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Complete N-point superstring disk amplitude II. Amplitude and hypergeometric function structure

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Abstract

Using the pure spinor formalism in part I (Mafra et al., preprint [1]) we compute the complete tree-level amplitude of N massless open strings and find a striking simple and compact form in terms of minimal building blocks: the full N-point amplitude is expressed by a sum over $(N-3)!$ Yang–Mills partial subamplitudes each multiplying a multiple Gaussian hypergeometric function. While the former capture the space–time kinematics of the amplitude the latter encode the string effects. This result disguises a lot of structure linking aspects of gauge amplitudes as color and kinematics with properties of generalized Euler integrals. In this part II the structure of the multiple hypergeometric functions is analyzed in detail: their relations to monodromy equations, their minimal basis structure, and methods to determine their poles and transcendentality properties are proposed. Finally, a Gröbner basis analysis provides independent sets of rational functions in the Euler integrals.

1. Introduction

During the last years remarkable progress has been accumulated in our understanding and in our ability to compute scattering amplitudes, both for theoretical and phenomenological purposes, cf. Ref. [2] for a recent account. Striking relations have emerged and simple structures have been discovered leading to a beautiful harmony between seemingly different structures and

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aspects of gauge and gravity scattering amplitudes cf. Ref. [3]. As an example we mention the duality between color and kinematics, which exhibits a new structure in gauge theory [4]. This property allows to rearrange the kinematical factors in the amplitude such, that the form of the amplitude becomes rather simple. Moreover, recently it has been shown [5], that the duality between color and kinematics allows to essentially interchange the role of color and kinematics in the full color decomposition of the amplitude. Many of the nice properties encountered in gauge amplitudes take over to graviton scattering.

The properties of scattering amplitudes in both gauge and gravity theories suggest a deeper understanding from string theory, cf. Ref. [6] for a recent review. In fact, many striking field-theory relations such as Bern–Carrasco–Johansson (BCJ) or Kawai–Lewellen–Tye (KLT) relations can be easily derived from and understood in string theory by tracing these identities back to the monodromy properties of the string world-sheet [7–9]. Furthermore, recently it has been shown, how superstring amplitudes can be used to efficiently provide numerators satisfying the color identities [10]. We shall demonstrate in this work, that the complete result for the N-point superstring amplitudes displays properties and symmetries inherent in field theory and reveals structures relevant to field theory. Moreover, we find a beautiful harmony of the string amplitudes with strong interrelations between field theory and string theory.

When computing amplitudes it is highly desirable to obtain results which are both simple and compact. In [1] we show how the pure spinor formalism [11] can be used to accomplish this for the complete N-point superstring disk amplitude, which is given by

$$\mathcal{A}(1,\ldots,N)=\sum_{\sigma\in S_{N-3}}\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N\big)F^\sigma\big(\alpha'\big), \tag{1.1}$$

where $\mathcal{A}_{YM}$ represent $(N-3)!$ color-ordered Yang–Mills (YM) subamplitudes, $F^\sigma(\alpha')$ are generalized Euler integrals encoding the full α'-dependence of the string amplitude and $i_\sigma=\sigma(i)$. The intriguing result (1.1) disguises a lot of structure linking aspects of gauge amplitudes as color and kinematics with properties of generalized Euler integrals. Both the Yang–Mills subamplitudes $\mathcal{A}_{YM}$ and the hypergeometric integrals F are reduced to a minimal basis of $(N-3)!$ elements each. Relations among the integrals F and relations among the string- or field-theory subamplitudes are found to be in one-to-one correspondence, hinting a duality between color and kinematics at the level of the full fledged superstring amplitude.

The pure spinor formalism proved to be crucial to arrive at the compact expression (1.1). It provides a manifestly space–time supersymmetric approach to superstring theory which can still be quantized covariantly [11]. Correlation functions of the world-sheet CFT in the pure spinor formalism can be efficiently organized in terms of so-called BRST building blocks [12, 13]. These are composite superfields which transform covariantly under the BRST operator and have the right symmetry properties to allow for an interpretation in terms of diagrams made of cubic vertices [14]. As shown in [1], manipulations of the BRST-covariant building blocks and the hypergeometric integrals reduce the number of distinct integrals in the N-point disk amplitude down to $(N-3)!$. At the same time, field-theory subamplitudes $\mathcal{A}_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma, N-1,N)$ build up as the associated kinematic factors.

So far, N-point superstring disk amplitudes have been computed up to seven open strings, i.e. $N\leqslant 7$. The scattering amplitude of four open superstrings has been known for a long time [15]. Five-point superstring disk amplitudes have been computed in the RNS formalism in Refs. [16,17], while in the pure spinor formalism in Refs. [14,18]. Furthermore, six open string amplitudes have been computed in Refs. [17,19–22] in the RNS formalism, while in pure spinor superspace in Refs. [12]. Finally, seven open string amplitudes with MHV helicity configurations have been computed in the RNS formalism in [21]. However, the result (1.1) represents the first superstring disk amplitude beyond $N \geqslant 7$ including the complete kinematics. In addition, in contrast to the previous results, Eq. (1.1) yields also very compact expressions for arbitrary N and independent on the chosen helicity configuration and the space–time dimension.

The organization of the present work is as follows. In Section 2 we discuss and explore the result (1.1) to reveal the various structures shared by this result. We find a complementarity between the system of equations derived by the monodromy relations (giving rise to relations between subamplitudes of different color ordering for the same kinematics) and the system of equations derived from partial fraction decomposition or partial integrations (giving rise to relations between functions referring to different kinematics for the same color ordering). We display the full color decomposition of the full string amplitude and comment on a possible string manifestation of the recently anticipated swapping symmetry between color and kinematics in the color decomposition of the full amplitude [5]. In Section 3 the module of multiple hypergeometric functions is analyzed in detail. We present a method to determine the leading poles of Euler integrals. Partial fraction expansion of these integrals can be made according to their leading pole structure. Furthermore, a Gröbner basis analysis provides an independent set of rational functions or monomials for the Euler integrals without poles. Any partial fraction decomposition of finite Euler integrals can be expressed in terms of this basis. In Section 4 we have some concluding remarks and comment on applications and implications of our result in view of effective D-brane action, recursion relation and graviton scattering amplitudes. In Appendix A we propose a method to analyze the transcendentality properties of Euler integrals. In Appendix B for the six open superstring amplitude we present the extended set of functions and its relation to the minimal basis set. Finally, in Appendix C we present α'-expansions of the basis functions F^σ for $N \geqslant 7$.

2. The structure of the N-point superstring disk amplitude

The complete superstring N-point disk subamplitude is given by [1]

$$\mathcal{A}(1,\ldots,N) = \sum_{\sigma\in S_{N-3}} \mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N\big) F^\sigma_{(1,\ldots,N)}\big(\alpha'\big). \tag{2.1}$$

In Eq. (2.1), $F^\sigma \equiv F^\sigma_{(1,\ldots,N)}$ denotes the set of $(N-3)!$ integrals which will be explicitly given in Subsection 2.4. The labels $(1,\ldots,N)$ in $F^\sigma_{(1,\ldots,N)}$ are related to the integration region of the integrals and are responsible for dictating which color ordering of the superstring subamplitude is being computed. The result (2.1) is valid in any space–time dimension D, for any compactification and any amount of supersymmetry. Furthermore, the expression (2.1) does not make any reference to any kinematical or helicity choices. In the following we explore the result (2.1) to illuminate the role of color and kinematics.

2.1. Basis representations: Kinematics vs. color

In field theory there are in total $(N-3)!$ independent YM color-ordered subamplitudes $\mathcal{A}_{YM}$ [4], see Refs. [8,9] for a string-theory derivation of this result. Hence, in field theory any subamplitude $\mathcal{A}_{YM}(1_\Pi,\ldots,N_\Pi)$, with $\Pi \in S_N$, can be expressed as

$$\mathcal{A}_{YM}(1_\Pi,\ldots,N_\Pi) = \sum_{\sigma\in S_{N-3}} K^\sigma_\Pi \mathcal{A}_{YM,\sigma}, \tag{2.2}$$

with $i_\Pi = \Pi(i)$, some universal and state-independent kinematic coefficients K_Π^σ generically depending on the kinematic invariants, cf. Eq. (2.7) for a straightforward derivation. Besides, we introduced the abbreviation:

$$\mathcal{A}_{YM,\sigma} := \mathcal{A}_{YM}\big(1, 2_\sigma, \ldots, (N-2)_\sigma, N-1, N\big). \tag{2.3}$$

One crucial property of (2.1) is the fact that the superstring N-point (sub)amplitude is decomposed in terms of a $(N-3)!$ basis of Yang–Mills color-ordered amplitudes $\mathcal{A}_{YM,\sigma}$, i.e. the whole superstring amplitude can be decomposed w.r.t. the kinematics described by the set of $\mathcal{A}_{YM,\sigma}$, $\sigma \in S_{N-3}$. Hence, by these results it is obvious, that in the sum of (2.1) only $(N-3)!$ terms and as many different multiple hypergeometric functions can appear since any additional kinematical term could be eliminated by redefining the functions F^σ thanks to the amplitude relations (2.2).

Moreover, the string subamplitudes (2.1) solve the system of relations given by

$$\begin{aligned}&\mathcal{A}(1,2,\ldots,N) + e^{i\pi s_{12}}\mathcal{A}(2,1,3,\ldots,N-1,N) + e^{i\pi(s_{12}+s_{13})}\mathcal{A}(2,3,1,\ldots,N-1,N)\\ &\quad + \cdots + e^{i\pi(s_{12}+s_{13}+\cdots+s_{1N-1})}\mathcal{A}(2,3,\ldots,N-1,1,N) = 0\end{aligned} \tag{2.4}$$

and permutations thereof. Throughout this work, we will be mostly using dimensionless Mandelstam invariants:

$$s_{ij} = \alpha'(k_i + k_j)^2. \tag{2.5}$$

The set of identities (2.4) has been derived from the monodromy properties of the disk worldsheet [8,9].

Furthermore, since there exists a basis of $(N-3)!$ YM building blocks allowing for the decomposition (2.2), we may express any string subamplitude by *one* specific set of YM amplitudes $\mathcal{A}_{YM,\sigma}$ referring e.g. to the string amplitude (2.1):

$$\mathcal{A}(1_\Pi, \ldots, N_\Pi) = \sum_{\sigma \in S_{N-3}} \mathcal{A}_{YM,\sigma} F_\Pi^\sigma\big(\alpha'\big), \tag{2.6}$$

with $\Pi \in S_N$. Inserting the set (2.6) into the monodromy relations yields a set of relations for the functions F_Π^σ for each given $\sigma \in S_{N-3}$. E.g. (2.4) gives the following set of identities:

$$\begin{aligned}&F^\sigma_{(1,\ldots,N)} + e^{i\pi s_{12}} F^\sigma_{(2,1,3,\ldots,N-1,N)} + e^{i\pi(s_{12}+s_{13})} F^\sigma_{(2,3,1,\ldots,N-1,N)}\\ &\quad + \cdots + e^{i\pi(s_{12}+s_{13}+\cdots+s_{1N-1})} F^\sigma_{(2,3,\ldots,N-1,1,N)} = 0, \quad \sigma \in S_{N-3}.\end{aligned} \tag{2.7}$$

Hence, for a given $\sigma \in S_{N-3}$ corresponding to the given YM amplitude $\mathcal{A}_{YM,\sigma}$ the set of functions F_Π^σ, $\Pi \in S_N$ enjoys the monodromy relations. As a consequence for each permutation $\sigma \in S_{N-3}$ or YM basis amplitude $\mathcal{A}_{YM,\sigma}$ there are $(N-3)!$ different functions F_Π^σ all related through Eqs. (2.7) and permutations thereof.

Note that the $\alpha' \to 0$ limit of Eq. (2.6) reproduces explicit expressions of the kinematic coefficients K_Π^σ introduced in (2.2) (which were already given in [4] for N-point field-theory amplitudes):

$$K_\Pi^\sigma = F_\Pi^\sigma\big(\alpha'\big)\big|_{\alpha'=0}. \tag{2.8}$$

This relation enables to compute the matrix elements K_Π^σ *directly* by means of extracting the field-theory limit of the string world-sheet integrals $F_\Pi^\sigma(\alpha')$ (by the method described in Subsection 3.3) rather than by solving the monodromy relations (2.4).

Further insights can be gained when looking at different representations for the same amplitude (2.1):

$$\mathcal{A}(1,\ldots,N)=\sum_{\pi\in S_{N-3}}\mathcal{A}_{YM,\pi}\,F^{\pi}_{(1,\ldots,N)}\big(\alpha'\big),\tag{2.9}$$

with some permutations $\pi\in S_{N-3}$ singling out a basis of $(N-3)!$ independent basis amplitudes $A_{YM,\pi}$. More precisely, in contrast to the set $\mathcal{A}_{YM,\sigma}$ in (2.3), the new set $\mathcal{A}_{YM,\pi}$ in (2.9) represents a more general basis of $(N-3)!$ independent subamplitudes $\mathcal{A}_{YM,\pi}$, where three legs i,j,k (possibly other than $1, N-1, N$) are fixed and the remaining ones are permuted by $\pi\in S_{N-3}$.

By applying the decomposition (2.2) and comparing the two expressions (2.9) and (2.1) we find the relation between the set of $(N-3)!$ new and old independent basis functions $F^{\pi}_{(1,\ldots,N)}$ and $F^{\sigma}_{(1,\ldots,N)}$:

$$F^{\sigma}_{(1,\ldots,N)}=\sum_{\pi\in S_{N-3}}\big(K^{-1}\big)^{\sigma}_{\pi}\,F^{\pi}_{(1,\ldots,N)},\quad \sigma\in S_{N-3}.\tag{2.10}$$

In this case the matrix K becomes a quadratic $(N-3)!\times(N-3)!$ matrix, cf. Subsection 2.6 for explicit examples. Hence, for a given *fixed* color ordering $(1,\ldots,N)$ any function F^{σ} may be expressed in terms of a basis of $(N-3)!$ functions F^{π} referring to the *same* color ordering. With (2.10) sets of systems of equations involving the kinematics functions F^{π} (of the *same* color ordering) can be generated. According to (2.8) the field-theory limits of the functions F^{σ}_{Π} are enough to determine the coefficients of these equations.

The relation (2.10) should be compared with (2.2): While in the first identity one specific color ordered amplitude is decomposed w.r.t. to a set of $(N-3)!$ independent color-ordered amplitudes all referring to the *same kinematics*, in the second identity one functions referring to one specific kinematics is decomposed to w.r.t. to a set of $(N-3)!$ independent kinematics functions all referring to the *same color ordering*. Moreover, as we shall show in Subsection 2.3, for a fixed color ordering $(1,\ldots,N)$ an explicit set of $(N-2)!$ functions $F^{\Pi}_{(1\ldots N)}$, $\Pi\in S_{N-2}$ can be given, which fulfills (2.10) – just as a set of $(N-2)!$ YM amplitudes $\mathcal{A}_{YM,\Pi}$ fulfills (2.2) for a fixed kinematics. Since the latter fact is a result of the (imaginary part) field-theory monodromy relations, also the relations (2.10) should follow from a system of equations for the $(N-2)!$ functions. Relations between functions $F^{\Pi}_{(1\ldots N)}$ of same color ordering are obtained by either partial fraction decomposition of their integrands or applying partial integration techniques within their $N-3$ integrals. The partial fraction expansion yields linear equations with integer coefficients for the functions F^{Π} – just like the (real part) field-theory monodromy relations yield linear identities (e.g. subcyclic identities) for the color-ordered subamplitudes $\mathcal{A}_{YM}$. On the other hand, the partial integration techniques applied to the $(N-2)!$ functions F^{Π} provides a system of equations of rank $(N-3)!$, whose solution is given by (2.10). Hence, we have found a complete analogy between the monodromy relations equating subamplitudes $\mathcal{A}_{YM,\Pi}$ of *different color orderings* $\Pi\in S_{N-2}$ at the *same kinematics* and a system of equations relating functions F^{Π} referring to *different kinematics* $\Pi\in S_{N-2}$ at the *same color ordering*.

To conclude, behind the expression (2.1) there are two sets of equations: one set, derived from the monodromy relations (2.4) and equating all subamplitudes of different color orderings and an other set, derived from the partial fraction decomposition and partial integration relations equating all kinematics functions F^{π}. Both systems are of rank $(N-3)!$ and allow to express all colored ordered subamplitudes in terms of a minimal basis or to express all kinematic functions in terms of a minimal basis.

2.2. Color decomposition of the full open superstring amplitude

The color decomposition of the full N-point open superstring amplitude $\mathcal{M}_N$ can be expressed by $(N-3)! \times (N-3)!$ different functions F_Π^σ with $(N-3)!$ YM building blocks $\mathcal{A}_{YM,\sigma}$. Firstly, the monodromy relations (2.4) allow to decompose each superstring subamplitude in an $(N-3)!$ element basis [8,9][1]

$$\mathcal{A}(1_\Pi,\ldots,N_\Pi) = \sum_{\pi\in S_{N-3}} \mathcal{K}_\Pi^\pi \mathcal{A}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big), \tag{2.11}$$

which generalizes the field-theory equation (2.2) in the sense that $\mathcal{K}_\Pi^\pi(\alpha')|_{\alpha'=0} = K_\Pi^\pi$. The basis expansion (2.11) simplifies the color dressed superstring amplitude to

$$\mathcal{M}_N = \sum_{\Pi\in S_{N-1}} \mathrm{Tr}\big(T^{a_1}T^{a_{2_\Pi}}\ldots T^{a_{N_\Pi}}\big) \sum_{\sigma\in S_{N-3}} \mathcal{A}_{YM,\sigma} \sum_{\pi\in S_{N-3}} \mathcal{K}_\Pi^\pi F_\pi^\sigma, \tag{2.12}$$

with:

$$F_\pi^\sigma := F^\sigma_{(1,\pi(2),\ldots,\pi(N-2),N-1,N)}\big(\alpha'\big). \tag{2.13}$$

In the sum (2.12) the same set of basis elements $\mathcal{A}_{YM,\sigma}$ is used for all color orderings Π. This enables to reorganize the color decomposition sum and to interchange the two sums over color and kinematics:

$$\mathcal{M}_N = \sum_{\sigma\in S_{N-3}} \mathcal{A}_{YM,\sigma} \sum_{\Pi\in S_{N-1}} \mathrm{Tr}\big(T^{a_1}T^{a_{2_\Pi}}\ldots T^{a_{N_\Pi}}\big) \sum_{\pi\in S_{N-3}} \mathcal{K}_\Pi^\pi F_\pi^\sigma. \tag{2.14}$$

Now in (2.14) the role of color and kinematics is swapped. While (2.12) represents a color decomposition in terms of $(N-1)!/2$ color-ordered subamplitudes, the sum (2.14) is a decomposition w.r.t. to $(N-3)!$ kinematical factors $\mathcal{A}_{YM,\sigma}$. The sum (2.14) could be the string-theory realization of the recently found observation, that in the color decomposition of a gauge-theory amplitude the role of color and kinematics may be swapped [5]. In these lines the sum over Π may represent the pre-version of a dual amplitude $\mathcal{A}_N^{\text{dual}}$, in which all kinematical factors $\mathcal{A}_{YM,\Pi}$ are replaced by color traces. However, further studies are necessary to establish a clear dictionary between Yang–Mills building blocks $\mathcal{A}_{YM,\Pi}$ and the kinematic analogue $\tau_{(12\ldots N)}$ of color traces: On the one hand, our $\mathcal{A}_{YM,\Pi}$ have the required cyclicity property, on the other hand, they still carry the kinematic poles which should ultimately be outsourced from the local $\tau_{(12\ldots N)}$ into the dual amplitudes $\mathcal{A}_N^{\text{dual}}$.

2.3. Yang–Mills subamplitudes

Compact expressions for $\mathcal{A}_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)$ in $D=10$ are derived in [13] and can be used to describe the YM subamplitudes of (2.1). On the other hand for $D=4$, compact forms for $\mathcal{A}_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)$ in the spinor helicity basis can be looked up

[1] In Ref. [23], systems of equations of this type are neatly rephrased in terms of the so-called momentum kernel matrix $\mathcal{S}_{\alpha'}[\pi|\sigma]$, which keeps track of relative monodromy phases between two S_{N-2} permutations π and σ. It has non-maximal rank $(N-2)!-(N-3)!$, so the linear relations between color-ordered superstring amplitudes can be compactly represented as $\sum_{\sigma\in S_{N-2}} \mathcal{S}_{\alpha'}[\pi|\sigma]\mathcal{A}(1,2_\sigma,3_\sigma,\ldots,(N-1)_\sigma,N)=0$, $\pi\in S_{N-2}$. On the level of the functions, this relation implies: $\sum_{\sigma\in S_{N-2}} \mathcal{S}_{\alpha'}[\pi|\sigma]F^\rho_{(1,2_\sigma,3_\sigma,\ldots,(N-1)_\sigma,N)}=0$, $\pi,\rho\in S_{N-2}$.

in the literature: In the maximal helicity violating (MHV) case, the subamplitudes reduce to the famous Parke–Taylor or Berends–Giele formula [24,25]. For the general NMHV case, the complete expressions for $\mathcal{A}_{YM}(1, 2_\sigma, \ldots, (N-2)_\sigma, N-1, N)$ can be found in [26].

Since in the sum (2.1) the kinematical factors $\mathcal{A}_{YM}$ and the functions F^σ encoding the string effects are multiplied together, supersymmetric Ward identities established in field theory [27–29] hold also for the full superstring amplitude, cf. also [21]. At any rate, after component expansion the pure spinor result provides the N-point amplitude involving any member of the SYM vector multiplet [30].

2.4. Minimal basis of multiple hypergeometric functions F^σ

The system of $(N-3)!$ multiple hypergeometric functions F^σ appearing in (2.1) are given as generalized Euler integrals [1][2]

$$F^{(23\ldots N-2)}(s_{ij}) = (-1)^{N-3} \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \left\{\prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}\right\}, \tag{2.15}$$

with permutations $\sigma \in S_{N-3}$ acting on all indices within the curly brace. Integration by parts admits to simplify the integrand in (2.15). As a result the length of the sum over m becomes shorter for $k > [N/2]$:

$$F^{(23\ldots N-2)}(s_{ij}) = (-1)^{N-3} \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \times \left(\prod_{k=2}^{[N/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}\right) \left(\prod_{k=[N/2]+1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}}\right). \tag{2.16}$$

In the above, $[\ldots]$ denotes the Gauss bracket $[x] = \max_{n\in\mathbf{Z}, n\leqslant x} n$, which picks out the nearest integer smaller than or equal to its argument.

The result (2.1) is manifestly gauge invariant as a consequence of gauge invariance of the YM subamplitudes $\mathcal{A}_{YM}$. Hence, gauge invariance does not impose further restrictions on the $(N-3)!$ functions $F^\sigma_{(1,\ldots,N)}$, which would further reduce the basis. The set (2.15) of $(N-3)!$ functions represents a minimal basis for the set of multiple Gaussian hypergeometric functions or Euler integrals appearing at N-point and referring to the same color ordering $(1, \ldots, N)$ or integration region $z_1 < \cdots < z_N$. Any function of this ordering can be expressed in terms of this basis.

The lowest terms of the α'-expansion of the functions F^σ assume the form

$$\begin{aligned} F^\sigma &= 1 + \alpha'^2 p_2^\sigma \zeta(2) + \alpha'^3 p_3^\sigma \zeta(3) + \cdots, \quad \sigma = (23\ldots N-2), \\ F^\sigma &= \alpha'^2 p_2^\sigma \zeta(2) + \alpha'^3 p_3^\sigma \zeta(3) + \cdots, \quad \sigma \neq (23\ldots N-2), \end{aligned} \tag{2.17}$$

with some polynomials p_n^σ of degree n in the dimensionful kinematic invariants $\hat{s}_{ij} = (k_i + k_j)^2 = s_{ij}/\alpha'$ and $\hat{s}_{i\ldots l} = (k_i + \cdots + k_l)^2 = s_{i\ldots l}/\alpha'$. Note that starting at $N \geqslant 7$ subsets of F^σ start at even higher order in α', i.e. $p_2^\sigma, \ldots, p_\nu^\sigma = 0$ for some $\nu \geqslant 2$, cf. Section 3 and

[2] In contrast to [1] here we use momenta redefined by a factor of i. As a consequence the signs of the kinematic invariants are flipped, e.g. $|z_{il}|^{-s_{il}} \to |z_{il}|^{s_{il}}$.

Appendix C for further details. Hence, only the first term of (2.1) contributes to the field-theory limit of the full N-point superstring amplitude. The power series expansions (2.17) in α' is such, that to each power α'^n a transcendental function of degree n shows up. More precisely, a set of multizeta values (MZVs) of fixed weight n appears at α'^n. The latter are multiplied by a polynomial p_n^σ of degree n in the kinematic invariants $\hat{s}$ with rational coefficients. We refer the reader to Subsection 3.1 and Appendix A for more details on α'-expansions and MZVs. From (2.17) we conclude, that the whole pole structure of the amplitude (2.1) is encoded in the YM subamplitudes $\mathcal{A}_{YM}$, while the functions F^σ are finite, i.e. do not have poles in the kinematic invariants. A detailed account on multiple Gaussian hypergeometric functions can be found in [31].

2.5. Extended set of multiple hypergeometric functions F^Π

A system of $(N-2)!$ functions F^Π, which fulfills (2.10) can be given as follows

$$F^{(23\ldots N-1)}(s_{ij}) = \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \left\{ \frac{(-1)^{N-3}}{z_{N-1}-z_1} \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\}, \tag{2.18}$$

with permutations $\Pi \in S_{N-2}$ acting on all indices within the curly brace. The set of $(N-2)!$ functions (2.18) can be expressed in terms of the basis (2.15) as a consequence of the relations (2.10). This allows to express $(N-2)! - (N-3)! = (N-3) \times (N-3)!$ functions of (2.18) in terms of (2.15). This will be demonstrated at some examples in the next subsection.

In contrast to the minimal set of functions F^σ, $\sigma \in S_{N-3}$, some elements of the extended set F^Π, $\Pi \in S_{N-2}$, might have poles in individual Mandelstam invariants.

