0, \dots, 0, v_p)).$$

If $p = 0$, we set $(\omega_{i_1,\dots,i_r}^0)_{v_0} := Z(s_{i_1,\dots,i_r}^{v_0})$.

Again we can rewrite the formula as follows: Denote by $q(t_1, \dots, t_p)$ an integral-curve to the tangent vectors $v_1, \dots, v_p$. We may choose as usual the very special one given by $q(t_1, \dots, t_p) = \sum t_i v_i$, since we are inside $U_{i_1,\dots,i_r} \cong \mathbb{R}^N$. Consider now $Q_{t_1,\dots,t_p} : \Delta_{r-1} \times [0,1]^p \longrightarrow M$ by

$$(x, u_1, \dots, u_p) \mapsto q(u_1 t_1, \dots, u_p t_p) = v_0 + \sum_{i=1}^{p} u_i t_i v_i.$$

Mainly this maps $\Delta_{r-1} \times [0,1]^p$ onto the resized integral curve q. Continue to define

$$(\omega_{i_1,\dots,i_r}^p)_{v_0}(v_1, \dots, v_p) := \frac{\partial^p}{\partial t_1 \dots \partial t_p} \log Z(Q_{t_1,\dots,t_p})|_{t_1=\dots=t_p=0}.$$

As above, the indices are hidden in the triangulation of Δ_{r-1}. What we have neglected here is that we actually need to reparametrize these maps to be constant on the boundary.

3.5.5 Example: For $d = 2$ we have the following three differential forms:

$$\begin{aligned} g_{ijk} &= \log Z(\varphi_{\Delta_2}) \\ A_{ij}(v_1) &= \frac{d}{dt} \log Z(Q_t)|_{t=0} \\ F_i(v_1, v_2) &= \frac{\partial^2}{\partial t_1 \partial t_2} \log Z(Q_{t_1,t_2})|_{t_1=t_2=0} \end{aligned}$$

These correspond to the equations $(10), (11)$ and (12) of [24].

3.5.6 Remark: We have to make sure, that our p-forms such defined really map under PT to the given parallel transport. This is an easy calculation, which we drop here.

3.5.2 Relations

The cocycle relations can be shown easily by the following idea, which is basically the same as in the holonomy case: The comparison of the forms $\omega^p_{i_1,\dots,i_r}$ and $\omega^{p+1}_{i_1,\dots,i_{r-1}}$ takes place at the »position« $(r, p+1)$. The corresponding simplex there is $s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_r}$. This is a $(d+1)$-dimensional simplex and we can consider its d-dimensional boundary. We will show that it decomposes as a sum of $\delta s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}}$ and $\Delta_{v^{p+1}} s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$, where the latter will eventually yield the contribution of $d\omega^p_{i_1,\dots,i_r}$. By the thin-invariance property, the $(p+1)$-th derivative of $\log Z(s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_r})$ will vanish, showing directly that the p-forms ω^p obey the cocycle relations. Let us make this idea precise.

3.5.7 CONSTRUCTION: Begin with the simplex $s^{v_0,\dots,v_{p+1}}_{i_1,\dots,i_r} = s^p_r = \Delta_{r-1} \times [0,1]^{p+1}$. As before we have the map $\varphi_{s^{p+1}_r}$ into M: Given tangent vectors $v_1, \dots, v_{p+1}$ at v_0, we consider

$$\varphi_{s^{p+1}_r} : (x, t_1, \dots, t_{p+1}) \mapsto v_0 + \sum t_i v_i.$$

The boundary of this simplex $s^{v_0,\dots,v_p}_{i_1,\dots,i_r}$ is

$$\partial(\Delta_{r-1} \times [0,1]^{p+1}) = \partial(\Delta_{r-1}) \times [0,1]^{p+1} \cup \Delta_{r-1} \times \partial[0,1]^{p+1}.$$

