



From $4d$ superconformal indices to $3d$ partition functions

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ABSTRACT

An exact formula for partition functions in $3d$ field theories was recently suggested by Jafferis, and Hama, Hosomichi, and Lee. These functions are expressed in terms of specific q -hypergeometric integrals whose key building block is the double sine function (or the hyperbolic gamma function). Elliptic hypergeometric integrals, discovered by the second author, define $4d$ superconformal indices. Using their reduction to the hyperbolic level, we describe a general scheme of reducing $4d$ superconformal indices to $3d$ partition functions which imply an efficient way of getting $3d$ $\mathcal{N} = 2$ supersymmetric dualities for both SYM and CS theories from the “parent” $4d$ $\mathcal{N} = 1$ dualities for SYM theories. As an example, we consider explicitly the duality pattern for $3d$ $\mathcal{N} = 2$ SYM and CS theories with $SP(2N)$ gauge group with the antisymmetric tensor matter.

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1. Introduction

Superconformal indices (SCIs) of supersymmetric Yang–Mills field theories in four dimensions [1] may be usefully identified with elliptic hypergeometric integrals which were discovered in [2]. This identification was made in [3] where, following earlier work of Romelsberger [1], exact matching between indices for standard Seiberg dual theories was demonstrated for classical gauge groups, as well as being extended to Kutasov–Schwimmer dualities for large N . This was the first instance where matching between SCIs for dual theories was explicitly shown for finite rank gauge groups, tests until then involving large N expansions, in the context of the AdS/CFT correspondence, for instance, in [1]. The relation between SCIs and the theory of elliptic hypergeometric functions was systematically investigated in [4,5], where many new $\mathcal{N} = 1$ dualities were discovered and a physical meaning of various mathematical properties of the elliptic hypergeometric integrals was recognized (see also [6,7]). SCIs provide perhaps currently the most rigorous and powerful mathematical tool for testing $4d$ supersymmetric dualities whereby indices for theories with quite different matter content may be shown to coincide due to non-trivial recently discovered special function identities.

SCIs of $3d$ field theories have an essentially more involved form (due to monopole contributions, that do not analogously arise for $4d$ theories) – see [8–12] and references therein. In [12] an attempt was made to find a connection between $4d$ and $3d$ SCIs, but no simple relation was found. In the present work we concentrate on the partition functions (PFs) of $3d$ theories [13–22]. More precisely, we demonstrate that certain of these partition functions as well as duality relations among different theories can be obtained by a reduction of $4d$ SCIs and corresponding duality relations.

The study of $3d$ partition functions using the localization technique was initiated by Kapustin, Willett, and Yaakov [14] inspired by [23]. In the work of Jafferis [18] and Hama, Hosomichi, and Lee [19] a general recipe for evaluating $3d$ PFs was suggested. It was found that these functions are expressed in terms of the q -hypergeometric integrals admitting the $|q| = 1$ regime [24,25] (which are referred also as the hyperbolic q -hypergeometric integrals) and having equal quasiperiods $\omega_1 = \omega_2$. In [20] this result was generalized to arbitrary values of the quasiperiods ω_1 and ω_2 .

Physically, the reduction of $4d$ supersymmetric field theories on $R^3 \times S^1$ to $3d$ theories on R^3 was discussed in detail by Seiberg and Witten [26]. Here we observe that there is an explicit connection between SCIs of $4d$ supersymmetric field theories on $S^3 \times S^1$, and the PFs of $3d$ theories on the squashed three-sphere S_b^3 , that allows also for a recipe for conjecturing $3d$ duals. Technically, this fact is realized by the reduction of elliptic hypergeometric integrals [2] to hyperbolic q -hypergeometric integrals, which was rigorously established by Rains

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in [27]. A detailed consideration of such limiting cases was given by van de Bult [28] (see also [29]). This suggests SCIs of 4d theories may be more fundamental objects than the PFs of 3d theories in that the properties of the latter are inherited from elliptic hypergeometric integrals, which, as functions, are more general, nevertheless having a simpler form. While the connection between 4d and 3d dualities for supersymmetric field theories was explored in [30] and studied further in the context of the three-dimensional analog of Seiberg duality in [31,32], here the connection between 4d SCIs and 3d partition functions gives a different perspective with strong predictive power.

In the following, we illustrate how a reduction in 4d SCIs lead to formulae equivalent to 3d PFs by considering particular examples. The same reduction may be applied essentially to other 4d SCIs and expressions for 3d PFs recovered so that these examples suffice to show the general procedure. Since 4d SCIs for dual theories are obtained from transformation formulae for these elliptic hypergeometric integrals, the same reduction applied to these integrals yields corresponding 3d partition functions for dual theories, from which matter fields content and their representations may be read off. While this procedure is applied to a few examples here, obviously if more examples were considered, due to the profusion of transformation formulae available for elliptic hypergeometric integrals, a whole plethora of new dualities for 3d theories would potentially be implied.