2.6. Examples

2.6.1. $N=4$

The unique integral appearing in (2.1) for the four-point amplitude is

$$\begin{aligned} F^{(2)} &= -\int_0^1 dz_2 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \\ &= 1 - \zeta(2)s_{12}s_{23} + \zeta(3)s_{12}s_{13}s_{23} + \mathcal{O}(\alpha'^4). \end{aligned} \tag{2.19}$$

The extended set of two functions consists of (2.19) (with $F^{(2)} \equiv F^{(23)}$) and the additional function (2.18):

$$\begin{aligned} F^{(32)} &= -\int_0^1 dz_2 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{21}} \frac{s_{13}}{z_{13}} = \frac{s_{13}}{s_{12}} \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \\ &= \frac{s_{13}}{s_{12}} - \zeta(2)s_{13}s_{23} + \zeta(3)s_{13}^2 s_{23} + \mathcal{O}(\alpha'^4). \end{aligned} \tag{2.20}$$

With this extended set of two functions we may explicitly verify the relation (2.10). For the new basis $\pi = \{(1,3,2,4)\}$ in Eq. (2.2) we have

$$K_\pi^\sigma = \frac{s_{12}}{s_{13}} \tag{2.21}$$

w.r.t. the reference basis $\sigma = \{(1,2,3,4)\}$ as a consequence of the field-theory relation $\mathcal{A}_{YM}(1,3,2,4) = \frac{s_{12}}{s_{13}}\mathcal{A}_{YM}(1,2,3,4)$. According to (2.10) the following identity indeed holds:

$$F^{(32)} = K^{-1}\, F^{(23)}. \tag{2.22}$$

2.6.2. $N = 5$

The set of two basis functions appearing in (2.1) and following from (2.15) is:

$$\begin{aligned}
F^{(23)} &= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}}\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right) \\
&= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}}\frac{s_{34}}{z_{34}} = 1 + \zeta(2)(s_1 s_3 - s_3 s_4 - s_1 s_5) \\
&\quad - \zeta(3)\big(s_1^2 s_3 + 2 s_1 s_2 s_3 + s_1 s_3^2 - s_3^2 s_4 - s_3 s_4^2 - s_1^2 s_5 - s_1 s_5^2\big) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(32)} &= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{13}}{z_{13}}\left(\frac{s_{12}}{z_{12}} + \frac{s_{32}}{z_{32}}\right) \\
&= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{13}}{z_{13}}\frac{s_{24}}{z_{24}} \\
&= \zeta(2) s_{13} s_{24} - \zeta(3) s_{13} s_{24}(s_1 + s_2 + s_3 + s_4 + s_5) + \mathcal{O}\big(\alpha'^4\big),
\end{aligned} \tag{2.23}$$

where $s_i \equiv \alpha'(k_i + k_{i+1})^2$ subject to cyclic identification $k_{i+N} \equiv k_i$.

The extended set of six functions consists of (2.23), with

$$F^{(234)} := F^{(23)}, \qquad F^{(324)} := F^{(32)}, \tag{2.24}$$

and the additional four functions (2.18):

$$\begin{aligned}
F^{(423)} &= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{31}}\frac{s_{14}}{z_{14}}\frac{s_{23}}{z_{23}}, \\
F^{(243)} &= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{31}}\frac{s_{12}}{z_{12}}\frac{s_{34}}{z_{43}}, \\
F^{(432)} &= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{21}}\frac{s_{14}}{z_{14}}\frac{s_{23}}{z_{32}}, \\
F^{(342)} &= \int\limits_{0<z_2<z_3<1} dz_2\, dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{21}}\frac{s_{13}}{z_{13}}\frac{s_{24}}{z_{42}}.
\end{aligned} \tag{2.25}$$

With this extended set of six functions we may explicitly verify the relation (2.10). For the new basis $\pi = \{(1,4,2,3,5), (1,2,4,3,5)\}$ in Eq. (2.2) we have

$$K_\pi^\sigma = -\frac{1}{s_{14}s_{35}}\begin{pmatrix} s_{12}\, s_{34} & -s_{13}(s_{34}+s_{45}) \\ s_{14}(s_{12}-s_{45}) & -s_{14}s_{13} \end{pmatrix} \tag{2.26}$$

w.r.t. the reference basis $\sigma = \{(1,2,3,4,5), (1,3,2,4,5)\}$. According to (2.10) the following identity indeed holds (with $K^* = (K^{-1})^t$):

$$\begin{pmatrix} F^{(423)} \\ F^{(243)} \end{pmatrix} = K^* \begin{pmatrix} F^{(234)} \\ F^{(324)} \end{pmatrix}. \tag{2.27}$$

On the other hand, for the new basis $\pi = \{(1,4,3,2,5), (1,3,4,2,5)\}$ we have

$$K_\pi^\sigma = -\frac{1}{s_{14}s_{25}} \begin{pmatrix} s_{12}(s_{14}+s_{34}) & s_{13}s_{24} \\ -s_{12}s_{14} & -s_{14}(s_{12}+s_{23}) \end{pmatrix}, \tag{2.28}$$

and the following relation can be checked:

$$\begin{pmatrix} F^{(432)} \\ F^{(342)} \end{pmatrix} = K^* \begin{pmatrix} F^{(234)} \\ F^{(324)} \end{pmatrix}. \tag{2.29}$$

Hence, the relations (2.27) and (2.29) allow to express the additional set of functions (2.25) in terms of the minimal basis (2.23).

2.6.3. $N=6$

The set of six basis functions appearing in (2.1) and following from (2.15) is

$$\begin{aligned}
F^{(234)} &= -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}} \frac{s_{45}}{z_{45}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right) \\
&= 1 - \zeta(2)(s_4 s_5 + s_1 s_6 - s_4 t_1 - s_1 t_3 + t_1 t_3) \\
&\quad + \zeta(3)\big(2s_1 s_2 s_4 + 2s_1 s_3 s_4 + s_4^2 s_5 + s_4 s_5^2 + s_1^2 s_6 + s_1 s_6^2 - 2s_3 s_4 t_1 \\
&\quad - s_4^2 t_1 - s_4 t_1^2 - 2s_1 s_4 t_2 - s_1^2 t_3 - 2s_1 s_2 t_3 + t_1^2 t_3 - s_1 t_3^2 + t_1 t_3^2\big) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(324)} &= -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{13}}{z_{13}} \frac{s_{45}}{z_{45}} \left(\frac{s_{12}}{z_{12}} + \frac{s_{32}}{z_{32}}\right) \\
&= -\zeta(2) s_{13} d_9 + \zeta(3) s_{13}\big(s_1 s_2 + s_2^2 - 2s_2 s_4 - 2s_3 s_4 - s_1 s_6 - s_6^2 \\
&\quad + s_2 t_1 - s_6 t_1 + 2s_4 t_2 + s_1 t_3 + 2s_2 t_3 + t_1 t_3 + t_3^2\big) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(432)} &= -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{14}}{z_{14}} \frac{s_{25}}{z_{25}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{43}}{z_{43}}\right) \\
&= -\zeta(2) s_{14} s_{25} + \zeta(3) s_{14} s_{25}(-s_2 - s_3 + s_5 + s_6 + t_1 + t_2 + t_3) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(342)} &= -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{13}}{z_{13}} \frac{s_{25}}{z_{25}} \left(\frac{s_{14}}{z_{14}} + \frac{s_{34}}{z_{34}}\right) \\
&= \zeta(2) s_{13} s_{25} + \zeta(3) s_{13} s_{25}(-s_1 + s_2 + 2s_3 - s_6 - t_1 - 2t_2 - t_3) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(423)} &= -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{14}}{z_{14}} \frac{s_{35}}{z_{35}} \left(\frac{s_{12}}{z_{12}} + \frac{s_{42}}{z_{42}}\right) \\
&= \zeta(2) s_{14} s_{35} + \zeta(3) s_{14} s_{35}(2s_2 + s_3 - s_4 - s_5 - t_1 - 2t_2 - t_3) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(243)} &= -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}} \frac{s_{35}}{z_{35}} \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}}\right) \\
&= -\zeta(2) s_{35} d_1 + \zeta(3) s_{35}\big(-2s_1 s_2 - 2s_1 s_3 + s_3^2 + s_3 s_4 - s_4 s_5 - s_5^2 \\
&\quad + 2s_3 t_1 + s_4 t_1 + t_1^2 + 2s_1 t_2 + s_3 t_3 - s_5 t_3 + t_1 t_3\big) + \mathcal{O}\big(\alpha'^4\big),
\end{aligned} \tag{2.30}$$

with $d_1 = s_3 - s_5 + t_1$, $d_9 = s_2 - s_6 + t_3$, and $t_i := t_{i,i+1,i+2} = \alpha'(k_i + k_{i+1} + k_{i+2})^2$, $i = 1, 2, 3$. The extended set of 24 functions consists of (2.30) with

$$F^{(2345)} := F^{(234)}, \qquad F^{(3245)} := F^{(324)}, \qquad F^{(4325)} := F^{(432)},$$
$$F^{(3425)} := F^{(342)}, \qquad F^{(4235)} := F^{(423)}, \qquad F^{(2435)} := F^{(243)}, \tag{2.31}$$

and the additional 18 functions (2.18) are given by:

$$F^{(2_\sigma 3_\sigma 5_\sigma 4)} = -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{41}} \frac{s_{12_\sigma}}{z_{12_\sigma}} \frac{s_{45_\sigma}}{z_{5_\sigma 4}} \left(\frac{s_{13_\sigma}}{z_{13_\sigma}} + \frac{s_{2_\sigma 3_\sigma}}{z_{2_\sigma 3_\sigma}}\right), \tag{2.32}$$

$$F^{(2_\sigma 4_\sigma 5_\sigma 3)} = -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{31}} \frac{s_{12_\sigma}}{z_{12_\sigma}} \frac{s_{35_\sigma}}{z_{5_\sigma 3}} \left(\frac{s_{14_\sigma}}{z_{14_\sigma}} + \frac{s_{2_\sigma 4_\sigma}}{z_{2_\sigma 4_\sigma}}\right), \tag{2.33}$$

$$F^{(3_\sigma 4_\sigma 5_\sigma 2)} = -\int\limits_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{1}{z_{21}} \frac{s_{13_\sigma}}{z_{13_\sigma}} \frac{s_{25_\sigma}}{z_{5_\sigma 2}} \left(\frac{s_{14_\sigma}}{z_{14_\sigma}} + \frac{s_{3_\sigma 4_\sigma}}{z_{3_\sigma 4_\sigma}}\right). \tag{2.34}$$

For the new basis $\pi = \{(1,2,3,5,4,6), (1,3,2,5,4,6), (1,5,3,2,4,6), (1,3,5,2,4,6), (1,5,2,3,4,6), (1,2,5,3,4,6)\}$ in Eq. (2.2) we have

$$K^\sigma_\pi = s_{46}^{-1} \times \begin{pmatrix} s_5 - t_1 & 0 & 0 & 0 & s_{14} & -d_1 \\ 0 & s_5 - t_1 & s_{14} & s_3 + s_{14} & 0 & 0 \\ \frac{s_1 s_4 d_0}{s_{15} t_{246}} & \frac{s_4 s_{13}(s_{25}-s_{46})}{s_{15} t_{246}} & \frac{-s_{13} s_{14} s_{25}}{s_{15} t_{246}} & \frac{-s_{13} s_{25}(s_3+s_{14})}{s_{15} t_{246}} & \frac{s_{14}(s_{46}-s_1) d_0}{s_{15} t_{246}} & \frac{s_1(s_3+s_4) d_0}{s_{15} t_{246}} \\ \frac{-s_1 s_4}{t_{246}} & \frac{-s_4(s_1+s_2)}{t_{246}} & \frac{s_{14} d_4}{t_{246}} & \frac{(s_{14}+s_3) d_4}{t_{246}} & \frac{s_{14}(s_1-s_{46})}{t_{246}} & \frac{-s_1(s_3+s_4)}{t_{246}} \\ \frac{s_1 s_4(s_{35}-s_{46})}{s_{15} t_{125}} & \frac{s_4 s_{13} d_3}{s_{15} t_{125}} & \frac{(s_{46}-s_{13}) d_3 s_{14}}{s_{15} t_{125}} & \frac{(s_4+s_{24}) s_{13} d_3}{s_{15} t_{125}} & \frac{-s_1 s_{14} s_{35}}{s_{15} t_{125}} & \frac{s_1 s_{35} d_1}{s_{15} t_{125}} \\ \frac{s_4(s_1-t_1)}{t_{125}} & \frac{-s_4 s_{13}}{t_{125}} & \frac{s_{14}(s_{13}-s_{46})}{t_{125}} & \frac{-s_{13}(s_4+s_{24})}{t_{125}} & \frac{-s_{14} d_2}{t_{125}} & \frac{d_1 d_2}{t_{125}} \end{pmatrix} \tag{2.35}$$

w.r.t. the reference basis $\sigma = \{(1,2,3,4,5,6), (1,3,2,4,5,6), (1,4,3,2,5,6), (1,3,4,2,5,6), (1,4,2,3,5,6), (1,2,4,3,5,6)\}$. According to (2.10) the following identity indeed holds:

$$\begin{pmatrix} F^{(2354)} \\ F^{(3254)} \\ F^{(5324)} \\ F^{(3524)} \\ F^{(5234)} \\ F^{(2534)} \end{pmatrix} = K^* \begin{pmatrix} F^{(2345)} \\ F^{(3245)} \\ F^{(4325)} \\ F^{(3425)} \\ F^{(4235)} \\ F^{(2435)} \end{pmatrix}. \tag{2.36}$$

In the above matrix (2.35) we have introduced $d_0 = s_{15} + s_{35}$, $d_2 = s_1 - s_4 - s_5$, $d_3 = s_3 - s_5 - t_3$, $d_4 = s_4 + s_5 - s_{13}$ and $t_{ijk} = \alpha'(k_i k_j + k_i k_k + k_j k_k)$. The other two sets of basis π and their relations (2.10) to the reference basis σ are displayed in Appendix B.

2.7. *Properties of the full amplitude*

The factorization properties of tree-level amplitudes are well studied in field theory [32]. These properties represent an important test of our string result.

2.7.1. *Soft limit*

According to Subsection 2.2 it is sufficient to focus on the N-gluon amplitude. We consider the limit $k_{N-2} \to 0$. In this limit the amplitude (2.1) behaves as[3]:

$$\mathcal{A}(1,\ldots,N) \to \left(\frac{\xi k_{N-2}}{k_{N-2}k} - \frac{\xi k_{N-3}}{k_{N-3}k}\right)\mathcal{A}(1,\ldots,N-1). \tag{2.37}$$

This can be proven by considering the limits of the individual summands of (2.1):

(i) $\sigma \in S_{N-4}$ with $(N-3)_\sigma = N-3$:

$$\begin{aligned}&\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-3)_\sigma,N-2,N-1,N\big)F^\sigma\big(\alpha'\big)\\&\quad\to \left(\frac{\xi k_{N-2}}{k_{N-2}k} - \frac{\xi k_{N-3}}{k_{N-3}k}\right)\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-3)_\sigma,N-2,N-1\big)\widetilde{F}^\sigma\big(\alpha'\big),\end{aligned} \tag{2.38}$$

(ii) $\sigma \in S_{N-4}$ with $(N-3)_\sigma \neq N-3$:

$$\begin{aligned}&\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-3)_\sigma,N-2,N-1,N\big)F^\sigma\big(\alpha'\big)\\&\quad\to \left(\frac{\xi k_{N-2}}{k_{N-2}k} - \frac{\xi k_{(N-3)_\sigma}}{k_{(N-3)_\sigma}k}\right)\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-3)_\sigma,N-2,N-1\big)\widetilde{F}^\sigma\big(\alpha'\big),\end{aligned}$$

(iii) $\sigma \in S_{N-4}$ with $N-3 \in \{2_\sigma,\ldots,i_\sigma\}$ and $i=2,\ldots,N-4$, i.e. $(N-3)_\sigma \neq N-3$:

$$\begin{aligned}&\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,i_\sigma,N-2,(i+1)_\sigma,\ldots,(N-3)_\sigma,N-1,N\big)F^\sigma\big(\alpha'\big)\\&\quad\to \left(\frac{\xi k_{(i+1)_\sigma}}{k_{(i+1)_\sigma}k} - \frac{\xi k_{i_\sigma}}{k_{i_\sigma}k}\right)\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,(N-3)_\sigma,N-2,N-1\big)\widetilde{F}^\sigma\big(\alpha'\big),\end{aligned}$$

(iv) $\sigma \in S_{N-4}$:

$$\mathcal{A}_{YM}\big(1,N-2,2_\sigma,\ldots,(N-3)_\sigma,N-1,N\big)F^\sigma\big(\alpha'\big) \to 0,$$

(v) $\sigma \in S_{N-4}$ with $N-3 \in \big\{(i+1)_\sigma,\ldots,(N-3)_\sigma\big\}$ and $i=2,\ldots,N-4$:

$$\mathcal{A}_{YM}\big(1,2_\sigma,\ldots,i_\sigma,N-2,(i+1)_\sigma,\ldots,(N-3)_\sigma,N-1,N\big)F^\sigma\big(\alpha'\big) \to 0.$$

The above functions $\widetilde{F}^\sigma$ refer to the $N-1$-point amplitude. While the $(N-5)!$ summands of case (i) already have the right form (2.37) and give rise to $(N-5)!$ terms of the $N-1$-point amplitude (2.1), the remaining non-vanishing limits (ii) and (iii) for a given $\sigma \in S_{N-4}$ with $(N-3)_\sigma \neq N-3$ conspire to comprise the remaining $(N-5)(N-5)!$ terms of (2.1) thanks to the relation:

[3] The vectors ξ and k refer to the transverse polarization and momentum of the soft gluon, respectively. Furthermore, k_j denote the external momenta of remaining legs. One could also express the kinematic dependent factor as soft or eikonal factor written e.g. in the $D=4$ spinor helicity basis [32,33].

$$\left(\frac{\xi k_{N-2}}{k_{N-2}k} - \frac{\xi k_{(N-3)_\sigma}}{k_{(N-3)_\sigma}k}\right) + \sum_{\substack{i=2 \\ N-3\in\{2_\sigma,\ldots,i_\sigma\}}}^{N-4} \left(\frac{\xi k_{(i+1)_\sigma}}{k_{(i+1)_\sigma}k} - \frac{\xi k_{i_\sigma}}{k_{i_\sigma}k}\right) = \left(\frac{\xi k_{N-2}}{k_{N-2}k} - \frac{\xi k_{N-3}}{k_{N-3}k}\right).$$

The remaining $\frac{1}{2}(N-3)!$ terms of the cases (iv) and (v) do not contribute in the soft limit.

2.7.2. Collinear limit

Again, according to Subsection 2.2 it is sufficient to focus on the N-gluon amplitude. The collinear limit is defined as two adjacent external momenta k_i and k_{i+1}, with $i+1$ mod N, becoming parallel. Due to cyclic symmetry, these can be chosen as k_{N-3} and k_{N-2}, with k_{N-3} carrying the fraction x of the combined momentum $k_{N-3}+k_{N-2} \to k_{N-3}$. Formally,

$$k_{N-3} \to x k_{N-3}, \qquad k_{N-2} \to (1-x)k_{N-3}, \tag{2.39}$$

where the momenta appearing in the limits describe the scattering amplitude of $N-1$ gluons. In this limit the amplitude (2.1) behaves as[4]

$$\mathcal{A}(1,\ldots,N) \to \frac{1}{k_{N-3}k_{N-2}} V^\mu \frac{\partial}{\partial \xi^\mu_{N-3}} \mathcal{A}(1,\ldots,N-1), \tag{2.40}$$

with the three-gluon vertex $V^\mu = (\xi_{N-3}\xi_{N-2})(k^\mu_{N-2} - k^\mu_{N-3}) + 2(\xi_{N-2}k_{N-3})\xi^\mu_{N-3} - 2(\xi_{N-3}k_{N-2})\xi^\mu_{N-2}$. This can be proven by considering the limits of the individual summands of (2.1). First, if the two states $N-3$ and $N-2$ are not neighbors, we have:

(i) $\sigma \in S_{N-4}$ with $2_\sigma \neq N-3$:

$$\mathcal{A}_{YM}\big(1, N-2, 2_\sigma, \ldots, (N-3)_\sigma, N-1, N\big) \to 0,$$

(ii) $\sigma \in S_{N-4}$ with $i_\sigma, (i+1)_\sigma \neq N-3$ and $i = 2, \ldots, N-4$:

$$\mathcal{A}_{YM}\big(1, 2_\sigma, \ldots, i_\sigma, N-2, (i+1)_\sigma, \ldots, (N-3)_\sigma, N-1, N\big) \to 0. \tag{2.41}$$

On the other hand, the remaining $2(N-4)!$ terms of (2.1) pair up into $(N-4)!$ tuples $(\sigma, \tilde\sigma)$ each giving rise to one of the $(N-4)!$ terms of the $N-1$-point amplitude (2.1):

$$\begin{aligned}
&\sigma, \tilde\sigma \in S_{N-4} \quad \text{with } i_\sigma = (i+1)_{\tilde\sigma} = N-3 \text{ and } i = 2, \ldots, N-4: \\
&\mathcal{A}_{YM}\big(1, 2_\sigma, \ldots, i_\sigma, N-2, (i+1)_\sigma, \ldots, (N-3)_\sigma, N-1, N\big) F^\sigma\big(\alpha'\big) \\
&\quad + \mathcal{A}_{YM}\big(1, 2_{\tilde\sigma}, \ldots, i_{\tilde\sigma}, N-2, (i+1)_{\tilde\sigma}, \ldots, (N-3)_{\tilde\sigma}, N-1, N\big) F^{\tilde\sigma}\big(\alpha'\big) \\
&\quad \to \frac{1}{k_{N-3}k_{N-2}} V^\mu \frac{\partial}{\partial \xi^\mu_{N-3}} \mathcal{A}_{YM}\big(1, 2_\sigma, \ldots, (N-3)_\sigma, N-2, N-1\big) F^\sigma\big(\alpha'\big).
\end{aligned} \tag{2.42}$$

Note that in the above combination the x-dependent parts of the two functions F^σ and $F^{\tilde\sigma}$, which stems from the limit (2.39), add up to zero.

[4] One could also express the kinematic dependent factor as splitting amplitude written e.g. in the $D = 4$ spinor helicity basis [32,33].

2.7.3. Cyclic invariance

While the YM constituent $\mathcal{A}_{YM}(1,\dots,N)$ of (2.1) is invariant under cyclic transformations of its labels $i \to i+1$ mod N, all others transform non-trivially. More precisely, the set $\{\mathcal{A}_{YM}(1,2_\sigma,\dots,(N-2)_\sigma,N-1,N) \mid \sigma \in S_{N-3}\}$ is mapped to the set $\{\mathcal{A}_{YM}(1,2,3_\sigma,\dots,(N-2)_\sigma,(N-1)_\sigma,N) \mid \sigma \in S_{N-3}\}$ by virtue of the cyclic properties of the $\mathcal{A}_{YM}$. The latter set belongs to the extended S_{N-2} family $\{\mathcal{A}_{YM}(1,2_\Pi,\dots,(N-1)_\Pi,N) \mid \Pi \in S_{N-2}\}$, which can be expanded in terms of the original basis $\mathcal{A}_{YM}(1,2_\sigma,\dots,(N-2)_\sigma,N-1,N)$ according to (2.2). The cyclic transformation properties of the minimal basis functions F^σ are such that the change of $\mathcal{A}_{YM,\sigma}$ into $\mathcal{A}_{YM,\Pi(\sigma)} = \sum_{\pi\in S_{N-3}} K^{\pi}_{\Pi(\sigma)}\mathcal{A}_{YM,\pi}$ is compensated:

$$F^\sigma\big|_{k_i\to k_{i+1}} = F^{\Pi(\sigma)} = \sum_{\rho\in S_{N-3}} \left(K^{-1}\right)_\rho^{\Pi(\sigma)} F^\rho. \tag{2.43}$$

The map $\Pi(\sigma)$ is defined by $(2_{\Pi(\sigma)},\dots,(N-1)_{\Pi(\sigma)}) = (2, 2_\sigma+1,\dots,(N-2)_\sigma+1)$.

3. The module of multiple hypergeometric functions

The functions F^σ describing the full N-point amplitude (2.1) have been introduced in Eqs. (2.15) and (2.18) and are given by generalized Euler integrals. Generalized Euler integrals appear in any higher-point open string amplitude computation. Therefore, we find it useful in this section to investigate the properties of these integrals on general grounds.

3.1. Generalized Euler integrals and multiple hypergeometric functions

For the color ordering $(1,\dots,N)$ the integrals of interest can be written

$$B_N[\tilde n] = V_{CKG}^{-1} \int\limits_{z_i<z_{i+1}} \left(\prod_{j=1}^N dz_j\right) \prod_{1\leqslant i<j\leqslant N} |z_{ij}|^{s_{ij}} z_{ji}^{\tilde n_{ij}}, \tag{3.1}$$

with some set $\tilde n$ of integers $\tilde n_{ij} \in \mathbf{Z}$ and the factor V_{CKG} accounting for the volume of the conformal Killing group of the disk after choosing the conformal gauge. The integers $\tilde n_{ij}$ must fulfill the conditions[5]

$$\sum_{i<j}^N \tilde n_{ij} + \sum_{i>j}^N \tilde n_{ji} = -2, \quad j=1,\dots,N, \tag{3.2}$$

as a result of conformal invariance on the string world-sheet. After fixing three of the vertex positions as

$$z_1 = 0, \qquad z_{N-1} = 1, \qquad z_N = \infty, \tag{3.3}$$

and parameterizing the integration region $z_2 < \dots < z_{N-2}$ as

$$z_k = \prod_{l=k-1}^{N-3} x_l, \quad k = 2,\dots,N-2, \tag{3.4}$$

with $0 < x_i < 1$ the integrand in (2.15) takes the generic form:

[5] Note that the integrands of (2.15) and (2.18) can always be completed to meet this condition.

$$B_N[n] = \left(\prod_{i=1}^{N-3} \int_0^1 dx_i \right) \prod_{j=1}^{N-3} x_j^{s_{12\ldots j+1}+n_j} \prod_{l=j}^{N-3} \left(1 - \prod_{k=j}^{l} x_k \right)^{s_{j+1,l+2}+n_{jl}}, \tag{3.5}$$

with the set of $\frac{1}{2}N(N-3)$ integers $n_j, n_{jl} \in \mathbf{Z}$ and $s_{i,j} \equiv s_{ij}$:

$$\begin{aligned} n_{jl} &= \tilde{n}_{j+1,l+2}, \quad j \leqslant l, \\ n_j &= j - 1 + \sum_{i<j}^{j+1} \tilde{n}_{il}, \quad 1 \leqslant j \leqslant N-3. \end{aligned} \tag{3.6}$$

The integrals represent generalized Euler integrals and integrate to multiple Gaussian hypergeometric functions [20].