Let us first consider the first component, $\partial\Delta_{r-1} \times [0,1]^{p+1}$. Now $\partial\Delta_{r-1}$, the boundary of the standard simplex Δ_{r-1}, is simply the alternating sum of its faces, $\partial\Delta_{r-1} = \sum_j (-1)^{j-1}\langle x_1, \dots, \hat{x}_j, \dots, x_r\rangle = \sum_j (-1)^{j-1}\Delta^{\hat{j}}_{r-2}$, where we denoted the vertices of Δ_{r-1} by x_j and the simplex Δ_{r-2} with non-standard labeling $i_1, \dots, \hat{i}_j, \dots, i_r$ by $\Delta^{\hat{j}}_{r-2}$. Remember that the sign in front of Δ_{r-1} only decides about the orientation of Δ_{r-1}, the map itself is not touched. But then $\partial\Delta_{r-1} \times [0,1]^{p+1}$, with the map $\varphi_{s^{p+1}_r}$ restricted to it, is nothing else than $\delta s^{v_1,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}}$. To see this, just observe that

$$\delta(s^{v_1,\dots,v_{p+1}}_{i_1,\dots,i_{r-1}})_{i_1,\dots,i_r} = \sum_j (-1)^{j-1} s^{v_0,\dots,v_{p+1}}_{i_1,\dots,\hat{i}_j,\dots,i_r} = \sum_j (-1)^{j-1}\Delta^{\hat{j}}_{r-2} \times [0,1]^{p+1}.$$

(Here again we should have written $\varphi_{s^{v_0,\dots,v_{r+1}}_{i_1,\dots,\hat{i}_j,\dots,i_r}}$ etc, making all formulas more cumbersome.) We have to take care of the other term $\Delta_{r-1} \times \partial[0,1]^{p+1}$. On this component

$$\Delta_{r-1} \times \partial[0,1]^{p+1} = \sum_j (-1)^{j+r-1}\Delta_{r-1} \times [0,1] \times \cdots \times \{0,1\} \times \cdots \times [0,1]$$

the map $\varphi_{s^{p+1}_r}$ takes the form

$$(y, t_1, \dots, 0, \dots, t_{p+1}) \mapsto v_0 + t_1 v_1 + \cdots + 0 + \cdots + t_{p+1} v_{p+1}$$
$$(y, t_1, \dots, 1, \dots, t_{p+1}) \mapsto v_0 + t_1 v_1 + \cdots + v_j + \cdots + t_{p+1} v_{p+1}$$

Now define a family of simplices in M simply by resizing these maps: For any parameters $t_1, \dots, t_{p+1}$ let $\hat{Q}_{t_1,\dots,t_{p+1}} : \partial(\Delta_{r-1} \times [0,1]^{p+1}) \longrightarrow M$ given by

$$\hat{Q}_{t_1,\dots,t_{p+1}} : (y, u_1, \dots, u_{p+1}) \mapsto v_0 + t_1 u_1 v_1 + \dots + t_{p+1} u_{p+1} v_{p+1}.$$

As written, this map is defined on $\Delta_{r-1} \times [0,1]^{p+1}$ and we have to restrict it to the boundary. We have on $\partial\Delta_{r-1} \times [0,1]^{p+1}$ the mapping

$$\hat{Q}_{t_1,\dots,t_{p+1}}|_{\partial\Delta_{r-1}\times[0,1]^{p+1}} : (y, u_1, \dots, u_{p+1}) \mapsto v_0 + \sum_{i=1}^{p+1} u_i t_i v_i$$

and on $\Delta_{r-1} \times \partial[0,1]^{p+1}$ the mapping

$$\hat{Q}_{t_1,\dots,t_{p+1}}|_{\Delta_{r-1}\times\partial[0,1]^{p+1}} : (y, u_1, \dots, \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \dots, u_{p+1}) \mapsto \sum_{i=1, i\neq j}^{p+1} u_i t_i v_i + \begin{Bmatrix} 0 \\ t_j v_j \end{Bmatrix} + v_0.$$

We need to compare this to the differential of ω. By the formula we have

$$\omega^p_{i_1,\dots,i_r}(v_1, \dots, v_p) := \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \log Z(Q_{t_1,\dots,t_p}).$$

Here $Q_{t_1,\dots,t_p}$ is the map $\Delta_{r-1} \times [0,1]^p \longrightarrow M$, given similarly as above by $(y, u_1, \dots, u_p) \mapsto \sum u_i t_i v_i$.