2. Reduction from 4d superconformal indices to 3d partition functions

Briefly reviewing the index, denote, as in [3], $\mathcal{N} = 1$ superconformal $SU(2, 2|1)$ group generators by $J_3, J_{\pm}, \bar{J}_3, \bar{J}_{\pm}$, (for the 4d Lorentz group $SO(3, 1) \sim SU(2) \times SU(2)$), $Q_{\alpha}, \bar{Q}_{\dot{\alpha}}, \alpha, \dot{\alpha} = 1, 2, P_{\mu}, \mu = 1, \dots, 4$ (for supertranslations), H (for dilations), $K_{\mu}, S_{\alpha}, \bar{S}_{\dot{\alpha}}$ (for special superconformal transformations), and R (for the R -symmetry group $U(1)_R$) so that, for $Q = \bar{Q}_1$ and $Q^{\dagger} = -\bar{S}_1$,

$$\{Q, Q^{\dagger}\} = 2\mathcal{H}, \quad \mathcal{H} = H - 2\bar{J}_3 - 3R/2. \tag{1}$$

Then the superconformal index [1] is constructed in terms a matrix integral involving generators commuting with Q as

$$I(p, q, f_k) = \int_G d\mu(g) \text{Tr}((-1)^{\mathcal{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} e^{\sum_a g_a G^a} e^{\sum_k f_k F^k}), \quad \mathcal{R} = R + 2\bar{J}_3, \tag{2}$$

where $d\mu(g)$ is the invariant measure of the gauge group, \mathcal{F} is the fermion number operator, and g_a, f_k are the chemical potentials associated with the gauge G and flavor F group generators G^a and F^k , respectively. Only those states in the cohomology of Q contribute and thus the \mathcal{H} eigenvalue dependence is trivial. Here, as in [3], $t = (pq)^{\frac{1}{2}}$ keeps track of the conformal dimensions of operators and thus $-\frac{1}{2} \ln pq$ may be identified with the circumference of the thermal circle, relevant in one limit below. (2) has been computed for free field theory on \mathbb{R}^4 via group characters [3], following from [33], and, for more general theories on $S^3 \times S^1$, using field theory arguments by Romelsberger [1].

We discuss now the reduction of 4d SCIs to PFs for 3d $\mathcal{N} = 2$ supersymmetric theories (SYM or CS) for a particular pair of dual $\mathcal{N} = 1$ SYM theories for which the SCIs match. This reduction may of course be applied generally to the SCIs for Seiberg duals considered in [3,5], and would be expected to further indicate/test 3d dualities. The chosen theories are of particular interest because they are related to the Selberg integral, an object of fundamental importance in various fields of mathematical physics. The complete set of related dualities was found only using the SCI technique in [4], and the novel 4d dualities discovered there yield evidence for novel 3d dualities, via reduction to PFs.

For the theories of interest, the electric theory has the gauge group $G = SP(2N)$ and global symmetry group $SU(8) \times U(1) \times U(1)_R$. They contain a vector superfield V in the adjoint representation of $SP(2N)$ and matter fields (collected in the table below) given by eight chiral multiplets Q forming the fundamental representation f of $SP(2N)$ and an antisymmetric $SP(2N)$ -tensor field X .

	$SP(2N)$	$SU(8)$	$U(1)$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	$\frac{1}{2}$
X	T_A	1	1	0

For $N = 1$, the field X is absent and the group $U(1)$ is decoupled.

The electric theory SCI is described by the following elliptic hypergeometric integral [4]:

$$I_E = \frac{(p; p)_{\infty}^N (q; q)_{\infty}^N}{2^N N!} \Gamma((pq)^s; p, q)^{N-1} \int_{\mathbb{T}^N} \prod_{1 \leq i < k \leq N} \frac{\Gamma((pq)^s z_i^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\Gamma((pq)^{r_Q} y_j z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \tag{3}$$

where $r_Q = (1 - (N - 1)s)/4$, s and y_i are chemical potentials for the groups $U(1)$ and $SU(8)$, respectively, with $\prod_{i=1}^8 y_i = 1$. Denoting $t = (pq)^s$ and $t_i = (pq)^{r_Q} y_i, i = 1, \dots, 8$, we have the constraints $|t|, |t_i| < 1$ and the balancing condition,

$$t^{2N-2} \prod_{i=1}^8 t_i = (pq)^2.$$

Here $(a; q)_{\infty} = \prod_{i=0}^{\infty} (1 - aq^i)$ and $\Gamma(z; p, q)$ is the elliptic gamma function,

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \quad |p|, |q| < 1,$$

used together with the conventions $\Gamma(a, b; p, q) \equiv \Gamma(a; p, q)\Gamma(b; p, q)$, $\Gamma(az^{\pm 1}; p, q) \equiv \Gamma(az; p, q)\Gamma(az^{-1}; p, q)$. Function (3) is a two-parameter generalization of the elliptic Selberg integral of [34] and a multidimensional extension of the elliptic analogue of the Gauss hypergeometric function of [2,35].