With (3.2) and (3.6) from a rational function

$$R(x_i) = \prod_{j=1}^{N-3} x_j^{n_j} \prod_{l=j}^{N-3} \left(1 - \prod_{k=j}^{l} x_k \right)^{n_{jl}}$$

in the $N-3$ variables x_i multiplying the integrand of (3.5) an other rational function

$$\widetilde{R}(z_{ij}) = \prod_{1 \leqslant i<j \leqslant N-1} z_{ji}^{\tilde{n}_{ij}}$$

depending on the $N-1$ variables z_i and multiplying the integrand of (3.1) can be computed. In the following we write this correspondence as:

$$R(x_i) \simeq \widetilde{R}(z_{ij}). \tag{3.7}$$

3.2. Partial fraction decomposition and finding a basis

There are many relations among integrals (3.1) with different sets $\tilde{n}$ of integers as a result[6] of partial fraction decomposition

$$\frac{1}{z_{ij} z_{jk}} + \frac{1}{z_{ik} z_{kj}} = \frac{1}{z_{ij} z_{ik}} \tag{3.8}$$

and partial integration of their integrands:

$$\begin{aligned} 0 &= \int \prod_{j=2}^{N-2} dz_j \frac{\partial}{\partial z_k} \prod_{1 \leqslant i<j \leqslant N-1} |z_{ij}|^{s_{ij}} z_{ji}^{\tilde{n}_{ij}} \\ &= \int \prod_{j=2}^{N-2} dz_j \prod_{1 \leqslant i<j \leqslant N-1} |z_{ij}|^{s_{ij}} z_{ji}^{\tilde{n}_{ij}} \left(\sum_{m \neq k} \frac{s_{km}}{z_{km}} + \sum_{m<k} \frac{\tilde{n}_{mk}}{z_{km}} + \sum_{m>k} \frac{\tilde{n}_{km}}{z_{mk}} \right). \end{aligned} \tag{3.9}$$

Note that in this way any integral (3.1) with powers $\tilde{n}_{ij} < -1$ can always be expressed by a chain of integrals with $\tilde{n}_{ij} \geqslant -1$. Hence, in the following it is sufficient to concentrate on those cases $\tilde{n}_{ij} \geqslant -1$. A quantitative handiness on finding a minimal set of functions can be obtained by performing

6 In fact, these tools have allowed to boil down the set of functions appearing in the open superstring N-point amplitude [1] to the set (2.15).

(i) a classification of the integrals (3.1) according to their pole structure in the kinematic invariants s_{ij}, and
(ii) a Gröbner basis analysis for those integrals (3.1) without poles.

Any partial fraction decomposition of an Euler integral with poles can be arranged according to its pole structure (modulo finite or subleading pieces) and the classification (i) yields a basis for them. This is achieved by performing a partial fraction expansion of the leading singularity in the kinematic invariants s_{ij}. On the other hand, the Gröbner basis analysis (ii) provides an independent set of rational functions or monomials in the Euler integrals and any integral (3.1) without poles can be expanded in terms of this set. Combining (i) and (ii) yields an independent set of integrals (3.1) and any partial fraction decomposition of Euler integrals (3.1) can be expressed in terms of the basis obtained this way. In Subsections 3.3 and 3.4 we explicitly construct this partial fraction basis for the cases $N=4,5$ and $N=6$ and verify its dimension $(N-2)!$.

The first classification (i) of the integrals (3.1) is done w.r.t. their pole structure in the kinematic invariants s_{ij}. The maximum number of possible simultaneous poles of an N-point amplitude is $N-3$. Integrals of this type play an important role, since they capture the field-theory limit of the full amplitude. They assume the following power series expansion in α':

$$\begin{aligned} B_N[\tilde{n}] &= \alpha'^{3-N} p_{3-N}[\tilde{n}] + \alpha'^{5-N} \sum_{m=0}^{\infty} \alpha'^m \sideset{}{'}\sum_{\substack{i_r \in \mathbf{N},\, i_1>1 \\ i_1+\cdots+i_d=m+2}} p^{\mathbf{i}}_{5-N+m}[\tilde{n}]\zeta(i_1,\ldots,i_d) \\ &= \alpha'^{3-N} p_{3-N}[\tilde{n}] + \alpha'^{5-N} p_{5-N}[\tilde{n}]\zeta(2) + \alpha'^{6-N} p_{6-N}[\tilde{n}]\zeta(3) + \cdots. \end{aligned} \tag{3.10}$$

The above rational functions or monomials $p^{\mathbf{i}}_{5-N+m}[\tilde{n}]$ are of degree $5-N+m$ in the dimensionful kinematic invariants $\hat{s}_{ij} = s_{ij}/\alpha'$ and depend on the integer set $\tilde{n}$. Furthermore, we have introduced the MZVs

$$\zeta(i_1,\ldots,i_d) = \sum_{n_1>\cdots>n_d>0} \prod_{r=1}^{d} n_r^{-i_r}, \quad i_r \in \mathbf{N},\ i_1>1$$

of transcendentality degree $\sum_{r=1}^{d} i_r = m+2$ and depth d, cf. e.g. [34] for more details and references. The prime at the sum (3.10) means, that the latter runs only over a basis of independent MZVs of weight $m+2$. In (3.10) at each order $5-N+m$ in α' a set of MZVs of a fixed transcendentality degree $m+2$ appears. We call such a power series expansion transcendental, cf. Appendix A for a detailed discussion. In Subsection 3.3 we present a method to extract the first term of (3.10) corresponding to integrals (3.1) with $N-3$ simultaneous poles. In fact, this method allows to extract any lowest order poles from integrals (3.1) with fewer simultaneous poles. However, as we shall demonstrate, their type of integrals generically does not assume the transcendental power series expansion (3.10). At any rate, the method of Subsection 3.3 determines the lowest order poles of the integral (3.1).

The second classification (ii) of the integrals (3.1) is appropriate, if the latter have no poles, i.e. their power series expansion in α' starts with some zeta constants. In Subsection 3.4 we introduce a Gröbner basis analysis, which allows to find an independent set of finite integrals (3.1), which serves as basis. Any other finite integral (3.1) is an $\mathbf{R}$-linear combination of this basis.

Note that the *individual* integrals entering the functions (2.15) and (2.18) are of both types – some of them have $N-3$ simultaneous poles and their α'-expansion assumes the form (3.10), others have no poles and start with some zeta constants. In either case our methods (i) or (ii) can be applied to further reduce them.

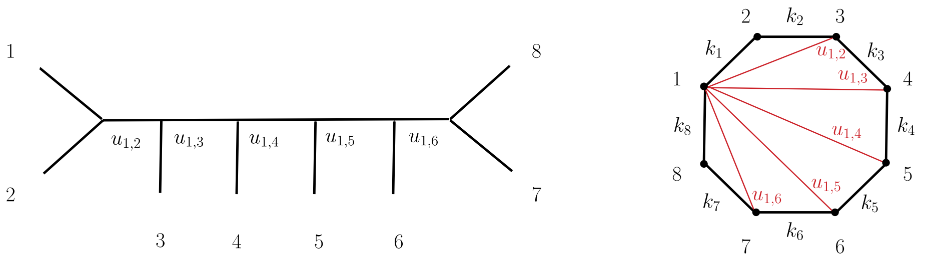

Fig. 1. Multiperipheral configuration and corresponding dual diagram for $N = 8$.

3.3. Structure of multiple resonance exchanges

Generically, an N-point scattering process has multiple resonance exchanges. As a result, the power series expansion in α' of the integrals (3.5) may have multiple poles in the Mandelstam variables. These poles come from regions of the integrand for which $x_i \to 0$ or $x_i \to 1$ for some of the variables x_i. To obtain information on the pole structure of the integrals (3.5) it is useful to transform the integrand to a different form, in which the poles can be easily extracted.

For an N-point scattering process there are $\frac{1}{2}N(N-3)$ planar channels $(i,j) \in \mathcal{P}$ associated to the Mandelstam variable $S_{i,j} = \alpha'(k_i + k_{i+1} + \cdots + k_j)^2$, with

$$\mathcal{P} = \{(1,j) \mid 2 \leqslant j \leqslant N-2\} \cup \{(p,q) \mid 2 \leqslant p < q \leqslant N-1\} \tag{3.11}$$

for the color ordering $(1,2,\ldots,N)$. The channels (i,j) with states from $i,\ldots,j$ and $(j+1, i-1)$ with states from $j+1,\ldots,N,1,\ldots,i-1$ are identical. The set of $N-3$ kinematic invariants, which can simultaneously appear in the denominator of the α'-expansion of the N-point amplitude, describe the allowed (planar) channels of the underlying field-theory diagram involving cubic vertices. Not all combinations of channels are allowed. E.g. adjacent channels as $(i,i+1)$ and $(i+1,i+2)$ cannot appear simultaneously in denominators (dual or incompatible channels). On the other hand, for non-dual channels coincident poles are possible. A geometric way to find all compatible channels is to draw a convex N-polygon of N sides representing momentum conservation. The number of ways of cutting this polygon into $N-2$ triangles with $N-3$ non-intersecting straight lines gives the number of distinct sets of allowed channels. According to Euler's polygon division problem this number is given by $C_{N-2} = \frac{2^{N-2}\,(2N-5)!!}{(N-1)!}$, with the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The $N-3$ diagonals of this polygon represent the momenta of possible intermediate states. To each of the $\frac{1}{2}N(N-3)$ channels (i,j) a variable $u_{i,j} \in (0,1)$ may be ascribed, with $u_{i,j} \equiv u_{j+1,i-1}$. For an account and references on the multiparticle dual model see [35].

For a given channel (i,j) with $u_{i,j} = 0$ all incompatible channels (p,q) are required to have $u_{p,q} = 1$. This property is described by the $\frac{1}{2}N(N-3)$ duality constraint equations

$$u_{i,j} = 1 - \prod_{\substack{1 \leqslant p < i \\ i \leqslant q < j}} u_{p,q} \prod_{\substack{i < r \leqslant j \\ j < s \leqslant N-1}} u_{r,s}, \quad 1 \leqslant i < j \leqslant N, \tag{3.12}$$

which are sufficient for excluding simultaneous poles in incompatible channels. We define $u_{i,i} = 0$, $u_{1,N-1} = 1$ and have $u_{k,N} = u_{1,k-1}$, $k \geqslant 3$. Only $\frac{1}{2}(N-2)(N-3)$ of these equations (3.12) are independent, leaving $N-3$ variables $u_{i,j}$ out of the set of $\frac{1}{2}N(N-3)$ variables free. The set of $N-3$ independent variables $u_{i,j}$ can be associated to the inner lines of one of the C_{N-2} sliced N-polygon. In particular, as a canonical choice we may define

$$u_{1,j+1}=x_j, \quad j=1,\ldots,N-3 \tag{3.13}$$

as a set of $N-3$ independent variables corresponding to Fig. 1. Hence, each of the internal lines of the polygon corresponds to an independent variable x_j in the integral (3.5). Choosing the inner lines of an other sliced N-polygon results in a different integral representation (3.5). As a consequence of (3.12) and (3.13) we have[7]:

$$\begin{aligned} 1-x_j &= \prod_{\substack{0<r\leqslant j\\ j<s\leqslant N-2}} u_{r+1,s+1}, \quad j=1,\ldots,N-3,\\ 1-\prod_{k=i}^{j} x_k &= \prod_{\substack{1\leqslant p\leqslant i\\ j+1\leqslant q\leqslant N-2}} u_{p+1,q+1}, \quad 1\leqslant i\leqslant j\leqslant N-3. \end{aligned} \tag{3.15}$$

With (3.15) and the Jacobian $\prod_{2\leqslant i<j\leqslant N-1} u_{i,j}^{j-i-1}$, the integral (3.5) translates into an integral over all $\frac{1}{2}N(N-3)$ variables $u_{\mathcal{P}}$ related to the partitions $\mathcal{P}$ given in (3.11)

$$B_N[n]=\prod_{(i,j)\in\mathcal{P}} \int_0^1 du_{i,j}\, u_{i,j}^{S_{i,j}+n_{i,j}} \prod_{\mathcal{P}'\notin(1,j)} \delta\left(u_{\mathcal{P}'}-1+\prod_{\tilde{\mathcal{P}}} u_{\tilde{\mathcal{P}}}\right), \tag{3.16}$$

with the assignments:

$$\begin{aligned} &n_{1,j+1}=n_j, \qquad n_{j+1,j+2}=n_{jj}, \quad j=1,\ldots,N-3,\\ &n_{i,j}=j-i-1+\sum_{i-1\leqslant k\leqslant l}^{j-2} n_{kl}, \quad 1<i<j<N. \end{aligned} \tag{3.17}$$

In (3.16) the integration is constrained by the duality conditions (3.12) resulting in a product of $\frac{1}{2}(N-2)(N-3)$ independent δ-functions. In this form (3.16) many properties of the integrals (3.5) like the pole structure or cyclicity become manifest. Later this will be elucidated with examples.

We can introduce a fundamental set of C_{N-2} integrals B_N

$$\bigcup_{(i_l,j_l)\in\mathcal{P}} \left\{ \prod_{(i,j)\in\mathcal{P}} \int_0^1 du_{i,j}\, u_{i,j}^{S_{i,j}} \left(\prod_{l=1}^{N-3} u_{i_l,j_l}\right)^{-1} \prod_{\mathcal{P}'\notin(1,j)} \delta\left(u_{\mathcal{P}'}-1+\prod_{\tilde{\mathcal{P}}} u_{\tilde{\mathcal{P}}}\right)\right\}, \tag{3.18}$$

[7] The inverse solution to the duality constraint (3.12) may be found as ($p=2,3,\ldots,N-2$; $q=3,4,\ldots,N-1$ and $p<q$):

$$u_{p,q}=\begin{cases} \dfrac{(1-\prod_{m=p}^{q-1}u_{1,m})(1-\prod_{n=p-1}^{q}u_{1,n})}{(1-\prod_{r=p-1}^{q-1}u_{1,r})(1-\prod_{s=p}^{q}u_{1,s})}, & q\neq N-1,\\ \dfrac{(1-\prod_{m=p}^{q-1}u_{1,m})}{(1-\prod_{r=p-1}^{q-1}u_{1,r})}, & q=N-1. \end{cases} \tag{3.14}$$

with (i_l, j_l) running over all C_{N-2} allowed channels.[8] The α'-expansion of each of the elements (3.18) assumes the form (3.10) with $\prod_{l=1}^{N-3} S_{i_l,j_l}^{-1}$ as its lowest order term. Any other integral (3.5) with $N-3$ simultaneous poles can be expressed as **R**-linear combination of the basis (3.18) modulo less singular terms. In case of a sum of $N-3$ simultaneous poles this is achieved by partial fraction decomposition of the polynomials according to their leading singular term and associating the latter with the basis (3.18).

A special role is played by the integral:

$$B_N[n=-1] = \prod_{(i,j)\in\mathcal{P}} \int_0^1 du_{i,j}\, u_{i,j}^{S_{i,j}-1} \prod_{\mathcal{P}'\notin(1,j)} \delta\Big(u_{\mathcal{P}'} - 1 + \prod_{\tilde{\mathcal{P}}} u_{\tilde{\mathcal{P}}}\Big). \tag{3.19}$$

By construction it is manifestly invariant under cyclic transformations $S_{i,j} \to S_{i+1,j+1}$, with $i \equiv i+N$, $j \equiv j+N$. Furthermore, it furnishes all C_{N-2} sets of allowed channels at the lowest order, i.e.

$$B_N[n=-1] = \sum_{(i_l,j_l)\in\mathcal{P}} \frac{1}{\prod_{l=1}^{N-3} S_{i_l,j_l}} + \cdots, \tag{3.20}$$

with the sum running over all C_{N-2} allowed channels. In terms of (3.5), Eq. (3.19) takes the form:

$$\begin{aligned} B_N\begin{bmatrix} n_i = -1 \\ n_{ii} = -1 \end{bmatrix} &= \Bigg(\prod_{i=1}^{N-3} \int_0^1 dx_i\Bigg) \prod_{j=1}^{N-3} x_j^{s_{12\ldots j+1}-1} (1-x_j)^{s_{j+1,j+2}-1} \\ &\quad \times \prod_{l=j+1}^{N-3} \Bigg(1 - \prod_{k=j}^{l} x_k\Bigg)^{s_{j+1,l+2}}. \end{aligned} \tag{3.21}$$

Obviously, (3.19) can be expanded in terms of the basis (3.18).

3.3.1. $N = 4$

In the case of $N=4$ we have the two planar channels $(1,2)$ and $(2,3) \equiv (1,4)$ related to the two variables $u_{1,2}$ and $u_{2,3}$, respectively. After choosing the independent variable $u_{1,2} = x_1 := x$ and following the steps (3.15) the integral (3.5)

$$B_4[n] = \int_0^1 dx\, x^{s_{12}+n_1} (1-x)^{s_{23}+n_{11}} \tag{3.22}$$

takes the form (3.16)

$$B_4[n] = \int_0^1 du_{1,2} \int_0^1 du_{2,3}\, u_{1,2}^{s_{12}+n_{1,2}} u_{2,3}^{s_{23}+n_{2,3}}\, \delta(u_{1,2}+u_{2,3}-1), \tag{3.23}$$

with (3.17), i.e. $n_{1,2} = n_1$ and $n_{2,3} = n_{11}$.

[8] As pointed out before, these integrals appear as constituents of some of the functions F^σ. The poles in their B_N combinations are canceled by the corresponding s_{ij} factors in the numerator of the F^σ such that they are rendered local.

The fundamental objects (3.18) correspond to the two rational functions

$$\frac{1}{u_{1,2}}, \qquad \frac{1}{u_{2,3}}, \tag{3.24}$$

which furnish the $C_2 = 2$ poles s_{12}^{-1} and s_{23}^{-1} as single poles, respectively. The cyclically invariant integral (3.19) is given by

$$B_4\begin{bmatrix} n_1 = -1 \\ n_{11} = -1 \end{bmatrix} = \int_0^1 dx\, x^{s_{12}-1}(1-x)^{s_{23}-1} = B(s_{12}, s_{23}) = \frac{1}{s_{12}} + \frac{1}{s_{23}} + \cdots \tag{3.25}$$

and exhibits both poles in its power series expansion.

3.3.2. $N = 5$

In this case we have the five planar channels $(1,2)$, $(2,3)$, $(3,4)$, $(1,3) \equiv (4,5)$ and $(2,4) \equiv (5,1)$ related to the five variables $u_{1,2}, u_{2,3}, u_{3,4}, u_{4,5} \equiv u_{1,3}$ and $u_{5,1} \equiv u_{2,4}$, respectively. The five-point integral (3.5) becomes

$$B_5[n] = \int_0^1 dx_1 \int_0^1 dx_2\, x_1^{s_1+n_1} x_2^{s_4+n_2} (1-x_1)^{s_2+n_{11}} (1-x_2)^{s_3+n_{22}} (1-x_1x_2)^{s_{24}+n_{12}}, \tag{3.26}$$

with $s_i = \alpha'(k_i + k_{i+1})^2$, $i = 1, \ldots, 5$, subject to the cyclic identification $i + 5 \equiv i$. To transform (3.26) into the form (3.16) according to (3.13) we choose the two independent variables $u_{1,2} = x_1$ and $u_{1,3} = x_2$. Then, with (3.15) the integral (3.26) takes the form

$$\begin{aligned} B_5[n] = {}& \int_0^1 du_{1,2} \int_0^1 du_{2,3} \int_0^1 du_{3,4} \int_0^1 du_{4,5} \int_0^1 du_{1,5}\, u_{1,2}^{s_1+n_{1,2}} u_{2,3}^{s_2+n_{2,3}} u_{3,4}^{s_3+n_{3,4}} u_{4,5}^{s_4+n_{1,3}} \\ & \times u_{1,5}^{s_5+n_{2,4}}\, \delta(u_{2,3} + u_{1,2}u_{3,4} - 1)\, \delta(u_{2,4} + u_{1,2}u_{4,5} - 1)\, \delta(u_{3,4} + u_{2,3}u_{4,5} - 1), \end{aligned} \tag{3.27}$$

with the assignment (3.17).

In what follows it is convenient to introduce

$$I_5(x, y) = x^{s_4} y^{s_1} (1-x)^{s_3} (1-y)^{s_2} (1-xy)^{s_{24}} \tag{3.28}$$

arising from (3.26) with the identifications $x_1 := y$ and $x_2 := x$. Furthermore, we use the following shorter notation for the dual variables $u_{i,j}$:

$$X_i = u_{i,i+1}, \quad i = 1, \ldots, 5,\ i + 5 \equiv i, \tag{3.29}$$

and define:

$$J_5(X) = \left(\prod_{i=1}^{5} X_i^{s_i}\right) \delta(X_2 + X_1X_3 - 1)\, \delta(X_3 + X_2X_4 - 1)\, \delta(X_5 + X_1X_4 - 1). \tag{3.30}$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 dx \int_0^1 dy\, \frac{I_5(x, y)}{(1-y)(1-xy)} \tag{3.31}$$

can be easily deduced after transforming it into the form (3.27)

$$\left(\prod_{i=1}^{5}\int_0^1 dX_i\right) J_5(X)\frac{1}{X_2X_5} = \frac{1}{s_2s_5} + \cdots. \tag{3.32}$$

Hence, the only simultaneous pole is at $X_2, X_5 \to 0$ with the product of δ-functions yielding the constraints for the three variables $X_1, X_3, X_4 \to 1$. In the sequel we list a few non-trivial examples:

rational function in Eq. (3.1)	*rational function in Eq.* (3.26)	*rational function in Eq.* (3.27)	*lowest order poles*
$\frac{z_{15}}{z_{12}z_{13}z_{14}z_{25}z_{35}z_{45}}$	$\frac{1}{xy}$	$\frac{X_5}{X_1X_4}$	$\frac{1}{s_1s_4}$,
$\frac{1}{z_{12}z_{13}z_{24}z_{35}z_{45}}$	$\frac{1}{xy(1-xy)}$	$\frac{1}{X_1X_4}$	$\frac{1}{s_1s_4}$,
$\frac{1}{z_{13}z_{14}z_{23}z_{25}z_{45}}$	$\frac{1}{x(1-y)}$	$\frac{1}{X_2X_4}$	$\frac{1}{s_2s_4}$,
$\frac{1}{z_{14}z_{15}z_{23}z_{25}z_{34}}$	$\frac{1}{(1-x)(1-y)}$	$\frac{1}{X_2X_3X_5}$	$\frac{1}{s_2s_5}+\frac{1}{s_3s_5}$,
$\frac{1}{z_{12}z_{15}z_{24}z_{34}z_{35}}$	$\frac{1}{(1-x)y(1-xy)}$	$\frac{1}{X_1X_3X_5}$	$\frac{1}{s_1s_3}+\frac{1}{s_3s_5}$.

(3.33)

The fundamental objects (3.18) correspond to the five rational functions

$$\frac{1}{X_1X_3}, \qquad \frac{1}{X_2X_4}, \qquad \frac{1}{X_3X_5}, \qquad \frac{1}{X_1X_4}, \qquad \frac{1}{X_2X_5}, \tag{3.34}$$

which furnish the $C_3 = 5$ poles

$$\frac{1}{s_1s_3}, \qquad \frac{1}{s_2s_4}, \qquad \frac{1}{s_3s_5}, \qquad \frac{1}{s_1s_4}, \qquad \frac{1}{s_2s_5}, \tag{3.35}$$

as single poles, respectively. In the basis (3.26) the rational functions become

$$\frac{1}{(1-x)y}, \qquad \frac{1}{x(1-y)}, \qquad \frac{1}{(1-x)(1-xy)}, \qquad \frac{1}{xy(1-xy)}, \qquad \frac{1}{(1-y)(1-xy)}, \tag{3.36}$$

respectively. The cyclically invariant integral (3.19) is given by

$$B_5\begin{bmatrix} n_i = -1 \\ n_{ii} = -1 \end{bmatrix} = \int_0^1 dx \int_0^1 dy \, \frac{I_5(x,y)}{x(1-x)y(1-y)}$$
$$= \frac{1}{s_1s_3} + \frac{1}{s_2s_4} + \frac{1}{s_3s_5} + \frac{1}{s_1s_4} + \frac{1}{s_2s_5} + \cdots, \tag{3.37}$$

and exhibits all five poles (3.35) in its power series expansion.

Finally, as we shall see in the next subsection there is one rational function without poles and its series expansion starts at $\zeta(2)$:

rational function in Eq. (3.1)	*rational function in Eq.* (3.26)	*monomial in Eq.* (3.27)	*lowest order*
$\frac{1}{z_{13}z_{14}z_{24}z_{25}z_{35}}$	$\frac{1}{(1-xy)}$	1	$\zeta(2)$.