Since the image of the whole family $\hat{Q}_{t_1,\dots,t_{p+1}}$ of simplices in M is contained in the volume that is spanned by the vectors $v_1, \dots, v_{p+1}$, we know by the thin invariance property of Z that the expression

$$\frac{\partial^{p+1}}{\partial t_1 \dots \partial t_p} \log Z(\hat{Q}_{t_1,\dots,t_{p+1}}) = 0.$$

vanishes.

Consider both components of the map $Z(\hat{Q}_{t_1,\dots,t_{p+1}})$ (we use the gluing axiom to split it into two factors): On $\partial\Delta_{r-1} \times [0,1]^{p+1}$ we have an equality of $\hat{Q}_{t_1,\dots,t_{p+1}}$ with $\delta Q_{t_1,\dots,t_{p+1}}$, since by the observation that the boundary of Δ_{r-1} splits into a sum of non-standard labeled $\Delta^{\hat{j}}_{r-2}$, both maps have the form $\Delta^{\hat{j}}_{r-2} \times [0,1]^{p+1} : (y, u_1, \dots, u_p) \mapsto \sum u_i v_i t_i$ and the same sign. Taking now the partial derivative and logarithm of $Z(\hat{Q}_{t_1,\dots,t_p})$ we therefore can rewrite it as

$$\begin{aligned} \frac{\partial^p}{\partial t_1 \dots \partial t_{p+1}} \log Z(\hat{Q}_{t_1,\dots,t_p}|_{\partial\Delta_{r-1}\times[0,1]^{p+1}}) &= \frac{\partial^p}{\partial t_1 \dots \partial t_{p+1}} \log Z(\delta Q_{t_1,\dots,t_{p+1}}) \\ &= \delta(\omega^{p+1})_{i_1,\dots,i_j}. \end{aligned}$$

Quite equally we can identify the other boundary component: The sign of $\Delta_{r-1} \times \partial[0,1]^{p+1}$ is just $(-1)^{r-1+j}$, when we take the boundary of the j^{th} interval. That is we have

$$\sum_j \hat{Q}_{t_1,\dots,t_{p+1}}|_{\Delta_{r-1}\times[0,1]^{j-1}\times\{1\}\times[0,1]^{p-j}} = \sum_j (-1)^{r-1+j}(Q^{v_0+t_jv_j,v_1,\dots,\hat{v}_j,\dots,v_{p+1}})_{t_1,\dots,t_p}.$$

Pulling out the global sign $(-1)^{r-1}$ and using the definition of the exterior derivative, we see that

$$\begin{aligned} d\omega^p(v_1,\dots,v_{p+1}) &= \sum_j (-1)^j \frac{\partial^{p+1}}{\partial t_1 \dots \partial t_{p+1}} \log(Q^{v_0+t_jv_j,v_1,\dots,\hat{v}_j,\dots,v_{p+1}}{}_{t_1,\dots,t_p})|_{t_i=0} \\ &= (-1)^{r-1}\sum_j \log Z(\hat{Q}_{t_1,\dots,t_{p+1}})|_{t_i=0} \end{aligned}$$

Here we have decorated Q with the vertices it depends on, to make clear how the map looks like. We also have neglected on the left hand side the maps on boundary point of the form $(x, u_1, \dots, 0, \dots, u_{p+1})$, for they vanish under taking the derivative. From this we directly deduce the following equation:

$$\begin{aligned} 0 &= \frac{\partial^{p+1}}{\partial t_1 \dots \partial t_p} \log Z(\hat{Q}_{t_1,\dots,t_{p+1}}) \\ &= (-1)^{r+1} d\omega^{p-1}(v_1,\dots,v_{p+1}) + 0 + \delta\omega^p(v_1,\dots,v_{p+1}), \end{aligned}$$

where we used the gluing axiom in the second line.