There are 72 dual theories associated with the orbit of $W(E_7)$ -Weyl group which are split in four different classes [4]. The first class of magnetic theories has the same gauge group and the flavor group $F = SU(4) \times SU(4) \times U(1)_B \times U(1)$. Its field content is described in the table below

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)$	$U(1)_R$
q	f	f	1	-1	$-\frac{N-1}{4}$	$\frac{1}{2}$
\tilde{q}	f	1	f	1	$-\frac{N-1}{4}$	$\frac{1}{2}$
x	T_A	1	1	0	1	0
M_J	1	T_A	1	2	$\frac{2J-N+1}{2}$	1
\tilde{M}_J	1	1	T_A	-2	$\frac{2J-N+1}{2}$	1

where $J = 0, \dots, N - 1$. Corresponding superconformal index has the form

$$I_M = \prod_{J=0}^{N-1} \prod_{1 \leq i < j \leq 4} \Gamma((pq)^{r_{M_J}} y_i y_j; p, q) \prod_{5 \leq i < j \leq 8} \Gamma((pq)^{r_{\tilde{M}_J}} y_i y_j; p, q) \Gamma((pq)^s; p, q)^{N-1} \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma((pq)^s z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{i=1}^4 \Gamma((pq)^{r_q} w^{-2} y_i z_j^{\pm 1}; p, q) \prod_{i=5}^8 \Gamma((pq)^{r_{\tilde{q}}} w^2 y_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \tag{4}$$

where $w = \sqrt[4]{y_1 y_2 y_3 y_4}$ and

$$r_q = r_{\tilde{q}} = \frac{1 - (N - 1)s}{4}, \quad r_{M_J} = r_{\tilde{M}_J} = \frac{1 - (N - 1 - 2J)s}{2}.$$

The equality of indices $I_E = I_M$ was established in [4] as a direct consequence of the integral identities proved in [35] for $N = 1$ and in [36] for arbitrary N .

Before describing the reduction to PFs for 3d theories, their construction is briefly indicated. According to [14,18–20], the partition function for 3d $\mathcal{N} = 2$ SYM theories has the form

$$Z(\underline{f}) = \int_{-\infty}^{i\infty} \prod_{j=1}^{\text{rank } G} du_j J(\underline{u}) Z^{\text{vec}}(\underline{u}) \prod_I Z_{\phi_I}^{\text{chir}}(\underline{f}, \underline{u}). \tag{5}$$

Here f_k are the chemical potentials for the flavor symmetry group F , u_j -variables are associated with the Weyl weights for the Cartan subalgebra of the gauge group G . For the CS theory one has $J(\underline{u}) = e^{-\pi i k \sum_{j=1}^{\text{rank } G} u_j^2}$, where k is the level of CS-term, and for SYM theories one has $J(\underline{u}) = e^{2\pi i \lambda \sum_{j=1}^{\text{rank } G} u_j}$, where λ is the Fayet-Illiopoulos term. The terms $Z^{\text{vec}}(\underline{u})$ and $Z_{\phi_I}^{\text{chir}}(\underline{f}, \underline{u})$ in (5) are coming from the vector superfield and the matter fields, correspondingly, and they are expressed in terms of the hyperbolic gamma functions, see below.

Now we describe the key degenerating limit from SCIs to obtain PFs. First, we parameterize variables as in [27],

$$t = e^{2\pi i v \tau}, \quad t_i = e^{2\pi i v \mu_i}, \quad i = 1, \dots, 8, \quad p = e^{2\pi i v \omega_1}, \quad q = e^{2\pi i v \omega_2},$$

with the balancing condition

$$2(N - 1)\tau + \sum_{i=1}^8 \mu_i = 2(\omega_1 + \omega_2), \tag{6}$$

and then take the limit $v \rightarrow 0$. It assumes $pq \rightarrow 1$, the limit of vanishing radius for S^1 , but it is not identical to that since, effectively, we are left with the squashed three-sphere S_b^3 instead of S^3 . Although one has also $p/q \rightarrow 1$, there still remain contributions in PF coming from the operator $(p/q)^{J_3}$ in the SCI, and the squashing parameter $b^2 = \omega_1/\omega_2$ survives this limit. This is most easily seen in terms of matching with the results for such PFs obtained in [20]. To obtain PFs on the usual round sphere S^3 , one has to set $b = 1$.