(3.38)

The function (3.38) together with (3.33) furnishes the six-dimensional partial fraction basis of $N = 5$ integrals. It may be added to (3.34) to give rise to another fundamental set

$$\frac{X_2}{X_1X_3}, \qquad \frac{X_3}{X_2X_4}, \qquad \frac{X_4}{X_3X_5}, \qquad \frac{X_5}{X_1X_4}, \qquad \frac{X_1}{X_2X_5}, \tag{3.39}$$

subject to the constraints (3.30) and with the same poles (3.35), respectively. In the basis (3.5) the latter rational functions correspond to

$$\frac{1-y}{(1-x)y(1-xy)}, \qquad \frac{1-x}{x(1-y)(1-xy)}, \qquad \frac{x}{(1-x)(1-xy)}, \qquad \frac{1}{xy}, \\ \frac{y}{(1-y)(1-xy)}, \tag{3.40}$$

respectively. Since we have

$$\begin{aligned}
\frac{X_3}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-x}{(1-xy)^2} \simeq \frac{z_{25}z_{34}}{z_{12}z_{13}z_{24}^2z_{35}^2z_{45}},\\
\frac{X_2}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-y}{(1-xy)^2} \simeq \frac{z_{14}z_{23}}{z_{12}z_{13}^2z_{24}^2z_{35}z_{45}},\\
\frac{X_3X_5}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-x}{(1-xy)} \simeq \frac{z_{15}z_{34}}{z_{12}z_{13}z_{14}z_{24}z_{35}^2z_{45}},\\
\frac{X_2X_5}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-y}{(1-xy)} \simeq \frac{z_{15}z_{23}}{z_{12}z_{13}^2z_{24}z_{25}z_{35}z_{45}},
\end{aligned}$$

the two rational functions $\frac{1}{X_1X_4}$ and $\frac{X_5}{X_1X_4}$ are the only possibilities to realize the poles $\frac{1}{s_1s_4}$ without double poles in the denominator of (3.1). Due to cyclicity these arguments take over to the other four poles (3.35) and their rational functions (3.34) and (3.39). Generally, rational functions other than the latter give rise to double powers in the denominator of (3.1), e.g.:

$$\begin{aligned}
\frac{1}{X_1} &\simeq \frac{1}{y(1-xy)} \simeq \frac{1}{z_{12}z_{14}z_{24}z_{35}^2},\\
\frac{X_1}{X_2} &\simeq \frac{y}{1-y} \simeq \frac{z_{12}}{z_{13}z_{14}^2z_{23}z_{25}^2}.
\end{aligned}$$

Similarly, as we shall see in the next subsection monomials in the variables X_i other than the trivial case (3.38) yield to double powers in the denominator of (3.1), e.g.:

$$\begin{aligned}
X_1X_4 &\simeq \frac{xy}{1-xy} \simeq \frac{z_{12}z_{45}}{z_{13}z_{14}^2z_{24}z_{25}^2z_{35}},\\
X_1 &\simeq \frac{y}{1-xy} \simeq \frac{z_{12}}{z_{13}^2z_{14}z_{24}z_{25}^2},\\
X_3X_5 &\simeq \frac{1-x}{1-xy} \simeq \frac{z_{15}z_{34}}{z_{13}z_{14}^2z_{24}z_{25}z_{35}^2},\\
X_2X_3 &\simeq \frac{(1-x)(1-y)}{(1-xy)^3} \simeq \frac{z_{23}z_{34}}{z_{13}^2z_{24}^3z_{35}^2}.
\end{aligned}$$

3.3.3. $N = 6$

In this case we have the nine planar channels $(1,2), (1,3), (1,4) \equiv (5,6), (2,3), (2,4), (2,5) \equiv (6,1), (3,4), (3,5)$ and $(4,5)$ related to the nine variables $u_{1,2}, u_{1,3}, u_{1,4} \equiv u_{5,6}, u_{2,3}, u_{2,4}, u_{2,5} \equiv u_{6,1}, u_{3,4}, u_{3,5}$ and $u_{4,5}$, respectively. The six-point integral (3.5) becomes

$$B_6[n] = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\, x_1^{s_1+n_1} x_2^{t_1+n_2} x_3^{s_5+n_3} (1-x_1)^{s_2+n_{11}} (1-x_2)^{s_3+n_{22}}$$
$$\times (1-x_3)^{s_4+n_{33}} (1-x_1x_2)^{s_{24}+n_{12}} (1-x_2x_3)^{s_{35}+n_{23}} (1-x_1x_2x_3)^{s_{25}+n_{13}}, \quad (3.41)$$

with $s_i = \alpha'(k_i + k_{i+1})^2$, $i = 1, \ldots, 6$, subject to the cyclic identification $i + 6 \equiv i$ and $t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2$, $j = 1, \ldots, 3$.

To bring (3.41) into the form (3.16) according to (3.13), we choose the three independent variables $u_{1,2} = x_1$, $u_{1,3} = x_2$ and $u_{1,4} = x_3$. Then, with (3.15) the integral (3.41) takes the form

$$B_6[n] = \int_0^1 du_{1,2} \int_0^1 du_{1,3} \int_0^1 du_{1,4} \int_0^1 du_{2,3} \int_0^1 du_{2,4} \int_0^1 du_{2,5} \int_0^1 du_{3,4} \int_0^1 du_{3,5} \int_0^1 du_{4,5}$$
$$\times u_{1,2}^{s_1+n_{1,2}} u_{2,3}^{s_2+n_{2,3}} u_{3,4}^{s_3+n_{3,4}} u_{4,5}^{s_4+n_{4,5}} u_{1,4}^{s_5+n_{1,4}} u_{2,5}^{s_6+n_{2,5}} u_{1,3}^{t_1+n_{1,3}} u_{2,4}^{t_2+n_{2,4}} u_{3,5}^{t_3+n_{3,5}}$$
$$\times \delta(u_{2,3} + u_{1,2}u_{3,4}u_{3,5} - 1)\, \delta(u_{2,4} + u_{1,2}u_{1,3}u_{3,5}u_{4,5} - 1)$$
$$\times \delta(u_{2,5} + u_{1,2}u_{1,3}u_{1,4} - 1)\, \delta(u_{3,4} + u_{1,3}u_{2,3}u_{4,5} - 1)$$
$$\times \delta(u_{3,5} + u_{1,3}u_{1,4}u_{2,3}u_{2,4} - 1)\, \delta(u_{4,5} + u_{1,4}u_{2,4}u_{3,4} - 1), \quad (3.42)$$

with the assignment (3.17).

Similarly as in the five-point case, it is convenient to introduce

$$I_6(x,y,z) = x^{s_5} y^{t_1} z^{s_1} (1-x)^{s_4} (1-y)^{s_3} (1-z)^{s_2} (1-xy)^{s_{35}} (1-yz)^{s_{24}} (1-xyz)^{s_{25}} \quad (3.43)$$

which arises from (3.41) with the identifications $x_1 := z$, $x_2 := y$ and $x_3 := x$. Furthermore, we define

$$X_i = u_{i,i+1}, \quad i = 1, \ldots, 6,\ i + 6 \equiv i, \qquad Y_j = u_{j,j+2}, \quad j = 1, \ldots, 3, \quad (3.44)$$

and

$$J_6(X,Y) = \left(\prod_{i=1}^{6} X_i^{s_i}\right)\left(\prod_{j=1}^{3} Y_j^{t_j}\right) \delta(X_2 + X_1X_3Y_3 - 1)\, \delta(Y_2 + X_1X_4Y_1Y_3 - 1)$$
$$\times \delta(X_6 + X_1X_5Y_1 - 1)\, \delta(X_3 + X_2X_4Y_1 - 1)$$
$$\times \delta(Y_3 + X_2X_5Y_1Y_2 - 1)\, \delta(X_4 + X_3X_5Y_2 - 1). \quad (3.45)$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \frac{I_6(x,y,z)}{(1-x)(1-xy)(1-xyz)} \quad (3.46)$$

can be easily deduced after transforming it into the form (3.42)

$$\left(\prod_{i=1}^{6} \int_0^1 dX_i\right)\left(\prod_{j=1}^{3} \int_0^1 dY_j\right) J_6(X,Y) \frac{Y_2}{X_4X_6Y_3} = \frac{1}{s_4s_6t_3} + \cdots. \quad (3.47)$$

Hence, the only simultaneous pole is at $X_4, X_6, Y_3 \to 0$ with the product of δ-functions yielding the constraints for the six variables $X_1, X_2, X_3, X_5, Y_1, Y_2 \to 1$. Note, that by construction a set of three poles in (3.42) does not necessarily yield a compatible set of channels, e.g. the integral

$$\begin{aligned}
&\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{(1-x)(1-y)} \\
&= \left(\prod_{i=1}^{6} \int_0^1 dX_i\right)\left(\prod_{j=1}^{3} \int_0^1 dY_j\right) \frac{J_6(X,Y)}{X_3X_4Y_3} \\
&= \frac{1}{s_3t_3} + \frac{1}{s_4t_3} + \frac{1}{s_3} + \frac{1}{s_4} - \frac{s_1}{s_3t_3} - \frac{s_1}{s_4t_3} - \frac{s_6}{s_3t_3} - \frac{s_6}{s_4t_3} + \cdots
\end{aligned} \tag{3.48}$$

does not give rise to a triple pole as $(3,4)$, $(4,5)$ and $(3,5)$ are not compatible channels. Similarly, for

$$\begin{aligned}
&\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{z(1-z)(1-xy)(1-xyz)} \\
&= \left(\prod_{i=1}^{6} \int_0^1 dX_i\right)\left(\prod_{j=1}^{3} \int_0^1 dY_j\right) \frac{J_6(X,Y)}{X_1X_2X_6} \\
&= \frac{1}{s_2s_6} + \frac{1}{s_2} + \frac{1}{s_6} - \frac{s_4}{s_2s_6} - \frac{t_2}{s_2s_6} + \frac{\zeta(2)}{s_1} + \cdots
\end{aligned} \tag{3.49}$$

the channels $(1,2)$, $(2,3)$ and $(6,1)$ are not compatible. In the following table we list a few non-trivial examples:

rational function in Eq. (3.1)	*rational function in Eq.* (3.41)	*rational function in Eq.* (3.42)	*lowest order poles*
$\frac{z_{16}^2}{z_{12}z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}z_{56}}$	$\frac{1}{xyz}$	$\frac{X_6^2Y_2Y_3}{X_1X_5Y_1}$	$\frac{1}{s_1s_5t_1}$,
$\frac{z_{16}}{z_{12}z_{13}z_{15}z_{26}z_{36}z_{45}z_{46}}$	$\frac{1}{(1-x)yz}$	$\frac{X_6Y_3}{X_1X_4Y_1}$	$-\frac{1}{s_1s_4t_1}$,
$\frac{1}{z_{13}z_{15}z_{23}z_{26}z_{45}z_{46}}$	$\frac{1}{(1-x)y(1-z)}$	$\frac{1}{X_2X_4Y_1}$	$\frac{1}{s_2s_4t_1}$,
$\frac{1}{z_{12}z_{14}z_{25}z_{34}z_{36}z_{56}}$	$\frac{1}{x(1-y)z(1-xyz)}$	$\frac{1}{X_1X_3X_5}$	$\frac{1}{s_1s_3s_5}$,
$\frac{z_{13}z_{45}}{z_{12}z_{14}^2z_{25}z_{34}z_{35}z_{36}z_{56}}$	$\frac{y(1-x)}{x(1-y)z(1-xy)(1-xyz)}$	$\frac{X_4Y_1}{X_1X_3X_5}$	$\frac{1}{s_1s_3s_5}$,
$\frac{1}{z_{14}z_{15}z_{23}z_{26}z_{34}z_{56}}$	$\frac{1}{x(1-y)(1-z)}$	$\frac{1}{X_2X_3X_5Y_2}$	$\frac{1}{s_2s_5t_2} + \frac{1}{s_3s_5t_2}$,
$\frac{1}{z_{12}z_{15}z_{26}z_{34}z_{36}z_{45}}$	$\frac{1}{z(1-x)(1-y)}$	$\frac{1}{X_1X_3X_4Y_3}$	$\frac{1}{s_1s_3t_3} + \frac{1}{s_1s_4t_3}$,
$\frac{1}{z_{15}z_{16}z_{24}z_{26}z_{34}z_{35}}$	$\frac{y}{(1-y)(1-xy)(1-yz)}$	$\frac{Y_1}{X_3X_6Y_2Y_3}$	$\frac{1}{s_3s_6t_2} + \frac{1}{s_3s_6t_3}$,
$\frac{1}{z_{15}z_{16}z_{23}z_{26}z_{34}z_{45}}$	$\frac{1}{(1-x)(1-y)(1-z)}$	$\frac{1}{X_2X_3X_4X_6Y_2Y_3}$	$-\frac{1}{s_2s_4s_6} - \frac{1}{s_2s_6t_2} - \frac{1}{s_3s_6t_2} - \frac{1}{s_3s_6t_3} - \frac{1}{s_4s_6t_3}$.

(3.50)

The fundamental objects (3.18) correspond to the 14 rational functions

$$\begin{aligned}&\frac{1}{X_1X_3X_5}, \quad \frac{1}{X_2X_4X_6}, \quad \frac{1}{X_1X_4Y_1}, \quad \frac{1}{X_2X_5Y_2}, \quad \frac{1}{X_3X_6Y_3}, \quad \frac{1}{X_2X_5Y_1},\\ &\frac{1}{X_3X_6Y_2}, \quad \frac{1}{X_1X_4Y_3}, \quad \frac{1}{X_2X_4Y_1}, \quad \frac{1}{X_3X_5Y_2}, \quad \frac{1}{X_4X_6Y_3}, \quad \frac{1}{X_1X_5Y_1},\\ &\frac{1}{X_2X_6Y_2}, \quad \frac{1}{X_1X_3Y_3},\end{aligned} \tag{3.51}$$

which furnish the $C_4 = 14$ poles

$$\begin{aligned}&\frac{1}{s_1s_3s_5}, \quad \frac{1}{s_2s_4s_6}, \quad \frac{1}{s_1s_4t_1}, \quad \frac{1}{s_2s_5t_2}, \quad \frac{1}{s_3s_6t_3}, \quad \frac{1}{s_2s_5t_1}, \quad \frac{1}{s_3s_6t_2},\\ &\frac{1}{s_1s_4t_3}, \quad \frac{1}{s_2s_4t_1}, \quad \frac{1}{s_3s_5t_2}, \quad \frac{1}{s_4s_6t_3}, \quad \frac{1}{s_1s_5t_1}, \quad \frac{1}{s_2s_6t_2}, \quad \frac{1}{s_1s_3t_3},\end{aligned} \tag{3.52}$$

as single poles in the denominator of (3.1), respectively. The cyclically invariant integral (3.19) is given by

$$\begin{aligned}B_6\begin{bmatrix} n_i = -1 \\ n_{ii} = -1 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{x(1-x)y(1-y)z(1-z)}\\ &= \frac{1}{s_1s_3s_5} + \frac{1}{s_2s_4s_6} + \frac{1}{s_1s_4t_1} + \frac{1}{s_2s_5t_2} + \frac{1}{s_3s_6t_3} + \frac{1}{s_2s_5t_1} + \frac{1}{s_3s_6t_2}\\ &\quad + \frac{1}{s_1s_4t_3} + \frac{1}{s_2s_4t_1} + \frac{1}{s_3s_5t_2} + \frac{1}{s_4s_6t_3} + \frac{1}{s_1s_5t_1} + \frac{1}{s_2s_6t_2}\\ &\quad + \frac{1}{s_1s_3t_3} + \cdots,\end{aligned} \tag{3.53}$$

and exhibits all fourteen poles (3.52) in its power series expansion. After triple poles, for a (transcendental) $N = 6$ integral the next leading order to start with are single poles. They always come with a $\zeta(2)$. In analogy to (3.51) for the latter we may introduce a fundamental set of rational functions[9] furnishing the six single poles $\frac{\zeta(2)}{s_i}$, $i = 1, \ldots, 6$:

rational function in Eq. (3.1)	rational function in Eq. (3.41)	rational function in Eq. (3.42)	lowest order poles
$\frac{1}{z_{12}z_{15}z_{24}z_{35}z_{36}z_{46}}$	$\frac{1}{(1-xy)z(1-yz)}$	$\frac{1}{X_1}$	$\frac{\zeta(2)}{s_1}$,
$\frac{1}{z_{14}z_{15}z_{23}z_{26}z_{35}z_{46}}$	$\frac{1}{(1-z)(1-xy)}$	$\frac{1}{X_2}$	$\frac{\zeta(2)}{s_2}$,
$\frac{1}{z_{13}z_{15}z_{25}z_{26}z_{34}z_{46}}$	$\frac{1}{(1-y)(1-xyz)}$	$\frac{1}{X_3}$	$\frac{\zeta(2)}{s_3}$,
$\frac{1}{z_{13}z_{15}z_{24}z_{26}z_{36}z_{45}}$	$\frac{1}{(1-x)(1-yz)}$	$\frac{1}{X_4}$	$\frac{\zeta(2)}{s_4}$,
$\frac{1}{z_{13}z_{14}z_{24}z_{26}z_{35}z_{56}}$	$\frac{1}{x(1-xy)(1-yz)}$	$\frac{1}{X_5}$	$\frac{\zeta(2)}{s_5}$,
$\frac{1}{z_{13}z_{16}z_{24}z_{25}z_{35}z_{46}}$	$\frac{1}{(1-xy)(1-yz)(1-xyz)}$	$\frac{1}{X_6}$	$\frac{\zeta(2)}{s_6}$.

(3.54)

[9] Note, that the rational functions $\frac{1}{Y_i}$ giving rise to the single poles $\sim t_i^{-1}$ have double poles in (3.1), i.e. $\tilde{n}_{ij} = -2$ for some z_{ij}.

All (transcendental) integrals with single poles can be decomposed w.r.t. the basis (3.54) modulo finite pieces to be discussed in a moment. Subject to (3.12) we have e.g.:

$$\frac{1}{z(1-xy)} \simeq \frac{X_6Y_2}{X_1} = \frac{1}{X_1} - Y_1, \qquad \frac{y}{(1-y)(1-xyz)} \simeq \frac{Y_1}{X_3} = \frac{1}{X_3} - X_6Y_2Y_3,$$
$$\frac{x}{(1-x)(1-xyz)} \simeq \frac{X_5Y_2}{X_4} = \frac{1}{X_4} - Y_3, \qquad \frac{1}{x(1-yz)} \simeq \frac{X_6Y_3}{X_5} = \frac{1}{X_5} - Y_1. \tag{3.55}$$

After single poles, for an $N = 6$ integral the next leading order to start with are constants. They always come with a $\zeta(2)$ or $\zeta(3)$, e.g.:

rational function in Eq. (3.1)	*rational function in Eq.* (3.41)	*monomial in Eq.* (3.42)	*lowest order*
$\frac{1}{z_{14}z_{15}z_{24}z_{26}z_{35}z_{36}}$	$\frac{y}{(1-xy)(1-yz)}$	Y_1	$2\zeta(3)$,
$\frac{1}{z_{13}z_{14}z_{25}z_{26}z_{35}z_{46}}$	$\frac{1}{(1-xy)(1-xyz)}$	Y_2	$2\zeta(3)$,
$\frac{1}{z_{13}z_{15}z_{24}z_{25}z_{36}z_{46}}$	$\frac{1}{(1-yz)(1-xyz)}$	Y_3	$2\zeta(3)$,
$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{25}z_{26}z_{36}z_{46}}$	$\frac{1}{1-xyz}$	$X_6Y_2Y_3$	$\zeta(3)$,
$\frac{z_{56}}{z_{14}z_{15}z_{25}z_{26}z_{35}z_{36}z_{46}}$	$\frac{xy}{(1-xy)(1-xyz)}$	$X_5Y_1Y_2$	$\zeta(3)$.

(3.56)

Again, we may add the functions (3.56) to (3.51) to obtain other fundamental sets subject to the constraints (3.45) and with the same poles (3.52), cf. the next subsection for more details.

3.3.4. $N = 7$

In this case we have the 14 planar channels $(1,2)$, $(1,3)$, $(1,4) \equiv (5,7)$, $(1,5) \equiv (6,7)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6) \equiv (7,1)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, $(4,6)$ and $(5,6)$ related to the 14 variables $u_{1,2}, u_{1,3}, u_{1,4} \equiv u_{5,7}, u_{1,5} \equiv u_{6,7}, u_{2,3}, u_{2,4}, u_{2,5} \equiv u_{6,1}, u_{2,6} \equiv u_{7,1}, u_{3,4}, u_{3,5}, u_{3,6} \equiv u_{7,2}, u_{4,5}, u_{4,6}$ and $u_{5,6}$, respectively. The seven-point integral (3.5) becomes

$$B_7[n] = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4\, x_1^{s_1+n_1} x_2^{t_1+n_2} x_3^{t_5+n_3} x_4^{s_6+n_4} (1-x_1)^{s_2+n_{11}}$$
$$\times (1-x_2)^{s_3+n_{22}} (1-x_3)^{s_4+n_{33}} (1-x_4)^{s_5+n_{44}} (1-x_1x_2)^{s_{24}+n_{12}} (1-x_2x_3)^{s_{35}+n_{23}}$$
$$\times (1-x_3x_4)^{s_{46}+n_{34}} (1-x_1x_2x_3)^{s_{25}+n_{13}} (1-x_2x_3x_4)^{s_{36}+n_{24}}$$
$$\times (1-x_1x_2x_3x_4)^{s_{26}+n_{14}}, \tag{3.57}$$

with $s_i = \alpha'(k_i + k_{i+1})^2$, $t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2$, $i, j = 1, \ldots, 7$ subject to the cyclic identifications $i + 7 \equiv i$ and $j + 7 \equiv j$, respectively.

To bring (3.57) into the form (3.16) according to (3.13) we choose the four independent variables $u_{1,2} = x_1$, $u_{1,3} = x_2$, $u_{1,4} = x_3$ and $u_{1,5} = x_4$. Then, with (3.15) the integral (3.57) assumes the form (3.16)

$$B_7[n] = \int_0^1 du_{i,j}\, u_{1,2}^{s_1+n_{1,2}} u_{2,3}^{s_2+n_{2,3}} u_{3,4}^{s_3+n_{3,4}} u_{4,5}^{s_4+n_{4,5}} u_{5,6}^{s_5+n_{5,6}} u_{1,5}^{s_6+n_{1,5}} u_{2,6}^{s_7+n_{2,6}} u_{1,3}^{t_1+n_{1,3}}$$
$$\times u_{2,4}^{t_2+n_{2,4}} u_{3,5}^{t_3+n_{3,5}} u_{4,6}^{t_4+n_{4,6}} u_{1,4}^{t_5+n_{1,4}} u_{2,5}^{t_6+n_{2,5}} u_{3,6}^{t_7+n_{3,6}}$$
$$\times \delta(u_{2,3} + u_{1,2}u_{3,4}u_{3,5}u_{3,6} - 1)\, \delta(u_{2,4} + u_{1,2}u_{1,3}u_{3,5}u_{3,6}u_{4,5}u_{4,6} - 1)$$

$$\times\,\delta(u_{2,5}+u_{1,2}u_{1,3}u_{1,4}u_{3,6}u_{4,6}u_{5,6}-1)\,\delta(u_{2,6}+u_{1,2}u_{1,3}u_{1,4}u_{1,5}-1)$$
$$\times\,\delta(u_{3,4}+u_{1,3}u_{2,3}u_{4,5}u_{4,6}-1)\,\delta(u_{3,5}+u_{1,3}u_{1,4}u_{2,3}u_{2,4}u_{4,6}u_{5,6}-1)$$
$$\times\,\delta(u_{3,6}+u_{1,3}u_{1,4}u_{1,5}u_{2,3}u_{2,4}u_{2,5}-1)\,\delta(u_{4,5}+u_{1,4}u_{2,4}u_{3,4}u_{5,6}-1)$$
$$\times\,\delta(u_{4,6}+u_{1,4}u_{1,5}u_{2,4}u_{2,5}u_{3,4}u_{3,5}-1)\,\delta(u_{5,6}+u_{1,5}u_{2,5}u_{3,5}u_{4,5}-1), \quad (3.58)$$

with the assignment (3.17).