Let us clearify the construction we did so far by examining the case $d = 2$ and recovering Picken's definitions.

3.5.8 EXAMPLE: To compare the forms g_{ijk} and A_{ij}, consider the boundary of $\Delta_2 \times [0,1]$. This is just the prism surface (see figure (12) in theorem 4.6 in [24]). We must carefully write down the maps involved. Start with $\hat{Q}^v_t : \partial(\Delta_2 \times [0,1]) \longrightarrow M$. On the boundary $\partial\Delta_2 \times [0,1] = (\langle x_1, x_2\rangle \star \langle x_2, x_3\rangle \star \langle x_3, x_1\rangle) \times [0,1]$ we have

$$\hat{Q}^v_t|_{\partial\Delta_2\times[0,1]} : (x,u) \mapsto v_0 + utv,$$

while on the other boundary component we have

$$\hat{Q}^v_t|_{\Delta_2\times\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}} : (x, \begin{Bmatrix}0\\1\end{Bmatrix}) \mapsto v_0 + \begin{Bmatrix}0\\tv\end{Bmatrix}.$$

On the other hand $Q_t^v : \Delta_1 \times [0,1] \longrightarrow M$ has the form

$$Q_t^v : (x,u) \mapsto v_0 + utv.$$

Consider first $\delta(Q_t^v)_{i_1 i_2 i_3}$. We have to replace $\Delta_1 = \langle x_1, x_2\rangle$ by $\langle x_1, x_2\rangle \star \langle x_2, x_3\rangle \star \langle x_3, x_1\rangle$, but then obviously we have $\delta Q_t^v = \hat{Q}_t^v|_{\partial\Delta_2\times[0,1]}$, that is we have after taking the derivative

$$\frac{d}{dt}\log Z(\hat{Q}_t^v|_{\partial\Delta_2\times[0,1]})|_{t=0} = \frac{d}{dt}\log Z(\delta Q_t^v)|_{t=0} = (\delta A)_{ijk}(v).$$

Next, we need to consider the definition of g_{ijk}, that is the map $\varphi_{\Delta_2} : \Delta_2 \longrightarrow M$. We have by definition

$$d\log g_{ijk}(v) = \frac{d}{dt}\log Z(t\mapsto v_0 + vt)|_{t=0} = \frac{d}{dt}\log Z(\hat{Q}_1^v)|_{t=0}.$$

Notice, that the other component of $\hat{Q}$, that is $\hat{Q}_0^v$ does not include any t, so will vanish, if we take the derivative. Altogether we have now:

$$\begin{aligned} 0 &= \frac{d}{dt}\log Z(\hat{Q}_t^v) \\ &= \frac{d}{dt}\log Z(\hat{Q}_t^v|_{\partial\Delta_2\times[0,1]} \star \hat{Q}_t^v|_{\Delta_2\times\{0\}} \star \hat{Q}_t^v|_{\Delta_2\times\{1\}})|_{t=0} \\ &= (\delta A)_{ijk}(v) + 0 + d\log g_{ijk}(v). \end{aligned}$$

This verifies the first relation.

Next we want to compare A_{ij} and F_i. For this we look at $\Delta_1 \times [0,1]^2$, which is a unit cube. The boundary decomposes into top and bottom and the four side faces. In this case the map $\hat{Q}_{t_1,t_2}^{v_1,v_2}$ takes the following form:

$$\begin{aligned} \hat{Q}_{t_1,t_2}^{v_1,v_2} :&\partial\Delta_1 \times [0,1] &&\longrightarrow M \\ &(x,u_1,u_2) &&\mapsto y + t_1u_1v_1 + t_2u_2v_2 \\ \hat{Q}_{t_1,t_2}^{v_1,v_2} :&\Delta_1 \times \partial[0,1]^2 &&\longrightarrow M \\ &(y,u_1,\begin{Bmatrix}0\\1\end{Bmatrix}) &&\mapsto v_0 + u_1t_1v_1 + \begin{Bmatrix}0\\t_2v_2\end{Bmatrix} \\ &(y,\begin{Bmatrix}0\\1\end{Bmatrix},u_2) &&\mapsto v_0 + \begin{Bmatrix}0\\t_1v_1\end{Bmatrix} + u_2t_2v_2 \end{aligned}$$