Using the notation of [29], the limiting SCI takes the following form,

$$I_E \Big|_{v \rightarrow 0} = e^{-\pi i (\omega_1 + \omega_2) (4sN^2 - 6(s-1)N + 2s - 1) / 12v \omega_1 \omega_2} I_E^r, \tag{7}$$

where

$$I_E^r(\mu_1, \dots, \mu_8, \tau; \omega_1, \omega_2) = \frac{1}{2^N N!} \gamma^{(2)}(\tau; \omega_1, \omega_2)^{N-1} \int_{-\infty}^{i\infty} \prod_{1 \leq i < k \leq N} \frac{\gamma^{(2)}(\tau \pm u_i \pm u_k; \omega_1, \omega_2)}{\gamma^{(2)}(\pm u_i \pm u_k; \omega_1, \omega_2)} \prod_{j=1}^N \frac{\prod_{i=1}^8 \gamma^{(2)}(\mu_i \pm u_j; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2u_j; \omega_1, \omega_2)} \prod_{j=1}^N \frac{du_j}{i\sqrt{\omega_1 \omega_2}}. \tag{8}$$

Here,

$$\gamma^{(2)}(u; \omega_1, \omega_2) = e^{-\pi i B_{2,2}(u; \omega_1, \omega_2) / 2} \frac{(e^{2\pi i u / \omega_1} \tilde{q}; \tilde{q})_\infty}{(e^{2\pi i u / \omega_2}; q)_\infty}, \tag{9}$$

with the redefined base parameter,

$$q = e^{2\pi i\omega_1/\omega_2}, \quad \tilde{q} = e^{-2\pi i\omega_2/\omega_1},$$

and for $B_{2,2}(u; \omega_1, \omega_2)$ denoting the second order Bernoulli polynomial,

$$B_{2,2}(u; \omega_1, \omega_2) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}. \tag{10}$$

The conventions, $\gamma^{(2)}(a, b; \omega_1, \omega_2) \equiv \gamma^{(2)}(a; \omega_1, \omega_2)\gamma^{(2)}(b; \omega_1, \omega_2)$, and $\gamma^{(2)}(a \pm u; \omega_1, \omega_2) \equiv \gamma^{(2)}(a + u; \omega_1, \omega_2)\gamma^{(2)}(a - u; \omega_1, \omega_2)$, are applied throughout the Letter.

A similar result can be obtained by considering the modified elliptic hypergeometric integrals constructed from the modified elliptic gamma function $G(u; \omega_1, \omega_2, \omega_3)$, [35], after taking the limit $\omega_3 \rightarrow \infty$. In this case, no diverging factors emerge in the reduction of integrals, see a detailed consideration of some examples in [25,29]. In Appendix A of [29] different forms of the function $\gamma^{(2)}(u)$ bearing different names are listed. In particular, $1/\gamma^{(2)}(u)$ is known as the double sine function. In [27,28] the hyperbolic gamma function $\Gamma_h(u)$ is used which is obtained after replacing in $\gamma^{(2)}(u)$ of u by iu and the quasiperiods ω_1, ω_2 by $i\omega_1, i\omega_2$.

In the discussed limit, a computation of the asymptotics of I_M yields the same diverging exponential as in (7), and one arrives at the relation $I_E^r = I_M^r$, where

$$\begin{aligned} I_M^r(\mu_1, \dots, \mu_8, \tau; \omega_1, \omega_2) &= \frac{1}{2^N N!} \gamma^{(2)}(\tau; \omega_1, \omega_2)^{N-1} \prod_{j=0}^{N-1} \prod_{1 \leq i < k \leq 4} \gamma^{(2)}(j\tau + \mu_i + \mu_k; \omega_1, \omega_2) \prod_{5 \leq i < k \leq 8} \gamma^{(2)}(j\tau + \mu_i + \mu_k; \omega_1, \omega_2) \\ &\times \int_{-\infty}^{i\infty} \prod_{1 \leq i < k \leq N} \frac{\gamma^{(2)}(\tau \pm u_i \pm u_k; \omega_1, \omega_2)}{\gamma^{(2)}(\pm u_i \pm u_k; \omega_1, \omega_2)} \prod_{j=1}^N \frac{\prod_{i=1}^8 \gamma^{(2)}(v_i \pm u_j; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2u_j; \omega_1, \omega_2)} \prod_{j=1}^N \frac{du_j}{i\sqrt{\omega_1\omega_2}}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} v_j &= \xi + \mu_j, \quad j = 1, 2, 3, 4, \quad v_j = -\xi + \mu_j, \quad j = 5, 6, 7, 8, \\ 2\xi &= \omega_1 + \omega_2 - (N - 1)\tau - \sum_{i=1}^4 \mu_i = -\omega_1 - \omega_2 + (N - 1)\tau + \sum_{i=5}^8 \mu_i. \end{aligned}$$