Using the identifications $x_1:=w$, $x_2:=z$, $x_3:=y$ and $x_4:=x$ in (3.57), it is convenient to introduce

$$I_7(x,y,z,w)=x^{s_6}y^{t_5}z^{t_1}w^{s_1}(1-x)^{s_5}(1-y)^{s_4}(1-z)^{s_3}(1-w)^{s_2}(1-xy)^{s_{46}}$$
$$\times(1-wz)^{s_{24}}(1-yz)^{s_{35}}(1-xyz)^{s_{36}}(1-yzw)^{s_{25}}(1-xyzw)^{s_{26}} \quad (3.59)$$

and use the following notation for the dual variables $u_{i,j}$

$$X_i=u_{i,i+1},\qquad Y_j=u_{j,j+2},\quad i,j=1,\ldots,7,\ i+7\equiv i,\ i,j=1,\ldots,7. \quad (3.60)$$

Furthermore, we define:

$$J_7(X,Y)=\left(\prod_{i=1}^{7}X_i^{s_i}\right)\left(\prod_{j=1}^{7}Y_j^{t_j}\right)\delta(X_2+X_1X_3Y_3Y_7-1)\delta(Y_2+X_1X_4Y_1Y_3Y_4Y_7-1)$$
$$\times\,\delta(Y_6+X_1X_5Y_1Y_4Y_5Y_7-1)\delta(X_7+X_1X_6Y_1Y_5-1)$$
$$\times\,\delta(X_3+X_2X_4Y_1Y_4-1)\delta(X_4+X_3X_5Y_2Y_5-1)$$
$$\times\,\delta(Y_4+X_3X_6Y_2Y_3Y_5Y_6-1)\delta(X_5+X_4X_6Y_3Y_6-1)$$
$$\times\,\delta(Y_3+X_2X_5Y_1Y_2Y_4Y_5-1)\delta(Y_7+X_2X_6Y_1Y_2Y_5Y_6-1). \quad (3.61)$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dw\,\frac{I_7(x,y,z,w)}{x(1-y)(1-wz)(1-yz)} \quad (3.62)$$

can be easily deduced after transforming it into the form (3.42)

$$\left(\prod_{i=1}^{7}\int_0^1 dX_i\right)\left(\prod_{j=1}^{7}\int_0^1 dY_j\right)\frac{J_7(X,Y)}{X_4X_6Y_3Y_6}=\frac{1}{s_4s_6t_3t_6}+\cdots. \quad (3.63)$$

Hence, the only simultaneous pole is at $X_4,X_6,Y_3,Y_6\to 0$ with the product of δ-functions yielding the constraints for the ten variables $X_1,X_2,X_3,X_5,X_7,Y_1,Y_2,Y_4,Y_5,Y_7\to 1$. Note, that by construction a set of four poles in (3.58) does not necessarily yield a compatible set of channels, e.g. the integral

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dw\,\frac{I_7(x,y,z,w)}{w(1-x)(1-z)(1-wyz)(1-wxyz)}$$
$$=\left(\prod_{i=1}^{7}\int_0^1 dX_i\right)\left(\prod_{j=1}^{7}\int_0^1 dY_j\right)\frac{J_7(X,Y)}{X_1X_3X_5X_7}=\frac{1}{s_1s_3s_5}+\cdots \quad (3.64)$$

does not give rise to a quadruple pole as (1, 2), (3, 4), (5, 6) and (7, 1) are not compatible channels. Subsequently, in the sequel we list a few non-trivial examples:

$$\begin{array}{cccc}
\underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.1)}} & \underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.57)}} & \underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.58)}} & \underline{\textit{lowest order poles}} \\
\frac{z_{17}^3}{z_{12}z_{13}z_{14}z_{15}z_{16}z_{27}z_{37}z_{47}z_{57}z_{67}} & \frac{1}{xyzw} & \frac{X_7^3Y_2Y_3Y_4Y_6^2Y_7^2}{X_1X_6Y_1Y_5} & \frac{1}{s_1s_6t_1t_5}, \\
\frac{z_{17}^2}{z_{12}z_{14}z_{15}z_{16}z_{27}z_{34}z_{37}z_{57}z_{67}} & \frac{1}{xy(1-z)w} & \frac{X_7^2Y_4Y_6Y_7}{X_1X_3X_6Y_5} & \frac{1}{s_1s_3s_6t_5}, \\
\frac{z_{17}^2}{z_{12}z_{13}z_{14}z_{16}z_{27}z_{37}z_{47}z_{56}z_{57}} & \frac{1}{(1-x)yzw} & \frac{X_7^2Y_2Y_3Y_6^2Y_7}{X_1X_5Y_1Y_5} & \frac{1}{s_1s_5t_1t_5}, \\
\frac{z_{17}z_{67}}{z_{12}z_{13}z_{16}z_{27}z_{37}z_{46}z_{47}z_{56}z_{57}} & \frac{x}{wz(1-x)(1-xy)} & \frac{X_6X_7Y_2Y_3Y_6^2}{X_1X_5Y_1Y_4} & \frac{1}{s_1s_5t_1t_4}, \\
\frac{z_{17}}{z_{12}z_{14}z_{16}z_{27}z_{34}z_{37}z_{56}z_{57}} & \frac{1}{yw(1-x)(1-z)} & \frac{X_7Y_6}{X_1X_3X_5Y_5} & \frac{1}{s_1s_3s_5t_5}, \\
\frac{z_{17}}{z_{12}z_{13}z_{16}z_{27}z_{37}z_{45}z_{47}z_{56}} & \frac{1}{zw(1-x)(1-y)} & \frac{X_7Y_3Y_6}{X_1X_4X_5Y_1Y_4} & \frac{1}{s_1s_4t_1t_4}+\frac{1}{s_1s_5t_1t_4}, \\
\frac{z_{17}}{z_{14}z_{15}z_{16}z_{23}z_{27}z_{34}z_{57}z_{67}} & \frac{1}{xy(1-z)(1-w)} & \frac{X_7Y_4Y_7}{X_2X_3X_6Y_2Y_5} & \frac{1}{s_2s_6t_2t_5}+\frac{1}{s_3s_6t_2t_5}, \\
\frac{z_{67}}{z_{12}z_{16}z_{27}z_{36}z_{37}z_{45}z_{47}z_{56}} & \frac{xy}{w(1-x)(1-y)(1-xyz)} & \frac{X_6Y_2Y_5Y_6}{X_1X_4X_5Y_4Y_7} & \frac{1}{s_1s_4t_4t_7}+\frac{1}{s_1s_5t_4t_7}, \\
\frac{1}{z_{12}z_{16}z_{27}z_{34}z_{37}z_{45}z_{56}} & \frac{1}{w(1-x)(1-y)(1-z)} & \frac{1}{X_1X_3X_4X_5Y_3Y_4Y_7} & \frac{1}{s_1s_3s_5t_7}+\frac{1}{s_1s_3t_3t_7} \\
 & & & +\frac{1}{s_1s_4t_3t_7}+\frac{1}{s_1s_4t_4t_7} \\
 & & & +\frac{1}{s_1s_5t_4t_7}.
\end{array} \tag{3.65}$$

After quadruple poles, for an $N = 7$ integral the next leading order to start with are double poles. They always come with a $\zeta(2)$, e.g.:

$$\begin{array}{cccc}
\underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.1)}} & \underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.57)}} & \underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.58)}} & \underline{\textit{lowest order poles}} \\
\frac{1}{z_{12}z_{15}z_{24}z_{35}z_{37}z_{46}z_{67}} & \frac{1}{wx(1-xy)(1-wz)(1-yz)} & \frac{1}{X_1X_6} & \frac{\zeta(2)}{s_1s_6}, \\
\frac{z_{14}}{z_{12}z_{13}z_{16}z_{24}z_{35}z_{46}z_{47}z_{57}} & \frac{1}{w(1-xy)z(1-wz)(1-yz)} & \frac{1}{X_1Y_1} & \frac{\zeta(2)}{s_1t_1}, \\
\frac{1}{z_{15}z_{16}z_{26}z_{27}z_{34}z_{37}z_{45}} & \frac{yz}{(1-y)(1-z)(1-wxyz)} & \frac{Y_1Y_5}{X_3X_4Y_3} & \frac{\zeta(2)}{s_3t_3}+\frac{\zeta(2)}{s_4t_3}.
\end{array} \tag{3.66}$$

After double poles, for an $N = 7$ integral the next leading order to start with are single poles. They are always accompanied by $\zeta(2)$ or $\zeta(3)$ factors:

$$\begin{array}{cccc}
\underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.1)}} & \underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.57)}} & \underline{\overline{\textit{rational function}}\atop \textit{in Eq. (3.58)}} & \underline{\textit{lowest order poles}} \\
\frac{1}{z_{12}z_{16}z_{24}z_{35}z_{37}z_{46}z_{57}} & \frac{1}{w(1-xy)(1-wz)(1-yz)} & \frac{1}{X_1} & \frac{2\zeta(2)}{s_1}, \\
\frac{z_{15}}{z_{12}z_{14}z_{16}z_{25}z_{35}z_{37}z_{46}z_{57}} & \frac{1}{w(1-xy)(1-yz)(1-wyz)} & \frac{Y_2}{X_1} & \frac{2\zeta(2)}{s_1}, \\
\frac{1}{z_{12}z_{16}z_{24}z_{35}z_{36}z_{47}z_{57}} & \frac{1}{w(1-wz)(1-yz)(1-xyz)} & \frac{Y_4}{X_1} & \frac{2\zeta(3)}{s_1}, \\
\frac{1}{z_{12}z_{15}z_{16}z_{24}z_{35}z_{37}z_{46}z_{47}} & \frac{y}{w(1-xy)(1-wz)(1-yz)} & \frac{Y_5}{X_1} & \frac{2\zeta(3)}{s_1}, \\
\frac{z_{15}z_{23}}{z_{12}z_{13}z_{16}z_{24}z_{25}z_{35}z_{37}z_{46}z_{57}} & \frac{1-w}{w(1-xy)(1-wz)(1-yz)(1-wyz)} & \frac{X_2Y_2}{X_1} & \frac{2\zeta(2)}{s_1}.
\end{array} \tag{3.67}$$

After single poles, for an $N = 7$ integral the next leading order to start with are the zeta constants $\zeta(2)$, $\zeta(3)$ or $\zeta(4)$. First, we display examples without poles and whose series expansion starts at $\zeta(2)$ or $\zeta(3)$:

rational function in Eq. (3.1)	*rational function in Eq.* (3.57)	*monomial in Eq.* (3.58)	*lowest order*
$\frac{z_{47}}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{37}z_{46}z_{57}}$	$\frac{z}{(1-xy)(1-yz)(1-wz)}$	Y_1	$2\zeta(2)+2\zeta(3)$,
$\frac{z_{14}z_{37}}{z_{13}z_{15}z_{16}z_{24}z_{27}z_{35}z_{36}z_{47}^2}$	$\frac{y}{(1-yz)(1-wz)(1-xyz)}$	Y_4Y_5	$\frac{3}{2}\zeta(2)+\frac{3}{2}\zeta(3)$,
$\frac{z_{15}z_{37}}{z_{13}z_{14}z_{16}z_{25}z_{27}z_{35}z_{36}z_{47}z_{57}}$	$\frac{1}{(1-yz)(1-xyz)(1-wyz)}$	Y_2Y_4	$\frac{5}{2}\zeta(4)+4\zeta(3)-2\zeta(3)$,
$\frac{1}{z_{13}z_{14}z_{25}z_{27}z_{36}z_{46}z_{57}}$	$\frac{1}{(1-xy)(1-wyz)(1-xyz)}$	$Y_2Y_3Y_6$	$3\zeta(3)$,

(3.68)

Finally, we give examples without poles and whose series expansion starts at $\zeta(4)$:

rational function in Eq. (3.1)	*rational function in Eq.* (3.57)	*monomial in Eq.* (3.58)	*lowest order*
$\frac{1}{z_{13}z_{16}z_{24}z_{27}z_{35}z_{46}z_{57}}$	$\frac{1}{(1-xy)(1-yz)(1-wz)}$	1	$\frac{27}{4}\zeta(4)$,
$\frac{1}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{36}z_{57}}$	$\frac{z}{(1-yz)(1-wz)(1-xyz)}$	Y_1Y_4	$\frac{17}{4}\zeta(4)$,
$\frac{z_{37}}{z_{13}z_{14}z_{26}z_{27}z_{35}z_{36}z_{47}z_{57}}$	$\frac{1}{(1-yz)(1-xyz)(1-wxyz)}$	$Y_2Y_4Y_6$	$3\zeta(4)$,
$\frac{1}{z_{13}z_{14}z_{25}z_{26}z_{37}z_{46}z_{57}}$	$\frac{1}{(1-xy)(1-wyz)(1-wxyz)}$	$Y_2Y_3Y_6Y_7$	$\frac{5}{2}\zeta(4)$,
$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{27}z_{36}z_{46}z_{57}}$	$\frac{1}{(1-xy)(1-xyz)(1-wxyz)}$	$Y_6^2Y_2Y_3$	$3\zeta(4)$.

(3.69)

Again, we may add the functions (3.69) to the 42 fundamental quadruple poles to obtain other fundamental sets subject to the constraints (3.61), cf. the next subsection for more details.

3.3.5. $N = 8$

In this case we have the 20 planar channels $(1,2)$, $(1,3)\equiv(4,8)$, $(1,4)\equiv(5,8)$, $(1,5)\equiv(6,8)$, $(1,6)\equiv(7,8)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(2,7)\equiv(8,1)$, $(3,4)$, $(3,5)$, $(3,6)$, $(3,7)$, $(4,5)$, $(4,6)$, $(4,7)$, $(5,6)$, $(5,7)$ and $(6,7)$ related to the 20 variables $u_{1,2}, u_{2,3}, u_{3,4}, u_{4,5}, u_{5,6}, u_{6,7}, u_{1,6}\equiv u_{7,8}, u_{2,7}, u_{1,3}, u_{2,4}, u_{3,5}, u_{4,6}, u_{5,7}, u_{1,5}\equiv u_{6,8}, u_{2,6}\equiv u_{7,1}, u_{3,7}\equiv u_{8,2}, u_{1,4}\equiv u_{5,8}, u_{2,5}, u_{3,6}, u_{1,3}=u_{4,8}$, respectively. The eight-point integral (3.5) becomes

$$\begin{aligned} B_8[n] = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \int_0^1 dx_5\, & x_1^{s_1+n_1} x_2^{t_1+n_2} x_3^{u_1+n_3} x_4^{t_6+n_4} x_5^{s_7+n_5} \\ & \times (1-x_1)^{s_2+n_{11}}(1-x_2)^{s_3+n_{22}}(1-x_3)^{s_4+n_{33}}(1-x_4)^{s_5+n_{44}}(1-x_5)^{s_6+n_{55}} \\ & \times (1-x_1x_2)^{s_{24}+n_{12}}(1-x_2x_3)^{s_{35}+n_{23}}(1-x_3x_4)^{s_{46}+n_{34}}(1-x_4x_5)^{s_{57}+n_{45}} \\ & \times (1-x_1x_2x_3)^{s_{25}+n_{13}}(1-x_2x_3x_4)^{s_{36}+n_{24}}(1-x_3x_4x_5)^{s_{47}+n_{35}} \\ & \times (1-x_1x_2x_3x_4)^{s_{26}+n_{14}}(1-x_2x_3x_4x_5)^{s_{37}+n_{25}}(1-x_1x_2x_3x_4x_5)^{s_{27}+n_{15}}, \end{aligned} \quad (3.70)$$

with $s_i = \alpha'(k_i + k_{i+1})^2$, $t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2$, $i, j = 1, \ldots, 8$, subject to the cyclic identifications $i + 8 \equiv i$, $j + 8 \equiv j$, respectively and $u_l = \alpha'(k_l + k_{l+1} + k_{l+2} + k_{l+3})^2$, for $l = 1, \ldots, 4$.

To bring (3.70) into the form (3.16) according to (3.13) we choose the five independent variables $u_{1,2} = x_1, u_{1,3} = x_2, u_{1,4} = x_3, u_{1,5} = x_4$ and $u_{1,6} = x_5$. Then, with (3.15) the integral (3.70) assumes the form (3.16)

$$\begin{aligned}
B_8[n] = \int_0^1 du_{i,j}\, & u_{1,2}^{s_1+n_{1,2}} u_{2,3}^{s_2+n_{2,3}} u_{3,4}^{s_3+n_{3,4}} u_{4,5}^{s_4+n_{4,5}} u_{5,6}^{s_5+n_{5,6}} u_{6,7}^{s_6+n_{6,7}} u_{1,6}^{s_7+n_{1,6}} \\
& \times u_{2,7}^{s_8+n_{2,7}} u_{1,3}^{t_1+n_{1,3}} u_{2,4}^{t_2+n_{2,4}} u_{3,5}^{t_3+n_{3,5}} u_{4,6}^{t_4+n_{4,6}} u_{5,7}^{t_5+n_{5,7}} u_{1,5}^{t_6+n_{1,5}} u_{2,6}^{t_7+n_{2,6}} \\
& \times u_{3,7}^{t_8+n_{3,7}} u_{1,4}^{u_1+n_{1,4}} u_{2,5}^{u_2+n_{2,5}} u_{3,6}^{u_3+n_{3,6}} u_{4,7}^{u_4+n_{4,7}} \\
& \times \delta(u_{2,3} + u_{1,2}u_{3,4}u_{3,5}u_{3,6}u_{3,7} - 1) \\
& \times \delta(u_{2,4} + u_{1,2}u_{1,3}u_{3,5}u_{3,6}u_{3,7}u_{4,5}u_{4,6}u_{4,7} - 1) \\
& \times \delta(u_{2,5} + u_{1,2}u_{1,3}u_{1,4}u_{3,6}u_{3,7}u_{4,6}u_{4,7}u_{5,6}u_{5,7} - 1) \\
& \times \delta(u_{2,7} + u_{1,2}u_{1,3}u_{1,4}u_{1,5}u_{1,6} - 1) \\
& \times \delta(u_{2,6} + u_{1,2}u_{1,3}u_{1,4}u_{1,5}u_{3,7}u_{4,7}u_{5,7}u_{6,7} - 1) \\
& \times \delta(u_{3,4} + u_{1,3}u_{2,3}u_{4,5}u_{4,6}u_{4,7} - 1) \\
& \times \delta(u_{3,5} + u_{1,3}u_{1,4}u_{2,3}u_{2,4}u_{4,6}u_{4,7}u_{5,6}u_{5,7} - 1) \\
& \times \delta(u_{6,7} + u_{1,6}u_{2,6}u_{3,6}u_{4,6}u_{5,6} - 1) \\
& \times \delta(u_{3,7} + u_{1,3}u_{1,4}u_{1,5}u_{1,6}u_{2,3}u_{2,4}u_{2,5}u_{2,6} - 1) \\
& \times \delta(u_{4,5} + u_{1,4}u_{2,4}u_{3,4}u_{5,6}u_{5,7} - 1) \\
& \times \delta(u_{4,6} + u_{1,4}u_{1,5}u_{2,4}u_{2,5}u_{3,4}u_{3,5}u_{5,7}u_{6,7} - 1) \\
& \times \delta(u_{5,6} + u_{1,5}u_{2,5}u_{3,5}u_{4,5}u_{6,7} - 1) \\
& \times \delta(u_{4,7} + u_{1,4}u_{1,5}u_{1,6}u_{2,4}u_{2,5}u_{2,6}u_{3,4}u_{3,5}u_{3,6} - 1) \\
& \times \delta(u_{5,7} + u_{1,5}u_{1,6}u_{2,5}u_{2,6}u_{3,5}u_{3,6}u_{4,5}u_{4,6} - 1) \\
& \times \delta(u_{3,6} + u_{1,3}u_{1,4}u_{1,5}u_{2,3}u_{2,4}u_{2,5}u_{4,7}u_{5,7}u_{6,7} - 1),
\end{aligned} \tag{3.71}$$

with the assignment (3.17).

In what follows it is convenient to introduce

$$\begin{aligned}
I_8(x, y, z, w, v) = {} & x^{s_7} y^{t_6} z^{u_1} w^{t_1} v^{s_1} (1-x)^{s_6} (1-y)^{s_5} (1-z)^{s_4} (1-w)^{s_3} (1-v)^{s_2} \\
& \times (1-xy)^{s_{57}} (1-yz)^{s_{46}} (1-wz)^{s_{35}} (1-vw)^{s_{24}} (1-xyz)^{s_{47}} \\
& \times (1-wyz)^{s_{36}} (1-vwz)^{s_{25}} (1-wxyz)^{s_{37}} (1-vwyz)^{s_{26}} \\
& \times (1-vwxyz)^{s_{27}}
\end{aligned} \tag{3.72}$$

arising from (3.70) with the identifications $x_1 := v$, $x_2 := w$, $x_3 := z$, $x_4 = y$ and $x_5 := x$. Similarly as in the previous subsections, the following shorter notation for the dual variables $u_{i,j}$ is used

$$\begin{aligned}
& X_i = u_{i,i+1}, \qquad Y_j = u_{j,j+2}, \quad i, j = 1, \ldots, 8,\ i+8 \equiv i,\ j+8 \equiv j, \\
& Z_k = u_{k,k+3}, \quad k = 1, \ldots, 4,
\end{aligned} \tag{3.73}$$

and we also define

$$\begin{aligned} J_8(X,Y,Z) = {} & \left(\prod_{i=1}^{8} X_i^{s_i}\right)\left(\prod_{j=1}^{8} Y_j^{t_j}\right)\left(\prod_{k=1}^{4} Z_k^{u_k}\right)\delta(X_2+X_1X_3Y_3Y_8Z_3-1) \\ & \times \delta(Y_2+X_1X_4Y_1Y_3Y_4Y_8Z_3Z_4-1)\,\delta(Z_2+X_1X_5Y_1Y_4Y_5Y_8Z_1Z_3Z_4-1) \\ & \times \delta(Y_7+X_1X_6Y_1Y_5Y_6Y_8Z_1Z_4-1)\,\delta(X_8+X_1X_7Y_1Y_6Z_1-1) \\ & \times \delta(X_3+X_2X_4Y_1Y_4Z_4-1)\,\delta(Y_3+X_2X_5Y_1Y_2Y_4Y_5Z_1Z_4-1) \\ & \times \delta(Z_3+X_2X_6Y_1Y_2Y_5Y_6Z_1Z_2Z_4-1)\,\delta(Y_8+X_2X_7Y_1Y_2Y_6Y_7Z_1Z_2-1) \\ & \times \delta(X_4+X_3X_5Y_2Y_5Z_1-1)\,\delta(Y_4+X_3X_6Y_2Y_3Y_5Y_6Z_1Z_2-1) \\ & \times \delta(Z_4+X_3X_7Y_2Y_3Y_6Y_7Z_1Z_2Z_3-1)\,\delta(X_5+X_4X_6Y_3Y_6Z_2-1) \\ & \times \delta(Y_5+X_4X_7Y_3Y_4Y_6Y_7Z_2Z_3-1)\,\delta(X_6+X_5X_7Y_4Y_7Z_3-1). \end{aligned} \tag{3.74}$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \int_0^1 dv \, \frac{I_8(x,y,z,w,v)}{w(1-v)(1-z)(1-xy)(1-yz)} \tag{3.75}$$

can be easily deduced after transforming it into the form (3.71)

$$\left(\prod_{i=1}^{8}\int_0^1 dX_i\right)\left(\prod_{j=1}^{8}\int_0^1 dY_j\right)\left(\prod_{k=1}^{4}\int_0^1 dZ_k\right)\frac{J_8(X,Y,Z)}{X_2X_4Y_1Y_4Z_4} = \frac{1}{s_2s_4t_1t_4u_4} + \cdots. \tag{3.76}$$

Hence, the only simultaneous pole is at $X_2, X_4, Y_1, Y_4, Z_4 \to 0$ with the product of δ-functions yielding the constraints for the 15 variables $X_1, X_3, X_5, X_6, X_7, X_8, Y_2, Y_3, Y_5, Y_6, Y_7, Y_8, Z_1, Z_2, Z_3 \to 1$. Subsequently, in the sequel we list a few non-trivial examples:

rational function in Eq. (3.1)	*rational function in Eq.* (3.70)	...
$\frac{z_{18}^4}{z_{12}z_{13}z_{14}z_{15}z_{16}z_{17}z_{28}z_{38}z_{48}z_{58}z_{68}z_{78}}$	$\frac{1}{xyzwv}$	...
$\frac{z_{18}^3}{z_{12}z_{13}z_{14}z_{16}z_{17}z_{28}z_{38}z_{45}z_{58}z_{68}z_{78}}$	$\frac{1}{xy(1-z)wv}$	...
$\frac{1}{z_{17}z_{18}z_{23}z_{24}z_{35}z_{46}z_{57}z_{68}}$	$\frac{1}{(1-v)(1-xy)(1-wz)(1-yz)(1-vw)}$	...
$\frac{1}{z_{12}z_{17}z_{28}z_{34}z_{36}z_{47}z_{56}z_{58}}$	$\frac{y}{v(1-y)(1-w)(1-xyz)(1-wyz)}$	...
$\frac{1}{z_{17}z_{18}z_{24}z_{26}z_{35}z_{37}z_{45}z_{68}}$	$\frac{w\,z}{(1-z)(1-vw)(1-wz)(1-vwyz)(1-wxyz)}$	...
$\frac{z_{18}^2}{z_{12}z_{15}z_{16}z_{17}z_{28}z_{34}z_{38}z_{45}z_{68}z_{78}}$	$\frac{1}{xyv(1-z)(1-w)}$	...
$\frac{1}{z_{12}z_{17}z_{24}z_{34}z_{38}z_{56}z_{57}z_{68}}$	$\frac{1}{vz(1-y)(1-w)(1-xy)(1-vw)}$	...
$\frac{1}{z_{13}z_{17}z_{23}z_{25}z_{45}z_{46}z_{68}z_{78}}$	$\frac{1}{xyw(1-z)(1-v)(1-yz)(1-vwz)}$	...

$$\begin{array}{lll}
\ldots & \underline{\text{rational function}}\atop\underline{\text{in Eq. (3.71)}} & \underline{\text{lowest order poles}} \\
\ldots & \frac{X_8^4 Y_2 Y_3 Y_4 Y_5 Y_7^3 Y_8^3 Z_2^2 Z_3^2 Z_4^2}{X_1 X_7 Y_1 Y_6 Z_1} & \frac{1}{s_1 s_7 t_1 t_6 u_1}, \\
\ldots & \frac{X_8^3 Y_2 Y_5 Y_7^2 Y_8^2 Z_2 Z_3^2 Z_4}{X_1 X_4 X_7 Y_1 Y_6} & \frac{1}{s_1 s_4 s_7 t_1 t_6}, \\
\ldots & \frac{1}{X_2 X_8 Y_2 Y_7 Z_2} & \frac{1}{s_2 s_8 t_2 t_7 u_2}, \\
\ldots & \frac{Y_6 Z_2}{X_1 X_3 X_5 Y_8 Z_3} & \frac{1}{s_1 s_3 s_5 t_8 u_3}, \\
\ldots & \frac{Y_1 Y_5 Z_1 Z_4}{X_4 X_8 Y_3 Y_7 Z_2} & \frac{1}{s_4 s_8 t_3 t_7 u_2}, \\
\ldots & \frac{X_8^2 Y_5 Y_7 Y_8 Z_4}{X_1 X_3 X_4 X_7 Y_3 Y_6} & \frac{1}{s_1 s_3 s_7 t_3 t_6} + \frac{1}{s_1 s_4 s_7 t_3 t_6}, \\
\ldots & \frac{1}{X_1 X_3 X_5 Y_2 Y_5 Z_1} & \frac{1}{s_1 s_3 s_5 t_5 u_1} + \frac{1}{s_3 s_5 t_2 t_5 u_1}, \\
\ldots & \frac{Y_5 Y_8}{X_3 X_4 X_7 Y_1 Y_4 Y_6 Y_7 Z_2} & \frac{1}{s_2 s_4 s_7 t_1 t_4} + \frac{1}{s_2 s_4 s_7 t_1 t_6} + \frac{1}{s_2 s_4 s_7 t_4 t_7} + \frac{1}{s_2 s_4 s_7 t_6 u_2} + \frac{1}{s_2 s_4 s_7 t_7 u_2}.
\end{array} \tag{3.77}$$

3.4. Polynomial relations and Gröbner basis reduction

For $n_{i,j} \geqslant 0$ the representation (3.16) in the dual variables $u_{i,j}$ gives rise to a polynomial ring $\mathbf{R}[u_{\mathcal{P}}]$ describing polynomials in $u_{i,j}$, $(i,j) \in \mathcal{P}$ with coefficients in $\mathbf{R}$. This ring is suited to perform a Gröbner basis analysis to find a minimal basis for the polynomials in the integrand. The set of integrals (3.16) with $n_{i,j} \geqslant 0$ describe all integrals without poles in their α'-expansion. Due to the constraints (3.12), which give rise to the δ-functions in (3.16), many polynomials in the variables $u_{i,j}$ referring to different choices of the integers $n_{i,j}$ yield to the same integral B_N. The constraints (3.12) define a monomial ideal I in the polynomial ring $\mathbf{R}[u_{\mathcal{P}}]$. Hence, we consider the quotient space $\mathbf{R}[u_{\mathcal{P}}]/I$ and the Gröbner basis method is well appropriate to choose a basis in the ideal I and generate independent sets of polynomials in the quotient ring $\mathbf{R}[u_{\mathcal{P}}]/I$. We are interested in simple representatives of equivalence classes for congruence modulo I. The properties of an ideal are reflected in the form of the elements of the Gröbner basis [36,37].

Given a monomial ordering[10] in the ring a Gröbner basis $G = \{g_1, \ldots, g_d\}$ comprises a finite subset of the ideal I such that the leading term[11] of any element of the ideal I is divisible by

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[10] As monomial ordering we may choose lexicographic order or graded lexicographic order. Then, a monomial ordering of two polynomials $f = \sum_\alpha a_\alpha x^\alpha$ and $g = \sum_\beta b_\alpha x^\beta$ can be defined as follows:

(i) *lexicographic order*: $\alpha >_{lex} \beta$, if in the vector difference $\alpha - \beta \in \mathbf{Z}^n$ the leftmost nonzero entry is positive ($x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$),

(ii) *graded lexicographic order*: $\alpha >_{grlex} \beta$, if $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta|$ and $\alpha >_{lex} \beta$ ($x^\alpha >_{grlex} x^\beta$, if $\alpha >_{grlex} \beta$).