Notice that the last map is not from $\Delta_1\times\partial[0,1]\times[0,1]$ to M, but from $-\Delta_1\times\partial[0,1]\times[0,1]$, just by the definition of ∂. This will account for the sign in the relation between the differential forms. Next, let us write down $Q^{v_1,v_2}_{t_1,t_2} : \Delta_0 \times [0,1]^2 \longrightarrow M$. By its definition we have

$$Q^{v_1,v_2}_{t_1,t_2} : (x, u_1, u_2) \mapsto v_0 + u_1 t_1 v_1 + u_2 t_2 v_2.$$

Taking the Čech-differential, we have two maps, the one going from $-\langle x_0 \rangle \times [0,1]^2$, the other from $\langle x_1 \rangle \times [0,1]^2$ to M. The map itself (i.e. between sets) does not change. Since we have $\partial\Delta_1 = \langle x_1 \rangle - \langle x_0 \rangle$, we see immediatly $\delta Q^{v_1,v_2}_{t_1,t_2} = \hat{Q}^{v_1,v_2}_{t_1,t_2}|_{\partial\Delta_1 \times [0,1]^2}$, that is

$$\frac{\partial^2}{\partial t_1 \partial t_2} \log Z(\hat{Q}^{v_1,v_2}_{t_1,t_2}|_{\partial\Delta_1 \times [0,1]^2})|_{t_1=t_2=0} = \frac{\partial^2}{\partial t_1 \partial t_2} \log Z(\delta Q^{v_1,v_2}_{t_1,t_2})|_{t_1=t_2=0} = (\delta F)_{ij}(v_1, v_2).$$

Finally, consider the exterior derivative of A_{ij}. Plugging the definition of A_{ij} into the derivative, we get:

$$\begin{aligned} dA_{ij}(v_1, v_2) = &\frac{\partial^2}{\partial t_1 \partial t_2} \log Z((t_1, t_2) \mapsto (v_0 + t_2 v_2) + u_1 t_1 v_1) \\ &- \log Z((t_1, t_2) \mapsto (v_0 + t_1 v_1) + u_2 t_2 v_2). \end{aligned}$$

But these are the maps $\hat{Q}^{v_1,v_2}_{t_1,t_2}|_{\Delta_1 \times \partial([0,1]^2)}$, so that we obtain

$$dA_{ij}(v_1, v_2) = \frac{\partial^2}{\partial t_1 \partial t_2} \log Z(-\hat{Q}^{v_1,v_2}_{t_1,t_2}|_{\Delta_1 \times \{1\} \times [0,1]}) - \log Z(\hat{Q}^{v_1,v_2}_{t_1,t_2}|_{\Delta_1 \times [0,1] \times \{1\}})|_{t_1=t_2=0}.$$

Notice again, that the other two component of $\hat{Q}^{v_1,v_2}_{t_1,t_2}$ (e.g. restricted to $\Delta_1 \times \{0\} \times [0,1]$) vanish. Altogether we have

$$0 = \frac{\partial^2}{\partial t_1 \partial t_2} \log Z(\hat{Q}^{v_1,v_2}_{t_1,t_2})|_{t_1=t_2=0} = (\delta F_i - dA_{ij} + 0)(v_1, v_2),$$

Hence we have verified that our construction above directly induces Pickens work.

3.5.3 Reconstruction

In this subsection we prove that the reconstruction above is an isomorphism. We have already shown that the map is a group morphism (see lemma 3.4.12) and is surjective. Just as it was the case in theorem 2.5.26, we prove now that the map PT is injective. Before we give the theorem in full generality, let us analyze again the case $d = 2$, which gives us a good understanding of the general case.