Applying the further limit,

$$\lim_{S \rightarrow \infty} I_E^r(\mu_1, \dots, \mu_6, \xi_1 + \alpha S, \xi_2 - \alpha S, \tau; \omega_1, \omega_2) e^{-\pi i N((\xi_2 - \alpha S - \omega)^2 - (\xi_1 + \alpha S - \omega)^2)/\omega_1\omega_2}, \tag{12}$$

where $\max\{\arg(\omega_1), \arg(\omega_2)\} - \pi < \arg(\alpha) < \min\{\arg(\omega_1), \arg(\omega_2)\}$ and $\omega = (\omega_1 + \omega_2)/2$, with fixed $\mu_7 = \xi_1 + \alpha S$ and $\mu_8 = \xi_2 - \alpha S$, carried out essentially in [28], leads to an expression coinciding exactly with the partition function for 3d $\mathcal{N} = 2$ SYM theory with $SP(2N)$ gauge group, six quarks and one chiral field in absolutely antisymmetric representation of a gauge group, namely,

$$Z_E^{3d} = \frac{1}{2^N N!} \gamma^{(2)}(\tau; \omega_1, \omega_2)^{N-1} \int_{-\infty}^{i\infty} \prod_{1 \leq i < k \leq N} \frac{\gamma^{(2)}(\tau \pm u_i \pm u_k; \omega_1, \omega_2)}{\gamma^{(2)}(\pm u_i \pm u_k; \omega_1, \omega_2)} \prod_{j=1}^N \frac{\prod_{i=1}^6 \gamma^{(2)}(\mu_i \pm u_j; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2u_j; \omega_1, \omega_2)} \prod_{j=1}^N \frac{du_j}{i\sqrt{\omega_1\omega_2}}. \tag{13}$$

Note that only six variables $\mu_i, i = 1, \dots, 6$, and τ have survived and, moreover, the balancing condition has disappeared.

The limit $S \rightarrow \infty$ is associated with the presence of an additional “twisted instanton” [30–32]. In these papers the connection of 3d and 4d dualities was discussed with the number of 3d flavors less by one in comparison to the 4d case resembling our situation, since we are “integrating out” two quarks (one flavor) and get the equality for PFs with six quarks (three flavors).

In [28], transformations of the integral in (13) forming the group $W(D_6)$ were deduced as a limit from (11). They lead to the representation,

$$\begin{aligned} Z_M^{3d} &= \frac{1}{2^N N!} \gamma^{(2)}(\tau; \omega_1, \omega_2)^{N-1} \prod_{j=0}^{N-1} \prod_{1 \leq i < k \leq 4} \gamma^{(2)}(j\tau + \mu_i + \mu_k; \omega_1, \omega_2) \\ &\times \prod_{j=0}^{N-1} \gamma^{(2)}\left(j\tau + \mu_5 + \mu_6, 4\omega - \sum_{i=1}^6 \mu_i - (2N - j - 2)\tau; \omega_1, \omega_2\right) \int_{-\infty}^{i\infty} \prod_{1 \leq i < k \leq N} \frac{\gamma^{(2)}(\tau \pm u_i \pm u_k; \omega_1, \omega_2)}{\gamma^{(2)}(\pm u_i \pm u_k; \omega_1, \omega_2)} \\ &\times \prod_{j=1}^N \frac{\prod_{i=1}^6 \gamma^{(2)}(v_i \pm u_j; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2u_j; \omega_1, \omega_2)} \prod_{j=1}^N \frac{du_j}{i\sqrt{\omega_1\omega_2}}, \end{aligned} \tag{14}$$

where the reflection identity,

$$\gamma^{(2)}(u, 2\omega - u; \omega_1, \omega_2) = 1,$$

has been applied, and the transformed variables are given by,

$$v_j = \xi + \mu_j, \quad j = 1, 2, 3, 4, \quad v_j = -\xi + \mu_j, \quad j = 5, 6, \quad 2\xi = \omega_1 + \omega_2 - (N - 1)\tau - \sum_{i=1}^4 \mu_i.$$

Interpreting these integrals in terms of the 3d field theories, we find the electric theory with the global symmetry group $SU(6) \times U(1) \times U(1)_A \times U(1)_R$ and the field content as tabulated below,

	$SP(2N)$	$SU(6)$	$U(1)$	$U(1)_A$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	1	$\frac{1}{2}$
X	T_A	1	1	0	0

These data may be directly read off from the partition function (13). The denominator terms in the integral kernel arise as the vector superfield contribution, while the numerator in the first product of the kernel, as well as the pre-factor terms in front of the integral, arise due the chiral superfield X , the antisymmetric representation of $SP(2N)$, the rest comes from the quarks in the fundamental representation. The parameters τ and μ_i , $i = 1, \dots, 6$, absorbed already the hypercharges for the Abelian part of the global group $\sum_{i=1}^3 r_i x_i$, with x_i , $i = 1, 2, 3$, being the chemical potentials of $U(1)$, $U(1)_A$, and $U(1)_R$, respectively, and r_i being their hypercharges. More precisely, $x_1 = \tau$, $x_2 = \sum_{i=1}^6 \mu_i/6$, and $x_3 = (\omega_1 + \omega_2)/2$.