[11] The leading term $LT(f)$ of a polynomial f is defined as follows [36]: For $f = \sum_\alpha a_\alpha x^\alpha$ a nonzero polynomial in $\mathbf{R}[x_1, \ldots, x_n]$ and $>$ a specific monomial order

(i) the *multidegree* of f is $multideg(f) := \mathrm{Max}\{\alpha \in \mathbf{Z}^n_{\geqslant 0} \mid a_\alpha \neq 0\}$,

(ii) the *leading coefficient* of f is: $LC(f) := a_{multideg(f)} \in \mathbf{R}$,

(iii) the *leading monomial* of f is $LM(f) = x^{multideg(f)}$, with coefficient 1, and

(iv) the *leading term* of f is

$$LT(f) = LC(f) LM(f).$$

As an example we consider $f = xyz + 2xy^2z^2 + 3z^3 - 7x^5y + 3x^2z^2$ with $>$ the lexicographic order. Then we have: $multideg(f) = (5,1,0)$, $LC(f) = -7$, $LM(f) = x^5y$ and $LT(f) = -7x^5y$.

a leading term $LT(g_i)$ of an element of the subset. Alternatively, a finite subset G of an ideal I in a polynomial ring represents a Gröbner basis, if $\langle LT(g_1),\ldots,LT(g_d)\rangle = \langle LT(I)\rangle$ [36,37]. Buchberger's algorithm generates the unique *reduced* Gröbner basis G, in which no monomial in a polynomial $p \in G$ of this basis is divisible by a leading term of the other polynomials in the basis and $LC(p) = 1$.

The main idea is, that after dividing a polynomial $p \in \mathbf{R}[x_1,\ldots,x_n]$ by a Gröbner basis $G = \{g_1,\ldots,g_d\}$ for the ideal $I \subset \mathbf{R}[x_1,\ldots,x_n]$ the remainder $\overline{p}^G$ is uniquely fixed by the polynomial p, cf. Chapter 5, §3 of [36]. More precisely according to Proposition 1 therein we have: For a given monomial ordering on $\mathbf{R}[x_1,\ldots,x_n]$ and an ideal $I \subset \mathbf{R}[x_1,\ldots,x_n]$,

(i) Every $f \in \mathbf{R}[x_1,\ldots,x_n]$ is congruent modulo I to a unique polynomial r, which is a $\mathbf{R}$-linear combination of the monomials in the complement of $\langle LT(I)\rangle$.

(ii) The elements $\{x^\alpha \mid x^\alpha \notin \langle LT(I)\rangle\}$ are linearly independent modulo I, i.e. if $\sum_\alpha c_\alpha x^\alpha = 0 \bmod I$, where the x^α are all in the complement of $\langle LT(I)\rangle$, then $c_\alpha = 0$ for all α. As a consequence, for any given $f \in \mathbf{R}[x_1,\ldots,x_n]$ the remainder $\overline{f}^G$ is a $\mathbf{R}$-linear combination of the monomials contained in the complement of $LT(I)$, i.e. $\overline{f}^G \in Span(x^\alpha \mid x^\alpha \notin \langle LT(I)\rangle)$:

$$f = x^a \equiv x_1^{a_1}\cdots x_n^{a_n} = \sum_{i=1}^{d} c_i g_i + \sum_{x^\alpha \notin \langle LT(I)\rangle} r_\alpha x^\alpha. \tag{3.78}$$

In the following with the Gröbner basis method we want to construct a basis for those polynomials, which are independent on the constraints (3.12). This basis is determined by the complement of $\langle LT(I)\rangle$ w.r.t. a Gröbner basis G. Note, that the representation of this basis (and also of $\langle LT(I)\rangle$ and the remainders) may depend on the chosen monomial ordering. At any rate, there is always the *same* number of monomials in the complement of $\langle LT(I)\rangle$. In addition, on the degree of the basis monomials we impose a condition to ensure, that in the denominator of the integrands of (3.1) the z_{ij} only appear with powers of at most one, i.e. $\tilde{n}_{ij} \geqslant -1$. This restriction is useful to take into account the relations stemming from partial integrations (3.8). We illustrate the method with the following examples.

3.4.1. N = 4

We work with the two coordinates $X_1 = u_{1,2}$ and $X_2 = u_{2,3}$ and consider the polynomial ring $\mathbf{R}[X_1, X_2]$. From (3.23) we can read off the constraints (3.12) giving rise to the monomial ideal:

$$I = \langle X_1 + X_2 - 1\rangle \subset \mathbf{R}[X_1, X_2]. \tag{3.79}$$

W.r.t. lexicographic order we find for the Gröbner basis of (3.79):

$$G = \{g_1\} = \{X_1 + X_2 - 1\}. \tag{3.80}$$

Hence w.r.t. lexicographic order the leading term of this monomial gives rise to:

$$LT(I) = X_1. \tag{3.81}$$

Therefore, the set of possible remainders modulo I is the set of all $\mathbf{R}$-linear combinations of the following monomials:

$$\{1, X_2, X_2^2, X_2^3, \ldots\}. \tag{3.82}$$

For some examples let us determine their remainders on dividing them by the Gröbner basis (3.80):

$$\begin{aligned}
&X_1 = g_1 + 1 - X_2 \simeq 1 - X_2,\\
&X_2 = 0g_1 + X_2 \simeq X_2,\\
&X_1X_2 = X_2g_1 + X_2 - X_2^2 \simeq X_2 - X_2^2,\\
&X_1^2 = (1 + X_1 - X_2)g_1 + 1 - 2X_2 + X_2^2 \simeq 1 - 2X_2 + X_2^2,\\
&X_1^2X_2 = X_2(1 + X_1 - X_2)g_1 + X_2 - 2X_2^2 + X_2^3 \simeq X_2 - 2X_2^2 + X_2^3.
\end{aligned} \tag{3.83}$$

Indeed, the remainders (displayed after the $\simeq$ sign) are generated by the basis (3.82).

In (3.23) the monomials $X_2^{n_{11}}$, $n_{11} = 0, 1, \ldots,$ of (3.82) give rise to the following integrals (3.22):

$$B_4[n] = \int_0^1 dx\, x^{s_{12}}(1 - x)^{s_{23}+n_{11}}. \tag{3.84}$$

The integrals (3.22) without poles in their field-theory expansions are given by the integers $n_1, n_{11} = 0, 1 \ldots.$ According to our construction all these integrals (3.22) can be generated by **R**-linear combinations of the basis (3.84). However according to (3.7) we have

$$(1 - x)^{n_{11}} \simeq \frac{z_{14}^{n_{11}} z_{23}^{n_{11}}}{z_{13}^{2+n_{11}} z_{24}^{2+n_{11}}}, \tag{3.85}$$

i.e. all finite integrals (3.84) in (3.1) imply some powers $\tilde{n}_{ij}$ with $\tilde{n}_{ij} < -1$. As a consequence the set of integrals (3.84) cannot serve as a basis and (3.24) are the only elements of the partial fraction basis. Note, that this basis is two-dimensional, i.e. $(N - 2)! = 2$ for $N = 4$.

3.4.2. $N = 5$

We work with the five coordinates (3.29) and consider the polynomial ring $\mathbf{R}[X_1, \ldots, X_5]$. From (3.30) we can read off the constraints (3.12) giving rise to the monomial ideal:

$$I = \langle X_2 + X_1X_3 - 1, X_3 + X_2X_4 - 1, X_5 + X_1X_4 - 1\rangle \subset \mathbf{R}[X_1, \ldots, X_5]. \tag{3.86}$$

W.r.t. lexicographic order we find for the (reduced) Gröbner basis of (3.86) the three elements:

$$G = \{g_1, g_2, g_3\} = \{X_1 + X_2X_5 - 1, X_3 + X_2X_4 - 1, X_4 + X_3X_5 - 1\}. \tag{3.87}$$

Hence w.r.t. lexicographic order the leading terms of these three monomials give rise to:

$$LT(I) = \{X_1, X_2X_4, X_3X_5\}. \tag{3.88}$$

Therefore, the set of possible remainders modulo I is the set of all **R**-linear combinations of the following monomials:

$$\bigcup_{m,n=0}^{\infty} \left\{X_2^mX_3^n, X_2^mX_5^n, X_3^mX_4^n, X_4^mX_5^n\right\}. \tag{3.89}$$

For some examples let us determine their remainders on dividing them by the Gröbner basis (3.87):

$$\begin{aligned}
&X_1 = g_3 + 1 - X_2X_5 \simeq 1 - X_2X_5,\\
&X_1X_4 = g_1 - X_5g_2 + X_4g_3 + 1 - X_5 \simeq 1 - X_5,\\
&X_3X_5 = g_1 + 1 - X_4 \simeq 1 - X_4,\\
&X_3X_5^2 = X_5g_1 + X_5 - X_4X_5 \simeq X_5 - X_4X_5,\\
&X_1X_2 = X_2g_3 + X_2 - X_2^2X_5 \simeq X_2 - X_2^2X_5,\\
&X_2X_3X_5 = X_2g_1 - g_2 - 1 + X_2 + X_3 \simeq -1 + X_2 + X_3.
\end{aligned} \tag{3.90}$$

Indeed, the remainders (displayed after the $\simeq$-sign) are generated by the basis (3.89).

We have the following dictionary

$$\begin{array}{ccc}
\underline{\textit{monomial}} & \underline{\textit{rational function}} & \underline{\textit{rational function}}\\
\underline{\textit{in Eq. (3.27)}} & \underline{\textit{in Eq. (3.26)}} & \underline{\textit{in Eq. (3.1)}}\\
1 & \frac{1}{1-xy} & \frac{1}{z_{13}z_{14}z_{24}z_{25}z_{35}},\\
X_2 & \frac{1-y}{(1-xy)^2} & \frac{z_{23}}{z_{13}^2z_{24}^2z_{25}z_{35}},\\
X_3 & \frac{1-x}{(1-xy)^2} & \frac{z_{34}}{z_{13}z_{14}z_{24}^2z_{35}^2},\\
X_4 & \frac{x}{(1-xy)} & \frac{z_{45}}{z_{14}^2z_{24}z_{25}z_{35}^2},\\
X_5 & 1 & \frac{z_{15}}{z_{13}z_{14}^2z_{25}^2z_{35}},\\
X_2X_3 & \frac{(1-x)(1-y)}{(1-xy)^3} & \frac{z_{23}z_{34}}{z_{13}^2z_{24}^3z_{35}^2},\\
X_2X_5 & \frac{1-y}{1-xy} & \frac{z_{15}z_{23}}{z_{13}^2z_{14}z_{24}z_{25}^2z_{35}},\\
X_3X_4 & \frac{x(1-x)}{(1-xy)^2} & \frac{z_{34}z_{45}}{z_{14}^2z_{24}^2z_{35}^3},\\
X_4X_5 & x & \frac{z_{15}z_{45}}{z_{14}^3z_{25}^2z_{35}^2},
\end{array} \tag{3.91}$$

between monomials in the integral (3.27), the polynomial in (3.26), and the representation (3.1). According to the list (3.91) from the generators (3.89) of the complement $\overline{\langle LT(I)\rangle}$ only the element 1 does not give rise to higher powers of z_{ij} in the denominator of the integrand (3.1), i.e. $\tilde{n}_{ij} \geqslant -1$. Therefore, we dismiss all other basis elements and the integral

$$\int_0^1 dx \int_0^1 dy\, \frac{I_5(x,y)}{1-xy} = \zeta(2) + \cdots \tag{3.92}$$

is left as the only basis element without poles. The integral (3.92) yields a transcendental power series in α', cf. Appendix A. Together with the fundamental set (3.34) we obtain a six-dimensional partial fraction basis, i.e. $(N-2)! = 6$ for $N = 5$.

3.4.3. $N = 6$

Using the coordinates (3.44) we consider the polynomial ring $\mathbf{R}[X_1, \ldots, X_6, Y_1, \ldots, Y_3]$. From (3.45) we can read off the constraints (3.12) giving rise to the monomial ideal:

$$\begin{aligned}
I = \langle &X_2 + X_1X_3Y_3 - 1, X_3 + X_2X_4Y_1 - 1, X_4 + X_3X_5Y_2 - 1,\\
&X_6 + X_1X_5Y_1 - 1, Y_2 + X_1X_4Y_1Y_3 - 1, Y_3 + X_2X_5Y_1Y_2 - 1\rangle.
\end{aligned} \tag{3.93}$$

W.r.t. lexicographic order we find for the (reduced) Gröbner basis of (3.93) the 13 elements:

$$\begin{aligned}G = \{&1 - Y_1 + X_6Y_1 - X_6Y_2 - X_6Y_3 + X_6^2Y_2Y_3, -1 + X_5Y_1 + X_6Y_3,\\ &1 - X_5 - X_6 + X_5X_6Y_2, -1 + X_4Y_3 + X_5Y_2, -1 + X_4Y_1 + X_3Y_2,\\ &X_4 - X_6 - X_4Y_1 + X_4X_6Y_1 + X_6Y_2 - X_4X_6Y_2, -1 + X_2Y_1 + X_3Y_3,\\ &X_3 - X_4 + X_6 - X_3Y_1 + X_4Y_1 - X_6Y_1 + X_3X_6Y_1 - X_3X_6Y_3 - X_6Y_2 + X_4X_6Y_2,\\ &-X_3 + X_3X_5 - X_6 + X_3X_6 + X_4X_6, 1 - X_2 - X_3Y_3 - X_6Y_3 + X_2X_6Y_3\\ &+ X_3X_6Y_3, -1 + X_2 + X_5 - X_2Y_3 + X_3Y_3 - X_5Y_3 + X_2X_5Y_3 + X_6Y_3\\ &- X_3X_6Y_3 - X_2X_5Y_2, -X_2 + X_3 + X_2X_4 - X_5 + X_2X_5 + X_6 - X_3X_6 - X_4X_6,\\ &-1 + X_1 + X_2X_6Y_2\}. \end{aligned} \tag{3.94}$$

Hence w.r.t. lexicographic order the leading terms of these 13 monomials give rise to:

$$\begin{aligned}LT(I) = \{&X_6^2Y_2Y_3, X_5Y_1, X_5X_6Y_2, X_4Y_3, X_4X_6Y_1, X_3Y_2, X_3X_6Y_1, X_3X_5, X_2Y_1,\\ &X_2X_6Y_3, X_2X_5Y_3, X_2X_4, X_1\}.\end{aligned} \tag{3.95}$$

We would like to mention that the Gröbner basis consists of 18 elements in the case of degree lexicographic order.

From the set (3.95) the monomials generating the complement $\overline{\langle LT(I)\rangle}$ can be determined. Most of these monomials yield to higher powers of z_{ij} in the denominator of the integrand (3.1), i.e. $\tilde{n}_{ij} = -2$ for some z_{ij}. In fact, only the following five monomials give rise to single powers in their denominators, i.e. $\tilde{n}_{ij} \geqslant -1$:

monomial in Eq. (3.42)	*rational function in Eq.* (3.41)	*rational function in Eq.* (3.1)
1	$\frac{1}{(1-xy)(1-yz)}$	$\frac{1}{z_{13}z_{15}z_{24}z_{26}z_{35}z_{46}}$,
Y_1	$\frac{y}{(1-xy)(1-yz)}$	$\frac{1}{z_{14}z_{15}z_{24}z_{26}z_{35}z_{36}}$,
Y_2	$\frac{1}{(1-xy)(1-xyz)}$	$\frac{1}{z_{13}z_{14}z_{25}z_{26}z_{35}z_{46}}$,
Y_3	$\frac{1}{(1-yz)(1-xyz)}$	$\frac{1}{z_{13}z_{15}z_{24}z_{25}z_{36}z_{46}}$,
$X_6Y_2Y_3$	$\frac{1}{(1-xyz)}$	$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{25}z_{26}z_{36}z_{46}}$.

(3.96)

Therefore, we dismiss all other basis elements of $\overline{\langle LT(I)\rangle}$. All (finite) integrals (3.1) with only single powers of z_{ij} in their denominators, i.e. $\tilde{n}_{ij} \geqslant -1$, are spanned by the following five integrals[12]:

$$G_0 = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \frac{I_6(x,y,z)}{(1-xy)(1-yz)} = 2\zeta(2) + \cdots,$$

$$G_1 = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \frac{yI_6(x,y,z)}{(1-xy)(1-yz)} = 2\zeta(3) + \cdots,$$

[12] Note, that although for degree lexicographic order the Gröbner basis consists of more elements than (3.94) the resulting list (3.96) of monomials is the same for any monomial ordering rule.

$$G_2 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{(1-xy)(1-xyz)} = 2\zeta(3) + \cdots,$$

$$G_3 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{(1-yz)(1-xyz)} = 2\zeta(3) + \cdots,$$

$$G_4 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{1-xyz} = \zeta(3) + \cdots. \tag{3.97}$$

E.g. we have

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{yz\, I_6(x,y,z)}{(1-yz)(1-xyz)} = G_3 - G_4,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{y(1-z)\, I_6(x,y,z)}{(1-xy)(1-yz)(1-xyz)} = G_1 - G_3 + G_4,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{(1-y) I_6(x,y,z)}{(1-xy)(1-yz)(1-xyz)} = -G_1 + G_2 + G_3 - G_4,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{(1-x)\, y\, I_6(x,y,z)}{(1-xy)(1-yz)(1-xyz)} = G_1 - G_2 + G_4,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{xy I_6(x,y,z)}{(1-xy)(1-xyz)} = G_2 - G_4 \tag{3.98}$$

as result from the identities between their corresponding monomials on dividing them by the Gröbner basis (3.94):

$$\begin{aligned}
&X_1Y_1Y_3 = Y_3 - X_6Y_2Y_3,\\
&X_2Y_1Y_2 = Y_1 - Y_3 + X_6Y_2Y_3,\\
&X_3Y_2Y_3 = -Y_1 + Y_2 + Y_3 - X_6Y_2Y_3,\\
&X_4Y_1Y_3 = Y_1 - Y_2 + X_6Y_2Y_3,\\
&X_5Y_1Y_2 = Y_2 - X_6Y_2Y_3.
\end{aligned} \tag{3.99}$$

To conclude: Any finite integral (3.1) with $\tilde{n}_{ij} \geqslant -1$ can be expressed as $\mathbf{R}$-linear combination of the basis (3.97) as a result of partial fraction decomposition of their integrands.

Except the first integral G_0, the other four integrals (3.97) yield a transcendental power series in α', cf. Appendix A. Any partial fraction decomposition, which involves G_0 must refer to a non-transcendental integral (3.1) and only partial fraction expansions involving the basis $G_1, \ldots, G_4$ comprise into a transcendental integral. In the previous subsection we have found a set of six transcendental integrals (3.54) with single poles. Together with the fundamental set (3.51) we obtain a partial fraction basis (of transcendental integrals (3.1)) with $4+6+14=24$ elements, i.e. $(N-2)!=24$ for $N=6$.

3.4.4. $N=7$

Using the coordinates (3.60) we consider the polynomial ring $\mathbf{R}[X_1,\ldots,X_7,Y_1,\ldots,Y_7]$. From (3.61) we can read off the constraints (3.12) giving rise to the monomial ideal:

$$\begin{aligned}I=\langle &X_2+X_1X_3Y_3Y_7-1,\, X_3+X_2X_4Y_1Y_4-1,\, X_4+X_3X_5Y_2Y_5-1,\\ &X_5+X_4X_6Y_3Y_6-1,\, Y_4+X_3X_6Y_3Y_2Y_5Y_6-1,\, Y_6+X_1X_5Y_1Y_4Y_5Y_7-1,\\ &Y_7+X_2X_6Y_1Y_2Y_5Y_6-1,\, X_7+X_1X_6Y_1Y_5-1,\, Y_2+X_1X_4Y_1Y_3Y_4Y_7-1,\\ &Y_3+X_2X_5Y_1Y_2Y_4Y_5-1\rangle.\end{aligned} \tag{3.100}$$

W.r.t. lexicographic order we find 84 elements in the (reduced) Gröbner basis of (3.100). On the other hand w.r.t. degree lexicographic order we have 184 basis elements. In the following, we determine the monomials generating the complement $\overline{\langle LT(I)\rangle}$ w.r.t. to degree lexicographic order as this ordering directly yields a cyclic invariant basis. Most of the monomials in the complement $\overline{\langle LT(I)\rangle}$ yield to higher powers of z_{ij} in the denominator of the integrand (3.1), i.e. $\tilde n_{ij}=-2$ for some z_{ij}. After disregarding those, only the following six monomials and their cyclic transformations give rise to single powers in their denominators, i.e. $\tilde n_{ij}\geqslant -1$:

$$\begin{array}{ccc} \underline{\textit{monomial in Eq. (3.58)}} & \underline{\textit{rational function in Eq. (3.57)}} & \underline{\textit{rational function in Eq. (3.1)}}\\ 1 & \frac{1}{(1-xy)(1-yz)(1-wz)} & \frac{1}{z_{13}z_{16}z_{24}z_{27}z_{35}z_{46}z_{57}},\\ Y_1Y_4 & \frac{z}{(1-yz)(1-wz)(1-xyz)} & \frac{1}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{36}z_{57}},\\ Y_1Y_3Y_6 & \frac{z}{(1-xy)(1-wz)(1-xyz)} & \frac{z_{47}}{z_{14}z_{15}z_{24}z_{27}z_{36}z_{37}z_{46}z_{57}},\\ Y_1Y_2Y_5 & \frac{yz}{(1-xy)(1-yz)(1-wyz)} & \frac{1}{z_{14}z_{16}z_{25}z_{27}z_{35}z_{37}z_{46}},\\ Y_2Y_4 & \frac{1}{(1-yz)(1-wyz)(1-xyz)} & \frac{z_{15}z_{37}}{z_{13}z_{14}z_{16}z_{25}z_{27}z_{35}z_{36}z_{47}z_{57}},\\ Y_1 & \frac{z}{(1-xy)(1-wz)(1-yz)} & \frac{z_{47}}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{37}z_{46}z_{57}}. \end{array} \tag{3.101}$$

Therefore, in total we have a basis of 36 elements and all (finite) integrals (3.1) with only single powers in their denominators z_{ij}, i.e. $\tilde n_{ij}\geqslant -1$, are spanned by the following six integrals

$$G_0=\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dw\,\frac{I_7(x,y,z,w)}{(1-xy)(1-yz)(1-wz)}=\frac{27}{4}\zeta(4)+\cdots,$$

$$G_{1a}=\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dw\,\frac{zI_7(x,y,z,w)}{(1-yz)(1-wz)(1-xyz)}=\frac{17}{4}\zeta(4)+\cdots,$$

$$G_{2b}=\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dw\,\frac{zI_7(x,y,z,w)}{(1-xy)(1-wz)(1-xyz)}=3\zeta(4)+\cdots,$$

$$G_{3a}=\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dw\,\frac{yzI_7(x,y,z,w)}{(1-xy)(1-yz)(1-wyz)}=3\zeta(3)+\cdots,$$

$$G_{4b} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{I_7(x,y,z,w)}{(1-yz)(1-wyz)(1-xyz)}$$
$$= \frac{5}{2}\zeta(4) + 4\zeta(3) - 2\zeta(2) + \cdots,$$
$$G_{5a} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{z\, I_7(x,y,z,w)}{(1-xy)(1-wz)(1-yz)}$$
$$= 2\zeta(3) + 2\zeta(2) + \cdots, \tag{3.102}$$

and their cyclic transformations $G_{ja}, G_{jb}, G_{jc}, G_{jd}, G_{je}, G_{jf}$, $j = 1, \ldots, 5$. E.g. we have

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{I_7(x,y,z,w)}{(1-xy)(1-wz)} = G_0 - G_{1b},$$
$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{yz^2 I_7(x,y,z,w)}{(1-yz)(1-wz)(1-xyz)} = -G_0 + G_{1a} + G_{1b} + G_{1d} - G_{2b},$$
$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{yz\, I_7(x,y,z,w)}{(1-yz)(1-wyz)(1-xyz)} = G_{5b} - G_{3c} - G_{3f} + G_{4b},$$
$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{zI_7(x,y,z,w)}{(1-wz)(1-xyz)} = G_0 - G_{1b} - G_{1d} + G_{2b},$$
$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{y\, I_7(x,y,z,w)}{(1-xy)(1-wyz)} = G_0 - G_{1b} - G_{1f} + G_{2f},$$
$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{yzI_7(x,y,z,w)}{(1-yz)(1-wxyz)}$$
$$= -G_{1b} + G_{1g} - G_{3d} - G_{3e} - 2G_{3g} + G_{4d} + G_{4g} + G_{5f},$$
$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{yzI_7(x,y,z,w)}{(1-wyz)(1-xyz)}$$
$$= 2G_0 - 2G_{1b} - G_{1d} - G_{1f} + G_{2a} + G_{2b} + G_{2f} + G_{3a} + G_{3b} + G_{3c} + G_{3f}$$
$$- G_{4a} - G_{4c} - G_{5c} - G_{5d}, \tag{3.103}$$

as results from the identities between their corresponding monomials on dividing them by the Gröbner basis of (3.100):

$$X_7 Y_3 Y_6 Y_7 = 1 - Y_1 Y_5,$$
$$Y_1^2 Y_4 Y_5 = -1 + Y_1 Y_4 + Y_1 Y_5 + Y_3 Y_6 - Y_1 Y_3 Y_6,$$

$$\begin{aligned}
&Y_1Y_2Y_4Y_5 = Y_3 + Y_2Y_4 - Y_2Y_3Y_6 - Y_3Y_4Y_7,\\
&X_7Y_1Y_3Y_4Y_6Y_7 = 1 - Y_1Y_5 - Y_3Y_6 + Y_1Y_3Y_6,\\
&X_7Y_2Y_3Y_5Y_6Y_7 = 1 - Y_1Y_5 - Y_3Y_7 + Y_3Y_5Y_7,\\
&X_7Y_1Y_2Y_4Y_5Y_6Y_7 = -Y_1Y_5 + Y_6 + Y_1Y_6 - Y_2Y_5Y_6 + Y_7 - Y_1Y_4Y_7 + Y_5Y_7 - 2Y_3Y_6Y_7,\\
&X_7Y_1Y_2Y_3Y_4Y_5Y_6Y_7 = 2 - Y_1Y_3 - Y_2 - Y_4 - 2Y_1Y_5 - Y_3Y_5 + Y_1Y_3Y_5 + Y_1Y_2Y_5\\
&\qquad + Y_1Y_4Y_5 - Y_3Y_6 + Y_1Y_3Y_6 + Y_2Y_3Y_6 - Y_3Y_7 + Y_3Y_4Y_7\\
&\qquad + Y_3Y_5Y_7.
\end{aligned} \tag{3.104}$$

Only G_0, G_1, G_2 out of the six integrals in (3.102) yield a transcendental power series in α', cf. Appendix A.