3.5.9 EXAMPLE: For $d = 2$ we have to show that all three forms F_i, A_{ij} and g_{ijk} are vanishing. First consider the 2-form F_i. Choose a point $v_0 \in M$ and two tangent vectors

v_1, v_2 in v_0. All we have to show is $F_i(v_1, v_2) = 0$. If we assume pt_ξ to be trivial, we know that also $\mathrm{pt}_\xi(Q_{t_1,t_2}) = 0$, with $Q_{t_1,t_2} : \Delta_0 \times [0,1]^2 \longrightarrow M$ is the simplex mapping the square to the integral curve spanned by v_1 and v_2 and having label i on the only 2-dimensional face $[0,1]^2$. We now use the explicit formula for pt_ξ:

$$1 = \mathrm{pt}_\xi(Q_{t_1,t_2}) = \exp(\int_{[0,1]^2} F_i),$$

and this is zero by the assumption, that pt_ξ is zero. Note that there are no other contributions for all 0 and 1-faces are external. Differentiating and taking the logarithm of this we obtain

$$0 = \frac{\partial^2}{\partial t_1 \partial t_2} \log \exp(\int_{[0,1]^2} F_i)|_{t_1=t_2=0} = F_i(v_1, v_2).$$

The other two cases are just the same, let us write them down for completeness. For the 1-form A_{ij} we consider $Q_t : \Delta_1 \times [0,1] = [0,1]^2 \longrightarrow M$, having as image the line segment spanned by v_1. Remeber that $[0,1]^2$ is here triangulated as follows: There are two faces f_i, f_j, labeled by i and j, and an edge e_{ij} and some edges and vertices on the boundary. Dropping these external contributions, the formula for the parallel transport gives us

$$0 = \mathrm{pt}_\xi(Q_t) = \exp(\int_{f_i} F_i + \int_{f_j} F_j + \int_{e_{ij}} A_{ij}),$$

where e_{ij} is the edge (barycentric midpoint of $)\Delta_1 \times [0,1]$. But the image of Q_t is one 1-dimensional, so both integrals over the faces do not contribute, and just in the same manner as above, by taking the logarithm and differentiating, we obtain $A_{ij}(v) = 0$.

Finally we have to consider the map φ_{Δ_2} from the 2-simplex Δ_2 to M, being mapped to a single point v_0. Since φ_{Δ_2} is the constant map and the only internal vertex is the barycentric midpoint, this leads to $g_{ijk} = 1$, if we take into account our choice of the triangulation (that is one gets 6 contributions from the three edges connected to the barycenter, and writing these out one sees that only $g_{ijk}(v_0)$ is non-zero).

3.5.10 THEOREM: *The map* PT *induces a group isomorphism between the group of Deligne d-cocycles and the group of d-dimensional, smooth, thin-invariant Picken-type topological quantum field theory, i.e.*

$$\mathrm{PT} : \check{Z}^d(M, \mathcal{D}^d) \longrightarrow \textit{TQFT}^{d,\infty}_{Picken}(M)$$

is an isomorphism of groups.

Proof. Let ξ^d be a Deligne cocycle such that its corresponding topological quantum field theory is trivial. By definition of the trivial topological quantum field theory, this means that $\mathrm{pt}_\xi(\Sigma, T_\Sigma)$ and $\mathrm{pt}_\xi(S, T_S, \tilde{T}_S)$ of any manifold is $1 \in U(1)$. We have to show that any of the p-forms ξ^p is zero, so we choose at an arbitrary point v_0 some tangent vectors $v_1, \dots, v_p$ and compute $\xi^p(v_1, \dots, v_p)$. Here we need to distinguish the case $p = 0$ and $p > 0$. So assume $p > 0$ and let $Q_{t_1,\dots,t_p} : \Delta_{r-1} \times [0,1]^p$ be the usual d-dimensional triangulated surface from the reconstruction. Then we know that $\mathrm{pt}_\xi(Q_{t_1,\dots,t_p}) = 0$, since we assumed pt_ξ to be trivial. Simply differentiate this with respect to $t_1, \dots, t_p$. What we obtain is exactly the p-forms from the reconstruction, that is we have

$$0 = \frac{\partial^p}{\partial t_1 \dots \partial t_p} \log \mathrm{pt}_\xi(Q_{t_1,\dots,t_p})|_{t_i=0}.$$