For the magnetic theory, the global symmetry group in the ultraviolet is $SU(4) \times SU(2) \times U(1)_B \times U(1) \times U(1)_A \times U(1)_R$, while the matter content may be similarly tabulated as,

	$SP(2N)$	$SU(4)$	$SU(2)$	$U(1)_B$	$U(1)$	$U(1)_A$	$U(1)_R$
q_1	f	f	1	-1	$-\frac{N-1}{4}$	-1	$\frac{1}{2}$
q_2	f	1	f	2	$-\frac{N-1}{4}$	-1	$\frac{1}{2}$
x	T_A	1	1	0	1	0	0
$M_{1,j}$	1	T_A	1	0	$\frac{2j-N+1}{2}$	2	1
$M_{2,j}$	1	1	T_A	0	$\frac{2j-N+1}{2}$	2	1
Y_j	1	1	1	0	$\frac{2j-N+1}{2}$	-6	1

where $j = 0, \dots, N - 1$. Analogously to [4], the above mentioned transformations are supposed to lead to $|W(D_6)/S_6| = 32$ different dual theories. We expect that integrals (8) and (11) also have proper 3d interpretation, which is a subject of separate consideration.

3. 3d $\mathcal{N} = 2$ SYM theory with $SP(2N)$ gauge group, 6f and T_A

As another example, here we discuss a 4d s -confining multiple duality considered in terms of indices in [4]. The flavor symmetry group is $F = SU(6) \times U(1)$ and the field content of both theories is presented as follows,

	$SP(2N)$	$SU(6)$	$U(1)$	$U(1)_R$
Q	f	f	$N - 1$	$\frac{1}{3}$
A	T_A	1	-3	0
A^k		1	-3k	0
$QA^m Q$		T_A	$2(N - 1) - 3m$	$\frac{2}{3}$

where $k = 2, \dots, N$ and $m = 0, \dots, N - 1$.

The electric superconformal index is given by the elliptic Selberg integral suggested by van Diejen and the second author in [34],

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(t; p, q)^{N-1} \int_{\mathbb{T}^N} \prod_{1 \leq i < k \leq N} \frac{\Gamma(tz_i^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi iz_j}, \quad (15)$$

while the magnetic index is,

$$I_M = \prod_{j=2}^N \Gamma(t^j; p, q) \prod_{j=1}^N \prod_{1 \leq m < s \leq 6} \Gamma(t^{j-1} t_m t_s; p, q), \quad (16)$$

where the balancing condition reads $t^{2N-2} \prod_{m=1}^6 t_m = pq$.

Integral (15) can be reduced to the hyperbolic level in the same way as before (see, e.g. [28]). This yields (up to some diverging factor which we skip for brevity) the partition function of an electric 3d $\mathcal{N} = 2$ SYM theory, with $SP(2N)$ gauge group, four quarks and one chiral field in the antisymmetric representation of the gauge group, given by the following formula,

$$Z_E^{3d} = \frac{1}{2^N N!} \gamma^{(2)}(\tau; \omega_1, \omega_2)^{N-1} \int_{-i\infty}^{i\infty} \prod_{1 \leq i < k \leq N} \frac{\gamma^{(2)}(\tau \pm u_i \pm u_k; \omega_1, \omega_2)}{\gamma^{(2)}(\pm u_i \pm u_k; \omega_1, \omega_2)} \prod_{j=1}^N \frac{\prod_{i=1}^4 \gamma^{(2)}(\mu_i \pm u_j; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2u_j; \omega_1, \omega_2)} \prod_{j=1}^N \frac{du_j}{i\sqrt{\omega_1 \omega_2}}. \quad (17)$$

Evidently, Z_E^{3d} for this case can be evaluated exactly, which can be seen by the direct consideration of the limit $v \rightarrow 0$ in the relation $I_E = I_M$, yielding,

$$Z_M^{3d} = \prod_{j=2}^N \gamma^{(2)}((j+1)\tau; \omega_1, \omega_2) \prod_{j=0}^{N-1} \frac{\prod_{1 \leq i < k \leq 4} \gamma^{(2)}(j\tau + \mu_i + \mu_k; \omega_1, \omega_2)}{\gamma^{(2)}((2N-2-j)\tau + \sum_{i=1}^4 \mu_i; \omega_1, \omega_2)}. \quad (18)$$

The equality $Z_E^{3d} = Z_M^{3d}$ was rigorously established for the first time by a different method in [25], which result was used as a motivation for a systematic consideration of the reduction procedure in [27].

The dual theories obtained from the equality of the partition functions both have the global symmetry group $SU(4) \times U(1) \times U(1)_A \times U(1)_R$. The spectrum of the electric theory may be tabulated as follows,

	$SP(2N)$	$SU(4)$	$U(1)$	$U(1)_A$	$U(1)_R$
Q	f	f	0	1	$\frac{1}{3}$
X	T_A	1	1	0	0

while that for the confining magnetic theory may be similarly tabulated as,

	$SU(4)$	$U(1)$	$U(1)_A$	$U(1)_R$
$M_j = X^j Q^2$	T_A	j	2	$\frac{2}{3}$
$N_k = X^k$	1	k	0	0
Y_j	1	$-(2N - 2 - j)$	-4	$\frac{2}{3}$

where $k = 2, \dots, N$ and $j = 0, \dots, N - 1$.