To conclude: Any finite integral (3.1) with $\tilde{n}_{ij} \geqslant -1$ can be expressed as $\mathbf{R}$-linear combination of the basis (3.102) as a result of partial fraction decomposition of their integrands.

As a concrete example let us discuss the function $F^{(3452)}$ from the set (2.15) of basis functions for $N = 7$. It is comprised by a sum of four integrals:

$$F^{(3524)} = s_{13}s_{46} \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \times \left(\frac{s_{15}s_{24}}{z_{13}z_{15}z_{24}z_{46}} + \frac{s_{15}s_{26}}{z_{13}z_{15}z_{26}z_{46}} + \frac{s_{24}s_{35}}{z_{13}z_{24}z_{35}z_{46}} + \frac{s_{26}s_{35}}{z_{13}z_{26}z_{35}z_{46}}\right). \tag{3.105}$$

Their corresponding rational functions in (3.57) and monomials in (3.58) are given in the following table:

rational function in Eq. (3.1)	*rational function in Eq.* (3.57)	*monomial in Eq.* (3.58)
$\frac{z_{17}}{z_{13}z_{15}z_{16}z_{24}z_{27}z_{37}z_{46}z_{57}}$	$\frac{1}{(1-xy)(1-wz)}$	$X_7Y_3Y_6Y_7$,
$\frac{z_{17}z_{67}}{z_{13}z_{15}z_{16}z_{26}z_{27}z_{37}z_{46}z_{47}z_{57}}$	$\frac{xy}{(1-xy)(1-wxyz)}$	$X_6X_7Y_2Y_3Y_5Y_6^2Y_7$,
$\frac{1}{z_{13}z_{16}z_{24}z_{27}z_{35}z_{46}z_{57}}$	$\frac{1}{(1-xy)(1-wz)(1-yz)}$	1,
$\frac{z_{67}}{z_{13}z_{16}z_{26}z_{27}z_{35}z_{46}z_{47}z_{57}}$	$\frac{xy}{(1-xy)(1-yz)(1-wxyz)}$	$X_6Y_2Y_5Y_6$.

(3.106)

Their polynomial reduction w.r.t. the Gröbner basis of (3.100) gives

$$\begin{aligned}
&X_7Y_3Y_6Y_7 = 1 - Y_1Y_5,\\
&X_6X_7Y_2Y_3Y_5Y_6^2Y_7 = 1 - Y_1Y_5 + Y_6 - Y_2Y_5Y_6 - Y_4Y_7 + Y_5Y_7 - Y_3Y_6Y_7,\\
&X_6Y_2Y_5Y_6 = 1 - Y_4Y_7,
\end{aligned} \tag{3.107}$$

respectively. The remaining monomials belong to the set (3.101) and cyclic transformations thereof. Hence, with the lowest expansion coefficients from (3.102) we compute:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{I_7(x,y,z,w)}{(1-xy)(1-wz)} = \frac{10}{4}\zeta(4) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{xyI_7(x,y,z,w)}{(1-xy)(1-wxyz)} = \frac{3}{4}\zeta(4) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{I_7(x,y,z,w)}{(1-xy)(1-wz)(1-yz)} = \frac{27}{4}\zeta(4) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{xyI_7(x,y,z,w)}{(1-xy)(1-yz)(1-wxyz)} = \frac{10}{4}\zeta(4) + \cdots.$$

Eventually for (3.105) we obtain:

$$F^{(3524)} = \frac{1}{4}\zeta(4)s_{13}s_{46}(10s_{15}s_{24} + 3s_{15}s_{26} + 27s_{24}s_{35} + 10s_{26}s_{35}) + \mathcal{O}\big(\alpha'^5\big). \tag{3.108}$$

A similar analysis can be done for the other three functions $F^{(5324)}$, $F^{(3542)}$ and $F^{(5342)}$ starting at $\zeta(4)$, cf. Appendix C.

4. Concluding remarks

In the first part of this work [1] we derived a strikingly short and compact expression for the N-point superstring amplitude involving any external massless open string state from the SYM vector multiplet. The final expression is given in (1.1) and gives rise to a beautiful harmony of the string amplitudes. We have elucidated their implications both from and to field theory in Section 2. Our result demonstrates how to efficiently compute tree-level superstring amplitudes with an arbitrary number of external states. The pure spinor cohomology techniques sketched in [14,12] proved to be crucial to derive (1.1). The methods presented in our work should be applicable to tackle any tree-level disk amplitude computation in any dimensions.

The availability of the compact expression (1.1) for the superstring N-point amplitude allows a detailed study of possible recursion relations allowing to construct the N-amplitude from amplitudes with fewer external states and some guiding principle. Due to the factorized form of (1.1), which separates the YM-part from the string part, the basic question is how to combine the field-theory recursions established in the YM sector [38] (see also [13]) to recursions working in the module of hypergeometric functions B_N. For the latter the following recurrence relations may be useful [39]

$$B_N = \sum B_{n_1} B_{n_2} \cdot \cdots \cdot B_{n_k}, \qquad \sum_{l=1}^{k} n_l = N + 3(k-1), \tag{4.1}$$

with some partition $\{n_1, \ldots, n_k\}$ into k smaller amplitudes B_{n_l}. Eq. (4.1) allows to write B_N in terms of products of $(N-3)$ functions B_4, cf. Fig. 2.

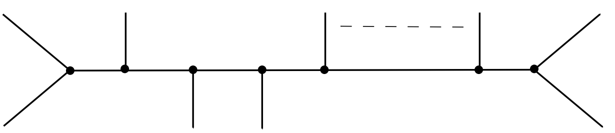

Fig. 2. Partition into products of four-point amplitudes B_4.

The amplitudes (1.1) give rise to higher order corrections in α' to the Yang–Mills action, therefore the YM amplitudes $\mathcal{A}_{YM}$ which appear in (1.1) serve as building blocks to construct the higher order terms in the effective action with the expansion coefficients encoded in the functions F^σ. Moreover, the field-theory amplitudes $\mathcal{A}_{YM}$ may be arranged such that only YM three-vertices contribute [4]. Hence, only the latter enter the full superstring amplitude (1.1). As a consequence it should be possible to describe the higher order α'-corrections in the effective action entirely in terms of the fundamental YM three-vertices dressed by the contributions from F^σ.

Together with the KLT relations [7], the open string N-point amplitudes (1.1) can be used to obtain compact expressions for the N-point closed string amplitudes [40]. The latter give rise to N-graviton scattering amplitudes. Their α'-expansions have been analyzed up to $N \leqslant 6$ through the order α'^8 in Ref. [34]. These findings proved to be crucial in constraining possible counterterms in $\mathcal{N} = 8$ supergravity in $D = 4$ up to seven loops [41]. Counterterms invariant under $\mathcal{N} = 8$ supergravity have an unique kinematic structure and the tree-level closed string amplitudes provide candidates for them, which are compatible with SUSY Ward identities and locality. The absence or restriction on higher order gravitational terms at the order α'^l together with their symmetries constrain the appearance of possible counterterms available at l-loop. With the present results it may now be possible to bolster up the results of [34] and to extend the research performed in [41,42].

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Appendix A. Degree of transcendentality in the α'-expansion

A.1. *Euler integrals and their power series expansions in α'*

The α'-dependence enters through the kinematic invariants s_{ij}, $s_{i\dots l}$ into the integrals (3.1) or (3.5). Hence, in their (integer) power series expansions in α', which may start at least at the order α'^{3-N}, each power α'^n is accompanied by some rational function or polynomial of degree n in the kinematic invariants $\hat{s}_{ij}$, $\hat{s}_{i\dots l}$. The latter have rational coefficients multiplied by multizeta values (MZVs) of certain weights. The maximal weight thereof appearing at a given order α'^n is related to the power n.

One important question is, whether the set of MZVs showing up at a given order n in α' is of a fixed weight. In this case we call the power series expansion *transcendental* (we may also call the integral transcendental). The power series (3.10) is of this kind. E.g. for $N = 6$ we may have the following integral and its power series expansion in α':

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{xyz}$$
$$= \frac{1}{s_1 s_5 t_1} - \zeta(2)\left(\frac{s_3}{s_1 s_5} + \frac{s_4}{s_1 t_1} + \frac{s_2}{s_5 t_1}\right)$$
$$+ \zeta(3)\left(\frac{s_3+s_4-t_3}{s_1} + \frac{s_2+s_3-t_2}{s_5} + \frac{s_3^2+s_3 t_1}{s_1 s_5} + \frac{s_4^2+s_4 s_5}{s_1 t_1} + \frac{s_2^2+s_1 s_2}{s_5 t_1}\right)$$
$$+ \mathcal{O}(\alpha'). \tag{A.1}$$

In (A.1) to each power α'^n in α' a Riemann zeta constant of fixed weight $n+3$ (with $n \geqslant -1$) appears. Hence, (A.1) represents a transcendental power series expansion. On the other hand, the following two integrals

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{(1-xyz)^2} = \zeta(2) + \zeta(2)(s_3+s_6-t_2-t_3)$$
$$- \zeta(3)(s_1+s_2+2s_3+s_4+s_5+2s_6+t_1-t_2-t_3)$$
$$+ \mathcal{O}(\alpha^2), \tag{A.2}$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{I_6(x,y,z)}{(1-xy)(1-yz)} = 2\zeta(2) + \left[2\zeta(2) - 4\zeta(3)\right](t_1+t_2+t_3)$$
$$- \left[2\zeta(2) - \zeta(3)\right](s_1+s_2+s_3+s_4+s_5+s_6)$$
$$+ \mathcal{O}(\alpha^2) \tag{A.3}$$

yield examples of non-transcendent power series.

It would be useful to have a criterion at hand, which allows to infer the transcendentality properties of an integral by inspecting its integrand before power series expanding the whole integral. In this subsection we present a criterion, which allows to deduce from the structure of the integrand, whether we should expect a transcendental power series expansion in α'. Although this is a mathematical question, it will turn out that superstring theory provides a satisfying answer to this.

Transforming the integrals from the representation (3.5) into the form (3.1) subject to (3.2) will prove to be useful in the following. Integrals (3.1), whose integrands are rational functions involving double or higher powers of z_{ij} in their denominators, i.e. $\tilde{n}_{ij} < -1$ for some z_{ij}, always give rise to non-transcendental power series. This can be seen by performing a partial integration within the integrals, e.g. for a double power we have:

$$\int z_{ij}^{s_{ij}-2} r(z_{kl}) = \frac{1}{s_{ij}-1} \int r(z_{kl}) \partial_{z_i} z_{ij}^{s_{ij}-1} = -\frac{1}{s_{ij}-1} \int z_{ij}^{s_{ij}-1} \partial_{z_i} r(z_{kl}). \tag{A.4}$$

Regardless of the transcendentality structure of the integral $\int z_{ij}^{s_{ij}-1} \partial_{z_i} r(z_{kl})$ the factor $\frac{1}{s_{ij}-1} = 1 + s_{ij} + s_{ij}^2 + \cdots$ always destroys any transcendentality. This explains, why the integral (A.2) with the corresponding rational functions (cf. Eq. (3.7))

$$\frac{1}{(1-xyz)^2} \simeq \frac{1}{z_{13}z_{14}z_{25}^2 z_{36}z_{46}}$$

yields a non-transcendental power series expansion. On the other hand, the non-transcendentality of the integral (3.5) with the rational function $R(x_i) = [(1-x)(1-y)(1-z)(1-xyz)]^{-1}$ can only be seen after transforming it into the representation (3.1), in which a rational function with a double power in the denominator appears, i.e.:

$$\frac{1}{(1-x)(1-y)(1-z)(1-xyz)} \simeq \frac{1}{z_{16}^2 z_{23}z_{25}z_{34}z_{45}}.$$

Let us now discuss the integrals (A.1) and (A.3) and elaborate their differences. W.r.t. to the two representations (3.5) and (3.1) we have the following correspondences

$$\begin{aligned} &\frac{1}{xyz} \simeq \frac{z_{16}^2}{z_{12}z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}z_{56}} \to \frac{1}{z_{12}z_{13}z_{14}z_{15}},\\ &\frac{1}{(1-xy)(1-yz)} \simeq \frac{1}{z_{13}z_{15}z_{24}z_{26}z_{35}z_{46}} \to \frac{1}{z_{13}z_{15}z_{24}z_{35}}, \end{aligned} \tag{A.5}$$

respectively. The last correspondence (denoted by the arrow) follows from the choice (3.3), with $z_6 = z_\infty = \infty$ and taking into account the z_∞^2 factor of the c-ghost factor $\langle c(z_1)c(z_5)c(z_6)\rangle = z_{15}z_\infty^2$. We may regard the rational functions (A.5) as originating from a CFT computation of a six-gluon amplitude. This fact will be exploited in the next subsection to infer the transcendentality properties of an integral (3.5) from the z_{ij}-representation of its integrand (3.1).

A.2. A transcendentality criterion from gluon amplitude computations

Gluon disk amplitudes in superstring theory provide transcendental power series when expanding them w.r.t. to α'. This fact follows from dimensional grounds and the underlying effective field-theory action describing the reducible and irreducible contributions of the power series expansions. As a consequence the individual constituents of a gluon amplitude describing some kinematical factor must be described by transcendental integrals (3.1). Recall that, in the NSR formalism with the choice (3.3) the color-ordered N-gluon amplitude $\mathcal{A}(1,\ldots,N)$ is computed from

$$\begin{aligned} &\mathrm{Tr}\big(T^{a_1}\ldots T^{a_N}\big)\mathcal{A}(1,\ldots,N)\\ &\quad = \big\langle c(z_1)c(z_{N-1})c(z_N)\big\rangle \left(\prod_{l=2}^{N-2}\int_{z_{l-1}}^{1} dz_l\right)\Big\langle V_g^{(-1)}(z_i)\,V_g^{(-1)}(z_j)\prod_{l\neq i,j}^{N} V_g^{(0)}(z_l)\Big\rangle, \end{aligned} \tag{A.6}$$

with the i-th and j-th gluon vertex operator put into the (-1)-ghost picture. The remaining $N-2$ vertex operators are in the zero-ghost picture in order to guarantee a total ghost charge of -2. The gluon vertex operator are given by

$$\begin{aligned} &V_g^{(-1)} = g_A T^a e^{-\phi}\xi_\mu \psi^\mu e^{ik_\rho X^\rho},\\ &V_g^{(0)} = T^a \frac{g_A}{(2\alpha')^{1/2}}\xi_\mu\big[i\partial X^\mu + 2\alpha'\big(k_\lambda\psi^\lambda\big)\psi^\mu\big]e^{ik_\rho X^\rho}, \end{aligned} \tag{A.7}$$

in the (-1)- and zero-ghost picture, respectively. Above we have the scalar field ϕ bosonizing the superghost system, the coupling constant g_A and the Chan–Paton factor T^a. In the following we always stick to the canonical color ordering $(1,\ldots,N)$. The assignment of the superghost charges is yet left unspecified. The interplay between the bosonic fields ∂X^μ and the fermionic parts $(k\psi)\psi^\mu$ of the $N-2$ zero-ghost vertices $V_g^{(0)}$ will play a crucial role for the following considerations.[13]

In a six-gluon amplitude (A.6) the integral (A.1) describes the space–time contraction $(\xi_1\xi_6)(\xi_2k_1)(\xi_3k_1)(\xi_4k_1)(\xi_5k_1)$, while the integral (A.3) characterizes the contraction $(\xi_2\xi_6)\times(\xi_1k_3)(\xi_3k_5)(\xi_4k_2)(\xi_5k_1)$. The crucial difference between the two encountered contractions is, that in (A.6) the first contraction can only be realized by contracting[14]

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^\lambda k_1^\sigma k_1^\rho k_1^\tau\langle\psi_1^{\mu_1}\psi_6^{\mu_6}\rangle\langle\partial X_2^{\mu_2}X_1^\lambda\rangle\langle\partial X_3^{\mu_3}X_1^\sigma\rangle\langle\partial X_4^{\mu_4}X_1^\rho\rangle\langle\partial X_5^{\mu_5}X_1^\tau\rangle,$$

with the first and sixth gluon vertex operator in the (-1)-ghost picture. Therefore, the integral (A.1) gives rise to a non-vanishing piece in the full amplitude. Since the full amplitude is only comprised by transcendental functions multiplying kinematical factors the contribution (A.1) must be a transcendental function. On the other hand, in (A.6) the second contraction can be obtained from:

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5}\langle\psi_2^{\mu_2}\psi_6^{\mu_6}\rangle\langle\partial X_1^{\mu_1}X_3^{\lambda_3}\rangle\langle\partial X_3^{\mu_3}X_5^{\lambda_5}\rangle\langle\partial X_4^{\mu_4}X_2^{\lambda_2}\rangle\langle\partial X_5^{\mu_5}X_1^{\lambda_1}\rangle,$$

with the second and sixth gluon vertex operator in the (-1)-ghost picture. Furthermore, we may also obtain the second contraction from the contraction involving fermionic correlators:

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5}\langle\psi_2^{\mu_2}\psi_6^{\mu_6}\rangle\langle\psi_1^{\mu_1}\psi_3^{\lambda_3}\rangle\langle\psi_3^{\mu_3}\psi_5^{\lambda_5}\rangle\langle\partial X_4^{\mu_4}X_2^{\lambda_2}\rangle\langle\psi_5^{\mu_5}\psi_1^{\lambda_1}\rangle.$$

In fact, after taking into account the anti-commutation symmetry of fermions the two contractions sum up to zero in the full amplitude (A.6):

$$\langle\partial X_1X_3\rangle\langle\partial X_3X_5\rangle\langle\partial X_5X_1\rangle-\langle\psi_1\psi_3\rangle\langle\psi_3\psi_5\rangle\langle\psi_5\psi_1\rangle=0.$$

Otherwise,[15] the latter would give rise to non-transcendental contributions to the full amplitude (A.6).

To summarize: in order to investigate the transcendentality properties of an Euler integral (3.5) we transform it into the form (3.1) subject to (3.2). If the rational function $\widetilde{R}$ of this integrand involves powers higher than one in the denominator the corresponding integral yields a non-transcendental power series. Otherwise, the rational function (more precisely its limit $z_N\to\infty$ with taking into account the c-ghost factor with the choice (3.3)) is mapped to a gluon contraction of the form[16] $(\xi_r\xi_N)(\xi_ik_j)\cdots(\xi_lk_m)$ arising from an N-gluon superstring computation (A.6)

[13] In the sequel, neither the normalization factors g_A of the gluon vertex operators nor the number of space–time dimensions play any role.

[14] According to Wicks rule the correlator in (A.6) decomposes into products of two-point correlators, given by: $\langle\partial X^\mu(z_1)X^\nu(z_2)\rangle=-\frac{2\alpha'\eta^{\mu\nu}}{z_{12}}$, $\langle\psi^\mu(z_1)\psi^\nu(z_2)\rangle=\frac{\eta^{\mu\nu}}{z_{12}}$.

[15] Alternatively, we could also consider the kinematics $(\xi_2\xi_6)(\xi_1k_5)(\xi_3k_1)(\xi_4k_2)(\xi_5k_3)$. Similar arguments as before would yield: $\langle\partial X_1X_5\rangle\langle\partial X_5X_3\rangle\langle\partial X_3X_1\rangle-\langle\psi_1\psi_5\rangle\langle\psi_5\psi_3\rangle\langle\psi_3\psi_1\rangle=0$.

[16] With no more than one $(\xi\xi)$ scalar product. Otherwise in (3.5) there may be double poles, of which not all disappear by the choice (3.3).

with the r-th and N-th gluon vertex operator in the (-1)-ghost picture. If the contraction under consideration can only be realized by the correlator $\langle\psi_r\psi_N\rangle\langle\partial X_i X_j\rangle\cdots\langle\partial X_l X_m\rangle$ the corresponding integral is transcendental. If on the other hand, the contraction under consideration can also be realized by correlators involving more fermionic contractions, the underlying integral is non-transcendental and the two contributions add up to zero. Hence, in the N-gluon amplitude computation (A.6) non-transcendental contributions referring to a given kinematics $(\xi_r\xi_N)(\xi_i k_j)\cdots(\xi_l k_m)$ are always accompanied by contributions involving a circle of fermionic contractions such, that all contributions add up to zero. Stated differently, integrals describing a kinematics[17] $(\xi_r\xi_N)(\xi_i k_j)\ldots(\xi_l k_m)$, which can be realized by several field contractions, describe non-transcendental functions.

In fact, this criterion rules out the double poles (A.4) to join into a transcendental integral. The latter can be realized by both bosonic and fermionic contractions. E.g. the power $1/z_{ij}^2$ describes the kinematical factor $(\xi_i k_j)(\xi_j k_i)$, which may stem from either $\xi_i^{\mu_i}\xi_j^{\mu_j}k_i^{\lambda_i}k_j^{\lambda_j}\langle\partial X_i^{\mu_i}X_j^{\lambda_j}\rangle\langle\partial X_j^{\mu_j}X_i^{\lambda_i}\rangle$ or from $\xi_i^{\mu_i}\xi_j^{\mu_j}k_i^{\lambda_i}k_j^{\lambda_j}\langle\psi_i^{\mu_i}\psi_j^{\lambda_j}\rangle\langle\psi_j^{\mu_j}\psi_i^{\lambda_i}\rangle$. Both contributions add up to zero:

$$\langle\partial X_i X_j\rangle\langle\partial X_j X_i\rangle-\langle\psi_i\psi_j\rangle\langle\psi_j\psi_i\rangle=0.$$

Note, that kinematics involving the product $(\xi_i\xi_j)$ are realized by both $\xi_i^{\mu_i}\xi_j^{\mu_j}\langle\partial X_i^{\mu_i}\partial X_j^{\mu_j}\rangle$ and $\xi_i^{\mu_i}\xi_j^{\mu_j}\langle\psi_i^{\mu_i}\psi_j^{\mu_j}\rangle k_i^{\lambda_i}k_j^{\lambda_j}\langle\psi_i^{\lambda_i}\psi_j^{\lambda_j}\rangle$ giving rise to $(1-2\alpha' k_i k_j)(\xi_i\xi_j)z_{ij}^{-2}$ in the end. According to (A.4) the non-transcendentality of the double pole integral is then compensated by the $1-s_{ij}$ factor in the numerator. Therefore, kinematics involving more than two pairs of $(\xi_i\xi_j)$ scalar products always involve double powers in the denominator. This is why kinematics with more than two pairs of $(\xi_i\xi_j)$ scalar products cannot provide information on the transcendentality property of the underlying integral. On the other hand, when mapping an integral to the kinematics $(\xi_r\xi_N)(\xi_i k_j)\cdots(\xi_l k_m)$ in (A.6) we put the r-th and N-th gluon vertex operator in the (-1)-ghost picture such that the double pole from the contraction $(\xi_r\xi_N)$ drops.

Let us mention, that the two integrals (3.48) and (3.49) have non-transcendent power series. Indeed our criterion confirms this: In the representation (3.1) the integral (3.48) gives rise to the rational function $\frac{1}{z_{13}z_{15}z_{26}^2z_{34}z_{45}}$ involving a double pole. As a consequence of the latter the α'-expansion in (3.48) is not transcendental. On the other hand, the integral (3.49) leads to the rational function $\frac{z_{13}z_{26}}{z_{12}z_{14}z_{16}z_{23}z_{25}z_{35}z_{36}z_{46}}\to\frac{z_{13}}{z_{12}z_{14}z_{23}z_{25}z_{35}}=\frac{1}{z_{12}z_{14}z_{25}z_{35}}+\frac{1}{z_{14}z_{23}z_{25}z_{35}}$. According to the previous statements the last two fractions correspond to the six-gluon kinematics $(\xi_1\xi_6)(\xi_2k_1)(\xi_3k_5)(\xi_4k_1)(\xi_5k_2)$ and $(\xi_1\xi_6)(\xi_4k_1)\underline{(\xi_2k_3)(\xi_3k_5)(\xi_5k_2)}$, respectively. The underlined part of the last kinematics may also be realized by contracting fermions along a circle. Hence the power series in (3.49) is non-transcendental.

A.3. Transcendentality criterion at work

Let us now apply our criterion for some $N=7$ integral examples. The following integrals can be associated to only one kinematical factor. Therefore, they represent integrals with transcendental power series expansions:

[17] Note, that this statement assumes the r-th and N-th gluon vertex operator in the (-1)-ghost picture to get rid of the double pole from the correlator $\langle e^{-\phi(z_r)}e^{-\phi(z_N)}\rangle\langle\psi_r\psi_N\rangle$.

rational function in Eq. (3.5)	*rational function in Eq.* (3.1)	*kinematics*	*transcendental power series*
$\frac{1}{(1-xy)(1-wz)(1-yz)}$	$\frac{1}{z_{13}z_{16}z_{24}z_{35}z_{46}}$	$(\xi_1\xi_7)(\xi_2k_4)(\xi_3k_1)(\xi_4k_6)(\xi_5k_3)(\xi_6k_1)$	yes,
$\frac{z}{(1-wz)(1-yz)(1-xyz)}$	$\frac{1}{z_{14}z_{16}z_{24}z_{35}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_4)(\xi_3k_6)(\xi_4k_1)(\xi_5k_3)(\xi_6k_1)$	yes,
$\frac{y}{(1-xy)(1-yz)(1-wyz)}$	$\frac{1}{z_{13}z_{16}z_{25}z_{35}z_{46}}$	$(\xi_1\xi_7)(\xi_2k_5)(\xi_3k_1)(\xi_4k_6)(\xi_5k_3)(\xi_6k_1)$	yes,
$\frac{1}{(1-yz)(1-xyz)(1-wxyz)}$	$\frac{1}{z_{13}z_{14}z_{26}z_{35}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_1)(\xi_4k_1)(\xi_5k_3)(\xi_6k_3)$	yes,
$\frac{z}{(1-wz)(1-wyz)(1-xyz)}$	$\frac{1}{z_{14}z_{16}z_{24}z_{25}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_4)(\xi_3k_6)(\xi_4k_1)(\xi_5k_2)(\xi_6k_1)$	yes,
$\frac{yz}{(1-yz)(1-wxyz)}$	$\frac{1}{z_{14}z_{15}z_{16}z_{26}z_{35}}$	$(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_5)(\xi_4k_1)(\xi_5k_1)(\xi_6k_1)$	yes,
$\frac{yz}{(1-wyz)(1-xyz)}$	$\frac{1}{z_{14}z_{15}z_{16}z_{25}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_5)(\xi_3k_6)(\xi_4k_1)(\xi_5k_1)(\xi_6k_1)$	yes,
$\frac{yz}{(1-y)(1-z)(1-wxyz)}$	$\frac{1}{z_{15}z_{16}z_{26}z_{34}z_{45}}$	$(\xi_3\xi_7)(\xi_1k_5)(\xi_2k_6)(\xi_4k_3)(\xi_5k_4)(\xi_6k_1)$	yes,
$\frac{1}{w(1-xy)(1-wz)(1-yz)}$	$\frac{1}{z_{12}z_{16}z_{24}z_{35}z_{46}}$	$(\xi_3\xi_7)(\xi_1k_6)(\xi_2k_1)(\xi_4k_2)(\xi_5k_3)(\xi_6k_4)$	yes,
$\frac{1}{w(1-wz)(1-yz)(1-xyz)}$	$\frac{1}{z_{12}z_{16}z_{24}z_{35}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_1)(\xi_3k_6)(\xi_4k_2)(\xi_5k_3)(\xi_6k_1)$	yes.