Now use the definition of pt_ξ, to get

$$0 = \frac{\partial^p}{\partial t_1 \dots \partial t_p} \log(\exp(\sum_{\substack{f_d \in k^d \\ f_d \not\subset \partial Q_t}} \int_{f_d} \xi^d_{\ell(f_d)} + \dots) \prod_{\substack{f_0 \subset \dots \subset f_d \\ f_o \not\subset \partial Q_t}} \xi^0_{\ell(f_d)\dots\ell(f_0)}(f_0))|_{t_i=0}.$$

But the simplex $\Delta_{r-1} \times [0,1]^p$ is triangulated by extending the canonical one on Δ_r constantly, so this triangulation does not have any face of dimension less than p in the interior (that is any such subface does not contribute, for it has the same label as the whole face). Therefore all these do not contribute to the parallel transport. Moreover, since $Q_{t_1,\dots,t_p}$ has a p-dimensional image, all the faces of higher dimension do not contribute. Hence in the local formula we used for the definition of the topological quantum field theory, all terms except the one in dimension p vanish, and we are only left with

$$0 = \frac{\partial^p}{\partial t_1 \dots \partial t_p} \log(\exp(\sum_{\substack{f_p \in k^p \\ f_p \not\subset \partial \Delta_{r-1} \times [0,1]^p}} \int_{f_p} \xi^p_{\ell(f_p)}))|_{t_i=0}.$$

By construction there is exactly one such p-dimensional face: the barycenter $\times\ [0,1]^p$, being mapped on $\langle v_1, \dots, v_p \rangle$, and the above equation reads then

$$0 = \xi^p(v_1, \dots, v_p).$$

Finally we have to take care of the case $p = 0$. By definition we have $\omega^0_{i_1,\dots,i_r} = \mathrm{pt}_\xi(\Delta_y)$, where Δ_y is the map from the standard simplex Δ_{r-1} to y and $\mathrm{pt}_\xi(\Delta_y)$ is hence the integral

$$\mathrm{pt}_\xi(\Delta_y) = \exp(\sum_{\substack{f_d \in k^d \\ f_d \not\subset \partial \Delta_y}} \int_{f_d} \xi^d_{\ell(f_d)} + \dots) \prod_{\substack{f_0 \subset \dots \subset f_d \\ f_o \not\subset \partial \Delta_y}} \xi^0_{\ell(f_d)\dots\ell(f_0)}(f_0)).$$

Since the image of Δ_y is just a point, all higher integrals vanish and we are just left with the contributions of the vertices. But the only vertex inside the standard simplex is the midpoint of the barycentric division, so the integral collapses to $\omega^0(v_0) = \mathrm{pt}_\xi(\Delta_y) = 1$ by assumption on the choice of triangulation.

This shows that the map PT is injective, and hence an isomorphism of groups, as stated. □

Conclusion

In this thesis we mainly have shown two things. First, smooth Deligne d-classes can be reconstructed by their holonomy maps, if the base manifold M is at least $(d{-}1)$-connected. This amounts to the same as saying that

$$\mathrm{HOL} : \check{H}^d(M, \mathcal{D}^d) \longrightarrow \mathrm{Hom}^\infty(\pi_d^d(M), U(1)).$$

is an isomorphism of groups. Second, smooth Deligne d-cocycles can be reconstructed by their associated thin-invariant, smooth d-dimensional Picken-type topological quantum field theories, so we have an isomorphism

$$\mathrm{PT} : \check{Z}^d(M, \mathcal{D}^d) \longrightarrow \mathsf{TQFT}^{d,\infty}_{Picken}(M).$$

These results show that the concepts of holonomy and of topological quantum field theories both capture the content of a Deligne class and a Deligne cocycle respectively.

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