4. Further dualities

Further dualities are implied by subsequent reduction of the PFs implemented by taking similar limits as in (12). In contrast to the four-dimensional case where similar reduction of SCIs corresponds to theories with fewer flavors, here such reduction corresponds to theories that, whilst also having fewer flavors than originally, have increased CS level. The technical details of the reduction of corresponding integrals are skipped here and only the final results for implied particular dualities, without an exhaustive list, are indicated.

One such reduction describes a $3d \mathcal{N} = 2$ CS theory with gauge group $SP(2N)_{k/2}$ and $SU(4 - k) \times U(1) \times U(1)_A \times U(1)_R$ global symmetry group with the spectrum involving, apart from the vector multiplet, $4 - k$ quarks $Q_i, i = 1, \dots, 4 - k$, and one chiral superfield X in the antisymmetric representation of the gauge group. This is a confining theory where the spectrum of the dual theory can be directly read from the expressions of the corresponding partition functions, following from the properties of the integrals presented in Section 5.6.3 of [28]. It involves singlets of the $SU(4 - k)$ flavor group $Y_j = X^{j+1}, j = 1, \dots, N - 1$, and baryons $M_j = X^j Q^2, j = 1, \dots, N$, described by the chiral superfield in the antisymmetric representation of $SU(4 - k)$, see the table below,

	$SP(2N)_{k/2}$	$SU(4 - k)$	$U(1)$	$U(1)_A$	$U(1)_R$
Q	f	f	0	1	$\frac{1}{3}$
X	T_A	1	1	0	0
$M_j = X^j Q^2$		T_A	j	2	$\frac{2}{3}$
$Y_l = X^l$		1	l	0	0

where $l = 2, \dots, N$ and $j = 0, \dots, N - 1$ and $k = 1, \dots, 4$. Note that for $k = 3, 4$ in the dual phase there are no fields M_j , for $k = 3$ there is only one quark in the electric theory, and for $k = 4$ the electric theory is just the CS theory with only one field X .

A particular example emerges from the above duality when $N = 1$ and $k = 3$ which gives a $\mathcal{N} = 2$ CS theory with $SP(2)_{3/2}$ gauge group and one quark. In the dual theory we have only the contribution coming from additional topological sector as suggested in [21] (where actually one has also some matter field on the magnetic side since the authors consider a different electric theory), reflected by the magnetic partition function involving only an exponent with some phase.

Continuing the reduction of (17) and (18) in a similar fashion, another limit can be identified with the confining phase description of $3d \mathcal{N} = 2$ SYM theory with $U(N)$ gauge group and the global symmetry and matter fields as described in the table below

	$U(N)$	$U(1)_l$	$U(1)_r$	$U(1)$	$U(1)_A$	$U(1)_j$	$U(1)_R$
Q	f	1	0	0	1	0	$\frac{1}{3}$
\tilde{Q}	f	0	1	0	1	0	$\frac{1}{3}$
X	T_A	0	0	1	0	0	0
$M_j = X^j Q \tilde{Q}$		1	1	j	2	0	$\frac{2}{3}$
$Y_l = X^l$		0	0	j	0	0	0
$W_{j,\pm}$		$-\frac{1}{2}$	$-\frac{1}{2}$	$-j$	0	± 1	$\frac{2}{3}$

where $j = 0, \dots, N - 1$ and $l = 2, \dots, N$. The exact evaluation of the corresponding partition functions follows from Theorem 5.6.8 in [28] (this theorem implies also dualities for CS theories with $U(N)$ gauge group and one chiral field in the adjoint representation and some number of flavors).

A whole set of dualities follows from appropriate reduction of the PFs of $\mathcal{N} = 2$ SYM theory with $SP(2N)$ gauge group, six flavors and one antisymmetric representation matter field, (13) and (14). These reductions are related to the integrals described in Fig. 5.8 of [28]. Corresponding models represent both SYM and CS theories with different number of flavors and different CS level. Specifically, the lines going to the left in Fig. 5.8 of [28] correspond in the field theory language to integrating out matter fields, reducing the number of flavors by 1 and increasing the CS-level by $1/2$ each time, and the lines going to the right correspond to a passage to $U(N)$ SYM or CS field theories with adjoint matter, see [37] for similar dualities involving adjoint matter fields.