(A.8)

Sometimes, before analyzing the integrands a partial fraction decomposition may be useful. E.g. according to (3.7) we have:

$$\frac{1}{(1-xy)(1-xyz)(1-wz)(1-wxyz)} \simeq \frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{27}z_{36}z_{46}z_{57}} \to \frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}}.$$

The partial fraction expansion yields:

$$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}} = \frac{1}{z_{13}z_{14}z_{15}z_{26}z_{46}} + \frac{1}{z_{14}z_{15}z_{26}z_{36}z_{46}}.$$

The two rational functions on the r.h.s. correspond to the two kinematical factors $(\xi_1\xi_7)(\xi_2k_6)\times(\xi_3k_1)(\xi_4k_1)(\xi_5k_1)(\xi_6k_4)$ and $(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_6)(\xi_4k_1)(\xi_5k_1)(\xi_6k_4)$, respectively. Both of them do not allow for additional fermionic contractions. Hence, the integral under consideration yields a transcendental series.

Furthermore, let us discuss some integrals with non-transcendental power series expansions. The rational functions of the following integrals describe kinematics, which can be realized in two ways. The second possibility involves contractions of several pairs of fermions. The latter are contracted along a circle and give rise to the underlined subset of the kinematics:

rational function in Eq. (3.5)	*rational function in Eq.* (3.1)	*kinematics*	*transcendental power series*
$\frac{z}{(1-xy)(1-wz)(1-yz)}$	$\frac{1}{z_{14}z_{16}z_{24}z_{35}z_{46}}$	$(\xi_5\xi_7)\underline{(\xi_1k_6)(\xi_4k_1)(\xi_6k_4)}(\xi_2k_4)(\xi_3k_5)$	no,
$\frac{1}{(1-xy)(1-wz)(1-wxyz)}$	$\frac{1}{z_{13}z_{15}z_{24}z_{26}z_{46}}$	$(\xi_1\xi_7)\underline{(\xi_2k_6)(\xi_6k_4)(\xi_4k_2)}(\xi_3k_1)(\xi_5k_1)$	no,
$\frac{xyz}{(1-xy)(1-wyz)(1-xyz)}$	$\frac{1}{z_{14}z_{16}z_{25}z_{36}z_{46}}$	$(\xi_2\xi_7)\underline{(\xi_1k_6)(\xi_6k_4)(\xi_4k_1)}(\xi_3k_6)(\xi_5k_2)$	no.

(A.9)

Sometimes, before analyzing the integrands a partial fraction decomposition may be useful. E.g. according to (3.7) we have:

$$\frac{z(1-xyz)}{(1-xy)(1-wz)(1-wyz)(1-xyz)} \simeq \frac{z_{26}z_{47}}{z_{14}z_{16}z_{24}z_{25}z_{27}z_{36}z_{37}z_{46}z_{57}} \to \frac{z_{26}}{z_{14}z_{16}z_{24}z_{25}z_{36}z_{46}}.$$

The partial fraction expansion yields:

$$\frac{z_{26}}{z_{14}z_{16}z_{24}z_{25}z_{36}z_{46}} = \frac{1}{z_{14}z_{16}z_{24}z_{25}z_{36}} + \frac{1}{z_{14}z_{16}z_{25}z_{36}z_{46}}.$$

The second term on the r.h.s. corresponds to one of the rational functions discussed in (A.9). Hence, the integral under consideration does not give rise to a transcendental series. An other example is:

$$\frac{1}{(1-yz)(1-wyz)(1-xyz)} \simeq \frac{z_{15}z_{37}}{z_{13}z_{14}z_{16}z_{25}z_{27}z_{35}z_{36}z_{47}z_{57}} \to \frac{z_{15}}{z_{13}z_{14}z_{16}z_{25}z_{35}z_{36}}.$$

The partial fraction expansion yields:

$$\frac{z_{15}}{z_{13}z_{14}z_{16}z_{25}z_{35}z_{36}} = \frac{1}{z_{13}z_{14}z_{16}z_{25}z_{36}} + \frac{1}{z_{14}z_{16}z_{25}z_{35}z_{36}}.$$

The two rational functions on the r.h.s. correspond to the two kinematical factors $(\xi_2\xi_7) \times \underline{(\xi_1 k_6)(\xi_3 k_1)(\xi_6 k_3)}(\xi_4 k_1)(\xi_5 k_2)$ and $(\xi_2\xi_7)(\xi_1 k_6)(\xi_3 k_5)(\xi_4 k_1)(\xi_5 k_2)(\xi_6 k_3)$, respectively. The first kinematics can also be realized by a fermionic contraction along a circle, which is underlined. Hence, the integral under consideration does not give rise to a transcendental series. Finally, the third integral with the integrand

$$\frac{y}{(1-wz)(1-yz)(1-xyz)} \simeq \frac{z_{14}z_{37}}{z_{13}z_{15}z_{16}z_{24}z_{27}z_{35}z_{36}z_{47}^2}$$

yields a non-transcendental power series due to the double pole.

The results (A.9) can be anticipated by explicitly computing the integrals:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{z\, I_7(x,y,z,w)}{(1-xy)(1-wz)(1-yz)} = 2\zeta(2) + 2\zeta(3) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{I_7(x,y,z,w)}{(1-xy)(1-wz)(1-wxyz)}$$
$$= 3\zeta(3) + \left(\frac{19}{4}\zeta(4) - 3\zeta(3)\right)s_7 + \frac{4}{5}\zeta(2)^2\,(s_1+s_6+t_1+t_5) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{xyz\, I_7(x,y,z,w)}{(1-xy)(1-wyz)(1-xyz)} = -2\zeta(2) + 4\zeta(3) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{I_7(x,y,z,w)}{(1-yz)(1-wyz)(1-xyz)} = \frac{5}{2}\zeta(4) + 4\zeta(3) - 2\zeta(2) + \cdots,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, \frac{yI_7(x,y,z,w)}{(1-wz)(1-yz)(1-xyz)} = \frac{3}{2}\zeta(2) + \frac{3}{2}\zeta(3) + \cdots. \tag{A.10}$$

Appendix B. Extended set of multiple hypergeometric functions for $N = 6$

For the six-point case we give here the relations (2.10) to the basis (2.15) of all additional 18 functions (2.18). In (2.36) we have displayed the relation (2.10) for one particular basis π. Here, we want to present the relations (2.10) for two other choices of basis. For the new basis $\pi = \{(1,2,4,5,3,6), (1,4,2,5,3,6), (1,5,4,2,3,6), (1,4,5,2,3,6), (1,5,2,4,3,6), (1,2,5,4,3,6)\}$ we have

$$K_\pi^\sigma = s_{36}^{-1} \times \begin{pmatrix} t_1 - s_1 & s_{13} & 0 & 0 & 0 & t_1 - s_1 + s_3 \\ 0 & 0 & s_3 + s_{13} & s_{13} & t_1 - s_1 + s_3 & 0 \\ \frac{s_1(t_3-s_4)d_{13}}{t_{145}s_{15}} & \frac{(s_{36}-s_1)s_{13}d_{13}}{t_{145}s_{15}} & \frac{-(s_3+s_{13})s_{14}s_{25}}{t_{145}s_{15}} & \frac{-s_{13}s_{14}s_{25}}{s_{145}s_{15}} & \frac{d_8 s_{14}s_{35}}{t_{145}s_{15}} & \frac{s_1 s_{35}d_{13}}{t_{145}s_{15}} \\ \frac{s_1(s_4-t_3)}{t_{145}} & \frac{(s_1-s_{36})s_{13}}{t_{145}} & \frac{(s_3+s_{13})d_5}{s_{145}} & \frac{s_{13}d_5}{t_{145}} & \frac{-(s_1+s_{24})s_{35}}{t_{145}} & \frac{-s_1 s_{35}}{t_{145}} \\ \frac{s_1 s_4(s_1-t_1)}{t_{125}s_{15}} & \frac{-s_1 s_4 s_{13}}{s_{125}s_{15}} & \frac{s_{14}(s_2+s_{35})d_3}{t_{125}s_{15}} & \frac{s_{13}d_3 d_7}{t_{125}s_{15}} & \frac{s_{14}s_{35}d_3}{t_{125}s_{15}} & \frac{s_1(s_4-s_{36})s_{35}}{s_{125}s_{15}} \\ \frac{(t_1-s_1)d_6}{t_{125}} & \frac{s_{13}d_6}{t_{125}} & \frac{-s_{14}(s_2+s_{35})}{t_{125}} & \frac{-d_7 s_{13}}{t_{125}} & \frac{-s_{14}s_{35}}{t_{125}} & \frac{d_1 s_{35}}{t_{125}} \end{pmatrix} \tag{B.1}$$

and the following relation can be checked:

$$\begin{pmatrix} F^{(2453)} \\ F^{(4253)} \\ F^{(5423)} \\ F^{(4523)} \\ F^{(5243)} \\ F^{(2543)} \end{pmatrix} = K^* \begin{pmatrix} F^{(2345)} \\ F^{(3245)} \\ F^{(4325)} \\ F^{(3425)} \\ F^{(4235)} \\ F^{(2435)} \end{pmatrix}. \tag{B.2}$$

On the other hand, for the third basis $\pi = \{(1,3,4,5,2,6), (1,4,3,5,2,6), (1,5,4,3,2,6), (1,4,5,3,2,6), (1,5,3,4,2,6), (1,3,5,4,2,6)\}$ we have

$$K_\pi^\sigma = s_{26}^{-1} \times \begin{pmatrix} s_1 & s_1 + s_2 & 0 & s_1 - s_3 + t_2 & 0 & 0 \\ 0 & 0 & s_1 - s_3 + t_2 & 0 & s_1 + s_{24} & s_1 \\ \frac{s_1(s_{26}-s_{13})d_{13}}{s_{145}s_{15}} & \frac{-d_9 s_{13}d_{13}}{s_{145}s_{15}} & \frac{d_{10}s_{14}s_{25}}{s_{145}s_{15}} & \frac{s_{13}s_{25}d_{13}}{s_{145}s_{15}} & \frac{-s_{14}(s_1+s_{24})s_{35}}{s_{145}s_{15}} & \frac{-s_1 s_{14}s_{35}}{s_{145}s_{15}} \\ \frac{s_1(s_{13}-s_{26})}{s_{145}} & \frac{d_9 s_{13}}{s_{145}} & \frac{-(s_3+s_{13})s_{25}}{s_{145}} & \frac{-s_{13}s_{25}}{s_{145}} & \frac{-d_{12}(s_1+s_{24})}{s_{145}} & \frac{-s_1 d_{12}}{s_{145}} \\ \frac{-s_1 s_4 s_{13}}{s_{246}s_{15}} & \frac{-(s_1+s_2)s_4 s_{13}}{s_{246}s_{15}} & \frac{s_{14}s_{25}d_0}{s_{246}s_{15}} & \frac{s_{13}s_{25}(s_4-s_{26})}{s_{246}s_{15}} & \frac{s_{14}(s_2+s_{25})d_0}{s_{246}s_{15}} & \frac{s_1(s_{26}-s_{14})d_0}{s_{246}s_{15}} \\ \frac{s_1 d_{11}}{s_{246}} & \frac{d_{11}(s_1+s_2)}{s_{246}} & \frac{-s_{14}s_{25}}{s_{246}} & \frac{-(s_3+s_{14})s_{25}}{s_{246}} & \frac{-s_{14}(s_2+s_{25})}{s_{246}} & \frac{s_1(s_{14}-s_{26})}{s_{246}} \end{pmatrix} \tag{B.3}$$

and the following relation can be checked:

$$\begin{pmatrix} F^{(3452)} \\ F^{(4352)} \\ F^{(5432)} \\ F^{(4532)} \\ F^{(5342)} \\ F^{(3542)} \end{pmatrix} = K^* \begin{pmatrix} F^{(2345)} \\ F^{(3245)} \\ F^{(4325)} \\ F^{(3425)} \\ F^{(4235)} \\ F^{(2435)} \end{pmatrix}. \tag{B.4}$$

Hence, the relations (2.36), (B.2) and (B.4) allow to express the additional set of 18 functions (2.32), (2.33) and (2.34) in terms of the minimal basis (2.30).

In the above matrices (B.1) and (B.3) we have introduced $d_5 = s_1 + s_{24} - s_{36}$, $d_6 = -s_1 + s_5 + s_{35}$, $d_7 = s_1 - s_5 + s_{24} - s_{35}$, $d_8 = s_6 - s_4 + s_{13} - s_{24}$, $d_{10} = s_1 - s_3 - s_4 + s_6$, $d_{11} = s_3 + s_{14} - s_{26}$, $d_{12} = s_{26} - s_3 - s_{13}$ and $d_{13} = s_{15} + s_{45}$.

Appendix C. Power series expansions in α' for $N \geqslant 7$

In this appendix we give the α'-expansions (2.17) of the functions F^σ. While for $N = 4, 5, 6$ the latter can be found in Subsection 2.5, here the cases $N \geqslant 7$ are dealt. The strategy how to compute the power series expansion in α' for any generalized Euler integral is described in [17, 20]. Generically, this task amounts to evaluate generalized Euler–Zagier sums involving many integer sums, which becomes quite tedious for $N \geqslant 6$. A complementary approach to determine the α'-expansion for the basis (2.15) can be set up by imposing the factorization properties discussed in Subsection 2.7.

Specifically, in the following we display the first orders of the 24 basis functions (2.15) specifying the $N = 7$ amplitude:

$$\begin{aligned}
F^{(2345)} &= 1 - \zeta(2)(s_5s_6 + s_1s_7 - t_1t_4 - s_5t_5 + t_4t_5 - s_1t_7 + t_1t_7) \\
&\quad + \zeta(3)\big(-2s_1s_3s_5 + s_5^2s_6 + s_5s_6^2 + s_1^2s_7 + s_1s_7^2 + 2s_3s_5t_1 + 2s_4s_5t_1 + 2s_1s_5t_2 \\
&\quad + 2s_1s_5t_3 - 2s_5t_1t_3 + 2s_1s_2t_4 + 2s_1s_3t_4 - 2s_3t_1t_4 - t_1^2t_4 - 2s_1t_2t_4 - t_1t_4^2 \\
&\quad - 2s_4s_5t_5 - s_5^2t_5 + t_4^2t_5 - s_5t_5^2 + t_4t_5^2 - 2s_1s_5t_6 - s_1^2t_7 - 2s_1s_2t_7 + t_1^2t_7 \\
&\quad - s_1t_7^2 + t_1t_7^2\big) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(2354)} &= -\zeta(2)s_{46}(s_4 - s_6 + t_5) + \zeta(3)s_{46}\big(2s_1s_3 + s_4^2 + s_4s_5 - s_5s_6 - s_6^2 - 2s_3t_1 \\
&\quad - 2s_4t_1 - 2s_1t_2 - 2s_1t_3 + 2t_1t_3 + s_4t_4 - s_6t_4 + 2s_4t_5 + s_5t_5 + t_4t_5 + t_5^2 + 2s_1t_6\big) \\
&\quad + \mathcal{O}\big(\alpha'^4\big), \\
F^{(2435)} &= \zeta(2)(s_3 + t_1 - t_5)(s_3 + t_4 - t_7) + \zeta(3)\big(2s_1s_2s_3 + 2s_1s_3^2 - s_3^3 + 2s_3^2s_5 + 2s_3s_4s_5 \\
&\quad - 2s_3^2t_1 + 2s_3s_5t_1 + 2s_4s_5t_1 - s_3t_1^2 - 2s_1s_3t_2 - 2s_3s_5t_3 - 2s_5t_1t_3 + 2s_1s_2t_4 \\
&\quad + 2s_1s_3t_4 - 2s_3^2t_4 - 3s_3t_1t_4 - t_1^2t_4 - 2s_1t_2t_4 - s_3t_4^2 - t_1t_4^2 - 2s_3s_5t_5 - 2s_4s_5t_5 \\
&\quad + 2s_5t_3t_5 + s_3t_4t_5 + t_4^2t_5 + s_3t_5^2 + t_4t_5^2 - 2s_1s_2t_7 - 2s_1s_3t_7 + s_3t_1t_7 + t_1^2t_7 \\
&\quad + 2s_1t_2t_7 + s_3t_5t_7 - t_5^2t_7 + s_3t_7^2 + t_1t_7^2 - t_5t_7^2\big) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(2453)} &= -\zeta(2)s_{36}(s_3 + t_1 - t_5) + \zeta(3)s_{36}\big(-2s_1s_2 - 2s_1s_3 - s_3^2 - 2s_3s_4 - 2s_4t_1 + t_1^2 \\
&\quad + 2s_1t_2 + 2s_3t_3 + 2t_1t_3 + s_3t_4 + t_1t_4 + 2s_3t_5 + 2s_4t_5 - 2t_3t_5 - t_4t_5 - t_5^2 + s_3t_7 \\
&\quad + t_1t_7 - t_5t_7\big) + \mathcal{O}\big(\alpha'^4\big), \\
F^{(2534)} &= \zeta(2)s_{46}(s_3 + s_6 - t_3 - t_5) + \zeta(3)s_{46}\big(2s_1s_3 + 2s_3^2 + s_3s_4 - s_3s_5 + s_3s_6 + s_4s_6 \\
&\quad - s_5s_6 - s_6^2 - 2s_1t_2 - 2s_1t_3 - 4s_3t_3 - s_4t_3 + s_5t_3 - s_6t_3 + 2t_3^2 - s_3t_4 - s_6t_4 \\
&\quad + t_3t_4 - 3s_3t_5 - s_4t_5 + s_5t_5 + 3t_3t_5 + t_4t_5 + t_5^2 + 2s_1t_6\big) + \mathcal{O}\big(\alpha'^4\big),
\end{aligned}$$

$$
\begin{aligned}
F^{(2543)} &= -\zeta(2)s_{36}(s_3+s_6-t_3-t_5)+\zeta(3)s_{36}\big(-2s_1s_3-s_3^2-s_3s_4-s_4s_6+s_6^2\\
&\quad+2s_1t_2+2s_1t_3+2s_3t_3+s_4t_3-t_3^2+s_3t_4+s_6t_4-t_3t_4+2s_3t_5+s_4t_5-2t_3t_5\\
&\quad-t_4t_5-t_5^2-2s_1t_6+s_3t_7+s_6t_7-t_3t_7-t_5t_7\big)+\mathcal{O}\big(\alpha'^4\big),\\
F^{(3245)} &= -\zeta(2)s_{13}(s_2-s_7+t_7)+\zeta(3)s_{13}\big(s_1s_2+s_2^2+2s_3s_5-s_1s_7-s_7^2+s_2t_1-s_7t_1\\
&\quad-2s_5t_2-2s_5t_3-2s_2t_4-2s_3t_4+2t_2t_4+2s_5t_6+s_1t_7+2s_2t_7+t_1t_7+t_7^2\big)\\
&\quad+\mathcal{O}\big(\alpha'^4\big),\\
F^{(3254)} &= -2\zeta(3)s_{13}s_{25}s_{46}+\mathcal{O}\big(\alpha'^4\big),\\
F^{(3425)} &= \zeta(2)s_{13}(s_3+s_7-t_2-t_7)+\zeta(3)s_{13}\big(-s_1s_3+s_2s_3+2s_3^2+2s_3s_5-s_1s_7+s_2s_7\\
&\quad+s_3s_7-s_7^2-s_3t_1-s_7t_1+s_1t_2-s_2t_2-4s_3t_2-2s_5t_2-s_7t_2+t_1t_2+2t_2^2\\
&\quad-2s_5t_3+2s_5t_6+s_1t_7-s_2t_7-3s_3t_7+t_1t_7+3t_2t_7+t_7^2\big)+\mathcal{O}\big(\alpha'^4\big),\\
F^{(3452)} &= \zeta(2)s_{13}s_{26}+\zeta(3)s_{13}s_{26}(-s_1+s_2-s_7-t_1+2t_3-2t_6-t_7)+\mathcal{O}\big(\alpha'^4\big),\\
F^{(3524)} &= \frac{1}{4}\zeta(4)s_{13}s_{46}(10s_{15}s_{24}+3s_{15}s_{26}+27s_{24}s_{35}+10s_{26}s_{35})+\mathcal{O}\big(\alpha'^5\big),\\
F^{(3542)} &= \frac{1}{4}\zeta(4)s_{13}s_{26}(-7s_{15}s_{24}-17s_{24}s_{35}+3s_{15}s_{46}+10s_{35}s_{46})+\mathcal{O}\big(\alpha'^5\big),\\
F^{(4235)} &= -\zeta(2)s_{14}(s_3+t_4-t_7)\\
&\quad+\zeta(3)s_{14}\big(-2s_2s_3-s_3^2-2s_3s_5-2s_4s_5+s_3t_1+2s_3t_2+2s_5t_3-2s_2t_4+t_1t_4\\
&\quad+2t_2t_4+t_4^2+s_3t_5+t_4t_5+2s_2t_7+2s_3t_7-t_1t_7-2t_2t_7-t_5t_7-t_7^2\big)+\mathcal{O}\big(\alpha'^4\big),\\
F^{(4253)} &= \zeta(2)s_{14}s_{36}+\zeta(3)s_{14}s_{36}(2s_2+3s_3+2s_4-t_1-2t_2-2t_3-t_4-t_5-t_7)\\
&\quad+\mathcal{O}\big(\alpha'^4\big),\\
F^{(4325)} &= -\zeta(2)s_{14}(s_3+s_7-t_2-t_7)+\zeta(3)s_{14}\big(-s_2s_3-s_3^2-2s_3s_5-s_2s_7+s_7^2+s_3t_1\\
&\quad+s_7t_1+s_2t_2+2s_3t_2+2s_5t_2-t_1t_2-t_2^2+2s_5t_3+s_3t_5+s_7t_5-t_2t_5-2s_5t_6\\
&\quad+s_2t_7+2s_3t_7-t_1t_7-2t_2t_7-t_5t_7-t_7^2\big)+\mathcal{O}\big(\alpha'^4\big),\\
F^{(4352)} &= -\zeta(2)s_{14}s_{26}+\zeta(3)s_{14}s_{26}(-s_2+s_3+s_7+t_1-t_2-2t_3+t_5+2t_6+t_7)\\
&\quad+\mathcal{O}\big(\alpha'^4\big),\\
F^{(4523)} &= \zeta(2)s_{14}s_{36}+\zeta(3)s_{14}s_{36}(2s_2+2s_4-t_1+t_2+t_3-t_4-t_5-3t_6-t_7)\\
&\quad+\mathcal{O}\big(\alpha'^4\big),\\
F^{(4532)} &= -\zeta(2)s_{14}s_{26}+\zeta(3)s_{14}s_{26}(-s_2-s_4+s_7+t_1-t_2-t_3+t_5+2t_6+t_7)\\
&\quad+\mathcal{O}\big(\alpha'^4\big),\\
F^{(5234)} &= \zeta(2)s_{15}s_{46}+\zeta(3)s_{15}s_{46}(s_4-s_5-s_6+2t_2-t_4-t_5-2t_6)+\mathcal{O}\big(\alpha'^4\big),\\
F^{(5243)} &= -\zeta(2)s_{15}s_{36}+\zeta(3)s_{15}s_{36}(s_3-s_4+s_6-2t_2-t_3+t_4+t_5+2t_6+t_7)\\
&\quad+\mathcal{O}\big(\alpha'^4\big),\\
F^{(5324)} &= \frac{1}{4}\zeta(4)s_{15}s_{46}(10s_{13}s_{24}+3s_{13}s_{26}-17s_{24}s_{35}-7s_{26}s_{35})+\mathcal{O}\big(\alpha'^5\big),\\
F^{(5342)} &= \frac{1}{4}\zeta(4)s_{15}s_{26}(-7s_{13}s_{24}+3s_{13}s_{46}+10s_{24}s_{35}-7s_{35}s_{46})+\mathcal{O}\big(\alpha'^5\big),
\end{aligned}
$$

$$F^{(5423)} = -\zeta(2)s_{15}s_{36} + \zeta(3)s_{15}s_{36}(-s_2 - s_4 + s_6 - t_2 - t_3 + t_4 + t_5 + 2t_6 + t_7) + \mathcal{O}(\alpha'^4),$$

$$F^{(5432)} = \zeta(2)s_{15}s_{26} + \zeta(3)s_{15}s_{26}(-s_6 - s_7 + t_2 + t_3 - t_5 - t_6 - t_7) + \mathcal{O}(\alpha'^4). \tag{C.1}$$

As anticipated after Eq. (2.17) there is one function starting only at $\zeta(3)\alpha'^3$ and a set of four functions starting not until at $\zeta(4)\alpha'^4$.

We also have the expressions for $N \geqslant 8$. However, it is too elaborate to present all expansions for $\geqslant 120$ basis functions (2.15). At any rate in [40] a detailed survey of the structure of the α'-expansions (2.17) is undertaken.

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