For example, one particular reduction of (13) and (14) leads to PFs for $\mathcal{N} = 2$ CS theory with $SP(2N)_{1/2}$ gauge group, whose global symmetry and the matter fields are described in the table below

	$SP(2N)_{1/2}$	$SU(5)$	$U(1)$	$U(1)_A$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	1	$\frac{1}{2}$
X	T_A	1	1	0	0

This is a self-dual gauge group theory obeying the multiple duality phenomenon arising from the $W(A_5)$ symmetry of the partition function. After application of a corresponding transformation formula, the PF for an $\mathcal{N} = 2$ CS theory arises whose global symmetry group $SU(4) \times U(1)_f \times U(1) \times U(1)_A \times U(1)_R$ differs from the original one (in a sense, this corresponds to the split of the original $SU(5)$ group to $SU(4) \times U(1)$). The dual matter fields are described in the following table

	$SP(2N)_{1/2}$	$SU(4)$	$U(1)_B$	$U(1)$	$U(1)_A$	$U(1)_R$
q_1	f	f	-1	$-\frac{N-1}{4}$	-1	$\frac{1}{2}$
q_2	f	1	4	$-\frac{N-1}{4}$	-1	$\frac{1}{2}$
x	T_A	1	0	1	0	0
$M_{1,j}$	1	T_A	0	$\frac{2j-N+1}{2}$	2	1

where $j = 0, \dots, N - 1$.

A series of dualities stems from the reduction of PFs of $\mathcal{N} = 2$ SYM or CS theories with $SP(2N)$ gauge group, N_f flavors and different CS levels using the results of Section 5.5 of [28]. These are the generalizations of the Giveon–Kutasov [38] type of dualities for $SP(2N)$ theories, see, e.g., [22] for a duality for $SP(2N)_{k/2}$ CS theory with N_f flavors.

5. Conclusion

This Letter demonstrates that there is a deep relation between superconformal indices for four-dimensional field theories and partition functions of three-dimensional supersymmetric field theories following from the reduction of elliptic hypergeometric integrals to hyperbolic q -hypergeometric integrals. It may be interesting to better understand from a field theory perspective how the various limits considered here could be realized, in particular for the more detailed reduction scheme of [28], and the extent to which it applies to generic CS theories with non-zero CS level.

The reduction procedure described in this Letter is *general* and can be applied to any SCI for 4d supersymmetric theories to give PFs for 3d models. Every 4d duality out of the large list described in [4,5], after appropriate reduction, yields a 3d analogue of the Seiberg duality similar to [30–32]. The approach here provides a powerful tool for indicating new 3d dualities. The absence of efficient physical checks of 3d dualities, such as 't Hooft anomaly matching conditions, which are useful in 4d, lends added significance to this approach. To understand the whole tree of reductions to PFs for 3d field theories one should investigate the degeneration of elliptic hypergeometric integrals to the q -hypergeometric ones using the rigorous procedure of [27]. For example, the reduction of SCIs for 4d $\mathcal{N} = 1$ SYM dual theories with $SP(2N)$ gauge group and $2N_f$ quarks [39,3,5] leads to 3d dualities for SYM theories of [32] and for CS theories of [38], and a number of new examples. The equality of PFs for these 3d $\mathcal{N} = 2$ SYM and CS theories and their duals was explicitly checked in [22] using the results from [28]. Starting from the equality for the corresponding hyperbolic q -hypergeometric integrals (interpreted as PFs) the dualities for 3d $\mathcal{N} = 2$ CS theory based on $SP(2N)_{k/2}$ gauge group with fundamental matter were derived. The case described in [21] should correspond to the reduction of 4d $\mathcal{N} = 1$ SYM theory with $SP(2)$ gauge group, $2N_f$ quarks, and one chiral superfield in the adjoint representation.

From the point of view of 3d PFs being obtained as limits in 4d SCIs, the results of Z -extremization for PFs for 3d theories [18] may not be so unexpected since the a -maximization of [40] seems to be related to certain automorphic properties of elliptic hypergeometric integrals describing 4d SCIs [41]. We hazard a guess that the reduction to hyperbolic q -hypergeometric integrals preserves some of the needed automorphic properties leading exactly to Z -extremization.

As shown in [42], elliptic hypergeometric integrals emerge in the context of Calogero–Sutherland type models either as wave functions or as normalization conditions of wave functions. It was conjectured there that all elliptic beta integrals and their higher order extensions should be associated with quantum integrable systems of such kind. Naturally, using the connection between SCIs of four-dimensional field theories and elliptic hypergeometric integrals, in [5] it was proposed that SCIs themselves have an added interpretation in terms of relativistic Calogero–Sutherland models. The hyperbolic q -hypergeometric integrals obtained in our Letter (i.e., 3d PFs obtained by reduction of 4d indices) should play a similar role in solvable hyperbolic Calogero–Sutherland models. Also, as discussed in [29], the hyperbolic q -hypergeometric integrals describe the star-triangle relations in some continuous spin models, i.e. they emerge as statistical sums of some two-dimensional integrable systems.

A recent intriguing conjecture made in [17] concerns a connection of the partition function for 3d $\mathcal{N} = 4$ SYM theory with $SU(2)$ gauge group with a kernel of the 2d Liouville field theory connecting conformal blocks in different channels [43] (see also [44]). This observation deserves further detailed investigation since from our perspective these kernels can be obtained by an appropriate reduction of the elliptic hypergeometric integrals pushing the 4d/3d correspondences of the present work down to a new 4d/2d correspondence.

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